Gibbs

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But, Captain, I cannot change the laws of physics

-Lt. Commander Montgomery "Scotty" Scott
USS *Enterprise*

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1 Abstract

The purpose of this report is to perform an analysis of the Gibbs and Herrick-Gibbs methods of orbital determination when subject to noise. The methods are explained and their derivations given. They are then computed for a sample orbit at various eccentricities. The Root Mean Square Error of both methods is obtained and the accuracy of the methods compared. Do More

2 Introduction

Watching the skys at night, the ancient Greeks noticed that some stars seemed to move in fixed path across then night sky. They called these stars *planetes*, meaning wanderers. The Greeks were not the first civilization to take notice of the planets, Babylonian astronomers were already recording the motion of the planets for five hundred years. But they were the first to develop methods of predicting the planet's motion. These methods were so advanced, that they would remain in use for over a thousand years until they were eclipsed the works of Copernicus [1].

Today the prediction of planet's orbits falls under what we would call orbital determination. Orbital determination is the estimation of an object's orbital elements from certain measurements of the object (position, velocity, angle), often taken from some ground station. Modern orbital determination employs various methods such as Lambert's method, Gauss' method, Gibbs' method, Herrick-Gibbs method. Some methods such as Gibb's and Herrick-Gibbs' require only three position measurements. Lambert's method requires two position and velocity measurements, and Gauss's method uses measurements of an object's right ascension and declination. A method of orbital determination is considered a preliminary orbit determination method if it just uses the equations of two body motion, and neglects other effects on the orbit.

Put a bit more on the other methods

Write up Page or two on orbit degetrimantion Angle only mtheod: gauss, ;laplace Velotto textbook on Orbit deterimantion 3 pos vect, 3 beartin time Thrre velc , IOD two (no eqn) Section on gibbs Frame problem Derive gibbs Show how you go through problem set up to the end Talk about now, assses how well gibbs does for ODE, diff ecc, diff noise level Talk about results Derivation of gibbs should be a page to two pages (plus figures) Harrit gibbs method (try to take a look at this) (it's a taylor series expansion of orbit around) Nice plots Get a pdf of report by last day of class Mean anonly, not true for circ

3 Gibbs method

Gibbs method was developed by Josiah Willard Gibbs [2], who was famous for his application of vectorized math to thermodynamics. Gibbs method works off having three measurements of the distance to an object at three different times. Thus, assume we have the following position vectors:

$$\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3} \tag{1}$$

which were observed at times $t_1, t_2,$ and t_3. Each of these position vectors has a corresponding velocity vector $\boldsymbol{v_1}, \boldsymbol{v_2},$ and $\boldsymbol{v_3}$. To obtain the orbital elements from one of the position vectors, the corresponding velocity vector is needed. Thus the focus of Gibbs method is searching for a velocity vector. This velocity vector may be found in just four equations:

$$\mathbf{N} = r_1 \left(\mathbf{r}_2 \times \mathbf{r}_3\right) + r_2 \left(\mathbf{r}_3 \times \mathbf{r}_1\right) + r_3 \left(\mathbf{r}_1 \times \mathbf{r}_2\right) \tag{2}$$

$$\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 \tag{3}$$

$$\mathbf{S} = \mathbf{r}_1 \left(r_2 - r_3\right) + \mathbf{r}_2 \left(r_3 - r_1\right) + \mathbf{r}_3 \left(r_1 - r_2\right) \tag{4}$$

$$\mathbf{v} = \sqrt{\frac{\mu}{ND}} \left(\frac{\mathbf{D} \times \mathbf{r}}{r} + \mathbf{S}\right) \tag{5}$$

This algorithm works well for a code to obtain orbital elements, and a MATLAB code for it is provided in the appendix.

3.1 Derivation of Gibbs Method

The algorithm above will now be expanded on, to explain how the steps were obtained. We will use the notation from Howard Curtis's Orbital Mechanics for Engineering Students [3].

Gibbs method beings with the conservation of momentum, which means that all the position vectors of an orbiting object must all be co planer. Or the unit vector normal to the plane of any two position vectors must be perpendicular to the unit vector of the third measurement. This may be expressed as:

$$\hat{u}_{r1} = \mathbf{r}_1 / r_1 \quad \hat{\mathrm{C}}_{23} = (\mathbf{r}_2 \times \mathbf{r}_3) \,/\, \|\mathbf{r}_2 \times \mathbf{r}_3\| \tag{6}$$

$$\hat{u}_{r2} = \mathbf{r}_2 / r_2 \quad \hat{\mathrm{C}}_{31} = (\mathbf{r}_3 \times \mathbf{r}_1) \,/\, \|\mathbf{r}_3 \times \mathbf{r}_1\| \tag{7}$$

$$\hat{u}_{r3} = \mathbf{r}_3 / r_3 \quad \hat{\mathrm{C}}_{12} = (\mathbf{r}_1 \times \mathbf{r}_2) \,/\, \|\mathbf{r}_1 \times \mathbf{r}_2\| \tag{8}$$

toDO, make these normal fractions, i like those better

To check that all the vectors are co planer as they should be, the dot product of

$$\hat{u}_{r1} \cdot \hat{C}_{23} = 0 \tag{9}$$

Additionally because the vectors are co planer, there must be scalar factors (c) that can be applied to two of the vectors to make their sum equal to the third vector:

$$\mathbf{r}_2 = c_1 \mathbf{r}_1 + c_3 \mathbf{r}_3 \tag{10}$$

We should note that convention has Gibbs method solve for the velocity vector associated with the second position vector. Gibbs method can be used to solve for the other vectors, the order of the position vectors must simply be rotated.

If these vectors are all co planer, we may define unit vector $\hat{\mathbf{w}}$ as a unit vector normal to the orbital plane, unit vector $\hat{\mathbf{p}}$ as being in the direction of the eccentricity vector, and $\hat{\mathbf{q}}$ as the unit vector that is normal to $\hat{\mathbf{w}}$ and $\hat{\mathbf{p}}$, such that:

$$\hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}} \tag{11}$$

The eccentricity and angular moment vectors may be written as

$$\mathbf{h} = h\hat{\mathbf{w}} \tag{12}$$

$$\mathbf{e} = e\hat{\mathbf{p}} \tag{13}$$

$$\mathbf{v} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right) \tag{14}$$

The relationship between position, velocity, momentum, and eccentricity may be expressed as equation 14. We rewrite equation 14 to solve for velocity as equation 15

$$\mathbf{v} = \frac{\mu}{h^2}\left(\frac{\mathbf{h} \times \mathbf{r}}{r} + \mathbf{h} \times \mathbf{e}\right) \tag{15}$$

and using equation 11, equation 15 may be rewritten as

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) \tag{16}$$

The relationship between eccentricity, angular momentum, and position vector may be written as

$$\mathbf{r} \cdot \mathbf{e} = \frac{h^2}{\mu} - r \tag{17}$$

Equation 17 has a relation of position, eccentricity, and momentum. To eliminat eccentricity, we first dot equation 10 with the eccentricity vector to get equation 18 and then insert equation 17 into equation 18.

$$(\mathbf{r}_2) \cdot \mathbf{e} = (c_1\mathbf{r}_1 + c_3\mathbf{r}_3) \cdot \mathbf{e} = c_1\mathbf{r}_1 \cdot \mathbf{e} + c_3\mathbf{r}_3 \cdot \mathbf{e} \tag{18}$$

$$\left(\frac{h^2}{\mu} - r_2\right) = c_1\left(\frac{h^2}{\mu} - r_1\right) + c_3\left(\frac{h^2}{\mu} - r_3\right) \tag{19}$$

Now we have a relationship between the coefficents from earlier, position, and momentum. If the coefficients c_1 and c_2 may be solved for, the angular momentum may be found.

Removing the coefficents gives the following equation

$$\frac{h^2}{\mu}\underbrace{\left(\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1\right)}_{D} = \underbrace{r_1\left(\mathbf{r}_2 \times \mathbf{r}_3\right) + r_2\left(\mathbf{r}_3 \times \mathbf{r}_1\right) + r_3\left(\mathbf{r}_1 \times \mathbf{r}_2\right)}_{N} \tag{20}$$

For simplicity we create the vectors N and D, which are:

$$\mathbf{N} = r_1\left(\mathbf{r}_2 \times \mathbf{r}_3\right) + r_2\left(\mathbf{r}_3 \times \mathbf{r}_1\right) + r_3\left(\mathbf{r}_1 \times \mathbf{r}_2\right) \tag{21}$$

$$\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 \tag{22}$$

$$N = \|\mathbf{N}\| \tag{23}$$

$$D = \|\mathbf{D}\| \tag{24}$$

Now using this notation equation 20 may be rewritten as

$$\mathbf{N} = \frac{h^2}{\mu}\mathbf{D} \tag{25}$$

Replacing the vectors with their norms and rearranging:

$$N = \frac{h^2}{\mu}D \tag{26}$$

$$h = \sqrt{\mu\frac{N}{D}} \tag{27}$$

From equation 27 the angular momentum may be found from just the position vectors.

Because $\hat{\mathbf{w}}$ was devined as being a unit vector normal to the orbital plane, and $\mathbf{N}$ and $\mathbf{D}$ are made up of vectors in the orbital plane then

$$\hat{\mathbf{w}} = \frac{\mathbf{N}}{N} = \frac{\mathbf{D}}{D} \tag{28}$$

Equation 11 may then be writen as

$$\hat{\mathbf{q}} = \frac{(\mathbf{D} \times \mathbf{e})}{De} \tag{29}$$

Fuly writing out the numerator:

$$\hat{\mathbf{q}} = \frac{[(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} + (\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} + (\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e}]}{De} \tag{30}$$

Using the bac-cab vector idendity, the following euqations ar eobtained

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{31}$$

$$(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} = \mathbf{r}_3 (\mathbf{r}_2 \cdot \mathbf{e}) - \mathbf{r}_2 (\mathbf{r}_3 \cdot \mathbf{e}) \tag{32}$$

$$(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} = \mathbf{r}_1 (\mathbf{r}_3 \cdot \mathbf{e}) - \mathbf{r}_3 (\mathbf{r}_1 \cdot \mathbf{e}) \tag{33}$$

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \mathbf{r}_2 (\mathbf{r}_1 \cdot \mathbf{e}) - \mathbf{r}_1 (\mathbf{r}_2 \cdot \mathbf{e}) \tag{34}$$

Instering equation 17 into equation 34

$$(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} = \mathbf{r}_3 \left(\frac{h^2}{\mu} - r_2\right) - \mathbf{r}_2 \left(\frac{h^2}{\mu} - r_3\right) = \frac{h^2}{\mu} (\mathbf{r}_3 - \mathbf{r}_2) + r_3\mathbf{r}_2 - r_2\mathbf{r}_3 \tag{35}$$

$$(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} = \mathbf{r}_1 \left(\frac{h^2}{\mu} - r_3\right) - \mathbf{r}_3 \left(\frac{h^2}{\mu} - r_1\right) = \frac{h^2}{\mu} (\mathbf{r}_1 - \mathbf{r}_3) + r_1\mathbf{r}_3 - r_3\mathbf{r}_1 \tag{36}$$

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \mathbf{r}_2 \left(\frac{h^2}{\mu} - r_1\right) - \mathbf{r}_1 \left(\frac{h^2}{\mu} - r_2\right) = \frac{h^2}{\mu} (\mathbf{r}_2 - \mathbf{r}_1) + r_2\mathbf{r}_1 - r_1\mathbf{r}_2 \tag{37}$$

NOte take out the midle part

The we write the sum of these equations as

$$\mathbf{S} = \mathbf{r}_1 (r_2 - r_3) + \mathbf{r}_2 (r_3 - r_1) + \mathbf{r}_3 (r_1 - r_2) \tag{38}$$

And equation 29 can now be written as

$$\hat{\mathbf{q}} = \frac{1}{De}\mathbf{S} \tag{39}$$

We can now do

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) = \frac{\mu}{\sqrt{\mu\frac{N}{D}}}\left[\frac{\frac{D}{D} \times \mathbf{r}}{r} + e\left(\frac{1}{De}\mathbf{S}\right)\right] \tag{40}$$

Cleaning things up

$$\mathbf{v} = \sqrt{\frac{\mu}{ND}}\left(\frac{\mathbf{D} \times \mathbf{r}}{r} + \mathbf{S}\right) \tag{41}$$

be described as These measurements

4 Herrick-Gibbs Method

The obvious problem with Gibbs method is what if the measured vectors are close to each other, for example in a single pass over a ground station. Herrick Gibbs is one such solution to this problem, it is a method of orbital determination that requires three position measurements, and their respective times. It then uses a Taylor series expansion. Because it is a taylor series, it is not as robust as Gibbs method.

Gibbs method is best for under 5(degree symbol)

5 Analysis of Methods

Put in Orbital Parameters The first parameter investigated was the separation of position measurements. Vallado [4] states that Herrik-Gibbs is only accurate for measurements taken within 5 degrees of the mean anomaly, however that seems Dependant on other factors. The other parameter was the noise, how accurate the position measurements were.

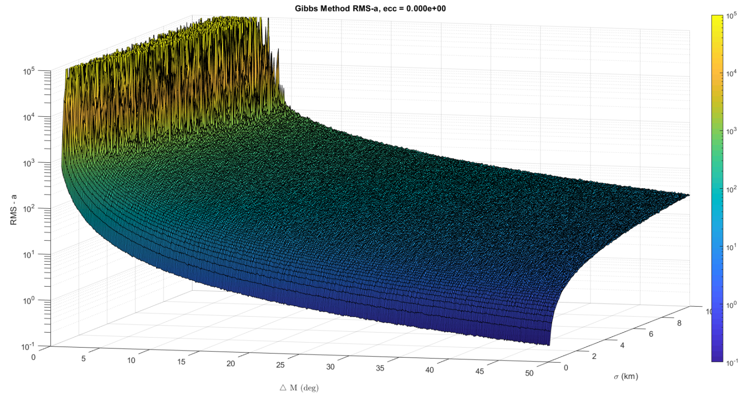

Figure 1: RMS error for the semi major axis via Gibbs method for an eccentricity of zero.

The process for each test goes as follows:

- The eccentricity is set and the orbital parameters calculated.
- The distance between measurements measured in degrees is selected from a range. The exact position vectors are calculated.
  - The noise level is selected from a range.
    * A large number of position vectors are generated from the exact position vector, and the noise level.
    * The code uses a method of orbital determination to compute the semi major axis corresponding to each "noisy" position vector.
    * The Root Mean Square error for the computed semi major axis vs the real semi major axis is calculated.
  - The next noise level is selected and the code repeates.
- The next distance between position measurements is selected.

For a perfectly circular orbit the following plots were created.

From figure 1, it can be seen that Gibbs method is incredibly inaccurate for position vectors taken within 10 degrees of each other. Herrick-Gibbs however excels in that range.

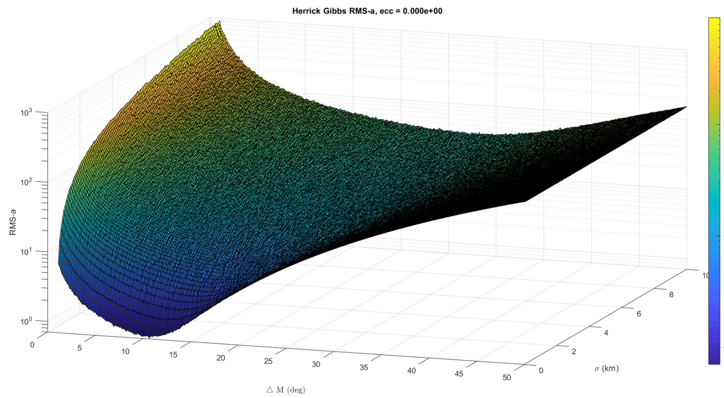

Figure 2: RMS error for the semi major axis via Herrick-Gibbs method for an eccentricity of zero.

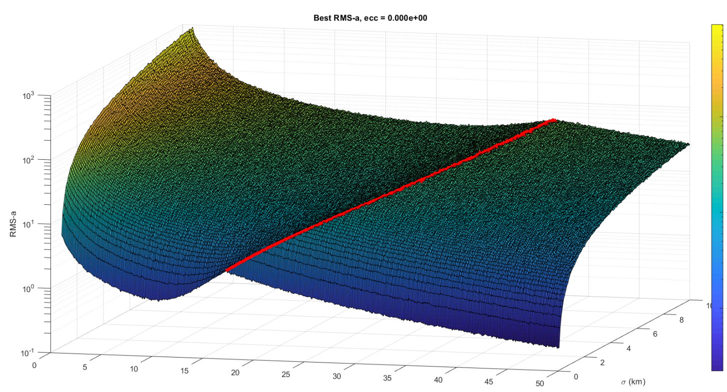

Figure 3: The best possible RMS error for the semi major axis. The red line indicates where Gibbs method becomes better than the Herrick Gibbs Method

At a low noise level, Gibbs method begins to overtake Herrick Gibbs at a separation of 15 degrees, at a high noise level it takes until 35 degrees of separation for Herrick-Gibbs to be overt taken.

6 Results

7 Comparison of Methods

8 Conclusion

The goal of this thesis was to create a program that can download the latest orbital elements needed to calculate the real time locations of space debris, given user inputted parameters. The information would be usable to someone who has taken a basic spaceflight mechanics class.

References

[1] Astro 110-01, lecture 7. `https://www.ifa.hawaii.edu/users/shadia/lectures/habbal_astro110-01_spring2009_lecture7.pdf`, Feb 2009.

[2] `https://www.aps.org/programs/outreach/history/historicsites/gibbs.cfm`.

[3] Howard D Curtis. *Orbital mechanics for engineering students*. Butterworth-Heinemann, 2013.

[4] David A Vallado. *Fundamentals of astrodynamics and applications*. Springer Science & Business Media, 3 edition, 2007.

Appendix 1 -derivation of gibbs

Appendix 2 - MATLAB code