# YORK UNIVERSITY

Faculty of Science and Engineering
MATH 2022 3.00 M
Test #2
Solutions

1. (10 pts) For each statement indicate whether it is always TRUE or sometimes FALSE.

   **Note:** For this question each correct answer is worth 1.25 points and each incorrect answer is worth $-0.5$ (negative half!) points. If the number of incorrect answers is more than two and half times greater than the number of correct ones, then the total mark will be zero. If you don't know the answer, don't write anything. For this question only, you do NOT need to explain your answer or show your work.

| **Statement** | **TRUE/FALSE** | |
|---|---|---|
| The standard basis of $\mathbb{P}_n$ contains $n$ vectors. | False | |
| The set of all invertible $2 \times 2$ matrices is a subspace of $\mathbb{M}_{22}$. | False | |
| If $U$ is a subspace of a vector space $V$, then $\dim U \leq \dim V$. | True | |
| If $U$ and $W$ are subspaces of a vector space $V$, then $U \cap W$ is a subspace of $V$. | True | |
| If $U$ is a subset of a vector space $V$, and $u \in U$, then $U = \operatorname{span}\{u\}$. | False | |
| If $T\colon V \to W$ is a linear transformation and for some vector $\mathbf{v} \in V$, $T(-\mathbf{v}) = T(\mathbf{v})$, then $\mathbf{v} = \mathbf{0}$. | False | |
| Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Then if $\ker T$ is a plane through the origin in $\mathbb{R}^3$, then $\operatorname{im} T$ is a line through the origin in $\mathbb{R}^3$. | True | |
| If $T\colon V \to W$ is a linear transformation and $V$ is finite dimensional, then a basis for $\ker T$ can be extended to a basis for $V$. | True | |

2. (6 pts) Find the least squares approximating line for the following points:

$$(-1,\ 2),\ (0,\ 3),\ (2,\ 2),\ (3,\ 6).$$

ANSWER:
¿From the data,

$$M^TM = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}$$

$$M^TY = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 22 \end{bmatrix}$$

So, the normal equations $(M^TM)Z = M^TY$ for $Z = [z_0\ z_1]^T$ will be

$$\begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \end{bmatrix}$$

Solving it for example, using Gaussian elimination, we obtain $z_0 = \frac{51}{20}$ and $z_1 = \frac{7}{10}$. Therefore, the least squares approximating line for the given points is $y = z_0 + z_1x = \frac{51}{20} + \frac{7}{10}x$.

3. (3 + 3 + 4 pts)

(a) Let $V$ be the set of all polynomials of the degree less than or equal to two. Determine whether $V$ is a vector space under the standard scalar multiplication and vector addition defined as $(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_1x + a_2x^2)$. Justify your answer.
ANSWER:
Let

$$\mathbf{u} = a_0 + a_1x + a_2x^2,\ \mathbf{v} = b_0 + b_1x + b_2x^2.$$

Then

$$\mathbf{u} + \mathbf{v} = a_0 + b_1x + a_2x^2$$

and

$$\mathbf{v} + \mathbf{u} = (b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) = b_0 + a_1x + b_2x^2.$$

Hence, $\mathbf{u} + \mathbf{v} \neq \mathbf{v} + \mathbf{u}$, i.e. Axiom A2 is not satisfied. Therefore, $V$ is not a vector space.

(b) Determine whether the vector $3x^2$ is in span of $S$, where $S = \{3x + 1, x^2 + 2x + 3\}$. Justify your answer.
ANSWER:
If vector $3x^2$ is in span of $S$, then there must exist scalars $s$ and $t$ such that

$$3x^2 = s(3x + 1) + t(x^2 + 2x + 3) = tx^2 + (2t + 3s)t + (3t + s).$$

Equating the coefficients of the corresponding terms, we obtain the following system of linear equations: $t = 3,\ 2t + 3s = 0,\ 3t + s = 0$. Substituting $t = 3$ in the 2nd equation, we obtain $s = -2$ and substituting $t = 3$ in the 3rd equation, we obtain $s = -9$. Hence, the system is inconsistent and therefore, $3x^2$ is not in span of $S$.

(c) Determine whether the set of vectors $\left\{\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}\right\}$ is linearly independent. Do they form a basis for $\mathbb{M}_{22}$? Justify your answer.

ANSWER:

Let

$$a\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix} = \mathbf{0}.$$

Then we obtain the following homogeneous system of four linear equations with two unknowns: $a+2b=0,\ -3a+2b=0,\ a-3b=0,\ 2a+4b=0$, which has only the trivial solution $a=b=0$. So, the set $\left\{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}\right\}$ is linearly independent.

On the other hand, $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}\right\}$.

Indeed, if

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} = r\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} + s\begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix},$$

we obtain the following inconsistent system of linear equations: $r+2s=2,\ -3r+2s=3,\ r-3s=1,\ 2r+4s=-2$.

Therefore, the original set is linearly independent.

Since $\dim \mathbb{M}_{22}=4$, the set of three vector is not a basis for $\mathbb{M}_{22}$.

4. (6 + 4 pts)

(a) Let $U$ be the set of all $2\times 2$ matrices of the form $\begin{bmatrix} a & b-a \\ c & a \end{bmatrix}$, where $a,b,c\in\mathbb{R}$. Is $U$ a subspace of $\mathbb{M}_{22}$? If so, find its basis, and determine $\dim U$.

ANSWER:

For all $a,b,c\in\mathbb{R}$,

$$\begin{bmatrix} a & b-a \\ c & a \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So, $U=\text{span}\left\{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$. Since the spanning set is a sebset of $\mathbb{M}_{22}$, we conclude that $U$ a subspace of $\mathbb{M}_{22}$.

On the other hand, $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is linearly independent because

$$r\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

implies that $r=s=t=0$.

Therefore, the set $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $U$ and consequently, $\dim U=3$.

(b) Let $W=\{f(x)\in\mathbb{P}_4 | f(x)=ax^4+bx^2,\ \text{where } a,b\in\mathbb{R}\}$. Is $W$ a subspace of $\mathbb{P}_4$? If so, find its basis, and determine $\dim W$.

ANSWER:

span $\{x^2,\, x^4\}=W$ and since the spanning set is a subset of $\mathbb{P}_4$, $W$ is a subspace of $\mathbb{P}_4$.

On the other hand, $\{x^2, x^4\}$ is linearly independent because $rx^2 + sx^4 = 0$ implies that $r = s = 0$. Therefore, $\{x^2, x^4\}$ is a basis for $W$ and $\dim W = 2$.

5. (3 + 4 pts)

   (a) Determine whether $T\colon \mathbb{P}_3 \to \mathbb{M}_{22}$ defined by $T(a + bx + cx^2 + dx^3) = \begin{bmatrix} 3c & 2a+4d \\ a+b+c+d & (b+c)^2 \end{bmatrix}$ is a linear transformation. Justify your answer.

   ANSWER:
   Let $a = c = d = 0$ and $b = 2$. Then $T(2x) = \begin{bmatrix} 0 & 0 \\ 2 & 2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$,
   while $2T(x) = 2\begin{bmatrix} 0 & 0 \\ 1 & 1^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$.
   Hence, $T(2x) \neq 2T(x)$ and Axiom 2 is not satisfied for $r = 2$.
   Therefore, $T$ is not a linear transformation.

   (b) Let $T\colon \mathbb{R}^3 \to \mathbb{M}_{22}$ be a linear transformation defined by $T(a, b, c) = \begin{bmatrix} a & b+c \\ a+2b & c \end{bmatrix}$.
   Then which of the following sets is a basis for im $T$?

      i. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$
      ii. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\}$
      iii. $\left\{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$
      iv. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$
      v. none of the abovementioned.

   Explain.
   ANSWER: iv.
   If $T(a, b, c) = \mathbf{0}$, then $\begin{bmatrix} a & b+c \\ a+2b & c \end{bmatrix} = \mathbf{0}$. Solving the corresponding system, we obtain $a = b = c = 0$. So, $\ker\ T = \{\mathbf{0}\}$ and $\dim\ (\ker\ T) = 0$.
   By the Dimension Theorem, dim $(\mathbb{R}^3)$ = dim (ker T)+ dim (im T).
   Hence, dim (im T)$= 3 - 0 = 3$. Since set iv is the set of three linearly independent vectors, it is a basis for im $T$.

6. (8 + 2 + 2 pts) Let $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ be a linear transformation defined by

$$T(c_0 + c_1x + c_2x^2) = c_0 + c_1(2x + 1) + c_2(2x + 1)^2.$$

   (a) Determine:
      i. a basis for $\operatorname{im} T$;
      ii. a basis for $\ker T$;
      iii. rank $(T)$ and nullity $(T)$.
      ANSWER:
      If $T(c_0 + c_1x + c_2x^2) = \mathbf{0}$, then $c_0 + c_1(2x + 1) + c_2(2x + 1)^2 = \mathbf{0}$, that is $(c_0 + c_1 + c_2) + (2c_1 + 4c_2)x + 4c_2x^2 = \mathbf{0}$. Equating the coefficients to zero, we obtain a

homogeneous linear system which has only the trivial solution $c_0 = c_1 = c_2 = 0$. So, $\ker T = \{\mathbf{0}\}$ and consequently, nullity $T = \dim(\ker T)$=0 and a basis of $\ker T = \emptyset$. By the Dimension Theorem, $\dim(\mathbb{P}_2) = \dim(\ker T) + \dim(\text{im } T)$. So, rank $T = \dim(\text{im } T) = 3 - 0 = 3$ and $\text{im } T = \mathbb{P}_2$, and its basis will be $\{1,\ x,\ x^2\}$.

(b) Is $T$ one-to-one? Explain.

ANSWER:

¿From part (a), $\ker T = \{\mathbf{0}.\}$ So, by Theorem 2 (Section 7.2) $T$ is one-to-one.

(c) Is $T$ onto? Explain.

ANSWER:

¿From part (a), $\text{im } T = \mathbb{P}_2$. So, $T$ is onto.

7. (3 + 3 + 4 pts)

(a) Which of the following can be added to $\{x^2 - 5,\ x^2 - 3x - 5\}$ to form a basis for $\mathbb{P}_3$?

i. $\{7x,\ x^3\}$

ii. $\{1,\ x,\ x^2\}$

iii. $\{4x^3 + 2x^2,\ 2x^3\}$

iv. $\{15x^3,\ x^3\}$

v. none of the abovementioned.

Explain.

ANSWER: iii.

$\dim(\mathbb{P}_3) = 4$.

$t_1(x^2 - 5) + t_2(x^2 - 3x - 5) + t_3(4x^3 + 2x^2) + t_4(2x^3) = \mathbf{0}$ implies $(-t_1 - t_2)5 - 3t_2x - (t_1 + t_2 + 2t_3)x^2 + (4t_3 + 2t_4)x^3 = \mathbf{0}$. Equating the coefficients to zero, we obtain a homogeneous linear system which has only the trivial solution $t_1 = t_2 = t_3 = t_4 = 0$. Hence,

$\{x^2 - 5,\ x^2 - 3x - 5,\ 4x^3 + 2x^2,\ 2x^3\}$ is a set of four linearly independent vectors that forms a basis for $\mathbb{P}_3$.

(b) Let $T\colon V \to W$ be a linear transformation, where $\dim V = 5$. Then which of the following statements is true?

i. If $T$ is one-to-one, then $\dim(\text{im } T) \leq \dim(\ker T)$

ii. If $\dim(\ker T) = 2$, then $\dim(\text{im } T) = 3$

iii. If $T$ is onto, then $\dim(\ker T) \geq 1$

iv. $T^2 = 1_V$

v. none of the abovementioned.

Explain.

ANSWER: ii.

By the Dimension Theorem, for linear transformation $T\colon V \to W$,

$\dim(V) = \dim(\ker T) + \dim(\text{im } T)$.

So, if $\dim(V) = 5$ and $\dim(\ker T) = 2$, then

$\dim(\text{im } T) = \dim(V) - \dim(\ker T) = 5 - 2 = 3$.

(c) Show that $\{A_1, A_2, \cdots, A_k\}$ is a basis for $\mathbb{M}_{mn}$ if and only if $\{A_1^T, A_2^T, \cdots, A_k^T\}$ is a basis for $\mathbb{M}_{nm}$.

ANSWER:
Let $T\colon \mathbb{M}_{mn} \to \mathbb{M}_{nm}$ be a linear transformation defined by $T(A) = A^T, \;\; \forall A \in \mathbb{M}_{mn}$.
Then $T(A) = \mathbf{0}$, i.e. a zero matrix of $[n \times m] \implies A = \mathbf{0}$, i.e. a zero matrix of $[m \times n]$.
Hence, ker $T = \{\mathbf{0}\}$ and consequently, a basis for ker $T$ is $\emptyset$.
On the other hand, dim $(\mathbb{M}_{mn}) = mn = nm = \dim(\mathbb{M}_{nm})$.
Therefore, if $\{A_1, A_2, \cdots, A_k\}$ is a basis for $\mathbb{M}_{mn}$ and $\emptyset$ is a basis for ker $T$, then $k = mn$ and by Theorem 5 (Section 7.2), $\{T(A_1), T(A_2), \cdots, T(A_k)\} = \{A_1^T, A_2^T, \cdots, A_k^T\}$ will be a basis for im $T = \mathbb{M}_{nm}$.
Similarly, since $(A^T)^T = A$, applying Theorem 5 (Section 7.2) to a linear transformation $S\colon \mathbb{M}_{nm} \to \mathbb{M}_{mn}$ defined by $S(A^T) = (A^T)^T = A, \;\; \forall A^T \in \mathbb{M}_{nm}$, we obtain that if $\{A_1^T, A_2^T, \cdots, A_k^T\}$ is a basis for $\mathbb{M}_{nm}$, then $\{A_1, A_2, \cdots, A_k\}$ is a basis for $\mathbb{M}_{mn}$.

The end.