Game Engine Architecture

Chapter 5

3D Math for Games

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Topics

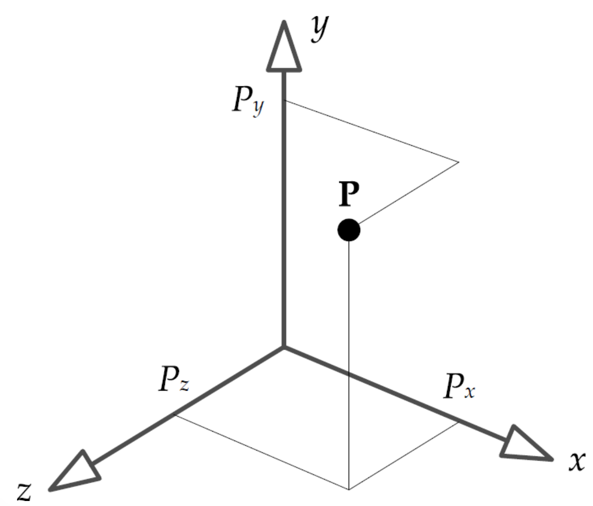
- Math for games
  - Points and Vectors
  - Matrices
  - Quaternions
  - Comparison of rotational representations
  - Other useful math objects
  - Hardware-accelerated SIMD Math
  - Random Number Generation

2D as a Start

- Most operations in 3D also make sense in 2D
- Always use 2D as a basis, it'll help

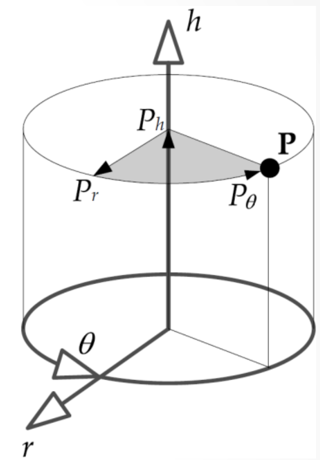
Points and vectors

- A point is a location in n-dimensional space
- Usually represented in Cartesian space
    - Two or three mutually perpendicular axes
    - A point is a triple of numbers (P_x, P_y, P_z)

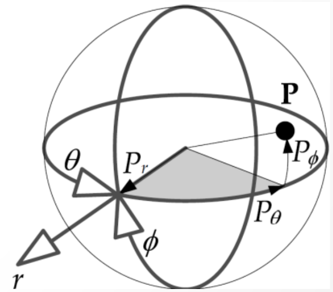


Other systems

- Cylindrical
  - Employs a height axis (h), a radial axis (r), and a yaw angle (θ)
  - Points represented as (P_h,P_r,P_θ)



- Spherical
  - Pitch(ϕ), yaw(θ), and radial (r)
  - Points represented as (P_r,P_ϕ,P_θ)

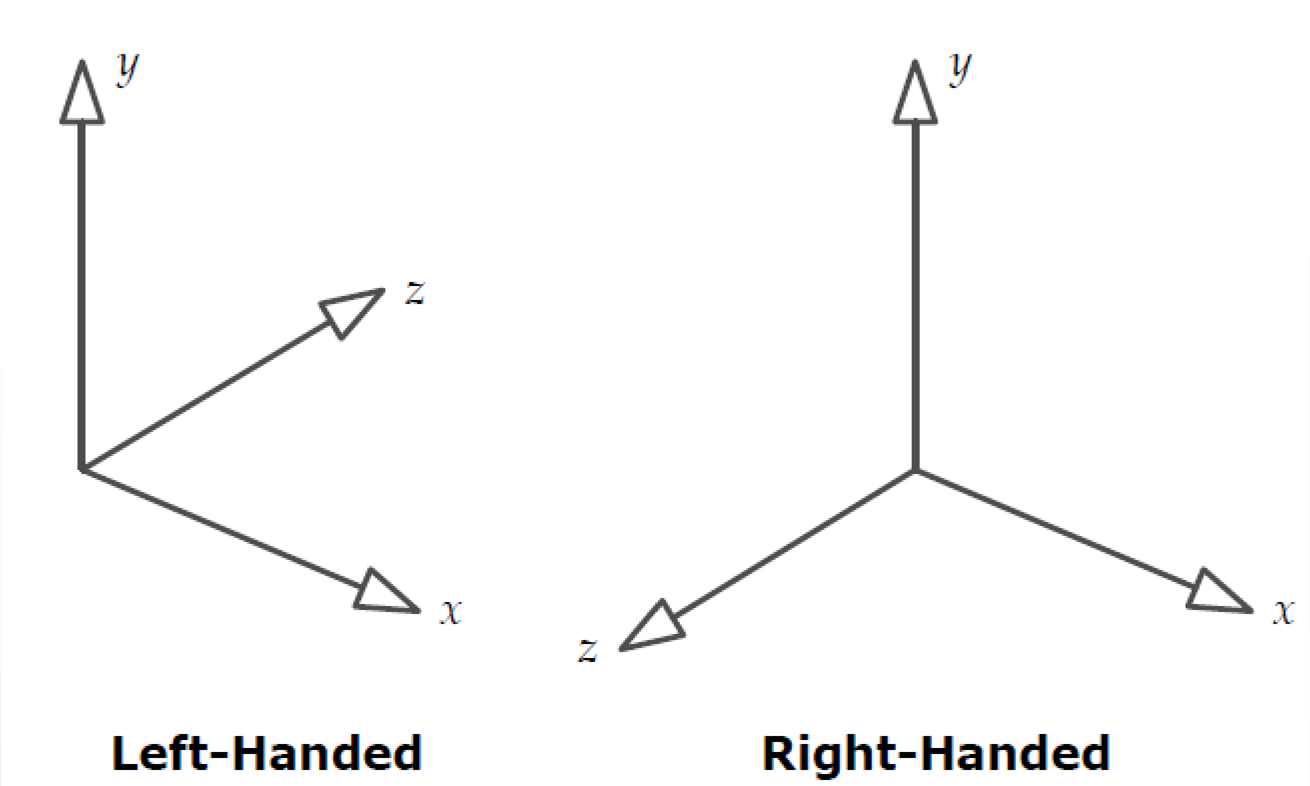


Picking a system

- Most people use Cartesian
- Sometimes it is easier to use something else
    - Swirling objects around a character are easier in cylindrical
    - Explosions may be easier in spherical

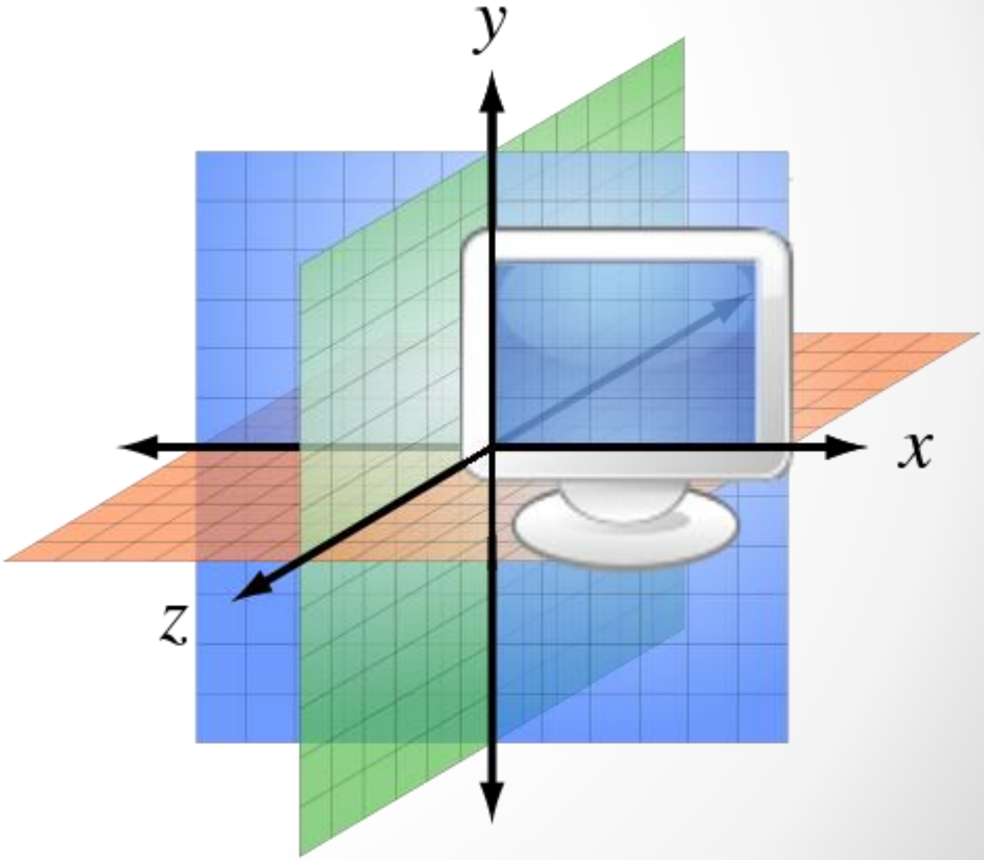
Left vs Right

- When using Cartesian coordinates, you get to choose either left or right handed coordinate systems



Ogre

- **Ogre** is using a **right-handed** coordinate system like OpenGL (DirectX by default is left-handed)



Left vs Right

- Converting is easy, flip the direction of any one axis.
- Graphics programmers typically pick left-handed with y pointing up, x to the right, and z pointing into the screen
- Helps with z-buffering

Vectors

- Vector have
  - Magnitude
  - Direction
- Extend from the tail toward the head
- Differs from a scalar because of the direction
- Technically represents an offset relative to a known point
- Can be used to represent a point if it is relative to the origin
  - Called a position vector versus direction vector

Basis Vectors

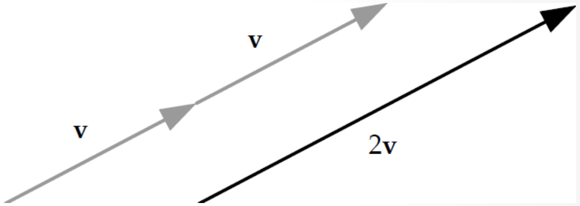
- Sometimes useful to define orthogonal unit vectors
  - **i** is along the x-axis
  - **j** is along the y-axis
  - **k** is along the z-axis
- Can now represent points as the sum of scalars multiplied by the unit vectors

$$(5,3,-2) = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

Vector Operations

- Multiply by a scalar - scales the magnitude without affecting direction

$$s\mathbf{a} = (sa_x, sa_y, sa_z)$$



- Non-uniform scaling

$$\mathbf{s}\otimes\mathbf{a} = (s_x a_x, s_y a_y, s_z a_z)$$

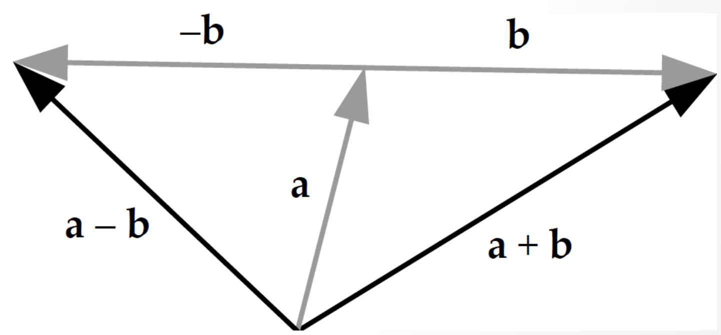
$$a\mathbf{S} = \begin{bmatrix} a_x & a_y & a_y \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} = \begin{bmatrix} a_x s_x & a_y s_y & a_z s_z \end{bmatrix}$$

Vector operations

- Addition and Subtraction

$$a + b = \left[(a_x + b_x), (a_y + b_y), (a_z + b_z)\right]$$
$$a - b = \left[(a_x - b_x), (a_y - b_y), (a_z - b_z)\right]$$

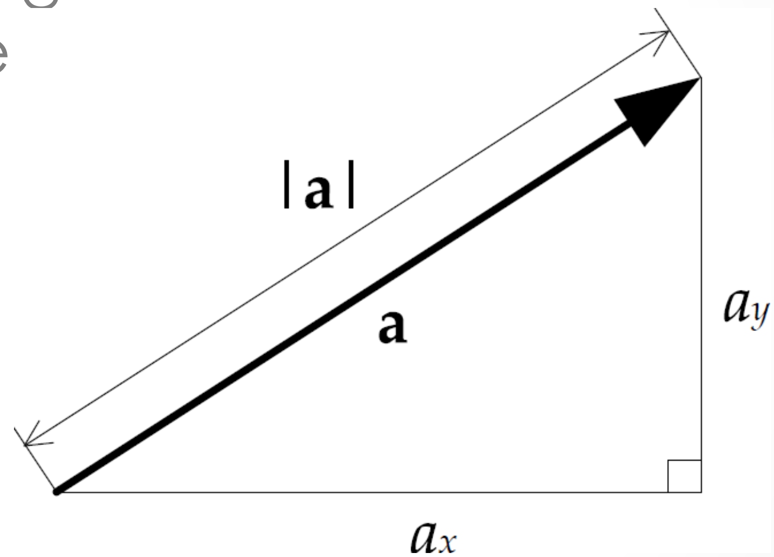
- Points and directions
  - direction + direction = direction
  - direction – direction = direction
  - point + direction = point
  - point – point = direction
  - point + point = crap



Vector Operations

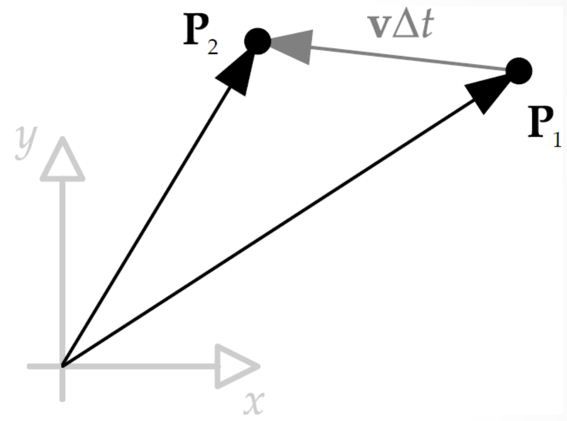
- Calculating magnitude – think of it as distance
- We can use the Pythagorean theorem to calculate a vector's magnitude

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

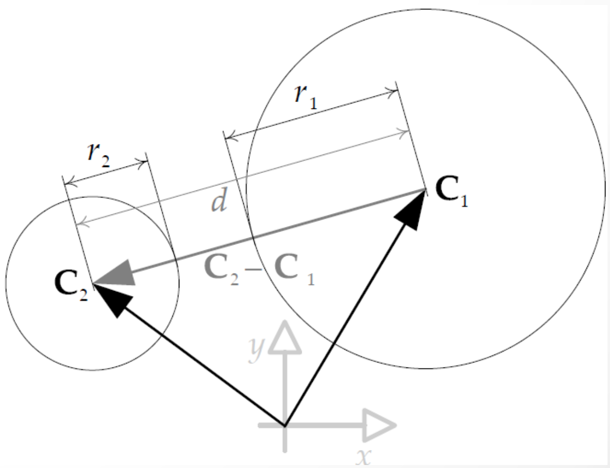


Use of Vector Operations

- Moving objects
  - $\mathbf{P_2} = \mathbf{P_1} + \mathbf{v}\Delta \boldsymbol{t}$



- Object collision
  - if $d < r_1 + r_2$ then they collide
  - Faster to compare $d^2 < (r_1 + r_2)^2$



Unit Vectors

- Sometimes you need a unit length vector in the same direction as the original – called normalization, not a normal vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{v}\mathbf{v}$
- A vector is said to be *normal* to a surface if it is *perpendicular* to that surface.
- Lighting calculations make heavy use of normal vectors to define the direction of surfaces relative to the direction of the light rays.
- Normal vectors are usually of unit length

DOT product

- Add the components of the vector

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

- Also written as

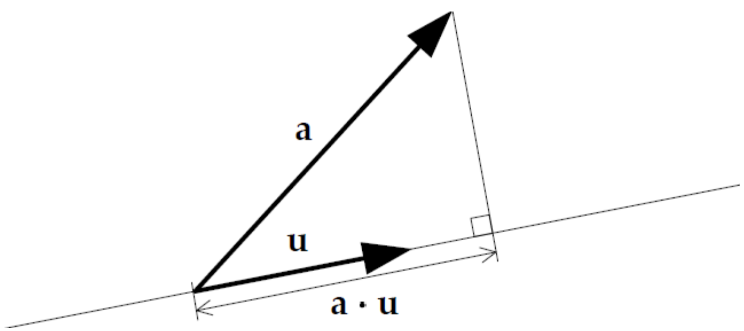
$$\mathbf{a} \cdot \mathbf{b} = |a||b| \cos(\theta)$$

- It is commutative, distributive, and works with scalar multiplication

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
$$s\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot s\mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

Vector Projection

- If **u** is a unit vector: |u| = 1
- then the dot product of **a** and **u** represents the length of the projection of **a** onto **u**

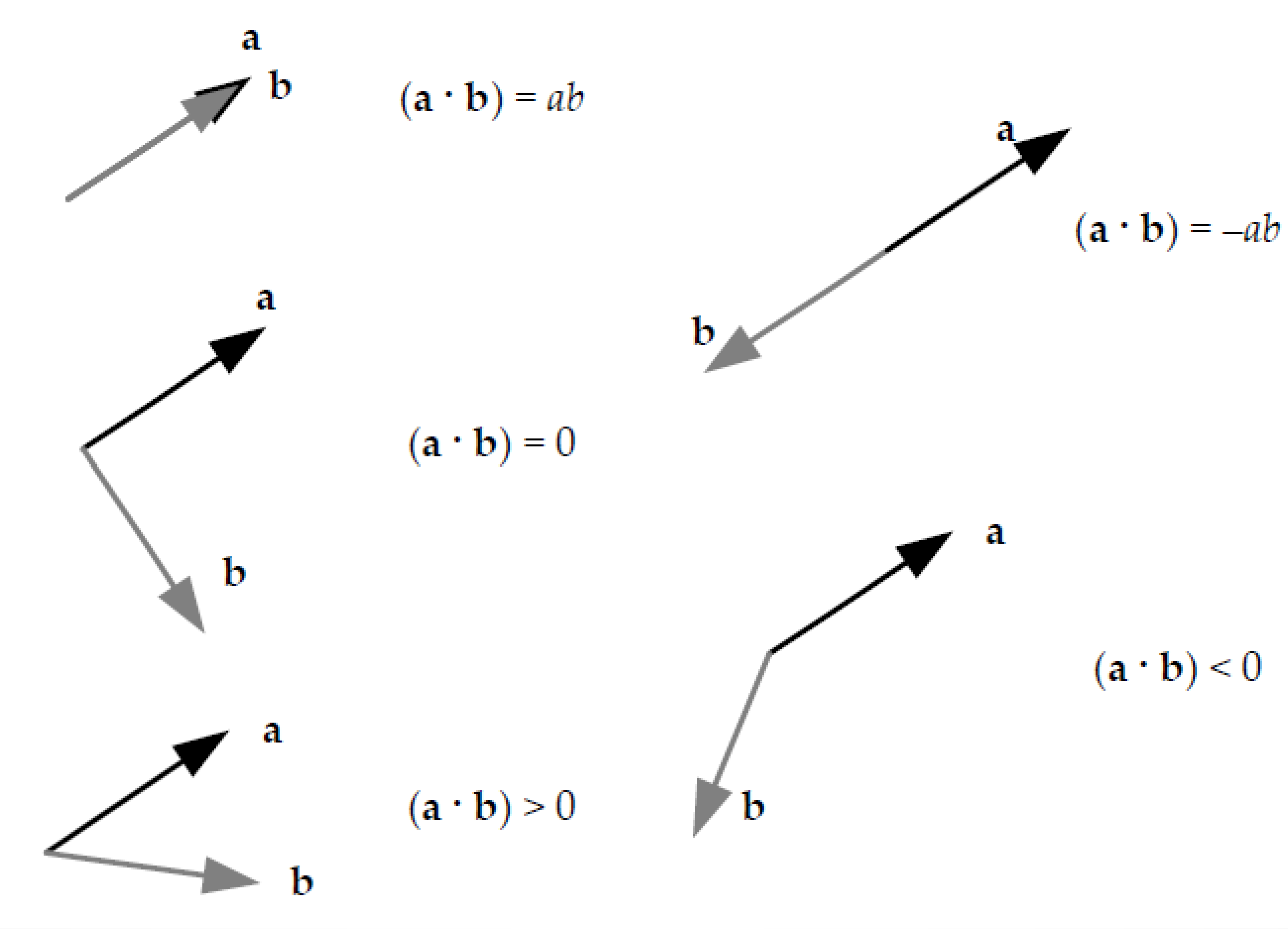


Magnitude

- You can use the dot product to find the magnitude as well

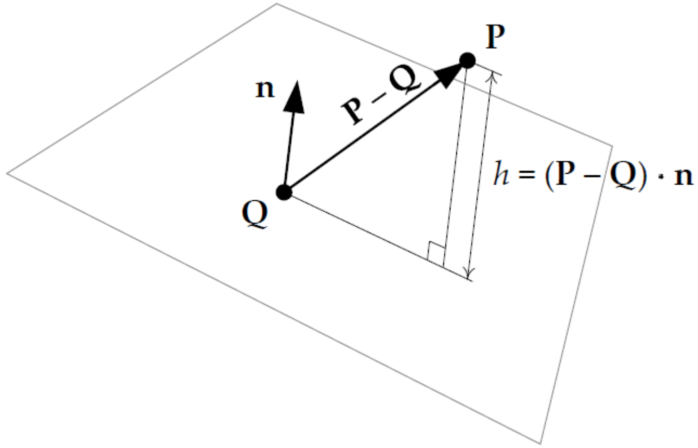
$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$
$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Useful Dot Product Tests



Other applications

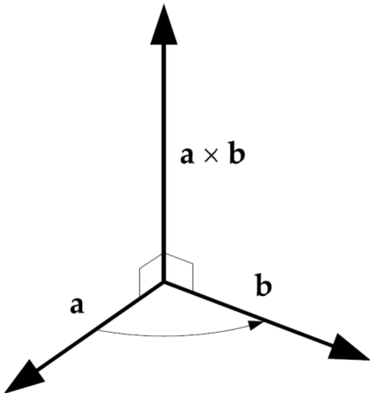
- Visibility – can player P see enemy E
  - **v** = E-P gives the relative position of E in relation to P
  - if **f** is the facing vector of P then when **v**·**f** is negative E is behind P
- If we define a plane as a point **Q** and a normal **n** then we can find the height *h* of a point P above the plane using projection



Cross Product

- Yields another vector that is perpendicular to the vectors being multiplied

$$\mathbf{a} \times \mathbf{b} = [(a_y b_z - a_z b_y), (a_z b_x - a_x b_z), (a_x b_y - a_y b_x)]$$
$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

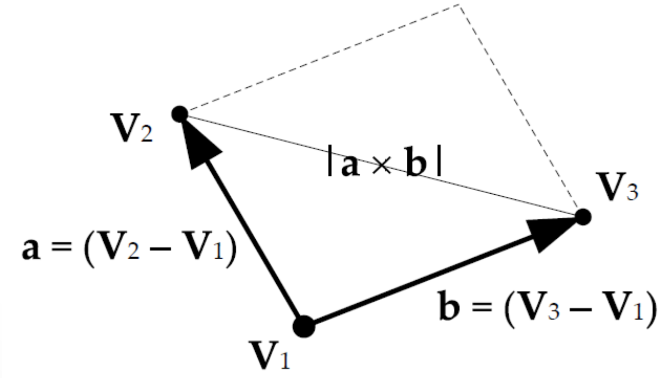
- 

Magnitude of X-product

- Magnitude of the cross product is the area of the parallelogram formed by the two vectors

$$|\mathbf{a} \times \mathbf{b}| = |a||b|\sin(\theta)$$

$$A_{triangle} = \frac{1}{2}|(V_2 - V_1) \times (V_3 - V_1)|$$



Properties of the cross product

- It is not commutative

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

- It is anti-commutative

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

- Distributive over addition

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + (\mathbf{a} \times \mathbf{c})$$

- Combine with scalar multiplication

$$(s\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s\mathbf{b}) = s(\mathbf{a} \times \mathbf{b})$$

- Cartesian basis vector are related by cross product

$$(\mathbf{i} \times \mathbf{j}) = \mathbf{k} \qquad (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \qquad (\mathbf{k} \times \mathbf{i}) = \mathbf{j}$$

Uses of the cross product

- Finding a vector that is perpendicular to two other vectors
- Finding the normal vector to a plane

$$\mathbf{n} = normalize\big((\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_1)\big)$$

- Calculate torque
    - Given a force **F** and a vector **r** from the center of mass the torque is

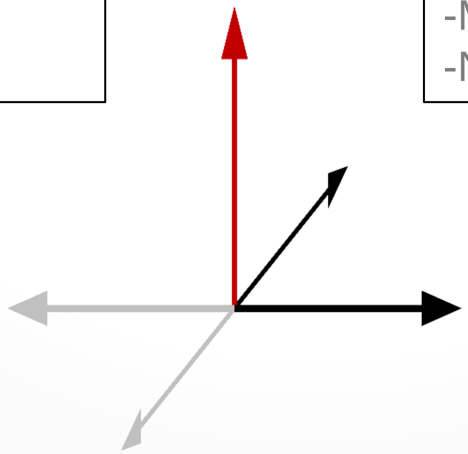
$$\mathbf{N} = \mathrm{r} \times \mathbf{F}$$

Pseudovectors

- Cross product doesn't actually produce a vector
  - Creates a pseudovector
- Difference is quite subtle – only apparent on reflection
  - Vectors reflect to mirror image
  - Pseudovectors reflect to mirror image, but also change direction

<u>Vectors</u>
-Position
-Velocity
-Acceleration

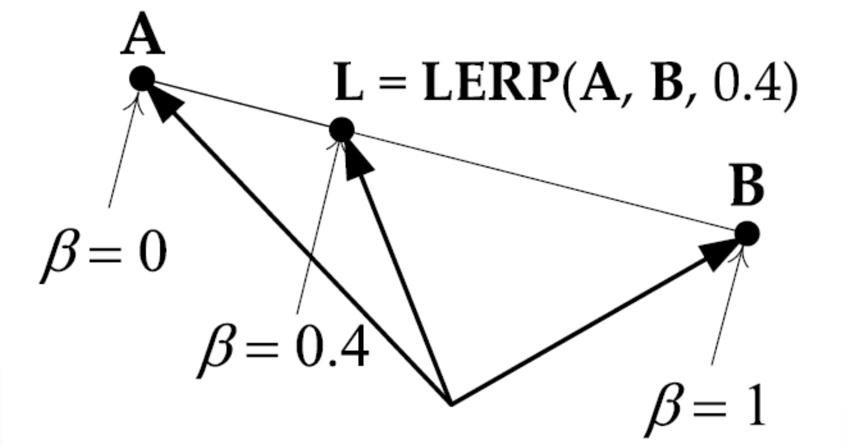
<u>Psuedovectors</u>
-Angular velocity
-Magnetic fields
-Normals to a surface



LERP

- A simple linear interpolation between 2 points
  (β ranges from 0 to 1)

$$\mathbf{L} = \mathrm{LERP}(\mathbf{A}, \mathbf{B}, \beta) = (1 - \beta)\mathbf{A} + \beta\mathbf{B}$$
$$= \left[(\mathbf{1} - \beta)\mathbf{A}_x + \beta\mathbf{B}_x, (\mathbf{1} - \beta)\mathbf{A}_y + \beta\mathbf{B}_y, (\mathbf{1} - \beta)\mathbf{A}_z + \beta\mathbf{B}_z\right]$$



Matrices

- A matrix is a rectangular array of *m* x *n* scalars

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

- They are convenient for representing translation, rotation, and scale operations
- When all rows and columns are of unit magnitude it is called isotropic or orthonormal – represent pure rotations

4x4 Matrix

- Sometimes 4x4 matrices are used to represent 3D transformations
  - They call them transformation matrices
- Affine matrix is a 4 x 4 transformation matrix that
  - Preserves parallelism of line
  - Relative distance ratio
  - Not necessarily absolute lengths and angles
- Affine matrix is any combination of translation, rotations, scale or shear

Matrix Multiplication

- The product of two matrices is written as **P**=**AB**, if **A** and **B** are transformation matrices then so is **P**
- Multiplication is done by taking the dot products of the rows of **A** and the columns of **B.** The inner dimensions must be the same $n_A=m_B$.

Multiplication

$$\boldsymbol{P} = \mathbf{AB}$$

$$\boldsymbol{P} = \begin{bmatrix} P_{11} = A_{row1} \cdot B_{col1} & P_{12} = A_{row1} \cdot B_{col2} & P_{13} = A_{row1} \cdot B_{col3} \\ P_{21} = A_{row2} \cdot B_{col1} & P_{22} = A_{row2} \cdot B_{col2} & P_{23} = A_{row2} \cdot B_{col3} \\ P_{31} = A_{row3} \cdot B_{col1} & P_{32} = A_{row3} \cdot B_{col2} & P_{33} = A_{row3} \cdot B_{col3} \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

Points and Vectors as Matrices

- Can represent points or vectors as row matrices (1 x n) or column matrices (n x 1)
- For example **v** = (3, 4, -1) can be

$$\mathbf{v}_1 = \begin{bmatrix} 3 & 4 & -1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \mathbf{v}_1^T$$

Vector X Matrix

- If multiplying a row vector (*1 x n*) by an *n x n* matrix, the vector appears on the left

$$\mathbf{v}'_{1\times n} = \mathbf{v}_{1\times n}\mathbf{M}_{n\times n}$$

- If multiplying an *n x n* matrix by a column vector (*n x 1*) the vector appears on the right

$$\mathbf{v}'_{n\times 1} = \mathbf{M}_{n\times n}\mathbf{v}_{n\times 1}$$

- $\mathbf{v}' = \left(\left((v\mathbf{A})\mathbf{B}\right)\mathbf{C}\right)$ row vectors: read left-to-right
- $\mathbf{v}'^{T} = \left(\mathbf{C}^{T}\left(\mathbf{B}^{T}\left(\mathbf{A}^{T}v^{T}\right)\right)\right)$ column vectors: read right-to-left

Identity matrix

- Usually donated by the symbol **I**
- Always square (n==m) with the diagonals = 1 and 0 everywhere else

$$\mathbf{I}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{AI} \equiv \mathbf{IA} \equiv \mathbf{A}$$

Inverse Matrix

- An inverse matrix **A** (denoted $\mathbf{A}^{-1}$) undoes the effects of matrix **A**
- When you multiply a matrix by its inverse, the result is always the identity matrix $\mathbf{A}(\mathbf{A}^{-1}) \equiv (\mathbf{A}^{-1})\mathbf{A} \equiv \mathbf{I}$
- Not all matrices have inverses. The ones in game development do.
- Undoing concatenated matrices requires special care

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

Transposition

- The transpose of matrix **M** is $\mathbf{M}^T$
- Found by reflecting across the diagonal

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

- Inverse of an orthonormal (pure rotation) matrix is equal to its transpose
- Transpose of concatenated matrices is handled like inverse

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

Homogeneous Coordinates

- A 2x2 matrix can be used to represent rotations in 2 dimensions

$$[r_x' \quad r_y'] = [r_x \quad r_y]\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

- The same is true for 3 dimensions

$$[r_x' \quad r_y' \quad r_z'] = [r_x \quad r_y \quad r_z]\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translations?

- Can you use a 3 x 3 to represent a translation?
- Nope $\mathbf{r}+\mathbf{t}=[r_x+t_x \quad r_y+t_y \quad r_z+t_z]$
- We can however use a 4 x 4

$$\mathbf{r}+\mathbf{t}=[r_x \quad r_y \quad r_z \quad 1]\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\t_x & t_x & t_x & 1\end{bmatrix}$$

$$=[r_x+t_x \quad r_y+t_y \quad r_z+t_z \quad 1]$$

Points versus Vectors

- Remember points and vectors are different
  - Points can be translated, vectors cannot
- We can achieve this by setting the last value in the homogenized vector to 1 for point and 0 for vectors
  - Eliminates the effects of translations

$$[\mathbf{v} \quad 0]\begin{bmatrix} U & 0 \\ \boldsymbol{t} & 1 \end{bmatrix} = [(\mathbf{vU} + 0\mathbf{t}) \quad 0] = [\mathbf{vU} \quad 0]$$

- Technically we can convert from 4D homogeneous to 3D non-homogeneous by dividing

$$[x \quad y \quad z \quad w] \equiv \begin{bmatrix} \frac{x}{w} & \frac{y}{w} & \frac{z}{w} \end{bmatrix}$$

- Now makes sense why we set w=0 for vectors

Atomic transformation matrices

- Notice that a 4 x 4 transformation matrix can be partitioned into 4 components

$$M_{affine} = \begin{bmatrix} \mathbf{U}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{t}_{1\times3} & 1 \end{bmatrix}$$

  - **U** represents rotation and/or scaling
  - **t** represents the translation
  - **0** =[0 0 0]^T
  - a scalar 1

Point times Atomic

- When you multiply a point by a matrix the result is

$$[\mathbf{r}'_{1\times 3} \quad 1] = [\mathbf{r}_{1\times 3} \quad 1]\begin{bmatrix} \mathbf{U}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{t}_{1\times 3} & 1 \end{bmatrix} = [\mathbf{rU} + \mathbf{t} \quad 1]$$

Atomic translation

Previous translation matrix

Now be written as

$$\mathbf{r} + \mathbf{t} = \begin{bmatrix} r_x & r_y & r_z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_x & t_x & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} r & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} r + t & 1 \end{bmatrix}$$

- The inversion is just the negation of **t**

Atomic Rotation

- Pure rotations have the form

$$\begin{bmatrix} r & 1 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} rR & 1 \end{bmatrix}$$

Rotations

- The 1 within the rotation matrix always appears on the axis of rotation, the sine and cosine are off axis
- Positive rotations go from
  - x to y - about z
  - y to z - about x
  - z to x - about y - this explains the transpose of the matrix
- The inverse of a pure rotation is its transpose
- $\text{Rotate}_{\text{x}}(\mathbf{r}, \alpha) = \begin{bmatrix} r_x & r_y & r_z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & sin\alpha & 0 \\ 0 & -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Scaling

Written as

$$\boldsymbol{rS} = [r_x \quad r_y \quad r_z \quad 1]\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shorthand of

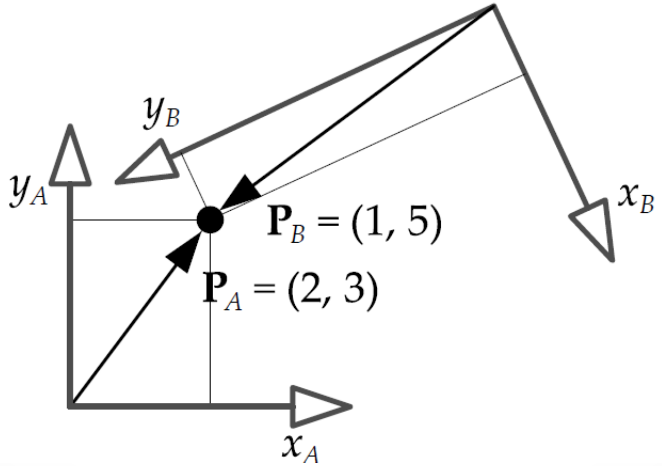
$$[r \quad 1]\begin{bmatrix} \mathbf{S}_{3\times3} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = [\boldsymbol{rS}_{3\times3} \quad 1]$$

- To invert scaling just use the reciprocals of the scaling factors
- If all the scaling factors are the same it is called uniform scaling

Coordinate Spaces

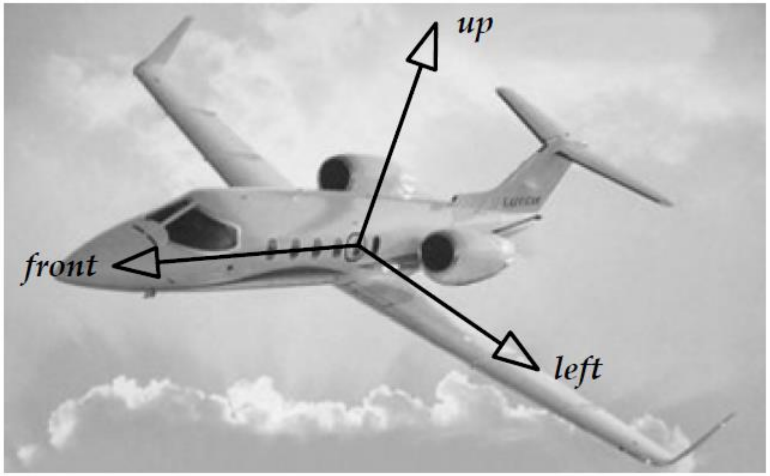
- We can think of a point as being a vector relative to a given set of axes

- The axes are just for a frame of reference and are referred to as a coordinate space



Model space

- When a model is created, the vertices are relative to a coordinate system called model space
- Model space origin is usually in the center of the object
- Model space axes are usually named something like front, left, and up



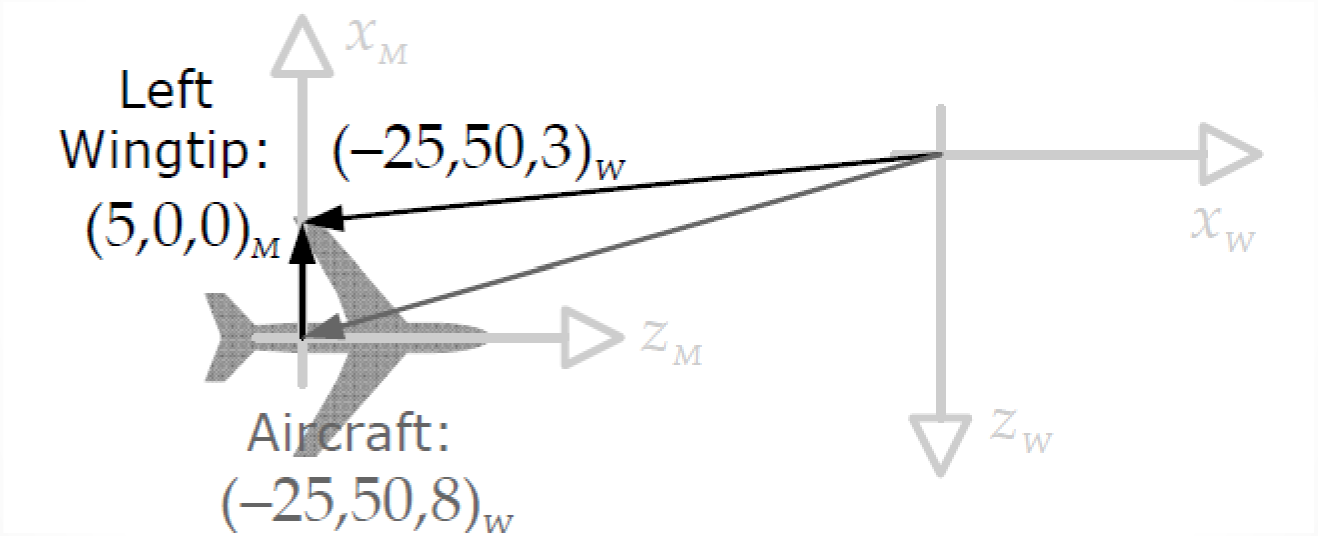
Mapping model to world

- The mapping can be arbitrary
  - $\text{Front} = \mathbf{k}, \text{Left} = \mathbf{i}, \text{Up} = \mathbf{j}$ or $\text{Front} = \mathbf{i}, \text{Right} = \mathbf{k}, \text{Up} = \mathbf{j}$
- Why use model space?
  - Consider pitch, yaw, and roll
  - They cannot be represented in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

World Space

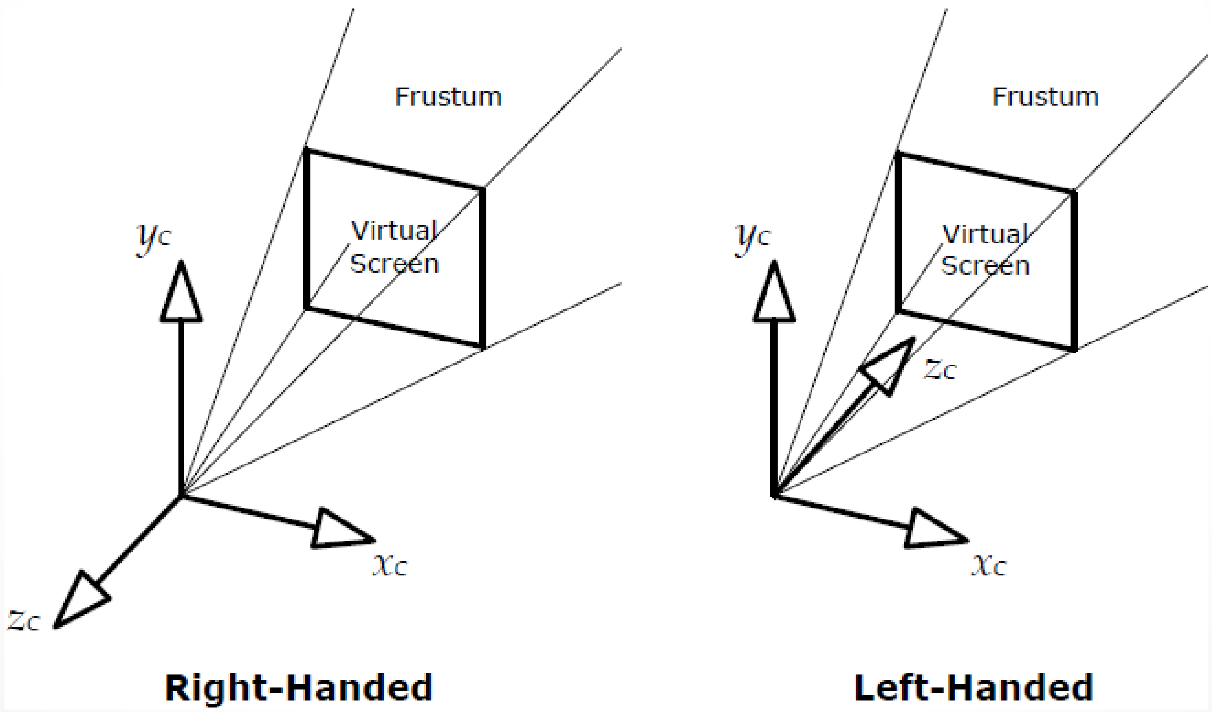
- A fixed coordinate system where the objects, orientations, and scales are defined
- Center usually placed at the center of the playable area
- The orientation is arbitrary but usually y or z is up

Model to world



View space

- A coordinate space fixed to the location of the camera



Change of basis

- All coordinate spaces are relative
  - You must specify them in relation to another space
- This implies a hierarchy of spaces
  - The world space is at the root
- The camera has a location in world space, so its view space has the world space as a parent

Change of basis matrix

- The matrix from transforming a child to parent space is called

$$\mathbf{M}_{C \to P}$$

- With the matrix any child-space position can be transformed into a parent-space position

$$\mathbf{P}_P = \mathbf{P}_C \mathbf{M}_{C \to P}$$

$$\mathbf{M}_{C \to P} = \begin{bmatrix} \mathbf{i}_C & 0 \\ \mathbf{j}_C & 0 \\ \mathbf{k}_C & 0 \\ \mathbf{t}_C & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{i}_{Cx} & \mathbf{i}_{Cy} & \mathbf{i}_{Cz} & 0 \\ \mathbf{j}_{Cx} & \mathbf{j}_{Cy} & \mathbf{j}_{Cz} & 0 \\ \mathbf{k}_{Cx} & \mathbf{k}_{Cy} & \mathbf{k}_{Cz} & 0 \\ \mathbf{t}_{Cx} & \mathbf{t}_{Cy} & \mathbf{t}_{Cz} & 1 \end{bmatrix}$$

$\mathbf{i}_C$ unit x-axis basis vector of child in parent space

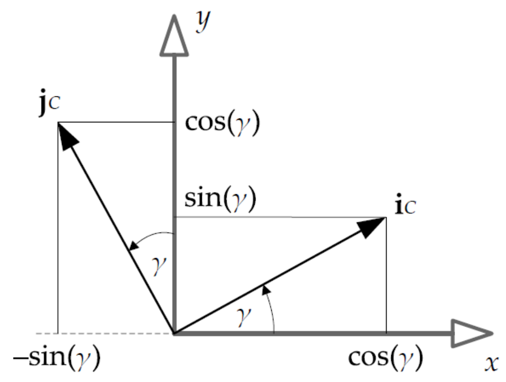
$\mathbf{j}_C$ unit y-axis basis vector of child in parent space

$\mathbf{k}_C$ unit z-axis basis vector of child in parent space

$\mathbf{t}_C$ translation of child relative to parent space

An example

- Consider a child space rotating by γ around the z-axis



- We can see that $i_C = [\cos\gamma \quad \sin\gamma \quad 0]$ and
$$j_C = [-\sin\gamma \quad \cos\gamma \quad 0]$$

Child to Parent

- By putting these into our mapping equation with $k_{\mathbf{C}} = [0 \quad 0 \quad 1]$ we get

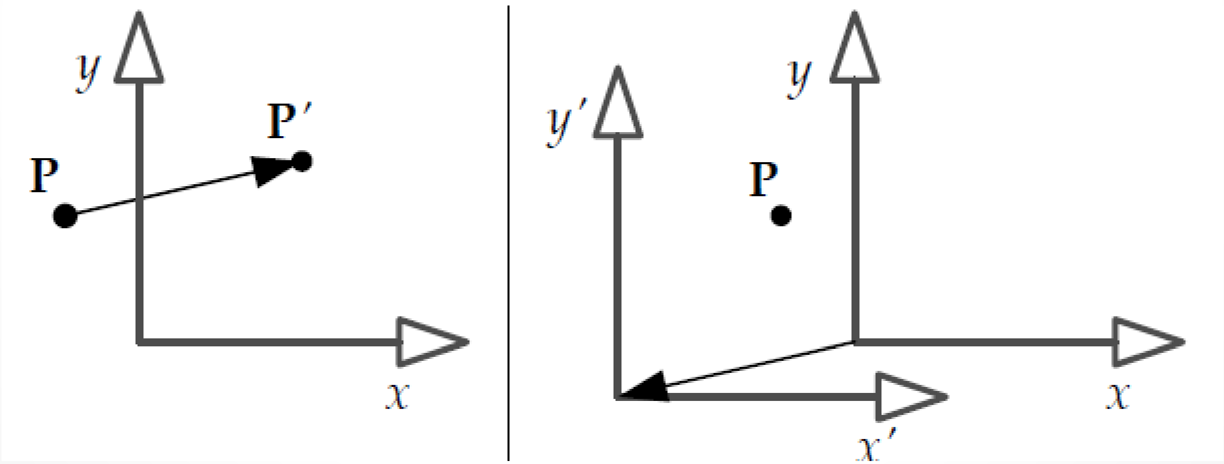
$$\mathbf{M}_{\mathbf{C}\rightarrow\mathbf{P}} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = rotate_z(r, \gamma)$$

Extracting the basis

- Given a 4 X 4 transformation matrix we can extract the unit basis vectors by grabbing the proper row
- Say you have a vehicle's model-to-world affine matrix and assume the vehicle always points in the positive z-axis
- We can extract $\mathbf{k}_c$ from the transformation to get the vehicle facing direction (by grabbing its third row)

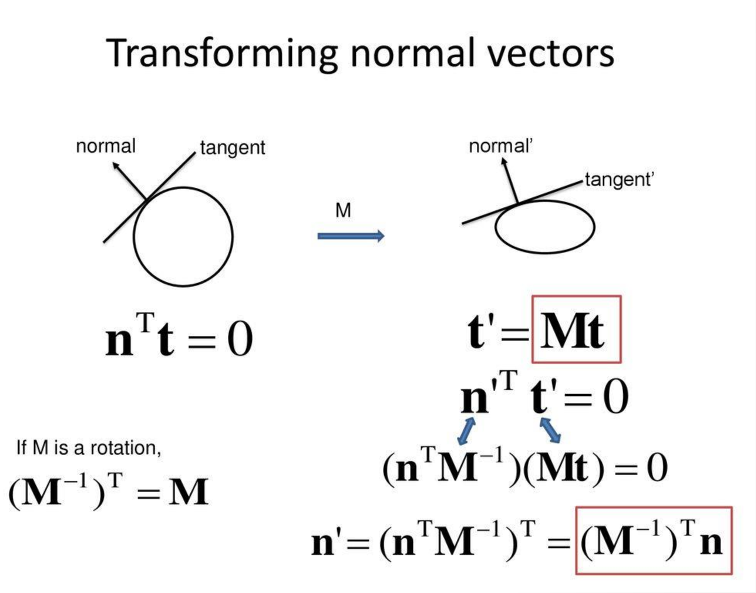
Confusion

- You can transform a coordinate system as well as a point, but in the opposite direction
- So, if a matrix transforms a point from child to parent space, then it also transforms coordinate axes from parent to child space



Transforming normal vectors

- Remember a normal vector has to remain perpendicular to the surface it is associated with
- In general if the point can be rotated from A to B with a 3X3 matrix $\mathbf{M}_{A\rightarrow B}$ then the normal vector will be transformed by $(M_{A\rightarrow B}^{-1})^T$



Storing Matrices in Memory

- `Ogre::Matrix3 rotMat = Matrix3(0.28, 0.96, -0.04, 0.58, -0.20, -0.79, -0.77, 0.20, -0.61);`

- syntax, the first subscript is the row and the second is the column,

```
Matrix3(Real fEntry00, Real fEntry01, Real fEntry02,
Real fEntry10, Real fEntry11, Real fEntry12,
Real fEntry20, Real fEntry21, Real fEntry22)
{
m[0][0] = fEntry00;
m[0][1] = fEntry01;
m[0][2] = fEntry02;
m[1][0] = fEntry10;
m[1][1] = fEntry11;
m[1][2] = fEntry12;
m[2][0] = fEntry20;
m[2][1] = fEntry21;
m[2][2] = fEntry22;
}
```

Quaternions

- We can use a 3X3 matrix to represent a rotations
- Not ideal
  - Nine floating point numbers for 3 DOF
  - Rotating a vector requires 9 multiplications and 6 additions
  - They are hard to interpolate
- We can use a quaternion

$$q = [q_x \quad q_y \quad q_z \quad q_w]$$

Quaternion

- Developed by Sir William Rowan Hamilton in 1843
  - Used to solve problems in mechanics
- The unit length quaternions can represent 3D rotations
  - All of them obey the constraint

$$q_x^2 + q_y^2 + q_z^2 + q_w^2 = 1$$

Quaternion

- Composed of two elements
  - A unit axis of rotation scaled by the sine of the half angle of rotation
  - The cosine of the half angle

$$
\begin{aligned}
\mathrm{q} &= \begin{bmatrix} \mathbf{q}_V & q_S \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},
\end{aligned}
$$

  - Can also be written as a four element vector

$$
\mathrm{q} = \begin{bmatrix} q_x & q_y & q_z & q_w \end{bmatrix}, \text{where}
$$

$$
\begin{aligned}
q_x &= q_{V_x} = a_x \sin\frac{\theta}{2}, \\
q_y &= q_{V_y} = a_y \sin\frac{\theta}{2}, \\
q_z &= q_{V_z} = a_z \sin\frac{\theta}{2}, \\
q_w &= q_S = \cos\frac{\theta}{2}.
\end{aligned}
$$

Operations

- Magnitude and vector addition work the same
- Adding two quaternions together is not the sum of two angles though
- Multiplication is the most important operations – represents one rotation followed by another

$$\mathrm{pq} = [(p_S\boldsymbol{q}_V + q_S\boldsymbol{p}_V + \mathbf{p}_V \times \boldsymbol{q}_V) \quad (p_S q_S - \mathbf{p}_V \cdot \boldsymbol{q}_V)]$$

Conjugate and Inverse

- The inverse of a quaternion q, denoted q^{-1}, has the following property

$$\mathrm{q}\mathrm{q}^{-1} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + 1$$

- To determine the inverse we need the conjugate

$$\mathrm{q}^{*} = [-\mathbf{q}_{V} \quad q_{S}]$$

- The inverse then is

$$\mathrm{q}^{-1} = \frac{\mathrm{q}^{*}}{|\mathrm{q}|^{2}}$$

It's simple

- Remember our quaternions are all unit length this means

$$q^{-1} = \mathrm{q}^* = [-\mathbf{q}_V \quad q_S]$$

- Other useful facts

$$(\mathrm{pq})^* = \mathrm{q}^*\mathrm{p}^*$$

$$(pq)^{-1} = \mathrm{q}^{-1}\mathrm{p}^{-1}$$

Rotating with quaternions

- To rotate a vector ***v*** by a quaternion q

1. We first write the vector in quaternion form
$$\mathrm{v} = [\mathbf{v} \quad 0] = [v_x \quad v_y \quad v_z \quad 0]$$
2. Then multiple q times v and by the inverse of q
$$\mathrm{v}' = rotate(\mathrm{q}, \mathbf{v}) = \mathrm{qvq}^{-1}$$
or
$$\mathrm{v}' = rotate(\mathrm{q}, \mathbf{v}) = \mathrm{qvq}^{*}$$
3. Finally extract the vector out of $\mathrm{v}' = [\mathbf{v}' \quad 0]$

Concatenation

- For matrices we followed the rule

$$\mathbf{R}_{net} = \mathbf{R}_1\mathbf{R}_2\mathbf{R}_3$$
$$\mathbf{v}' = \mathbf{v}\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3 = \mathbf{v}\mathbf{R}_{net}$$

- We do something similar with quaternion rotation

$$\mathrm{q}_{net} = \mathrm{q}_3\mathrm{q}_2\mathrm{q}_1$$
$$\mathrm{v}' = \mathrm{q}_3\mathrm{q}_2\mathrm{q}_1\mathrm{v}\mathrm{q}_1^{-1}\mathrm{q}_2^{-1}\mathrm{q}_3^{-1}$$
$$= \mathrm{q}_{net}v\mathrm{q}_{net}^{-1}$$

Quaternion to matrix

- If

  $$q = [\mathbf{q}_V \quad q_s] = [q_{V_x} \quad q_{V_y} \quad q_{V_z} \quad q_s] = [x \quad y \quad z \quad w]$$

- Then

  $$R = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy + 2zw & 2xz - 2yw \\ 2xy - 2zw & 1 - 2x^2 - 2z^2 & 2yz + 2xw \\ 2xz + 2yw & 2yz - 2xw & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

- Much harder to go the other way

Some useful (normalized) quaternions

- $[w, x, y, z] = [\cos(\frac{a}{2}), \ \sin(\frac{a}{2})\, n_x, \ \sin(\frac{a}{2}) n_y, \ \sin(\frac{a}{2}) n_z]$
- Where **a** is the angle of rotation and $(n_x, ny, n_z)$is the axis of rotation

w	x	y	z	Description
1	0	0	0	Identity quaternion, no rotation
0	1	0	0	180° turn around X axis
0	0	1	0	180° turn around Y axis
0	0	0	1	180° turn around Z axis
sqrt(0.5)	sqrt(0.5)	0	0	90° rotation around X axis
sqrt(0.5)	0	sqrt(0.5)	0	90° rotation around Y axis
sqrt(0.5)	0	0	sqrt(0.5)	90° rotation around Z axis
sqrt(0.5)	-sqrt(0.5)	0	0	-90° rotation around X axis
sqrt(0.5)	0	-sqrt(0.5)	0	-90° rotation around Y axis
sqrt(0.5)	0	0	-sqrt(0.5)	-90° rotation around Z axis

How to use Quaternions in Ogre

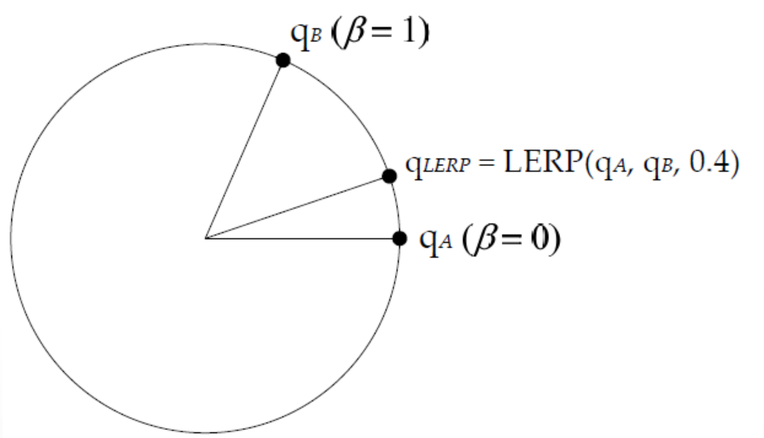
```
cbool BasicTutorial1::keyPressed(const KeyboardEvent& evt)
{
switch (evt.keysym.sym)

{
case 119: //ASCII code for "w"

{
Quaternion q(Degree(-45), Vector3::UNIT_X); //rotate around the X axis 45 degrees.
ogreNode->rotate(q);
}
break;
case 115: //ASCII code for "s"

{
Quaternion q(Degree(45), Vector3::UNIT_X); //rotate around the X axis -45 degrees.
ogreNode->rotate(q);
}
break;
case 97: //ASCII code for "a"

{
Quaternion q(Degree(-45), Vector3::UNIT_Y); //rotate around the Y axis -45 degrees.
ogreNode->rotate(q);
}
break;
case 100: //ASCII code for "d"

{
Quaternion q(Degree(45), Vector3::UNIT_Y); //rotate around the Y axis 45 degrees.
ogreNode->rotate(q);
}
}
```

Rotational Interpolation

- Given two quaternions q_A and q_B we can find an intermediate rotation β percent between them as

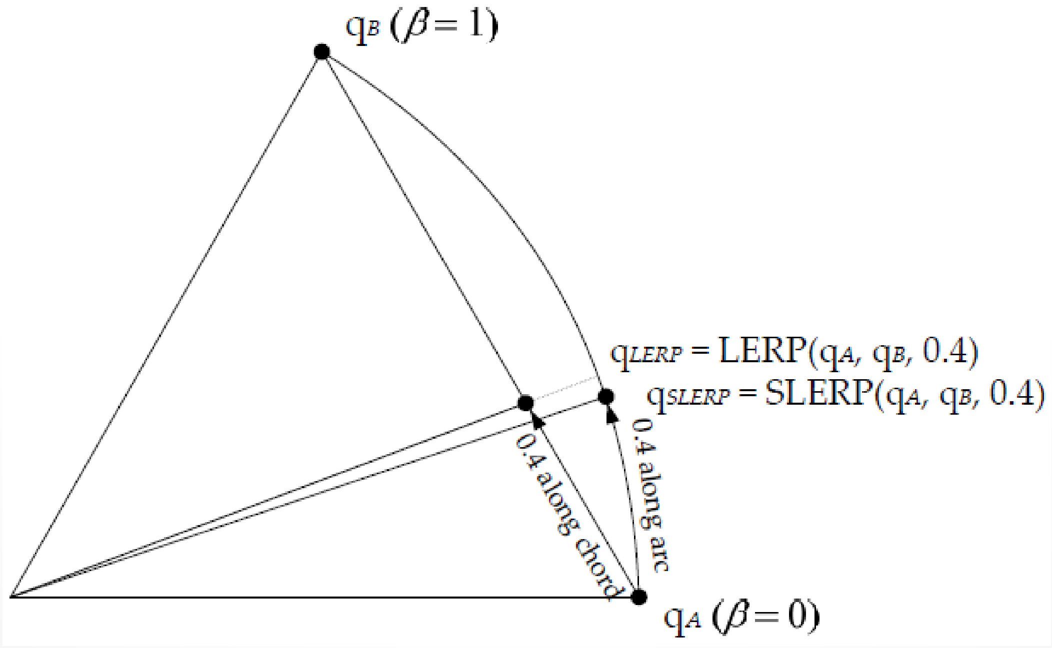
$$q_{LERP} = LERP(q_A, q_B, \beta) = \frac{(1-\beta)q_A + \beta q_B}{|(1-\beta)q_A + \beta q_B|}$$

$$= normalize\left(\begin{bmatrix} (1-\beta)q_{A_x} + \beta q_{B_x} \\ (1-\beta)q_{A_y} + \beta q_{B_y} \\ (1-\beta)q_{A_z} + \beta q_{B_z} \\ (1-\beta)q_{A_w} + \beta q_{B_w} \end{bmatrix}^T\right)$$



Problems with lerp

- Because LERP interpolate along a chord instead of the actual surface of the 4D hyper-sphere, the angular speed is not constant
  - The beginning and end are slower than the middle



Slerp

- Spherical Linear Interpolation (SLERP) fixes this
- SLERP is like LERP, but the weighting of the two quaternions is modified based on the angle between them

$$\mathrm{SLERP}(\mathsf{p}, \mathsf{q}, \beta) = w_p \mathsf{p} + w_q \mathsf{q},$$

where

$$w_p = \frac{\sin(1-\beta)\theta}{\sin\theta},$$

$$w_q = \frac{\sin\beta\theta}{\sin\theta}.$$

Find the angle

- To find the angle θ we use the following

$$\cos(\theta) = \mathrm{p} \cdot \mathrm{q} = p_x q_x + p_y q_y + p_z q_z + p_w q_w$$

$$\theta = cos^{-1}(\mathrm{p} \cdot \mathrm{q})$$

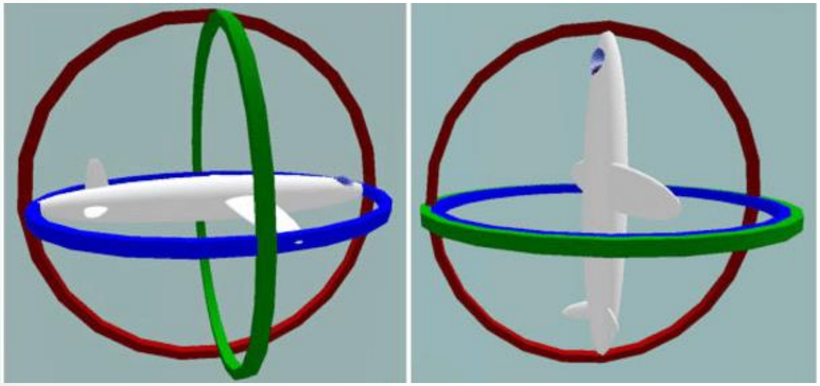
Should you slerp?

- It depends
- If you can get it to be fast enough, then use it
- Otherwise LERP is fine and likely the user can't tell anyway

Comparison of Rotational Representations

- Euler Angles
  - Simple: yaw, pitch and roll
  - represented by a 3D vector $[q_Y\ q_P\ q_R]$.
  - Easy to visualize
  - Easy to interpolate unless it is around an arbitrary axis
  - Prone to gimbal lock
  - Order of the rotations matters

PYR ≠ YPR ≠ RYP



3X3 Matrix

- Overall pretty good
- Not subject to gimbal lock
- Uses lots of computation and memory
- Not very intuitive to read

Axis/Angle

- We can use a representation similar to quaternions called axis+angle $[\mathbf{a} \quad \theta]$
- Easy to understand
- Compact
- Cannot interpolate rotations
- Cannot be easily applied to points and vectors

Quaternions

- Small
- Can be easily applied to vectors and points
- Easy to interpolate
- Hard to read

SQT Transforms

- Quaternions can only represent rotations
- We can combine them with translation and scaling into a SQT transform
- Good for most things
  - Easy to understand
  - Easy to interpolate
  - Compact representation

$$SQT = [\mathbf{s} \quad \mathrm{q} \quad \mathbf{t}]$$

Dual Quaternions

- A *rigid transformation* is a transformation involving a rotation and a translation—a "corkscrew" motion.
- A rigid transformation can be represented using a mathematical object known as a *dual quaternion*
- A dual quaternion is like an ordinary quaternion, except that its four components are *dual numbers* instead of regular real-valued numbers
- A dual number can be written as the sum of a non-dual part and a dual part as follows: $a + \varepsilon b$.
- Here ε is a magical number called the *dual unit*
- $\varepsilon^2 = 0$

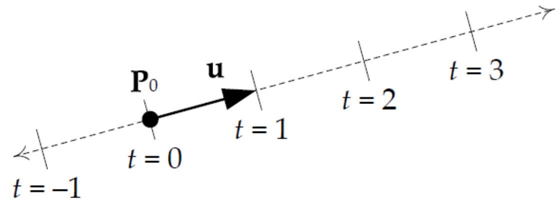
Rotations and Degrees of Freedom

- DOF refers to the number of mutually independent ways in which an object's physical state (position and orientation) can change.
- a three-dimensional object has three degrees of freedom in its translation (along the x-, *y*- and z-axes) and three degrees of freedom in its rotation (about the x-, *y*- and z-axes) ➔ 6 DOF
- For example, Euler angles require three floats, but axis+angle and quaternion representations use four floats, and a 3X3 matrix

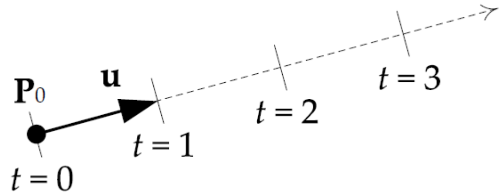
Other useful object

- Line – can be represented using a parametric equation

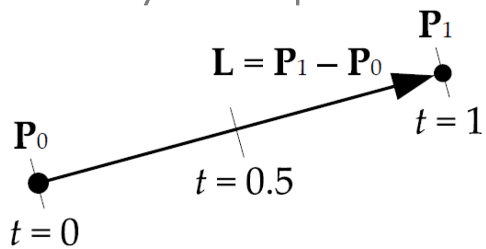
$$P(t) = P_0 + t\mathbf{u} \quad where \quad -\infty < t < +\infty$$



- Ray – only extends in one direction so $t \geq 0$

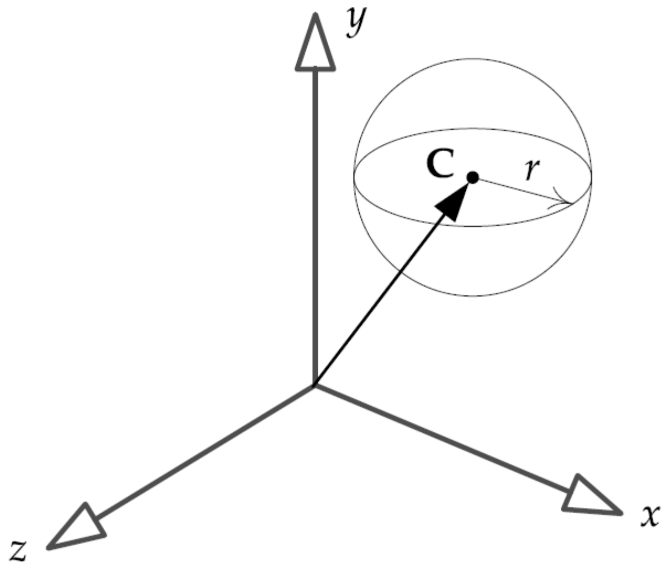


- Segment – bounded by two points $0 \leq t \leq L$



Spheres

- Defined using a center point C and a radius r
- Fits nicely in a 4 element vector



Planes

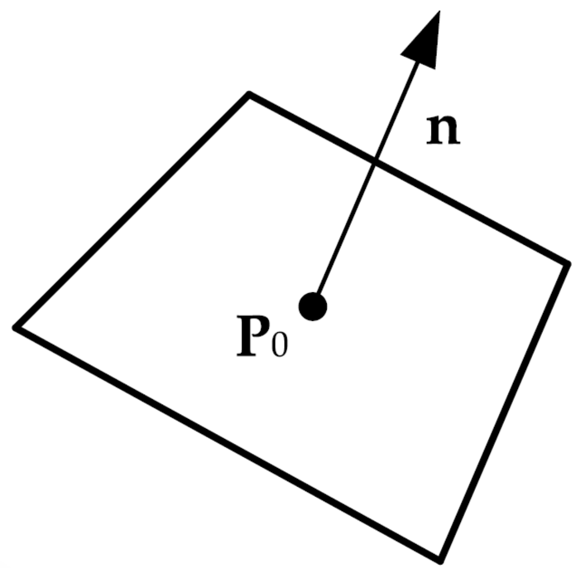
- Plane is a 2D surface in 3D space

- Only satisfied when **P**=[x y z] lies on the plane

$$Ax + By + Cz + D = 0$$

Point-normal

- Planes can also be represented by
  - A point $\mathbf{P}_0$
  - A unit vector **n** that is normal to the plane



Different, but the same

- If A,B,C from the traditional equation are interpreted as a vector
- The vector lies in the direction of the plane normal
- The normalized version $[a \quad b \quad c] = \mathbf{n}$
- The normalized parameter $d = D/\sqrt{A^2 + B^2 + C^2}$ is the distance to the origin

AABB

- Axis aligned bounding box
- Useful for collision detection
- Represented using 6 values

$$[x_{min}, x_{max}, y_{min}, y_{max}, z_{min}, z_{max}]$$

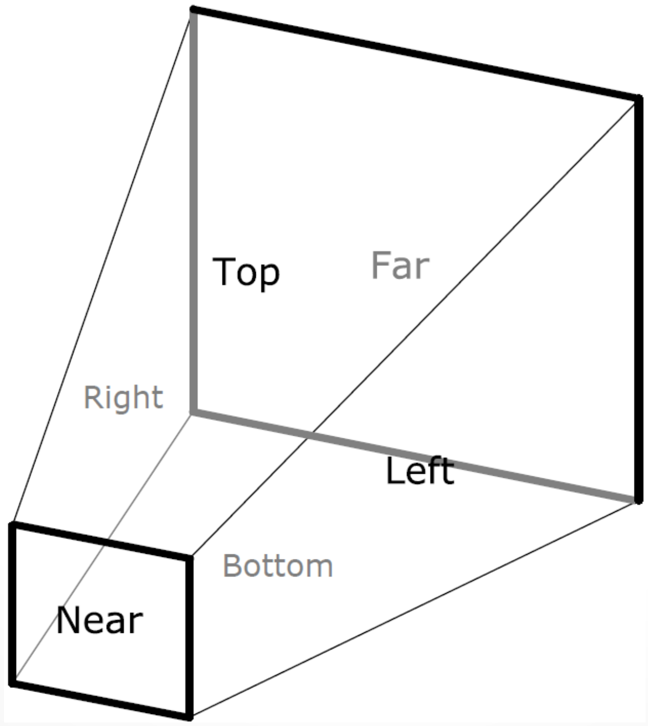
- Or two points **P**$_{min}$ and **P**$_{max}$
- Easy to test if a point is within the bounds

oBB

- Oriented bounding box
- Differs from AABB because it aligns with the objects local coordinate space, not the world space
- Testing for intersection usually involves transforming the point into the OBB coordinate system

Frusta

- A group of 6 planes that create a truncated pyramid shape
- Usually represented as an array of six planes in point-normal form

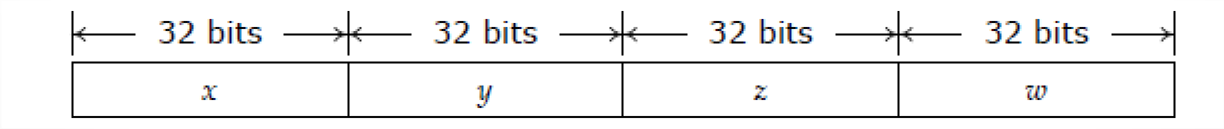


Hardware-accelerated SIMD Math

- Stands for Single Instruction Multiple Data
- Allows a single operation to be performed on multiple data items in parallel
- For example you can multiple four pairs of floating point numbers with a single command

History

- Intel introduced MMX instruction in the first Pentiums
- Could do SIMD on 8-eight bit, 4-16 bit or 2-32 bit integers
- Later enhancements resulted in the Streaming SIMD Extensions (SSE)
  - Uses special 128-bit registers
  - Can perform parallel operation on 32-bit floating point numbers



SSE Registers

- In packed 32-bit floating point mode holds 4 -32 floating point numbers
- Elements are called $[x \quad y \quad z \quad w]$
- Consider the SIMD instruction

addps xmm0, xmm1

- This performs the following operations

$$\text{xmm0}.x = \text{xmm0}.x + \text{xmm1}.x;$$
$$\text{xmm0}.y = \text{xmm0}.y + \text{xmm1}.y;$$
$$\text{xmm0}.z = \text{xmm0}.z + \text{xmm1}.z;$$
$$\text{xmm0}.w = \text{xmm0}.w + \text{xmm1}.w;$$

Using SIMD

- Visual Studio has a special datatype __m128
  - Complier understands how to use it

```
__m128 v= _mm_set_ps(-1.0f, 2.0f, 0.5f, 1.0f);
```

- GNU C/C++ uses the keyword vector

```
vector float = (vector float) (-1.0f, 2.0f, 0.5f, 1.0f);
```

Coding with SSE

- Can use inline assembly

```
__m128 addWithAssembly(const __m128 a, const __m128 b)
{
        __asm   addps   xmm0, xmm1
}
```

- Can use intrinsics (# include <xmmintrin.h>)

```
__m128 addWithIntrinsics(const __m128 a, const __m128 b)
{
        return _mm_add_ps (a,b);
}
```

Vector-matrix multiplication

- Say you want to multiple a 1X4 vector **v** with a 4X4 matrix **M** to get the solution **r**

$$\mathbf{r} = \mathbf{vM}$$

$$\begin{bmatrix} r_x & r_y & r_z & r_w \end{bmatrix} = \begin{bmatrix} v_x & v_y & v_z & v_w \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}$$

$$= \begin{bmatrix} v_x M_{11} + v_y M_{21} + v_z M_{31} + v_w M_{41} \\ v_x M_{12} + v_y M_{22} + v_z M_{32} + v_w M_{42} \\ v_x M_{13} + v_y M_{23} + v_z M_{33} + v_w M_{43} \\ v_x M_{14} + v_y M_{24} + v_z M_{34} + v_w M_{44} \end{bmatrix}$$

Attempt 1

- We could store **v** in an SSE and each column of **M** in an SSE
- The multiple each **M** SSE by the **v** SSE
- We get

$$
\begin{aligned}
vMcol1 &= [v_x M_{11} \quad v_y M_{21} \quad v_z M_{31} \quad v_w M_{41}]; \\
vMcol2 &= [v_x M_{12} \quad v_y M_{22} \quad v_z M_{32} \quad v_w M_{42}]; \\
vMcol3 &= [v_x M_{13} \quad v_y M_{23} \quad v_z M_{33} \quad v_w M_{43}]; \\
vMcol4 &= [v_x M_{14} \quad v_y M_{24} \quad v_z M_{34} \quad v_w M_{44}];
\end{aligned}
$$

- Now we have to add across the vectors
  - Not very fast

Attempt 2

- We could create SSEs with replicated values from **v** and multiple each row of **M** by the appropriate **v** SSE
- We get
- Now we can add the vectors
  - Much better

Random Number Generator

- Linear Congruent Generators
  - Found in the c rand() function
  - Not particular good
- Mersenne Twister
  - Very large period (before repeating a sequence)
  - Very high order of equidistribution
  - Passes numerous test for randomness
  - Fast
- Mother-of-All
  - Published by George Marsaglia
  - Reasonable period
  - Faster than Mersenne Twister