Call Pricing

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1 Call Price Derivation

This relationship needs to incorporate risk-free rates; however, until I figure out how to do that I'll see where the math takes this.

Let $K \geq 0$ be the option's strike price, $p(x)$ a probability density function over the underlying's price at expiration, $c \geq 0$ be the option's premium (what we are trying to solve for), and $f(x, K, c)$ be the option's payout function give the price of the underlying at expiration and option's premium.

If the premium of a call is to ensure that the bet has a net payout of zero then the following relationship should hold.

$$\int_0^\infty f(x, K, c)p(x)dx = 0 \tag{1}$$

Notice that the payout of call is only positive past the break even $(K + c)$ and only negative before it, thus.

$$\int_0^{K+c} f(x, K, c)p(x)dx = -\int_{K+c}^\infty f(x, K, c)p(x)dx \tag{2}$$

While $p(x)$ is very hard to obtain, $f(x, K, c)$ has a known form.

$$f(x, K, c) = \begin{cases} 100(x - K - c) & x \geq K \\ -100c & x < K \end{cases}$$

Equation 2 becomes:

$$\int_0^K -100cp(x)dx + \int_K^{K+c} 100(x - K - c)p(x)dx = -\int_{K+c}^\infty 100(x - K - c)p(x)dx$$

$$c\int_0^K p(x)dx - \int_K^{K+c} (x - K - c)p(x)dx = \int_{K+c}^\infty (x - K - c)p(x)dx$$

$$c\int_0^K p(x)dx = \int_K^\infty (x - K - c)p(x)dx$$

$$
\begin{aligned}
c\int_0^K p(x)dx &= \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx - c\int_K^\infty p(x)dx \\
c\int_0^\infty p(x)dx &= \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx \\
c &= \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx \qquad (3)
\end{aligned}
$$

or

$$c = \int_K^\infty (x-K)p(x)dx$$

The jupyter notebook and python files that use a discrete approximation of $p(x)$ via Monte-Carlo simulation will utilize Equation 2 and a simple algorithm to calculate call price. I call Equations 1, 2, and 3 model-free since they give us call prices for any model of $p(x)$. Notice Equation 3 is reminiscent of the black-scholes-merton call price equation.

2 LogNormal Model

Given a lognormal model of $p(x)$ parameterized by μ and σ, we can simplify Equation 3 even further.

$$p(x) = \frac{e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}$$

$$c = \frac{1}{2}(e^{\frac{\sigma^2}{2}+\mu}\text{erf}(\frac{\ln x - \sigma^2 - \mu}{\sqrt{2}\sigma}) - K\text{erf}(\frac{\ln x - \mu}{\sqrt{2}\sigma}))\Big|_K^\infty$$

$$c = \frac{1}{2}(e^{\frac{\sigma^2}{2}+\mu}(1 - \text{erf}(\frac{\ln k - \sigma^2 - \mu}{\sqrt{2}\sigma})) - K(1 - \text{erf}(\frac{\ln k - \mu}{\sqrt{2}\sigma}))) \qquad (4)$$

$p(x)$ is not a distribution we can measure, as it is in the future. However, if we are given a log-normal distribution for the return of the underlying asset at some interval (2 minutes in the code), then we can use that to approximate $p(x)$.

Let $X_i \sim \mathcal{N}(\mu_{\text{return}}, \sigma^2_{\text{return}})$, $Z_i = e^{X_i}$ representing the return at each interval, s_0 is the spot price of the asset (constant), and s_T is a random variable taken from the distribution of the underlying asset's price at time T (or simply $p(x)$). We can find the μ and σ of $p(x)$ from observing the following:

$$s_T = s_0 \prod_{i=1}^{T} Z_i$$

$$s_T = s_0 \exp(\sum_{i=1}^{T} X_i)$$

$$s_T = \exp(\ln s_0 + \sum_{i=1}^{T} X_i)$$

$$\mu = T * \mu_{\text{return}} + \ln s_0$$

$$\sigma = \sqrt{T} * \sigma_{\text{return}}$$

Where T is the number of return intervals (NOT DAYS) until expiration. Thus, we can approximate call price without monte-carlo simulation!