

# MA 305 Spring 2K Homework 4 Solutions

**1.** Define the set  $P_3 = \{a + bx + cx^2 + dx^3 : a, b, c, d \in \mathbb{R}\}$ , the set of polynomials of degree 3 or less with real coefficients. Show that  $P_3$  is a vector space over the real numbers with the usual scalar multiplication and addition over the polynomials.

There are two ways to do this. One is to show that it is a vector space by showing it satisfies the definition of a vector space. The other way is to show it's a subspace of a known vector space.

**(a.)** Showing  $P_3$  satisfies the definition of a vector space.

Consider three arbitrary polynomials in  $P_3$ :

$$\begin{aligned} u &= a_1 + b_1x + c_1x^2 + d_1x^3 \\ v &= a_2 + b_2x + c_2x^2 + d_2x^3 \\ w &= a_3 + b_3x + c_3x^2 + d_3x^3 \end{aligned}$$

and two arbitrary real numbers  $\alpha$  and  $\beta$ . Then, let us prove the eight axioms that compose the definition of a vector space.

**(A1)**  $\forall u, v \in P_3 : u + v = v + u$ .

$$u + v = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3$$

and

$$\begin{aligned} v + u &= (a_2 + a_1) + (b_2 + b_1)x + (c_2 + c_1)x^2 + (d_2 + d_1)x^3 \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3 \\ &= u + v \end{aligned}$$

Thus the first axiom is satisfied.

**(A2)**  $\forall u, v, w \in P_3 : (u + v) + w = u + (v + w)$ .

$$(u + v) + w = ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3)x + ((c_1 + c_2) + c_3)x^2 + ((d_1 + d_2) + d_3)x^3$$

and

$$\begin{aligned} u + (v + w) &= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3))x + (c_1 + (c_2 + c_3))x^2 + (d_1 + (d_2 + d_3))x^3 \\ &= ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3)x + ((c_1 + c_2) + c_3)x^2 + ((d_1 + d_2) + d_3)x^3 \\ &= (u + v) + w \end{aligned}$$

Thus the second axiom is satisfied.

**(A3)**  $\exists 0 \in P_3 : \forall v \in P_3 : 0 + v = v$ .

We can see the zero polynomial  $0 = 0 + 0x + 0x^2 + 0x^3$  is in  $P_3$ . Let us denote it by  $0_{P_3}$  to keep from confusing it with the real number zero. Then,

$$\begin{aligned} 0_{P_3} + v &= (0 + a_2) + (0 + b_2)x + (0 + c_2)x^2 + (0 + d_2)x^3 \\ &= a_2 + b_2x + c_2x^2 + d_2x^3 \\ &= v \end{aligned}$$

Thus the third axiom is satisfied with the zero polynomial  $0_{P_3}$ .

$$(A4) \quad \forall v \in P_3 : \exists (-v) \in P_3 : v + (-v) = 0_{P_3}.$$

For the arbitrary polynomial  $v$ , we see that the polynomial  $(-a_2) + (-b_2)x + (-c_2)x^2 + (-d_2)x^3 = -a_2 - b_2x - c_2x^2 - d_2x^3$  is also in  $P_3$ , so let us denote it as  $-v$ . Then,

$$\begin{aligned} v + (-v) &= (a_2 - a_2) + (b_2 - b_2)x + (c_2 - c_2)x^2 + (d_2 - d_2)x^3 \\ &= 0 + 0x + 0x^2 + 0x^3 \\ &= 0_{P_3} \end{aligned}$$

Thus, the fourth axiom is satisfied.

$$(A5) \quad \forall \alpha \in \mathbb{R} : \forall u, v \in P_3 : \alpha(u + v) = \alpha u + \alpha v.$$

$$\begin{aligned} \alpha(u + v) &= \alpha((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3) \\ &= (\alpha(a_1 + a_2)) + (\alpha(b_1 + b_2))x + (\alpha(c_1 + c_2))x^2 + (\alpha(d_1 + d_2))x^3 \\ &= ((\alpha a_1) + (\alpha a_2)) + ((\alpha b_1) + (\alpha b_2))x + ((\alpha c_1) + (\alpha c_2))x^2 + ((\alpha d_1) + (\alpha d_2))x^3 \\ &= ((\alpha a_1) + (\alpha b_1)x + (\alpha c_1)x^2 + (\alpha d_1)x^3) + ((\alpha a_2) + (\alpha b_2)x + (\alpha c_2)x^2 + (\alpha d_2)x^3) \\ &= (\alpha u) + (\alpha v) \end{aligned}$$

Thus the fifth axiom is satisfied.

$$(A6) \quad \forall \alpha, \beta \in \mathbb{R} : \forall v \in P_3 : (\alpha + \beta)v = \alpha v + \beta v.$$

$$\begin{aligned} (\alpha + \beta)v &= ((\alpha + \beta)a_2) + ((\alpha + \beta)b_2)x + ((\alpha + \beta)c_2)x^2 + ((\alpha + \beta)d_2)x^3 \\ &= ((\alpha a_2) + (\beta a_2)) + ((\alpha b_2) + (\beta b_2))x + ((\alpha c_2) + (\beta c_2))x^2 + ((\alpha d_2) + (\beta d_2))x^3 \\ &= ((\alpha a_2) + (\alpha b_2)x + (\alpha c_2)x^2 + (\alpha d_2)x^3) + ((\beta a_2) + (\beta b_2)x + (\beta c_2)x^2 + (\beta d_2)x^3) \\ &= (\alpha v) + (\beta v) \end{aligned}$$

Thus the sixth axiom is satisfied.

$$(A7) \quad \forall \alpha, \beta \in \mathbb{R} : \forall v \in P_3 : (\alpha\beta)v = \alpha(\beta v).$$

$$\begin{aligned} \alpha(\beta v) &= \alpha((\beta a_2) + (\beta b_2)x + (\beta c_2)x^2 + (\beta d_2)x^3) \\ &= (\alpha(\beta a_2)) + (\alpha(\beta b_2))x + (\alpha(\beta c_2))x^2 + (\alpha(\beta d_2))x^3 \\ &= ((\alpha\beta)a_2) + ((\alpha\beta)b_2)x + ((\alpha\beta)c_2)x^2 + ((\alpha\beta)d_2)x^3 \\ &= (\alpha\beta)v \end{aligned}$$

Thus the seventh axiom is satisfied.

$$(A8) \quad \forall v \in P_3 : 1v = v.$$

$$\begin{aligned}
1v &= 1(a_2 + b_2x + c_2x^2 + d_2x^3) \\
&= a_2 + b_2x + c_2x^2 + d_2x^3 \\
&= v
\end{aligned}$$

Thus the eighth, and final, axiom is satisfied.

Because  $P_3$  satisfies all eight axioms, it is a vector space over the real numbers.

**(b.)** Showing  $P_3$  satisfies the definition of a subspace.

The other way of showing  $P_3$  is a vector space is to show it is a subspace of a known vector space. We know from lecture 17 that the set of all polynomials with real coefficients

$$\mathbb{R}[x] = \{a_0 + a_1x + \cdots + a_dx^d : a_i \in \mathbb{R}, d \in \mathbb{Z}_{\geq 0}\}$$

is a vector space, and  $P_3 \subseteq \mathbb{R}[x]$ . So, we only need to show  $P_3$  satisfies the two axioms of a subspace. Thus, let us consider two arbitrary polynomials in  $P_3$ :

$$\begin{aligned}
u &= a_1 + b_1x + c_1x^2 + d_1x^3 \\
v &= a_2 + b_2x + c_2x^2 + d_2x^3
\end{aligned}$$

and an arbitrary real number  $\alpha$ .

$$(S1) \quad \forall u, v \in P_3 : u + v \in P_3.$$

$$u + v = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3$$

Here, we know  $a_1 + a_2 \in \mathbb{R}$ ,  $b_1 + b_2 \in \mathbb{R}$ ,  $c_1 + c_2 \in \mathbb{R}$ , and  $d_1 + d_2 \in \mathbb{R}$ , so  $u + v \in P_3$ .

Thus, the first axiom is satisfied.

$$(S2) \quad \forall \alpha \in \mathbb{R} : \forall v \in P_3 : \alpha v \in P_3.$$

$$\alpha v = (\alpha a_1) + (\alpha b_1)x + (\alpha c_1)x^2 + (\alpha d_1)x^3 + (\alpha a_2) + (\alpha b_2)x + (\alpha c_2)x^2 + (\alpha d_2)x^3$$

Here, we know  $\alpha a_2 \in \mathbb{R}$ ,  $\alpha b_2 \in \mathbb{R}$ ,  $\alpha c_2 \in \mathbb{R}$ , and  $\alpha d_2 \in \mathbb{R}$ , so  $\alpha v \in P_3$ .

Thus, the second, and final, axiom is satisfied.

Because it satisfies both axioms of being a subspace,  $P_3$  is a subspace of  $\mathbb{R}[x]$ , and is thus a vector space by itself.

**2.** For the vector space  $P_3$  defined above, do the following.

**(a.)** Find a basis for  $P_3$ .

To do this, we must first guess at a basis for  $P_3$  and then prove that it is, in fact, a basis. Let us first consider the arbitrary polynomial  $u = a + bx + cx^2 + dx^3 \in P_3$ , and the arbitrary vector

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4.$$

We know the elementary vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for  $\mathbb{R}^4$ , so we might guess that the corresponding polynomials  $\{1, x, x^2, x^3\}$  form a basis for  $P_3$ . The four polynomials are certainly contained in  $P_3$ , so we must show they are linearly independent and span the space.

**(Linearly Independence.)**

Suppose a linear combination of the four polynomials is equal to the zero polynomial,

$$c_1 1 + c_2 x + c_3 x^2 + c_4 x^3 = 0_{P_3}.$$

Then we know the coefficients must all be zero:  $c_1 = c_2 = c_3 = c_4 = 0$ , and thus by definition the polynomials are linearly independent.

**(Span  $P_3$ .)**

Suppose we have an arbitrary polynomial  $u = a + bx + cx^2 + dx^3 \in P_3$ . We want to show we can write  $u$  as a linear combination of the four polynomials:

$$u = c_1 1 + c_2 x + c_3 x^2 + c_4 x^3,$$

but this is trivial to see if we pick  $c_1 = a$ ,  $c_2 = b$ ,  $c_3 = c$ ,  $c_4 = d$ . Thus, by definition, the polynomials span  $P_3$ .

Therefore,  $\{1, x, x^2, x^3\}$  is a basis for  $P_3$ .

**(b.)** Give the dimension of  $P_3$ .

We have found a basis for  $P_3$ , and the dimension of a vector space is the number of elements in its basis, so the dimension of  $P_3$  is 4.

**3.** Find a basis for the right null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 4 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

We are just trying to solve the equation  $Ax = 0$ , so we can set up the augmented coefficient matrix

$$\left[ \begin{array}{cccc|c} 1 & 3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{array} \right]$$

and reduce it to row echelon form

$$\left[ \begin{array}{cccc|c} 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -8 & 0 \end{array} \right].$$

The, back substitution gives us the answer

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \\ 0 \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, the basis for the null space  $N(A)$  is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4. Consider the vector space  $\mathbb{R}^4$  of four-dimensional vectors, and the two vectors,

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a.) Show these two vectors are linearly independent.

To show the vectors are linearly independent, we put them into a matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

and we reduce it to row echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because there is a pivot in every column, we know that if we solved  $Ax = b$ , there would be no free variables, and thus the only solution would be the zero vector. This means the only linear combination of these two vectors that yields the zero vector is the combination with zeros for both coefficients. Thus, the vectors are linearly independent.

(b.) Extend these two vectors to a basis for  $\mathbb{R}^4$ .

To extend these vectors to a basis, we first extend them to a set that spans  $\mathbb{R}^4$ , and then pare this set down to a linearly independent set. Because we know the elementary vectors mentioned in problem 2(a) form a basis for  $\mathbb{R}^4$ , we know the two vectors above plus the elementary vectors span  $\mathbb{R}^4$ . Thus, we put them all into a matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and reduce it to row echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Because there are pivot elements in the first four columns, we know the first four columns of the original matrix are linearly independent and form a basis for  $\mathbb{R}^4$ .

Therefore, the set of four vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is an extension of the two original vectors to a basis for  $\mathbb{R}^4$ .