Exercises Chapter III.

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1. Let $L(x_1, x_2, x_3) = x_1 - x_2 + x_3$.

a. Show that L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}$.

$$\begin{aligned} L(\boldsymbol{x}+\boldsymbol{y}) &= L(x_1+y_1, x_2+y_2, x_3+y_3) = (x_1+y_1) - (x_2+y_2) + (x_3+y_3) \\ &= (x_1 - x_2 + x_3) + (y_1 - y_2 + y_3) \\ &= L(\boldsymbol{x}) + L(\boldsymbol{y}) \end{aligned}$$

$$\begin{aligned} L(\alpha\boldsymbol{x}) &= L(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_1) - (\alpha x_2) + (\alpha x_3) \\ &= \alpha(x_1 - x_2 - x_3) \\ &= \alpha L(\boldsymbol{x}) \end{aligned}$$

b. Find a 1×3 matrix A such that $L(\boldsymbol{x}) = A\boldsymbol{x}^T$ for every $\boldsymbol{x}$ in $\mathbb{R}^3$.

$$A = [1\ {-1}\ \ 1]$$

c. Compute $L(\boldsymbol{e}_k)$ for $k = 1, 2, 3$.

$$L(\boldsymbol{e}_1) = 1, \quad L(\boldsymbol{e}_2) = -1, \quad L(\boldsymbol{e}_3) = 1$$

d. Find a basis for the subspace $K = \{\boldsymbol{x} \colon\ A\boldsymbol{x}^T = 0\}$.

$$\{(1,1,0),\ (-1,0,1)\}$$

2. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that $L(\boldsymbol{e}_1) = (-1, 6)$, $L(\boldsymbol{e}_2) = (0, 2)$, $L(\boldsymbol{e}_3) = (8, 1)$.

a. $L(1, 2, -6) = ?$

$$\begin{aligned} L(1,2,-6) &= L(\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 6\boldsymbol{e}_3) = L(\boldsymbol{e}_1) + 2L(\boldsymbol{e}_2) - 6L(\boldsymbol{e}_3) \\ &= (-1,6) + 2(0,2) - 6(8,1) = (-49, 4) \end{aligned}$$

b. $L(x_1, x_2, x_3) = ?$

$$\begin{aligned} L(x_1,x_2,x_3) &= L(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = x_1L(\boldsymbol{e}_1) + x_2L(\boldsymbol{e}_2) + x_3L(\boldsymbol{e}_3) \\ &= x_1(-1,6) + x_2(0,2) + x_3(8,1) = (-x_1 + 8x_3, 6x_1 + 2x_2 + x_3) \end{aligned}$$

c. Find a matrix A such that $L(\boldsymbol{x}) = A\boldsymbol{x}^T$.

$$A = \begin{bmatrix} -1 & 0 & 8 \\ 6 & 2 & 1 \end{bmatrix}$$

4. For each of the following functions f determine an appropriate V and W. Then decide if f is a linear transformation from V to W.

a. $f(x_1, x_2) = (x_1, 0, 1)$

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{not linear} \quad f(2,0) \neq 2f(1,0)$$

b. $f(x_1, x_2) = (x_1 - x_2, x_1)$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{linear}$$

c. $f(x) = (x, x)$

$$f : \mathbb{R} \to \mathbb{R}^2 \quad \text{linear}$$

d. $f(x_1, x_2, x_3) = (x_1, x_2, x_2, x_3, x_3, x_1)$

$$f : \mathbb{R}^3 \to \mathbb{R}^6 \quad \text{linear}$$

10. Let $S = cI_2$, be an arbitrary 2×2 scalar matrix. Describe the geometrical effect that the linear transformation $S\boldsymbol{x}^T$ has on $\mathbb{R}^2$.

If $c > 1$, then S stretches each direction by a factor of c. If $0 < c < 1$, then S contracts every direction by the factor c. If $c < 0$, then besides the scaling done by $|c|$, vectors are also flipped through the origin. The last two cases $c = 0$ and $c = 1$ are obvious.

16. Let $V = C[0, 1]$.

a. Let $L\colon\ V \to V$ be defined by $L[\boldsymbol{f}](x) = \boldsymbol{f}(x)\sin x$. Is L a linear transformation?

L is a linear transformation.

b. Let $L\colon\ V \to V$ be defined by $L[\boldsymbol{f}](x) = \sin x + \boldsymbol{f}(x)$. Is L a linear transformation?

L is not a linear transformation.

1. Let $L(x_1, x_2) = (3x_1 + 6x_2, -2x_1 + x_2)$

a. Find the matrix representation of L using the standard bases.

$$\begin{aligned} L(\boldsymbol{e}_1) &= (3, -2) = 3\boldsymbol{e}_1 - 2\boldsymbol{e}_2 \quad L(\boldsymbol{e}_2) = (6, 1) = 6\boldsymbol{e}_1 + \boldsymbol{e}_2 \\ A &= \begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

b. Find the matrix representation of L using the basis $F = \{(-4, 1),\ (2, 3)\}$.

$$\begin{aligned} L(\boldsymbol{f}_1) &= L(-4, 1) = (-6, 9) = \frac{18}{7}\boldsymbol{f}_1 + \frac{15}{7}\boldsymbol{f}_2 \\ L(\boldsymbol{f}_2) &= L(2, 3) = (24, -1) = \frac{-37}{7}\boldsymbol{f}_1 + \frac{10}{7}\boldsymbol{f}_2 \\ A &= \begin{bmatrix} \frac{18}{7} & \frac{-37}{7} \\ \frac{15}{7} & \frac{10}{7} \end{bmatrix} \end{aligned}$$

4. Let $V = \mathbb{R}^3$ and let $F = \{(1, 2, -3), (1, 0, 0), (0, 1, 0)\}$. Suppose that the matrix A represents a linear transformation $L\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the basis F, where

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

a. $L(1, 2, -3) =?$ b. $L(1, 0, 0) =?$ c. $L(x_1, x_2, x_3) =?$

a.

$$L(1, 2, -3) = L(\boldsymbol{f}_1) = 2\boldsymbol{f}_2 + 5\boldsymbol{f}_3 = (2, 5, 0)$$

b.

$$L(1, 0, 0) = L(\boldsymbol{f}_2) = \boldsymbol{f}_1 + \boldsymbol{f}_2 = (2, 2, -3)$$

c.

$$
\begin{aligned}
L(x_1,x_2,x_3) &= L\left((-\frac{x_3}{3})\boldsymbol{f}_1 + (x_2+\frac{x_3}{3})\boldsymbol{f}_2 + (x_1+\frac{2x_3}{3})\boldsymbol{f}_3\right)\\
&= (-\frac{x_3}{3})L(\boldsymbol{f}_1) + (x_1+\frac{x_3}{3})L(\boldsymbol{f}_2) + (x_2+\frac{2x_3}{3})L(\boldsymbol{f}_3)\\
&= (-\frac{x_3}{3})(2,5,0) + (x_1+\frac{x_3}{3})(2,2,-3) + (x_2+\frac{2x_3}{3})(-2,-3,6)\\
&= \left(2x_1-2x_2-\frac{4x_3}{3}, 2x_1-3x_2-3x_3, -3x_1+6x_2+3x_3\right).
\end{aligned}
$$

10. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. Let $F=\{(1,1,1),(0,1,1),\ (1,1,0)\}=\{\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3\}$. Let $G=\{(1,2),(2,3)\}=\{\boldsymbol{g}_1,\boldsymbol{g}_2\}$. Suppose that the matrix representation of L with respect to these bases is $\begin{bmatrix}2 & 0 & 4\\ 1 & -2 & 1\end{bmatrix}$.

a. For $k=1,2,3$, $[L(\boldsymbol{f}_k)]_G=?$

$$
[L(\boldsymbol{f}_1)]_G=\begin{bmatrix}2\\1\end{bmatrix},\quad [L(\boldsymbol{f}_2)]_G=\begin{bmatrix}0\\-2\end{bmatrix},\quad [L(\boldsymbol{f}_3)]_G=\begin{bmatrix}4\\1\end{bmatrix}
$$

b. Compute $L(\boldsymbol{f}_k)$ for $k=1,2,3$.

$$
\begin{aligned}
[L(\boldsymbol{f}_1)] &= 2\boldsymbol{g}_1+\boldsymbol{g}_2 = 2(1,2)+(2,3)=(4,7)\\
[L(\boldsymbol{f}_2)] &= -2\boldsymbol{g}_2 = -2(2,3)=(-4,-6)\\
[L(\boldsymbol{f}_3)] &= 4\boldsymbol{g}_1+\boldsymbol{g}_2 = 4(1,2)+(2,3)=(6,11)
\end{aligned}
$$

c. Find the matrix representation of L using the standard basis S in $\mathbb{R}^3$ and the basis G in $\mathbb{R}^2$.

Let P_F be the change of basis matrix ($\mathbb{R}^3$) such that $[\boldsymbol{x}]_S=P_F[\boldsymbol{x}]_F$. Then we have

$$
\begin{aligned}
[L(\boldsymbol{x})]_G &= \begin{bmatrix}2 & 0 & 4\\ 1 & -2 & 1\end{bmatrix}[\boldsymbol{x}]_F = \begin{bmatrix}2 & 0 & 4\\ 1 & -2 & 1\end{bmatrix}P_F^{-1}[\boldsymbol{x}]_S\\
&= \begin{bmatrix}2 & 0 & 4\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 0\\ 0 & 1 & -1\end{bmatrix}[\boldsymbol{x}]_S
\end{aligned}
$$

Thus, the matrix representation of L is

$$
\begin{bmatrix}2 & 0 & 4\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 0\\ 0 & 1 & -1\end{bmatrix}=\begin{bmatrix}2 & 2 & -2\\ 3 & -2 & 0\end{bmatrix}.
$$

d. Find the matrix representation of L using the standard basis S in both $\mathbb{R}^3$ and $\mathbb{R}^2$. Let P_G be the change of basis matrix ($\mathbb{R}^2$) such that $[\boldsymbol{x}]_S = P_G[\boldsymbol{x}]_G$. Then we have

$$[L(\boldsymbol{x})]_S = P_G\,[L(\boldsymbol{x})]_G = P_G \begin{bmatrix} 2 & 0 & 4 \\ 1 & -2 & 1 \end{bmatrix} [\boldsymbol{x}]_F = P_G \begin{bmatrix} 2 & 0 & 4 \\ 1 & -2 & 1 \end{bmatrix} P_F^{-1}[\boldsymbol{x}]_S, .$$

Thus, the matrix representation equals

$$P_G \begin{bmatrix} 2 & 0 & 4 \\ 1 & -2 & 1 \end{bmatrix} P_F^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -2 & -2 \\ 13 & -2 & -4 \end{bmatrix}$$

11. Let $L\colon\ \mathbb{R}^2 \to \mathbb{R}^2$. Let $F = \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ be a basis of $\mathbb{R}^2$. Suppose that

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

is the matrix representation of L with respect to the basis F. What is $L(\boldsymbol{f}_k)$ for $k = 1, 2$?

$$L[\boldsymbol{f}_1] = -2\boldsymbol{f}_1\,, \qquad L(\boldsymbol{f}_2] = 3\boldsymbol{f}_2\,.$$

15 Define $L[\boldsymbol{p}](t) = \int_0^t \boldsymbol{p}(s)ds$, for each t in [0,1]. Then $L\colon\ P_n \to P_{n+1}$. Find a matrix representation for L using the standard bases.

The standard basis for P_n is $\{1, t, \cdots, t^n\}$, and $L(t^i) = (t^{i+1})/(i+1)$. The matrix representation A is an $(n+2) \times (n+1)$ matrix, and has the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1/3 & \cdots & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & & 1/(n+1) \end{bmatrix}$$

1. Each of the matrices below represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Determine the values of n and m for each matrix. Then determine their kernels and ranges and find a basis for each of these subspaces.

a. $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

a. $\mathbb{R}^3$ to $\mathbb{R}^1$. Range is all of $\mathbb{R}^1$, and

$\ker = \{(x_1, x_2, x_3) : x_1 + x_3 = 0\} = \mathrm{Sp}[(0,1,0), (-1,0,1)]$.

b. $\mathbb{R}^1$ to $\mathbb{R}^3$. Range $= \mathrm{Sp}[(1,0,1)]$, and the kernel is the zero vector.

c. $\mathbb{R}^2$ to $\mathbb{R}^2$. Range is all of $\mathbb{R}^2$, and the kernel is the zero vector.

d. $\mathbb{R}^3$ to $\mathbb{R}^4$. Range is the span of the columns of the matrix, and the kernel is the zero vector.

3. Let A be the coefficient matrix of the system of equations below. If L is the linear transformation defined by A, what is the range of L and what is its kernel? Does this particular equation have a solution; i.e., is $(-2, 1)$ in the range of L?

$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 &= -2 \\ x_1 + 3x_2 \quad\quad &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -5 & 3 \\ 1 & 3 & 0 \end{bmatrix} \text{ is row equialent to } \begin{bmatrix} 1 & 0 & 9/11 \\ 0 & 1 & -3/11 \end{bmatrix}$$

The kernel equals $\mathrm{Sp}[(-9/11, 3/11, 1)]$. This subspace has dimension 1. Thus, the range of L has dimension $3-1 = 2$, which implies that the range of L is all of $\mathbb{R}^2$, which means the equation has a solution.

4. For each of the matrices below determine the dimension of its range and the dimension of its kernel. Then decide if the linear transformations represented by these matrices are onto and/or one-to-one.

a. $\begin{bmatrix} 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a. This represents a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^1$. It has a non-trivial kernel of dimension 1, which means its range also has dimension 1. Thus, the transformation is not one-to-one, but it is onto.

b. This represents a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. It's kernel is just the zero vector, so the transformation is one-to-one, but it is not onto as its range has dimension 2, and cannot fill up all of $\mathbb{R}^3$.

c. This represents a linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^2$. It's kernel is just the zero vector, so the transformation is one-to-one, but it is not onto as its range has dimension 1, and cannot fill up all of $\mathbb{R}^2$

1. Calculate the rank of each of the following matrices:

a. $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

a. rank = 1 b. rank = 1 c. rank = 2 d. rank = 3.

4. For each matrix below, determine the dimensions of the range and kernel. Then decide if the linear transformation it represents is onto and/or one-to-one.

a. $\begin{bmatrix} 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

a. The matrix has rank = 1, and is 1×2. Thus, the linear transformation maps $\mathbb{R}^2$ into $\mathbb{R}^1$. Since the dimension of the range is one, the map is onto. The dimension of the kernel is $2 - 1 = 1$, which means the transformation is not one-to-one.

b. The transformation maps $\mathbb{R}^2$ into $\mathbb{R}^3$, the matrix has rank = 2, so the transformation is not onto, and the kernel has dimension $2 - 2 = 0$, so the mapping is one-to-one.

c The transformation maps $\mathbb{R}^1$ into $\mathbb{R}^2$, the matrix has rank = 1, so the transformation is not onto, and the kernel has dimension $1 - 1 = 0$, so the mapping is one-to-one.

9. Consider the following system of linear equations:

$$\begin{aligned} 4x_1 \qquad\quad + 2x_3 + \ x_4 &= 0 \\ 2x_1 - \ x_2 + \ x_3 + 3x_4 &= 1 \\ -8x_1 - 2x_2 - 4x_3 + 3x_4 &= 2 \end{aligned}$$

Let A be the coefficient matrix of this system.

a. Compute the rank of A.

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & -1 & 1 & 3 \\ -8 & -2 & -4 & 3 \end{bmatrix} \quad \text{and it is row equivalent to } \hat{A} = \begin{bmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, A has rank equal to 2.

b. $\dim(\ker(A)) =?$ Find a basis for $\ker(A)$.

$\dim \ker(A) = 4 - 2$. x_3 and x_4 are free variables. Hence a basis for the kernel of A is

$$\{(-1/2, 0, 1, 0),\ (-1/4, 5/2, 0, 1)\}$$

c. $\dim(\mathrm{Rg}(A)) =?$ Find a basis for $\mathrm{Rg}(A)$.

$\dim(\mathrm{Rg}(A)) =$ rank of A. Since the first two columns of $\hat{A}$ form a basis for its column space, the first two columns of A form a basis for the column space of A. Thus, a basis for the range of A is:

$$\left\{(4, 2, -8)^T,\ (0, -1, -2)^T\right\}\ .$$

d. Is A a one-to-one linear transformation?

A is not one-to-one as its kernel (null space) contains non-zero vectors.

e. Is A onto?

A is not onto as the dimension of its codomain is 3, but the dimension of its range is 2.

f. Does the above system of equations have a solution? If yes, characterize the solution set in terms of the kernel of A and a particular solution.

This system does have a solution. The augmented matrix, which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1/2 & 1/4 & 0 \\ 0 & 1 & 0 & -5/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has the same rank as A. Thus, we know there is a solution, and the general solution has the form:

$$\boldsymbol{x} = (0, -1, 0, 0) + c_1(-1/2, 0, 1, 0) + c_2(-1/4, 5/2, 0, 1)$$

16. For each of the following matrices A compute the rank of A, A^T, AA^T, and A^TA. You should get rank A^TA equals rank A and rank AA^T equals rank A^T. Hence all four numbers are equal.

a. $A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$. Rank of A equals 1, which is the same as the rank of A^T.

$$AA^T = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix} \text{ and } A^TA = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Both of these matrices have rank 1.

b. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Rank of A is 2, as is the rank of A^T.

$$AA^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix} \text{ and } A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 13 \end{bmatrix}.$$

It's clear that AA^T has rank 2. A little work is needed to verify that AA^T also has rank 2.

c. $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$. A and A^T have rank 2.

$$AA^T = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 13 \end{bmatrix} \text{ and } A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 10 \end{bmatrix}.$$

Both AA^T and A^TA have rank equal to 2.

3. Let $L\colon\ \mathbb{R}^3 \to \mathbb{R}^3$. Let $F = \{(1,2,-3),(1,0,0),(0,1,0)\}$. Suppose

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

is the matrix representation of L with respect to the basis F. Use (3.12) to find the matrix representation of L with respect to the standard basis; cf. problem 4 of Section 3.2.

Set $P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix}$. Then P is the change of basis matrix that satisfies the equation $[\boldsymbol{x}]_S = P[\boldsymbol{x}]_F$. Thus, the matrix representation, B, of L with respect to the standard basis is

$$B = PAP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/3 \\ 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4/3 \\ 2 & -3 & -3 \\ -3 & 6 & 3 \end{bmatrix}.$$

4. Suppose $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ is the matrix representation of a linear transformation with respect to the standard basis. Let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and set $B = PAP^{-1}$. Then B can be thought of as the matrix representation of L with respect to some other basis. What is this basis?

The matrix P can be thought of as a change of basis matrix between the standard basis and the basis $F = \{\boldsymbol{e}_2,\ \boldsymbol{e}_1\}$. That is, $[\boldsymbol{x}]_F = P[\boldsymbol{x}]_S$. Thus, we can think of PAP^{-1} as the matrix representation of the linear transformation with respect to the basis F.

7. Let $F = \{(1,2),(1,0)\}$ and $G = \{(1,-1),(0,1)\}$. Let $L(x_1,x_2) = (x_1+x_2, 2x_1-x_2)$. Find the matrix representations of L with respect to bases F and G. Verify (3.12).

$L[(1,2)] = 3(1,0),\ L([(1,0)] = (1,2)$. representation with respect to $F = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$L[(1,-1)] = 3(0,1),\ L[(0,1)] = (1,-1)$. representation with respect to $G = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

The matrix $P = \begin{bmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix}$, satisfies $[\boldsymbol{x}]_F = P[\boldsymbol{x}]_G$, and

$$P \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}.$$

12. Let $L\colon\ P_2 \to P_2$ be a linear transformation. Let $F = \{t^2+t-1, t^2+2, t-6\}$. Suppose the matrix representation of L with respect to F is

$$A = \begin{bmatrix} -14 & -2 & -18 \\ 23 & 11 & 18 \\ 11 & 2 & 15 \end{bmatrix}$$

Find the matrix representation of L with respect to the standard basis of P_2.

Let $P = \begin{bmatrix} -1 & 2 & -6 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Then P is the change of basis matrix between the basis F and the standard basis $\{1, t, t^2\}$. That is, $[\boldsymbol{x}]_S = P[\boldsymbol{x}]_F$. Thus, the matrix representation of L with respect to the standard basis is

$$\begin{aligned} PAP^{-1} &= \begin{bmatrix} -1 & 2 & -6 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -14 & -2 & -18 \\ 23 & 11 & 18 \\ 11 & 2 & 15 \end{bmatrix} \begin{bmatrix} 1/3 & 2 & -2/3 \\ -1/3 & -2 & 5/3 \\ -1/3 & -1 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \end{aligned}$$

2. Let $A = \begin{bmatrix} 2 & 6 \\ 1 & -2 \\ 3 & 5 \end{bmatrix}$ be the matrix representation of a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^P$.

a. $k = ?p = ?$

L is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

b. Determine the dimensions of $\ker(A)$ and $\mathrm{Rg}(A)$.

Since the matrix A has rank equal to 2, we know the dimension of the range of A is 2, and since the dimension of the domain of A is also 2, the dimension of its kernel is 0.

6. Describe, geometrically, the following linear transformations, and in each case determine the kernel and range:

a. $L(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$.

The matrix representation of this linear transformation with respect to the standard basis is:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The matrix is a rotation of -45^o, i.e., in a clockwise direction, and the scalar factor $\sqrt{2}$ is a stretching. So any vector is rotated 45 degrees clockwise, and then lengthened by a factor of $\sqrt{2}$. A has rank equal to 2. Thus, the range of A has dimension 2 and is all of $\mathbb{R}^2$. The dimension of its kernel is 0, so the kernel of A is just the zero vector space.

b. $L(x_1, x_2) = (2x_1 - x_2, 4x_1 - 2x_2)$

$$L(x_1, x_2) = (2x_1 - x_2, 4x_1 - 2x_2) = (2x_1 - x_2)(1, 2).$$

Thus, the linear transformation L maps all of $\mathbb{R}^2$ onto the span of the vector $(1, 2)$, which has dimension 1. The kernel of L is that subspace of $\mathbb{R}^2$ for which $2x_1 - x_2 = 0$.

11. Let $V = \{\sum_{k=1}^{n} a_k \sin(kx)e^{-k^2t}, n$ any positive integer and a_k arbitrary numbers$\}$.

a. Under ordinary addition and multiplication show that V is a vector space. Define $L\colon\ V \to V$ by $L[u] = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}$.

Suppose that $\boldsymbol{x} = \sum_{k=1}^{n} a_k \sin(kx)e^{-k^2t}$, and $\boldsymbol{y} = \sum_{k=1}^{n} b_k \sin(kx)e^{-k^2t}$. Then

$$\boldsymbol{x} + \boldsymbol{y} = \sum_{k=1}^{n} (a_k + b_k) \sin(kx)e^{-k^2t} \in V \text{ and } \alpha\boldsymbol{x} = \sum_{k=1}^{n} (\alpha a_k) \sin(kx)e^{-k^2t} \in V$$

Thus, V, which is not empty, is closed under vector addition and scalar multiplication, and therefore a vector space.

b. Show L is a linear transformation.

Since taking derivatives is a linear operation the transformation L is linear. The fact that it maps V into V follows from the fact that for every integer k we have

$$\left[\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right]\left(a_k \sin(kx)e^{-k^2t}\right) = 0,$$

which is in V. So L must map sums of such terms into V.

c. Find the kernel and range of L.

The above equation tells us that the range of L is the zero subspace of V, and the kernel is all of V.