1. Assume there're two sinusoids $A_1 \cos(\omega t + \phi_1)$, $A_2 \cos(\omega t + \phi_2)$, $\omega = 2\pi f$ have same frequency.

$$A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) = \frac{A_1}{2}\left(e^{j\omega t + j\phi_1} + e^{-j\omega t - j\phi_1}\right) + \frac{A_2}{2}\left(e^{j\omega t + j\phi_2} + e^{-j\omega t - j\phi_2}\right)$$

$$= \frac{1}{2} e^{j\omega t}\left(A_1 e^{j\phi_1} + A_2 e^{j\phi_2}\right) + \frac{1}{2} e^{-j\omega t}\left(A_1 e^{-j\phi_1} + A_2 e^{-j\phi_2}\right)$$

$A_1 e^{j\phi_1} + A_2 e^{j\phi_2}$ is still a complex exponential. Convert them to Cartesian coordinate:

$$A_1 e^{j\phi_1} : (A_1 \cos\phi_1 , A_1 \sin\phi_1)$$

$$A_2 e^{j\phi_2} : (A_2 \cos\phi_2 , A_2 \sin\phi_2)$$

$$\Rightarrow A_1 e^{j\phi_1} + A_2 e^{j\phi_2} : (A_1 \cos\phi_1 + A_2 \cos\phi_2 , A_1 \sin\phi_1 + A_2 \sin\phi_2)$$

Convert back to complex exponential:

$$A_1 e^{j\phi_1} + A_2 e^{j\phi_2} = A_3 e^{j\phi_3} \quad \text{where } A_3 = \sqrt{(A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2}$$

$$\phi_3 = \arctan\left(\frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}\right)$$

Similarly, $A_1 e^{-j\phi_1} + A_2 e^{-j\phi_2} = A_3 e^{-j\phi_3}$

$\therefore$ $A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$

$$= \frac{1}{2} e^{j\omega t} \cdot A_3 e^{j\phi_3} + \frac{1}{2} e^{-j\omega t} \cdot A_3 e^{-j\phi_3}$$

$$= \frac{A_3}{2}\left(e^{j\omega t + j\phi_3} + e^{-j\omega t - j\phi_3}\right)$$

$= A_3 \cos(\omega t + \phi_3)$ is still a sinusoid of same frequency.

2. Let $x(t) = \begin{cases} t & , \text{ if } 0 \le t \le \frac{T_0}{2} \\ \frac{-2}{T_0} t + 2 & , \text{ if } \frac{T_0}{2} < t \le T_0 \\ 0 & , \text{ otherwise} \end{cases}$, $T_0 = 0.04$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\,dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} t\,dt = \frac{2}{T_0} \cdot \frac{t^2}{2} \Big|_0^{\frac{T_0}{2}} = \frac{2}{T_0} \cdot \frac{T_0^2}{8} = \frac{T_0}{4} = 0.01$$

A: $a_0 = 0.01$

3. $$\mathcal{F}(e^{-at}u(t)) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-j\omega t}\,dt$$
$$= \int_0^{\infty} e^{-at} \cdot e^{-j\omega t}\,dt$$
$$= \int_0^{\infty} e^{(-a-j\omega)t}\,dt = \frac{1}{-a-j\omega} e^{(-a-j\omega)t} \Big|_0^{\infty} = \frac{1}{-a-j\omega}(0-1) = \frac{1}{a+j\omega}$$

4. $$\mathcal{F}(e^{\frac{-t^2}{2\sigma^2}}) = \int_{-\infty}^{\infty} e^{\frac{-t^2}{2\sigma^2}} \cdot e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} e^{-(\frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j)^2 - \frac{1}{2}\sigma^2\omega^2}\,dt$$
$$= e^{-\frac{1}{2}\sigma^2\omega^2} \int_{-\infty}^{\infty} e^{-(\frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j)^2}\,dt \qquad \left(u = \frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j\ ,\ du = \frac{dt}{\sqrt{2}\sigma} \right)$$
$$= e^{-\frac{1}{2}\sigma^2\omega^2} \int_{-\infty}^{\infty} e^{-u^2} \cdot \sqrt{2}\sigma\,du$$
$$= e^{-\frac{\sigma^2\omega^2}{2}} \cdot \sqrt{2}\sigma \cdot \int_{-\infty}^{\infty} e^{-u^2}\,du$$
$$= e^{-\frac{\sigma^2\omega^2}{2}} \sqrt{2}\sigma \cdot \sqrt{\pi} \qquad \left(\text{Gaussian integral},\ \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \right)$$

5. $x(t) = v(t)\cos(2\pi f_c t) \qquad f_c = 700\,Hz$

$v(t) = 5 + \cos(40\pi t)$

Let $\omega_c = 2\pi f_c$, $\mathcal{F}(x(t)) = \frac{1}{2\pi}\mathcal{F}(v(t)) * \mathcal{F}(\cos(\omega_c t))$

$$\mathcal{F}(\cos(\omega_c t)) = \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$$

$$\mathcal{F}(v(t)) = \int_{-\infty}^{\infty}(5+\cos(40\pi t))e^{-j\omega t}dt$$

$$= 5\int_{-\infty}^{\infty}e^{-j\omega t} + \int_{-\infty}^{\infty}\cos(40\pi t)e^{-j\omega t}dt$$

$$= 10\pi\delta(\omega) + \frac{1}{2}\int_{-\infty}^{\infty}e^{j40\pi t - j\omega t}dt + \frac{1}{2}\int_{-\infty}^{\infty}e^{-j40\pi t - j\omega t}dt$$

$$= 10\pi\delta(\omega) + \frac{1}{2}\int_{-\infty}^{\infty}e^{-j(\omega-40\pi)t}dt + \frac{1}{2}\int_{-\infty}^{\infty}e^{-j(\omega+40\pi)t}dt$$

$$= 10\pi\delta(\omega) + \pi\delta(\omega-40\pi) + \pi\delta(\omega+40\pi)$$