Call Pricing

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1 Call Price Derivation

This relationship needs to incorporate risk-free rates; however, until I figure out how to do that I'll see where the math takes this.

Let $K \geq 0$ be the option's strike price, $p(x)$ a probability density function over the underlying's price at expiration, $c \geq 0$ be the option's premium (what we are trying to solve for), and $f(x, K, c)$ be the option's payout function given the price of the underlying at expiration and option's premium.

If the premium of a call is to ensure that the bet has a net payout of zero then the following relationship should hold.

$$\int_0^\infty f(x, K, c)p(x)dx = 0 \tag{1}$$

Notice that the payout of call is only positive past the break even $(K + c)$ and only negative before it, thus.

$$\int_0^{K+c} f(x, K, c)p(x)dx = -\int_{K+c}^\infty f(x, K, c)p(x)dx \tag{2}$$

While $p(x)$ is very hard to obtain, $f(x, K, c)$ has a known form.

$$f(x, K, c) = \begin{cases} 100(x - K - c) & x \geq K \\ -100c & x < K \end{cases}$$

Equation 2 becomes:

$$\int_0^K -100cp(x)dx + \int_K^{K+c} 100(x-K-c)p(x)dx = -\int_{K+c}^\infty 100(x-K-c)p(x)dx$$

$$c\int_0^K p(x)dx - \int_K^{K+c} (x - K - c)p(x)dx = \int_{K+c}^\infty (x - K - c)p(x)dx$$

$$c\int_0^K p(x)dx = \int_K^\infty (x - K - c)p(x)dx$$

$$c\int_0^K p(x)dx = \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx - c\int_K^\infty p(x)dx$$
$$c\int_0^\infty p(x)dx = \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx$$
$$c = \int_K^\infty xp(x)dx - K\int_K^\infty p(x)dx \tag{3}$$

or

$$c = \int_K^\infty (x-K)p(x)dx$$

The jupyter notebook and python files that use a discrete approximation of $p(x)$ via Monte-Carlo simulation will utilize Equation 2 and a simple algorithm to calculate call price. I call Equations 1, 2, and 3 model-free since they give us call prices for any model of $p(x)$. Notice Equation 3 is reminiscent of the black-scholes-merton call price equation.

2 LogNormal Model

Given a lognormal model of $p(x)$ parameterized by μ and σ, we can simplify Equation 3 even further.

$$p(x) = \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}$$
$$c = \frac{1}{2}(e^{\frac{\sigma^2}{2}+\mu}\text{erf}(\frac{\ln x - \sigma^2 - \mu}{\sqrt{2}\sigma}) - K\text{erf}(\frac{\ln x - \mu}{\sqrt{2}\sigma}))\Big|_K^\infty$$
$$c = \frac{1}{2}(e^{\frac{\sigma^2}{2}+\mu}(1-\text{erf}(\frac{\ln k - \sigma^2 - \mu}{\sqrt{2}\sigma})) - K(1-\text{erf}(\frac{\ln k - \mu}{\sqrt{2}\sigma}))) \tag{4}$$

$p(x)$ is not a distribution we can measure, as it is in the future. However, if we are given a log-normal distribution for the return of the underlying asset at some interval (1 minute in the code), then we can use that to approximate $p(x)$.

Let $X_i \sim \mathcal{N}(\mu_{\text{return}}, \sigma^2_{\text{return}})$, $Z_i = e^{X_i}$ representing the return at each interval, s_0 is the spot price of the asset (constant), and s_T is a random variable taken from the distribution of the underlying asset's price at time T (or simply $p(x)$). We can find the μ and σ of $p(x)$ from observing the following:

$$s_T = s_0\prod_{i=1}^T Z_i$$

$$s_T = s_0\exp(\sum_{i=1}^T X_i)$$

$$s_T = \exp(\ln s_0 + \sum_{i=1}^{T} X_i)$$
$$\mu = T * \mu_{\text{return}} + \ln s_0$$
$$\sigma = \sqrt{T} * \sigma_{\text{return}}$$

Where T is the number of return intervals (NOT DAYS) until expiration. Thus, we can approximate call price without monte-carlo simulation! Note that if we assume martingality of stock price, then $\mu_{\text{return}} = 0$ and $\mu = \ln s_0$.

3 Kelly for Buying Call Options

This derivation is dubious and intuitive. This is NOT a proof. I am simply listing my thoughts. Here, c is the call premium offered by the market, NOT the call premium we calculate earlier.

$$E_{\text{binary}} = P(\text{Loss}) \ln(1 - f) + P(\text{Win}) \ln(1 + fb)$$

For a call option:

$$E = \ln(1 - f) \int_0^K p(x)dx + \int_K^\infty \ln(1 + f\frac{x - K - c}{c})p(x)dx$$

Thus,

$$\frac{dE}{df} = -\frac{\int_0^K p(x)dx}{1 - f} + \int_K^\infty \frac{p(x)}{1 + f\frac{x-K-c}{c}} \frac{x - K - c}{c} dx$$

We want to find the f^* that sets $\frac{dE}{df}(f^*) = 0$.

$$(1 - f^*) \int_K^\infty \frac{p(x)}{1 + f^*\frac{x-K-c}{c}} \frac{x - K - c}{c} dx - \int_0^K p(x)dx = 0$$

The leftmost integral is intractable even with the assumption that $p(x)$ is log-normal, so I solve the integral using numerical methods.

4 Kelly for Stocks

The derivation is also dubious. This will use a return distribution (say 1 minute) I will denote as $r(x)$.

$$E = \int_0^\infty r(x) \ln(1 + (x - 1)f)dx$$
$$\frac{dE}{df} = \int_0^\infty (x - 1)\frac{r(x)}{1 + (x - 1)f} dx$$

Again, this integral is intractable so I solve for f^* numerically.

$$\int_0^\infty (x-1)\frac{r(x)}{1+(x-1)f^*}dx = 0$$

The formula indicates that Kelly has nothing to do with the asset price itself, which makes sense since we are trying to find a ratio of our bankroll we are willing to spend on the owning the asset!