

# Call Pricing

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## 1 Call Price Derivation

This relationship needs to incorporate risk-free rates; however, until I figure out how to do that I'll see where the math takes this.

Let  $K \geq 0$  be the option's strike price,  $p(x)$  a probability density function over the underlying's price at expiration,  $c \geq 0$  be the option's premium (what we are trying to solve for), and  $f(x, K, c)$  be the option's payout function give the price of the underlying at expiration and option's premium.

If the premium of a call is to ensure that the bet has a net payout of zero then the following relationship should hold.

$$\int_0^{\infty} f(x, K, c)p(x)dx = 0 \quad (1)$$

Notice that the payout of call is only positive past the break even ( $K + c$ ) and only negative before it, thus.

$$\int_0^{K+c} f(x, K, c)p(x)dx = - \int_{K+c}^{\infty} f(x, K, c)p(x)dx \quad (2)$$

While  $p(x)$  is very hard to obtain,  $f(x, K, c)$  has a known form.

$$f(x, K, c) = \begin{cases} 100(x - K - c) & x \geq K \\ -100c & x < K \end{cases}$$

Equation 2 becomes:

$$\begin{aligned} \int_0^K -100cp(x)dx + \int_K^{K+c} 100(x - K - c)p(x)dx &= - \int_{K+c}^{\infty} 100(x - K - c)p(x)dx \\ c \int_0^K p(x)dx - \int_K^{K+c} (x - K - c)p(x)dx &= \int_{K+c}^{\infty} (x - K - c)p(x)dx \\ c \int_0^K p(x)dx &= \int_K^{\infty} (x - K - c)p(x)dx \end{aligned}$$

$$\begin{aligned}
c \int_0^K p(x)dx &= \int_K^\infty xp(x)dx - K \int_K^\infty p(x)dx - c \int_K^\infty p(x)dx \\
c \int_0^\infty p(x)dx &= \int_K^\infty xp(x)dx - K \int_K^\infty p(x)dx \\
c &= \int_K^\infty xp(x)dx - K \int_K^\infty p(x)dx
\end{aligned} \tag{3}$$

or

$$c = \int_K^\infty (x - K)p(x)dx$$

The jupyter notebook and python files that use a discrete approximation of  $p(x)$  via Monte-Carlo simulation will utilize Equation 2 and a simple algorithm to calculate call price. I call Equations 1, 2, and 3 model-free since they give us call prices for any model of  $p(x)$ . Notice Equation 3 is reminiscent of the black-scholes-merton call price equation.

## 2 LogNormal Model

Given a lognormal model of  $p(x)$  parameterized by  $\mu$  and  $\sigma$ , we can simplify Equation 3 even further.

$$\begin{aligned}
p(x) &= \frac{e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \\
c &= \frac{1}{2} \left( e^{\frac{\sigma^2}{2} + \mu} \operatorname{erf}\left(\frac{\ln x - \sigma^2 - \mu}{\sqrt{2}\sigma}\right) - K \operatorname{erf}\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right) \right) \Big|_K^\infty \\
c &= \frac{1}{2} \left( e^{\frac{\sigma^2}{2} + \mu} \left( 1 + \operatorname{erf}\left(\frac{\ln k - \sigma^2 - \mu}{\sqrt{2}\sigma}\right) \right) - K \left( 1 + \operatorname{erf}\left(\frac{\ln k - \mu}{\sqrt{2}\sigma}\right) \right) \right)
\end{aligned} \tag{4}$$