“Abstract” Vector Spaces

Math 152

Lecture 7

Algebraic Properties of $\mathbb{R}^n$ and Subspaces of $\mathbb{R}^n$

There are three important "algebraic structures" on $\mathbb{R}^n$ (and any subspace $W \subset \mathbb{R}^n$) that allow us to study linear algebra on $\mathbb{R}^n$ (or on W). Namely:

- **Addition**: For any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we can add them $\vec{v} + \vec{w}$ to obtain another element of $\mathbb{R}^n$.

Algebraic Properties of $\mathbb{R}^n$ and Subspaces of $\mathbb{R}^n$

There are three important "algebraic structures" on $\mathbb{R}^n$ (and any subspace $W \subset \mathbb{R}^n$) that allow us to study linear algebra on $\mathbb{R}^n$ (or on W). Namely:

- Addition: For any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we can add them $\vec{v} + \vec{w}$ to obtain another element of $\mathbb{R}^n$.
- **Scaling**: For any vector $\vec{v} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$, I can multiply $c \cdot \vec{v}$ to obtain another element of $\mathbb{R}^n$.

Algebraic Properties of $\mathbb{R}^n$ and Subspaces of $\mathbb{R}^n$

There are three important "algebraic structures" on $\mathbb{R}^n$ (and any subspace $W \subset \mathbb{R}^n$) that allow us to study linear algebra on $\mathbb{R}^n$ (or on W). Namely:

- Addition: For any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we can add them $\vec{v} + \vec{w}$ to obtain another element of $\mathbb{R}^n$.
- Scaling: For any vector $\vec{v} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$, I can multiply $c \cdot \vec{v}$ to obtain another element of $\mathbb{R}^n$.
- **Zero Element**: There is a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{0} + \vec{v} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$.

Algebraic Properties of $\mathbb{R}^n$ and Subspaces of $\mathbb{R}^n$

There are three important "algebraic structures" on $\mathbb{R}^n$ (and any subspace $W \subset \mathbb{R}^n$) that allow us to study linear algebra on $\mathbb{R}^n$ (or on W). Namely:

- Addition: For any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we can add them $\vec{v} + \vec{w}$ to obtain another element of $\mathbb{R}^n$.
- Scaling: For any vector $\vec{v} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$, I can multiply $c \cdot \vec{v}$ to obtain another element of $\mathbb{R}^n$.
- Zero Element: There is a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{0} + \vec{v} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$.

These structures satisfy several rules - and the importance of the zero element is so that we can state one of the rules.

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. **Associativity of Addition**: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. **Commutativity Addition**: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. Commutativity Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. **Additive inverse**: $\vec{v} + (-1) \cdot \vec{v} = \vec{0}$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. Commutativity Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. Additive inverse: $\vec{v} + (-1) \cdot \vec{v} = \vec{0}$
4. **Distributive Property for Vectors**: $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. Commutativity Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. Additive inverse: $\vec{v} + (-1) \cdot \vec{v} = \vec{0}$
4. Distributive Property for Vectors: $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$
5. **Distributive Property for Scalars**: $(c + k) \cdot \vec{v} = c \cdot \vec{v} + k \cdot \vec{v}$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. Commutativity Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. Additive inverse: $\vec{v} + (-1) \cdot \vec{v} = \vec{0}$
4. Distributive Property for Vectors: $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$
5. Distributive Property for Scalars: $(c + k) \cdot \vec{v} = c \cdot \vec{v} + k \cdot \vec{v}$
6. **Associativity of Scaling**: $c \cdot (k \cdot \vec{v}) = (c \cdot k) \cdot \vec{v}$

Rules that Addition, Scaling, and Zero Satisfy

Let $\vec{v}, \vec{w}, \vec{u}$ be vectors in $\mathbb{R}^n$, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
2. Commutativity Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. Additive inverse: $\vec{v} + (-1) \cdot \vec{v} = \vec{0}$
4. Distributive Property for Vectors: $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$
5. Distributive Property for Scalars: $(c + k) \cdot \vec{v} = c \cdot \vec{v} + k \cdot \vec{v}$
6. Associativity of Scaling: $c \cdot (k \cdot \vec{v}) = (c \cdot k) \cdot \vec{v}$
7. **1 is Identity**: $1 \cdot \vec{v} = \vec{v}$

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5.

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5. For example, the polynomials

$$p(x) = 4x^4 + 2x^3 + x^2 - 10 \qquad q(x) = x^5 + x + 10 \qquad n(x) = 0$$

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5. For example, the polynomials

$$p(x) = 4x^4 + 2x^3 + x^2 - 10 \qquad q(x) = x^5 + x + 10 \qquad n(x) = 0$$

P_5 has the three algebraic structures analagous to those for $\mathbb{R}^n$. That is:

- **Addition**: We can add any two polynomials in P_5 to obtain a polynomial in P_5. For example,

$$p(x) + q(x) = x^5 + 4x^4 + 2x^3 + x^2 + x$$

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5. For example, the polynomials

$$p(x) = 4x^4 + 2x^3 + x^2 - 10 \qquad q(x) = x^5 + x + 10 \qquad n(x) = 0$$

P_5 has the three algebraic structures analagous to those for $\mathbb{R}^n$. That is:

- Addition: We can add any two polynomials in P_5 to obtain a polynomial in P_5.
- **Scaling**: We can multiply any polynomial in P_5 by a real number to obtain a polynomial in P_5. For example,

$$2 \cdot q(x) = 2x^5 + 2x + 20$$

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5. For example, the polynomials

$$p(x) = 4x^4 + 2x^3 + x^2 - 10 \qquad q(x) = x^5 + x + 10 \qquad n(x) = 0$$

P_5 has the three algebraic structures analagous to those for $\mathbb{R}^n$. That is:

- Addition: We can add any two polynomials in P_5 to obtain a polynomial in P_5.
- Scaling: We can multiply any polynomial in P_5 by a real number to obtain a polynomial in P_5.
- **Zero Element**: The zero polynomial $n(x) = 0$ is a zero element. For example,

$$p(x) + n(x) = 4x^4 + 2x^3 + x^2 - 10 + 0 = p(x)$$

Polynomials of degree 5

Consider the set P_5 of polynomials with degree less than or equal to 5. For example, the polynomials

$$p(x) = 4x^4 + 2x^3 + x^2 - 10 \qquad q(x) = x^5 + x + 10 \qquad n(x) = 0$$

P_5 has the three algebraic structures analagous to those for $\mathbb{R}^n$. That is:

- Addition: We can add any two polynomials in P_5 to obtain a polynomial in P_5.
- Scaling: We can multiply any polynomial in P_5 by a real number to obtain a polynomial in P_5.
- Zero Element: The zero polynomial $n(x) = 0$ is a zero element.

There is nothing special about 5. If d is any whole number, then the collection P_d denotes the set of polynomials with degree $\leq d$. It has addition, scaling, and a zero element.

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. **Associativity of Addition**:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. **Commutativity Addition**: $p(x) + q(x) = q(x) + p(x)$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. Commutativity Addition: $p(x) + q(x) = q(x) + p(x)$
3. **Additive inverse**: $p(x) + (-1) \cdot p(x) = \vec{0}$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition: $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. Commutativity Addition: $p(x) + q(x) = q(x) + p(x)$
3. Additive inverse: $p(x) + (-1) \cdot p(x) = \vec{0}$
4. **Distributive Property for Vectors**: $c \cdot (p(x) + q(x)) = c \cdot p(x) + c \cdot q(x)$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. Commutativity Addition: $p(x) + q(x) = q(x) + p(x)$
3. Additive inverse: $p(x) + (-1) \cdot p(x) = \vec{0}$
4. Distributive Property for Vectors:
   $c \cdot (p(x) + q(x)) = c \cdot p(x) + c \cdot q(x)$
5. **Distributive Property for Scalars**:
   $(c + k) \cdot p(x) = c \cdot p(x) + k \cdot p(x)$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. Commutativity Addition: $p(x) + q(x) = q(x) + p(x)$
3. Additive inverse: $p(x) + (-1) \cdot p(x) = \vec{0}$
4. Distributive Property for Vectors:
   $c \cdot (p(x) + q(x)) = c \cdot p(x) + c \cdot q(x)$
5. Distributive Property for Scalars: $(c + k) \cdot p(x) = c \cdot p(x) + k \cdot p(x)$
6. **Associativity of Scaling**: $c \cdot (k \cdot p(x)) = (c \cdot k) \cdot p(x)$

Properties for Addition, Scaling, and Zero in P_d

Let $p(x), q(x), r(x)$ be polynomials in P_d, and $c, k \in \mathbb{R}$ scalars. Then we have the following properties:

1. Associativity of Addition:
   $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
2. Commutativity Addition: $p(x) + q(x) = q(x) + p(x)$
3. Additive inverse: $p(x) + (-1) \cdot p(x) = \vec{0}$
4. Distributive Property for Vectors:
   $c \cdot (p(x) + q(x)) = c \cdot p(x) + c \cdot q(x)$
5. Distributive Property for Scalars: $(c + k) \cdot p(x) = c \cdot p(x) + k \cdot p(x)$
6. Associativity of Scaling: $c \cdot (k \cdot p(x)) = (c \cdot k) \cdot p(x)$
7. **1 is Identity**: $1 \cdot p(x) = p(x)$

Definition of Vector Space

A **vector space** (over $\mathbb{R}$) is a set V along with the three algebraic structures:

- Addition: If $v, w \in V$, there is an element $v + w \in V$.
- Scaling: If $v \in V$ and $c \in \mathbb{R}$, then there is an element $c \cdot v \in V$.
- Zero element: There is an element $0 \in V$ such that $0 + v = v$ for all $v \in V$.

These three structures must satisfy the 7 rules given before.

Definition of Vector Space

A **vector space** (over $\mathbb{R}$) is a set V along with the three algebraic structures:

- Addition: If $v, w \in V$, there is an element $v + w \in V$.
- Scaling: If $v \in V$ and $c \in \mathbb{R}$, then there is an element $c \cdot v \in V$.
- Zero element: There is an element $0 \in V$ such that $0 + v = v$ for all $v \in V$.

These three structures must satisfy the 7 rules given before.

The elements v of a vector space V are called **vectors**.

Difference between V and $\mathbb{R}^n$

There are two main differences between the behavior of an abstract vector space V and $\mathbb{R}^n$:

Difference between V and $\mathbb{R}^n$

There are two main differences between the behavior of an abstract vector space V and $\mathbb{R}^n$:

- The dimension of $\mathbb{R}^n$ is n. But the dimension of V can be infinite. We will not study such vector spaces closely.

Difference between V and $\mathbb{R}^n$

There are two main differences between the behavior of an abstract vector space V and $\mathbb{R}^n$:

- The dimension of $\mathbb{R}^n$ is n. But the dimension of V can be infinite. We will not study such vector spaces closely.
- V does not have a standard basis, so you can't take it's standard coordinates. For $\vec{v} \in \mathbb{R}^n$, the i-th standard coordinate of $\vec{v}$ is v_i. For example, the second standard coordinate of

$$\vec{v} = \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}$$

is 7.

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

We are interested in the solution set. I.e. the set of all functions that satisfy that equation:

$$V = \{f(x) : \text{ the fourth derivative of } f(x) \text{ equals } f(x)\}$$

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

We are interested in the solution set. I.e. the set of all functions that satisfy that equation:

$$V = \{f(x) : \text{ the fourth derivative of } f(x) \text{ equals } f(x)\}$$

For example, $f(x) = 7e^x + \cos x \in V$ is a solution.

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

We are interested in the solution set. I.e. the set of all functions that satisfy that equation:

$$V = \{f(x) : \text{ the fourth derivative of } f(x) \text{ equals } f(x)\}$$

For example, $f(x) = 7e^x + \cos x \in V$ is a solution.

This is another example of a vector space. This is because *differentiation is linear*. That is,

$$(f+g)'(x) = f'(x) + g'(x)$$

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

We are interested in the solution set. I.e. the set of all functions that satisfy that equation:

$$V = \{f(x) : \text{ the fourth derivative of } f(x) \text{ equals } f(x)\}$$

For example, $f(x) = 7e^x + \cos x \in V$ is a solution.

This is another example of a vector space. This is because *differentiation is linear*. That is,

$$(f+g)'(x) = f'(x) + g'(x)$$

Thus, if f and g are solutions, then

$$(f+g)^{(4)}(x) = f^{(4)}(x) + g^{(4)}(x) = f(x) + g(x) = (f+g)(x)$$

Solutions to $f^{(4)} = f$

Consider the differential equation

$$f^{(4)} = f$$

We are interested in the solution set. I.e. the set of all functions that satisfy that equation:

$$V = \{f(x) : \text{ the fourth derivative of } f(x) \text{ equals } f(x)\}$$

For example, $f(x) = 7e^x + \cos x \in V$ is a solution.

This is another example of a vector space. This is because *differentiation is linear*. That is,

$$(f+g)'(x) = f'(x) + g'(x)$$

Thus, if f and g are solutions, then

$$(f+g)^{(4)}(x) = f^{(4)}(x) + g^{(4)}(x) = f(x) + g(x) = (f+g)(x)$$

Hence, $f+g$ is also a solution. Similarly, multiplying a solution by a real number is also a solution. This example illustrates that there aren't standard coordinates for any vector space.

The Space of Matrices

Let $\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ matrices. Then $\mathbb{R}^{m \times n}$ is a vector space.

The Space of Matrices

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices. Then $\mathbb{R}^{m\times n}$ is a vector space.

- Addition is given entry by entry

$$A + B = (a_{ij}) + (b_{ij}) = \left(a_{ij} + b_{ij}\right)$$

The Space of Matrices

Let $\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ matrices. Then $\mathbb{R}^{m \times n}$ is a vector space.

- Addition is given entry by entry

$$A + B = (a_{ij}) + (b_{ij}) = \left(a_{ij} + b_{ij}\right)$$

- Scaling is given entry by entry

$$c \cdot A = c \cdot (a_{ij}) = (c \cdot a_{ij})$$

- The zero element is the matrix with all 0's.

A Flood of Definitions in Linear Algebra on $\mathbb{R}^n$ and for V

The following have been defined for $\mathbb{R}^n$, and will also be defined for any vector space V:

A Flood of Definitions in Linear Algebra on $\mathbb{R}^n$ and for V

The following have been defined for $\mathbb{R}^n$, and will also be defined for any vector space V:

- Subspace
- Linear Independence/Dependence
- Span
- Basis
- Coordinates for a given basis $\mathcal{B}$
- Matrix of a linear transformation in terms of a given basis
- Linear Transformation
- Kernel
- Image

Definitions in Linear Algebra on $\mathbb{R}^n$ and for V

The following have been defined for $\mathbb{R}^n$, but CANNOT be defined for every vector space V:

Definitions in Linear Algebra on $\mathbb{R}^n$ and for V

The following have been defined for $\mathbb{R}^n$, but CANNOT be defined for every vector space V:

- The standard basis
- Standard coordinates
- The matrix of a linear transformation. (I.e., the matrix of a linear transformation in terms of the standard basis.)

Definitions in Linear Algebra on $\mathbb{R}^n$ and for V

The following have been defined for $\mathbb{R}^n$, but CANNOT be defined for every vector space V:

- The standard basis
- Standard coordinates
- The matrix of a linear transformation. (I.e., the matrix of a linear transformation in terms of the standard basis.)

Note that a matrix will be given for a linear transformation, but only in terms of a basis.

Subspace

A **subspace** of a vector space V is a subset $W \subset V$ which is itself a vector space, with the addition, scaling, and zero objecte the same as in V.

Subspace

A **subspace** of a vector space V is a subset $W \subset V$ which is itself a vector space, with the addition, scaling, and zero objecte the same as in V.

In practice, this means showing that W contains 0, and W is **closed** under addition and scaling. That is,

- If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$
- If $w \in W$ and $c \in \mathbb{R}$ a scalar, then $c \cdot w \in W$.

Examples of Subspaces

- Any line or plane containing $\vec{0}$ in $\mathbb{R}^n$.

Examples of Subspaces

- Any line or plane containing $\vec{0}$ in $\mathbb{R}^n$.
- The set $W \subset P_5$ of all polynomials $f(x)$ of degree 5 such that $f(2) = 0$ is a subspace of P_5.

Examples of Subspaces

- Any line or plane containing $\vec{0}$ in $\mathbb{R}^n$.
- The set $W \subset P_5$ of all polynomials $f(x)$ of degree 5 such that $f(2) = 0$ is a subspace of P_5.
- The set $W \subset \mathbb{R}^{2\times 2}$ of matrices A such that

$$A\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

  is a subspace of $\mathbb{R}^{2\times 2}$.

Examples of Subspaces

- Any line or plane containing $\vec{0}$ in $\mathbb{R}^n$.
- The set $W \subset P_5$ of all polynomials $f(x)$ of degree 5 such that $f(2) = 0$ is a subspace of P_5.
- The set $W \subset \mathbb{R}^{2\times 2}$ of matrices A such that
$$A\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$
is a subspace of $\mathbb{R}^{2\times 2}$.
- The solution set of
$$f'(x) = x \cdot f(x)$$
is a subspace of the solution set of
$$f''(x) = (x^2 + 1) \cdot f(x)$$

Span

A **linear combination** of $v_1, \cdots, v_n$ in a vector space V is an element of the form

$$c_1 v_1 + \cdots + c_n v_n$$

Span

A **linear combination** of $v_1, \cdots, v_n$ in a vector space V is an element of the form

$$c_1 v_1 + \cdots + c_n v_n$$

The **span** of $v_1, \cdots, v_n$ is the set of linear combinations of $v_1, \cdots, v_n$.

Span

A **linear combination** of $v_1, \cdots, v_n$ in a vector space V is an element of the form

$$c_1 v_1 + \cdots + c_n v_n$$

The **span** of $v_1, \cdots, v_n$ is the set of linear combinations of $v_1, \cdots, v_n$.

The span of a set of vectors is a subspace.

Example of Span

Consider $x, x^2 + x^4 \in P_5$. The element

$$p(x) = 5x - x^2 - x^4 = (5) \cdot x + (-1) \cdot (x^2 + x^4)$$

is a linear combination of x and $x^2 + x^4$

Example of Span

Consider $x, x^2 + x^4 \in P_5$. The element

$$p(x) = 5x - x^2 - x^4 = (5) \cdot x + (-1) \cdot (x^2 + x^4)$$

is a linear combination of x and $x^2 + x^4$ Hence,

$$p(x) \in \mathrm{span}\big(x, x^2 + x^4\big) = \big\{ax + bx^2 + bx^4 : a, b \in \mathbb{R}\big\}$$

Linear Independence/Dependence

A set of vectors $v_1, \cdots, v_n$ in V are **linearly dependent** if there exists a **relation**

$$c_1 v_1 + \cdots c_n v_n = 0$$

for scalars c_i which is **nontrivial**.

Linear Independence/Dependence

A set of vectors $v_1, \cdots, v_n$ in V are **linearly dependent** if there exists a **relation**

$$c_1 v_1 + \cdots c_n v_n = 0$$

for scalars c_i which is **nontrivial**. Nontrivial means that at least one of the $c_i \neq 0$. Every set of vectors satisfies the trivial relation.

Linear Independence/Dependence

A set of vectors $v_1, \cdots, v_n$ in V are **linearly dependent** if there exists a **relation**

$$c_1 v_1 + \cdots c_n v_n = 0$$

for scalars c_i which is **nontrivial**. Nontrivial means that at least one of the $c_i \neq 0$. Every set of vectors satisfies the trivial relation.

The vectors $v_1, \cdots, v_n$ are **linearly independent** if they are not linearly dependent.

Linear Independence/Dependence

A set of vectors $v_1, \cdots, v_n$ in V are **linearly dependent** if there exists a **relation**

$$c_1 v_1 + \cdots c_n v_n = 0$$

for scalars c_i which is **nontrivial**. Nontrivial means that at least one of the $c_i \neq 0$. Every set of vectors satisfies the trivial relation.

The vectors $v_1, \cdots, v_n$ are **linearly independent** if they are not linearly dependent. To check that vectors are linearly independent, you must check that every relation between them is trivial.

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

They are linearly dependent, because $I_3 = A_1 + A_2$.

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

They are linearly dependent, because $I_3 = A_1 + A_2$. This corresponds to the relation

$$A_1 + A_2 - I_3 = 0$$

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

They are linearly dependent, because $I_3 = A_1 + A_2$. This corresponds to the relation

$$A_1 + A_2 - I_3 = 0$$

But A_1 and A_2 are linearly independent.

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

They are linearly dependent, because $I_3 = A_1 + A_2$. This corresponds to the relation

$$A_1 + A_2 - I_3 = 0$$

But A_1 and A_2 are linearly independent. Indeed, if

$$c_1 A_1 + c_2 A_2 = 0$$

Example of Linear Independence/Dependence

Consider the set of vectors in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3$$

They are linearly dependent, because $I_3 = A_1 + A_2$. This corresponds to the relation

$$A_1 + A_2 - I_3 = 0$$

But A_1 and A_2 are linearly independent. Indeed, if

$$c_1 A_1 + c_2 A_2 = 0$$

Then

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_1 = c_2 = 0$$

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.
- They are linearly independent.

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.
- They are linearly independent.

We say that V is **finite dimensional** if such a basis exists.

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.
- They are linearly independent.

We say that V is **finite dimensional** if such a basis exists. (Note: one can define what it means for an infinite list of vectors to be a basis, so that one can also study infinite dimensional vector spaces. From now on, we will always assume V is finite dimensional.)

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.
- They are linearly independent.

We say that V is **finite dimensional** if such a basis exists. (Note: one can define what it means for an infinite list of vectors to be a basis, so that one can also study infinite dimensional vector spaces. From now on, we will always assume V is finite dimensional.)

To say that $v_1, \cdots, v_n$ are a basis is equivalent to saying that every element of V <u>can</u> be written <u>uniquely</u> as a linear combination of the vectors $v_1, \cdots, v_n$. (The "can" corresponds to the "span" condition, the "uniquely" corresponds to the "linear independence" condition.)

Basis

A **basis** of V is a set of vectors $v_1, \cdots, v_n$ such that:

- They span V. That is, their span is all of V.
- They are linearly independent.

We say that V is **finite dimensional** if such a basis exists. (Note: one can define what it means for an infinite list of vectors to be a basis, so that one can also study infinite dimensional vector spaces. From now on, we will always assume V is finite dimensional.)

To say that $v_1, \cdots, v_n$ are a basis is equivalent to saying that every element of V <u>can</u> be written <u>uniquely</u> as a linear combination of the vectors $v_1, \cdots, v_n$. (The "can" corresponds to the "span" condition, the "uniquely" corresponds to the "linear independence" condition.)

Just as with subspaces of $\mathbb{R}^n$, the number of vectors in any basis of V is always the same. This number is called the **dimension** of V.

Example of Basis

Consider the space P_3 of polynomials of degree ≤ 3. Every element can be written uniquely as a linear combination of the monomials $1, x, x^2, x^3$. Hence, they form a basis of P_3. Thus, it is 4-dimensional.

Example of Basis

Consider the space P_3 of polynomials of degree ≤ 3. Every element can be written uniquely as a linear combination of the monomials $1, x, x^2, x^3$. Hence, they form a basis of P_3. Thus, it is 4-dimensional.

Now, consider the space $\mathbb{R}^{3\times 2}$ of 3×2 matrices. Then the matrices

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_{3,1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad E_{3,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of $\mathbb{R}^{3\times 2}$. Hence, it is 6-dimensional.

Example of Basis

By general theory of ordinary differential equations (which you don't need to know), the space of solutions to

$$f'' = f$$

is two dimensional.

Example of Basis

By general theory of ordinary differential equations (which you don't need to know), the space of solutions to

$$f'' = f$$

is two dimensional.

Two solutions are given by $f(x) = e^x$ and $f(x) = e^{-x}$.

Example of Basis

By general theory of ordinary differential equations (which you don't need to know), the space of solutions to

$$f'' = f$$

is two dimensional.

Two solutions are given by $f(x) = e^x$ and $f(x) = e^{-x}$.
They aren't scalar multiples of each other, and thus are linearly independent. Thus, e^x, e^{-x} is a basis of the solution set - i.e., a "fundamental system of solutions."

Coordinates

Fix a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of V. Every vector w of V can be written uniquely as

$$w = c_1 v_1 + \cdots + c_n v_n$$

The $\mathcal{B}$-**coordinates** of w are the scalars $c_1, \cdots, c_n$.

Coordinates

Fix a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of V. Every vector w of V can be written uniquely as

$$w = c_1 v_1 + \cdots + c_n v_n$$

The $\mathcal{B}$-**coordinates** of w are the scalars $c_1, \cdots, c_n$.

The $\mathcal{B}$-**coordinate vector** of w is the column vector of $\mathbb{R}^n$

$$[w]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \ddots \\ c_n \end{pmatrix}$$

Coordinates

Fix a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of V. Every vector w of V can be written uniquely as

$$w = c_1 v_1 + \cdots + c_n v_n$$

The $\mathcal{B}$-**coordinates** of w are the scalars $c_1, \cdots, c_n$.

The $\mathcal{B}$-**coordinate vector** of w is the column vector of $\mathbb{R}^n$

$$[w]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \ddots \\ c_n \end{pmatrix}$$

The $\mathcal{B}$-**coordinate transformation** $L_{\mathcal{B}}$ is the function from V to $\mathbb{R}^n$ given by

$$L_{\mathcal{B}} \colon V \to \mathbb{R}^n \qquad w \mapsto [w]_{\mathcal{B}}$$

Example of Coordinates

Let $\mathcal{B}$ be the basis $1, x, x^2, x^3, x^4$ of P_4. Then

$$L_{\mathcal{B}}(3x^4 - 1) = [3x^4 - 1]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

Example of Coordinates

Let $\mathcal{B}$ be the basis $1, x, x^2, x^3, x^4$ of P_4. Then

$$L_{\mathcal{B}}(3x^4 - 1) = [3x^4 - 1]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

Note that $L_{\mathcal{B}}$ took the degree 4 polynomial $3x^4 - 1$ and gave a column vector in $\mathbb{R}^5$.

Linearity of the $\mathcal{B}$-Coordinate Transformation

The $\mathcal{B}$-coordinate transformation for a basis $\mathcal{B}$ gives our first example of an "abstract" linear transformation. That is, it satisfies for any vectors $v, w \in V$ and scalar $c \in \mathbb{R}$,

- $[v + w]_{\mathcal{B}} = [v]_{\mathcal{B}} + [w]_{\mathcal{B}}$
- $[c \cdot v]_{\mathcal{B}} = c \cdot [v]_{\mathcal{B}}$

Linear Transformation

Let V and W be two vector spaces.

Linear Transformation

Let V and W be two vector spaces. A function T from V to W

$$T : V \to W$$

is called a **linear transformation** if

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

for all vectors v_1, v_2 of V,

Linear Transformation

Let V and W be two vector spaces. A function T from V to W

$$T \colon V \to W$$

is called a **linear transformation** if

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

for all vectors v_1, v_2 of V, and

$$T(c \cdot v) = c \cdot T(v)$$

for all vectors v of V and scalars $c \in \mathbb{R}$.

Examples of Linear Transformation

The function

$$\int_0^x : P_5 \to P_6 \qquad f(x) \mapsto \int_0^x f(t)\, dt$$

is a linear transformation.

Examples of Linear Transformation

The function

$$\int_0^x : P_5 \to P_6 \qquad f(x) \mapsto \int_0^x f(t)\, dt$$

is a linear transformation.

The function

$$\frac{d}{dx} : P_8 \to P_8$$

taking a polynomial to its derivative is also a linear transformation.

Examples of Linear Transformation

The function

$$\int_0^x : P_5 \to P_6 \qquad f(x) \mapsto \int_0^x f(t)\,dt$$

is a linear transformation.

The function

$$\frac{d}{dx} : P_8 \to P_8$$

taking a polynomial to its derivative is also a linear transformation.

Let B be any matrix $k \times m$ matrix. The map

$$M_B^L : \mathbb{R}^{m \times n} \to \mathbb{R}^{k \times n} \qquad B \mapsto AB$$

is a linear transformation.

Examples of Linear Transformation

The function

$$\int_0^x : P_5 \to P_6 \qquad f(x) \mapsto \int_0^x f(t)\, dt$$

is a linear transformation.

The function

$$\frac{d}{dx} : P_8 \to P_8$$

taking a polynomial to its derivative is also a linear transformation.

Let B be any matrix $k \times m$ matrix. The map

$$M_B^L : \mathbb{R}^{m\times n} \to \mathbb{R}^{k\times n} \qquad B \mapsto AB$$

is a linear transformation. M_B^L is my notation for multiply on the left by B.

Kernel

If T is a linear transformation from V to W, the set of elements v in V such that $T(v)$ is called the **kernel** of T, and is denoted $\ker T$.

Kernel

If T is a linear transformation from V to W, the set of elements v in V such that $T(v)$ is called the **kernel** of T, and is denoted $\ker T$.

$\ker T$ is a subspace of V.

Image

If T is a linear transformation from V to W, the set of elements w in W such that $w = T(v)$ for some v in V is called the **image** of T, and is denoted $\operatorname{im} T$.

Image

If T is a linear transformation from V to W, the set of elements w in W such that $w = T(v)$ for some v in V is called the **image** of T, and is denoted $\operatorname{im} T$.

$\operatorname{im} T$ is a subspace of W.

Isomorphism/Isomorphic

A linear transformation T from V to W is called an **isomorphism** if T is invertible as a function.

Isomorphism/Isomorphic

A linear transformation T from V to W is called an **isomorphism** if T is invertible as a function.

Its inverse $T^{-1} \colon W \to V$ is necessarily linear.

Isomorphism/Isomorphic

A linear transformation T from V to W is called an **isomorphism** if T is invertible as a function.

Its inverse $T^{-1}\colon W \to V$ is necessarily linear.

Isomorphism means "same shape." By identifying vectors of V with those of W via T, studying linear algebra on V becomes equivalent to studying linear algebra on W.