

Call Pricing

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1 Call Price Derivation

This relationship needs to incorporate risk-free rates; however, until I figure out how to do that I'll see where the math takes this.

Let $K \geq 0$ be the option's strike price, $\rho(x)$ a probability density function over the underlying's price at expiration, $c \geq 0$ be the option's premium (what we are trying to solve for), and $f(x, K, c)$ be the option's payout function given the price of the underlying at expiration and option's premium.

If the premium of a call is to ensure that the bet has a net payout of zero then the following relationship should hold.

$$\int_0^{\infty} f(x, K, c) \rho(x) dx = 0 \quad (1)$$

Notice that the payout of call is only positive past the break even ($K + c$) and only negative before it, thus.

$$\int_0^{K+c} f(x, K, c) \rho(x) dx = - \int_{K+c}^{\infty} f(x, K, c) \rho(x) dx \quad (2)$$

While $\rho(x)$ is very hard to obtain, $f(x, K, c)$ has a known form.

$$f(x, K, c) = \begin{cases} 100(x - K - c) & x \geq K \\ -100c & x < K \end{cases}$$

Equation 2 becomes:

$$\begin{aligned} \int_0^K -100c \rho(x) dx + \int_K^{K+c} 100(x - K - c) \rho(x) dx &= - \int_{K+c}^{\infty} 100(x - K - c) \rho(x) dx \\ c \int_0^K \rho(x) dx - \int_K^{K+c} (x - K - c) \rho(x) dx &= \int_{K+c}^{\infty} (x - K - c) \rho(x) dx \\ c \int_0^K \rho(x) dx &= \int_K^{\infty} (x - K - c) \rho(x) dx \end{aligned}$$

$$\begin{aligned}
c \int_0^K \rho(x) dx &= \int_K^\infty x \rho(x) dx - K \int_K^\infty \rho(x) dx - c \int_K^\infty \rho(x) dx \\
c \int_0^\infty \rho(x) dx &= \int_K^\infty x \rho(x) dx - K \int_K^\infty \rho(x) dx \\
c &= \int_K^\infty x \rho(x) dx - K \int_K^\infty \rho(x) dx
\end{aligned} \tag{3}$$

or

$$c = \int_K^\infty (x - K) \rho(x) dx$$

The jupyter notebook and python files that use a discrete approximation of $\rho(x)$ via Monte-Carlo simulation will utilize Equation 2 and a simple algorithm to calculate call price. I call Equations 1, 2, and 3 model-free since they give us call prices for any model of $\rho(x)$. Notice Equation 3 is reminiscent of the black-scholes-merton call price equation. We can use similar methods to derive an equation for pricing puts, p , using $\rho(x)$.

$$p = - \int_0^K (x - K) \rho(x) dx \tag{4}$$

2 LogNormal Model

Given a lognormal model of $\rho(x)$ parameterized by μ and σ , we can simplify Equation 3 even further.

$$\begin{aligned}
\rho(x) &= \frac{e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \\
c &= \frac{1}{2} \left(e^{\frac{\sigma^2}{2} + \mu} \operatorname{erf}\left(\frac{\ln x - \sigma^2 - \mu}{\sqrt{2}\sigma}\right) - K \operatorname{erf}\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right) \right) \Big|_K^\infty \\
c &= \frac{1}{2} \left(e^{\frac{\sigma^2}{2} + \mu} (1 - \operatorname{erf}\left(\frac{\ln k - \sigma^2 - \mu}{\sqrt{2}\sigma}\right)) - K (1 - \operatorname{erf}\left(\frac{\ln k - \mu}{\sqrt{2}\sigma}\right)) \right)
\end{aligned} \tag{4}$$

$\rho(x)$ is not a distribution we can measure, as it is in the future. However, if we are given a log-normal distribution for the return of the underlying asset at some interval (1 minute in the code), then we can use that to approximate $\rho(x)$.

Let $X_i \sim \mathcal{N}(\mu_{\text{return}}, \sigma_{\text{return}}^2)$, $Z_i = e^{X_i}$ representing the return at each interval, s_0 is the spot price of the asset (constant), and s_T is a random variable taken from the distribution of the underlying asset's price at time T (or simply $\rho(x)$). We can find the μ and σ of $\rho(x)$ from observing the following:

$$s_T = s_0 \prod_{i=1}^T Z_i$$

$$\begin{aligned}
s_T &= s_0 \exp\left(\sum_{i=1}^T X_i\right) \\
s_T &= \exp\left(\ln s_0 + \sum_{i=1}^T X_i\right) \\
\mu &= T * \mu_{\text{return}} + \ln s_0 \\
\sigma &= \sqrt{T} * \sigma_{\text{return}}
\end{aligned}$$

Where T is the number of return intervals (NOT DAYS) until expiration. Thus, we can approximate call price without monte-carlo simulation! Note that if we assume martingality of stock price, then $\mu_{\text{return}} = 0$ and $\mu = \ln s_0$.

3 Kelly for Buying Call Options

This derivation is dubious and intuitive. This is NOT a proof. I am simply listing my thoughts. Here, c is the call premium offered by the market, NOT the call premium we calculate earlier.

$$E_{\text{binary}} = P(\text{Loss}) \ln(1 - f) + P(\text{Win}) \ln(1 + fb)$$

For a call option:

$$E = \ln(1 - f) \int_0^K \rho(x) dx + \int_K^\infty \ln\left(1 + f \frac{x - K - c}{c}\right) \rho(x) dx$$

Thus,

$$\frac{dE}{df} = -\frac{\int_0^K \rho(x) dx}{1 - f} + \int_K^\infty \frac{\rho(x)}{1 + f \frac{x - K - c}{c}} \frac{x - K - c}{c} dx$$

We want to find the f^* that sets $\frac{dE}{df}(f^*) = 0$.

$$(1 - f^*) \int_K^\infty \frac{\rho(x)}{1 + f^* \frac{x - K - c}{c}} \frac{x - K - c}{c} dx - \int_0^K \rho(x) dx = 0$$

The leftmost integral is intractable even with the assumption that $\rho(x)$ is log-normal, so I solve the integral using numerical methods. It may just be easier to find the minimum of E in the interval $(0, 1)$ via line search, since the equation above may have multiple zeros. Again, using similar methods I derive a Kelly Equations for puts.

$$\begin{aligned}
(1 - f^*) \int_0^K \frac{\rho(x)}{1 + f^* \frac{x - K + p}{p}} \frac{x - K + p}{p} dx + \int_K^\infty \rho(x) dx &= 0 \\
E_{\text{put}} = \ln(1 - f) \int_0^K \rho(x) dx + \int_0^K \ln\left(1 - f \frac{x - K + p}{p}\right) \rho(x) dx &
\end{aligned}$$

4 Kelly for Stocks

The derivation is also dubious. This will use a return distribution (say 1 minute) I will denote as $r(x)$.

$$E = \int_0^\infty r(x) \ln(1 + (x-1)f) dx$$

$$\frac{dE}{df} = \int_0^\infty (x-1) \frac{r(x)}{1 + (x-1)f} dx$$

Again, this integral is intractable so I solve for f^* numerically.

$$\int_0^\infty (x-1) \frac{r(x)}{1 + (x-1)f^*} dx = 0$$

The formula indicates that Kelly has nothing to do with the asset price itself, which makes sense since we are trying to find a ratio of our bankroll we are willing to spend on the owning the asset!