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1. Assume there're two sinusoids $A_1\cos(\omega t+\phi_1)$, $A_2\cos(\omega t+\phi_2)$, $\omega=2\pi f$ have same frequency.

$$A_1\cos(\omega t+\phi_1)+A_2\cos(\omega t+\phi_2)=\frac{A_1}{2}\left(e^{j\omega t+j\phi_1}+e^{-j\omega t-j\phi_1}\right)+\frac{A_2}{2}\left(e^{j\omega t+j\phi_2}+e^{-j\omega t-j\phi_2}\right)$$
$$=\frac{1}{2}e^{j\omega t}\left(A_1e^{j\phi_1}+A_2e^{j\phi_2}\right)+\frac{1}{2}e^{-j\omega t}\left(A_1e^{-j\phi_1}+A_2e^{-j\phi_2}\right)$$

$A_1e^{j\phi_1}+A_2e^{j\phi_2}$ is still a complex exponential. Convert them to Cartesian coordinate:

$A_1e^{j\phi_1}$: $(A_1\cos\phi_1,\ A_1\sin\phi_1)$

$A_2e^{j\phi_2}$: $(A_2\cos\phi_2,\ A_2\sin\phi_2)$

$\Rightarrow$ $A_1e^{j\phi_1}+A_2e^{j\phi_2}$: $(A_1\cos\phi_1+A_2\cos\phi_2,\ A_1\sin\phi_1+A_2\sin\phi_2)$

Convert back to complex exponential:

$$A_1e^{j\phi_1}+A_2e^{j\phi_2}=A_3e^{j\phi_3} \quad \text{where } A_3=\sqrt{(A_1\cos\phi_1+A_2\cos\phi_2)^2+(A_1\sin\phi_1+A_2\sin\phi_2)^2}$$
$$\phi_3=\arctan\left(\frac{A_1\sin\phi_1+A_2\sin\phi_2}{A_1\cos\phi_1+A_2\cos\phi_2}\right)$$

Similarly, $A_1e^{-j\phi_1}+A_2e^{-j\phi_2}=A_3e^{-j\phi_3}$

$\therefore$ $A_1\cos(\omega t+\phi_1)+A_2\cos(\omega t+\phi_2)$

$$=\frac{1}{2}e^{j\omega t}\cdot A_3e^{j\phi_3}+\frac{1}{2}e^{-j\omega t}\cdot A_3e^{-j\phi_3}$$
$$=\frac{A_3}{2}\left(e^{j\omega t+j\phi_3}+e^{-j\omega t-j\phi_3}\right)$$
$$=A_3\cos(\omega t+\phi_3)$$ is still a sinusoid of same frequency.

2. Let $x(t) = \begin{cases} \cancel{t}\ \frac{2t}{T_0}, & \text{if } 0 \le t \le \frac{T_0}{2} \\ -\frac{2}{T_0}t + 2, & \text{if } \frac{T_0}{2} < t \le T_0 \\ 0, & \text{otherwise} \end{cases}$, $T_0 = 0.04$

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$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} \frac{2}{T_0} t\,dt = \frac{2}{T_0}\cdot\frac{t^2}{2}\Big|_0^{\frac{T_0}{2}} = \frac{2}{T_0}\cdot\frac{T_0^2}{8} = \frac{T_0}{4} = 0.01$$

$0.01 \times \frac{2}{T_0} = \frac{0.02}{0.04} = \frac{1}{2}$ #

A: $a_0 = \cancel{0.01}$

3. $$\mathcal{F}(e^{-at}u(t)) = \int_{-\infty}^{\infty} e^{-at}u(t)\cdot e^{-j\omega t}\,dt$$

$$= \int_0^{\infty} e^{-at}\cdot e^{-j\omega t}\,dt$$

$$= \int_0^{\infty} e^{(-a-j\omega)t}\,dt = \frac{1}{-a-j\omega}\,e^{(-a-j\omega)t}\Big|_0^{\infty} = \frac{1}{-a-j\omega}(0-1) = \frac{1}{a+j\omega}$$

4. $$\mathcal{F}(e^{\frac{-t^2}{2\sigma^2}}) = \int_{-\infty}^{\infty} e^{\frac{-t^2}{2\sigma^2}}\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-(\frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j)^2 - \frac{1}{2}\sigma^2\omega^2}\,dt$$

$$= e^{-\frac{1}{2}\sigma^2\omega^2}\int_{-\infty}^{\infty} e^{-(\frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j)^2}\,dt \qquad \left(u = \frac{t}{\sqrt{2}\sigma} + \frac{\sqrt{2}}{2}\sigma\omega j,\ du = \frac{dt}{\sqrt{2}\sigma}\right)$$

$$= e^{-\frac{1}{2}\sigma^2\omega^2}\int_{-\infty}^{\infty} e^{-u^2}\cdot\sqrt{2}\sigma\,du$$

$$= e^{-\frac{\sigma^2\omega^2}{2}}\cdot\sqrt{2}\sigma\cdot\int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= e^{-\frac{\sigma^2\omega^2}{2}}\sqrt{2}\sigma\cdot\sqrt{\pi} \qquad \left(\text{Gaussian integral},\ \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}\right)$$

5. $x(t) = v(t)\cos(2\pi f_c t) \quad f_c = 700\,\mathrm{Hz}$

$v(t) = 5 + \cos(40\pi t)$

Let $\omega_c = 2\pi f_c$, $\mathcal{F}(x(t)) = \frac{1}{2\pi}\mathcal{F}(v(t)) * \mathcal{F}(\cos(\omega_c t))$

$$\mathcal{F}(\cos(\omega_c t)) = \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$$

$$\mathcal{F}(v(t)) = \int_{-\infty}^{\infty}(5+4\cos(40\pi t))e^{-j\omega t}\,dt$$

$$= 5\int_{-\infty}^{\infty}e^{-j\omega t}\,dt + \int_{-\infty}^{\infty}4\cos(40\pi t)e^{-j\omega t}\,dt$$

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$$= 10\pi\delta(\omega) + \frac{4}{2}\int_{-\infty}^{\infty}e^{j40\pi t - j\omega t}\,dt + \frac{4}{2}\int_{-\infty}^{\infty}e^{-j40\pi t - j\omega t}\,dt$$

$$= 10\pi\delta(\omega) + \frac{4}{2}\int_{-\infty}^{\infty}e^{-j(\omega-40\pi)t}\,dt + \frac{4}{2}\int_{-\infty}^{\infty}e^{-j(\omega+40\pi)t}\,dt$$

$$= 10\pi\delta(\omega) + 4\pi\delta(\omega-40\pi) + 4\pi\delta(\omega+40\pi)$$