Homework I, Advanced Algorithms 2022

Due on Wednesday March 30 at 17.00 (upload one solution per group on moodle). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Michael Kapralov. You are, however, allowed to discuss problems in groups of up to three students; it is sufficient to hand in one solution per group.

1 *(34 pts)* **Extreme points.** For a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}_+$ consider the following linear program:

$$\begin{aligned} \min \quad & \textstyle\sum_{e \in E} w_e \cdot x_e \\ \text{s.t.} \quad & \\ & \textstyle\sum_{e \in \delta(u)} x_e \geq 1 \qquad \text{for all } u \in V \\ & x_e \geq 0 \qquad\qquad\quad \text{for all } e \in E, \end{aligned}$$

where $\delta(u)$ stands for the set of edges in E incident on u. Prove that every extreme point solution is supported on a disjoint union of odd length cycles and stars.

Hint: first show that no extreme point solution can contain an even length cycle in its support; then consider an odd length cycle with a path attached to it, and then two odd length cycles connected by a path.

2 *(33 pts)* **Partitioning into disjoint bases.** Give an algorithm that, taking as input a matrix $A \in \mathbb{R}^{n \times kn}$ for integers $n, k \geq 1$, outputs YES if the columns of A can be partitioned into k sets such that every set forms a basis of $\mathbb{R}^n$, and outputs NO otherwise.

Hint: use matroid intersection.

3 *(33 pts)* **Implementation.** The objective of this problem is to successfully solve the problem *Minimum spanning tree dual program* on our online judge. You will find detailed instructions on how to do this on Moodle.

Hint: you may find Section 5 of these notes useful: `https://theory.epfl.ch/courses/topicstcs/Lecture52015.pdf`.