Game Theory

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A Multi-Leveled Approach

Springer

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Voor Lenie, Nina and Remco

Preface

This book is a compilation of much of the material I used for various game theory courses over, roughly, the past two decades. The first part, *Thinking Strategically*, is intended for undergraduate students in economics or business, but can also serve as an introduction for the subsequent parts of the book. The second and third parts go deeper into the various topics treated in the first part. These parts are intended for more mathematically oriented undergraduate students, or for graduate students in (for instance) economics. Part II is on noncooperative games and Part III on cooperative games. Part IV is only a mathematical tools chapter. Every chapter has a final section with problems. Selected hints, answers, and solutions to these problems are given at the end of the book. Complete solutions can be obtained from the author.

The book claims neither originality nor completeness. As to originality, the material draws heavily on game theory texts developed by many others, often in collaboration. I mention in particular Jean Derks, Thijs Jansen, Andrés Perea, Ton Storcken, Frank Thuijsman, Stef Tijs, Dries Vermeulen, and Koos Vrieze. I am also greatly indebted to a large number of introductory, intermediate, and advanced texts and textbooks on game theory, and hope I have succeeded in giving sufficient credit to the authors of these works in all relevant places. As to completeness, the book is far from achieving this quality but I trust that it presents at least the basics of game theory. When writing and compiling the material I had ambitious plans for many more chapters, and only hope that the phrase that all good things must come to an end applies here.

How to Use this Book

Part I of the book is intended, firstly, for undergraduate students in economics and business and, secondly, as preparation and background for Parts II–IV. Part I is preceded by Chap. 1, which is a general introduction to game theory by means of examples. The first chapter of Part I, Chap. 2 of the book, is on zero-sum games. This chapter is included, not only for historical reasons – the minimax theorem of von Neumann [140] was one of the first formal results in game theory – but also because zero-sum games (all parlor games) require basic, strictly competitive, game-theoretic thinking. The heart of Part I consists of Chaps. 3–6 on noncooperative games and applications, and Chap. 9 as a basic introduction to cooperative games. These chapters can serve as a basics course in game theory. Chapters 7 and 8 on repeated games and evolutionary games can serve as extra material, as well as Chap. 10 on cooperative game models and Chap. 11, which is an introduction to the related area of social choice theory.

Although this book, in particular Part I, can be used for self-study, it is not intended to replace the teacher. Part I is meant for students who are knowledgeable in basic calculus, and does not try to avoid the use of mathematics on that basic level. There is, after all, a reason why we teach calculus courses to (for instance) undergraduate students in economics. Nevertheless, the mathematics in Part I does not go beyond such things as maximizing a quadratic function or elementary matrix notation. In my own experience, the difficulties that students encounter when studying formal models are usually conceptual rather than mathematical, but they are often confused with mathematical difficulties. A well recognized example is the distinction between parameters and variables in a model. Another example is the use of mathematical notation. Even in Part I of the book (almost) all basic game theory models are described in a formally precise manner, although I am aware that some students may have a blind spot for mathematical notation that goes beyond simple formulas for functions and equations. This formal presentation is included especially because many students have always been asking questions about it: leaving it out may lead to confusion and ambiguities. On the other hand, a teacher may decide to drop these more formal parts and go directly to the examples of concretely specified games. For example, in Chap. 5, the game theory teacher may decide to skip the formal Sect. 5.1 and go directly to the worked out examples of games with incomplete information – and perhaps later return to Sect. 5.1.

I can be much shorter on Parts II–IV, which require more mathematical sophistication and are intended for graduate students in economics, or for an elective game theory course for students in (applied) mathematics. In my experience again, it works well to couple the material in these parts to related chapters in Part I. In particular, one can combine Chaps. 2 and 12 on zero-sum games, Chaps. 3 and 13 on finite games, Chaps. 4, 5, and 14 on games with incomplete information and games in extensive form, and Chaps. 8 and 15 on evolutionary games.[1] For cooperative game theory, one can combine Chap. 9 with Part III.

Each chapter concludes with a problems section. Partial hints, answers and solutions are provided at the end of the book. For a complete set of solutions for teachers, please contact the author by email.[2]

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[1] Chapter 7 is on repeated games but no advanced chapter on this topic is included. See, e.g., [39], [75], or [88].

[2] H.Peters@ke.unimaas.nl.

References

This is a textbook and not a handbook, and consequently the list of references is limited and far from complete. In most cases references in the text are indicated by a number and only in some cases also by the name(s) of the author(s), for instance when a concept is named after a person.

Notation

I do not have much to say on notation. Bold letters are used to indicate vectors – while working on the book I came to regret this convention very much because of the extra work, but had already passed the point of no return. Transpose signs for vectors and matrices are only used when absolutely necessary. Vector inequalities use the symbols $>$ for all coordinates strictly larger and $\geq$ for all coordinates at least as large – and of course their reverses.

Errors

All errors are mine, and I would appreciate any feedback. All comments, not only those on errors, are most welcome.

Maastricht *Hans Peters*
May 2008

Contents

Chapter 1
Introduction

The best introduction to game theory is by way of examples. In this chapter we start with a global definition of the field in Sect. 1.1, collect some historical facts in Sect. 1.2, and present examples in Sect. 1.3. In Sect. 1.4 we briefly comment on the distinction between cooperative and noncooperative game theory.

1.1 A Definition

Game theory is a formal, mathematical discipline which studies situations of competition and cooperation between several involved parties. This is a broad definition but consistent with the large number of applications. These applications range from strategic questions in warfare to understanding economic competition, from economic or social problems of fair distribution to behavior of animals in competitive situations, from parlor games to political voting systems – and this list is certainly not exhaustive.

Although game theory is an official mathematical discipline (AMS[1] Classification code 90D) it is applied mostly by economists. Many articles and books on game theory and applications are found under the JEL[2] codes C7x. The list of references at the end of this book contains many textbooks and other books on game theory.

1.2 Some History

In terms of applications, game theory is a broad discipline, and it is therefore not surprising that 'game-theoretic' situations can be recognized in the Bible (see [17]) or the Talmud (see [7]). Also the literature on strategic warfare contains many

[1] American Mathematical Society.

[2] Journal of Economic Literature.

situations that could have been modelled using game theory: a very early reference, over 2,000 years old, is the work of the Chinese warrior-philosopher Sun Tzu (see [130]). Early works dealing with economic problems are the work of A. Cournot on quantity competition (see [21]) and J. Bertrand on price competition (see [11]). Some of the work of C.L. Dodgson (better known as Lewis Carroll, the writer of *Alice's Adventures in Wonderland*) is an early application of zero-sum games to the political problem of parliamentary representation, see [32] and [12].

One of the first more formal works on game theory is the article of the logician Zermelo, see [150]. He proved that in the game of chess either White has a winning strategy (i.e., can always win), or Black has a winning strategy, or each player can always enforce a draw.[3] Up to the present, however, it is still not known which of these three cases is the true one. A milestone in the history of game theory is the work of von Neumann on zero-sum games [140], in which he proved the famous minimax theorem for zero-sum games. This article was the basis for the book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern [141], by many regarded as the starting point of game theory. In this book the authors extended von Neumann's work on zero-sum games and laid the groundwork for the study of cooperative (coalitional) games.[4]

The title of the book of von Neumann and Morgenstern reveals the intention of the authors that game theory was to be applied to economics. Nevertheless, in the fifties and sixties the further development of game theory was mainly the domain of mathematicians. Seminal articles in this period were the papers by John F. Nash[5] on Nash equilibrium and on bargaining (see [91] and [90]) and Shapley on the Shapley value and the core for games with transferable utility (see [121] and [122][6]). Apart from these articles, the foundations of much that was to follow later were laid in the contributed volumes [68], [69], [33], [73], and [34].

In the late sixties and seventies of the previous century game theory became accepted as a new formal language for economics in particular. This development was stimulated by the work of John Harsanyi on modelling games with incomplete information (see [50]) and Reinhard Selten [117, 118] on (sub)game perfect Nash equilibrium.[7] From the eighties on, large parts of economics have been rewritten and further developed using the ideas, concepts and formal language of game theory. Articles on game theory and applications can be found in many economic journals. Journals focusing on game theory are the *International Journal of Game Theory*, *Games and Economic Behavior*, and *International Game Theory Review*. Game theorists are organized within the *Game Theory Society*, see http://www.gametheorysociety.org/.

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[3] See Sect. 13.2.5.

[4] See [31] for a comprehensive history of game theory up to 1945.

[5] See [89] for a biography, and the later movie with the same title *A Beautiful Mind*.

[6] See also [16].

[7] In 1994, Nash, Harsanyi and Selten received the Nobel prize in economics for the mentioned work in game theory.

1.3 Examples

Every example in this section is based on a 'story'. Each time this story is presented first and, next, it is translated into a formal mathematical model. Such a mathematical model is an alternative description, capturing the essential ingredients of the story with the omission of details that are considered unimportant: the mathematical model is an 'abstraction' of the story. After having established the model, we spend some lines on how to 'solve' it: we try to say something about how the players should or would act. In more philosophical terms, these 'solutions' can be normative or positive in nature, or somewhere in between, but often such considerations are left as food for thought for the reader. As a general remark, a basic distinction between optimization theory and game theory is that in optimization it is usually clear what is meant by the word 'optimal', whereas in game theory we deal with human (or, more generally, animal) behavior and then it is less clear what 'optimal' means.[8] Each example is concluded by further comments, possibly including a short preview on the treatment of the exemplified game in the book.

The examples are grouped in subsections on zero-sum games, nonzero-sum games, extensive form games, cooperative games, and bargaining games.

1.3.1 Zero-Sum Games

The first example is taken from [106].

The Battle of the Bismarck Sea

Story The game is set in the South-Pacific in 1943. The Japanese admiral Imamura has to transport troops across the Bismarck Sea to New Guinea, and the American admiral Kenney wants to bomb the transport. Imamura has two possible choices: a shorter Northern route (2 days) or a larger Southern route (3 days), and Kenney must choose one of these routes to send his planes to. If he chooses the wrong route he can call back the planes and send them to the other route, but the number of bombing days is reduced by 1. We assume that the number of bombing days represents the payoff to Kenney in a positive sense and to Imamura in a negative sense.

Model The Battle of the Bismarck Sea problem can be modelled as in the following table:

$$\begin{array}{c} \\ \text{North} \\ \text{South} \end{array} \begin{array}{c} \begin{array}{cc} \text{North} & \text{South} \end{array} \\ \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \end{array}.$$

[8] Feyerabend's [35] 'anything goes' adage reflects a workable attitude in a young science like game theory.

This table represents a game with two players, namely Kenney and Imamura. Each player has two possible choices; Kenney (player 1) chooses a row, Imamura (player 2) chooses a column, and these choices are to be made independently and simultaneously. The numbers represent the payoffs to Kenney. For instance, the number 2 up left means that if Kenney and Imamura both choose North, the payoff to Kenney is 2 and to Imamura -2. (The convention is to let the numbers denote the payments *from* player 2 (the column player) *to* player 1 (the row player).) This game is an example of a zero-sum game because the sum of the payoffs is always equal to zero.

Solution In this particular example, it does not seem difficult to predict what will happen. By choosing North, Imamura is always at least as well off as by choosing South, as is easily inferred from the above table of payoffs. So it is safe to assume that Imamura chooses North, and Kenney, being able to perform this same kind of reasoning, will then also choose North, since that is the best reply to the choice of North by Imamura. Observe that this game is easy to analyze because one of the players has a weakly dominant choice, i.e., a choice which is always at least as good (giving always at least as high a payoff) as any other choice, no matter what the opponent decides to do.

Another way to look at this game is to observe that the payoff 2 resulting from the combination (North, North) is maximal in its column ($2 \geq 1$) and minimal in its row ($2 \leq 2$). Such a position in the matrix is called a *saddlepoint*. In such a saddlepoint, neither player has an incentive to deviate unilaterally.[9] Also observe that, in such a saddlepoint, the row player maximizes his minimal payoff (because $2 = \min\{2,2\} \geq 1 = \min\{1,3\}$), and the column player (who has to pay according to our convention) minimizes the maximal amount that he has to pay (because $2 = \max\{2,1\} \leq 3 = \max\{2,3\}$). The resulting payoff of 2 from player 2 to player 1 is called the *value* of the game.

Comments Two-person zero-sum games with finitely many choices, like the one above, are also called *matrix games* since they can be represented by a single matrix. Matrix games are studied in Chaps. 2 and 12. The combination (North, North) in the example above corresponds to what happened in reality back in 1943. See the memoirs of Winston Churchill [20].[10]

Matching Pennies

Story In the two-player game of *matching pennies*, both players have a coin and simultaneously show heads or tails. If the coins match, player 2 gives his coin to player 1; otherwise, player 1 gives his coin to player 2.

Model This is a zero-sum game with payoff matrix

[9] As will become clear later, this implies that the combination (North, North) is a Nash equilibrium.
[10] In 1953, Churchill received the Nobel prize in literature for this work.

$$
\begin{array}{c} \\ \text{Heads} \\ \text{Tails} \end{array}
\begin{array}{c}
\begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\
\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\end{array}.
$$

Solution Observe that in this game no player has a (weakly) dominant choice, and that there is no *saddlepoint*: there is no position in the matrix at which there is simultaneously a minimum in the row and a maximum in the column. Thus, there does not seem to be a natural way to solve the game. Von Neumann [140] proposed to solve games like this – and zero-sum games in general – by allowing the players to randomize between their choices. In the present example of matching pennies, suppose player 1 chooses heads or tails both with probability $\frac{1}{2}$. Suppose furthermore that player 2 plays heads with probability q and tails with probability $1-q$, where $0 \leq q \leq 1$. In that case the expected payoff for player 1 is equal to

$$\frac{1}{2}[q \cdot 1 + (1-q) \cdot -1] + \frac{1}{2}[q \cdot -1 + (1-q) \cdot 1]$$

which is independent of q, namely, equal to 0. So by randomizing in this way between his two choices, player 1 can guarantee to obtain 0 in expectation (of course, the actually realized outcome is always $+1$ or -1). Analogously, player 2, by playing heads or tails each with probability $\frac{1}{2}$, can guarantee to pay 0 in expectation. Thus, the amount of 0 plays a role similar to that of a saddlepoint. Again, we will say that 0 is the *value* of this game.

Comments The randomized choices of the players are usually called *mixed strategies*. Randomized choices are often interpreted as *beliefs* of the other player(s) about the choice of the player under consideration. See, e.g., Sect. 3.1.

Von Neumann [140] proved that every two-person matrix game has a value if the players can use mixed strategies: this is the minimax theorem.

1.3.2 Nonzero-Sum Games

Prisoners' Dilemma

Story Two prisoners (players 1 and 2) have committed a crime together and are interrogated separately. Each prisoner has two possible choices: he may 'cooperate' (C) which means 'not betray his partner' or he may 'defect' (D), which means 'betray his partner'. The punishment for the crime is 10 years of prison. Betrayal yields a reduction of 1 year for the traitor. If a prisoner is not betrayed, he is convicted to 1 year for a minor offense.

Model This situation can be summarized as follows:

$$
\begin{array}{c} \\ C \\ D \end{array}
\begin{array}{c}
\begin{array}{cc} C & D \end{array} \\
\begin{pmatrix} -1,-1 & -10,0 \\ 0,-10 & -9,-9 \end{pmatrix}
\end{array}.
$$

This table must be read in the same way as before, but now there are two payoffs at each position: by convention the first number is the payoff for player 1 and the second number is the payoff for player 2. Observe that the game is no longer zero-sum, and we have to write down both numbers at each matrix position.

Solution Observe that for both players D is a strictly dominant choice: for each player, D is (strictly) the best choice, whatever the other player does. So it is natural to argue that the outcome of this game will be the pair of choices (D,D), leading to the payoffs $-9,-9$. Thus, due to the existence of strictly dominant choices, the Prisoners' Dilemma game is easy to analyze.

Comments The payoffs $(-9,-9)$ are inferior: they are not 'Pareto optimal', the players could obtain the higher payoff of -1 for each by cooperating, i.e., both playing C. There is a large literature on how to establish cooperation, e.g., by reputation effects in a repeated play of the game. See, in particular, Axelrod [8]. If the game is played repeatedly, other (higher) payoffs are possible, see Chap. 7.

The Prisoners' Dilemma is a metaphor for many economic situations. An outstanding example is the so-called *tragedy of the commons* ([47]; see also [45], p. 27, and Problem 6.26 in this book).

Battle of the Sexes

Story A man and a woman want to go out together, either to a soccer match or to a ballet performance. They forgot to agree where they would go to that night, are in different places and have to decide on their own where to go; they have no means to communicate. Their main concern is to be together, the man has a preference for soccer and the woman for ballet.

Model A table reflecting the situation is as follows.

$$\begin{array}{c} \\ \text{Soccer} \\ \text{Ballet} \end{array} \begin{array}{c} \begin{array}{cc} \text{Soccer} & \text{Ballet} \end{array} \\ \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{array} .$$

Here, the man chooses a row and the woman a column.

Solution Observe that no player has a dominant choice. The players have to coordinate without being able to communicate. Now it may be possible that the night before they discussed soccer at length; each player remembers this, may think that the other remembers this, and so this may serve as a 'focal point' (see Schelling[11] [115] on the concept of focal points). In the absence of such considerations it is hard to give a unique prediction for this game. We can, however, say that the combinations (Soccer, Soccer) and (Ballet, Ballet) are special in the sense that the players' choices are 'best replies' to each other; if the man chooses Soccer (Ballet), then it is optimal for the woman to choose Soccer (Ballet) as well, and vice versa. In

[11] One of the winners of the 2005 Nobel prize in economics; the other one was R.J. Aumann.

literature, such choice combinations are called *Nash equilibria*. The concept of Nash equilibrium [91] is no doubt the main solution concept developed in game theory.

Comments The Battle of the Sexes game is metaphoric for problems of coordination.

Matching Pennies

Every zero-sum game is, trivially, a special case of a nonzero-sum game. For instance, the Matching Pennies game discussed in Sect. 1.3.1 can be represented as a nonzero-sum game as follows:

$$\begin{array}{cc} & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}.$$

Clearly, no player has a dominant choice and there is no combination of a row and a column such that each player's choice is optimal given the choice of the other player – there is no Nash equilibrium. If mixed strategies are allowed, then it can be checked that if player 2 plays Heads and Tails each with probability $\frac{1}{2}$, then for player 1 it is optimal to do so too, and vice versa. Such a combination of mixed strategies is again called a Nash equilibrium. Nash [91] proved that every game in which each player has finitely many choices – zero-sum or nonzero-sum – has a Nash equilibrium in mixed strategies. See Chaps. 3 and 13.

A Cournot Game

Story Two firms produce a similar ('homogenous') product. The market price of this product is equal to $p = 1 - Q$ or zero (whichever is larger), where Q is the total quantity produced. There are no production costs.

Model The two firms are the players, 1 and 2. Each player $i = 1,2$ chooses a quantity $q_i \geq 0$, and makes a profit of $K_i(q_1,q_2) = q_i(1 - q_1 - q_2)$ (or zero if $q_1 + q_2 \geq 1$).

Solution Suppose player 2 produces $q_2 = \frac{1}{3}$. Then player 1 maximizes his own profit $q_1(1 - q_1 - \frac{1}{3})$ by choosing $q_1 = \frac{1}{3}$. Also the converse holds: if player 1 chooses $q_1 = \frac{1}{3}$ then $q_2 = \frac{1}{3}$ maximizes profit for player 2. This combination of strategies consists of mutual best replies and is therefore again called a Nash equilibrium.

Comments Situations like this were first analyzed by Cournot [21]. The Nash equilibrium is often called Cournot equilibrium. It is easy to check that the Cournot equilibrium in this example is again not 'Pareto optimal': if the firms each would produce $\frac{1}{4}$, then they would both be better off.

The main difference between this example and the preceding ones is, that each player here has infinitely many choices, also without including mixed strategies.

See further Chap. 6.

1.3.3 Extensive Form Games

All examples in Sects. 1.3.1 and 1.3.2 are examples of 'one-shot games'. The players choose only once, independently and simultaneously. In parlor games as well as in games derived from real-life economic or political situations, this is often not what happens. Players may move sequentially, and observe or partially observe each others' moves. Such situations are better modelled by 'extensive form games'.

Sequential Battle of the Sexes

Story The story is similar to the story in Sect. 1.3.2, but we now assume that the man chooses first and the woman can observe the choice of the man.

Model This situation can be represented by the decision tree in Fig. 1.1. Player 1 (the man) chooses first, player 2 (the woman) observes player 1's choice and then makes her own choice. The first number in each pair of numbers is the payoff to player 1, and the second number is the payoff to player 2. Filled circles denote decision nodes (of a player) or end nodes (followed by payoffs).

Solution An obvious way to analyze this game is to work backwards. If player 1 chooses S, then it is optimal for player 2 to chooses S as well, and if player 1 chooses B, then it is optimal for player 2 to choose B as well. Given this choice behavior of player 2 and assuming that player 1 performs this line of reasoning about the choices of player 2, player 1 should choose S.

Comments What this simple example shows is that in such a so-called extensive form game, there is a distinction between a play plan of a player and an actual move or choice of that player. Player 2 has the plan to choose S (B) if player 1 has chosen S (B). Player 2's actual choice is S – assuming as above that player 1 has chosen

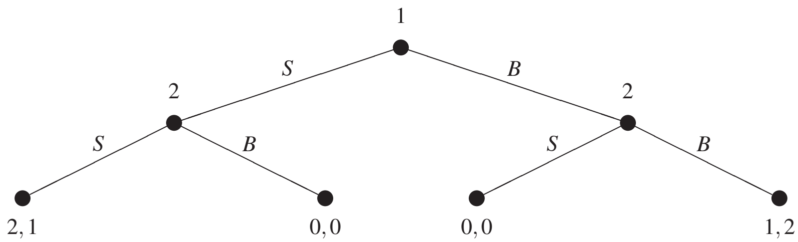


Fig. 1.1 The decision tree of sequential Battle of the Sexes

S. We use the word *strategy* to denote a play plan, and the word *action* to denote a particular move. In a one-shot game there is no difference between the two, and then the word 'strategy' is used.

Games in extensive form are studied in Chaps. 4 and 14. The solution described above is an example of a so-called backward induction (or subgame perfect) (Nash) equilibrium. Such equilibria were first explicitly studied in [117]. There are other equilibria as well. Suppose player 1 chooses B and player 2's plan (strategy) is to choose B always, independent of player 1's choice. Observe that, given the strategy of the opponent, no player can do better, and so this combination is a Nash equilibrium, although player 2's plan is only partly 'credible': if player 1 would choose S instead of B, then player 2 would be better off by changing her choice to S.

Sequential Cournot

Story The story is similar to the story in Sect. 1.3.2, but we now assume that firm 1 chooses first and firm 2 can observe the choice of firm 1.

Model Since each player $i = 1,2$ has infinitely many actions $q_i \geq 0$, we cannot draw a picture like Fig. 1.1 for the sequential Battle of the Sexes. Instead of straight lines we use zigzag lines to denote a continuum of possible actions. For this example we obtain Fig. 1.2.

Player 1 moves first and chooses $q_1 \geq 0$. Player 2 observes player 1's choice of q_1 and then chooses $q_2 \geq 0$.

Solution Like in the sequential Battle of the Sexes game, an obvious way to solve this game is by working backwards. Given the observed choice q_1, player 2's optimal (profit maximizing) choice is $q_2 = \frac{1}{2}(1-q_1)$ or $q_2 = 0$, whichever is larger. Given this 'reaction function' of player 2, the optimal choice of player 1 is obtained by maximizing the profit function $q_1 \mapsto q_1\left(1-q_1-\frac{1}{2}(1-q_1)\right)$. The maximum is obtained for $q_1 = \frac{1}{2}$. Consequently, player 2 chooses $q_2 = \frac{1}{4}$.

Comments The solution described here is another example of a backward induction or subgame perfect equilibrium. It is also called 'Stackelberg equilibrium'. See [142] and Chap. 6.

Entry Deterrence

Story An old question in industrial organization is whether an incumbent monopolist can maintain his position by threatening to start a price war against any new

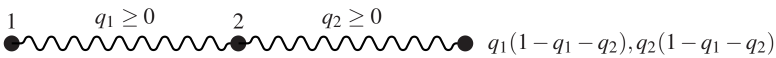


Fig. 1.2 The extensive form of sequential Cournot

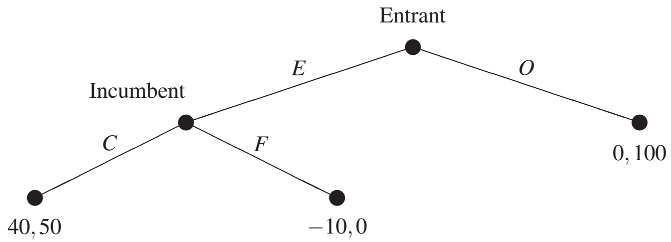


Fig. 1.3 The game of entry deterrence. Payoffs: entrant, incumbent

firm that enters the market. In order to analyze this question, consider the following situation. There are two players, the entrant and the incumbent. The entrant decides whether to Enter (E) or to Stay Out (O). If the entrant enters, the incumbent can Collude (C) with him, or Fight (F) by cutting the price drastically. The payoffs are as follows. Market profits are 100 at the monopoly price and 0 at the fighting price. Entry costs 10. Collusion shares the profits evenly.

Model This situation can be represented by the decision tree in Fig. 1.3.

Solution By working backward, we find that the entrant enters and the incumbent colludes.

Comments Also here there exists another Nash equilibrium. If the entrant stays out and the incumbent's plan is to fight if the entrant would enter, then this is again a combination where no player can do better given the strategy of the other player. Again, one might argue that the 'threat' of the incumbent firm to start a price war in case the potential entrant would enter, is not credible since the incumbent hurts himself by carrying out the threat.

Entry Deterrence with Incomplete Information

Story Consider the following variation on the foregoing entry deterrence model. Suppose that with 50% probability the incumbent's payoff from Fight (F) is equal to some amount x rather than the 0 above, that both firms know this, but that the true payoff is only observed by the entrant. This situation might arise, for instance, if the technology or cost structure of the entrant firm is private information but both firms would make the same estimate about the associated probabilities.

Model This situation can be modelled by including a chance move in the game tree. Moreover, the tree should express the asymmetric information between the players. Consider the following game tree. In this tree, there is first a chance move. The entrant learns the outcome of the chance move and decides to enter or not. If he enters, then the incumbent decides to collude or fight, without however knowing the outcome of the chance move: this is indicated by the dashed line. Put otherwise, the incumbent has two decision nodes where he should choose, but he does not know at which node he actually is. Thus, he can only choose between 'collude' and 'fight',

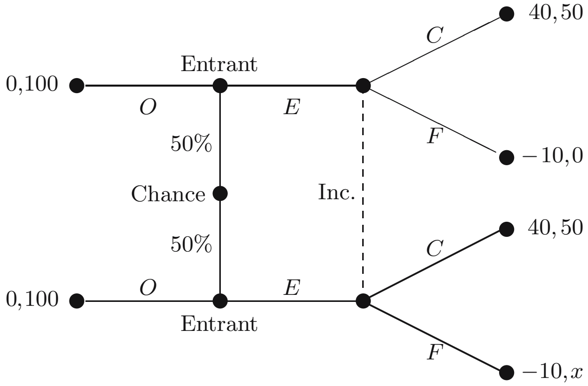


Fig. 1.4 Entry deterrence with incomplete information

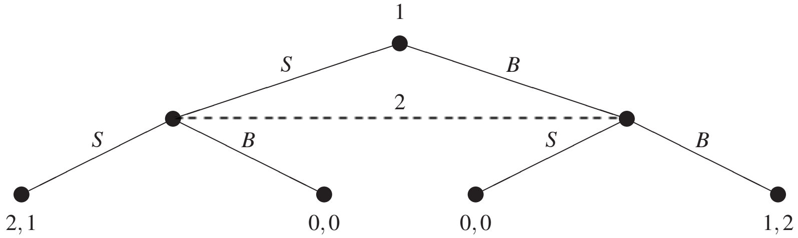


Fig. 1.5 Simultaneous Battle of the Sexes in extensive form

without making this choice contingent on the outcome of the chance move. See Fig. 1.4.

Solution Clearly, if $x \leq 50$ then an obvious solution is that the incumbent colludes and the entrant enters. Also the combination of strategies where the entrant stays out no matter what the outcome of the chance move is, and the incumbent fights, is a Nash equilibrium. A complete analysis is more subtle and may include considering probabilistic information that the incumbent might derive from the action of the entrant in a so-called 'perfect Bayesian equilibrium', see Chaps. 5 and 14.

Comments The collection of the two nodes of the incumbent, connected by the dashed line, is usually called an 'information set'. Information sets are used in general to model imperfect information. In the present example imperfect information arises since the incumbent does not know the outcome of the chance move. Imperfect information can also arise if some player does not observe some move of some other player. As a simple example, consider again the simultaneous move Battle of the Sexes game of Sect. 1.3.2. This can be modelled as a game in extensive form as in Fig. 1.5.

Hence, player 2, when he moves, does not know what player 1 has chosen. This is equivalent to players 1 and 2 moving independently and simultaneously.

1.3.4 Cooperative Games

In a cooperative game the focus is on payoffs and coalitions, rather than on strategies. The prevailing analysis has an axiomatic flavor, in contrast to the equilibrium analysis of noncooperative theory.

Three Cooperating Cities

Story Cities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in the following picture. Each city can hire any of the transmission links. If the cities cooperate in hiring the links they save on the hiring costs (the links have unlimited capacity). The situation is represented in Fig. 1.6.

Model The players in this situation are the three cities. Denote the player set by $N = \{1,2,3\}$. These players can form coalitions: any subset S of N is called a *coalition*. Table 1.1 presents the costs as well as the savings of each coalition. The numbers $c(S)$ are obtained by calculating the cheapest routes connecting the cities in the coalition S with the power source.[12] The cost savings $v(S)$ are determined by

$$v(S) := \sum_{i \in S} c(\{i\}) - c(S) \quad \text{for each nonempty} \quad S \subseteq N.$$

The cost savings $v(S)$ for coalition S are equal to the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together. The pair (N, v) is called a *cooperative game*.

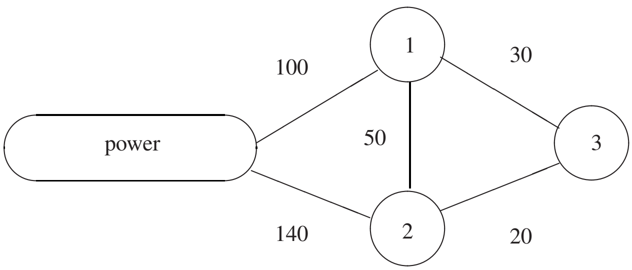


Fig. 1.6 Situation leading to the three cities game

Table 1.1 The three cities game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

[12] Cf. Bird [14].

Solution Basic questions in a cooperative game (N,v) are: which coalitions will actually be formed, and how should the proceeds (savings) of such a coalition be distributed among its members? To form a coalition the consent of every member is needed, but it is likely that the willingness of a player to participate in a coalition depends on what the player obtains in that coalition. Therefore, the second question seems to be the more basic one, and in this book attention is focussed on that question. Specifically, it is usually assumed that the 'grand' coalition N of all players is formed, and the question is then reduced to the problem of distributing the amount $v(N)$ among the players. In the present example, how should the amount 220 $(=v(N))$ be distributed among the three cities? In other words, we look for vectors $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ such that $x_1+x_2+x_3 = 220$, where player $i \in \{1,2,3\}$ obtains x_i. One obvious candidate is to choose $x_1 = x_2 = x_3 = 220/3$, but this does not really reflect the asymmetry of the situation: some coalitions save more than others. The literature offers many quite different solutions to this distribution problem, among which are the *core*, the *Shapley value*, and the *nucleolus*. The core, for instance, consists of those payoff distributions that cannot be improved upon by any smaller coalition. For the three cities example, this means that the core consists of those vectors (x_1,x_2,x_3) such that $x_1+x_2+x_3 = 220$, $x_1,x_2,x_3 \geq 0$, $x_1+x_2 \geq 90$, $x_1+x_3 \geq 100$, and $x_2+x_3 \geq 120$. Hence, this is quite a big set and therefore rather indeterminate as a solution to the game. In contrast, the Shapley value consists by definition of one point (vector), in this example the distribution $(65,75,80)$. Also the nucleolus consists of one point, in this case the vector $(56\frac{2}{3},76\frac{2}{3},86\frac{2}{3})$.[13]

Comments The implicit assumptions in a game like this are, first, that a coalition can make binding agreements on the distribution of its payoff and, second, that any payoff distribution that distributes (or, at least, does not exceed) the savings or, more generally, *worth* of the coalition is possible. For these reasons, such games are called *cooperative games with transferable utility*. See Chaps. 9 and 16–20.

The Glove Game

Story Assume there are three players, 1, 2, and 3. Players 1 and 2 each possess a right-hand glove, while player 3 has a left-hand glove. A pair of gloves has worth 1. The players cooperate in order to generate a profit.

Model The associated cooperative game is described by Table 1.2.

Table 1.2 The glove game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	1	1

[13] The reader should take these claims for granted. Definitions of these concepts are provided in Chap. 9. See also Chaps. 16–20.

Table 1.3 Preferences for dentist appointments

	Mon	Tue	Wed
Adams	2	4	8
Benson	10	5	2
Cooper	10	6	4

Table 1.4 The dentist game: a permutation game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	5	4	14	18	9	24

Solution The core of this game consists of exactly one vector (see Problem 1.5). The Shapley value assigns $2/3$ to player 3 and $1/6$ to both player 1 and player 2. The nucleolus is the unique element of the core.

A Permutation Game

Story (From [22], p. 54) Mr. Adams, Mrs. Benson, and Mr. Cooper have appointments with the dentist on Monday, Tuesday, and Wednesday, respectively. This schedule not necessarily matches their preferences, due to different urgencies and other factors. These preferences (expressed in numbers) are given in Table 1.3.

Model This situation gives rise to a game in which the coalitions can gain by reshuffling their appointments. For instance, Adams (player 1) and Benson (player 2) can change their appointments and obtain a total of 14 instead of 7. A complete description of the resulting game is given in the Table 1.4.

Solution The core of this game is the convex hull of the vectors $(15,5,4)$, $(14,6,4)$, $(8,6,10)$, and $(9,5,10)$. The Shapley value is the vector $(9\frac{1}{2},6\frac{1}{2},8)$, and the nucleolus is the vector $(11\frac{1}{2},5\frac{1}{2},7)$.[14]

A Voting Game

(From [98], p. 247: The United Nations Security Council.) The United nations Security Council consists of five permanent members (United States, Russia, Britain, France, and China) and ten other members. Motions must be approved by nine members, including all the permanent members. This situation gives rise to a 15-player so called voting game (N,v) with $v(S)=1$ if the coalition S contains the five permanent members and at least four nonpermanent members, and $v(S)=0$ otherwise. Such games are also called *simple games*. Coalitions with worth equal to 1 are called winning, the other coalitions are called losing. Simple games are studied in Chap. 16.

[14] See Chap. 20 for an analysis of permutation games.

A solution to such a voting game is interpreted as representing the power of a player, rather than payoff (money) or utility.

1.3.5 Bargaining Games

Bargaining theory focusses on agreements between individual players.

A Division Problem

Story Consider the following situation. Two players have to agree on the division of one unit of a perfectly divisible good, say a liter of wine. If they reach an agreement, say (α,β) where $\alpha,\beta \geq 0$ and $\alpha+\beta \leq 1$, then they split up the one unit according to this agreement; otherwise, they both receive nothing. The players have preferences for the good, described by utility functions.

Model To fix ideas, assume that player 1 has a utility function $u_1(\alpha) = \alpha$ and player 2 has a utility function $u_2(\alpha) = \sqrt{\alpha}$. Thus, a distribution $(\alpha, 1-\alpha)$ of the good leads to a corresponding pair of utilities $(u_1(\alpha), u_2(1-\alpha)) = (\alpha, \sqrt{1-\alpha})$. By letting α range from 0 to 1 we obtain all utility pairs corresponding to all feasible distributions of the good, as in Fig. 1.7. It is assumed that also distributions summing to less than the whole unit are possible. This yields the whole shaded region.

Solution Nash [90] proposed the following way to 'solve' this bargaining problem: maximize the product of the players' utilities on the shaded area. Since this maximum will be reached on the boundary, the problem is equivalent to

$$\max_{0 \leq \alpha \leq 1} \alpha\sqrt{1-\alpha}.$$

The maximum is obtained for $\alpha = \frac{2}{3}$. So the 'solution' of the bargaining problem in utilities equals $(\frac{2}{3}, \frac{1}{3}\sqrt{3})$, which is the point z in the picture above. This implies

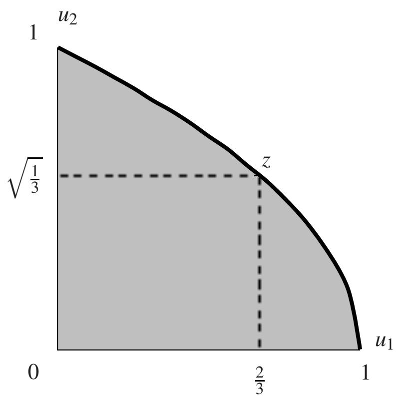


Fig. 1.7 A bargaining game

that player 1 obtains $\frac{2}{3}$ of the 1 unit of the good, whereas player 2 obtains $\frac{1}{3}$. As described here, this solution comes out of the blue. Nash, however, provided an axiomatic foundation for this solution (which is usually called the *Nash bargaining solution*).

Comments The bargaining literature includes many noncooperative, strategic approaches to the bargaining problem, including an attempt by Nash himself [92]. An important, seminal article in this literature is Rubinstein [110], in which the bargaining problem is modelled as an alternating offers extensive form game. Binmore et al. [13] observed the close relationship between the Nash bargaining solution and the strategic approach of Rubinstein. See Chap. 10.

The bargaining game can be seen as a special case of a cooperative game without transferable utility. Also games with transferable utility form a subset of the more general class of games without transferable utility. See also Chap. 21.

1.4 Cooperative vs. Noncooperative Game Theory

The usual distinction between cooperative and noncooperative game theory is that in a cooperative game binding agreements between players are possible, whereas this is not the case in noncooperative games. This distinction is informal and also not very sharp: for instance, the core of a cooperative game has a clear noncooperative flavor; a concept like correlated equilibrium for noncooperative games (see Sect. 13.7) has a clear cooperative flavor. Moreover, quite some game-theoretic literature is concerned with viewing problems both from a cooperative and a noncooperative angle. This approach is sometimes called the *Nash program*; the bargaining problem discussed above is a typical example. In a much more formal sense, the theory of *implementation* is concerned with representing outcomes from cooperative solutions as equilibrium outcomes of specific noncooperative solutions.

A more workable distinction between cooperative and noncooperative games can be based on the 'modelling technique' that is used: in a noncooperative game players have explicit strategies, whereas in a cooperative game players and coalitions are characterized, more abstractly, by the outcomes and payoffs that they can reach. The examples in Sects. 1.3.1–1.3.3 are examples of noncooperative games, whereas those in Sects. 1.3.4 and 1.3.5 are examples of cooperative games.

Problems

1.1. *Battle of the Bismarck Sea*

(a) Represent the 'Battle of the Bismarck Sea' as a game in extensive form.

(b) Now assume that Imamura moves first, and Kenney observes Imamura's move and moves next. Represent this situation in extensive form and solve by working backwards.

(c) Answer the same questions as under (b) with the roles of the players reversed.

1.2. *Variant of Matching Pennies*

Consider the following variant of the 'Matching Pennies' game

$$\begin{array}{c c} & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{pmatrix} x & -1 \\ -1 & 1 \end{pmatrix} \end{array},$$

where x is a real number. For which value(s) of x does this game have a saddlepoint, if any?

1.3. *Mixed Strategies*

Consider the following zero-sum game:

$$\begin{array}{c c} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \end{array}.$$

(a) Show that this game has no saddlepoint.

(b) Find a mixed strategy (randomized choice) of (the row) player 1 that makes his expected payoff independent of player 2's strategy.

(c) Find a mixed strategy of player 2 that makes his expected payoff independent of player 1's strategy.

(d) Consider the expected payoffs found under (b) and (c). What do you conclude about how the game could be played if randomized choices are allowed?

1.4. *Three Cooperating Cities*

Show that the Shapley value and the nucleolus of the 'Three Cooperating Cities Game' are elements of the core of this game.

1.5. *Glove Game*

(a) Compute the core of the glove game.

(b) Is the Shapley value an element of the core?

1.6. *Dentist Appointments*

For the permutation (dentist appointments) game, check if the Shapley value and the nucleolus are in the core of the game.

1.7. *Nash Bargaining*

Verify the computation of the Nash bargaining solution for the division problem.

1.8. *Variant of Glove Game*

Suppose there are $n = \ell + r$ players, where ℓ players own a left-hand glove and r players own a right-hand glove. Let N be the set of all players and let $S \subseteq N$ be a coalition. As before, each pair of gloves has worth 1. Find an expression for $v(S)$, i.e., the maximal profit that S can generate by cooperation of its members.

Chapter 2
Finite Two-Person Zero-Sum Games

This chapter deals with two-player games in which each player chooses from finitely many pure strategies or randomizes among these strategies, and the sum of the players' payoffs or expected payoffs is always equal to zero. Games like the 'Battle of the Bismarck Sea' and 'Matching Pennies', discussed in Sect. 1.3.1 belong to this class.

In Sect. 2.1 the basic definitions and theory are discussed. Section 2.2 shows how to solve $2 \times n$ and $m \times 2$ games, and larger games by elimination of strictly dominated strategies.

2.1 Basic Definitions and Theory

Since all data of a finite two-person zero-sum game can be summarized in one matrix, such a game is usually called a 'matrix game'.

Definition 2.1 (Matrix game). A *matrix game* is an $m \times n$ matrix A of real numbers, where the number of rows m and the number of columns n are integers greater than or equal to 1. A (*mixed*) *strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\}.$$

Similarly, a (*mixed*) *strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\}.$$

A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$.

The interpretation of such a matrix game A is as follows. If player 1 plays row i (i.e., pure strategy $\mathbf{e}^i$) and player 2 plays column j (i.e., pure strategy $\mathbf{e}^j$), then player 1 receives payoff a_{ij} and player 2 pays a_{ij} (and, thus, receives $-a_{ij}$), where a_{ij} is the number in row i and column j of matrix A. If player 1 plays strategy[1] $\mathbf{p}$ and player 2 plays strategy $\mathbf{q}$, then player 1 receives the expected payoff[2]

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij},$$

and player 2 receives $-\mathbf{p}A\mathbf{q}$.

For 'solving' matrix games, i.e., establishing what clever players would or should do, the concepts of maximin and minimax strategies are important, as will be explained below. First we give the formal definitions.

Definition 2.2 (Maximin and minimax strategies). A strategy $\mathbf{p}$ is a *maximin strategy* of player 1 in matrix game A if

$$\min\{\mathbf{p}A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \geq \min\{\mathbf{p}'A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \text{ for all } \mathbf{p}' \in \Delta^m.$$

A strategy $\mathbf{q}$ is a *minimax strategy* of player 2 in matrix game A if

$$\max\{\mathbf{p}A\mathbf{q} \mid \mathbf{p} \in \Delta^m\} \leq \max\{\mathbf{p}A\mathbf{q}' \mid \mathbf{p} \in \Delta^m\} \text{ for all } \mathbf{q}' \in \Delta^n.$$

In words: a maximin strategy of player 1 maximizes the minimal (with respect to player 2's strategies) payoff of player 1, and a minimax strategy of player 2 minimizes the maximum (with respect to player 1's strategies) that player 2 has to pay to player 1. Of course, the asymmetry in these definitions is caused by the fact that, by convention, a matrix game represents the amounts that player 2 has to pay to player 1.[3]

In order to check if a strategy $\mathbf{p}$ of player 1 is a maximin strategy it is sufficient to check that the first inequality in Definition 2.2 holds with $\mathbf{e}^j$ for every $j = 1, \ldots, n$ instead of every $\mathbf{q} \in \Delta^n$. This is not difficult to see but the reader is referred to Chap. 12 for a more formal treatment. A similar observation holds for minimax strategies. In other words, to check if a strategy is maximin (minimax) it is sufficient to consider its performance against every pure strategy, i.e., column (row).

Why would we be interested in such strategies? At first glance, such strategies seem to express a very conservative or pessimistic, worst-case scenario attitude. The reason for considering maximin/minimax strategies is provided by von

[1] Observe that here, by a 'strategy' we mean a mixed strategy: we add the adjective 'pure' if we wish to refer to a pure strategy.

[2] Since no confusion is likely to arise, we do not use transpose notations like $\mathbf{p}^T A\mathbf{q}$ or $\mathbf{p}A\mathbf{q}^T$.

[3] It can be proved by basic mathematical analysis that maximin and minimax strategies always exist.

Neumann [140]. Von Neumann shows[4] that for every matrix game A there is a real number $v = v(A)$ with the following properties:

1. A strategy $\mathbf{p}$ of player 1 guarantees a payoff of at least v to player 1 (i.e., $\mathbf{p}A\mathbf{q} \geq v$ for all strategies $\mathbf{q}$ of player 2) if and only if $\mathbf{p}$ is a maximin strategy.
2. A strategy $\mathbf{q}$ of player 2 guarantees a payment of at most v by player 2 to player 1 (i.e., $\mathbf{p}A\mathbf{q} \leq v$ for all strategies $\mathbf{p}$ of player 1) if and only if $\mathbf{q}$ is a minimax strategy.

Hence, player 1 can obtain a payoff of at least v by playing a maximin strategy, and player 2 can guarantee to pay not more than v – hence secure a payoff of at least $-v$ – by playing a minimax strategy. For these reasons, the number $v = v(A)$ is also called the *value* of the game A – it represents the worth to player 1 of playing the game A – and maximin and minimax strategies are called *optimal strategies* for players 1 and 2, respectively.

Therefore, 'solving' the game A means, naturally, determining the optimal strategies and the value of the game. In the 'Battle of the Bismarck Sea' in Sect. 1.3.1, the pure strategies N of both players guarantee the same amount 2. Therefore, this is the value of the game and N is optimal for both players. The analysis of that game is easy since it has a 'saddlepoint', namely position $(1,1)$ with $a_{11} = 2$. The formal definition of a saddlepoint is as follows.

Definition 2.3 (Saddlepoint). A position (i, j) in a matrix game A is a *saddlepoint* if

$$a_{ij} \geq a_{kj} \text{ for all } k = 1, \ldots, m \text{ and } a_{ij} \leq a_{ik} \text{ for all } k = 1, \ldots, n,$$

i.e., if a_{ij} is maximal in its column j and minimal in its row i.

Clearly, if (i, j) is a saddlepoint, then player 1 can guarantee a payoff of at least a_{ij} by playing the pure strategy row i, since a_{ij} is minimal in row i. Similarly, player 2 can guarantee a payoff of at least $-a_{ij}$ by playing the pure strategy column j, since a_{ij} is maximal in column j. Hence, a_{ij} must be the value of the game A: $v(A) = a_{ij}$, $\mathbf{e}^i$ is an optimal (maximin) strategy of player 1, and $\mathbf{e}^j$ is an optimal (minimax) strategy of player 2.

2.2 Solving $2 \times n$ Games and $m \times 2$ Games

In this section we show how to solve matrix games where at least one of the players has two pure strategies. We also show how the idea of strict domination can be of help in solving matrix games.

[4] See Chap. 12 for a more rigorous treatment of zero-sum games and a proof of von Neumann's result.

2.2.1 $2 \times n$ Games

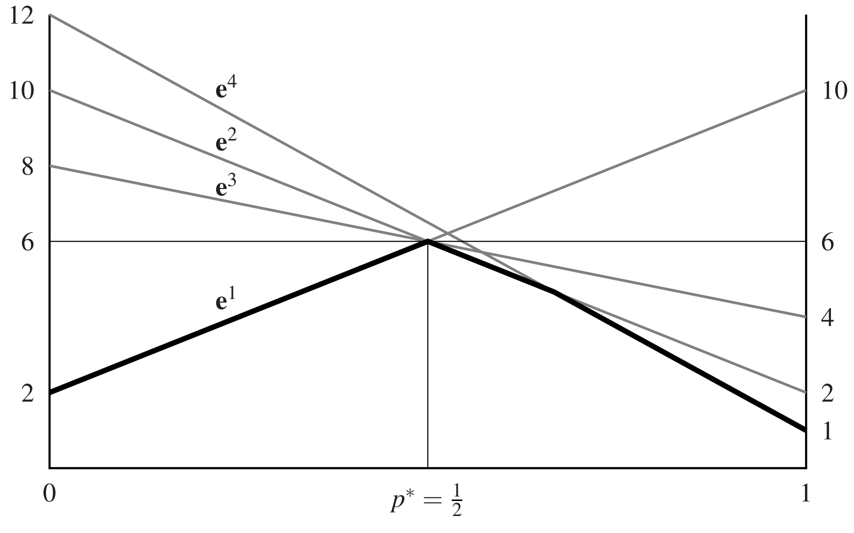
We demonstrate how to solve a matrix game with 2 rows and n columns graphically, by considering the following 2×4 example:

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{array} \\ A = & \begin{pmatrix} 10 & 2 & 4 & 1 \\ 2 & 10 & 8 & 12 \end{pmatrix}. \end{array}$$

We have labelled the columns of A, i.e., the pure strategies of player 2 for reference below. Let $\mathbf{p} = (p, 1-p)$ be an arbitrary strategy of player 1. The expected payoffs to player 1 if player 2 plays a pure strategy are equal to:

$$\begin{aligned} \mathbf{p}A\mathbf{e}^1 &= 10p + 2(1-p) = 8p + 2 \\ \mathbf{p}A\mathbf{e}^2 &= 2p + 10(1-p) = 10 - 8p \\ \mathbf{p}A\mathbf{e}^3 &= 4p + 8(1-p) = 8 - 4p \\ \mathbf{p}A\mathbf{e}^4 &= p + 12(1-p) = 12 - 11p. \end{aligned}$$

We plot these four linear functions of p in one diagram:



In this diagram the values of p are plotted on the horizontal axis, and the four straight gray lines plot the payoffs to player 1 if player 2 plays one of his four pure strategies, respectively. Observe that for every $0 \leq p \leq 1$ the minimum payoff that player 1 may obtain is given by the lower envelope of these curves, the thick black curve in the diagram: for any p, any combination (q_1, q_2, q_3, q_4) of the points on $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and $\mathbf{e}^4$ with first coordinate p would lie on or above this lower envelope. Clearly, the lower envelope is maximal for $p = p^* = \frac{1}{2}$, and the maximal value is 6. Hence, we have established that player 1 has a unique optimal (maximin) strategy, namely $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and that the value of the game, $v(A)$, is equal to 6.

What are the optimal or minimax strategies of player 2? From the theory of the previous section we know that a minimax strategy $\mathbf{q} = (q_1, q_2, q_3, q_4)$ of player 2 should guarantee to player 2 to have to pay at most the value of the game. From the diagram it is clear that q_4 should be equal to zero since otherwise the payoff to player 1 would be larger than 6 if player 1 plays $(\frac{1}{2}, \frac{1}{2})$, and thus $\mathbf{q}$ would not be a minimax strategy. So a minimax strategy has the form $\mathbf{q} = (q_1, q_2, q_3, 0)$. Any such strategy, plotted in the diagram, gives a straight line that is a combination of the lines associated with $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ and which passes through the point $(\frac{1}{2}, 6)$ since all three lines pas through this point. Moreover, for no value of p should this straight line exceed the value 6, otherwise $\mathbf{q}$ would not guarantee a payment of at most 6 by player 2. Consequently, this straight line has to be horizontal. Summarizing this argument, we look for numbers $q_1, q_2, q_3 \geq 0$ such that

$$\begin{aligned}
2q_1 + 10q_2 + 8q_3 &= 6 \quad \text{(left endpoint should be } (0,6)\text{)} \\
10q_1 + 2q_2 + 4q_3 &= 6 \quad \text{(right endpoint should be } (1,6)\text{)} \\
q_1 + q_2 + q_3 &= 1 \quad (\mathbf{q} \text{ is a probability vector).}
\end{aligned}$$

This system of equations is easily reduced[5] to the two equations

$$\begin{aligned}
3q_1 - q_2 &= 1 \\
q_1 + q_2 + q_3 &= 1.
\end{aligned}$$

The first equation implies that if $q_1 = \frac{1}{3}$ then $q_2 = 0$ and if $q_1 = \frac{1}{2}$ then $q_2 = \frac{1}{2}$. Clearly, q_1 and q_2 cannot be larger since then their sum exceeds 1. Hence the set of optimal strategies of player 2 is

$$\{\mathbf{q} = (q_1, q_2, q_3, q_4) \in \Delta^4 \mid \frac{1}{3} \leq q_1 \leq \frac{1}{2},\ q_2 = 3q_1 - 1,\ q_4 = 0\}.$$

2.2.2 *m* × 2 *Games*

The solution method to solve $m \times 2$ games is analogous. Consider the following example:

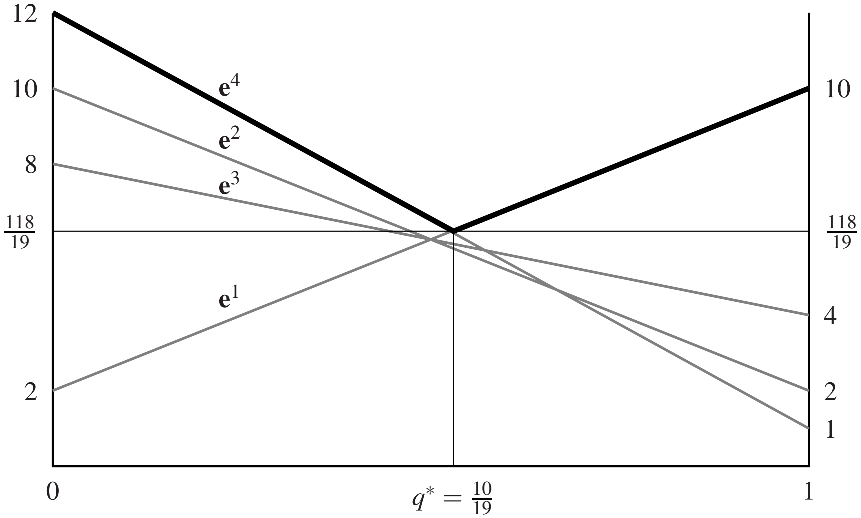
$$A = \begin{array}{c} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \\ \mathbf{e}^4 \end{array} \begin{pmatrix} 10 & 2 \\ 2 & 10 \\ 4 & 8 \\ 1 & 12 \end{pmatrix}.$$

[5] For instance, by substitution. In fact, one of the two first equations could be omitted to begin with, since we already know that any combination of the three lines passes through $(\frac{1}{2}, 6)$, and two points are sufficient to determine a straight line.

Let $\mathbf{q} = (q, 1-q)$ be an arbitrary strategy of player 2. Again, we make a diagram in which now the values of q are put on the horizontal axis, and the straight lines indicated by $\mathbf{e}^i$ for $i = 1,2,3,4$ are the payoffs to player 1 associated with his four pure strategies (rows) as functions of q. The equations of these lines are given by:

$$\begin{aligned}
\mathbf{e}^1 A\mathbf{q} &= 10q + 2(1-q) = 8q + 2 \\
\mathbf{e}^2 A\mathbf{q} &= 2q + 10(1-q) = 10 - 8q \\
\mathbf{e}^3 A\mathbf{q} &= 4q + 8(1-q) = 8 - 4q \\
\mathbf{e}^4 A\mathbf{q} &= q + 12(1-q) = 12 - 11q.
\end{aligned}$$

The resulting diagram is as follows.



Observe that the maximum payments that player 2 has to make are now located on the upper envelope, represented by the thick black curve. The minimum is reached at the point of intersection of $\mathbf{e}^1$ and $\mathbf{e}^4$ in the diagram, which has coordinates $(\frac{10}{19}, \frac{118}{19})$. Hence, the value of the game is $\frac{118}{19}$, and the unique optimal (minimax) strategy of player 2 is $\mathbf{q}^* = (\frac{10}{19}, \frac{9}{19})$.

To find the optimal strategy or strategies $\mathbf{p} = (p_1, p_2, p_3, p_4)$ of player 1, it follows from the diagram that $p_2 = p_3 = 0$, otherwise for $q = \frac{10}{19}$ the value $\frac{118}{19}$ of the game is not reached, so that $\mathbf{p}$ is not a maximin strategy. So we look for a combination of $\mathbf{e}^1$ and $\mathbf{e}^4$ that gives at least $\frac{118}{19}$ for every q, hence it has to be equal to $\frac{118}{19}$ for every q. This gives rise to the equations $2p_1 + 12p_4 = 10p_1 + p_4 = \frac{118}{19}$ and $p_1 + p_4 = 1$, with unique solution $p_1 = \frac{11}{19}$ and $p_4 = \frac{8}{19}$. So the unique optimal strategy of player 1 is $(\frac{11}{19}, 0, 0, \frac{8}{19})$.

2.2.3 Strict Domination

The idea of strict domination can be used to eliminate pure strategies before the graphical analysis of a matrix game. Consider the game

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{array} \\ A = & \begin{pmatrix} 10 & 2 & 5 & 1 \\ 2 & 10 & 8 & 12 \end{pmatrix}, \end{array}$$

which is almost identical to the game in Sect. 2.2.1, except that a_{13} is now 5 instead of 4. Consider a strategy $(\alpha, 1-\alpha, 0, 0)$ of player 2. The expected payments from this strategy from player 2 to player 1 are $8\alpha + 2$ if player 1 plays the first row and $10 - 8\alpha$ if player 1 plays the second row. For any value $\frac{1}{4} < \alpha < \frac{3}{8}$, the first number is smaller than 5 and the second number is smaller than 8. Hence, this is strictly better for player 2 than playing his pure strategy $\mathbf{e}^3$, no matter what player 1 does. But then, for any strategy $\mathbf{q} = (q_1, q_2, q_3, q_4)$ of player 2 with $q_3 > 0$, the expected payoff to player 2 would become strictly larger (his payment to player 1 strictly smaller) by transferring the probability q_3 to the first two pure strategies in some right proportion α, i.e., by playing $(q_1 + \alpha q_3, q_2 + (1-\alpha)q_3, 0, q_4)$ for some $\frac{1}{4} < \alpha < \frac{3}{8}$, instead of $\mathbf{q}$. Hence, in an optimal (minimax) strategy we must have $q_3 = 0$. This implies that, in order to solve the above game, we can start by deleting the third column of the matrix. In the diagram in Sect. 2.2.1, we do not have to draw the line corresponding to $\mathbf{e}^3$. The value of the game is still 6, player 1 still has a unique optimal strategy $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and player 2 now also has a unique optimal strategy, namely the one where $q_3 = 0$, which is the strategy $(\frac{1}{2}, \frac{1}{2}, 0, 0)$.

In general, strictly dominated pure strategies in a matrix game are not played with positive probability in any optimal strategy and can therefore be deleted before solving the game. Sometimes, this idea can also be used to solve matrix games in which each player has more than two pure strategies ($m, n > 2$). Moreover, the idea can be applied iteratively, that is, after deletion of a strictly dominated pure strategy, in the smaller game perhaps another strictly dominated pure strategy can be deleted, etc., until no more pure strategies are strictly dominated.[6]

We first provide a formal definition of strict domination for completeness' sake, and then discuss another example where iterated elimination of strictly dominated strategies is applied.

Definition 2.4 (Strict domination). Let A be an $m \times n$ matrix game and i a row. The pure strategy $\mathbf{e}^i$ is *strictly dominated* if there is a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ with $p_i = 0$ such that $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A \mathbf{e}^j$ for every $j = 1, \ldots, n$. Similarly, let j be a column. The pure strategy $\mathbf{e}^j$ is *strictly dominated* if there is a strategy $\mathbf{q} = (q_1, \ldots, q_n) \in \Delta^n$ with $q_j = 0$ such that $\mathbf{e}^i A \mathbf{q} < \mathbf{e}^i A \mathbf{e}^j$ for every $i = 1, \ldots, m$.[7]

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[6] Of course, all this needs to be proved formally, but in this chapter it is just assumed. See Chap. 13 for a more formal treatment.

[7] An equivalent definition is obtained if the conditions $p_i = 0$ and $q_j = 0$ are omitted.

Example 2.5. Consider the following 3×3 matrix game:

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

For player 1, the third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{p} = (\frac{7}{12}, \frac{5}{12}, 0)$, since $\mathbf{p}A = (3\frac{1}{2}, 2\frac{1}{12}, 2\frac{5}{6})$ has every coordinate strictly larger than $\mathbf{e}^3 A = (3, 2, 1)$. Hence, in any optimal strategy player 1 puts zero probability on the third row. Elimination of this row results in the matrix

$$B = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \end{pmatrix}.$$

Now, player 2's third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{q} = (\frac{1}{4}, \frac{3}{4}, 0)$, since $B\mathbf{q} = (\frac{3}{2}, 3\frac{3}{4})$, which has every coordinate strictly smaller than $B\mathbf{e}^3 = (2, 4)$. Hence, in any optimal strategy player 2 puts zero probability on the third column. Elimination of this column results in the matrix

$$C = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}.$$

This is a 2×2 matrix game, which can be solved by the method in Sect. 2.2.1 or Sect. 2.2.2. See Problem 2.1(a).

Problems

2.1. *Solving Matrix Games*

Solve the following matrix games, i.e., determine the optimal strategies and the value of the game. Each time, start by checking if the game has a saddlepoint.

(a)

$$\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

What are the optimal strategies in the original matrix game A in Example 2.5?

(b)

$$\begin{pmatrix} 2 & -1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 16 & 12 & 2 \\ 2 & 6 & 16 \\ 8 & 8 & 6 \\ 0 & 7 & 8 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

2.2. *Saddlepoints*

(a) Let A be an arbitrary $m \times n$ matrix game. Show that any two saddlepoints must have the same value. In other words, if (i, j) and (k, l) are two saddlepoints, show that $a_{ij} = a_{kl}$.

(b) Let A be a 4×4 matrix game in which $(1, 1)$ and $(4, 4)$ are saddlepoints. Show that A has at least two other saddlepoints.

(c) Give an example of a 4×4 matrix game with exactly three saddlepoints.

2.3. *Rock–Paper–Scissors*

In the famous Rock–Paper–Scissors two-player game each player has three pure strategies: Rock, Paper, and Scissors. Here, Scissors beats Paper, Paper beats Rock, Rock beats Scissors. Assign a 1 to winning, 0 to a draw, and -1 to losing. Model this game as a matrix game, try to guess its optimal strategies, and then show that these are the unique optimal strategies. What is the value of this game?

Chapter 3
Finite Two-Person Games

In this chapter we consider two-player games where each player chooses from finitely many pure strategies or randomizes among these strategies. In contrast to Chap. 2 it is no longer required that the sum of the players' payoffs is zero (or, equivalently, constant). This allows for a much larger class of games, including many games relevant for economic or other applications. Famous examples are the 'Prisoners' Dilemma' and the 'Battle of the Sexes' discussed in Sect. 1.3.2.

In Sect. 3.1 we introduce the model and the concept of 'Nash equilibrium'. Section 3.2 shows how to compute Nash equilibria in pure strategies for arbitrary games, all Nash equilibria in games where both players have exactly two pure strategies, and how to use the concept of strict domination to facilitate computation of Nash equilibria and to compute equilibria also of larger games. The structure of this chapter thus parallels the structure of Chap. 2. For a deeper and more comprehensive analysis of finite two-person games see Chap. 13.

3.1 Basic Definitions and Theory

The data of a finite two-person game can be summarized by two matrices. Usually, these matrices are written as one matrix with two numbers at each position. Therefore, such games are often called 'bimatrix games'. The formal definition is as follows.

Definition 3.1 (Bimatrix game). A *bimatrix game* is a pair of $m \times n$ matrices (A, B), where m and n are integers greater than or equal to 1.

The interpretation of such a bimatrix game (A, B) is that, if player 1 (the row player) plays row i and player 2 (the column player) plays column j, then player 1 receives payoff a_{ij} and player 2 receives b_{ij}, where these numbers are the corresponding entries of A and B, respectively. Definitions and notations for pure and mixed strategies, strategy sets and expected payoffs are similar to those for matrix games, see

Sect. 2.1, but for easy reference we repeat them here. A *(mixed) strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\} .$$

Similarly, a *(mixed) strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\} .$$

A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$. If player 1 plays $\mathbf{p}$ and player 2 plays $\mathbf{q}$ then the payoff to player 1 is the expected payoff

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} ,$$

and the payoff to player 2 is the expected payoff

$$\mathbf{p}B\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j b_{ij} .$$

As mentioned, the entries of A and B are usually grouped together in one (bi)matrix, by putting the pair a_{ij}, b_{ij} at position (i, j) of the matrix. Cf. the examples in Sect. 1.3.2.

Central to noncooperative game theory is the idea of best reply. It says that a rational selfish player should always maximize his (expected) payoff, given his knowledge of or conjecture about the strategies chosen by the other players.

Definition 3.2 (Best reply). A strategy $\mathbf{p}$ of player 1 is a *best reply* to a strategy $\mathbf{q}$ of player 2 in an $m \times n$ bimatrix game (A, B) if

$$\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q} \quad \text{for all} \quad \mathbf{p}' \in \Delta^m .$$

Similarly, $\mathbf{q}$ is a *best reply* of player 2 to $\mathbf{p}$ if

$$\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}' \quad \text{for all} \quad \mathbf{q}' \in \Delta^n .$$

In a 'Nash equilibrium', each player's strategy is a best reply to the other player's strategy.

Definition 3.3 (Nash equilibrium). A pair of strategies $(\mathbf{p}^*, \mathbf{q}^*)$ in a bimatrix game (A, B) is a *Nash equilibrium* if $\mathbf{p}^*$ is a best reply of player 1 to $\mathbf{q}^*$ and $\mathbf{q}^*$ is a best reply of player 2 to $\mathbf{p}^*$. A Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$ is called *pure* if both $\mathbf{p}^*$ and $\mathbf{q}^*$ are pure strategies.

The concept of a Nash equilibrium can be extended to quite arbitrary games, including games with arbitrary numbers of players, strategy sets, and payoff functions. We will see a lot of examples in later chapters of this book. For finite two-person games, Nash [91] proved that every game has a Nash equilibrium in mixed strategies, i.e., a Nash equilibrium according to Definition 3.3. See Sect. 13.1 for such a proof. Generally speaking, the main concern with Nash equilibrium is not its existence but rather the opposite, namely its abundance, as well as its interpretation. In many games, there are many Nash equilibria, and then the questions of equilibrium selection and equilibrium refinement are relevant. With respect to interpretation, an old question is how in reality the players would come to play a Nash equilibrium. The definition of Nash equilibrium does not say anything about this.

For a Nash equilibrium in mixed strategies as in Definition 3.3, an additional question is what the meaning of such a mixed strategy is. Does it mean that the players actually randomize when playing the game? A different and quite common interpretation is that a mixed strategy of a player, say player 1, represents the belief, or conjecture, of the other player, player 2, about what player 1 will do. Thus, it embodies the 'strategic uncertainty' of the players in a game, a term coined by von Neumann and Morgenstern [141].

For now, we just leave these questions aside and take the definition of Nash equilibrium at face value. We show how to compute pure Nash equilibria in general, and all Nash equilibria in games where both players have two pure strategies. Just like in Chap. 2, we also consider the role of strict domination.

3.2 Finding Nash Equilibria

To find all Nash equilibria of an arbitrary bimatrix game is a difficult task. We refer to Sect. 13.2.3 for more discussion on this problem. Here we restrict ourselves to, first, the much easier problem of finding all Nash equilibria in pure strategies of an arbitrary bimatrix game and, second, to showing how to find all Nash equilibria in 2×2 games graphically. It is also possible to solve 2×3 and 3×2 games graphically, see Sect. 13.2.2. For larger games, graphical solutions are impractical or, indeed, impossible.

3.2.1 Pure Nash Equilibria

To find the pure Nash equilibria in a bimatrix game, one can first determine the pure best replies of player 2 to every pure strategy of player 1, and next determine the pure best replies of player 1 to every pure strategy of player 2. Those pairs of pure strategies that are mutual best replies are the pure Nash equilibria of the game. To illustrate this method, consider the bimatrix game

$$\begin{array}{c c} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{pmatrix} 2,2 & 4,0 & 1,1 & 3,2 \\ 0,3 & 1,5 & 4,4 & 3,4 \\ 2,0 & 2,1 & 5,1 & 1,0 \end{pmatrix} \end{array}.$$

First we determine the pure best replies of player 2 to every pure strategy of player 1, indicated by the stars at the corresponding entries. This yields:

$$\begin{array}{c c} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{pmatrix} 2,2^* & 4,0 & 1,1 & 3,2^* \\ 0,3 & 1,5^* & 4,4 & 3,4 \\ 2,0 & 2,1^* & 5,1^* & 1,0 \end{pmatrix} \end{array}.$$

Next, we determine the pure best replies of player 1 to every pure strategy of player 2, again indicated by the stars at the corresponding entries. This yields:

$$\begin{array}{c c} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{pmatrix} 2^*,2 & 4^*,0 & 1,1 & 3^*,2 \\ 0,3 & 1,5 & 4,4 & 3^*,4 \\ 2^*,0 & 2,1 & 5^*,1 & 1,0 \end{pmatrix} \end{array}.$$

Putting the two results together yields:

$$\begin{array}{c c} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{pmatrix} 2^*,2^* & 4^*,0 & 1,1 & 3^*,2^* \\ 0,3 & 1,5^* & 4,4 & 3^*,4 \\ 2^*,0 & 2,1^* & 5^*,1^* & 1,0 \end{pmatrix} \end{array}.$$

We conclude that the game has three Nash equilibria in pure strategies, namely (T,W), (T,Z), and (B,Y). In mixed strategy notation, these are the pairs $\left(\mathbf{e}^1,\mathbf{e}^1\right)$, $\left(\mathbf{e}^1,\mathbf{e}^4\right)$, and $\left(\mathbf{e}^3,\mathbf{e}^3\right)$, respectively. In more extensive notation: $((1,0,0),(1,0,0,0))$, $((1,0,0),(0,0,0,1))$, and $((0,0,1),(0,0,1,0))$, respectively.

Strictly speaking, one should also consider mixed best replies to a pure strategy in order to establish whether this pure strategy can occur in a Nash equilibrium, but it is not hard to see that any mixed best reply is a combination of pure best replies and, thus, can never lead to a higher payoff. For instance, in the example above, any strategy of the form $(q,0,0,1-q)$ played against T yields to player 2 a payoff of 2 $(= 2q+2(1-q))$ and is therefore a best reply, but does not yield a payoff higher than W or Z. However, the reader can check that all strategy pairs of the form $(T,(q,0,0,1-q))$ $(0<q<1)$ are also Nash equilibria of this game.

It is also clear from this example that a Nash equilibrium does not have to result in 'Pareto optimal' payoffs[1]: the payoff pair $(4,4)$, resulting from (M,Y), is better for both players than the equilibrium payoffs $(2,2)$, resulting from (T,W). We know this phenomenon already from the 'Prisoners' Dilemma' game in Sect. 1.3.2.

3.2.2 2×2 Games

The best way to demonstrate the graphical solution method for 2×2 games is by means of an example. Consider the bimatrix game

$$(A,B) \;=\; \begin{array}{c} \\ T \\ B \end{array} \!\!\begin{array}{c} \begin{array}{cc} L & \;\;R \end{array} \\ \begin{pmatrix} 2,2 & 0,1 \\ 1,1 & 3,3 \end{pmatrix} \end{array} .$$

Observe that this game has two Nash equilibria in pure strategies, namely (T,L) and (B,R), cf. Sect. 3.2.1. To find all Nash equilibria we determine the best replies of both players.

First consider the strategy $(q,1-q)$ of player 2. The best reply of player 1 to this strategy is T or, equivalently, $(1,0)$, if the expected payoff from playing T is higher than the expected payoff from playing B, since then it is also higher than the expected payoff from playing any combination $(p,1-p)$ of T and B. Hence, the best reply is T if

$$2q + 0(1-q) > 1q + 3(1-q)\,,$$

so if $q > \frac{3}{4}$. Similarly, we find that B is the best reply if $q < \frac{3}{4}$, and that T and B are both best replies if $q = \frac{3}{4}$. In the last case, since T and B yield the same payoff to player 1 against $(q,1-q)$, it follows that any $(p,1-p)$ is a best reply. Summarizing, if we denote the set of best replies of player 1 against $(q,1-q)$ by $\beta_1(q,1-q)$, we have

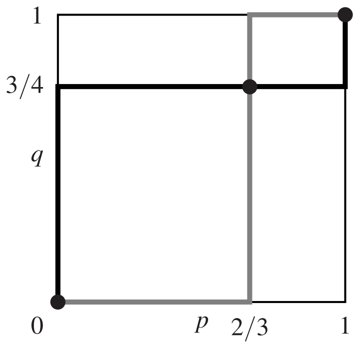
$$\beta_1(q,1-q) = \begin{cases} \{(1,0)\} & \text{if } \frac{3}{4} < q \le 1 \\ \{(p,1-p) \mid 0 \le p \le 1\} & \text{if } q = \frac{3}{4} \\ \{(0,1)\} & \text{if } 0 \le q < \frac{3}{4}\,. \end{cases} \tag{3.1}$$

By completely analogous arguments, we find that for a strategy $(p,1-p)$ the best replies $\beta_2(p,1-p)$ of player 2 are given by

$$\beta_2(p,1-p) = \begin{cases} \{(1,0)\} & \text{if } \frac{2}{3} < p \le 1 \\ \{(q,1-q) \mid 0 \le q \le 1\} & \text{if } p = \frac{2}{3} \\ \{(0,1)\} & \text{if } 0 \le p < \frac{2}{3}\,. \end{cases} \tag{3.2}$$

[1] Formally, a pair of payoffs is *Pareto optimal* if there is no other pair of payoffs which are at least as high for both players and strictly higher for at least one player.

By definition, the Nash equilibria of the game are the strategy combinations $(\mathbf{p}^*, \mathbf{q}^*)$ such that $\mathbf{p}^* \in \beta_1(\mathbf{q}^*)$ and $\mathbf{q}^* \in \beta_2(\mathbf{p}^*)$, i.e., the points of intersection of the best reply functions in (3.1) and (3.2). A convenient way to find these points is by drawing the graphs of $\beta_1(q, 1-q)$ and $\beta_2(p, 1-p)$. We put p on the horizontal axis and q on the vertical axis and obtain the following diagram.



The solid black curve is the best reply function of player 1 and the solid grey curve is the best reply function of player 2. The solid circles indicate the three Nash equilibria of the game: $((1,0),(1,0))$, $((2/3,1/3),(3/4,1/4))$, and $((0,1),(0,1))$.

3.2.3 Strict Domination

The graphical method discussed in Sect. 3.2.2 is suited for 2×2 games. It can be extended to 2×3 and 3×2 games as well, see Sect. 13.2.2.

In general, for the purpose of finding Nash equilibria the size of a game can sometimes be reduced by iteratively eliminating strictly dominated strategies. We look for a strictly dominated (pure) strategy of a player, eliminate the associated row or column, and continue this procedure for the smaller game until there is no more strictly dominated strategy. In fact, it can be shown (see Sect. 13.3) that no pure strategy that is eliminated by this procedure is ever played with positive probability in a Nash equilibrium of the original game. Thus, no Nash equilibrium of the original game is eliminated. Also, no Nash equilibrium is added. It follows, in particular, that the order in which strictly dominated strategies are eliminated does not matter.

For completeness we first repeat the definition of strict domination, formulated for a bimatrix game, and then present an example.

Definition 3.4 (Strict domination). Let (A,B) be an $m \times n$ bimatrix game and i a row. The pure strategy $\mathbf{e}^i$ is *strictly dominated* if there is a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ with $p_i = 0$ such that $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A\mathbf{e}^j$ for every $j = 1, \ldots, n$. Similarly, let j be a column. The pure strategy $\mathbf{e}^j$ is *strictly dominated* if there is a strategy $\mathbf{q} = (q_1, \ldots, q_n) \in \Delta^n$ with $q_j = 0$ such that $\mathbf{e}^i B\mathbf{q} > \mathbf{e}^i B\mathbf{e}^j$ for every $i = 1, \ldots, m$.

Consider the following bimatrix game

$$
(A,B) = \begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,1 & 2,2 & 0,0 \\ 1,0 & 4,1 & 2,4 & 1,5 \\ 0,4 & 3,1 & 3,0 & 3,3 \end{pmatrix} \end{array}.
$$

Observe, first, that no pure strategy (row) of player 1 is strictly dominated by another pure strategy of player 1, and that no pure strategy (column) of player 2 is strictly dominated by another pure strategy of player 2. Consider the second pure strategy, X, of player 2. Then X is strictly dominated by any strategy of the form $(q,0,1-q,0)$ with $\frac{1}{4} < q < \frac{3}{4}$. So X can be eliminated, to obtain

$$
\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} W & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,2 & 0,0 \\ 1,0 & 2,4 & 1,5 \\ 0,4 & 3,0 & 3,3 \end{pmatrix} \end{array}.
$$

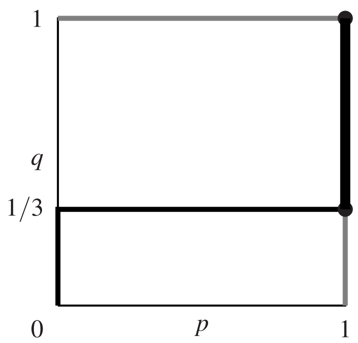
Next, observe that, in this reduced game for player 1, pure strategy M is strictly dominated by any strategy of the form $(p,0,1-p)$ with $\frac{1}{2} < p < \frac{2}{3}$. So M can be eliminated to obtain

$$
\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{ccc} W & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,2 & 0,0 \\ 0,4 & 3,0 & 3,3 \end{pmatrix} \end{array}.
$$

Here, finally, Z can be eliminated since it is strictly dominated by W, and we are left with the 2×2 game

$$
\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} W & Y \end{array} \\ \begin{pmatrix} 2,2 & 2,2 \\ 0,4 & 3,0 \end{pmatrix} \end{array}.
$$

This game can be solved by using the graphical method of Sect. 3.2.2. Doing so results in the diagram



The solid black (grey) curve is the graph of player 1's (2's) best reply function. In this case, they overlap in infinitely many points, resulting in the set of Nash equilibria $\{((1,0),(q,1-q)) \mid 1/3 \leq q \leq 1\}$. In the original 3×4 game, the set of all Nash equilibria is therefore equal to

$$\{((1,0,0),(q,0,1-q,0)) \mid 1/3 \leq q \leq 1\} .$$

Problems

3.1. *Some Applications*

In each of the following situations, set up the corresponding bimatrix game and solve for all Nash equilibria.

(a) *Pure coordination* ([106], p. 35). Two firms (Smith and Brown) decide whether to design the computers they sell to use large or small floppy disks. Both players will sell more computers if their disk drives are compatible. If they both choose for large disks the payoffs will be 2 for each. If they both choose for small disks the payoffs will be 1 for each. If they choose different sizes the payoffs will be -1 for each.

(b) *The welfare game* ([106], p. 70). This game models a government that wishes to aid a pauper if he searches for work but not otherwise, and a pauper who searches for work only if he cannot depend on government aid, and who may not succeed in finding a job even if the tries. The payoffs are $3,2$ (for government, pauper) if the government aids and the pauper tries to work; $-1,1$ if the government does not aid and the pauper tries to work; $-1,3$ if the government aids and the pauper does not try to work; and $0,0$ in the remaining case.

(c) *Wage game* ([45], p. 51; [80]). Each of two firms has one job opening. Suppose that firm i ($i = 1,2$) offers wage w_i, where $0 < \frac{1}{2}w_1 < w_2 < 2w_1$ and $w_1 \neq w_2$. Imagine that there are two workers, each of whom can apply to only one firm. The workers simultaneously decide whether to apply to firm 1 or firm 2. If only one worker applies to a given firm, that worker gets the job; if both workers apply to one firm, the firm hires one worker at random (with probability $\frac{1}{2}$) and the other worker is unemployed (and has a payoff of zero).

(d) *Marketing game*. Two firms sell a similar product. Each percent of market share yields a net payoff of 1. Without advertising both firms have 50% of the market. The cost of advertising is equal to 10 but leads to an increase in market share of 20% at the expense of the other firm. The firms make their advertising decisions simultaneously and independently. The total market for the product is of fixed size.

(e) *Voting game.* Two political parties, *I* and *II*, each have three votes that they can distribute over three party-candidates each. A committee is to be elected, consisting of three members. Each political party would like to see as many as possible of their own candidates elected in the committee. Of the total of six candidates those three that have most of the votes will be elected; in case of ties, tied candidates are drawn with equal probabilities.

3.2. *Matrix Games*

(a) Since a matrix game is a special case of a bimatrix game, it may be 'solved' by the method graphical method of Sect. 3.2.2. Do this for the game in Problem 2.1(a). Compare your answer with what you found previously.

(b) Argue that a pair consisting of a maximin and a minimax strategy in a matrix game is a Nash equilibrium; and that any Nash equilibrium in a matrix game must be a pair consisting of a maximin and a minimax strategy. (You may give all your arguments in words.)

(c) A maximin strategy for player 1 in the bimatrix game (A,B) is a maximin strategy in the matrix game A. Which definition is appropriate for player 2 in this respect? With these definitions, find examples showing that a Nash equilibrium in a bimatrix game does not have to consist of maximin strategies, and that a maximin strategy does not have to be part of a Nash equilibrium.

3.3. *Strict Domination*

Consider the bimatrix game

$$(A,B) \;=\; \begin{array}{c} \\ T \\ B \end{array} \!\!\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 6,6 & 4,4 & 1,2 & 8,5 \\ 4,5 & 6,6 & 2,8 & 4,4 \end{pmatrix} \end{array}.$$

(a) Which pure strategy of player 1 or player 2 is strictly dominated by a pure strategy?

(b) Describe all combinations of strategies W and Y of player 2 that strictly dominate X.

(c) Find all equilibria of this game.

3.4. *Iterated Elimination (1)*

Consider the bimatrix game (from [145], p. 65)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \!\!\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 5,4 & 4,4 & 4,5 & 12,2 \\ 3,7 & 8,7 & 5,8 & 10,6 \\ 2,10 & 7,6 & 4,6 & 9,5 \\ 4,4 & 5,9 & 4,10 & 10,9 \end{pmatrix} \end{array}.$$

(a) Find a few different ways in which strictly dominated strategies can be iteratedly eliminated in this game.

(b) Find the Nash equilibria of this game.

3.5. *Iterated Elimination (2)*

Consider the bimatrix game

$$\begin{pmatrix} 2,0 & 1,1 & 4,2 \\ 3,4 & 1,2 & 2,3 \\ 1,3 & 0,2 & 3,0 \end{pmatrix}.$$

Find the Nash equilibria of this game.

3.6. *Weakly Dominated Strategies*

A pure strategy i of player 1 in an $m \times n$ bimatrix game (A,B) is *weakly dominated* if there a strategy $\mathbf{p} = (p_1,\ldots,p_m) \in \Delta^m$ with $p_i = 0$ such that $\mathbf{p}A\mathbf{e}^j \geq \mathbf{e}^i A\mathbf{e}^j$ for every $j = 1,\ldots,n$, and $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A\mathbf{e}^j$ for at least one j. The definition of a weakly dominated strategy of player 2 is similar. In words, a pure strategy is weakly dominated if there is some pure or mixed strategy that is always at least as good, and that is better against at least one pure strategy of the opponent. Instead of iterated elimination of strictly dominated strategies one might also consider iterated elimination of weakly dominated strategies. The advantage is that in games where no strategy is strictly dominated it might still be possible to eliminate strategies that are weakly dominated. The main disadvantages are that some Nash equilibria of the original game may be eliminated as well, and also that the order of elimination may matter. These issues are illustrated by the following examples.

(a) Consider the bimatrix game

$$\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 11,10 & 6,9 & 10,9 \\ 11,6 & 6,6 & 9,6 \\ 12,10 & 6,9 & 9,11 \end{pmatrix} \end{array}.$$

First, determine the pure Nash equilibria of this game. Next, apply iterated elimination of weakly dominated strategies to reduce the game to a 2×2 game and determine the unique Nash equilibrium of this game.

(b) Consider the bimatrix game

$$\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 1,2 & 1,2 & 1,1 \\ 0,0 & 1,1 & 1,1 \end{pmatrix} \end{array}.$$

Show that different orders of eliminating weakly dominated strategies may result in different Nash equilibria.

3.7. *A Parameter Game*

Consider the bimatrix game

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 1,1 & a,0 \\ 0,0 & 2,1 \end{pmatrix}, \end{array}$$

where $a \in \mathbb{R}$. Determine the Nash equilibria of this game for every possible value of a.

3.8. *Equalizing Property of Mixed Equilibrium Strategies*

(a) Consider again the game of Problem 3.3, which has a unique Nash equilibrium in mixed strategies. In this equilibrium, player 1 puts positive probability p^* on T and $1-p^*$ on B, and player 2 puts positive probability q^* on W and $1-q^*$ on Y. Show that, if player 2 plays this strategy, then both T and B give player 1 the same expected payoff, equal to the equilibrium payoff. Also show that, if player 1 plays his equilibrium strategy, then both W and Y give player 2 the same expected payoff, equal to the equilibrium payoff, and higher than the expected payoff from X or from Z.

(b) Generalize the observations made in (a), more precisely, give an argument for the following statement:

Let (A,B) be an $m \times n$ bimatrix game and let $(\mathbf{p}^*, \mathbf{q}^*)$ be a Nash equilibrium. Then each row played with positive probability in this Nash equilibrium has the same expected payoff for player 1 against $\mathbf{q}^*$ and this payoff is at least as high as the payoff from any other row. Each column played with positive probability in this Nash equilibrium has the same expected payoff for player 2 against $\mathbf{p}^*$ and this payoff is at least as high as the payoff from any other column.

You may state your argument in words, without using formulas.

Chapter 4
Finite Extensive Form Games

Most games derived from economic or political situations have in common with most parlor games (like card games and board games) that they are not ‘one-shot’: players move sequentially, and one and the same player may move more often than once. Such games are best described by drawing a decision tree which tells us whose move it is and what a player’s information is when that player has to make a move.

In this chapter these so-called ‘games in extensive form’ are studied. Attention here is restricted to games with finitely many players (usually two), finitely many decision moments and finitely many moves. See Sect. 1.3.3 for a few examples. We also assume that each player has ‘complete information’, which – formally – boils down to either there being no chance move in the game or, if a chance move occurs, each player becoming informed about the outcome of that chance move. This excludes, for instance, the game of entry deterrence with incomplete information in Sect. 1.3.3: for the analysis of games with incomplete information see Chap. 5. Chapter 14 extends the analysis of the present and the next chapter.

The first section of this chapter introduces games in extensive form. In order to avoid a load of cumbersome notation the treatment will be somewhat informal but – hopefully – not imprecise. In Sect. 4.2 we define strategies and the ‘strategic form’ of a game: the definition of Nash equilibrium for extensive form games is then practically implied. The focus in this chapter is on pure Nash equilibrium.

In the third section the concept of Nash equilibrium is refined by considering subgame perfection (first introduced in [117] and [118]) and backward induction. A further important refinement, called ‘perfect Bayesian equilibrium’, is treated in the fourth section.

4.1 The Extensive Form

A *game in extensive form* is described by a *game tree*. Such a game tree is characterized by *nodes* and *edges*. Each node is either a *decision node* of a player, or a *chance node*, or an *end node*. Each edge corresponds to either an *action* of a player or a choice made by chance, sometimes called a ‘move of Nature’.

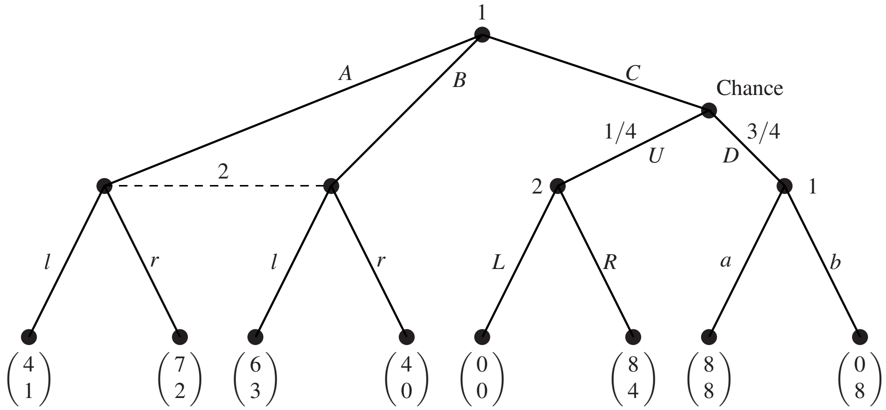


Fig. 4.1 A game in extensive form

Figure 4.1 illustrates these and other concepts.

The upper node in the tree, the *root* of the tree, is a decision node of player 1 and the starting point of the game. Player 1 chooses between three actions, namely A, B, and C. Player 2 learns that player 1 has chosen either A or B, or C. The first event is indicated by the dashed line connecting the two left decision nodes of player 2. In that case, player 2 has two actions, namely l and r. We call the two connected nodes an *information set* of player 2: player 2 knows that the play of the game has arrived at one of these nodes but he does not know at which one. The fact that player 2 has the same set of actions at each of the two nodes in this information set is a necessary consequence: if this were not the case, player 2 would know at which node he was (i.e., would know whether player 1 would have played A or B) by simply examining the set of available actions, which would go against the interpretation of an information set. This last argument is one consequence of the more general assumption that the whole game tree is common knowledge between the players: each player knows it, knows that the other player(s) know(s) it, knows that the other player(s) know(s) that he knows it, etc.

If player 1 plays C, then there is a chance move, resulting with probability $1/4$ in a decision node of player 2 (following U) and with probability $3/4$ in a decision node of player 1 (following D). At player 2's decision node this player has two actions, namely L and R. At player 1's decision node this player has also two actions, namely a and b. All the remaining nodes are end nodes, indicated by payoff pairs, where the upper number is the payoff to player 1 and the lower number the payoff to player 2. In this diagram, the payoffs are written as column vectors, but we also write them as row vectors, whatever is convenient in a given situation.

We note that also the singleton decision nodes are called information sets. So in this game, each player has two different information sets. Call an information set *nontrivial* if it consists of at least two nodes. Games with nontrivial information sets are called games with *imperfect information*. If a game has only trivial information sets, then we say that it has *perfect information*. In the present example, if player 2

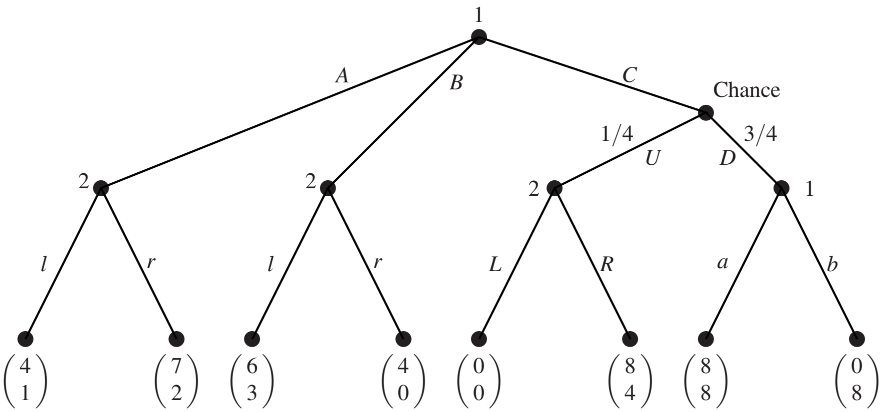


Fig. 4.2 The game of Fig. 4.1, now with perfect information

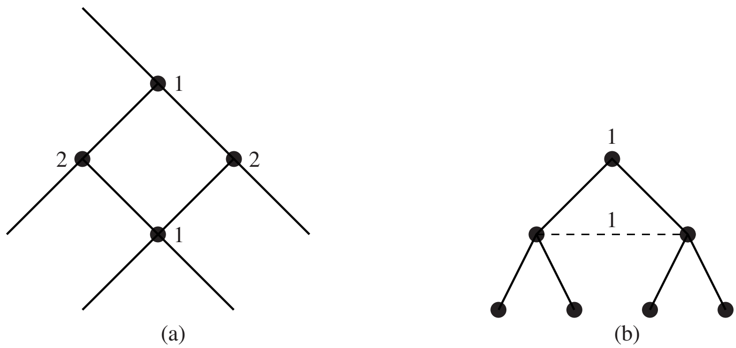


Fig. 4.3 An example of a cycle (**a**), and of a game without perfect recall (**b**)

observes whether player 1 chooses *A* or *B*, then the game has perfect information. See Fig. 4.2.

The chance move in our example is not a particularly interesting one, since the players learn what the outcome of the chance move is.[1]

As mentioned before, we do not give a formal definition of a *game in extensive form*: the examples in Figs. 4.1 and 4.2 illustrate the main ingredients of such a game.[2] An important condition is that the game tree should be a *tree* indeed: it should have a single root and no 'cycles'. This means that a situation like for instance in Fig. 4.3a is not allowed.

We also consider only games in extensive form that have *perfect recall*: each player remembers what he did in the past. For instance, a situation like in Fig. 4.3b,

[1] The situation is different if at least one player is not completely informed about the outcome of a chance move and if this lack of information has strategic consequences. In that case, we talk about games with 'incomplete' information, see Chap. 5.

[2] See Chap. 14 for a formal definition.

where player 1 at his lower information set does not recall which action he took earlier, is not allowed.[3]

4.2 The Strategic Form

In a game in extensive form, it is extremely important to distinguish between 'actions' and 'strategies'. An action is a possible move of a player at an information set. In the games in Figs. 4.1 and 4.2 player 1 has the actions A, B, and C, and a and b; and player 2 has the actions l and r, and L and R. In contrast:

> A *strategy* is a *complete plan to play the game*.

This is one of the most important concepts in game theory. In the games in Figs. 4.1 and 4.2, a possible strategy for player 1 is:

> Start by playing C; if the chance move of the game results in D, then play b.

Another strategy of player 1 is:

> Start by playing A; if the chance move of the game results in D, then play b.

The last strategy might look strange since player 1's first action A excludes him having to take a further action. Nevertheless, also this plan is regarded as a possible strategy.[4]

A possible strategy for player 2 in the game of Fig. 4.1 is:

> Play l if player 1 plays A or B, and play L if player 1 plays C and the chance move results in U.

Note that player 2 cannot make his action contingent on whether player 1 plays A or B, since player 2 does not have that information. In the perfect information game of Fig. 4.2, however, player 2's strategy should tell what player 2 plays after A and what he plays after B. A possible strategy would then be:

> Play l if player 1 plays A, play r if player 1 plays B, and play L if player 1 plays C and the chance move results in U.

A more formal definition of a strategy of a player is:

> A strategy is a *list of actions, exactly one at each information set of that player*.

In both our examples, a strategy of player 1 is therefore a list of two actions since player 1 has two information sets. The number of possible strategies of player 1 is the number of different lists of actions. Since player 1 has three possible actions at

[3] The assumption of perfect recall plays a particular role for the relation between mixed and behavioral strategies, see Chap. 14.

[4] Although there is not much lost if we would exclude such strategies – as some authors do.

his first information set and two possible actions at his second information set, this number is equal to $3 \times 2 = 6$. The strategy set of player 1 can be denoted, thus, as

$$\{Aa, Ab, Ba, Bb, Ca, Cb\}\ .$$

Similarly, in the imperfect information game in Fig. 4.1 player 2 has $2 \times 2 = 4$ different strategies, and his strategy set can be denoted as

$$\{lL, lR, rL, rR\}\ .$$

In the perfect information game in Fig. 4.2 player 2 has three information sets and two actions at each information set, so $2 \times 2 \times 2 = 8$ different strategies, and his strategy set can be denoted as

$$\{llL, llR, lrL, lrR, rlL, rlR, rrL, rrR\}\ ,$$

where the first letter is the action to be played if player 1 plays A and the second letter is the action to be played if player 1 plays B.

There are several important reasons why we are interested in strategies. The main reason is that by considering strategies the extensive form game is effectively reduced to a one-shot game. Once we fix a profile (in the present example, pair) of strategies we can compute the payoffs by following the path followed in the game tree. Consider for instance the strategy pair (Cb, rL) in the game in Fig. 4.1. Then player 1 starts by playing C, and this is followed by a chance move; if the result of this move is U, then player 2 plays L; if the result is D, then player 1 plays b. Hence, with probability $1/4$ the resulting payoff pair is $(0,0)$ and with probability $3/4$ the resulting payoff pair is $(0,8)$. So the expected payoffs are 0 for player 1 and 6 for player 2. In this way, we can compute the payoffs in the game of Fig. 4.1 resulting from each of the 6×4 possible strategy combinations. Similarly, for the game in Fig. 4.2 we compute 6×8 payoff pairs. We next write these payoff pairs in a bimatrix, as in Chap. 3. The resulting bimatrix games are presented in Fig. 4.4. Such a bimatrix game is called the *strategic form* of the extensive form game. The definition of Nash equilibrium of an extensive form game is then almost implied:

> A *Nash equilibrium* of a game in extensive form is a Nash equilibrium of the strategic form.

This definition holds for pure Nash equilibria and, more generally, Nash equilibria in mixed strategies, but in this chapter we restrict attention to pure strategies and pure strategy Nash equilibria.

The pure strategy Nash equilibria of the bimatrix games in Fig. 4.4 can be found by using the method of Sect. 3.2.1. The equilibria correspond to the double-starred entries. Thus, the imperfect information game has six different Nash equilibria in pure strategies, and the perfect information game has ten different Nash equilibria in pure strategies.

In the next two sections we examine these Nash equilibria more closely and discuss ways to distinguish between them.

$$\begin{array}{cc} & \begin{array}{cccc} lL & lR & rL & rR \end{array} \\ \begin{array}{c} Aa \\ Ab \\ Ba \\ Bb \\ Ca \\ Cb \end{array} & \begin{pmatrix} 4,1 & 4,1 & 7^*,2^* & 7,2^* \\ 4,1 & 4,1 & 7^*,2^* & 7,2^* \\ 6^*,3^* & 6,3^* & 4,0 & 4,0 \\ 6^*,3^* & 6,3^* & 4,0 & 4,0 \\ 6^*,6 & 8^*,7^* & 6,6 & 8^*,7^* \\ 0,6 & 2,7^* & 0,6 & 2,7^* \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccccc} llL & llR & lrL & lrR & rlL & rlR & rrL & rrR \end{array} \\ \begin{array}{c} Aa \\ Ab \\ Ba \\ Bb \\ Ca \\ Cb \end{array} & \begin{pmatrix} 4,1 & 4,1 & 4,1 & 4,1 & 7^*,2^* & 7,2^* & 7^*,2^* & 7,2^* \\ 4,1 & 4,1 & 4,1 & 4,1 & 7^*,2^* & 7,2^* & 7^*,2^* & 7,2^* \\ 6^*,3^* & 6,3^* & 4,0 & 4,0 & 6,3^* & 6,3^* & 4,0 & 4,0 \\ 6^*,3^* & 6,3^* & 4,0 & 4,0 & 6,3^* & 6,3^* & 4,0 & 4,0 \\ 6^*,6 & 8^*,7^* & 6^*,6 & 8^*,7^* & 6,6 & 8^*,7^* & 6,6 & 8^*,7^* \\ 0,6 & 2,7^* & 0,6 & 2,7^* & 0,6 & 2,7^* & 0,6 & 2,7^* \end{pmatrix} \end{array}$$

Fig. 4.4 The 6×4 strategic form of the game in Fig. 4.1 and the 6×8 strategic form of the game in Fig. 4.2

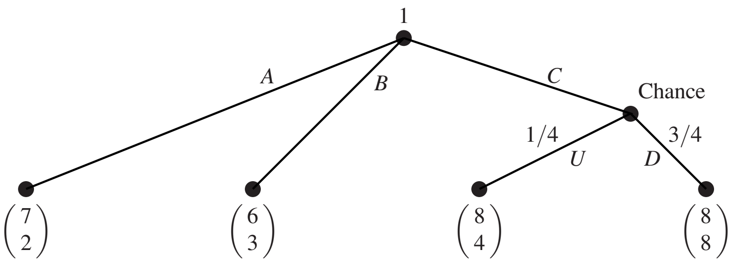


Fig. 4.5 The reduced game of Fig. 4.2

4.3 Backward Induction and Subgame Perfection

We first consider the perfect information game of Fig. 4.2. This game can be analyzed using the principle of *backward induction*. This means that we start with the nodes preceding the end nodes, and turn them into end nodes with payoffs resulting from choosing the optimal action(s). For the game under consideration this yields the reduced game of Fig. 4.5. Note that player 2's strategy has already been completely determined: it is the strategy rlR. Player 1 has chosen a at his lower information set. Next, in this reduced game, player 1 chooses the action(s) that yield(s) the highest payoff. Since A yields a payoff of 7, B a payoff of 6, and C a(n expected) payoff of $1/4 \times 8 + 3/4 \times 8 = 8$, it is optimal for player 1 to choose C. Hence, we obtain the strategy combination (Ca, rlR) with payoffs $(8, 7)$. This is one of the ten Nash equilibria of the game (see Fig. 4.4). It is called *backward induction equilibrium*. It can be shown that *applying the backward induction principle always* *results in a (pure) Nash equilibrium.*[5] As a by-product, we obtain that *a game of perfect information has at least one Nash equilibrium in pure strategies, which can be obtained by backward induction.*

It is important to distinguish between backward induction equilibrium (in this game, (Ca, rlR)) and *backward induction outcome*. The latter refers to the actual play of the game or, equivalently, the equilibrium path, in this case (Ca, R). Observe that there are other Nash equilibria in this game that generate the same outcome or path, namely (Ca, llR), (Ca, lrR), and (Ca, rrR): they all generate the path (Ca, R), but differ in the left part of the tree, where player 2 makes at least one suboptimal decision. Hence, the principle of backward induction ensures that every player always takes an optimal action, even in parts of the game tree that are not actually reached when the game is played.

A more general way to do this is to use the idea of *subgame perfection* (first explicitly formulated by Selten [117]). The definition of a subgame is as follows:

> A *subgame* is any part of the game tree, starting at a single decision node (trivial information set) of a player or a chance node, which is not connected to the tree by any later information set.

The game in Fig. 4.2 has six different subgames, namely: the entire game; the game starting from the chance move; and the four games starting from the four nodes preceding the end nodes. The definition of a subgame perfect equilibrium is as follows:

> A *subgame perfect equilibrium* is a strategy combination that induces a Nash equilibrium in every subgame.

To see what this means, consider again the game in Fig. 4.2. In order for a strategy combination to be a subgame perfect equilibrium, it has to induce a Nash equilibrium in every subgame. Since the entire game is a subgame, a subgame perfect equilibrium has to be a Nash equilibrium in the entire game, and, thus, the ten Nash equilibria in this game are the candidates for a subgame perfect equilibrium. This is the case for any game, and therefore *a subgame perfect equilibrium is always a Nash equilibrium.* A subgame perfect equilibrium also has to induce an equilibrium in each of the four one-player subgames preceding the end nodes: although we have not defined Nash equilibria for one-person games, the only reasonable definition is that a player should choose the action that is optimal. In the example, this means that (from left to right) the actions r, l, R, and a, should be chosen. This implies that the players choose optimally also in the subgame starting from the chance node. Summarizing, we look for the Nash equilibrium or equilibria that generate the mentioned actions, and the only Nash equilibrium that does this is again (Ca, rlR). Hence, the unique subgame perfect equilibrium in this game is (Ca, rlR). It is not surprising that this is also the backward induction equilibrium: *in games of perfect information, backward induction equilibria and subgame perfect equilibria coincide.*

[5] This result is intuitive but nevertheless not that easy to prove formally. See, e.g., [102], Chap. 3.

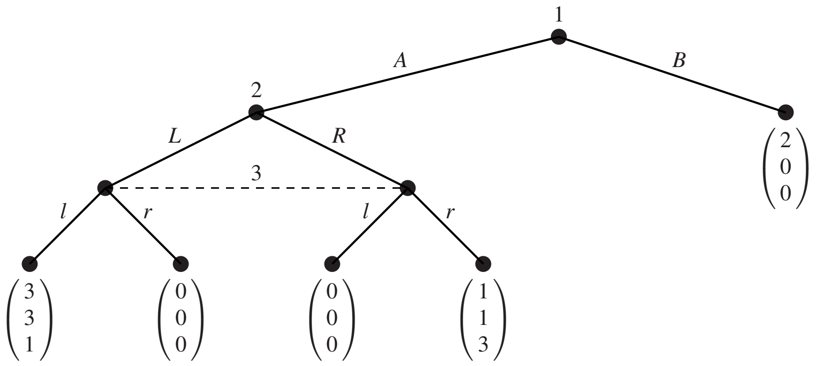


Fig. 4.6 A three-player game

Let us now consider the imperfect information version of the game in Fig. 4.1. In this case, backward induction cannot be applied to the left part of the game tree: since player 2 does not know whether player 1 has played A or B when he has to choose an action in his left information set, he cannot choose between l and r. In terms of subgames, the only subgames are now: the entire game; the two subgames following U and D; and the subgame starting from the chance move. Hence, the restrictions imposed by subgame perfection are that player 1 should play a, player 2 should play R, and the strategy combination should be a Nash equilibrium in the entire game. Of the six Nash equilibria of the game (see Fig. 4.4), this leaves the two equilibria (Ca, lR) and (Ca, rR). So these are the subgame perfect equilibria of the game in Fig. 4.1.

We conclude this section with an example which shows more clearly than the preceding example that subgame perfection can be more generally applied than the backward induction principle. Consider the game in Fig. 4.6, which is a three player game (for a change). Clearly, backward induction cannot be applied here. For subgame perfection, notice that this game has only two subgames: the entire game, and the game starting with player 2's decision node. The latter game is a game between players 2 and 3 with strategic form

$$\begin{array}{cc} & \begin{array}{cc} l & r \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 3,1 & 0,0 \\ 0,0 & 1,3 \end{pmatrix} \end{array}$$

which has two pure Nash equilibria, namely (L,l) and (R,r). Hence, a subgame perfect equilibrium has to induce one of these two equilibria in the subgame. Note that if the first equilibrium is played, then player 1 should play A, yielding him a payoff of 3 rather than the payoff of 2 obtained by playing B. If the other equilibrium is played in the subgame, then player 1 should obviously play B since A now yields only 1. So the two subgame perfect equilibria are (A,L,l) and (B,R,r).

Alternatively, one can first compute the (pure) Nash equilibria of the entire game. The strategic form of the game can be represented as follows, where the left matrix results from player 1 playing A and the right matrix from player 1 playing B.

$$1:A \quad \begin{array}{c} \\ L \\ R \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 3^*,3^*,1^* & 0,0,0 \\ 0,0,0 & 1,1^*,3^* \end{pmatrix} \end{array} \qquad 1:B \quad \begin{array}{c} \\ L \\ R \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 2,0^*,0^* & 2^*,0^*,0^* \\ 2^*,0^*,0^* & 2^*,0^*,0^* \end{pmatrix} \end{array}.$$

Best replies are marked by asterisks (for player 1 one has to compare the corresponding payoffs over the two matrices), and the pure Nash equilibria are (A,L,l), (B,L,r), (B,R,l), and (B,R,r). The subgame perfect equilibria are those where the combination (L,l) or (R,r) is played, resulting in the two equilibria found above.

4.4 Perfect Bayesian Equilibrium

A further refinement of Nash equilibrium and of subgame perfect equilibrium is provided by the concept of 'perfect Bayesian equilibrium'. Consider an information set of a player in an extensive form game. A *belief* of that player is simply a probability distribution over the nodes of that information set or, equivalently, over the actions leading to that information set. Of course, if the information set is trivial (consists of a single node) then also the belief is trivial, namely attaching probability 1 to the unique node. Our somewhat informal definition of a perfect Bayesian equilibrium is as follows.

A *perfect Bayesian equilibrium* in an extensive form game is a combination of strategies and a specification of beliefs such that two conditions are satisfied:

1. The beliefs are consistent with the strategies under consideration.
2. The players choose optimally given the beliefs.

The first condition is a version of 'Bayesian consistency of beliefs' and the second condition is 'sequential rationality'.[6] The first condition says that the beliefs should satisfy Bayesian updating with respect to the strategies whenever possible. The second condition says that a player should maximize his expected payoff given his beliefs. In order to see what these conditions mean exactly, we consider some examples.

Consider the game in Fig. 4.1. This game has one nontrivial information set. Suppose player 2's belief at this information set is given by the probabilities α at the left node and $1-\alpha$ at the right node, where $0 \le \alpha \le 1$. That is, if this information set is reached then player 2 attaches probability α to player 1 having played A and probability $1-\alpha$ to player 1 having played B. All the other information sets

[6] In the expression 'perfect Bayesian equilibrium' the word 'Bayesian' refers to the meaning of 'consistency', namely consistent with respect to Bayesian updating. There is also a stronger version of consistency, resulting in 'sequential equilibrium'. See [66] and Chap. 14.

are trivial and therefore the beliefs attach probability 1 to each of the corresponding nodes. Given these beliefs, applying condition (2) means that player 2 should choose R and player 1 should choose a at the corresponding information sets. At the nontrivial information set, player 2 should choose the action that maximizes his expected payoff. The expected payoff from l is equal to $\alpha \cdot 1 + (1-\alpha) \cdot 3 = 3 - 2\alpha$ and the expected payoff from r is $\alpha \cdot 2 + (1-\alpha) \cdot 0 = 2\alpha$. Hence, l is optimal if $3 - 2\alpha \geq 2\alpha$, i.e., if $\alpha \leq 3/4$ and r is optimal if $\alpha \geq 3/4$.

In this game, it is always optimal for player 1 to play C, given the actions R and a following the chance move: C yields 8 whereas A or B yield at most 7. But if player 1 does not play A or B, then condition (1) above does not put any restriction on the belief α of player 2. More precisely, if player 1 plays C then the nontrivial information set of player 2 is reached with zero probability, and therefore the probability α cannot by determined by Bayesian updating, that is, by computing the conditional probability of reaching the left (or right) node in player 2's information set. This means that α can be chosen in any way we like, but given α player 2 should choose optimally, as computed before. Hence, we have essentially two perfect Bayesian equilibria, namely (Ca, lL) with beliefs $\alpha \leq 3/4$ and (Ca, rL) with beliefs $\alpha \geq 3/4$. Note that these are also the subgame perfect equilibria, now 'backed up' by a belief of player 2 on his nontrivial information set.

It is not difficult to see that a perfect Bayesian equilibrium is always subgame perfect, and therefore also a Nash equilibrium.[7] In fact, by assigning probabilities to nodes in an information set, we essentially make it possible to apply backward induction again, as is clear from the example.

In order to show that the perfect Bayesian equilibrium requirement can have an additional impact compared to subgame perfection, consider the variation on the game of Fig. 4.1, obtained by replacing the payoffs $(4,1)$ after A and l by the payoffs $(4,3)$. One can check that the subgame perfect equilibria are still (Ca, lL) and (Ca, rL). Obviously, a rational player 2 would never play r at his nontrivial information set since l is always better, but subgame perfection does not rule this out. But clearly, there is no belief that player 2 could have at this information set that would make r optimal: if we denote player 2's belief by $(\alpha, 1-\alpha)$ as before, then r yields 2α whereas l yields 3, which is always larger than 2α. Hence, the only perfect Bayesian equilibrium is (Ca, lL), with arbitrary belief of player 2 at his nontrivial information set.

As another example, consider again the game of Fig. 4.6, reproduced in Fig. 4.7 with belief $(\alpha, 1-\alpha)$ attached to the nodes in the information set of player 3.

There are two ways to find the perfect Bayesian equilibria of this game. One can consider the subgame perfect equilibria and find appropriate beliefs. Alternatively, one can start from scratch and apply a form of backward induction. To illustrate the last method, start with player 3. If player 3 plays l then his (expected) payoff is α. If player 3 plays r then his (expected) payoff is $3 - 3\alpha$. Therefore, l is optimal if $\alpha \geq 3/4$ and r is optimal if $\alpha \leq 3/4$.

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[7] For a more formal treatment see Chap. 14.

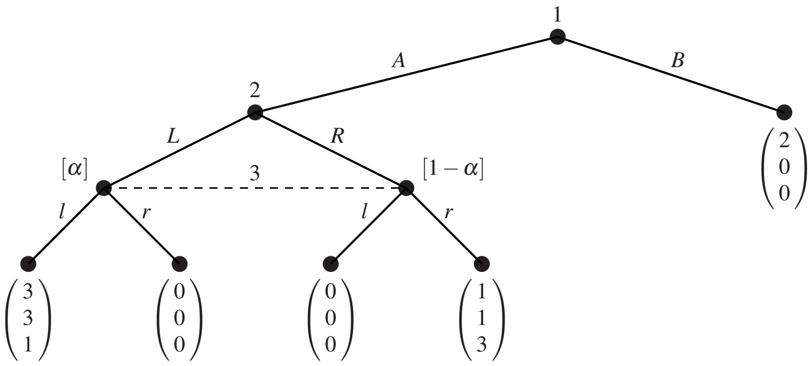

Fig. 4.7 The three-player game of Fig. 4.6 with belief of player 3

Now suppose player 3 plays l. Then it is optimal for player 2 to play L. If player 2 plays L, then (1) in the definition of a perfect Bayesian equilibrium implies $\alpha = 1$: that is, player 3 should indeed believe that player 2 has played L. Since $1 \geq 3/4$, l is the optimal action for player 3. Player 1, finally, should play A, yielding payoff 3 instead of the payoff 2 resulting from B. So we have a perfect Bayesian equilibrium (A, L, l) with belief $\alpha = 1$.

If player 3 plays r, then it is optimal for player 2 to play R, resulting in $\alpha = 0$, and thus making it optimal for player 3 to play r indeed. In this case, player 1 should play B. Hence, we have a perfect Bayesian equilibrium (B, R, r) with belief $\alpha = 0$.

Problems

4.1. *Counting Strategies*

Consider the following simplified chess game. White moves first (in accordance with the usual rules). Black observes White's move and then makes its move. Then the game ends in a draw. Determine the strategy sets of White and Black. How many strategies does Black have?

4.2. *Extensive vs. Strategic Form*

Each game in extensive form leads to a unique game in strategic form. The converse, however, is not true. Consider the following bimatrix game and find two different games in extensive form with this bimatrix game as strategic form:

$$\begin{pmatrix} a_1, a_2 & b_1, b_2 & e_1, e_2 & f_1, f_2 \\ c_1, c_2 & d_1, d_2 & g_1, g_2 & h_1, h_2 \end{pmatrix}.$$

4.3. *Entry Deterrence*

Consider the entry deterrence game of Chap. 1, of which the extensive form is reproduced in Fig. 4.8.

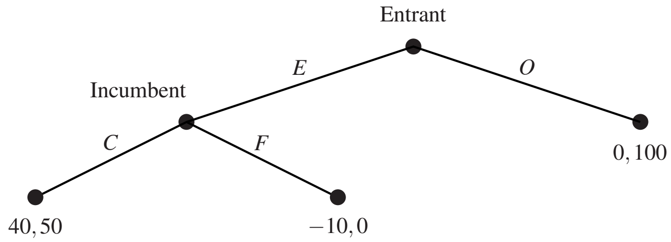

Fig. 4.8 Entry deterrence, Problem 4.3

(a) Write down the strategic form of this game.

(b) Determine the Nash equilibria (in pure strategies). Which one is the backward induction equilibrium? Which one is subgame perfect? In which sense is the other equilibrium based on an 'incredible threat'?

4.4. *Choosing Objects*

Four objects O_1, O_2, O_3, and O_4 have different worths for two players 1 and 2, given by the following table:

	O_1	O_2	O_3	O_4
Worth for player 1:	1	2	3	4
Worth for player 2:	2	3	4	1

Player 1 starts with choosing an object. After him player 2 chooses an object, then player 1 takes his second object, and finally player 2 gets the object that is left.

(a) Draw the decision tree for this extensive form game.

(b) How many strategies does each player have?

(c) Determine the backward induction or subgame perfect equilibria (in pure strategies).

4.5. *An Extensive Form Game*

For the game in Fig. 4.9, write down the strategic form and compute all Nash equilibria, subgame perfect equilibria, and perfect Bayesian equilibria in pure strategies.

4.6. *Another Extensive Form Game*

For the game in Fig. 4.10, write down the strategic form and compute all Nash equilibria, subgame perfect equilibria, and perfect Bayesian equilibria in pure strategies.

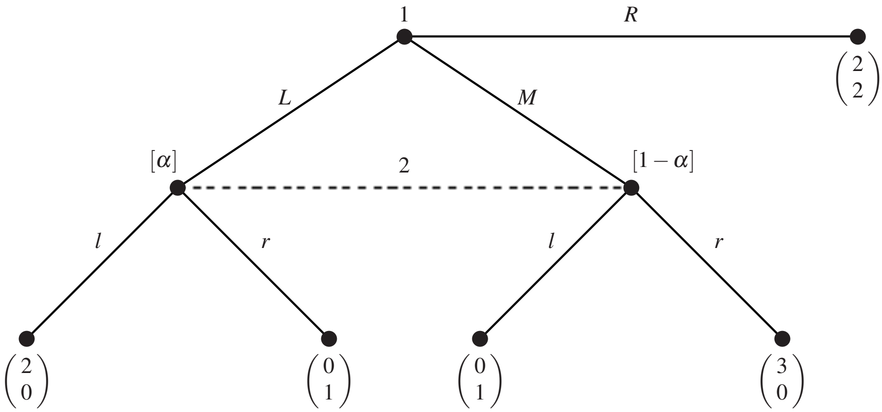

Fig. 4.9 Extensive form game of Problem 4.5

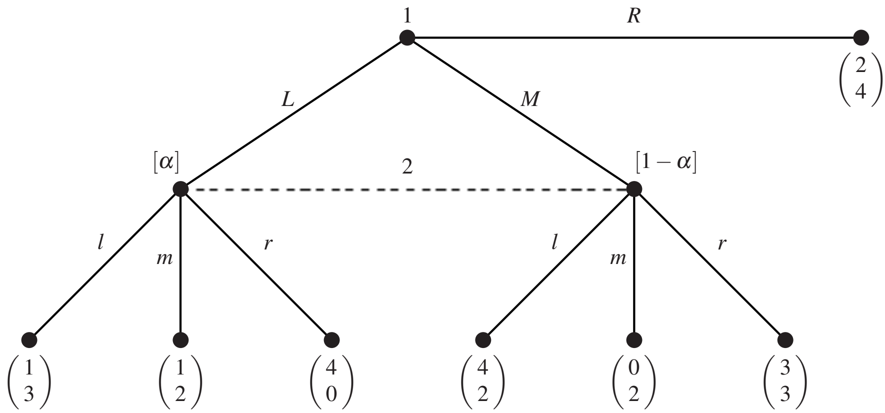

Fig. 4.10 Extensive form game of Problem 4.6

4.7. *A Centipede Game*

In the centipede game, the two players move alternatingly. On each move, a player can stop (S) or continue (C). On any move, a player is better off stopping the game than continuing if the other player stops immediately afterward, but is worse off stopping than continuing if the other player continues, regardless of the subsequent actions. The game ends after a finite number of periods. Consider an example of this game in Fig. 4.11.

(a) Determine the backward induction or subgame perfect equilibrium of this game. What is the associated outcome?

(b) Show that there are other Nash equilibria, but that these always result in the same outcome as the subgame perfect equilibrium.

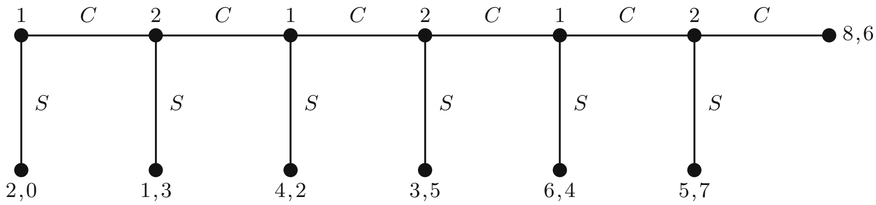

Fig. 4.11 The centipede game of Problem 4.7

4.8. *Finitely Repeated Games*

Consider the prisoners' dilemma game of Chap. 1:

$$\begin{array}{c} \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} -1,-1 & -10,0 \\ 0,-10 & -9,-9 \end{pmatrix} \end{array}.$$

Suppose that this game is played twice. After the first play of the game the players learn the outcome of that play.

(a) How many strategies does each player have in this game?

(b) Determine the subgame perfect equilibrium or equilibria of this game. What if the game is repeated more than twice but still finitely many times?

Consider, next, the following bimatrix game:

$$\begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,1 & 1,0 \\ 5,1 & 4,4 \end{pmatrix} \end{array}.$$

Suppose again that the game is played twice, and that after the first play of the game the players learn the outcome of that play.

(c) Determine the subgame perfect equilibrium or equilibria of this game. What if the game is repeated more than twice but still finitely many times?

(d) Exhibit a Nash equilibrium (of the twice repeated game) where (B,L) is played in the first round.

Consider the following bimatrix game:

$$\begin{array}{c} \\ T \\ C \\ B \end{array}\begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} 8,8 & 0,9 & 0,0 \\ 9,0 & 0,0 & 3,1 \\ 0,0 & 1,3 & 3,3 \end{pmatrix} \end{array}.$$

(e) For the twice repeated version of this game, describe a subgame perfect equilibrium in which (T,L) is played in the first round.

Consider, finally, the bimatrix game (cf. [9])

$$\begin{array}{c c} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} T \\ C \\ B \end{array} & \begin{pmatrix} 5,3 & 0,0 & 2,0 \\ 0,0 & 2,2 & 0,0 \\ 0,0 & 0,0 & 0,0 \end{pmatrix} \end{array}.$$

(f) For the twice repeated version of this game, describe a subgame perfect equilibrium in which (B,R) is played in the first round.

Chapter 5
Finite Games with Incomplete Information

In a game of *imperfect* information players may be uninformed about the moves made by other players. Every one-shot, simultaneous move game is a game of imperfect information. In a game of *incomplete* information players may be uninformed about certain characteristics of the game or of the players. For instance, a player may have incomplete information about actions available to some other player, or about payoffs of other players. Following Harsanyi [50], we model incomplete information by assuming that every player can be of a number of different types. A type of a player summarizes all relevant information (in particular, actions and payoffs) about that player. Furthermore, it is assumed that each player knows his own type and, given his own type, has a probability distribution over the types of the other players. Often, these probability distributions are assumed to be consistent in the sense that they are the marginal probability distributions derived from a basic commonly known distribution over all combinations of player types.

In this chapter we consider games with finitely many players, finitely many types, and finitely many strategies. These games can be either static (simultaneous, one-shot) or dynamic (extensive form games). A Nash equilibrium in this context is also called 'Bayesian equilibrium', and in games in extensive form an appropriate refinement is perfect Bayesian equilibrium. As will become clear, in essence the concepts studied in Chaps. 3 and 4 are applied again.

In Sect. 5.1 we present a brief introduction to the concept of player types in a game. Section 5.2 considers static games of incomplete information, and Sect. 5.3 discusses so-called signaling games, which is the most widely applied class of extensive form games with incomplete information.

5.1 Player Types

Consider a set of players, say $N = \{1,2,\ldots,n\}$. For each player $i \in N$, there is a finite set of *types* T_i which that player can have. If we denote by $T = T_1 \times T_2 \times \ldots \times T_n$ the set

$$T = \{(t_1,t_2,\ldots,t_n) \mid t_1 \in T_1, t_2 \in T_2, \ldots, t_n \in T_n\} ,$$

H. Peters, *Game Theory – A Multi-Leveled Approach.*

i.e., the set of all possible combinations of types, then a *game with incomplete information* specifies a separate game for every possible combination $t = (t_1, t_2, \ldots, t_n) \in T$, in a way to be explained in the next sections. We assume that each player i knows his own type t_i and, given t_i, attaches probabilities

$$p(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n | t_i)$$

to all type combinations $t_1 \in T_1$, ..., $t_{i-1} \in T_{i-1}$, $t_{i+1} \in T_{i+1}$, ..., $t_n \in T_n$ of the other players.

Often, these probabilities are derived from a common probability distribution p over T, where $p(t)$ is the probability that the type combination is t. This is also what we assume in this chapter. Moreover, we assume that every player i, apart from his type own type t_i, also knows the probability distribution p. Hence, if player i has type t_i, then he can compute the probability that the type combination of the other players is the vector $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. Formally, this probability is equal to the conditional probability

$$p(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n | t_i) = \frac{p(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)}{\sum p(t'_1, \ldots, t'_{i-1}, t_i, t'_{i+1}, \ldots, t'_n)}$$

where the sum in the denominator is taken over all possible types of the other players, i.e., over all possible $t'_1 \in T_1, \ldots, t'_{i-1} \in T_{i-1}, t'_{i+1} \in T_{i+1}, \ldots, t'_n \in T_n$. Hence, the sum in the denominator is the probability that player i has type t_i.

Thus, a player in a game of incomplete information can make his actions dependent on his own type but not on the types of the other players. However, since he knows the probabilities of the other players' types, he can compute the expected payoffs from taking specific actions. In the next two sections we will see how this works in static and in extensive form games.

5.2 Static Games of Incomplete Information

Instead of giving formal definitions we discuss a few examples.

Battle-of-the-Sexes with One-Sided Incomplete Information

The first example (taken from [96], p. 273) is a variant of the Battle-of-the-Sexes (see Sect. 1.3.2) in which player 1 (the man) does not know whether player 2 (the woman) wants to go out with him or avoid him. More precisely, player 1 does not know whether he plays the game y or the game n, where these games are as follows:

$$y: \begin{array}{c c} & \begin{array}{cc} S & B \end{array} \\ \begin{array}{c} S \\ B \end{array} & \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{array} \qquad n: \begin{array}{c c} & \begin{array}{cc} S & B \end{array} \\ \begin{array}{c} S \\ B \end{array} & \begin{pmatrix} 2,0 & 0,2 \\ 0,1 & 1,0 \end{pmatrix} \end{array}.$$

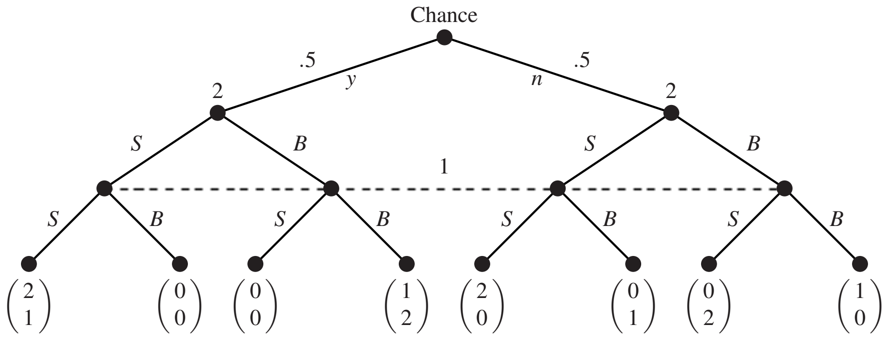

Fig. 5.1 An extensive form representation of the Battle-of-the-Sexes game with incomplete information

$$
\begin{array}{c} \\ S \\ B \end{array}
\begin{array}{c}
\begin{array}{cccc} SS & SB & BS & BB \end{array} \\
\begin{pmatrix} 2^*,0.5 & 1^*,1.5^* & 1^*,0 & 0,1 \\ 0,0.5 & 0.5,0 & 0.5,1.5^* & 1^*,1 \end{pmatrix}
\end{array}
$$

Fig. 5.2 The strategic form of the game in Fig. 5.1. Player 1 is the row player

Player 1 attaches probability $1/2$ to each of these games, and player 2 knows this. In the terminology of types, this means that player 1 has only one type, simply indicated by '1', and that player 2 has two types, namely y and n. So there are two type combinations, namely $(1,y)$ and $(1,n)$, each occurring with probability $1/2$. Player 2 knows player 1's type with certainty, and also knows her own type, that is, knows which game is actually being played. Player 1 attaches probability $1/2$ to each type of player 2.

What would be a Nash equilibrium in a game like this? To see this, it is helpful to model the game as a game in extensive form, using the tree representation of Chap. 4. Such a tree representation is given in Fig. 5.1.

The game starts with a chance move which selects which of the two bimatrix games is going to be played. In the terminology of types, it selects the type of player 2. Player 2 is informed but player 1 is not. Player 2 has four different strategies but player 1 only two. From this strategic form it is apparent that every Nash equilibrium is subgame perfect, since there are no nontrivial subgames. Also, every Nash equilibrium is perfect Bayesian, since the only nontrivial information set (of player 1) is reached with positive probability for any strategy of player 2, and thus the beliefs are completely determined by player 2's strategy through Bayesian updating.

The strategic form of the game is given in Fig. 5.2. There, the first letter in a strategy of player 2 says what player 2 plays if y is chosen by the Chance move, and the second letter says what player 2 plays if n is chosen. Also the best replies are indicated.

From the strategic form it is apparent that the game has a unique Nash equilibrium in pure strategies, namely (S,SB).[1] In this equilibrium player 1 plays S, type y of player 2 plays S and type n of player 2 plays B. Such an equilibrium is also called *Bayesian equilibrium* but, clearly, it is a Nash equilibrium of an appropriately specified extensive form (or strategic form) game.

The (pure) Nash equilibrium or equilibria of a game like this can also be found without drawing the extensive form and computing the strategic form. Suppose first that player 1 plays S in an equilibrium. Then the best reply of player 2 is to play S if her type is y and B if here type is n. The expected payoff to player 1 is then 1; playing B against this strategy of player 2 yields only 0.5. So (S,SB) is a Nash equilibrium. If, on the other hand, player 1 plays B, then the best reply of player 2 if her type is y, is B and if her type is n it is S. This yields a payoff of 0.5 to player 1, whereas playing S against this strategy of player 2 yields payoff 1. Hence, there is no equilibrium where player 1 plays B. Of course, this is also apparent from the strategic form, but the argument can be made without complete computation of the strategic form.

Battle-of-the-Sexes with Two-Sided Incomplete Information

The next example (cf. [96], p. 277) is a further variation of the Battle-of-the-Sexes game in which neither player knows whether the other player wants to be together with him/her or not. It is based on the four bimatrix games in Fig. 5.3. These four bimatrix games correspond to the four possible type combinations of players 1 and 2. The probabilities of these four different combinations are given in Table 5.1. One way to find the Nash equilibria of this game is to draw the extensive form

$$y_1y_2: \begin{array}{c c} & \begin{array}{cc} \text{S} & \text{B} \end{array} \\ \begin{array}{c} \text{S} \\ \text{B} \end{array} & \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{array} \qquad y_1n_2: \begin{array}{c c} & \begin{array}{cc} \text{S} & \text{B} \end{array} \\ \begin{array}{c} \text{S} \\ \text{B} \end{array} & \begin{pmatrix} 2,0 & 0,2 \\ 0,1 & 1,0 \end{pmatrix} \end{array}$$

$$n_1y_2: \begin{array}{c c} & \begin{array}{cc} \text{S} & \text{B} \end{array} \\ \begin{array}{c} \text{S} \\ \text{B} \end{array} & \begin{pmatrix} 0,1 & 2,0 \\ 1,0 & 0,2 \end{pmatrix} \end{array} \qquad n_1n_2: \begin{array}{c c} & \begin{array}{cc} \text{S} & \text{B} \end{array} \\ \begin{array}{c} \text{S} \\ \text{B} \end{array} & \begin{pmatrix} 0,0 & 2,2 \\ 1,1 & 0,0 \end{pmatrix} \end{array}$$

Fig. 5.3 Payoffs for Battle-of-the-Sexes with two types per player

Table 5.1 Type probabilities for Battle-of-the-Sexes with two types per player

t	y_1y_2	y_1n_2	n_1y_2	n_1n_2
$p(t)$	$2/6$	$2/6$	$1/6$	$1/6$

[1] One may use the graphical method of Chap. 3 to find possible other, mixed strategy equilibria. Here we focus on pure Nash equilibrium.

and compute the associated strategic form: see Problem 5.1. Alternatively, we can systematically examine the sixteen possible strategy pairs, as follows.

The conditional type probabilities can easily be computed from Table 5.1. For instance,

$$p(y_2|y_1) = \frac{p(y_1y_2)}{p(y_1y_2)+p(y_1n_2)} = \frac{2/6}{(2/6)+(2/6)} = 1/2.$$

The other conditional probabilities are computed in the same way, yielding:

$$p(n_2|y_1) = 1/2, \quad p(y_2|n_1) = 1/2, \quad p(n_2|n_1) = 1/2,$$

$$p(y_1|y_2) = 2/3, \quad p(n_1|y_2) = 1/3, \quad p(y_1|n_2) = 2/3, \quad p(n_1|n_2) = 1/3.$$

Suppose player 1 plays the strategy SS, meaning that he plays S (the first letter) if his type is y_1 and also S (the second letter) if his type is n_1. (Throughout this argument the first letter of a strategy refers to the y-type and the second letter to the n-type.) Then the expected payoff for type y_2 of player 2 if she plays S is $(2/3)\cdot 1+(1/3)\cdot 1 = 1$ and if she plays B it is $(2/3)\cdot 0+(1/3)\cdot 0 = 0$. Hence the best replay of type y_2 is S. Similarly, for type n_2 of player 2, playing S yields 0 and playing B yields 2, so that B is the best reply. Hence, player 2's best reply against SS is SB. Suppose, now, that player 2 plays SB. Then playing S yields type y_1 of player 1 an expected payoff of $(1/2)\cdot 2+(1/2)\cdot 0 = 1$ and playing B yields $(1/2)\cdot 0+(1/2)\cdot 1 = 1/2$, so that S is the best reply for type y_1 of player 1. Similarly, for type n_1 playing S yields $(1/2)\cdot 0+(1/2)\cdot 2 = 1$ whereas playing B yields $1/2$. Hence, S is the best reply for type n_1. Hence, player 1's best reply against SB is SS. We conclude that (SS,SB) is a Nash equilibrium.

Next, suppose player 1 plays SB. Similar computations as before yield that player 2 has two best replies, namely SB and BB. Against SB player 1's best reply is SS (as established in the previous paragraph) and not SB, so this does not result in a Nash equilibrium. Against BB player 1's best reply is BS and not SB, so also this combination is not a Nash equilibrium.

Third, suppose that player 1 plays BS. Then player 2 has two best replies, namely BS and BB. Against BS the best reply of player 1 is SS and not BS, so this combination is not a Nash equilibrium. Against BB, player 1's best reply is BS, so the combination (BS,BB) is a Nash equilibrium.

Finally, suppose player 1 plays BB. Then player 2's best reply is BS. Against this, player 1's best reply is SS and not BB. So BB of player 1 is not part of a Nash equilibrium.

We conclude that the game has two Nash equilibria in pure strategies, namely: (1) both types of player 1 play S, type y_2 of player 2 also plays S but type n_2 of player 2 plays B; (2) type y_1 of player 1 plays B, type n_1 plays S, and both types of player 2 play B. Again, these equilibria are also called Bayesian Nash equilibria.

5.3 Signaling Games

The extensive form can be used to examine a static game of incomplete information, usually by letting the game start with a chance move that picks the types of the players (see Sect. 5.2). More generally, the extensive form can be used to describe incomplete information games where players move sequentially. An important class of such games is the class of signaling games. One of the first examples is the Spence [128] job market signaling model (see Problem 5.4).

A (finite) signaling game starts with a chance move that picks the type of player 1. Player 1 is informed about his type but player 2 is not. Player 1 moves first, player 2 observes player 1's action and moves next, and then the game ends. Such a game is called a *signaling game* because the action of player 1 may be a signal about his type: that is, from the action of player 1 player 2 may be able to infer something about the type of player 1.

An Example

Consider the example in Fig. 5.4. (The numbers between square brackets at player 2's decision nodes are the beliefs of player 2, which are used in a perfect Bayesian equilibrium below.) In this game, player 1 learns the result of the chance move but player 2 does not. In the terminology of Sect. 5.1, there are two type combinations, namely $(t,2)$ and $(t',2)$, each one occurring with probability $1/2$.

In order to analyze this game and find the (pure strategy) Nash equilibria, one possibility is to first compute the strategic form. Both players have four strategies. Player 1 has strategy set

$$\{LL, LR, RL, RR\},$$

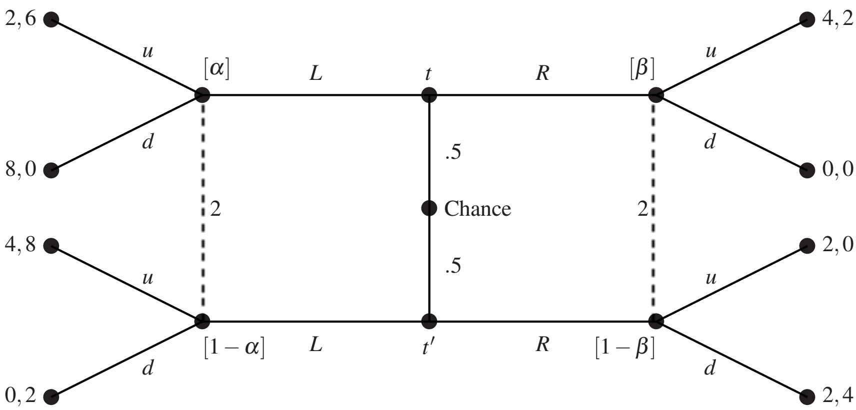

Fig. 5.4 A signaling game

$$\begin{array}{c} \\ LL \\ LR \\ RL \\ RR \end{array} \begin{array}{c} \begin{array}{cccc} uu & ud & du & dd \end{array} \\ \begin{pmatrix} 3,7^* & 3^*,7^* & 4,1 & 4,1 \\ 2,3 & 2,5^* & 5^*,0 & 5^*,2 \\ 4^*,5^* & 2,4 & 2,2 & 0,1 \\ 3,1 & 1,2^* & 3,1 & 1,2^* \end{pmatrix} \end{array}$$

Fig. 5.5 The strategic form of the game in Fig. 5.4

where the first letter refers to the action of type t and the second letter to the action of type t'. Player 2 has strategy set

$$\{uu, ud, du, dd\},$$

where the first letter refers to the action played if player 1 plays L (hence, at player 2's left information set) and the second letter to the action played if player 1 plays R (hence, at player 2's right information set). The (expected) strategic form of the game can be computed in the usual way and is presented in Fig. 5.5. The (pure) best replies are marked with an asterisk. This shows that the game has two Nash equilibria, namely (RL, uu) and (LL, ud). What else can be said about these equilibria? Observe that the only subgame of the game is the entire game, so that both equilibria are trivially subgame perfect. Are they also perfect Bayesian?[2]

First consider the equilibrium (RL, uu). The consistency requirement on the beliefs (see Sect. 4.4) requires $\alpha = 0$ and $\beta = 1$. Given these beliefs, uu is indeed the best reply of player 2. This should not come as a surprise, since it is implicit in the computation of the strategic form. Thus, the pair (RL, uu) is a perfect Bayesian equilibrium with beliefs $\alpha = 0$ and $\beta = 1$. Such an equilibrium is called *separating*: it separates the two types of player 1, since these types play different actions. In this equilibrium, the action of player 1 is a signal for his type, and the equilibrium is 'information revealing'.

Next, consider the Nash equilibrium (LL, ud). Consistency of beliefs forces $\alpha = 1/2$: since each type of player 1 plays L, the conditional probabilities of the two decision nodes in the left information set of player 2 are both equal to $1/2$. Given $\alpha = 1/2$ it follows that u is optimal at player 2's left information set (in fact, in this game u is optimal for any α), but again this already follows from the computation of the strategic form. The beliefs $(\beta, 1-\beta)$, however, are not restricted by the consistency requirement since, in equilibrium, the right information set is reached with probability 0; but they should be such that player 2's action d is optimal at player 2's right information set, by the sequential rationality requirement. Hence, the expected payoff to player 2 from playing d should be at least as large as the expected payoff from playing u, so $4(1-\beta) \geq 2\beta$, which is equivalent to $\beta \leq 2/3$. Thus, (LL, ud) is a perfect Bayesian equilibrium with beliefs $\alpha = 1/2$ and $\beta \leq 2/3$. Such an equilibrium is called *pooling*, since it 'pools' the two types of player 1. In this equilibrium, the action of player 1 does not reveal any information about his type.

[2] See Sect. 4.4.

The Intuitive Criterion

In a perfect Bayesian equilibrium the only requirement on a player's beliefs is the consistency requirement, saying that whenever possible a belief should assign the conditional probabilities to the nodes in an information set derived from the combination of strategies under consideration. In the literature, there are proposals for further restrictions on beliefs in order to exclude unreasonable or implausible ones. An important restriction is the *intuitive criterion* proposed in [19]. In words, this criterion says the following. Consider a perfect Bayesian equilibrium in a signaling game. Then a belief of player 2 (the uninformed player) on an information set should attach probability zero to any type of player 1 that could never possibly gain (compared to the equilibrium payoff) by playing the action leading to this information set. In the case, however, that this would exclude all possible types of player 1, then the criterion does not put any restriction on player 2's belief.

Let us apply this criterion to the perfect Bayesian equilibrium (LL, ud) of the game in Fig. 5.4. The equilibrium payoff to type t of player 1 is equal to 2. Note that type t of player 1 could have a reason to deviate to R, since then he could possibly get a payoff of 4. The equilibrium payoff to type t' of player 1 is 4. Type t', however, could get at most 2 (in fact, would always get 2 in this game) by deviating to R. The intuitive criterion now says that it is not reasonable for player 2 to assume that type t' would ever deviate to R. Formally, the intuitive criterion implies $\beta = 1$. With this belief, however, (LL, ud) is no longer a perfect Bayesian equilibrium.

Applied to a separating equilibrium like (RL, uu), the intuitive criterion has no bite, since there the beliefs are completely determined by the actions, which are optimal in equilibrium.

Computing Perfect Bayesian Equilibria in the Extensive Form

The perfect Bayesian equilibria can also be found without first computing the strategic form.

First, assume that there is an equilibrium where player 1 plays LL. Then $\alpha = 1/2$ by the consistency requirement, and player 2's optimal action at the left information set following L is u. At the right information set, player 2's optimal action is u if $\beta \geq 2/3$ and d if $\beta \leq 2/3$. If player 2 would play u after R, then type t of player 1 would improve by playing R instead of L, so this cannot be an equilibrium. If player 2 plays d after R, then no type of player 1 would want to play R instead of L. We have established that (LL, ud) with beliefs $\alpha = 1/2$ and $\beta \leq 2/3$ is a (pooling) perfect Bayesian equilibrium. (We have already seen above that it does not satisfy the intuitive criterion.)

Second, assume player 1 plays LR in equilibrium. Then player 2's beliefs are $\alpha = 1$ and $\beta = 0$, and player 2's best reply is ud. But then type t of player 1 would gain by playing R instead of L, so this cannot be an equilibrium.

Third, assume player 1 plays RL in equilibrium. Then $\alpha = 0$, $\beta = 1$, and player 2's best reply is uu. Against uu, RL is player 1's best reply, so that (RL, uu) is a (separating) perfect Bayesian equilibrium with beliefs $\alpha = 0$ and $\beta = 1$.

Fourth, suppose player 1 plays RR in equilibrium. Then $\beta = 1/2$ and player 2's best reply after R is d. After L, player 2's best reply is u for any value of α. Against ud, however, type t of payer 1 would gain by playing L instead of R. So RR is not part of an equilibrium.

Of course, these considerations can also be based on the strategic form, but we do not need the entire strategic form to find the perfect Bayesian equilibria.

Problems

5.1. *Battle-of-the-Sexes*

Draw the extensive form of the Battle-of-the-Sexes game in Sect. 5.2 with payoffs in Fig. 5.3 and type probabilities in Table 5.1. Compute the strategic form and find the pure strategy Nash equilibria of the game.

5.2. *A Static Game of Incomplete Information*

Compute all pure strategy Nash equilibria in the following static game of incomplete information:

1. Chance determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.
2. Player 1 learns which game has been chosen but player 2 does not.

The two bimatrix games are:

$$\text{Game 1:}\quad \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{pmatrix} \end{array} \qquad \text{Game 2:}\quad \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 0,0 & 0,0 \\ 0,0 & 2,2 \end{pmatrix} \end{array}$$

5.3. *Another Static Game of Incomplete Information*

Player 1 has two types, t_1 and t_1', and player 2 has two types, t_2 and t_2'. The conditional probabilities of these types are:

$$p(t_2|t_1) = 1, \quad p(t_2|t_1') = 3/4, \quad p(t_1|t_2) = 3/4, \quad p(t_1|t_2') = 0\,.$$

(a) Show that these conditional probabilities can be derived from a common distribution p over the four type combinations, and determine p.

As usual suppose that each player learns his own type and knows the conditional probabilities above. Then player 1 chooses between T and B and player 2 between

L and R, where these actions may be contingent on the information a player has. The payoffs for the different type combinations are given by the bimatrix games

$$t_1t_2: \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,2 & 0,0 \\ 3,0 & 1,1 \end{pmatrix} \end{array} \quad t_1't_2: \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,2 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array} \quad t_1't_2': \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,2 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array},$$

where the type combination (t_1,t_2') is left out since it has zero probability.

(b) Compute all pure strategy Nash equilibria for this game.

5.4. *Job-Market Signaling*

(Cf. [128].) A worker can have either high or low ability, each with probability $1/2$. A worker knows his ability, but a firm which wants to hire the worker does not. The worker, whether a high or a low ability type, can choose between additional education or not. Choosing additional education does not enlarge the worker's productivity but may serve as a signal to the firm: a high ability worker can choose education without additional costs, whereas for a low ability worker the cost of education equals $e > 0$. The firm chooses either a high or a low wage, having observed whether the worker took additional education or not. The payoff to the firm equals the productivity of the worker minus the wage. The payoff to the worker equals the wage minus the cost of education; if, however, this payoff is lower than the worker's reservation utility, he chooses not to work at all and to receive his reservation utility, leaving the firm with 0 payoff. Denote the productivities of the high and low ability worker by p^H and p^L, respectively, and denote the high and low wages by w^h and w^l. Finally, let r^H and r^L denote the reservation utilities of both worker types. (All these numbers are fixed.)

(a) Determine the extensive form of this game.

Choose $p^H = 10$, $p^L = 8$, $w^h = 6$, $w^l = 4$, $r^H = 3$, $r^L = 2$, $e = 3$.

(b) Compute the strategic form of this game, and determine the pure strategy Nash equilibria. Also compute the perfect Bayesian equilibrium or equilibria in pure strategies, determine whether they are separating or pooling and whether they satisfy the intuitive criterion.

5.5. *A Joint Venture*

(Cf. [106], p. 65.) Software Inc. and Hardware Inc. are in a joint venture together. The parts used in the joint product can be defective or not; the probability of defective parts is 0.7, and this is commonly known before the start of the game. Each can exert either high or low effort, which is equivalent to costs of 20 and 0. Hardware moves first, but software cannot observe his effort. Revenues are split equally at the end. If both firms exert low effort, total profits are 100. If the parts are defective, the total profit is 100; otherwise (i.e., if the parts are defective), if both exert high effort, profit is 200, but if only one player does, profit is 100 with probability 0.9 and 200

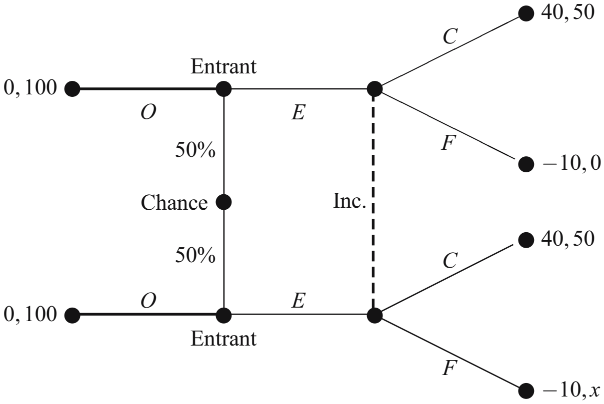

Fig. 5.6 The entry deterrence game of Problem 5.6

with probability 0.1. Hardware discovers the truth about the parts by observation before he chooses effort, but software does not.

(a) Determine the extensive form of this game. Is this a signaling game?

(b) Determine the strategic form of this game.

(c) Compute the (pure) Nash equilibria? Which one(s) is (are) subgame perfect? Perfect Bayesian?

5.6. *Entry Deterrence*

The entry deterrence game of Chap. 1 is reproduced in Fig. 5.6. For this game, compute the pure strategy perfect Bayesian equilibria for every value of $x \in \mathbb{R}$. Which one(s) is (are) pooling or separating? Satisfy the intuitive criterion?

5.7. *The Beer–Quiche Game*

(Cf. [19].) Consider the following two-player signaling game. Player 1 is either 'weak' or 'strong'. This is determined by a chance move, resulting in player 1 being 'weak' with probability $1/10$. Player 1 is informed about the outcome of this chance move but player 2 is not. Player 1 has two actions: either have quiche (Q) or have beer (B) for breakfast. Player 2 observes the breakfast of player 1 and then decides to duel (D) or not to duel (N) with player 1. The payoffs are as follows. If player 1 is weak and eats quiche then D and N give him payoffs of 1 and 3, respectively; if he is weak and drinks beer, then these payoffs are 0 and 2, respectively. If player 1 is strong, then the payoffs are 0 and 2 from D and N, respectively, if he eats quiche; and 1 and 3 from D and N, respectively, if he drinks beer. Player 2 has payoff 0 from not duelling, payoff 1 from duelling with the weak player 1, and payoff -1 from duelling with the strong player 1.

(a) Draw a diagram modelling this situation.

(b) Compute all the pure strategy Nash equilibria of the game. Find out which of these Nash equilibria are perfect Bayesian equilibria. Give the corresponding beliefs and determine whether these equilibria are pooling or separating, and which ones satisfy the intuitive criterion.

5.8. *Issuing Stock*

The following financing problem is studied in [86]. The players are a manager (M) and an existing shareholder (O). The manager is informed about the current value of the firm a and the NPV (net present value) of a potential investment opportunity b, but the shareholder only knows that high values and low values each have probability $1/2$. More precisely, either $(a,b) = (\overline{a},\overline{b})$ or $(a,b) = (\underline{a},\underline{b})$, each with probability $1/2$, where $\underline{a} < \overline{a}$ and $\underline{b} < \overline{b}$. The manager moves first and either proposes to issue new stock E (where E is fixed) to undertake the investment opportunity, or decides not to issue new stock. The existing shareholder decides whether to approve of the new stock issue or not. The manager always acts in the interest of the existing shareholder: their payoffs in the game are always equal.

If the manager decides not to issue new stock, then the investment opportunity is foregone, and the payoff is either $\overline{a}$ or $\underline{a}$. If the manager proposes to issue new stock but this is not approved by the existing shareholder, then again the investment opportunity is foregone and the payoff is either $\overline{a}$ or $\underline{a}$. If the manager proposes to issue new stock E and the existing shareholder approves of this, then the payoff to the existing shareholder is equal to $[M/(M+E)](\overline{a}+\overline{b}+E)$ in the good state $(\overline{a},\overline{b})$ and $[M/(M+E)](\underline{a}+\underline{b}+E)$ in the bad state $(\underline{a},\underline{b})$; here, $M = (1/2)[\overline{a}+\overline{b}] + (1/2)[\underline{a}+\underline{b}]$ is the price of the existing shares if the investment is undertaken.

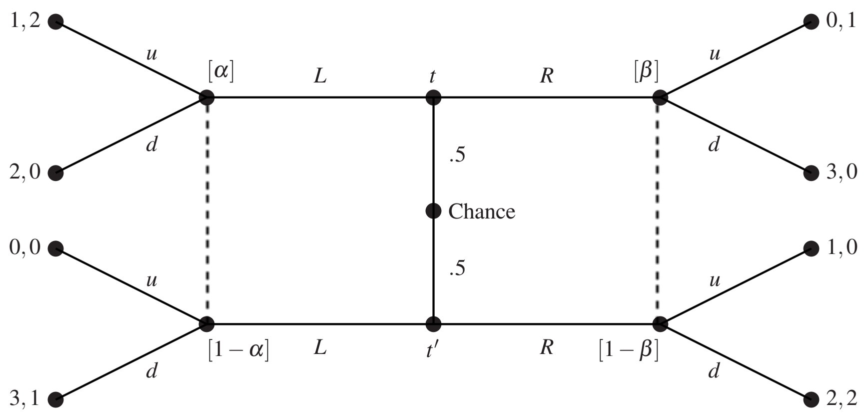

Fig. 5.7 The signaling game of Problem 5.9(a)

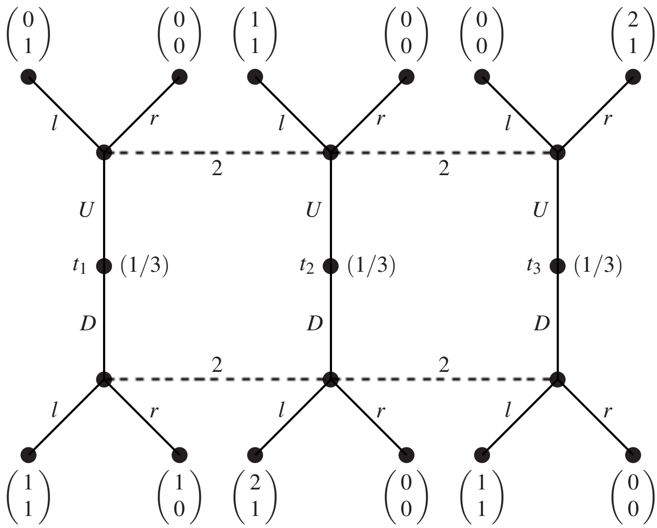

Fig. 5.8 The signaling game of Problem 5.9(b). Each type of player 1 has probability 1/3

(a) Set up the extensive form of this signaling game.

(b) Take $\overline{a} = 150$, $\underline{a} = 50$, $\overline{b} = 20$, $\underline{b} = 10$, and $E = 100$. Compute the pure strategy perfect Bayesian equilibria of this game. Are they pooling, separating? How about the intuitive criterion? Try to interpret the results from an economic point of view.

(c) Repeat the analysis of (b) for $\overline{b} = 100$.

5.9. *More Signaling Games*

(a) Compute the pure strategy perfect Bayesian equilibria and test for the intuitive criterion in the signaling game in Fig. 5.7.

(b) Consider the signaling game in Fig. 5.8, where the chance move is not explicitly drawn in order to keep the diagram simple. Compute the pure strategy perfect Bayesian equilibria and test for the intuitive criterion.

Chapter 6
Noncooperative Games: Extensions

In Chaps. 2–5 we have studied noncooperative games in which the players have finitely many (pure) strategies. The reason for the finiteness restriction is that in such games special results hold, like the existence of a value and optimal strategies for two-person zerosum games, and the existence of a Nash equilibrium in mixed strategies for finite nonzerosum games.

The basic game-theoretical concepts discussed in these chapters can be applied to much more general games. Once, in a game-theoretic situation, the players, their possible strategies, and the associated payoffs are identified, the concepts of best reply and of Nash equilibrium can be applied. Also the concepts of backward induction, subgame perfection, and perfect Bayesian equilibrium carry over to quite general extensive form games. In games of incomplete information, the concept of player types and the associated Nash equilibrium (Bayesian Nash equilibrium) can be applied also if the game has infinitely many strategies.

The bulk of this chapter consists of various, diverse examples verifying these claims. The main objective of the chapter is, indeed, to show how the basic game-theoretic apparatus can be applied to various different conflict situations; and, of course, to show these applications themselves.

In Sect. 6.1 we generalize some of the concepts of Chaps. 2–3. This section serves only as background and general framework for the examples in the following sections. Concepts specific to extensive form games and to incomplete information games are adapted later, when they are applied. In Sects. 6.2–6.7 we discuss, respectively, Cournot competition with complete and incomplete information, Bertrand competition, Stackelberg equilibrium, auctions with complete and incomplete information, mixed strategies with objective probabilities, and sequential bargaining. Variations on these topics and various other topics are treated in the problem section.

6.1 General Framework: Strategic Games

An *n-person strategic game* is a $2n+1$-tuple

$$G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n) \ ,$$

where

- $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, $n \geq 1$, is the set of *players*;
- for every $i \in N$, S_i is the *strategy set* of player i;
- for every $i \in N$, $u_i : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ is the *payoff function* of player i; i.e., for every strategy combination $(s_1, \ldots, s_n) \in S$ where $s_1 \in S_1$, …, $s_n \in S_n$, $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ is player i's payoff.

A *best reply* of player i to the strategy combination $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ of the other players is a strategy $s_i \in S_i$ such that

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $s_i' \in S_i$.

A *Nash equilibrium* of G is a strategy combination $(s_1^*, \ldots, s_n^*) \in S$ such that for each player i, s_i^* is a best reply to $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$.

A strategy $s_i' \in S_i$ of player i is *strictly dominated* by $s_i \in S_i$ if

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) > u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$, i.e., for all strategy combinations of players other than i. Clearly, a strictly dominated strategy is never used in a Nash equilibrium.

For completeness we also define weak domination. A strategy $s_i' \in S_i$ of player i is *weakly dominated* by $s_i \in S_i$ if

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$, such that at least once this inequality is strict.

The reader should verify that matrix games (Chap. 2) and bimatrix games (Chap. 3) are special cases of this general framework. The same is true for the concepts of Nash equilibrium and domination discussed in these chapters.

6.2 Cournot Quantity Competition

6.2.1 Simple Version with Complete Information

In the simplest version of the famous Cournot [21] model, two firms producing a homogenous good are competing in quantity. Each firm offers a quantity of this good on the market. The price of the good depends on the total quantity offered: the higher this quantity is, the lower the price of the good. The profit for each firm is equal to total revenue (price times quantity) minus total cost. This gives rise to a two-person game in which the players' strategies are the quantities offered and the payoff functions are the profit functions. In a simple version, the price depends linearly on total quantity and marginal cost is constant while there are no fixed costs. Specifically, we study the following game:

(a) The set of players is $N = \{1,2\}$.
(b) Each player $i = 1,2$ has set of strategies $S_i = [0,\infty)$, with typical element q_i.
(c) The payoff function of player i is $\Pi_i(q_1,q_2) = q_i P(q_1,q_2) - cq_i$, for all $q_1,q_2 \geq 0$, where

$$P(q_1,q_2) = \begin{cases} a - q_1 - q_2 & \text{if } q_1 + q_2 \leq a \\ 0 & \text{if } q_1 + q_2 > a \end{cases}$$

is the market price of the good and c is marginal cost, with $a > c \geq 0$.

A Nash equilibrium in this game is a pair (q_1^C, q_2^C), with $q_1^C, q_2^C \geq 0$, of mutually best replies, that is,

$$\Pi_1(q_1^C,q_2^C) \geq \Pi_1(q_1,q_2^C), \quad \Pi_2(q_1^C,q_2^C) \geq \Pi_2(q_1^C,q_2) \quad \text{for all } q_1,q_2 \geq 0 .$$

For obvious reasons, this equilibrium is also called *Cournot equilibrium*. To find the equilibrium, we first compute the best reply functions, also called *reaction functions*. The reaction function $\beta_1(q_2)$ of player 1 is found by solving the maximization problem

$$\max_{q_1 \geq 0} \ \Pi_1(q_1,q_2)$$

for each given value of $q_2 \geq 0$. For $q_2 \leq a$ this means maximizing the function

$$q_1(a - q_1 - q_2) - cq_1 = q_1(a - c - q_1 - q_2)$$

for $q_1 \geq 0$. For $q_2 \leq a - c$, the maximum is obtained by setting the derivative with respect to q_1, namely the function $a - 2q_1 - q_2 - c$, equal to zero, yielding $q_1 = (a - c - q_2)/2$.[1] For $a - c < q_2 \leq a$, it is optimal to take $q_1 = 0$ since otherwise the profit is negative. Note that, indeed, $q_1 + q_2 \leq a$ in these cases. If $q_2 > a$, then $P(q_1,q_2) = 0$ independent of the choice of q_1, and then player 1 maximizes profit $-cq_1$ by choosing $q_1 = 0$ if $c > 0$ and $q_1 \in [0,\infty)$ if $c = 0$. Summarizing, we have

$$\beta_1(q_2) = \begin{cases} \{\frac{a-c-q_2}{2}\} & \text{if } q_2 \leq a - c \\ \{0\} & \text{if } a - c < q_2 \leq a \\ \{0\} & \text{if } a < q_2 \text{ and } c > 0 \\ [0,\infty) & \text{if } a < q_2 \text{ and } c = 0. \end{cases} \tag{6.1}$$

(Since, in all cases of interest, this reaction function is single-valued – if $c = 0$ and $a < q_2$ we may just as well take $q_1 = 0$ since this does not occur in equilibrium

[1] The second derivative is equal to -2 so that, indeed, we have a maximum.

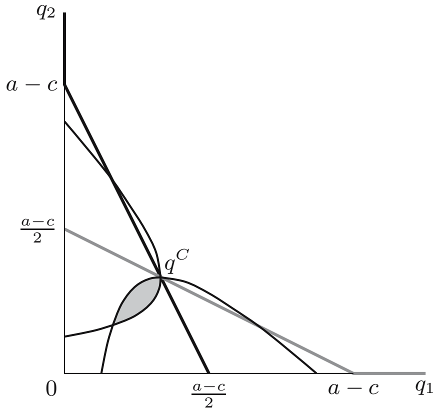

Fig. 6.1 The Cournot model: the *solid black curve* is the reaction function of player 1 and the *solid gray curve* the reaction function of player 2. The point q^C is the Nash–Cournot equilibrium. The two isoprofit curves of the players through the Nash equilibrium are drawn, and the *shaded area* consists of the quantity combinations that Pareto dominate the equilibrium

anyway – usually the braces are omitted.) By completely symmetric arguments we obtain for the reaction function of player 2:

$$\beta_2(q_1) = \begin{cases} \{\frac{a-c-q_1}{2}\} & \text{if } q_1 \le a-c \\ \{0\} & \text{if } a-c < q_1 \le a \\ \{0\} & \text{if } a < q_1 \text{ and } c > 0 \\ [0,\infty) & \text{if } a < q_1 \text{ and } c = 0. \end{cases} \tag{6.2}$$

These reaction functions are drawn in Fig. 6.1. The Nash equilibrium is the point of intersection of the reaction functions. It is obtained by simultaneously solving the two equations $q_1 = (a-c-q_2)/2$ and $q_2 = (a-c-q_1)/2$, resulting in

$$(q_1^C, q_2^C) = (\frac{a-c}{3}, \frac{a-c}{3}) .$$

Pareto Optimality

A pair (q_1, q_2) of strategies is *Pareto optimal* if there is no other pair (q_1', q_2') such that the associated payoffs are at least as good for both players and strictly better for at least one player. Not surprisingly, the equilibrium (q_1^C, q_2^C) is not Pareto optimal. For instance, both players can strictly benefit from joint profit maximization, by solving the problem

$$\max_{q_1,q_2 \ge 0} \ \Pi_1(q_1,q_2) + \Pi_2(q_1,q_2) .$$

The first-order conditions, obtained by setting the partial derivatives with respect to q_1 and q_2 equal to zero, result in the equation $a - 2q_1 - 2q_2 - c = 0$, so that any pair

$(q_1,q_2) \geq 0$ with $q_1+q_2=(a-c)/2$ is a solution (check again the second-order conditions). Taking $q_1=q_2=(a-c)/4$ yields each player a profit of $(a-c)^2/8$, whereas in the Nash equilibrium each player obtains $(a-c)^2/9$. See also Fig. 6.1, where all points in the gray-shaded area 'Pareto dominate' the Nash equilibrium: the associated payoffs are at least as good for both agents and better for at least one agent.

6.2.2 Simple Version with Incomplete Information

Consider the Cournot model of Sect. 6.2.1 but now assume that the marginal cost of firm 2 is either high, c_H, or low, c_L, where $c_H > c_L \geq 0$. Firm 2 knows its marginal cost but firm 1 only knows that it is c_H with probability ϑ or c_L with probability $1-\vartheta$. The cost of firm 1 is c and this is commonly known. In the terminology of Sect. 5.1, player 1 has only one type but player 2 has two types, c_H and c_L. The associated game is as follows:

(a) The player set is $\{1,2\}$.
(b) The strategy set of player 1 is $[0,\infty)$ with typical element q_1, and the strategy set of player 2 is $[0,\infty)\times[0,\infty)$ with typical element (q_H,q_L). Here, q_H is the chosen quantity if player 2 is of type c_H, and q_L is the chosen quantity if player 2 is of type c_L.
(c) The payoff functions of the players are the expected payoff functions. These are

$$\Pi_i(q_1,q_H,q_L) = \vartheta\Pi_i(q_1,q_H) + (1-\vartheta)\Pi_i(q_1,q_L) ,$$

for $i=1,2$, where $\Pi_i(\cdot,\cdot)$ is the payoff function from the Cournot model of Sect. 6.2.1.

To find the (Bayesian) Nash equilibrium, we first compute the best reply function or reaction function of player 1, by maximizing $\Pi_1(q_1,q_H,q_L)$ over $q_1 \geq 0$, with q_H and q_L regarded as given. Hence, we solve the problem

$$\max_{q_1\geq 0} \ \vartheta\,[q_1(a-c-q_1-q_H)] + (1-\vartheta)\,[q_1(a-c-q_1-q_L)] \ .$$

Assuming $q_H,q_L \leq a-c$ (this has to be checked later for the equilibrium), this problem is solved by setting the derivative with respect to q_1 equal to zero, which yields

$$q_1 = q_1(q_H,q_L) = \frac{a-c-\vartheta q_H-(1-\vartheta)q_L}{2} \ . \tag{6.3}$$

Observe that, compared to (6.1), we now have the expected quantity $\vartheta q_H + (1-\vartheta)q_L$ instead of q_2: this is due to the linearity of the model.

For player 2, we consider, for given q_1, the problem

$$\max_{q_H,q_L\geq 0} \ \vartheta\,[q_H(a-c_H-q_1-q_H)] + (1-\vartheta)\,[q_L(a-c_L-q_1-q_L)] \ .$$

Since the first term in this function depends only on q_H and the second term only on q_L, solving this problem amounts to maximizing the two terms separately. In other words, we determine the best replies of types c_H and c_L separately.[2] Assuming $q_1 \le a - c_H$ (and hence $q_1 \le a - c_L$) this results in

$$q_H = q_H(q_1) = \frac{a - c_H - q_1}{2} \tag{6.4}$$

and

$$q_L = q_L(q_1) = \frac{a - c_L - q_1}{2} . \tag{6.5}$$

The Nash equilibrium is obtained by simultaneously solving (6.3–6.5), using substitution or linear algebra. The solution is the triple

$$\begin{aligned} q_1^C &= \frac{a - 2c + \vartheta c_H + (1-\vartheta)c_L}{3} \\ q_H^C &= \frac{a - 2c_H + c}{3} + \frac{1-\vartheta}{6}(c_H - c_L) \\ q_L^C &= \frac{a - 2c_L + c}{3} - \frac{\vartheta}{6}(c_H - c_L) . \end{aligned}$$

Assuming that the parameters of the game are such that these three values are nonnegative and that $q_1 \le a - c_H$ and $q_H, q_L \le a - c$, this is the Bayesian Nash–Cournot equilibrium of the game. This solution should be compared with the Nash equilibrium in the complete information model with asymmetric costs, see Problem 6.1. The high cost type of firm 2 produces more than it would in the complete information case: it benefits from the fact that firm 1 is unsure about the cost of firm 2 and therefore produces less than it would if it knew for sure that firm 2 had high costs. Similarly, the low cost firm 2 produces less.

6.3 Bertrand Price Competition

Consider two firms who compete in the price of a homogenous good.[3] Specifically, assume that the demand q for the good is given by $q = q(p) = \max\{a - p, 0\}$ for every $p \ge 0$. The firm with the lower price serves the whole market; if prices are equal the firms share the market equally. Each firm has the same marginal cost $0 \le c < a$, and no fixed cost. If firm 1 sets a price p_1 and firm 2 sets a price p_2, then the profit of firm 1 is

[2] This is generally so in a Bayesian, incomplete information game: maximizing the expected payoff of a player over all his types is equivalent to maximizing the payoff per type.

[3] See Problem 6.3(d)–(f) for an example of price competition with heterogenous goods.

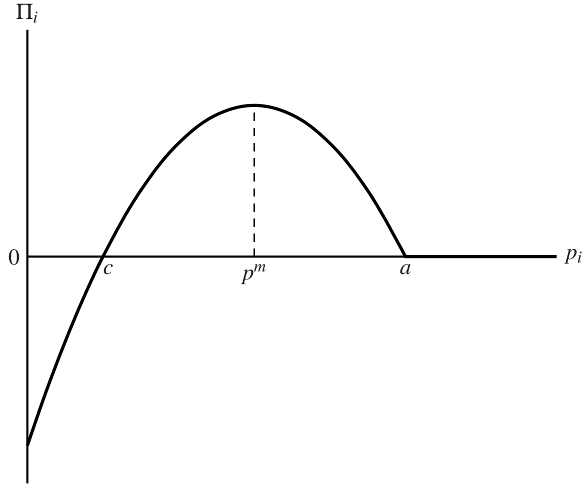

Fig. 6.2 The profit function of firm i in the monopoly situation

$$\Pi_1(p_1,p_2) = \begin{cases} (p_1-c)(a-p_1) & \text{if } p_1 < p_2 \text{ and } p_1 \le a \\ \frac{1}{2}(p_1-c)(a-p_1) & \text{if } p_1 = p_2 \text{ and } p_1 \le a \\ 0 & \text{in all other cases.} \end{cases}$$

Similarly, the profit of firm 2 is

$$\Pi_2(p_1,p_2) = \begin{cases} (p_2-c)(a-p_2) & \text{if } p_2 < p_1 \text{ and } p_2 \le a \\ \frac{1}{2}(p_2-c)(a-p_2) & \text{if } p_1 = p_2 \text{ and } p_2 \le a \\ 0 & \text{in all other cases.} \end{cases}$$

So, in terms of the game, these are the payoff functions of players 1 and 2; their strategy sets are $[0,\infty)$ for each, with typical elements p_1 and p_2. To find a Nash equilibrium (*Bertrand equilibrium*) we first compute the best reply functions (*reaction functions*). An important role is played by the price that maximizes profit if there is only one firm in the market, i.e., the monopoly price $p^m = (a+c)/2$. Also note that the profit function is a quadratic function, and that profit increases as the price gets closer to the monopoly price. See Fig. 6.2.

To determine player 1's best reply function $\beta_1(p_2)$ we distinguish several cases.

If $p_2 < c$, then any $p_1 \le p_2$ yields player 1 a negative payoff, whereas any $p_1 > p_2$ yields a payoff of zero. Hence, the set of best replies in this case is the interval (p_2,∞).

If $p_2 = c$, then any $p_1 < p_2$ yields a negative payoff for player 1, and any $p_1 \ge p_2$ yields zero payoff. So the set of best replies in this case is the interval $[c,\infty)$.

If $c < p_2 \le p^m$, then the best reply of player 1 would be a price below p_2 (to obtain the whole market) and as close to the monopoly price as possible (to maximize payoff) but such a price does not exist: for any price $p_1 < p_2$, a price in between

p_1 and p_2 would still be better. Hence, in this case the set of best replies of player 1 is empty.[4]

If $p_2 > p^m$ then the unique best reply of player 1 is the monopoly price p^m.

Summarizing we obtain

$$\beta_1(p_2) = \begin{cases} \{p_1 \mid p_1 > p_2\} & \text{if } p_2 < c \\ \{p_1 \mid p_1 \geq c\} & \text{if } p_2 = c \\ \emptyset & \text{if } c < p_2 \leq p^m \\ \{p^m\} & \text{if } p_2 > p^m. \end{cases}$$

For player 2, similarly,

$$\beta_2(p_1) = \begin{cases} \{p_2 \mid p_2 > p_1\} & \text{if } p_1 < c \\ \{p_2 \mid p_2 \geq c\} & \text{if } p_1 = c \\ \emptyset & \text{if } c < p_1 \leq p^m \\ \{p^m\} & \text{if } p_1 > p^m. \end{cases}$$

The point(s) of intersection of these best reply functions can be found by making a diagram or by direct inspection. We follow the latter method and leave the diagram method to the reader. If $p_2 < c$ then a best reply p_1 satisfies $p_1 > p_2$. But then, according to $\beta_2(p_1)$, we always have $p_2 \geq p_1$ or $p_2 = p^m$, a contradiction. Therefore, in equilibrium, we must have $p_2 \geq c$. If $p_2 = c$, then $p_1 \geq c$; if however, $p_1 > c$ then the only possibility is $p_2 = p^m$, a contradiction. Hence, $p_1 = c$ as well and, indeed, $p_1 = p_2 = c$ is a Nash equilibrium. If $p_2 > c$, then the only possibility is $p_1 = p^m$ but then p_2 is never a best reply. We conclude that the unique Nash equilibrium (Bertrand equilibrium) is $p_1 = p_2 = c$.

It is also possible to establish this result without completely computing the best reply function. Suppose, in equilibrium, that $p_1 \neq p_2$, say $p_1 < p_2$. If $p_1 < p^m$ then player 1 can increase his payoff by setting a higher price still below p_2. If $p_1 \geq p^m$ then player 2 can increase his payoff by setting a price below p_1, e.g., slightly below p^m if $p_1 = p^m$ and equal to p^m if $p_1 > p^m$. Hence, we must have $p_1 = p_2$ in equilibrium. If this common price is below c then each player can improve by setting a higher price. If this common price is above c then each player can improve by setting a slightly lower price. Hence, the only possibility that remains is $p_1 = p_2 = c$, and this is indeed an equilibrium, as can be verified directly.

6.4 Stackelberg Equilibrium

In the Cournot model of Sect. 6.2.1, the two firms move simultaneously. Consider now the situation where firm 1 moves first, and firm 2 observes this move and moves next. This situation has already been mentioned in Chap. 1. The corresponding

[4] If prices are in smallest monetary units this somewhat artificial consequence is avoided. See Problem 6.7.

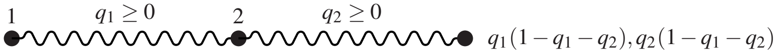

Fig. 6.3 Extensive form representation of the Stackelberg game with firm 1 as the leader

extensive form game is reproduced in Fig. 6.3. In this game, player 1 has infinite action/strategy set $[0,\infty)$, with typical element q_1. In the diagram, we use a zigzag line to express the fact that the number of actions is infinite. Player 2 has the infinite set of actions $[0,\infty)$ with typical element q_2, again represented by a zigzag line. A strategy of player 2 assigns to each information set, hence to each decision node – the game has perfect information – an action. Hence, a strategy of player 2 is a function $s_2 : [0,\infty) \to [0,\infty)$. Obviously, the number of strategies of player 2 is infinite as well.[5] The appropriate solution concept is the *backward induction* or *subgame perfect* equilibrium. The subgames of this game are the entire game and the infinite number of one-player games starting at each decision node of player 2, i.e., following each choice q_1 of player 1. Hence, the subgame perfect equilibrium can be found by backward induction, as follows. In each subgame for player 2, that is, after each choice q_1, player 2 should play optimally. This means that player 2 should play according to the reaction function $\beta_2(q_1)$ as derived in (6.2). Then, going back to the beginning of the game, player 1 should choose $q_1 \geq 0$ so as to maximize $\Pi_1(q_1,\beta_2(q_1))$. In other words, player 1 takes player 2's optimal reaction into account when choosing q_1. Assuming $q_1 \leq a-c$ – it is easy to verify that $q_1 > a-c$ is not optimal – player 1 maximizes the expression

$$q_1\left(a-c-q_1-\frac{a-c-q_1}{2}\right) .$$

The maximum is obtained for $q_1 = (a-c)/2$, and thus $q_2 = \beta_2((a-c)/2) = (a-c)/4$. Hence, the subgame perfect equilibrium of the game is

$$q_1 = (a-c)/2, \quad q_2 = \beta_2(q_1) .$$

The subgame perfect equilibrium *outcome* is by definition the resulting play of the game, that is, the actions chosen on the equilibrium path in the extensive form. In this case, the equilibrium outcome is

$$q_1^S = (a-c)/2, \quad q_2^S = (a-c)/4 .$$

The letter 'S' here is the first letter of 'Stackelberg', after whom this equilibrium is named (see [142]). More precisely, this subgame perfect equilibrium (or outcome) is called the *Stackelberg equilibrium* (or *outcome*) with player 1 as the *leader* and player 2 as the *follower*. Check that player 1's profit in this equilibrium is higher and player 2's profit is lower than in the Cournot equilibrium $q_1^C = q_2^C = (a-c)/3$. See also Problem 6.9.

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[5] In mathematical notation the strategy set of player 2 is the set $[0,\infty)^{[0,\infty)}$.

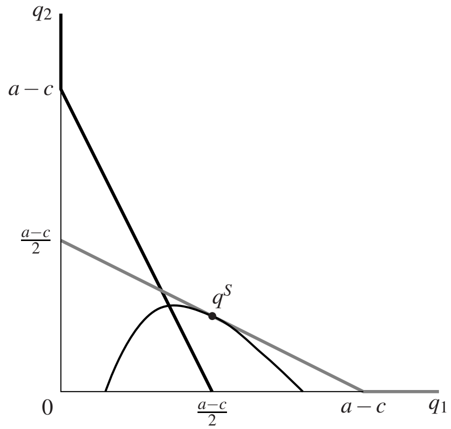

Fig. 6.4 As before, the *solid black curve* is the reaction function of player 1 and the *solid gray curve* the reaction function of player 2. The point $q^S = (\frac{a-c}{2}, \frac{a-c}{4})$ is the Stackelberg equilibrium: it is the point on the reaction curve of player 2 where player 1 maximizes profit. The associated isoprofit curve of player 1 is drawn

The Stackelberg equilibrium is depicted in Fig. 6.4. Observe that player 1, the leader, picks the point on the reaction curve of player 2 which has maximal profit for player 1. Hence, player 2 is on his reaction curve but player 1 is not.

6.5 Auctions

An auction is a procedure to sell goods among various interested parties, such that the prices are determined in the procedure. Examples range from selling a painting through an ascending bid auction (English auction) and selling flowers through a descending bid auction (Dutch auction) to tenders for public projects and selling mobile telephone frequencies. For a recent overview, see [78].

In this section we consider a few simple, classical auction models. We start with first and second-price sealed-bid auctions under complete information, and end with a first-price sealed bid auction with incomplete information. Some variations and extensions are discussed in Problems 6.10–6.13.

6.5.1 Complete Information

Consider n individuals who are interested in one indivisible object. Each individual i has valuation $v_i > 0$ for the object. We assume without loss of generality $v_1 \geq v_2 \geq \cdots \geq v_n$. In a *first-price sealed-bid auction* each individual submits a bid $b_i \geq 0$ for the object: the bids are simultaneous and independent ('sealed bids'). The individual with the highest bid wins the auction and obtains the object at a price equal to his own bid ('first price'). In case there are more highest bidders, the bidder among these with the lowest number wins the auction and pays his own bid.

This gives rise to a game with player set $N = \{1, 2, \ldots, n\}$, where each player i has strategy set $S_i = [0, \infty)$ with typical element b_i. The payoff function to player i is

$$u_i(b_1, b_2, \ldots, b_i, \ldots, b_n) = \begin{cases} v_i - b_i & \text{if } i = \min\{k \in N \mid b_k \geq b_j \text{ for all } j \in N\} \\ 0 & \text{otherwise.} \end{cases}$$

One Nash equilibrium in this game is the strategy combination $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$. To check this one should verify that no player has a better bid, given the bids of the other players: see Problem 6.10. In this equilibrium, player 1 obtains the object and pays v_2, the second-highest valuation. Check that this is also the outcome one would approximately expect in an auction with ascending bids (English auction) or descending bids (Dutch auction).

This game has many Nash equilibria. In each of these equilibria, however, a player with a highest valuation obtains the object. Bidding one's true valuation as well as bidding higher than one's true valuation are weakly dominated strategies. Bidding lower than one's true valuation is not weakly dominated (see Sect. 6.1 for the definition of weak domination). Problem 6.10 is about proving all these statements.

A *second-price sealed-bid auction* differs from a first-price sealed-bid auction only in that the winner now pays the bid of the second highest bidder. In the case that two or more players have the highest bid the player with the lowest number wins and pays his own bid. The main property of this auction is that for each player i, the strategy of bidding v_i weakly dominates all other strategies. This and other properties are collected in Problem 6.11.

6.5.2 Incomplete Information

We consider the same setting as in Sect. 6.5.1 but now assume that each bidder knows his own valuation but has only a probabilistic estimate about the valuations of the other bidders. In the terminology of types (cf. Sect. 5.1), a bidder's valuation is his true type, and each bidder holds a probability distribution over the type combinations of the other bidders. To keep things simple, we assume that every bidder's type is drawn independently from the uniform distribution over the interval $[0, 1]$, that this is common knowledge, and that each bidder learns his true type. The auction is a first-price sealed-bid auction. Of course, we can no longer fix the ordering of the valuations, but we can still employ the same tie-breaking rule in case of more than one highest bid.

We discuss the case of two bidders and postpone the extension to $n > 2$ bidders until Problem 6.13. In the associated two-person game, a strategy of player $i \in \{1, 2\}$ should assign a bid to each of his possible types. Since the set of possible types is the interval $[0,1]$ and it does not make sense to ever bid more than 1, a strategy is a function $s_i : [0,1] \to [0,1]$. Hence, if player i's type is v_i, then $b_i = s_i(v_i)$ is his bid according to the strategy s_i. The payoff function u_i of player i assigns to each strategy pair (s_i, s_j) (where j is the other player) player i's expected payoff if these strategies are played. In a (Bayesian) Nash equilibrium of the game, player i maximizes this payoff function given the strategy of player j, and vice versa. For this, it is sufficient that each type of player i maximizes expected payoff given the strategy of player j, and vice versa.

We claim that $s_1^*(v_1) = v_1/2$ and $s_2^*(v_2) = v_2/2$ is a Nash equilibrium of this game. To prove this, first consider type v_1 of player 1 and suppose that player 2 plays strategy s_2^*. If player 1 bids b_1, then the probability that player 1 wins the auction is equal to the probability that the bid of player 2 is smaller than or equal to b_1. This probability is equal to the probability that $v_2/2$ is smaller than or equal to b_1, i.e., to the probability that v_2 is smaller than or equal to $2b_1$. We may assume without loss of generality that $b_1 \le 1/2$, since according to s_2^* player 2 will never bid higher than $1/2$. Since v_2 is uniformly distributed over the interval $[0,1]$ and $2b_1 \le 1$, the probability that v_2 is smaller than or equal to $2b_1$ is just equal to $2b_1$. Hence, the probability that the bid b_1 of player 1 is winning is equal to $2b_1$ if player 2 plays s_2^*, and therefore the expected payoff from this bid is equal to $2b_1(v_1 - b_1)$ (if player 1 loses his payoff is zero). This is maximal for $b_1 = v_1/2$. Hence, $s_1^*(v_1) = v_1/2$ is a best reply to s_2^*. The converse is almost analogous – the only difference being that for player 2 to win player 1's bid must be strictly smaller due to the tie-breaking rule employed, but this does not change the associated probability under the uniform distribution. Hence, we have proved the claim.

Thus, in this equilibrium, each bidder bids half his true valuation, and a player with the highest valuation wins the auction.

How about the second-price sealed-bid auction with incomplete information? This is more straightforward since bidding one's true valuation ($s_i(v_i) = v_i$ for all $v_i \in [0,1]$) is a strategy that weakly dominates every other strategy, for each player i. Hence, these strategies still form a (Bayesian) Nash equilibrium. See Problem 6.11.

6.6 Mixed Strategies and Incomplete Information

Consider the bimatrix game (cf. Chap. 3)

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,1 & 2,0 \\ 3,0 & 1,3 \end{pmatrix} \end{array},$$

which has a unique Nash equilibrium $((p^*, 1-p^*), (q^*, 1-q^*))$ with $p^* = 3/4$ and $q^* = 1/2$. The interpretation of mixed strategies and of a mixed strategy Nash equilibrium in particular is an old issue in the game-theoretic literature. One obvious interpretation is that a player actually plays according to the equilibrium probabilities. Although there is some empirical evidence that this may occur in reality (see [144]), this interpretation may not be entirely convincing, in particular since in a mixed strategy Nash equilibrium a player is indifferent between all pure strategies played with positive probability in equilibrium (see Problem 3.8). An alternative interpretation – also mentioned in Sect. 3.1 – is that a mixed strategy of a player represents the belief(s) of the other player(s) about the strategic choice of that player. For instance, in the above equilibrium, player 2 believes that player 1 plays T with probability 3/4. The drawback of this interpretation is that these beliefs are *subjective*, and it is not explained how they are formed. Harsanyi [51] proposed a way to obtain a mixed strategy Nash equilibrium as the limit of pure (Bayesian) Nash equilibria in games obtained by adding some *objective* uncertainty about the payoffs. In this way, the strategic uncertainty of players as expressed by their beliefs is replaced by the objective uncertainty of a chance move.

In the above example, suppose that the payoff to player 1 from (T,L) is the uncertain amount $2+\alpha$ and the payoff to player 2 from (B,R) is the uncertain amount $3+\beta$. Assume that both α and β are (independently) drawn from a uniform distribution over the interval $[0,x]$, where $x>0$. Moreover, player 1 learns the true value of α and player 2 learns the true value of β, and all this is common knowledge among the players. In terms of types, player 1 knows his type α and player 2 knows his type β. The new payoffs are given by

$$\begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} \begin{array}{cc} L & \quad R \end{array} \\ \begin{pmatrix} 2+\alpha,1 & 2,0 \\ 3,0 & 1,3+\beta \end{pmatrix} \end{array} .$$

A (pure) strategy of a player assigns an action to each of his types. Hence, for player 1 it is a map $s_1 : [0,x] \to \{T,B\}$ and for player 2 it is a map $s_2 : [0,x] \to \{L,R\}$.

To find an equilibrium of this incomplete information game, suppose that player 2 has the following rather simple strategy: play L if β is small and R if β is large. Specifically, let $b \in [0,x]$ such that each type $\beta \le b$ plays L and each type $\beta > b$ plays R. Call this strategy $s_2(b)$. What is player 1's best reply against $s_2(b)$? Suppose the type of player 1 is α. If player 1 plays T, then his expected payoff is equal to $2+\alpha$ times the probability that player 2 plays L plus 2 times the probability that player 2 plays R. The probability that player 2 plays L, given the strategy $s_2(b)$, is equal to the probability that β is at most equal to b, and this is equal to b/x since β is uniformly distributed over $[0,x]$. Hence, the expected payoff to player 1 from playing T is

$$(2+\alpha)\cdot\frac{b}{x}+2(1-\frac{b}{x}) = 2+\alpha\cdot\frac{b}{x} .$$

Similarly, the expected payoff to player 1 from playing B is

$$3\cdot\frac{b}{x}+1(1-\frac{b}{x}) = 1+2\cdot\frac{b}{x} .$$

From this, it easily follows that T is at least as good as B if $\alpha \ge (2b-x)/b$. Hence, the following strategy of player 1 is a best reply against the assumed strategy of player 2: play T if $\alpha \geq a$ and play B if $\alpha < a$, where $a = (2b - x)/b$. Call this strategy $s_1(a)$.

For the converse, assume that player 1 plays $s_1(a)$. To find player 2's best reply against $s_1(a)$ we proceed similarly as above. If type β of player 2 plays L then the expected payoff is 1 times the probability that player 1 plays T, hence 1 times $(x-a)/x$. If type β of player 2 plays R then his expected payoff is equal to $3+\beta$ times the probability that player 1 plays B, hence $(3+\beta)a/x$. So L is at least as good as R if $\beta \leq (x-4a)/a$. Hence, player 2's best reply against $s_1(a)$ is the strategy $s_2(b)$ with $b = (x-4a)/a$.

Summarizing these arguments, we have that $(s_1(a), s_2(b))$ is a Nash equilibrium for

$$a = (2b - x)/b, \quad b = (x-4a)/a\,.$$

Solving these two equations simultaneously for solutions $a, b \in [0, x]$ yields:

$$a = (1/4)(x + 4 - \sqrt{x^2+16}), \quad b = (1/2)(x - 4 + \sqrt{x^2+16})\,.$$

In this equilibrium, the a priori probability that player 1 will play T, that is, the probability of playing T before he learns his type, is equal to $(x-a)/x$, hence to $(\sqrt{x^2+16} + 3x - 4)/4x$. Similarly, the a priori probability that player 2 plays L is equal to b/x, hence to $(x - 4 + \sqrt{x^2+16})/2x$. What happens to these probabilities as the amount of uncertainty decreases, i.e., for x approaching 0? For player 1,

$$\lim_{x\to 0} \frac{\sqrt{x^2+16} + 3x - 4}{4x} = \lim_{x\to 0} \frac{x/\sqrt{x^2+16} + 3}{4} = \frac{3}{4}\,,$$

where the first identity follows from l'Hôpital's rule. Similarly for player 2:

$$\lim_{x\to 0} \frac{x - 4 + \sqrt{x^2+16}}{2x} = \lim_{x\to 0} \frac{1 + x/\sqrt{x^2+16}}{2} = \frac{1}{2}\,.$$

In other words, these probabilities converge to the mixed strategy Nash equilibrium of the original game.

6.7 Sequential Bargaining

In its simplest version, the bargaining problem involves two parties who have to agree on one alternative within a set of feasible alternatives. If they fail to reach an agreement, a specific 'disagreement' alternative is implemented. In the game-theoretic literature on bargaining there are two main strands, namely the cooperative, axiomatic approach as initiated by Nash in [90], and the noncooperative, strategic approach, with Nash [92] and Rubinstein [110] as seminal articles. These two approaches are not entirely disconnected and, in fact, there is a close relationship between the so-called Nash bargaining solution as proposed in [90] and the

Rubinstein subgame perfect equilibrium outcome in [110]. In this section the focus is on the strategic approach, but in Sect. 6.7.2 we also mention the connection with the Nash bargaining solution. For an introduction to the axiomatic bargaining approach see Sect. 10.1.

6.7.1 Finite Horizon Bargaining

Consider the example in Sect. 1.3.5, where two players bargain over the division of one unit of a perfectly divisible good, e.g., one liter of wine. If they do not reach an agreement, we assume that no one gets anything. To keep the problem as simple as possible, assume that the preferences of the players are represented by $u_1(\alpha) = u_2(\alpha) = \alpha$ for every $\alpha \in [0,1]$. That is, obtaining an amount α of the good has utility α for each player. Observe that the picture of the feasible set in Sect. 1.3.5 would become a triangle in this case.

To model the bargaining process we consider the following alternating offers procedure. There are $T \in \mathbb{N}$ rounds. In round $t = 0$ player 1 makes a proposal, say $(\alpha, 1-\alpha)$, where $\alpha \in [0,1]$, meaning that he claims an amount α for himself, so that player 2 obtains $1-\alpha$. Player 2 can either accept this proposal, implying that the proposal is implemented and the game is over, or reject the proposal. In the latter case the next round $t = 1$ starts, and the first round is repeated with the roles of the players interchanged: player 2 makes the proposal and player 1 accepts or rejects it. If player 1 accepts the proposal then it is implemented and the game is over; if player 1 rejects the proposal then round $t = 2$ starts, and the roles of the players are interchanged again. So at even moments, player 1 proposes; at odd moments, player 2 proposes. The last possible round is round T: if this round is reached, then the disagreement alternative $(0,0)$ is implemented.

As assumed above, receiving an amount α at time (round) t has a utility of α for each player, but this is the utility at time t. We assume that utilities are discounted over time, reflecting the fact that receiving the same amount earlier is more valuable. Specifically, there is a common discount factor $0 < \delta < 1$, such that receiving α at time t has utility $\delta^t \alpha$ at time 0.

In Fig. 6.5 this bargaining procedure is represented as a game in extensive form. Here, we assume that T is odd.

We look for a subgame perfect equilibrium of this game, which can be found by backward induction. Note that subgames start at each decision node of a player, and that, except at player 1's decision node at $t = 0$, there are infinitely many subgames starting at each decision node since there are infinitely many possible proposals and therefore infinitely many possible paths leading to a decision node.

To start the analysis, at the final decision node, player 2 accepts if $\alpha < 1$ and is indifferent between acceptance and rejection if $\alpha = 1$. In the subgame starting at round $T-1$ with a proposal of player 1, the only equilibrium therefore is for player 1 to propose $\alpha = 1$ and for player 2 to accept any proposal: if player 2 would reject $\alpha = 1$ then player 1 could improve by proposing $0 < \alpha < 1$, and given that player 2

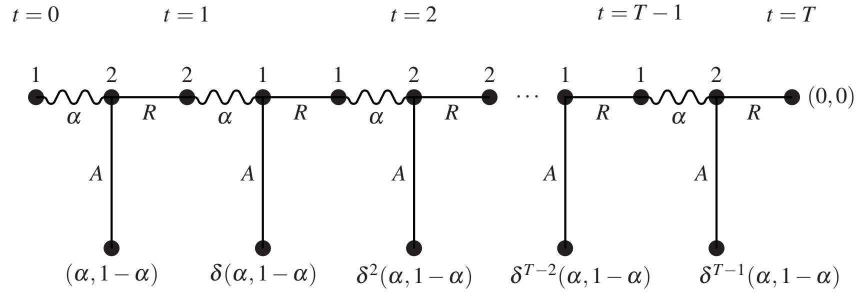

Fig. 6.5 The extensive form representation of the finite horizon bargaining procedure. The number of rounds T is odd, α denotes the proposed amount for player 1, A is acceptance and R is rejection

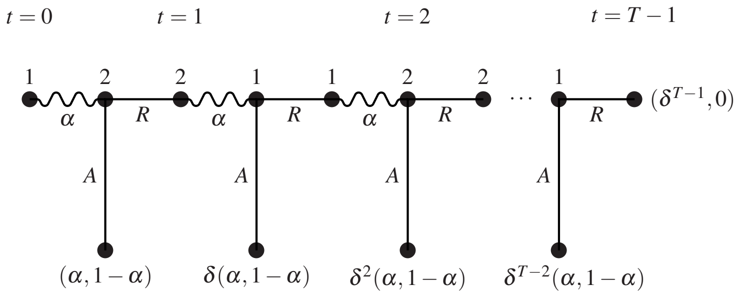

Fig. 6.6 The game of Fig. 6.5 reduced by replacing rounds $T-1$ and T by the equilibrium payoffs of the associated subgames

accepts any proposal, $\alpha = 1$ is optimal for player 1. Hence, we can replace the part of the game from round $T-1$ on by the pair of payoffs $(\delta^{T-1}, 0)$, as in Fig. 6.6.

Similarly, in this reduced game, in the complete subgame starting at round $T-2$ the only backward induction equilibrium is player 2 proposing $\alpha = \delta$ and player 1 accepting this proposal or any higher α and rejecting any lower α, since player 1 can always obtain $\delta^{T-1} = \delta^{T-2}\delta$ by rejecting player 2's proposal. Hence, we can replace this whole subgame by the pair of payoffs $(\delta^{T-1}, \delta^{T-2}(1-\delta))$. Continuing this line of reasoning, in the subgame starting at round $T-3$, player 1 proposes $\alpha = 1-\delta(1-\delta)$, which will be accepted by player 2. This results in the payoffs $(\delta^{T-3}(1-\delta(1-\delta)), \delta^{T-2}(1-\delta))$. This can be written as $\left(\delta^{T-3}(1-\delta+\delta^2), \delta^{T-3}(\delta-\delta^2)\right)$. By backtracking all the way to round 0 (see Table 6.1), we find that player 1 proposes $1-\delta+\delta^2-\cdots+\delta^{T-1}$ and player 2 accepts this proposal, resulting in the payoffs $1-\delta+\delta^2-\cdots+\delta^{T-1}$ for player 1 and $\delta-\delta^2+\cdots-\delta^{T-1}$ for player 2. This is the subgame perfect equilibrium *outcome* of the game and the associated payoffs. This outcome is the path of play, induced by the following subgame perfect *equilibrium*:

- At even rounds t, player 1 proposes $\alpha = 1-\delta+\cdots+\delta^{T-1-t}$ and player 2 accepts this proposal or any smaller α, and rejects any larger α.

Table 6.1 The proposals made in the subgame perfect equilibrium

Round	Proposer	Share for player 1	Share for player 2
T		0	0
$T-1$	1	1	0
$T-2$	2	δ	$1-\delta$
$T-3$	1	$1-\delta+\delta^2$	$\delta-\delta^2$
$T-4$	2	$\delta-\delta^2+\delta^3$	$1-\delta+\delta^2-\delta^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	1	$1-\delta+\delta^2-\cdots+\delta^{T-1}$	$\delta-\delta^2+\cdots-\delta^{T-1}$

- At odd rounds t, player 2 proposes $\alpha = \delta - \delta^2 + \cdots + \delta^{T-1-t}$ and player 1 accepts this proposal or any larger α, and rejects any smaller α.

In Problem 6.15 some variations on this finite horizon bargaining game are discussed.

6.7.2 Infinite Horizon Bargaining

In this subsection we consider the same bargaining problem as in the previous subsection, but now we assume $T = \infty$: the number of rounds may potentially be infinite. If no agreement is ever reached, then no player obtains anything. This game, like the finite horizon game, has many Nash equilibria: see Problem 6.15(f).

One way to analyze the game is to consider the finite horizon case and take the limit as T approaches infinity: see Problem 6.15(e). In fact, the resulting distribution is the uniquely possible outcome of a subgame perfect equilibrium, as can be seen by comparing the answer to Problem 6.15(e) with the result presented below. Of course, this claim is not proved by just taking the limit.

Note that a subgame perfect equilibrium cannot be obtained by backward induction, since the game has no final decision nodes. Here, we will just describe a pair of strategies and show that they are a subgame perfect equilibrium of the game. For a proof that the associated outcome is the unique outcome resulting in any subgame perfect equilibrium see [110].

Let $\mathbf{x}^* = (x_1^*, x_2^*)$ and $\mathbf{y}^* = (y_1^*, y_2^*)$ such that $x_1^*, x_2^*, y_1^*, y_2^* \geq 0$, $x_1^* + x_2^* = y_1^* + y_2^* = 1$, and moreover

$$x_2^* = \delta y_2^*, \quad y_1^* = \delta x_1^* \,. \tag{6.6}$$

It is not difficult to verify that $\mathbf{x}^* = (1/(1+\delta), \delta/(1+\delta))$ and $y^* = (\delta/(1+\delta), 1/(1+\delta))$. Consider the following strategies for players 1 and 2, respectively:

(σ_1^*) At $t = 0, 2, 4, \ldots$ propose $\mathbf{x}^*$; at $t = 1, 3, 5, \ldots$ accept a proposal $\mathbf{z} = (z_1, z_2)$ of player 2 if and only if $z_1 \geq \delta x_1^*$.

(σ_2^*) At $t = 1,3,5,\ldots$ propose $\mathbf{y}^*$; at $t = 0,2,4,\ldots$ accept a proposal $\mathbf{z} = (z_1, z_2)$ of player 1 if and only if $z_2 \geq \delta y_2^*$.

These strategies are stationary: players always make the same proposals. Moreover, a player accepts any proposal that offers him at least the discounted value of his own demand. According to (6.6), player 2 accepts the proposal $\mathbf{y}^*$ and player 1 accepts the proposal $\mathbf{x}^*$. Hence, play of the strategy pair (σ_1^*, σ_2^*) results in player 1's proposal $\mathbf{x}^* = (1/(1+\delta), \delta/(1+\delta))$ being accepted at round 0, so that these are also the payoffs. We are going to show that (σ_1^*, σ_2^*) is a subgame perfect equilibrium of the game.

To show this, note that there are two kinds of subgames: subgames where a player has to make a proposal; and subgames where a proposal is on the table and a player has to choose between accepting and rejecting the proposal.

For the first kind of subgame we may without loss of generality consider the entire game, i.e., the game starting at $t = 0$. We have to show that (σ_1^*, σ_2^*) is a Nash equilibrium in this game. First, suppose that player 1 plays σ_1^*. By accepting player 1's proposal at $t = 0$, player 2 has a payoff of $\delta/(1+\delta)$. By rejecting this proposal, the most player 2 can obtain against σ_1^* is $\delta/(1+\delta)$, by proposing $\mathbf{y}^*$ in round $t = 1$. Proposals $\mathbf{z}$ with $z_2 > y_2^*$ and thus $z_1 < y_1^*$ are rejected by player 1. Hence, σ_2^* is a best reply against σ_1^*. Similarly, if player 2 plays σ_2^*, then the best player 1 can obtain is x_1^* at $t = 0$ with payoff $1/(1+\delta)$, since player 2 will reject any proposal that gives player 1 more than this, and also does not offer more.

For the second kind of subgame, we may without loss of generality take $t = 0$ and assume that player 1 has made some proposal, say $\mathbf{z} = (z_1, z_2)$ – the argument for t odd, when there is a proposal of player 2 on the table, is analogous. First, suppose that in this subgame player 1 plays σ_1^*. If $z_2 \geq \delta y_2^*$, then accepting this proposal yields player 2 a payoff of $z_2 \geq \delta y_2^* = \delta/(1+\delta)$. By rejecting, the most player 2 can obtain against σ_1^* is $\delta/(1+\delta)$ by proposing $\mathbf{y}^*$ at $t = 1$, which will be accepted by player 1. If, on the other hand, $z_2 < \delta y_2^*$, then player 2 can indeed better reject $\mathbf{z}$ and obtain $\delta/(1+\delta)$ by proposing $\mathbf{y}^*$ at $t = 1$. Hence, σ_2^* is a best reply against σ_1^*. Next, suppose player 2 plays σ_2^*. Then $\mathbf{z}$ is accepted if $z_2 \geq \delta y_2^*$ and rejected otherwise. In the first case it does not matter how player 1 replies, and in the second case the game starts again with player 2 as the first proposer, and by an argument analogous to the argument in the previous paragraph, player 1's best reply is σ_1^*.

We have, thus, shown that (σ_1^*, σ_2^*) is a subgame perfect equilibrium of the game. In Problem 6.16 some variations on this game are discussed.

Note, finally, that nothing in the whole analysis changes if we view the number δ not as a discount factor but as the probability that the game continues to the next round. Specifically, if a proposal is rejected, then assume that with probability $1-\delta$ the game stops and each player receives 0, and with probability δ the game continues in the usual way. Under this alternative interpretation, the game ends with probability 1 (Problem 6.16(e)), which makes the infinite horizon assumption more acceptable.

Problems

6.1. *Cournot with Asymmetric Costs*

Consider the Cournot model of Sect. 6.2.1 but now assume that the firms have different marginal costs $c_1, c_2 \geq 0$. Compute the Nash equilibrium: distinguish different cases with respect to the values of c_1 and c_2.

6.2. *Cournot Oligopoly*

Consider the Cournot model of Sect. 6.2.1 but now assume that there are n firms instead of two. Each firm $i = 1, 2, \ldots, n$ offers $q_i \geq 0$ and the market price is

$$P(q_1, q_2, \ldots, q_n) = \max\{a - q_1 - q_2 - \cdots - q_n, 0\} .$$

Each firm still has marginal cost c with $a > c \geq 0$ and no fixed costs.

(a) Set up the game associated with this situation.

(b) Derive the reaction functions of the players.

(c) Derive a Nash equilibrium of the game by trying equal quantities offered. What happens if the number of firms becomes large?

(d) Show that the Nash equilibrium found in (c) is unique.

6.3. *Quantity Competition with Heterogenous Goods*

Suppose, in the Cournot model, that the firms produce heterogenous goods, which have different market prices. Specifically, suppose that these market prices are given by

$$p_1 = \max\{5 - 3q_1 - 2q_2, 0\}, \quad p_2 = \max\{4.5 - 1.5q_1 - 3q_2, 0\} .$$

The firms still compete in quantities.

(a) Formulate the game corresponding to this situation. In particular, write down the payoff functions.

(b) Solve for the reaction functions and the Nash equilibrium of this game. Also compute the corresponding prices.

(c) Compute the quantities at which joint profit is maximized. Also compute the corresponding prices.

In (d)–(f), we assume that the firms compete in prices.

(d) Derive the demands for q_1 and q_2 as a function of the prices. Set up the associated game where the prices p_1 and p_2 are now the strategic variables.

(e) Solve for the reaction functions and the Nash equilibrium of this game. Also compute the corresponding quantities.

(f) Compute the prices at which joint profit is maximized. Also compute the corresponding quantities.

(g) Compare the results found under (b) and (c) with those under (e) and (f).[6]

6.4. *A Numerical Example of Cournot with Incomplete Information*

Redo the model of Sect. 6.2.2 for the following values of the parameters: $a = 1$, $c = 0$, $\vartheta = 1/2$, $c_L = 0$, $c_H = 1/4$. Compute the Nash equilibrium and compare with what was found in the text. Also compare with the complete information case by using the answer to Problem 6.1.

6.5. *Cournot with Two-Sided Incomplete Information*

Consider the Cournot game of incomplete information of Sect. 6.2.2 and assume that also firm 1 can have high costs or low costs, say c_h with probability π and c_l with probability $1-\pi$. Set up the associated game and compute the (four) reaction functions. (Assume that the parameters of the game are such that the Nash equilibrium quantities are positive and the relevant parts of the reaction functions can be found by differentiating the payoff functions (i.e., no corner solutions).) How can the Nash equilibrium be computed? (You do not actually have to compute it explicitly.)

6.6. *Incomplete Information about Demand*

Consider the Cournot game of incomplete information of Sect. 6.2.2 but now assume that the incomplete information is not about the cost of firm 2 but about market demand. Specifically, assume that the number a can be either high, a_H, with probability ϑ, or low, a_L, with probability $1-\vartheta$. Firm 2 knows the value for sure but firm 1 only knows these probabilities. Set up the game and compute the reaction functions and the Nash equilibrium (make appropriate assumptions on the parameters a_H, a_L, ϑ, and c to avoid corner solutions).

6.7. *Variations on Two-Person Bertrand*

(a) Assume that the two firms in the Bertrand model of Sect. 6.3 have different marginal costs, say $c_1 \le c_2 < a$. Derive the best reply functions and find the Nash–Bertrand equilibrium or equilibria, if any.

(b) Reconsider the questions in (a) for the case where prices and costs are restricted to integer values, i.e., $p_1, p_2, c_1, c_2 \in \{0, 1, 2, \ldots\}$. Distinguish between the cases $c_1 = c_2$ and $c_1 < c_2$. (This reflects the assumption that there is a smallest monetary unit.)

6.8. *Bertrand with More Than Two Firms*

Suppose that there are $n > 2$ firms in the Bertrand model of Sect. 6.3. Assume again that all firms have equal marginal cost c, and that the firm with the lowest price gets

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[6] The goods in the model of this problem are *strategic substitutes*. In *duopoly models* like this the distinction between strategic substitutes and *strategic complements* is important for the differences between quantity and price competition. See, e.g., [137].

the whole market. In case of a tie, each firm with the lowest price gets an equal share of the market. Set up the associated game and find all its Nash equilibria.

6.9. *Variations on Stackelberg*

(a) Suppose, in the model in Sect. 6.4, that the firms have different marginal costs c_1 and c_2 (cf. Problem 6.1). Compute the Stackelberg equilibrium and outcome with firm 1 as a leader and with firm 2 as a leader.

(b) Give a logical argument why the payoff of the leader in a Stackelberg equilibrium is always at least as high as his payoff in the Cournot equilibrium. Can you generalize this to arbitrary games?

(c) Consider the situation in Sect. 6.4, but now assume that there are n firms, firm 1 moves first, firm 2 second, etc. Assume again perfect information, and compute the subgame perfect equilibrium.

6.10. *First-Price Sealed-Bid Auction*

Consider the game associated with the first-price sealed-bid auction in Sect. 6.5.1.

(a) Show that $(b_1,\ldots,b_n) = (v_2,v_2,v_3,\ldots,v_n)$ is a Nash equilibrium in this game.

(b) Show that, in any Nash equilibrium of the game, a player with the highest valuation obtains the object. Exhibit at least two other Nash equilibria in this game, apart from the equilibrium in (a).

(c) Show that bidding one's true valuation as well as bidding higher than one's true valuation are weakly dominated strategies. Also show that bidding lower than one's true valuation is not weakly dominated. (Note: to show that a strategy is weakly dominated one needs to exhibit some other strategy that is always – that is, whatever the other players do – at least as good as the strategy under consideration and at least once – that is, for at least one strategy combination of the other players – strictly better.)

(d) Show that, in any Nash equilibrium of this game, at least one player plays a weakly dominated strategy.

6.11. *Second-Price Sealed-Bid Auction*

Consider the game associated with the second price sealed bid auction in Sect. 6.5.1.

(a) Formulate the payoff functions in this game.

(b) Show that $(b_1,\ldots,b_n) = (v_1,\ldots,v_n)$ is a Nash equilibrium in this game.

(c) Show, for each player, that bidding one's true valuation weakly dominates any other action (show that this holds even if each player only knows his own valuation).

(d) Show that $(b_1,\ldots,b_n) = (v_2,v_1,0,\ldots,0)$ is a Nash equilibrium in this game. What about $(b_1,\ldots,b_n) = (v_1,0,0,\ldots,0)$?

(e) Determine *all* Nash equilibria in the game with two players ($n = 2$). (Hint: compute the best reply functions and make a diagram.)

6.12. *Third-Price Sealed-Bid Auction*

In the auction of Sect. 6.5.1, assume that there are at least three bidders and that the highest bidder wins and pays the third highest bid.

(a) Show that for any player i bidding v_i weakly dominates any lower bid but does not weakly dominate any higher bid.

(b) Show that the strategy combination in which each player i bids his true valuation v_i is in general not a Nash equilibrium.

(c) Find some Nash equilibria of this game.

6.13. *n-Player First-Price Sealed-Bid with Incomplete Information*

Consider the setting of Sect. 6.5.2 but now assume that the number of bidders/players is $n \geq 2$. Show that $(s_1^*, \ldots, s_n^*)$ with $s_i^*(v_i) = (1 - 1/n)v_i$ for every player i is a Nash equilibrium of this game. (Hence, for large n, each bidder almost bids his true valuation.)

6.14. *Mixed Strategies and Objective Uncertainty*

Consider the bimatrix game

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 4,1 & 1,3 \\ 1,2 & 3,0 \end{pmatrix} \end{array}.$$

(a) Compute the Nash equilibrium of this game.

(b) Add some uncertainty to the payoffs of this game and find a pure (Bayesian) Nash equilibrium of the resulting game of incomplete information, such that the induced a priori mixed strategies converge to the Nash equilibrium of the original game as the amount of uncertainty shrinks to 0.

6.15. *Variations on Finite Horizon Bargaining*

(a) Adapt the arguments and the results of Sect. 6.7.1 for the case where T is even and the case where player 2 proposes at even rounds.

(b) Let $T = 3$ in Sect. 6.7.1 and suppose that the players have different discount factors δ_1 and δ_2. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.

(c) Consider again the model of Sect. 6.7.1, let $T = 3$, but now assume that the utility function of player 2 is $u_2(\alpha) = \sqrt{\alpha}$ for all $\alpha \in [0,1]$. Hence, the utility of receiving α at time t for player 2 is equal to $\delta^t \sqrt{\alpha}$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.

(d) Suppose, in the model of Sect. 6.7.1, that at time T the 'disagreement' distribution is $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 \leq 1$, rather than $(0,0)$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.

(e) In (d), compute the limits of the equilibrium shares for T going to infinity. Do these limits depend on $\mathbf{s}$?

(f) Show, in the game in Sect. 6.7.1, that subgame perfection really has a bite. Specifically, for every $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$, exhibit a Nash equilibrium of the game in Fig. 6.5 resulting in the distribution s.

6.16. *Variations on Infinite Horizon Bargaining*

(a) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when the players have different discount factors δ_1 and δ_2.

(b) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when player 2 proposes at even rounds and player 1 at odd rounds.

(c) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when the 'disagreement' distribution is $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 \leq 1$, rather than $(0, 0)$, in case the game never stops.

(d) Consider the game in Sect. 6.7.2, but now assume that the utility function of player 2 is $u_2(\alpha) = \sqrt{\alpha}$ for all $\alpha \in [0, 1]$. Hence, the utility of receiving α at time t for player 2 is equal to $\delta^t \sqrt{\alpha}$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome. (Hint: first determine for this situation the values for $\mathbf{x}^*$ and $\mathbf{y}^*$ analogous to (6.6).)

(e) Interpret, as at the end of Sect. 6.7.2, the discount factor as the probability that the game continues to the next round. Show that the game ends with probability equal to 1.

6.17. *A Principal–Agent Game*

There are two players: a worker (the agent) and an employer (the principal). The worker has three choices: either reject the contract offered to him by the employer, or accept this contract and exert high effort, or accept the contract and exert low effort. If the worker rejects the contract then the game ends with a payoff of zero to the employer and a payoff of 2 to the worker (his reservation payoff). If the worker accepts the contract he works for the employer: if he exerts high effort the revenues for the employer will be 10 with probability 0.8 and 6 with probability 0.2; if he exerts low effort then these revenues will be 10 with probability 0.2 and 6 with probability 0.8. The employer can only observe the revenues but not the effort exerted by the worker: in the contract he specifies a high wage w_H in case the revenues equal 10 and a low wage w_L in case the revenues are equal to 6. These wages are the respective choices of the employer. The final payoff to the employer if the worker accepts the contract will be equal to revenues minus wage. The worker will receive his wage; his payoff equals this wage minus 3 if he exerts high effort and this wage minus 0 if he exerts low effort.

(a) Set up the extensive form of this game. Does this game have incomplete or imperfect information? What is the associated strategic form?

(b) Determine the subgame perfect equilibrium or equilibria of the game.

6.18. *The Market for Lemons*

(Cf. [1].) A buyer wants to buy a car but does not know whether the particular car he is interested in has good or bad quality (a lemon is a car of bad quality). About half of the market consists of good quality cars. The buyer offers a price p to the seller, who is informed about the quality of the car; the seller may then either accept of reject this price. If he rejects, there is no sale and the payoff will be 0 to both. If he accepts, the payoff to the seller will be the price minus the value of the car, and to the buyer it will be the value of the car minus the price. A good quality car has a value of 15,000, a lemon has a value of 5,000.

(a) Set up the extensive as well as strategic form of this game.

(b) Compute the subgame perfect equilibrium or equilibria of this game.

6.19. *Corporate Investment and Capital Structure*

(Cf. [45], p. 205.) Consider an entrepreneur who has started a company but needs outside financing to undertake an attractive new project. The entrepreneur has private information about the profitability of the existing company, but the payoff of the new project cannot be disentangled from the payoff of the existing company – all that can be observed is the aggregate profit of the firm. Suppose the entrepreneur offers a potential investor an equity stake in the firm in exchange for the necessary financing. Under what circumstances will the new project be undertaken, and what will the equity stake be? In order to model this as a game, assume that the profit of the existing company can be either high or low: $\pi = L$ or $\pi = H$, where $H > L > 0$. Suppose that the required investment for the new project is I, the payoff will be R, the potential investor's alternative rate of return is r, with $R > I(1+r)$. The game is played as follows:

1. Nature determines the profit of the existing company. The probability that $\pi = L$ is p.
2. The entrepreneur learns π and then offers the potential investor an equity stake s, where $0 \leq s \leq 1$.
3. The investor observes s (but not π) and then decides either to accept or to reject the offer.
4. If the investor rejects then the investor's payoff is $I(1+r) - I$ and the entrepreneur's payoff is π. If he accepts his payoff is $s(\pi + R) - I$ and the entrepreneur's is $(1-s)(\pi + R)$.

(a) Set up the extensive form and the strategic form of this signaling game.

(b) Compute the weak sequential equilibrium or equilibria, if any.

6.20. *A Poker Game*

(Cf. [134].) Consider the following game. There are two players, I and II. Player I deals II one of three cards – Ace, King, or Queen – at random and face down. II looks at the card. If it is an Ace, II must say 'Ace', if a King he can say 'King' or 'Ace', and if a Queen he can say 'Queen' or 'Ace'. If II says 'Ace' player I can either believe him and give him \$1 or ask him to show his card. If it is an Ace, I must pay II \$2, but if it is not, II pays I \$2. If II says 'King' neither side looses anything, but if he says 'Queen' II must pay player I \$1.

(a) Set up the extensive form and the strategic form of this zerosum game.

(b) Determine its value and optimal strategies (cf. Chap. 2).

6.21. *A Hotelling Location Problem*

Consider n players each choosing a location in the interval $[0,1]$. One may think of n shops choosing locations in a street, n firms choosing product characteristics on a continuous scale from 0 to 1, or n political parties choosing positions on the ideological scale. We assume that customers or voters are uniformly distributed over the interval, with a total of 1. The customers go to (voters vote for) the nearest shop (candidate). E.g., if $n = 2$ and the chosen positions are $x_1 = 0.2$ and $x_2 = 0.6$, then 1 obtains 0.4 and 2 obtains 0.6 customers (votes). In case two or more players occupy the same position they share the customers or voters for that position equally.

In the first scenario, the players care only about winning or loosing in terms of the number of customers or votes. This scenario may be prominent for presidential elections, as an example. For each player the best alternative is to be the unique winner, the second best alternative is to be one of the winners, and the worst alternative is not to win. For this scenario, answer questions (a) and (b).

(a) Show that there is a unique Nash equilibrium for $n = 2$.

(b) Exhibit a Nash equilibrium for $n = 3$.

In the second scenario, the payoffs of the players are given by the total numbers of customers (or voters) they acquire. For this scenario, answer questions (c) and (d).

(c) Show that there is a unique Nash equilibrium for $n = 2$.

(d) Is there a Nash equilibrium for $n = 3$? How about $n = 4$?

6.22. *Median Voting*

Of the n persons in a room, each person has an ideal room temperature, and the further away (lower or higher) the room temperature is from the ideal, the worse it is for this person. Specifically, if the ideal temperature of person i is t_i and the room temperature is x, then person i's utility is equal to $-|x - t_i|$. In order to find a compromise, the janitor asks each person to propose a room temperature, and based on the proposed temperatures a compromise is determined. Let the ideal temperatures be given by $t_1 \leq t_2 \leq \cdots \leq t_n$. The proposed temperatures are not necessarily equal to the ideal temperatures. Only temperatures (proposed and ideal) in the interval 0°C–30°C are possible.

(a) Suppose the janitor announces that he will take the average of the proposed temperatures as the compromise temperature. Formulate this situation as an n person game, that is, give the strategy sets of the players and the payoff functions. Does this game have a Nash equilibrium?

(b) Suppose n is odd, and suppose the janitor announces that he will take the median of the proposed temperatures as the compromise temperature. Formulate this situation as an n person game, that is, give the strategy sets of the players and the payoff functions. Show that, for each player, proposing his ideal temperature weakly dominates any other strategy: thus, in particular, $(t_1, \ldots, t_n)$ is a Nash equilibrium of this game. Does the game have any other Nash equilibria?

6.23. *The Uniform Rule*

An amount $M \geq 0$ of a good (labor, green pea soup,...) is to be distributed completely among n persons. Each person i considers an amount $t_i \geq 0$ as the ideal amount, and the further away the allocated amount is from this ideal, the worse it is. Specifically, if the amount allocated to person i is x_i, then person i's utility is equal to $-|x - t_i|$. In order to find a compromise, each person is asked to report an amount, and based on the reported amounts a compromise is determined. Let the ideal amounts be given by $t_1 \leq t_2 \leq \cdots \leq t_n$. The reported amounts are not necessarily equal to the ideal amounts.

(a) Suppose M is distributed proportionally to the reported amounts, that is, if the reported amounts are $(r_1, \ldots, r_n)$, then person i receives $x_i = \left(r_i / \sum_{j=1}^n r_j\right) M$. (If all r_j are zero then take $x_i = M/n$.) Formulate this situation as a game. Does this game have a Nash equilibrium?

Consider the following division rule, called the *uniform* rule, see [129]. Let $(r_1, \ldots, r_n)$ denote the reported amounts. If $M \leq \sum_{j=1}^n r_j$, then each person i receives

$$x_i = \min\{r_i, \lambda\} ,$$

where λ is such that $\sum_{j=1}^n x_j = M$. If $M \geq \sum_{j=1}^n r_j$, then each person i receives

$$x_i = \max\{r_i, \lambda\} ,$$

where, again, λ is such that $\sum_{j=1}^n x_j = M$.

(b) Suppose that $n = 3$ and $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. Apply the uniform rule for $M = 4$, $M = 5$, $M = 5.5$, $M = 6$, $M = 6.5$, $M = 7$, $M = 8$, $M = 9$.

(c) Suppose, for the general case, that the uniform rule is used to distribute the amount M. Formulate this situation as a game. Show that reporting one's ideal amount weakly dominates any other strategy: thus, in particular, $(t_1, \ldots, t_n)$ is a Nash equilibrium of this game. Does the game have any other Nash equilibria?

6.24. *Reporting a Crime*

There are n individuals who witness a crime. Everybody would like the police to be called. If this happens, each individual derives satisfaction $v > 0$ from it. Calling the police has a cost of c, where $0 < c < v$. The police will come if at least one person calls. Hence, this is an n-person game in which each player chooses from $\{C,N\}$: C means 'call the police' and N means 'do not call the police'. The payoff to person i is 0 if nobody calls the police, $v-c$ if i (and perhaps others) call the police, and v if the police is called but not by person i.

(a) What are the Nash equilibria of this game in pure strategies? In particular, show that the game does not have a symmetric Nash equilibrium in pure strategies (a Nash equilibrium is symmetric if every player plays the same strategy).

(b) Compute the symmetric Nash equilibrium or equilibria in mixed strategies. (Hint: suppose, in such an equilibrium, every person plays C with probability $0 < p < 1$. Use the fact that each player must be indifferent between C and N.)

(c) For the Nash equilibrium/equilibria in (b), compute the probability of the crime being reported. What happens to this probability if n becomes large?

6.25. *Firm Concentration*

(From [145], p. 102) Consider a market with 10 firms. Simultaneously and independently, the firms choose between locating downtown and locating in the suburbs. The profit of each firm is influenced by the number of other firms that locate in the same area. Specifically, the profit of a firm that locates downtown is given by $5n - n^2 + 50$, where n denotes the number of firms that locate downtown. Similarly, the profit of a firm that locates in the suburbs is given by $48 - m$, where m denotes the number of firms that locate in the suburbs. In equilibrium how many firms locate in each region and what is the profit of each?

6.26. *Tragedy of the Commons*

(Cf. [47], and [45], p. 27) There are n farmers, who use a common piece of land to graze their goats. Each farmer i chooses a number of goats g_i – for simplicity we assume that goats are perfectly divisible. The value to a farmer of grazing a goat when the total number of goats is G, is equal to $v(G)$ per goat. We assume that there is a number $\bar{G}$ such that $v(G) > 0$ for $G < \bar{G}$ and $v(G) = 0$ for $G \geq \bar{G}$. Moreover, v is twice differentiable with $v'(G) < 0$ and $v''(G) < 0$ for $G < \bar{G}$. The payoff to farmer i if each farmer j chooses g_j, is equal to

$$g_i v(g_1 + \cdots + g_{i-1} + g_i + g_{i+1} + \cdots + g_n) - cg_i \,,$$

where $c \geq 0$ is the cost per goat.

(a) Interpret the conditions on the function v.

(b) Show that the total number of goats in a Nash equilibrium $(g_1^*, \ldots, g_n^*)$ of this game, $G^* = g_1^* + \cdots + g_n^*$, satisfies

$$v(G^*) + (1/n)G^* v'(G^*) - c = 0.$$

(c) The socially optimal number of goats G^{**} is obtained by maximizing $Gv(G) - cG$ over $G \geq 0$. Show that G^{**} satisfies

$$v(G^{**}) + G^{**} v'(G^{**}) - c = 0.$$

(d) Show that $G^* > G^{**}$. (Hence, in a Nash equilibrium too many goats are grazed.)

Chapter 7
Repeated Games

In the famous prisoners' dilemma game the bad (Pareto inferior) outcome, resulting from each player playing his dominant action, cannot be avoided in a Nash equilibrium or subgame perfect Nash equilibrium even if the game is repeated a finite number of times, cf. Problem 4.8(a)–(c). As we will see in this chapter, this bad outcome can be avoided if the game is repeated an infinite number of times. This, however, is going to have a price, namely the existence of a multitude of outcomes attainable in equilibrium. Such an *embarrassment of richness* is expressed by a so-called *Folk theorem*.

As was illustrated in Problem 4.8(d)–(g), also *finite* repetitions of a game may sometimes lead to outcomes that are better than (repeated) Nash equilibria of the original game. See also [9] and [37].

In this chapter we consider two-person *infinitely* repeated games and formulate Folk theorems both for subgame perfect and for Nash equilibrium. The approach is somewhat informal, and mainly based on examples. In Sect. 7.1 we consider subgame perfect equilibrium and in Sect. 7.2 we consider Nash equilibrium.

7.1 Subgame Perfect Equilibrium

7.1.1 The Prisoners' Dilemma

Consider the prisoners' dilemma game

$$G_p = \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & \quad D \end{array} \\ \begin{pmatrix} 50,50 & 30,60 \\ 60,30 & 40,40 \end{pmatrix} \end{array}.$$

(This is the 'marketing game' of Problem 3.1(d).) In G_p each player has a strictly dominant action, namely D, and (D,D) is the unique Nash equilibrium of the game, also if mixed strategies are allowed.

We assume now that G_p is played infinitely many times, at times $t = 0,1,2,\ldots$, and that after each play of G_p the players learn what has been played,[1] i.e., they learn which element of the set $\{(C,C), (C,D), (D,C), (D,D)\}$ has occurred.[2] These realizations induce an infinite stream of associated payoffs, and we assume that there is a common discount factor $0 < \delta < 1$ such that the final payoff to each player is the δ-discounted value of the infinite stream of payoffs. That is, player i ($i = 1,2$) obtains

$$\sum_{t=0}^{\infty} \delta_i^t \cdot (\text{payoff from } t\text{-th play of the stage game}),$$

where $\delta_i = \delta$ for $i = 1,2$. Here, the expression *stage game* is used for the one-shot game G_p in order to distinguish the one-shot game from the repeated game.

As always, a strategy of a player is a complete plan to play the game. This means that, at each moment t, this plan should prescribe an action of a player – a mixed or pure strategy in G_p – for each possible history of the game up to time t, that is, an action for each sequence of length t of elements from the set $\{(C,C),(C,D),(D,C),(D,D)\}$. Clearly, such a strategy can be quite complicated and the number of possible different strategies is enormous. We will be able, however, to restrict attention to quite simple strategies.

The infinite extensive form game just defined is denoted by $G_p^\infty(\delta)$. A natural solution concept for this game is the concept of subgame perfect (Nash) equilibrium. Each subgame in $G_p^\infty(\delta)$ is, basically, equal to the game $G_p^\infty(\delta)$ itself: the difference between two subgames is the difference between the two histories leading to those subgames. For instance, at $t = 6$, there are 4^6 possible histories of play and therefore there are 4^6 different subgames; each of these subgames, however, looks exactly like $G_p^\infty(\delta)$.

We will now exhibit a few subgame perfect equilibria of $G_p^\infty(\delta)$. First consider the simple strategy:

> D^∞: play D at each moment $t = 0,1,2,\ldots$, independent of the history of the game.

First observe that D^∞ is a well-defined strategy. If both players play D^∞ then the resulting payoff is

$$\sum_{t=0}^{\infty} 40\,\delta^t = 40/(1-\delta)$$

for each player. We claim that (D^∞, D^∞) is a subgame perfect equilibrium in $G_p^\infty(\delta)$. Consider any $t = 0,1,\ldots$ and any subgame starting at time t. Then (D^∞, D^∞) induces a Nash equilibrium in this subgame: given that player 2 always plays D, player 1 cannot do better than always playing D as well, and vice versa. Hence, (D^∞, D^∞) is

[1] In the marketing game, one can think of the game being played once per period – a week, month – each player observing in each period whether his opponent has advertised or not.

[2] Hence, a player does not learn the exact, possibly mixed, action of his opponent, but only its realization.

a subgame perfect equilibrium. In this subgame perfect equilibrium, the players just play the Nash equilibrium of the stage game at every time t.

We next exhibit another subgame perfect equilibrium. Consider the following strategy:

> $Tr(C)$: at $t = 0$ and at every time t such that in the past only (C,C) has occurred in the stage game: play C. Otherwise, play D.

Strategy $Tr(C)$ is a so-called *trigger strategy*. A player playing this strategy starts playing C and keeps on playing C as long as both players have only played C in the past; after any deviation from this, however, this player plays D and keeps on playing D forever. Again, $Tr(C)$ is a well-defined strategy, and if both players play $Tr(C)$, then each player obtains the payoff

$$\sum_{t=0}^{\infty} 50\,\delta^t = 50/(1-\delta).$$

Is $(Tr(C), Tr(C))$ also a subgame perfect equilibrium? The answer is a qualified yes: if δ is large enough, then it is. The crux of the argument is as follows. At each stage of the game, a player has an incentive to deviate from C and play his dominant action D, thereby obtaining a momentary gain of 10. Deviating, however, triggers eternal 'punishment' by the other player, who is going to play D forever. The best reply to this punishment is to play D as well, entailing an eternal loss of 10 from the next moment on. So the discounted value of this loss is equal to $10\delta/(1-\delta)$, and to keep a player from deviating this loss should be at least as large as the momentary gain of 10. This is the case if and only if $\delta \geq 1/2$.

More formally, we can distinguish two kinds of subgames that are relevant for the strategy combination $(Tr(C), Tr(C))$. One kind are those subgames where not always (C,C) has been played in the past. In such a subgame, $Tr(C)$ tells a player to play D forever, and therefore the best reply of the other player is to do so as well, i.e., to play according to $Tr(C)$. So in this kind of subgame, $(Tr(C), Tr(C))$ is a Nash equilibrium. In the other kind of subgame, no deviation has occurred so far: in the past always (C,C) has been played. Consider this subgame at time T and suppose that player 2 plays $Tr(C)$. If player 1 plays $Tr(C)$ as well, his payoff is equal to

$$\sum_{t=0}^{T-1} 50\,\delta^t + \sum_{t=T}^{\infty} 50\,\delta^t.$$

If, instead, he deviates at time T to D, he obtains

$$\sum_{t=0}^{T-1} 50\,\delta^t + 60\,\delta^T + \sum_{t=T+1}^{\infty} 40\,\delta^t.$$

Hence, to avoid deviation (and make $Tr(C)$ player 1's best reply in the subgame) we need that the first payoff is at least as high as the second one, resulting in the inequality

$$50\,\delta^T/(1-\delta) \geq 60\,\delta^T + 40\,\delta^{T+1}/(1-\delta)$$

or, equivalently, $\delta \geq 1/2$ – as found before. We conclude that for every $\delta \geq 1/2$, $(Tr(C), Tr(C))$ is a subgame perfect equilibrium of the game $G_p^\infty(\delta)$. The existence of this equilibrium is a major reason to study infinitely repeated games. In popular terms, it shows that cooperation is sustainable if deviations can be credibly punished, which is the case if the future is sufficiently important (i.e., δ large enough).

To exhibit a subgame perfect equilibrium different from (D^∞, D^∞) and $(Tr(C), Tr(C))$, consider the following strategies:

Tr_1: As long as the sequence (C,D), (D,C), (C,D), (D,C), (C,D), (D,C), $\ldots$ has occurred in the past from time 0 on, play C at $t \in \{0,2,4,6,\ldots\}$; play D at $t \in \{1,3,5,7,\ldots\}$. Otherwise, play D.

Tr_2: As long as the sequence (C,D), (D,C), (C,D), (D,C), (C,D), (D,C), $\ldots$ has occurred in the past from time 0 on, play D at $t \in \{0,2,4,6,\ldots\}$; play C at $t \in \{1,3,5,7,\ldots\}$. Otherwise, play D.

Note that these are again 'trigger strategies': the players 'tacitly' agree on a certain sequence of play, but revert to playing D forever after a deviation. If player 1 plays Tr_1 and player 2 plays Tr_2, then the sequence (C,D), (D,C), (C,D), (D,C), ..., results. To see why (Tr_1, Tr_2) might be a subgame perfect equilibrium, note that on average a player obtains 45 per stage, which is more than the 40 that would be obtained from deviating from this sequence and playing D forever. More precisely, suppose player 2 plays Tr_2 and suppose player 1 considers a deviation from Tr_1. It is optimal to deviate at an even moment, say at $t = 0$. This yields a momentary gain of 10 and a discounted future loss of

$$20(\delta^1 + \delta^3 + \delta^5 + \cdots) - 10(\delta^2 + \delta^4 + \delta^6 + \cdots).$$

To keep player 1 from deviating this loss should be at least as large as the gain of 10, hence

$$\frac{20\,\delta}{1-\delta^2} - \frac{10\,\delta^2}{1-\delta^2} \geq 10$$

which yields $\delta \geq 1/2$. We conclude (the formal argument is analogous to the one before) that for each $\delta \geq 1/2$, (Tr_1, Tr_2) is a subgame perfect equilibrium in $G_p^\infty(\delta)$.

More generally, by playing appropriate sequences of elements from the set of possible outcomes $\{(C,C),(C,D),(D,C),(D,D)\}$ of the stage game G_p, the players can on average reach any convex combination of the associated payoffs in the long run. That is, take any such combination

$$\alpha_1(50,50) + \alpha_2(30,60) + \alpha_3(60,30) + \alpha_4(40,40),$$

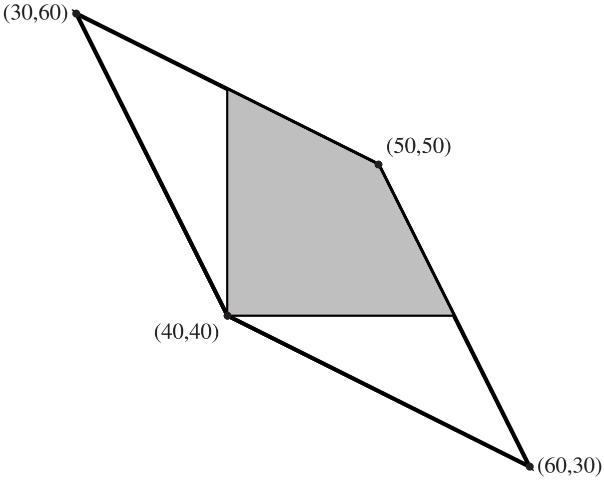

Fig. 7.1 For every payoff pair in the *shaded area* there is a δ large enough such that this payoff pair can be obtained as the limiting (long run) average in a subgame perfect equilibrium of $G_p^\infty(\delta)$

where $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$ for every $i = 1, \ldots, 4$, and $\sum_{i=1}^4 \alpha_i = 1$. By choosing a sequence of possible outcomes such that (C,C) occurs (in the long run) in a fraction α_1 of the stages, (C,D) in a fraction α_2, (D,C) in a fraction α_3, and (D,D) in a fraction α_4, then the above payoffs are reached as averages in the limit. As long as these limiting average payoffs exceed 40 for each player, associated trigger strategies can be formulated that lead to these payoffs and that trigger eternal play of (D,D) after a deviation, similar to the strategies $Tr(C)$, Tr_1 and Tr_2 above. For δ sufficiently high, such strategies form again a subgame perfect equilibrium in $G_p^\infty(\delta)$. Figure 7.1 shows the payoffs that can be reached this way as limiting average payoffs in a subgame perfect equilibrium of $G_p^\infty(\delta)$ for δ high enough.

7.1.2 Some General Observations

For the prisoners' dilemma game we have established that each player playing always D is a subgame perfect equilibrium of $G_p^\infty(\delta)$ for every $0 < \delta < 1$. The following proposition follows from exactly the same simple logic.

Proposition 7.1. *Let G be any arbitrary (not necessarily finite) n-person game, and let the strategy combination* $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ *be a Nash equilibrium in G. Let* $0 < \delta < 1$. *Then each player i playing* s_i *at every moment t is a subgame perfect equilibrium in* $G^\infty(\delta)$.[3]

[3] We assume that $G^\infty(\delta)$ is well-defined, in particular that the discounted payoff sums are finite.

In particular, this proposition holds for any bimatrix game (see Definition 3.1) and any (at least any or pure) Nash equilibrium in this bimatrix game.

Let $G = (A,B)$ be an $m \times n$-bimatrix game. Let $P(G)$ be the convex hull of the set $\{(a_{ij},b_{ij}) \in \mathbb{R}^2 \mid i = 1,\ldots,m,\ j = 1,\ldots,n\}$. That is,

$$P(G) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}(a_{ij},b_{ij}) \;\middle|\; \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} = 1,\ \forall\, i,j:\ \alpha_{ij} \geq 0 \right\}.$$

For the prisoners' dilemma game G_p, $P(G_p)$ is the quadrangle with $(40,40)$, $(30,60)$, $(60,30)$, and $(50,50)$ as vertices, see Fig. 7.1. The elements (payoff pairs) of $P(G)$ can be obtained as limiting (long run) average payoffs in the infinitely repeated game G by an appropriate sequence of play.[4] The following proposition says that every payoff pair in $P(G)$ that strictly dominates the payoffs associated with a Nash equilibrium of G can be obtained as limiting average payoffs in a subgame perfect equilibrium of $G^\infty(\delta)$ for δ large enough. Such a proposition is known as a *Folk theorem*.[5] Its proof (omitted here) is somewhat technical but basically consists of formulating trigger strategies in a similar way as for the prisoners' dilemma game above. In these strategies, after a deviation from the pattern leading to the desired limiting average payoffs, players revert to the Nash equilibrium under consideration of the stage game forever.[6]

Proposition 7.2 (Folk theorem for subgame perfect equilibrium). *Let* $(\mathbf{p}^*,\mathbf{q}^*)$ *be a Nash equilibrium of G, and let* $\mathbf{x} = (x_1,x_2) \in P(G)$ *such that* $x_1 > \mathbf{p}^*A\mathbf{q}^*$ *and* $x_2 > \mathbf{p}^*B\mathbf{q}^*$. *Then there is a* $0 < \delta_x^* < 1$ *such that for every* $\delta \geq \delta_x^*$ *there is a subgame perfect equilibrium in* $G^\infty(\delta)$ *with limiting average payoffs* $\mathbf{x}$.[7]

7.2 Nash Equilibrium

In this section we consider the consequences of relaxing the subgame perfection requirement for a Nash equilibrium in an infinitely repeated game. When thinking of trigger strategies as in Sect. 7.1, this means that deviations can be punished more severely, since the equilibrium does not have to induce a Nash equilibrium in the 'punishment subgame'.

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[4] $P(G)$ is sometimes called the *cooperative payoff space*. Its elements are also attainable if the players could agree to pick payoff pairs randomly, i.e., to agree on some probability distribution – called *correlated strategy* – over the payoff pairs in the set $\{(a_{ij},b_{ij})\}$. See Sect. 13.7.

[5] This expression refers to the fact that results like this had been known among game theorists even before they were formulated and written down explicitly. They belonged to the *folklore* of game theory.

[6] For a stronger result see [38], where it is shown that Proposition 7.3 below also holds for subgame perfect Nash equilibrium if the dimension of the cooperative payoff space is equal to the number of players. This, however, requires more sophisticated strategies. See also [39] for further references.

[7] Formally, if $\xi_0, \xi_1, \xi_2, \ldots$ is a sequence of real numbers, then the limiting average is the number $\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \xi_t$, assuming that this limit exists.

For the infinitely repeated prisoners' dilemma game this has no consequences. In the game $G_p^\infty(\delta)$, each player can guarantee to obtain at least 40 at each stage, so that more severe punishments are not possible. In the following subsection we consider a different example.

7.2.1 An Example

Consider the bimatrix game

$$G_1 = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & \quad R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & -1,4 \end{pmatrix} \end{array}.$$

The set $P(G_1)$ is the triangle with vertices $(1,1)$, $(0,0)$, and $(-1,4)$. In the game G_1 the strategy U is a strictly dominant strategy for player 1. The unique Nash equilibrium is (U,L). Player 1 always playing U and player 2 always playing L is a subgame perfect equilibrium in $G_1^\infty(\delta)$ for every $0 < \delta < 1$, cf. Proposition 7.1. Proposition 7.2 does not add anything to this observation, since $P(G_1)$ does not contain any payoff pair larger than $(1,1)$ for each player.

Now consider the following strategies in the infinitely repeated game $G_1^\infty(\delta)$:

N_1: At $t = 0$ play D. After a history where (D,R) was played at $t = 0,4,8,12,\ldots$ and (U,L) at all other times: play D at $t = 0,4,8,12,\ldots$ and play U at all other times. After any other history play the mixed action $(\frac{4}{5},\frac{1}{5})$, that is, play U with probability $\frac{4}{5}$ and D with probability $\frac{1}{5}$.

N_2: At $t = 0$ play R. After a history where (D,R) was played at $t = 0,4,8,12,\ldots$ and (U,L) at all other times: play R at $t = 0,4,8,12,\ldots$ and play L at all other times. After any other history play R.

Note that these strategies are again trigger strategies. They induce a sequence of play in which within each four times, (D,R) is played once and (U,L) is played thrice. After a deviation player 1 plays the mixed action $(\frac{4}{5},\frac{1}{5})$ and player 2 the pure action R forever. Thus, in a subgame following a deviation the players do not play a Nash equilibrium: if player 2 plays R always, then player 1's best reply is to play U always. Hence, (N_1,N_2) is not a subgame perfect equilibrium.

We claim, however, that (N_1,N_2) is a Nash equilibrium if δ is sufficiently large.

First observe that player 2 can never gain from deviating since, if player 1 plays N_1, then N_2 requires player 2 to play a best reply in the stage game at every moment t. Moreover, after any deviation player 1 plays $(\frac{4}{5},\frac{1}{5})$ at any moment t, so that both L and R have an expected payoff of $\frac{4}{5}$ for player 2, which is less than 1 and less than 4.

Suppose player 2 plays N_2. If player 1 wants to deviate from N_1, the best moment to do so is one where he is supposed to play D, so at $t = 0,4,\ldots$. Without loss of generality suppose player 1 deviates at $t = 0$. Then (U,R) results at $t = 0$, yielding payoff 0 to player 1. After that, player 2 plays R forever, and the best reply of player 1 to this is to play U forever, again yielding 0 each time. So his total payoff from deviating is 0. Without deviation player 1's total discounted payoff is equal to

$$-1(\delta^0+\delta^4+\delta^8+\cdots)+1(\delta^1+\delta^2+\delta^3+\delta^5+\delta^6+\delta^7+\cdots).$$

In order to keep player 1 from deviating this expression should be at least 0, i.e.

$$\frac{-1}{1-\delta^4}+\left[\frac{1}{1-\delta}-\frac{1}{1-\delta^4}\right]\geq 0$$

which holds if and only if $\delta \geq \delta^*$ with $\delta^* \approx 0.54$.[8] Hence, for these values of δ, (N_1,N_2) is a Nash equilibrium in $G_1^\infty(\delta)$. The limiting average payoffs in this equilibrium are equal to $\frac{3}{4}(1,1)+\frac{1}{4}(-1,4)$, hence to $(\frac{1}{2},\frac{7}{4})$.

The actions played in this equilibrium after a deviation are, in fact, the actions that keep the opponent to his maximin payoff. To see this, first consider the action of player 2, R. The payoff matrix of player 1 is the matrix A with

$$A=\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}.$$

The value of the matrix game A, cf. Chap. 2, is equal to 0 – in fact, (U,R) is a saddlepoint of A – and, thus, player 1 can always obtain at least 0. By playing R, which is player 2's optimal strategy in A, player 2 can hold player 1 down to 0. Hence, this is the most severe punishment that player 2 can inflict upon player 1 after a deviation.

Similarly, if we view the payoff matrix B for player 2 as a zerosum game with payoffs to player 2 and, following convention, convert this to a matrix game giving the payoffs to player 1, we obtain

$$-B=\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \end{array}.$$

In this game, $(\frac{4}{5},\frac{1}{5})$ is an (the) optimal strategy for player 1, yielding the value of the game, which is equal to $-\frac{4}{5}$. Hence, player 2 can guarantee to obtain a payoff of $\frac{4}{5}$, but player 1 can make sure that player 2 does not obtain more than this by playing $(\frac{4}{5},\frac{1}{5})$. Again, this is the most severe punishment that player 1 can inflict upon player 2 after a deviation.

By using these punishments in a trigger strategy, the same logic as in Sect. 7.1 tells us that any pair of payoffs in $P(G_1)$ that strictly dominates the pair $(v(A),-v(-B))=(0,\frac{4}{5})$ can be obtained as limiting average payoffs in a Nash equilibrium of the game $G_1^\infty(\delta)$ for δ sufficiently large. This is illustrated in Fig. 7.2.

[8] Found by solving a third degree equation.

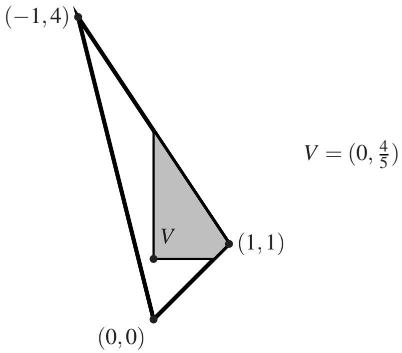

Fig. 7.2 For every payoff pair in the *shaded area* there is a δ large enough such that this payoff pair can be obtained as the limiting (long run) average in a Nash equilibrium of $G_1^\infty(\delta)$

7.2.2 A Folk Theorem for Nash Equilibrium

Let $G=(A,B)$ be an arbitrary $m\times n$ bimatrix game. Let $v(A)$ be the value of the matrix game A and let $v(-B)$ be the value of the matrix game $-B$. As in Sect. 7.1 let $P(G)$ be the cooperative payoff space. The following proposition generalizes what we found above for the game G_1.

Proposition 7.3 (Folk theorem for Nash equilibrium). *Let* $\mathbf{x}=(x_1,x_2)\in P(G)$ *such that* $x_1>v(A)$ *and* $x_2>-v(-B)$*. Then there is a* $0<\delta_x^*<1$ *such that for every* $\delta\geq\delta_x^*$ *there is Nash equilibrium in* $G^\infty(\delta)$ *with limiting average payoffs* $\mathbf{x}$.

Problems

7.1. *On Discounting and Limiting Average*

(a) In a repeated game, interpret the discount factor $0<\delta<1$ as the probability that the game will continue, i.e., that the stage game will be played again. Show that, with this interpretation, the repeated game will end with probability 1 (cf. Problem 6.16(e)).

(b) Can you give an example in which the limit that defines the long run average payoffs, does not exist? (cf. Footnote 7).

7.2. *Nash and Subgame Perfect Equilibrium in a Repeated Game*

Consider the following bimatrix game:

$$G=(A,B)=\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 2,3 & 1,5 \\ 0,1 & 0,1 \end{pmatrix} \end{array}.$$

(a) Determine all Nash equilibria of this game. Also determine the value $v(A)$ of the matrix game A and the value $v(-B)$ of the matrix game $-B$. Determine the optimal strategies of player 2 in A and of player 1 in $-B$.

(b) Consider the repeated game $G^\infty(\delta)$. Which limiting average payoff(s) can be obtained in a subgame perfect equilibrium of this repeated game according to Proposition 7.1 or Proposition 7.2? Does this depend on δ?

(c) Which limiting average payoffs can be obtained in a Nash equilibrium in $G^\infty(\delta)$ according to Proposition 7.3? Describe a pair of Nash equilibrium strategies that result in the long run average payoffs $(2,3)$. What is the associated minimum value of δ?

7.3. *Nash and Subgame Perfect Equilibrium in Another Repeated Game*

Consider the following bimatrix game:

$$G = (A,B) = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{array}.$$

(a) Which payoffs can be reached as limiting average payoffs in subgame perfect equilibria of the infinitely repeated game $G^\infty(\delta)$ for suitable choices of δ according to Propositions 7.1 and 7.2?

(b) Which payoffs can be reached as limiting average payoffs in Nash equilibria of the infinitely repeated game $G^\infty(\delta)$ for suitable choices of δ according to Proposition 7.3?

(c) Describe a subgame perfect Nash equilibrium of $G^\infty(\delta)$ resulting in the limiting average payoffs $(\frac{3}{2},\frac{3}{2})$. Also give the corresponding restriction on δ.

7.4. *Repeated Cournot and Bertrand*

(a) Reconsider the duopoly (Cournot) game of Sect. 6.2.1. Suppose that this game is repeated infinitely many times, and that the two firms discount the streams of payoffs by a common discount factor δ. Describe a subgame perfect Nash equilibrium of the repeated game that results in each firm receiving half of the monopoly profits on average. Also give the corresponding restriction on δ. What could be meant by the expression 'tacit collusion'?

(b) Answer the same questions as in (a) for the Bertrand game of Sect. 6.3.

Chapter 8
An Introduction to Evolutionary Games

In an evolutionary game, players are interpreted as populations – of animals or individuals. The probabilities in a mixed strategy of a player in a bimatrix game are interpreted as shares of the population. Individuals within the same part of the population play the same pure strategy. The main 'solution' concept is the concept of an evolutionary stable strategy.

Evolutionary game theory originated in the work of the biologists Maynard Smith and Price [77]. Taylor and Jonker [133] and Selten [120], among others, played an important role in applying the developed evolutionary biological concepts to boundedly rational human behavior, and to establish the connection with dynamic systems and with game-theoretic concepts like Nash equilibrium. A relatively recent and comprehensive overview can be found in Weibull [147].

This chapter presents a short introduction to evolutionary game theory. For a somewhat more advanced continuation see Chap. 15.

In Sect. 8.1 we consider symmetric two-player games and evolutionary stable strategies. Evolutionary stability is meant to capture the idea of *mutation* from the theory of evolution. We also establish that an evolutionary stable strategy is part of a symmetric Nash equilibrium. In Sect. 8.2 the connection with the so-called *replicator dynamics* is studied. Replicator dynamics intends to capture the evolutionary idea of selection based on fitness. In Sect. 8.3 asymmetric games are considered. Specifically, a connection between replicator dynamics and strict Nash equilibrium is discussed.

8.1 Symmetric Two-Player Games and Evolutionary Stable Strategies

A famous example from evolutionary game theory is the *Hawk–Dove game*:

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} 0,0 & 3,1 \\ 1,3 & 2,2 \end{pmatrix} \end{array}.$$

This game models the following situation. Individuals of the same large population meet randomly, in pairs, and behave either aggressively (Hawk) or passively (Dove) – the fight is about nest sites or territories, for instance. This behavior is genetically determined, so an individual does not really choose between the two modes of behavior. The payoffs reflect (Darwninian) fitness, e.g., the number of offspring. In this context, players 1 and 2 are just two different members of the same population who meet: indeed, the game is symmetric – see below for the formal definition. A mixed strategy $\mathbf{p} = (p_1, p_2)$ (of player 1 or player 2) is naturally interpreted as expressing the population shares of individuals characterized by the same type of behavior. In other words, $p_1 \times 100\%$ of the population are Hawks and $p_2 \times 100\%$ are Doves. In view of this interpretation, in what follows we are particularly interested in symmetric Nash equilibria, i.e., Nash equilibria in which the players have the same strategy. The Hawk–Dove game has three Nash equilibria, only one of which is symmetric namely $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.[1] The formal definitions of a symmetric game and a symmetric Nash equilibrium are as follows.

Definition 8.1. Let $G = (A, B)$ be an $m \times n$-bimatrix game. Then G is *symmetric* if $m = n$ and $B = A^T$, where A^T denotes the transpose of A.[2] A Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$ of G is *symmetric* if $\mathbf{p}^* = \mathbf{q}^*$.

We state the following fact without a proof (see Chap. 15).

Proposition 8.2. *Every symmetric bimatrix game G has a symmetric Nash equilibrium.*

With the interpretation above it is meaningful to consider symmetric Nash equilibria. In fact, we will require a little bit more.

Let $G = (A, B)$ be a symmetric game. Knowing that the game is symmetric, it is sufficient to give the payoff matrix A, since then $B = A^T$. In what follows, when we talk about a symmetric game A we mean the game $G = (A, A^T)$. Let A be an $m \times m$-matrix. Recall (cf. Chap. 2) that Δ^m denotes the set of mixed strategies (for player 1 or player 2). The following definition is due to Maynard Smith and Price [77].

Definition 8.3. A strategy $\mathbf{x} \in \Delta^m$ is an *evolutionary stable strategy* (ESS) in A if for every strategy $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$, there exists some $\varepsilon_{\mathbf{y}} \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_{\mathbf{y}})$ we have

$$\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}). \tag{8.1}$$

The interpretation of an ESS $\mathbf{x}$ is as follows. Consider any small *mutation* $\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}$ of $\mathbf{x}$. Condition (8.1) then says that against such a small mutation, the original strategy $\mathbf{x}$ is better than the *mutant* strategy $\mathbf{y}$. In other words, if the population $\mathbf{x}$

[1] The Hawk–Dove game can also be interpreted as a *Game of Chicken*. Two car drivers approach each other on a road, each one driving in the middle. The driver who is the first to return to his own lane (Dove) 'loses' the game, the one who stays in the middle 'wins' (Hawk). With this interpretation also the asymmetric equilibria are of interest.

[2] Hence, $b_{ij} = a_{ji}$ for all $i, j = 1, \ldots, m$.

is *invaded* by a small part of the *mutant* population $\mathbf{y}$, then $\mathbf{x}$ survives since it fares better against this small mutation than the mutant $\mathbf{y}$ itself does. In [77] evolutionary stability was regarded as expressing stability of a population against mutations. We will see below that evolutionary stability has game-theoretic as well as dynamic consequences.

Before applying Definition 8.3 to the Hawk–Dove game, it is convenient to first establish the following propositions, which show that an ESS results in a symmetric Nash equilibrium with a special additional property. Readers not interested in the proofs may also skip these propositions and continue with the summary following Proposition 8.5.

Proposition 8.4. *Let A be an $m \times m$-matrix and let $\mathbf{x} \in \Delta^m$ be an ESS in A. Then $(\mathbf{x}, \mathbf{x})$ is a Nash equilibrium in $G = (A, A^T)$.*

Proof. Let $\mathbf{y} \in \Delta^m$, then it is sufficient to show $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. Let $\varepsilon_{\mathbf{y}}$ be as in Definition 8.3, then

$$\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$$

for all $0 < \varepsilon < \varepsilon_{\mathbf{y}}$ by (8.1). By letting ε go to zero, this implies $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. □

This proposition shows that, indeed, evolutionary stable strategies result in symmetric Nash equilibria. Hence, to find ESS's, it is sufficient to restrict attention to symmetric Nash equilibria. A second convenient property of ESS's is given in the next proposition.

Proposition 8.5. *Let A be an $m \times m$-matrix. If $\mathbf{x} \in \Delta^m$ is an ESS in A, then, for all $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ we have:*

$$\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \Rightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}. \tag{8.2}$$

Conversely, if $(\mathbf{x}, \mathbf{x}) \in \Delta^m \times \Delta^m$ is a Nash equilibrium in $G = (A, A^T)$ and (8.2) holds, then $\mathbf{x}$ is an ESS.

Proof. Let $\mathbf{x} \in \Delta^m$ be an ESS. Let $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$. Suppose that $\mathbf{y}A\mathbf{y} \geq \mathbf{x}A\mathbf{y}$. Then, for any $\varepsilon \in [0,1]$, $\mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) \geq \mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$, contradicting (8.1).

Conversely, let $(\mathbf{x}, \mathbf{x}) \in \Delta^m \times \Delta^m$ be a Nash equilibrium in $G = (A, A^T)$ and let (8.2) hold for $\mathbf{x}$. If $\mathbf{x}A\mathbf{x} > \mathbf{y}A\mathbf{x}$, then also $\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$ for small enough ε. If $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$, then $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$, hence (8.1) holds for any $\varepsilon \in (0,1]$. □

The two preceding propositions state that evolutionary stable strategies are those strategies $\mathbf{x}$ that (1) occur in a symmetric Nash equilibrium and (2) perform strictly better against any alternative best reply $\mathbf{y}$ than that alternative best reply performs against itself.

Thus, the evolutionary stable strategies for an $m \times m$ matrix A can be found as follows. First, compute the symmetric Nash equilibria of the game $G = (A, B)$ with

$B = A^T$. This can be done using the methods developed in Chap. 3. Second, for each such equilibrium $(\mathbf{x}, \mathbf{x})$, check whether (8.2) holds. If it does, then $\mathbf{x}$ is an evolutionary stable strategy.

We apply this method to the Hawk–Dove game. For this game,

$$A = \begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} \end{array}.$$

The unique symmetric equilibrium strategy was $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$. Let $\mathbf{y} = (y, 1-y)$ be an arbitrary strategy, then the condition $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ in (8.2) is always satisfied. This can be seen by direct computation but it also follows from the fact that $(\mathbf{x}, \mathbf{x})$ is a Nash equilibrium (how?). Hence, we have to check if

$$\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$$

for all $\mathbf{y} = (y, 1-y) \neq \mathbf{x}$. This inequality reduces (check!) to

$$4y^2 - 4y + 1 > 0,$$

which is true for all $y \neq \frac{1}{2}$. Thus, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ is the unique ESS in A.

8.2 Replicator Dynamics and Evolutionary Stability

Central in the theory of evolution are the concepts of *mutation* and *selection*. While the idea of mutation is meant to be captured by the concept of evolutionary stability, the idea of selection is captured by the so-called replicator dynamics. We illustrate the concept of replicator dynamics, introduced in [133], by considering again the Hawk–Dove game

$$\begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} 0,0 & 3,1 \\ 1,3 & 2,2 \end{pmatrix} \end{array}.$$

Consider a mixed strategy or, in the present context, vector of population shares $\mathbf{x} = (x, 1-x)$. Consider an arbitrary individual of the population. Playing 'Hawk' against the population $\mathbf{x}$ yields an expected payoff or 'fitness' of

$$0 \cdot x + 3 \cdot (1-x) = 3(1-x)$$

and playing 'Dove' yields

$$1 \cdot x + 2 \cdot (1-x) = 2 - x.$$

Hence, the *average fitness* of the population is

$$x \cdot 3(1-x) + (1-x) \cdot (2-x) = 2 - 2x^2.$$

We now assume that the population shares develop over time, i.e., that x is a function of time t, and that the change in x, described by the time derivative $\dot{x} = \dot{x}(t) = dx(t)/dt$, is proportional to the difference with the average fitness. That is, we assume that $\dot{x}$ is given by the following equation

$$\dot{x}(t) = dx(t)/dt = x(t)\left[3(1-x(t)) - (2-2x(t)^2)\right]. \tag{8.3}$$

Equation (8.3) is the *replicator dynamics* for the Hawk–Dove game. The equation says that the population of Hawks changes continuously (described by $dx(t)/dt$), and that this change is proportional to the difference of the fitness at time t (which is equal to $3(1-x(t))$) with the average fitness of the population (which is equal to $2-2x(t)^2$). Simplifying (8.3) and writing x instead of $x(t)$ yields

$$\dot{x} = dx/dt = x(x-1)(2x-1).$$

This makes it possible to make a diagram of dx/dt as a function of x (a so-called *phase diagram*). See Fig. 8.1. We see that this replicator dynamics has three different roots, the so-called *rest points*[3] $x = 0$, $x = \frac{1}{2}$, and $x = 1$. For these values of x, the derivative dx/dt is equal to zero, so the population shares do not change: the system is at rest. In case $x = 0$ all members of the species are Doves, their fitness is equal to the average fitness, and so nothing changes. This rest point, however, is not stable. A slight disturbance, e.g., a genetic mutation resulting in a Hawk, makes the number of Hawks increase because dx/dt becomes positive: see Fig. 8.1. This increase will go on until the rest point $x = \frac{1}{2}$ is reached. A similar story holds for the rest point $x = 1$, where the population consists of only Hawks. Now suppose the system is at the rest point $x = \frac{1}{2}$. Note that, after a disturbance in either direction, the system will move back again to the state where half the population consists of Doves. Thus, of the three rest points, only $x = \frac{1}{2}$ is stable.

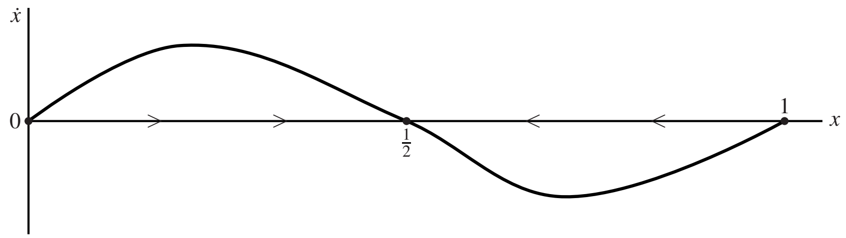

Fig. 8.1 Replicator dynamics for the Hawk–Dove game

[3] Also called equilibrium points, critical points, stationary points.

Recall from the previous section that $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ is also the unique evolutionary stable strategy of the Hawk–Dove game. That this is no coincidence follows from the next proposition, which we state here without a proof (see Chap. 15 for a proof).

Proposition 8.6. *Let A be a* 2×2*-matrix. Then:*

(1) *A has at least one evolutionary stable strategy.*
(2) $\mathbf{x} = (x, 1-x)$ *is an evolutionary stable strategy of A, if and only if* $\mathbf{x}$ *is a stable rest point of the replicator dynamics.*

For general $m \times m$-matrices the set of completely mixed[4] rest points of the replicator dynamics coincides with the set of completely mixed (symmetric) Nash equilibrium strategies. There are also connections between stability of rest points and further properties of Nash equilibria. See Chap. 15 for more details.

8.3 Asymmetric Games

The evolutionary approach to game theory is not necessarily restricted to symmetric situations, i.e., bimatrix games of the form (A, A^T) in which the row and column players play identical strategies. In biology as well as economics one can find many examples of asymmetric situations. Think of two different species competing about territory in biology. As to economics, see for instance [42], Chap. 8 for some applications. The following example is also taken from [42]. Here only the formal analysis is discussed.

Consider the 2×2-bimatrix game

$$(A,B) = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0,0 & 2,2 \\ 1,5 & 1,5 \end{pmatrix} \end{array}.$$

Think of two populations, the row population and the column population. In each population there are two different types: T and B in the row population and L and R in the column population. Individuals of one population are continuously and randomly matched with individuals of the other population, and we are interested again in the development of the population shares. To start with, assume the shares of T and B types in the row population are x and $1-x$, respectively, and the shares of L and R types in the column population are y and $1-y$. The expected payoff of a T type individual is given by

$$0 \cdot y + 2 \cdot (1-y) = 2 - 2y.$$

For a B type individual it is

$$1 \cdot y + 1 \cdot (1-y) = 1.$$

[4] I.e., all coordinates positive.

For an L type individual it is

$$0 \cdot x + 5 \cdot (1-x) = 5 - 5x.$$

And for an R type individual:

$$2 \cdot x + 5 \cdot (1-x) = 5 - 3x.$$

The average of the row types is therefore:

$$x[2(1-y)] + 1 \cdot (1-x)$$

and the replicator dynamics for the population share $x(t)$ of T individuals is given by

$$dx/dt = x[2(1-y) - x[2(1-y)] - (1-x)] = x(1-x)(1-2y). \tag{8.4}$$

Here, we write x and y instead of $x(t)$ and $y(t)$. Similarly one can calculate the replicator dynamics for the column population (check this result!):

$$dy/dt = y(1-y)(-2x). \tag{8.5}$$

We are interested in the rest points of the dynamical system described by (8.4) and (8.5), and, in particular, by the stable rest points. The easiest way is to make a diagram of the possible values of x and y, see Fig. 8.2. In this diagram, the black lines are the values of x and y for which the derivative in (8.4) is equal to 0, i.e., for which the row population is at rest. The light gray lines are the values of x and y for which the derivative in (8.5) is equal to 0: there, the column population is at rest. The points of intersection are the points where the whole system is at rest; this is the set

$$\{(0,y) \mid 0 \le y \le 1\} \cup \{(1,0)\} \cup \{(1,1)\}. \tag{8.6}$$

In Fig. 8.2, arrows indicate the direction in which x and y move. For instance, if $1 > y \ge \frac{1}{2}$ and $0 < x < 1$ we have $dx/dt < 0$ and $dy/dt < 0$, so that in that region

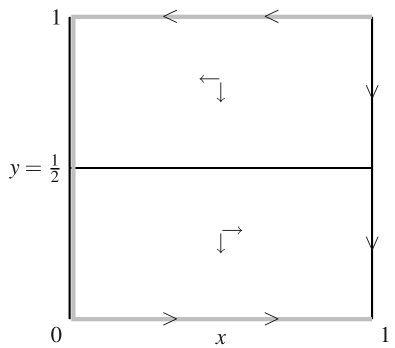

Fig. 8.2 Phase diagram of the asymmetric evolutionary game

x as well as y decrease. A stable rest point is a rest point such that, if the system is slightly disturbed and changes to some point close to the rest point in question, then it should move back again to this rest point. In terms of the arrows in Fig. 8.2 this means that a stable rest point is one where all arrows in the neighborhood point towards that point. It is obvious that in our example the point $(1,0)$ is the only such point. So the situation where the row population consists only of T type individuals ($x = 1$) and the column population consists only of R type individuals ($y = 0$) is the only stable situation with respect to the replicator dynamics.

Is there a relation with Nash equilibrium? One can check (!) that the set of Nash equilibria in this example is the set:

$$\{(T,R),(B,L)\} \cup \{(B,(q,1-q)) \mid \frac{1}{2} \le q \le 1\}.$$

So the stable rest point (T,R) is a Nash equilibrium. Furthermore, it has a special characteristic, namely, it is the only *strict* Nash equilibrium of the game. A *strict* Nash equilibrium in a game is a Nash equilibrium where each player not only does not gain but in fact strictly loses by deviating. For instance, if the row player deviates from T in the Nash equilibrium (T,R) then he obtains strictly less than 2. All the other equilibria in this game do not have this property. For instance, if the column player deviates from L to R in the Nash equilibrium (B,L), then he still obtains 5.

The observation that the stable rest point of the replicator dynamics coincides with a strict Nash equilibrium is not a coincidence. The following proposition is stated without a proof (see [119] or [57]).

Proposition 8.7. *In a* 2×2 *bimatrix game a pair of strategies is a stable rest point of the replicator dynamics if, and only if, it is a strict Nash equilibrium. For larger games, any stable rest point of the replicator dynamics is a strict Nash equilibrium, but the converse does not necessarily hold.*

In the literature the concept of evolutionary stable strategy is extended to asymmetric games. See the cited references for details.

Problems

8.1. *Symmetric Games*

Compute the evolutionary stable strategies for the following payoff matrices A.

(a) $A = \begin{pmatrix} 4 & 0 \\ 5 & 3 \end{pmatrix}$ (Prisoners' Dilemma)

(b) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (Coordination game)

8.2. *More Symmetric Games*

For each of the following two matrices, determine the replicator dynamics, rest points and stable rest points, and evolutionary stable strategies. Include phase diagrams for the replicator dynamics. For the evolutionary stable strategies, provide independent arguments to show evolutionary stability by using Propositions 8.4 and 8.5; that is, without using the replicator dynamics.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}.$$

8.3. *Asymmetric Games*

For each of the following two asymmetric situations, determine the replicator dynamics, rest points and stable rest points, including phase diagrams. Also determine all Nash and strict Nash equilibria.

$$(A, A^T) = \begin{pmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{pmatrix}, \quad (B, B^T) = \begin{pmatrix} 2,2 & 0,1 \\ 1,0 & 0,0 \end{pmatrix}.$$

8.4. *Frogs Call for Mates* ([42], p. 215)

Consider the following game played by male frogs who Call or Don't Call their mates.

$$\begin{array}{l} \\ \text{Call} \\ \text{Don't Call} \end{array} \begin{array}{c} \begin{array}{cc} \text{Call} & \text{Don't Call} \end{array} \\ \begin{pmatrix} P-z, P-z & m-z, 1-m \\ 1-m, m-z & 0,0 \end{pmatrix} \end{array}$$

The payoffs are in units of 'fitness', measured by the frog's offspring. Here z denotes the cost of Calling (danger of becoming prey, danger of running out of energy); and m is the probability that the male who calls in a pair of males, the other of whom is not calling, gets a mate. Typically, $m \geq \frac{1}{2}$. Next, if no male calls then no female is attracted, and if both call returns diminish and they each attract P females with $0 < P < 1$.

(a) Show that there are several possible evolutionary stable strategies for this game, depending on the parameters (m, z, P).

(b) Set $m = 0.6$ and $P = 0.8$. Find values for z for each of the following situations: (1) Don't Call is an evolutionary stable strategy (ESS); (2) Call is an ESS; (3) A mixture of Call and Don't Call is an ESS.

(c) Suppose there are two kinds of frogs in *Frogs call for mates*. Large frogs have a larger cost of calling (z_1) than do small frogs (z_2). Determine the corresponding asymmetric bimatrix game. Determine the possible stable rest points of the replicator dynamics.

8.5. *Video Market Game* ([42], pp. 229, 233).

Two boundedly rational video companies are playing the following asymmetric game:

$$\begin{array}{l c} & \begin{array}{cc} \text{Open system} & \text{Lockout system} \end{array} \\ \begin{array}{l} \text{Open system} \\ \text{Lockout system} \end{array} & \begin{pmatrix} 6,4 & 5,5 \\ 9,1 & 10,0 \end{pmatrix} \end{array}$$

Company I (the row company) has to decide whether to have an open system or a lockout system. Company II (the column company) has to decide whether to create its own system or copy that of company I. What is a rest point of the replicator dynamics for this system?

Chapter 9
Cooperative Games with Transferable Utility

The implicit assumption in a cooperative game is that players can form coalitions and make binding agreements on how to distribute the proceeds of these coalitions. A cooperative game is more abstract than a noncooperative game in the sense that strategies are not explicitly modelled: rather, the game describes what each possible coalition can earn by cooperation. In a cooperative game with *transferable utility* it is assumed that the earnings of a coalition can be expressed by one number. One may think of this number as an amount of money, which can be distributed among the players in any conceivable way – including negative payments – if the coalition is actually formed. More generally, it is an amount of *utility* and the implicit assumption is that it makes sense to transfer this utility among the players – for instance, due to the presence of a medium like money, assuming that individual utilities can be expressed in monetary terms.

This chapter presents a first acquaintance with the theory of cooperative games with transferable utility.[1] A few important solution concepts – the core, the Shapley value, and the nucleolus – are briefly discussed in Sects. 9.2–9.4. We start with examples and preliminaries in Sect. 9.1.

9.1 Examples and Preliminaries

In Chap. 1 we have seen several examples of cooperative games with transferable utilities: the three cities game, a glove game, a permutation game, and a voting game. For the stories giving rise to these games the reader is referred to Sect. 1.3.4. Here we reconsider the resulting games.

In the three cities game, cooperation between cities leads to cost savings expressed in amounts of money, as in Table 9.1. In the first line of this table all possible coalitions are listed. It important to note that the term 'coalition' is used for any subset of the set of players. So a coalition is not necessarily formed. The

[1] See Chaps. 16–20 for a more advanced treatment.

Table 9.1 The three cities game

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	90	100	120	220

Table 9.2 A glove game

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	0	1	1	1

Table 9.3 A permutation game

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	2	5	4	14	18	9	24

empty subset (empty coalition) has been added for convenience: it just is assigned the number 0 by convention. The numbers in the second line of the table are called the 'worths' of the coalitions. For instance, coalition $S = \{1,2\}$ has worth 90. In this particular example, 90 is the amount of cost savings earned by cities 1 and 2 if they cooperate. It is assumed that this amount can be split between the two players (cities) if the coalition is actually formed. That is, player 1 may receive $x_1 \in \mathbb{R}$ and player 2 may receive $x_2 \in \mathbb{R}$ such that $x_1 + x_2 = 90$ or, more generally, $x_1 + x_2 \leq 90$.

In the glove game in Sect. 1.3.4 coalitions may make pairs of gloves. The game is described in Table 9.2. In this game the worth 1 of the 'grand coalition' $\{1,2,3\}$, for instance, means that this coalition can earn 1 by producing one pair of gloves. One can think of this number as expressing the monetary value of this pair of gloves. Alternatively, one can think of one pair of gloves having 'utility' equal to 1. Again, it is assumed that the players can split up this amount in any way they like. So a possible distribution takes the form $(x_1,x_2,x_2) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 \leq 1$. For $i = 1,2,3$, the number x_i may represent the money that player i receives, or (if nonnegative) the percentage of time that player i is allowed to wear the gloves.

The permutation game (dentist game) of Sect. 1.3.4 is reproduced in Table 9.3. In this game, one could think of the worth of a coalition as expressing, for instance, savings of opportunity costs by having dentist appointments on certain days. What is important is that, again, these worths can be distributed in any way among the players of the coalitions.

For the voting game related to the UN Security Council, a table could be constructed as well, but this table would be huge: there are $2^{15} = 32{,}768$ possible coalitions (cf. Problem 9.1). Therefore, it is more convenient to describe this game as follows. Let the permanent members be the players $1,\ldots,5$ and let the other members be the players $6,\ldots,15$. Denote by $N = \{1,2,\ldots,15\}$ the grand coalition of all players and by $v(S)$ the worth of a coalition $S \subseteq N$. Then

$$v(S) := \begin{cases} 1 & \text{if } \{1,\ldots,5\} \subseteq S \text{ and } |S| \geq 9, \\ 0 & \text{otherwise,} \end{cases}$$

where $|S|$ denotes the number of players in S. In this case the number 1 indicates that the coalition is 'winning' and the number 0 that the coalition is 'losing'. In analyzing games like this the resulting numbers – e.g., nonnegative numbers $x_1, \ldots, x_{15}$ summing to 1 – are usually interpreted as power indices, expressing the power of a player in some way or another.

We summarize the concepts introduced informally in the preceding examples formally within the following definition.

Definition 9.1. A *cooperative game with transferable utility* or *TU-game* is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ with $n \in \mathbb{N}$ is the set of *players*, and v is a function assigning to each *coalition* S, i.e., to each subset $S \subseteq N$ a real number $v(S)$, such that $v(\emptyset) = 0$. The function v is called the *characteristic function* and $v(S)$ is called the *worth* of S. The coalition N is called the *grand coalition*. A *payoff distribution for* coalition S is a vector of real numbers $(x_i)_{i \in S}$.

When analyzing a TU-game there are two important questions to answer: which coalitions are formed; and how are the worths of these coalitions distributed among their members? In this chapter we assume that the grand coalition is formed and we concentrate on the second question. This is less restrictive than it might seem at first sight, since coalition formation depends, naturally, on how the proceeds of a coalition are going to be distributed among its members. Thus, also if smaller coalitions are formed the distribution question has to be considered for these coalitions.

9.2 The Core

Consider the three cities game in Table 9.1, suppose that the grand coalition gets together, and suppose that there is a proposal $x_1 = 40$, $x_2 = 40$, and $x_3 = 140$ for distribution of the savings $v(N) = 220$ on the bargaining table. One can imagine, for instance, that player 3 made such a proposal. In that case, players 1 and 2 could protest successfully, since they can save $v(\{1, 2\}) = 90 > 80 = x_1 + x_2$ without player 3. We express this by saying that $\mathbf{x} = (x_1, x_2, x_3)$ is not in the 'core' of this game. More generally, the core of the three cities game is the set of payoff distributions for $N = \{1, 2, 3\}$ such that the sum of the payoffs is equal to $v(N) = 220$ and each nonempty coalition S obtains at least its own worth. Formally, it is the set

$$
\begin{aligned}
C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid\ & x_1, x_2, x_3 \geq 0 \\
& x_1 + x_2 \geq 90,\ x_1 + x_3 \geq 100,\ x_2 + x_3 \geq 120 \\
& x_1 + x_2 + x_3 = 220\}\ .
\end{aligned}
$$

To obtain a better idea of how this set looks like, we can make a diagram. Although C is a subset of $\mathbb{R}^3$, the constraint $x_1 + x_2 + x_3 = 220$ makes that the C is contained in a two-dimensional subset of $\mathbb{R}^3$, i.e., the plane through the points $(220, 0, 0)$, $(0, 220, 0)$, and $(0, 0, 220)$. The triangle formed by these three points is represented

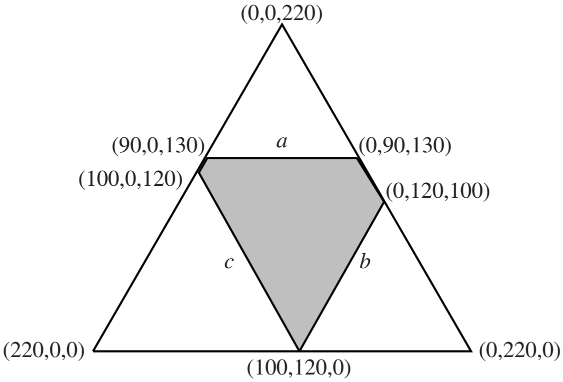

Fig. 9.1 The set C – the core – of the three cities game. The line segment a corresponds to the constraint $x_1+x_2 \geq 90$, the line segment b corresponds to the constraint $x_1+x_3 \geq 100$, and the line segment c corresponds to the constraint $x_2+x_3 \geq 120$

in Fig. 9.1. Note that C is a subset of this triangle by the constraints $x_i \geq 0$ for every $i = 1,2,3$, derived from the conditions $x_i \geq v(\{i\})$ for $i = 1,2,3$. The set C is the shaded area bounded by the three constraints for the two-person coalitions.
Hence, the core of the three cities game is the polygon with vertices $(100,120,0)$, $(0,120,100)$, $(0,90,130)$, $(90,0,130)$, and $(100,0,120)$.

We now give the formal definition of the core and of some other related concepts. It will be convenient to use the notation $x(S) := \sum_{i \in S} x_i$ for a payoff distribution $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and a nonempty coalition $S \subseteq N = \{1,\ldots,n\}$.

Definition 9.2. For a TU-game (N,v), a payoff distribution $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ is

- *Efficient* if $x(N) = v(N)$
- *Individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$
- *Coalitionally rational* if $x(S) \geq v(S)$ for all nonempty coalitions S

The *core* of (N,v) is the set

$$C(N,v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } \emptyset \neq S \subseteq N\} .$$

So the core of (N,v) is the set of all efficient and coalitionally rational payoff distributions.

The core of a game can be a large set, like in the three cities game; a ‘small’ set, like in the glove game (see Problem 9.2); or it can be empty (see again Problem 9.2). Games with nonempty cores were characterized in [16] and [122]: see Chap. 16. In general, core elements can be computed by linear programming techniques. For games with two or three players the core can be computed graphically, as we did for the three cities game. Sometimes, the core can be computed by using the special structure of a specific game under consideration.

9.3 The Shapley Value

The Shapley value [121] is a solution concept for TU-games that is quite different from the core. Whereas the core is a (possibly empty) set, the Shapley value assigns a unique payoff distribution for the grand coalition to every TU-game. The Shapley value is not so much based on strategic considerations but, rather, assigns to each player his 'average marginal contribution' in the game.

Consider again the three cities game[2] of Table 9.1. Imagine a setting where the players enter a bargaining room one by one, and upon entering each player demands and obtains his marginal contribution. Suppose that player 1 enters first, player 2 enters next, and player 3 enters last. Player 1 enters an empty room and can take his 'marginal contribution' $v(\{1\}) - v(\emptyset) = 0 - 0 = 0$. When player 2 enters, player 1 is already present, and player 2 obtains his marginal contribution $v(\{1,2\}) - v(\{1\}) = 90 - 0 = 90$. When, finally, player 3 enters, then the coalition $\{1,2\}$ is already present. So player 3 obtains his marginal contribution $v(\{1,2,3\}) - v(\{1,2\}) = 220 - 90 = 130$. Hence, this procedure results in the payoff distribution $(0,90,130)$, which is called a *marginal vector*. Of course, this payoff distribution does not seem quite fair since it depends on the order in which the players enter the room, and this order is quite arbitrary: there are five other possible orders. The Shapley value takes the marginal vectors of all six orders into consideration, and assigns to a TU-game their average. See Table 9.4.

For an arbitrary TU-game (N,v) with player set $N = \{1,\ldots,n\}$ the Shapley value can be computed in the same way, by first computing the marginal vectors corresponding to the $n!$ different orders of the players, and then taking the average – that is, summing all marginal vectors and dividing the result by $n!$. If the number of players is large, then this is a huge task. In the UN security council voting game of Sect. 9.1, for instance, this would mean computing $15! > 13 \times 10^{11}$ marginal vectors. Fortunately, there is a more clever way to compute the total marginal contribution of a player.

Table 9.4 Computation of the Shapley value for the three cities game. The Shapley value is obtained by dividing the totals of the marginal contributions by 6

Order of entry	1	2	3
1,2,3	0	90	130
1,3,2	0	120	100
2,1,3	90	0	130
2,3,1	100	0	120
3,1,2	100	120	0
3,2,1	100	120	0
Total	390	450	480
Shapley value	65	75	80

[2] Cf. Bird [14].

For instance, let (N,v) be a TU-game with 10 players. Consider player 7 and the coalition $\{3,5,9\}$. The marginal contribution $v(\{3,5,9,7\}) - v(\{3,5,9\})$ accruing to player 7 occurs in more than one marginal vector. In how many marginal vectors does it occur? To compute this, note that first players 3, 5, and 9 must enter, and this can happen in 3! different orders. Then player 7 enters. Finally, the other six players enter, and this can happen in 6! different orders. Therefore, the total number of marginal vectors in which player 7 obtains the marginal contribution $v(\{3,5,9,7\}) - v(\{3,5,9\})$ is equal to $3! \times 6!$. By counting in this way the number of computations is greatly reduced.

We now repeat this argument for an arbitrary TU-game (N,v), an arbitrary player $i \in N$, and an arbitrary coalition S that does not contain player i. By the same argument as in the preceding paragraph, the total number of marginal vectors in which player i receives the marginal contribution $v(S \cup \{i\}) - v(S)$ is equal to the number of different orders in which the players of S can enter first, $|S|!$, multiplied by the number of different orders in which the players not in $S \cup \{i\}$ can enter after player i, which is $(n - |S| - 1)!$. Hence, the total contribution obtained by player i by entering after the coalition S is equal to $|S|!(n-|S|-1)![v(S \cup \{i\}) - v(S)]$. The Shapley value for player i is then obtained by summing over all coalitions S not containing player i, and dividing by $n!$. In fact,we use this alternative computation as the definition of the Shapley value.

Definition 9.3. The *Shapley value* of a TU-game (N,v) is denoted by $\Phi(N,v)$. Its i-th coordinate, i.e., the Shapley value payoff to player $i \in N$, is given by

$$\Phi_i(N,v) = \sum_{S \subseteq N:\ i \notin S} \frac{|S|!(n-|S|-1)!}{n!} [v(S \cup \{i\}) - v(S)] \, .$$

Especially for larger TU-games it is easier to work with the formula in Definition 9.3 than to use the definition based on marginal vectors. For some purposes, however, it is easier to use the latter definition (see, e.g., Problem 9.5).

The Shapley value of the three cities game is an element of the core of that game (check this). In general, however, this does not have to be the case even if the core is nonempty – see, e.g., Problem 9.6.

The definition of the Shapley value as assigning to each player in a game his average marginal contribution, can be regarded as a justification of this solution concept by itself. In literature there are, moreover, a number of axiomatic characterizations of the Shapley value. In an axiomatic characterization one proceeds as follows. Consider an arbitrary map, which (like the Shapley value) assigns to each game with player set N a payoff vector. Next, define 'reasonable' properties or *axioms* for this map. Such axioms limit the possible maps (i.e., solution concepts), and if the axioms are strong enough, they admit only one solution concept. This so-called *axiomatic approach* is quite commonplace in cooperative game theory. An axiomatic characterization of the Shapley value has already been given in [121]. Problem 9.13 preludes to this. For details, see Chap. 17.

9.4 The Nucleolus

In order to define the last solution concept to be considered in this chapter, call a TU-game (N,v) *essential* if $v(N) \geq \sum_{i \in N} v(\{i\})$. Hence, for an essential game there are payoff distributions for the grand coalition that are both efficient and individually rational. Such payoff distributions are called *imputations*, following [141]. The set

$$I(N,v) = \{\mathbf{x} \in \mathbb{R}^N \mid x(N) = v(N),\ x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

is called the *imputation set* of the TU-game (N,v). So a game (N,v) is essential if and only if $I(N,v) \neq \emptyset$.[3]

Let (N,v) be an essential TU-game, let $\mathbf{x} \in I(N,v)$, and let S be a nonempty coalition unequal to N. The *excess of S at* $\mathbf{x}$, denoted by $e(S,\mathbf{x})$, is defined by

$$e(S,\mathbf{x}) = v(S) - x(S)\ .$$

The excess can be seen as a measure of the dissatisfaction of the coalition S with the imputation $\mathbf{x}$: the larger $e(S,\mathbf{x})$, the less S obtains at $\mathbf{x}$ relative to its worth $v(S)$. In particular, if this excess is positive then S obtains less than its own worth.

In words, the *nucleolus* of an essential TU-game (N,v) is defined as follows. First, for every imputation $\mathbf{x}$ compute all excesses. Then select those imputations for which the maximal excesses are smallest. If this is the case at a unique imputation, then that imputation is the nucleolus of the game. If not, then consider the second maximal excesses of the selected imputations and make a further selection by taking those imputations for which these second maximal excesses are smallest. If this happens at a unique imputation, then that is the nucleolus. Otherwise, continue with the third maximal excesses, etc., until a unique imputation is found: this is the nucleolus.

Thus, the idea behind the nucleolus is to make the largest dissatisfaction as small as possible. If there is more than one possibility to do this, then we also make the second largest dissatisfaction as small as possible, etc., until a unique distribution is reached. In this sense, the nucleolus is similar in spirit to the main principle of distributive justice proposed in [107], namely to maximize the lot of the worst off people in society. The nucleolus was introduced in [116].

We illustrate this procedure by means of the three cities game, reproduced in Table 9.5. The third line of the table gives the excesses at the imputation $(70,70,80)$. The choice of this particular imputation is arbitrary: we use it as a starting point to find the nucleolus. The largest excess at this imputation is -30, namely for the coalition $\{2,3\}$.[4] Clearly, we can decrease this excess by giving players 2 and 3 more at the expense of player 1. Doing so implies that the excess of $\{1,2\}$ or of $\{1,3\}$ or of both will increase. Consider the imputation $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$. At this imputation, the excesses of the three two-player coalitions are equal, see Table 9.5.,

[3] Check the statements made in this paragraph.

[4] The excess of the grand coalition at any imputation is zero, and therefore does not play a role in finding the nucleolus. These excesses are omitted from the table.

Table 9.5 Heuristic determination of the nucleolus of the three cities game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	90	100	120	220
$e(S,(70,70,80))$	-70	-70	-80	-50	-50	-30	
$e(S,(56\frac{2}{3},76\frac{2}{3},86\frac{2}{3}))$	$-56\frac{2}{3}$	$-76\frac{2}{3}$	$-86\frac{2}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	

and these are also the maximal excesses. Now first observe that at this imputation the maximal excess must be smallest. This is so since the sum of the excesses of the three two-player coalitions at any imputation must be the same, namely equal to -130, as follows from

$$\begin{aligned} e(\{1,2\},\mathbf{x})+e(\{1,3\},\mathbf{x})+e(\{2,3\},\mathbf{x}) &= v(\{1,2\})+v(\{1,3\})+v(\{2,3\}) \\ &\quad -2(x_1+x_2+x_3) \\ &= 310-2\cdot 220 \\ &= -130\,. \end{aligned}$$

This implies that none of these excesses can be decreased without increasing at least one other excess. Second, the imputation at which these three excesses are equal is unique, since the system

$$\begin{aligned} 90-x_1-x_2 &= 100-x_1-x_3 \\ 100-x_1-x_3 &= 120-x_2-x_3 \\ x_1+x_2+x_3 &= 220 \\ x_1,x_2,x_3 &\geq 0 \end{aligned}$$

has a unique solution – namely, indeed, $(56\frac{2}{3},76\frac{2}{3},86\frac{2}{3})$. So this imputation must be the nucleolus of the three cities game.

This example suggests that, at least for a three player TU-game, it is easy to find the nucleolus, namely simply by equating the excesses of the three two-player coalitions. Unfortunately, this is erroneous. It works if the worths of the two-player coalitions are large relative to the worths of the single player coalitions, but otherwise it may fail to result in the nucleolus. Consider the three-player TU-game in Table 9.6, which is identical to the three cities game except that now $v(\{1\}=20$. The third line of the table shows the excesses at $(56\frac{2}{3},76\frac{2}{3},86\frac{2}{3})$ in this TU-game (observe that this vector is still an imputation). The maximal excess is now $-36\frac{2}{3}$ for the single-player coalition $\{1\}$. Clearly, the original imputation is no longer the nucleolus: the excess of $\{1\}$ can be decreased by giving player 1 more at the expense of players 2 and/or 3. Suppose we equalize the excesses of $\{1\}$ and $\{2,3\}$ by solving the equation $20-x_1=120-x_2-x_3$. Together with $x_1+x_2+x_3=220$ this yields $x_1=60$ and $x_2+x_3=160$. Trying the imputation $(60,75,85)$, obtained by taking away the same amount from players 2 and 3, yields the excesses in the fourth line of Table 9.6. We claim that $(60,75,85)$ is the nucleolus of this TU-game.

Table 9.6 Heuristic determination of the nucleolus in the three cities game with the worth of coalition $\{1\}$ changed to 20

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	20	0	0	90	100	120	220
$e(S,(56\frac{2}{3},76\frac{2}{3},86\frac{2}{3}))$	$-36\frac{2}{3}$	$-76\frac{2}{3}$	$-86\frac{2}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	
$e(S,(60,75,85))$	-40	-75	-85	-45	-45	-40	

The maximal excess is -40, reached by the coalitions $\{1\}$ and $\{2,3\}$, and this cannot be decreased: decreasing the excess for one of the two coalitions implies increasing the excess for the other coalition. Hence, x_1 has to be equal to 60 in the nucleolus. The second maximal excess is -45, reached by the coalitions $\{1,2\}$ and $\{1,3\}$. Since x_1 has already been fixed at 60, a decrease in the excess for one of these two coalitions implies an increase of the excess for the other coalition. Hence, also x_2 and x_3 are fixed, at 75 and 85, respectively.

These two examples indicate that it is not easy to compute the nucleolus. In general, it can be computed by solving a series of linear programs. The arguments used above to show that a particular imputation is indeed the nucleolus implicitly use a general property of the nucleolus called the Kohlberg criterion. See Chap. 19 for a more detailed study of the nucleolus.

In spite of its not being easy to compute the nucleolus is an attractive solution. It assigns a unique imputation to every essential game and, moreover, if a game has a nonempty core, the nucleolus assigns a core element: see Problem 9.9.

Problems

9.1. *Number of Coalitions*

Show that a set of $n \in \mathbb{N}$ elements has 2^n different subsets.

9.2. *Computing the Core*

(a) Compute the core of the glove game of Table 9.2.

(b) Compute the core of the dentist game of Table 9.3.

(c) Compute the core of the UN security council voting game in Sect. 9.1.

(d) Consider the two-person game $(\{1,2\},v)$ given by $v(\{1\}) = a$, $v(\{2\}) = b$, and $v(\{1,2\}) = c$, where $a,b,c \in \mathbb{R}$. Give a necessary and sufficient condition on a, b, and c for the core of $(\{1,2\},v)$ to be nonempty. Compute the core.

9.3. *A Condition for Non-Emptiness of the Core of a Three-Person Game*

Let $(\{1,2,3\},v)$ be a three-person game which has a nonempty core. Show that $2v(\{1,2,3\}) \geq v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})$. (Hint: Take a core element $\mathbf{x} = (x_1,x_2,x_3)$ and write down the core constraints.)

9.4. *'Non-Monotonicity' of the Core*

Consider the following four-person game: $v(\{i\}) = 0$ for every $i = 1,\ldots,4$, $v(\{1,2\}) = v(\{3,4\}) = 0$, $v(S) = 1$ for all other two-person coalitions and for all three-person coalitions, and $v(N) = 2$.

(a) Show that $C(N,v) = \{(\alpha,\alpha,1-\alpha,1-\alpha) \in \mathbb{R}^4 \mid 0 \leq \alpha \leq 1\}$.

(b) Consider the game (N,v') equal to (N,v) except for $v'(\{1,3,4\}) = 2$. Show that the core of (N,v') consists of a single element. What about the payoff to player 1 if core elements in (N,v) and (N,v') are compared? Conclude that the core is not 'monotonic' (cf. [82]).

9.5. *Efficiency of the Shapley Value*

Let (N,v) be an arbitrary TU-game. Show that the Shapley value $\Phi(N,v)$ is efficient. (Hint: take an order $i_1,i_2,\ldots,i_n$ of the players and show that the sum of the coordinates of the corresponding marginal vector is equal to $v(N)$; use this to conclude that $\Phi(N,v)$ is efficient.)

9.6. *Computing the Shapley Value*

(a) Compute the Shapley value of the glove game of Table 9.2. Is it an element of the core?

(b) Compute the Shapley value of the dentist game of Table 9.3. Is it an element of the core?

(c) Compute the Shapley value of the UN security council voting game in Sect. 9.1. (Hint: observe the – more or less – obvious fact that the Shapley value assigns the same payoff to all permanent members and also to all nonpermanent members. Use the formula in Definition 9.3.) Is it an element of the core?

9.7. *The Shapley Value and the Core*

For every real number a the three-player TU-game v_a is given by: $v_a(\{i\}) = 0$ for $i = 1,2,3$, $v_a(\{1,2\}) = 3$, $v_a(\{1,3\}) = 2$, $v_a(\{2,3\}) = 1$, $v_a(\{1,2,3\}) = a$.

(a) Determine the minimal value of a so that the TU-game v_a has a nonempty core.

(b) Calculate the Shapley value of v_a for $a = 6$.

(c) Determine the minimal value of a so that the Shapley value of v_a is a core distribution.

9.8. *Shapley Value in a Two-Player Game*

Let (N,v) be a two-player TU-game, i.e., $N = \{1,2\}$. Compute the Shapley value (expressed in $v(\{1\})$, $v(\{2\})$, and $v(\{1,2\})$), and show that it is in the core of the game provided the core is nonempty.

9.9. *The Nucleolus and the Core*

Let (N,v) be an essential TU-game, and suppose that it has a nonempty core. Show that the nucleolus of (N,v) is a core element.

9.10. *Computing the Nucleolus*

(a) Compute the nucleolus of the glove game of Table 9.2.

(b) Compute the nucleolus of the dentist game of Table 9.3.

(c) Compute the nucleolus of the UN security council voting game in Sect. 9.1. (Hint: use Problem 9.9.)

(d) Compute the nucleolus of the games (N,v) and (N,v') in Problem 9.4.

9.11. *Nucleolus of Two-Player Games*

Let (N,v) be an essential two-player TU-game. Compute the nucleolus.

9.12. *Computing the Core, the Shapley Value, and the Nucleolus*

(a) Compute the Shapley value and the nucleolus in the three-player TU-game given by: $v(\{i\}) = 1$ for $i = 1,2,3$, $v(\{1,2\}) = 2$, $v(\{1,3\}) = 3$, $v(\{2,3\}) = 4$, $v(\{1,2,3\}) = 6$. Is the Shapley value a core element in this game?

(b) Compute the core of this game. Make a picture.

(c) Suppose we increase $v(\{1\})$. What is the maximal value of $v(\{1\})$ such that the game still has a nonempty core?

9.13. *Properties of the Shapley Value*

The properties of the Shapley value described in (a)–(c) below are called symmetry, additivity, and dummy property, respectively. It can be shown (see Chap. 17) that the Shapley value is the unique solution concept that assigns exactly one payoff vector to each TU-game and has these three properties together with efficiency (cf. Problem 9.5). In other words, a solution concept has these four properties if, and only if, it is the Shapley value. In this exercise you are asked to show the 'easy' part of this statement, namely the if-part.
(Hint: in each case, decide which of the two formulas for the Shapley value is most convenient to use.)

(a) Let (N,v) be a TU-game, and suppose players i and j are *symmetric* in this game, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions S which do not contain i or j. Show that i and j obtain the same payoff from the Shapley value.

(b) Let (N,v) and (N,w) be two TU-games with the same player set N. Define the sum of these TU-games as the TU-game with player set N where the worth of each coalition S is given by $v(S) + w(S)$; denote this TU-game by $(N, v+w)$. Show that the Shapley value assigns to this sum TU-game the payoff vector which is the sum of the Shapley values of (N,v) and (N,w).

(c) Call player i a *dummy* in the TU-game (N,v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every coalition S to which player i does not belong. Show that the Shapley value assigns exactly the payoff $v(\{i\})$ to player i.

Chapter 10
Cooperative Game Theory Models

The common features of a *cooperative game theory model* – like the model of a game with transferable utility in Chap. 9 – include: the abstraction from a detailed description of the strategic possibilities of a player; instead, a detailed description of what players and coalitions can attain in terms of outcomes or utilities; solution concepts based on strategic considerations and/or considerations of fairness, equity, efficiency, etc.; if possible, an axiomatic characterization of such solution concepts. For instance, one can argue that the core for TU-games is based on strategic considerations whereas the Shapley value is based on a combination of efficiency and symmetry or fairness with respect to contributions. The latter is made precise by an axiomatic characterization as in Problem 9.13.

In this chapter a few other cooperative game theory models are discussed: bargaining problems in Sect. 10.1, exchange economies in Sect. 10.2, matching problems in Sect. 10.3, and house exchange in Sect. 10.4.

10.1 Bargaining Problems

An example of a bargaining problem is the division problem in Sect. 1.3.5. A noncooperative, strategic approach to such a bargaining problem can be found in Sect. 6.7, see also Problems 6.15 and 6.16. In this section we treat the bargaining problem from a cooperative, axiomatic perspective. Surprisingly, there is a close relation between this approach and the strategic approach, as we will see below. In Sect. 10.1.1 we discuss the Nash bargaining solution and in Sect. 10.1.2 its relation with the Rubinstein bargaining procedure of Sect. 6.7.

H. Peters, *Game Theory – A Multi-Leveled Approach.*

10.1.1 The Nash Bargaining Solution

We start with the general definition of a two-person bargaining problem.[1]

Definition 10.1. A two-person *bargaining problem* is a pair (S,d), where

(1) $S \subseteq \mathbb{R}^2$ is a convex, closed and bounded set.[2]
(2) $\mathbf{d} = (d_1,d_2) \in S$ such that there is some point $\mathbf{x} = (x_1,x_2) \in S$ with $x_1 > d_1$ and $x_2 > d_2$.

S is the *feasible set* and $\mathbf{d}$ is the *disagreement point.*

The interpretation of a bargaining problem $(S,\mathbf{d})$ is as follows. The two players bargain over the feasible outcomes in S. If they reach an agreement $\mathbf{x} = (x_1,x_2) \in S$, then player 1 receives utility x_1 and player 2 receives utility x_2. If they do not reach an agreement, then the game ends in the disagreement point $\mathbf{d}$, yielding utility d_1 to player 1 and d_2 to player 2. Note that this is an interpretation: the bargaining procedure is not spelled out, so formally there is only the pair $(S,\mathbf{d})$.

For the example in Sect. 1.3.5, the feasible set and the disagreement point are given by

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \leq x_1,x_2 \leq 1,\ x_2 \leq \sqrt{1-x_1}\}, \quad d_1 = d_2 = 0.$$

See also Fig. 1.7. In general, a bargaining problem may look as in Fig. 10.1. The set of all such bargaining problems is denoted by $\mathscr{B}$.

Nash [90] raised the following question: for any given bargaining problem $(S,\mathbf{d})$, what is a good compromise? In formal terms, he was looking for a map $F : \mathscr{B} \to \mathbb{R}^2$ which assigns a feasible point to every bargaining problem, i.e., satisfies $F(S,\mathbf{d}) \in S$ for every $(S,\mathbf{d}) \in \mathscr{B}$. Such a map is called a (two-person) *bargaining solution.* According to Nash, a bargaining solution should satisfy four conditions, namely:

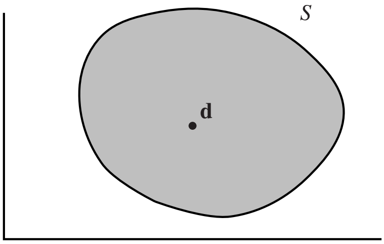

Fig. 10.1 A two-person bargaining problem

[1] We restrict attention here to two-person bargaining problems. For n-person bargaining problems and, more generally, NTU-games, see Remark 10.3 and Chap. 21.

[2] A subset of $\mathbb{R}^k$ is convex if with each pair of points in the set also the line segment connecting these points is in the set. A set is closed if it contains its boundary or, equivalently, if for every sequence of points in the set that converges to a point that limit point is also in the set. It is bounded if there is a number $M > 0$ such that $|x_i| \leq M$ for all points $\mathbf{x}$ in the set and all coordinates i.

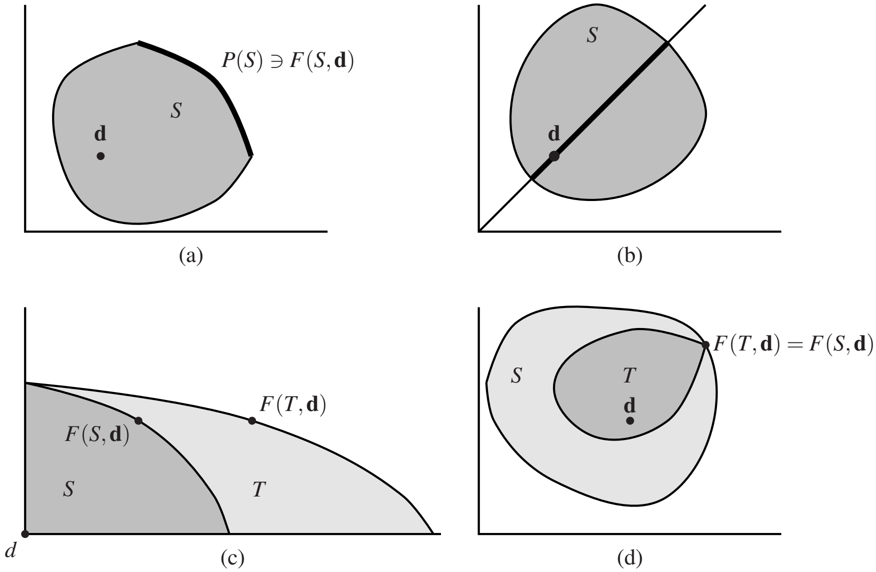

Fig. 10.2 Illustration of the four conditions ('axioms') determining the Nash bargaining solution – cf. Theorem 10.2. In (**a**) the Pareto optimal subset of S is the thick black curve. The bargaining problem $(S,\mathbf{d})$ in (**b**) is symmetric, and symmetry of F means that F should assign a point of the thick black line segment. In (**c**), which illustrates scale covariance, we took $\mathbf{d}$ to be the origin, and T arises from S by multiplying all first coordinates by 2: then scale covariance implies that $F_1(T,\mathbf{d}) = 2F_1(S,\mathbf{d})$. The independence of irrelevant alternatives axiom is illustrated in (**d**)

Pareto optimality, symmetry, scale covariance, and independence of irrelevant alternatives. We discuss each of these conditions in detail. The conditions are illustrated in Fig. 10.2a–d.

For a bargaining problem $(S,\mathbf{d}) \in \mathscr{B}$, the Pareto optimal points of S are those where the utility of no player can be increased without decreasing the utility of the other player. Formally,

$$P(S) = \{\mathbf{x} \in S \mid \text{for all } \mathbf{y} \in S \text{ with } y_1 \geq x_1,\, y_2 \geq x_2, \text{ we have } \mathbf{y} = \mathbf{x}\}$$

is the *Pareto optimal* (sub)set of S. The bargaining solution F is *Pareto optimal* if $F(S,\mathbf{d}) \in P(S)$ for all $(S,\mathbf{d}) \in \mathscr{B}$. Hence, a Pareto optimal bargaining solution assigns a Pareto optimal point to each bargaining problem. See Fig. 10.2a for an illustration.

A bargaining problem $(S,\mathbf{d}) \in \mathscr{B}$ is *symmetric* if $d_1 = d_2$ and if S is symmetric with respect to the 45°-line through $\mathbf{d}$, i.e., if

$$S = \{(x_2,x_1) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\}.$$

In a symmetric bargaining problem there is no way to distinguish between the players other than by the arbitrary choice of axes. A bargaining solution is *symmetric* if $F_1(S,\mathbf{d}) = F_2(S,\mathbf{d})$ for each symmetric bargaining problem $(S,\mathbf{d}) \in \mathscr{B}$. Hence, a symmetric bargaining solution assigns the same utility to each player in a symmetric bargaining problem. See Fig. 10.2b.

Observe that, for a symmetric bargaining problem $(S,\mathbf{d})$, Pareto optimality and symmetry of F would completely determine the solution point $F(S,\mathbf{d})$, since there is a unique symmetric Pareto optimal point in S.

The condition of scale covariance says that a bargaining solution should not depend on the choice of the origin or on a positive multiplicative factor in the utilities. For instance, in the wine division problem in Sect. 1.3.5, it should not matter if the utility functions were $\bar{u}_1(\alpha) = a_1\alpha + b_1$ and $\bar{u}_2(\alpha) = a_2\sqrt{\alpha} + b_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1, a_2 > 0$. Saying that this should not matter means that the final outcome of the bargaining problem, the division of the wine, should not depend on this. One can think of $\bar{u}_1, \bar{u}_2$ expressing the same preferences about wine as u_1, u_2 in different units.[3] Formally, a bargaining solution F is *scale covariant* if for all $(S,\mathbf{d}) \in \mathscr{B}$ and all $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1, a_2 > 0$ we have

$$F\left(\{(a_1x_1 + b_1, a_2x_2 + b_2) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\}, (a_1d_1 + b_1, a_2d_2 + b_2)\right)$$
$$= (a_1F_1(S,\mathbf{d}) + b_1, a_2F_2(S,\mathbf{d}) + b_2).$$

For a simple case, this condition is illustrated in Fig. 10.2c.

The final condition imposed by Nash [90] is also regarded as the most controversial one. Consider a bargaining problem $(S,\mathbf{d})$ with solution outcome $\mathbf{z} = F(S,\mathbf{d}) \in S$. In a sense, $\mathbf{z}$ can be regarded as the best compromise in S according to F. Now consider a smaller bargaining problem $(T,\mathbf{d})$ with $T \subseteq S$ and $\mathbf{z} \in T$. Since $\mathbf{z}$ was the best compromise in S, it is should certainly be regarded as the best compromise in T: $\mathbf{z}$ is available in T and every point of T is also available in S. Thus, we should conclude that $F(T,\mathbf{d}) = \mathbf{z} = F(S,\mathbf{d})$. As a less abstract example, suppose that in the wine division problem the wine is split fifty–fifty, with utilities $(1/2, \sqrt{1/2})$. Suppose now that no player wants to drink more than $3/4$ liter of wine: more wine does not increase utility. In that case, the new feasible set is

$$T = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \le x_1 \le 3/4,\ 0 \le x_2 \le \sqrt{3/4},\ x_2 \le \sqrt{1-x_1}\}.$$

According to the argument above, the wine should still be split fifty–fifty: $T \subseteq S$ and $(1/2, \sqrt{1/2}) \in T$. This may seem reasonable but it is not hard to change the example in such a way that the argument is, at the least, debatable. For instance, suppose that player 1 still wants to drink as much as possible but player 2 does not want to drink more than 1/2 liter. In that case, the feasible set becomes

$$T' = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le \sqrt{1/2},\ x_2 \le \sqrt{1-x_1}\},$$

[3] The usual assumption (as in [90]) is that the utility functions are expected utility functions, which uniquely represent preferences up to choice of origin and scale.

and we would still split the wine fifty–fifty. In this case player 2 would obtain his maximal feasible utility: $(1/2, \sqrt{1/2})$ no longer seems a reasonable compromise since only player 1 makes a concession. This critique was formalized in [63], see Chap. 21.

Formally, a bargaining solution F is *independent of irrelevant alternatives* if for all $(S,\mathbf{d}),(T,\mathbf{d}) \in \mathscr{B}$ with $T \subseteq S$ and $F(S,\mathbf{d}) \in T$, we have $F(T,\mathbf{d}) = F(S,\mathbf{d})$. See Fig. 10.2d for an illustration.

Nash [90] proved that these four conditions determine a unique bargaining solution F^{Nash}, defined as follows. For $(S,\mathbf{d}) \in \mathscr{B}$, $F^{\text{Nash}}(S,\mathbf{d})$ is equal to the unique point $\mathbf{z} \in S$ with $z_i \geq d_i$ for $i = 1,2$ and such that

$$(z_1 - d_1)(z_2 - d_2) \geq (x_1 - d_1)(x_2 - d_2) \quad \text{for all } \mathbf{x} \in S \text{ with } x_i \geq d_i,\ i = 1,2.$$

The solution F^{Nash} is called the *Nash bargaining solution*. Formally, the result of Nash is as follows.

Theorem 10.2. *The Nash bargaining solution F^{Nash} is the unique bargaining solution which is Pareto optimal, symmetric, scale covariant, and independent of irrelevant alternatives.*

For a proof of this theorem and the fact that F^{Nash} is well defined – i.e., the point $\mathbf{z}$ above exists and is unique – see Chap. 21.

10.1.2 Relation with the Rubinstein Bargaining Procedure

In the Rubinstein bargaining procedure the players make alternating offers. See Sect. 6.7.2 for a detailed discussion of this noncooperative game, and Problem 6.16d for the application to the wine division problem of Sect. 1.3.5. Here, we use this example to illustrate the relation with the Nash bargaining solution.

The Nash bargaining solution assigns to this bargaining problem the point $\mathbf{z} = (2/3, \sqrt{1/3})$. This means that player 1 obtains 2/3 of the wine and player 2 obtains 1/3. According to the Rubinstein infinite horizon bargaining game with discount factor $0 < \delta < 1$ the players make proposals $\mathbf{x} = (x_1, x_2) \in P(S)$ and $\mathbf{y} = (y_1, y_2) \in P(S)$ such that

$$x_2 = \delta y_2, \quad y_1 = \delta x_1, \tag{10.1}$$

and these proposals are accepted in (subgame perfect) equilibrium. Setting $x_1 = \alpha$ for a moment, we obtain $y_1 = \delta\alpha$ and thus $\sqrt{1-\alpha} = x_2 = \delta y_2 = \sqrt{1-\delta\alpha}$. This is an equation in δ and α with solution (check!)

$$x_1 = \alpha = \frac{1-\delta^2}{1-\delta^3} = \frac{1+\delta}{1+\delta+\delta^2}.$$

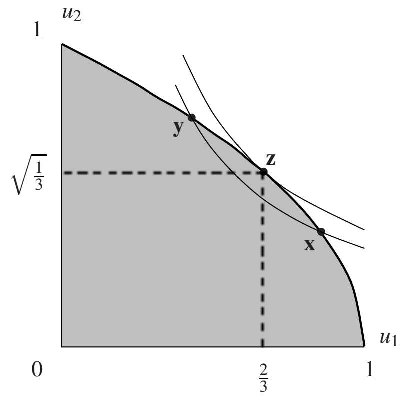

Fig. 10.3 The wine division problem. The disagreement point is the origin, and **z** is the Nash bargaining solution outcome. The points **x** and **y** are the proposals of players 1 and 2, respectively, in the subgame perfect equilibrium of the Rubinstein bargaining game for $\delta = 0.5$

For $\delta = 0.5$ the corresponding proposals **x** and **y** are represented in Fig. 10.3. In general, it follows from (10.1) that

$$(x_1 - d_1)(x_2 - d_2) = x_1 x_2 = (y_1/\delta)(\delta y_2) = y_1 y_2 = (y_1 - d_1)(y_2 - d_2)$$

hence the Rubinstein proposals **x** and **y** have the same 'Nash product': see Fig. 10.3, where the level curve of this product through **x** and **y** is drawn. As δ increases to 1, this level curve shifts up until it passes through the point **z**, since $\mathbf{z} = F^{\text{Nash}}(S,d)$ maximizes this product on the set S: see again Fig. 10.3.

We conclude that the subgame perfect equilibrium outcome of the infinite horizon Rubinstein bargaining game converges to the Nash bargaining solution outcome as the discount factor δ approaches 1.

Remark 10.3. Two-person bargaining problems and TU-games (Chap. 9) are both special cases of the general model of *cooperative games without transferable utility*, so-called NTU-games. In an NTU-game, a *set* of feasible utility vectors $V(T)$ is assigned to each coalition T of players. For a TU-game (N,v) and a coalition T, this set takes the special form $V(T) = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in T} x_i \leq v(T)\}$, i.e., a coalition T can attain any vector of utilities such that the sum of the utilities for the players in T does not exceed the worth of the coalition. In a two-player bargaining problem $(S,\mathbf{d})$, one can set $V(\{1,2\}) = S$ and $V(\{i\}) = \{\alpha \in \mathbb{R} \mid \alpha \leq d_i\}$ for $i = 1,2$. See also Chap. 21.

10.2 Exchange Economies

In an exchange economy with n agents and k goods, each agent is initially endowed with a bundle of goods. Each agent has preferences over different bundles of goods, expressed by some utility function over these bundles. By exchanging goods among each other, it is in general possible to increase the utilities of all agents. One way to arrange this exchange is to introduce prices. For given prices the endowment of each agent represents the agent's income, which can be spent on buying a bundle of the goods that maximizes the agent's utility. If prices are such that the market for each good clears – total demand is equal to total endowment – while each agent maximizes utility, then the prices are in equilibrium: such an equilibrium is called Walrasian or competitive equilibrium. Alternatively, reallocations of the goods can be considered which are in the *core* of the exchange economy. A reallocation of the total endowment is in the core of the exchange economy if no coalition of agents can improve the utilities of its members by, instead, reallocating the total endowment of its own members among each other. It is well known that a competitive equilibrium allocation is an example of a core allocation.

This section is a first acquaintance with exchange economies. Attention is restricted to exchange economies with two agents and two goods. We work out an example of such an economy. Some variations are considered in Problem 10.4.

There are two agents, A and B, and two goods, 1 and 2. Agent A has an endowment $\mathbf{e}^A = (e_1^A, e_2^A) \in \mathbb{R}_+^2$ of the goods, and a utility function $u^A : \mathbb{R}_+^2 \to \mathbb{R}$, representing the preferences of A over bundles of goods.[4] Similarly, agent B has an endowment $\mathbf{e}^B = (e_1^B, e_2^B) \in \mathbb{R}_+^2$ of the goods, and a utility function $u^B : \mathbb{R}_+^2 \to \mathbb{R}$. (Note that we use superscripts to denote the agents and subscripts to denote the goods.) This is a complete description of the exchange economy.

For our example we take $\mathbf{e}^A = (2,3)$, $\mathbf{e}^B = (4,1)$, $u^A(x_1,x_2) = x_1^2 x_2$ and $u^B(x_1,x_2) = x_1 x_2^2$. Hence, the total endowment in the economy is $\mathbf{e} = (6,4)$, and the purpose of the exchange is to reallocate this bundle of goods such that both agents are better off.

Let $\mathbf{p} = (p_1, p_2)$ be a vector of positive prices of the goods. Given these prices, both agents want to maximize their utilities. Agent A has an income of $p_1 e_1^A + p_2 e_2^A$, i.e., the monetary value of his endowment. Then agent A solves the maximization problem

$$\begin{array}{l} \text{maximize } u^A(x_1,x_2) \\ \text{subject to } p_1 x_1 + p_2 x_2 = p_1 e_1^A + p_2 e_2^A,\ x_1, x_2 \geq 0. \end{array} \tag{10.2}$$

The income constraint is called the *budget equation*. The solution of this maximization problem is a bundle $\mathbf{x}^A(\mathbf{p}) = (x_1^A(\mathbf{p}), x_2^A(\mathbf{p}))$, called agent A's *demand function*. Problem (10.2) is called the *consumer problem* (of agent A). Similarly, agent B's consumer problem is

$$\begin{array}{l} \text{maximize } u^B(x_1,x_2) \\ \text{subject to } p_1 x_1 + p_2 x_2 = p_1 e_1^B + p_2 e_2^B,\ x_1, x_2 \geq 0. \end{array} \tag{10.3}$$

For our example, (10.2) becomes

$$\begin{array}{l} \text{maximize } x_1^2 x_2 \\ \text{subject to } p_1 x_1 + p_2 x_2 = 2p_1 + 3p_2,\ x_1, x_2 \geq 0, \end{array}$$

[4] $\mathbb{R}_+^2 := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid x_1,\ x_2 \geq 0\}$.

which can be solved by using Lagrange's method or by substitution. By using the latter method the problem reduces to

$$\text{maximize } x_1^2\,((2p_1+3p_2-p_1x_1)/p_2)$$

subject to $x_1 \geq 0$ and $2p_1+3p_2-p_1x_1 \geq 0$. Setting the derivative with respect to x_1 equal to 0 yields

$$2x_1\left(\frac{2p_1+3p_2-p_1x_1}{p_2}\right)-x_1^2\left(\frac{p_1}{p_2}\right)=0,$$

which after some simplifications yields the demand function $x_1 = x_1^A(\mathbf{p}) = (4p_1+6p_2)/3p_1$. By using the budget equation, $x_2^A(\mathbf{p}) = (2p_1+3p_2)/3p_2$. Similarly, solving (10.3) for our example yields $x_1^B(\mathbf{p}) = (4p_1+p_2)/3p_1$ and $x_2^B(\mathbf{p}) = (8p_1+2p_2)/3p_2$ (check this).

The prices $\mathbf{p}$ are *Walrasian equilibrium* prices if the markets for both goods clear. For the general model, this means that $x_1^A(\mathbf{p})+x_1^B(\mathbf{p}) = e_1^A+e_1^B$ and $x_2^A(\mathbf{p})+x_2^B(\mathbf{p}) = e_2^A+e_2^B$. For the example, this means

$$(4p_1+6p_2)/3p_1+(4p_1+p_2)/3p_1 = 6 \quad \text{and}$$

$$(2p_1+3p_2)/3p_2+(8p_1+2p_2)/3p_2 = 4.$$

Both equations result in the same condition, namely $10p_1-7p_2=0$. This is no coincidence, since prices are only relative, as is easily seen from the budget equations. In fact, the prices represent the rate of exchange between the two goods, and are meaningful even if money does not exist in the economy. Thus, $\mathbf{p}=(7,10)$ (or any positive multiple thereof) are the equilibrium prices in this exchange economy. The associated equilibrium demands are $\mathbf{x}^A(7,10) = (88/21, 22/15)$ and $\mathbf{x}^B(7,10) = (38/21, 38/15)$.

We now turn to the *core* of an exchange economy. A reallocation of the total endowments is in the core if no coalition can improve upon it. Basically, this is the same definition as in Chap. 9 for TU-games (Definition 9.2). In a two-person exchange economy, there are only three coalitions (excluding the empty coalition), namely $\{A\}$, $\{B\}$, and $\{A,B\}$. Consider an allocation $(\mathbf{x}^A,\mathbf{x}^B)$ with $x_1^A+x_1^B = e_1^A+e_1^B$ and $x_2^A+x_2^B = e_2^A+e_2^B$. To avoid that agents A or B can improve upon $(\mathbf{x}^A,\mathbf{x}^B)$ we need that

$$u^A(\mathbf{x}^A) \geq u^A(\mathbf{e}^A), \quad u^B(\mathbf{x}^B) \geq u^B(\mathbf{e}^B), \tag{10.4}$$

which are the *individual rationality* constraints. To avoid that the grand coalition $\{A,B\}$ can improve upon $(\mathbf{x}^A,\mathbf{x}^B)$ we need that

For *no* $(\mathbf{y}^A,\mathbf{y}^B)$ with $y_1^A+y_1^B = e_1^A+e_1^B$ and $y_2^A+y_2^B = e_2^A+e_2^B$ we have: $u^A(\mathbf{y}^A) \geq u^A(\mathbf{x}^A)$ and $u^B(\mathbf{y}^B) \geq u^B(\mathbf{x}^B)$ with at least one inequality strict. (10.5)

In words, (10.5) says that there should be no other reallocation of the total endowments such that no agent is worse off and at least one agent is strictly better off. This is the *efficiency* or *Pareto optimality* constraint.

We apply (10.4) and (10.5) to our example. The individual rationality constraints are

$$(x_1^A)^2 x_2^A \geq 12, \quad x_1^B (x_2^B)^2 \geq 4.$$

The Pareto optimal allocations, satisfying (10.5), can be computed as follows. Fix the utility level of one of the agents, say B, and maximize the utility of A subject to the utility level of B being fixed. By varying the fixed utility level of B we find all Pareto optimal allocations. In the example, we solve the following maximization problem for $c \in \mathbb{R}$:

$$\begin{aligned}&\text{maximize } (x_1^A)^2 x_2^A \\ &\text{subject to } x_1^A + x_1^B = 6,\ x_2^A + x_2^B = 4,\ x_1^B (x_2^B)^2 = c,\ x_1^A, x_2^A, x_1^B, x_2^B \geq 0.\end{aligned}$$

By substitution this problem reduces to

$$\begin{aligned}&\text{maximize } (x_1^A)^2 x_2^A \\ &\text{subject to } (6 - x_1^A)(4 - x_2^A)^2 = c,\ x_1^A, x_2^A \geq 0.\end{aligned}$$

The associated Lagrange function is $(x_1^A)^2 x_2^A - \lambda\,[(6 - x_1^A)(4 - x_2^A)^2 - c]$ and the first-order conditions are

$$2x_1^A x_A^2 + \lambda(4 - x_2^A)^2 = 0, \quad (x_1^A)^2 + 2\lambda(6 - x_1^A)(4 - x_2^A) = 0.$$

Extracting λ from both equations and simplifying yields

$$x_2^A = \frac{4x_1^A}{24 - 3x_1^A}.$$

Thus, for any value of x_1^A between 0 and 6 this equation returns the corresponding value of x_2^A, resulting in a Pareto optimal allocation with $x_1^B = 6 - x_1^A$ and $x_2^B = 4 - x_2^A$.

It is straightforward to check by substitution that the Walrasian equilibrium allocation $\mathbf{x}^A(7,10) = (88/21, 22/15)$ and $\mathbf{x}^B(7,10) = (38/21, 38/15)$ found above, is Pareto optimal. This is no coincidence: the *First Welfare Theorem* states that in an exchange economy like the one under consideration, a Walrasian equilibrium allocation is Pareto optimal.[5]

Combining the individual rationality constraint for agent A with the Pareto optimality constraint yields $4(x_1^A)^3/(24 - 3x_1^A) \geq 12$, which holds for x_1^A larger than approximately 3.45. For agent B, similarly, the individual rationality and Pareto optimality constraints imply

$$(6 - x_1^A)\left(\frac{96 - 16x_1^A}{24 - 3x_1^A}\right)^2 \geq 4,$$

[5] For a proof of this theorem, see for instance [59].

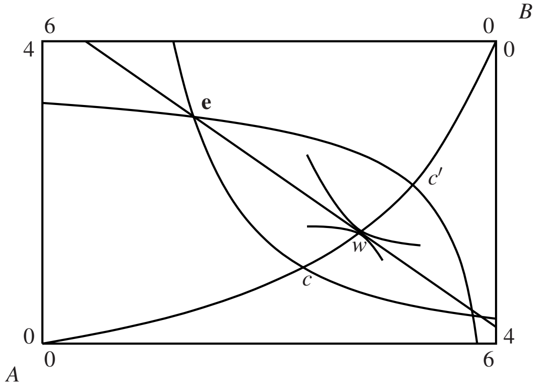

Fig. 10.4 The contract curve is the curve through c and c'. The point c is the point of intersection of the contract curve and the indifference curve of agent A through the endowment point **e**. The point c' is the point of intersection of the contract curve and the indifference curve of agent B through the endowment point **e**. The core consists of the allocations on the contract curve between c and c'. The straight line ('budget line') through **e** is the graph of the budget equation for A at the equilibrium prices, i.e., $7x_1 + 10x_2 = 44$, and its point of intersection with the contract curve, w, is the Walrasian equilibrium allocation. At this point the indifference curves of the two agents are both tangential to the budget line

which holds for x_1^A smaller than approximately 4.88. Hence, the core of the exchange economy in the example is the set

$$\{(x_1^A, x_2^A, x_1^B, x_2^B) \in \mathbb{R}^4 \mid 3.45 \leq x_1^A \leq 4.88,$$
$$x_2^A = \frac{4x_1^A}{24 - 3x_1^A},\ x_1^B = 6 - x_1^A,\ x_2^B = 4 - x_2^A\}.$$

Clearly, the Walrasian equilibrium allocation is in the core, since $3.45 \leq 88/21 \leq 4.88$, and also this is no coincidence (see for instance [59]). Thus, decentralization of the reallocation process through prices leads to an allocation that is in the core.

For an exchange economy with two agents and two goods a very useful pictorial device is the *Edgeworth box*, see Fig. 10.4. The Edgeworth box consists of all possible reallocations of the two goods. The origin for agent A is the South West corner and the origin for agent B the North East corner. In the diagram, the indifference curves of the agents through the endowment point are plotted, as well as the *contract curve*, i.e., the set of Pareto optimal allocations. The core is the subset of the contract curve between the indifference curves of the agents through the endowment point.

10.3 Matching Problems

In a matching problem there is a group of agents that have to form couples. Examples are: students who have to be coupled with schools; hospitals that have to be coupled with doctors; workers who have to be coupled with firms; men who have

Table 10.1 A matching problem

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_2	m_1
w_1	w_2	w_2	m_3	m_1	m_3
	w_3		m_2	m_3	m_2

to be coupled with women; etc. In this section we consider so-called one-to-one matching problems.[6]

The agents are divided in two equally large (finite and nonempty) sets, denoted M and W. Each agent in M has a strict preference over those agents in W which he prefers over staying single. Similarly, each agent in W has a strict preference over those agents in M which she prefers over staying single. In such a *matching problem*; a *matching* assigns to each agent in M at most one agent in W, and vice versa; thus, no two agents in M are assigned the same agent in W, and vice versa.

Such matching problems are also called *marriage problems*, and the agents of M and W are called *men* and *women*, respectively. While the problem may indeed refer to the 'marriage market', this terminology is of course adopted for convenience. Other examples are matching tasks and people, or matching roommates.

As an example, consider the matching problem in Table 10.1. The set of men is $M = \{m_1, m_2, m_3\}$ and the set of women is $W = \{w_1, w_2, w_3\}$. The columns in the table represent the preferences. For instance m_1 prefers w_2 over w_1 over staying single, but prefers staying single over w_3. An example of a matching in this particular matching problem is (m_1, w_1), (m_3, w_2), m_2, w_3, meaning that m_1 is married to w_1 and m_3 to w_2, while m_2 and w_3 stay single.[7] Observe that this matching does not seem very 'stable': m_1 and w_2 would prefer to be married to each other instead of to their partners in the given matching. Moreover, m_2 and w_3 would prefer to be married to each other instead of being single. Also, for instance, any matching in which m_1 would be married to w_3 would not be plausible, since m_1 would prefer to stay single.

The obvious way to formalize these considerations is to require that a matching should be in the *core* of the matching problem. A matching is in the core of there is no subgroup (coalition) of men and/or women who can do better by marrying (or staying single) among each other. For a matching to be in the core, the following two requirements are certainly necessary:

(c1) Each person prefers his/her partner over being single.
(c2) If $m \in M$ and $w \in W$ are not matched to each other, then it is *not* the case that both m prefers w over his current partner if m is married or over being single if m is not married; and w prefers m over her current partner if w is married or over being single if w is not married.

[6] This section is largely based on Sect. 8.7 in [96].

[7] Check that there are 34 possible different matchings for this problem.

Obviously, if (c1) were violated then the person in question could improve by divorcing and becoming single; if (c2) were violated then m and w would both be better off by marrying each other. A matching satisfying (c1) and (c2) is called *stable*. Hence, any matching in the core must be stable. Interestingly, the converse is also true: any stable matching is in the core. To see this, suppose there is a matching outside the core and satisfying (c1) and (c2). Then there is a coalition of agents each of whom can improve by marrying or staying single within that coalition. If a member of the coalition improves by becoming single, then (c1) is violated. If two coalition members improve by marrying each other, then (c2) is violated. This contradiction establishes the claim that stable matchings must be in the core. Thus, the core of a matching problem is the set of all stable matchings.

How can stable matchings be computed? A convenient procedure is the *deferred acceptance procedure* proposed in [41]. In this procedure, the members of one of the two parties propose and the members of the other party accept or reject proposals. Suppose men propose. In the first round, each man proposes to his favorite woman (or stays single if he prefers that) and each woman, if proposed to at least once, chooses her favorite man among those who have proposed to her (which may mean staying single). This way, a number of couples may form, and the involved men and women are called 'engaged'. In the second round, the non-engaged men (in particular, rejected men) propose to their second-best woman (or stay single); then each woman again picks here favorite among the men who proposed to her including possibly the man to whom she is currently engaged. The procedure continues until all proposals are accepted. Then all currently engaged couples marry and a matching is established.

It is not hard to verify that this matching is stable. A man who stays single was rejected by all women he preferred over staying single and therefore can find no woman who prefers him over her husband or over being single. A woman who stays single was never proposed to by any man whom she prefers over staying single. Consider, finally, an $m \in M$ and a $w \in W$ who are married but not to each other. If m prefers w over his current wife, then w must have rejected him for a better partner somewhere in the procedure. If w prefers m over her current husband, then m has never proposed to her and, thus, prefers his wife over her.

Of course, the deferred acceptance procedure can also be applied with women as proposers, resulting in a stable matching that is different in general.

Table 10.2 shows how the deferred acceptance procedure with the men proposing works, applied to the matching problem in Table 10.1.

Table 10.2 The deferred acceptance procedure applied to the matching problem of Table 10.1. The resulting matching is (m_1, w_1), (m_2, w_2), (m_3, w_3)

Stage 1	Stage 2	Stage 3	Stage 4
m_1: $\rightarrow w_2$	rejected	$\rightarrow w_1$	
m_2 $\rightarrow w_1$ rejected	$\rightarrow w_2$		
m_3 $\rightarrow w_1$		rejected	$\rightarrow w_2$ rejected

There may be other stable matchings than those found by applying the deferred acceptance procedure with the men and with the women proposing. It can be shown that the former procedure – with the men proposing – results in a stable matching that is optimal, among all stable matchings, from the point of view of the men, whereas the latter procedure – with the women proposing – produces the stable matching optimal from the point of view of the women. See also Problems 10.6 and 10.7.

10.4 House Exchange

In a house exchange problem each one of finitely many agents owns a house, and has a preference over all houses. The purpose of the exchange is to make the agents better off. A house exchange problem is an exchange economy with as many goods as there are agents, and where each agent is endowed with one unit of a different, indivisible good.[8]

Formally, the player set is $N = \{1,\ldots,n\}$, and each player $i \in N$ owns house h_i, and has a strict preference of the set of all (n) houses. In a *core* allocation, each player obtains exactly one house, and there is no coalition that can make each of its members strictly better off by exchanging their *initially owned* houses among themselves.

As an example, consider the house exchange problem in Table 10.3. In this problem there are six possible different allocations of the houses. Table 10.4 lists these allocations and also which coalition could improve by exchanging their own houses.

Table 10.3 A house exchange problem with three players. For instance, player 1 prefers the house of player 3 over the house of player 2 over his own house

Player 1	Player 2	Player 3
h_3	h_1	h_2
h_2	h_2	h_3
h_1	h_3	h_1

Table 10.4 Analysis of the house exchange problem of Table 10.3. There are two core allocations

Player 1	Player 2	Player 3	Improving coalition(s)
h_1	h_2	h_3	$\{1,2\}$, $\{1,2,3\}$
h_1	h_3	h_2	$\{2\}$, $\{1,2\}$
h_2	h_1	h_3	None: core allocation
h_2	h_3	h_1	$\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,2,3\}$
h_3	h_1	h_2	None: core allocation
h_3	h_2	h_1	$\{3\}$

[8] This section is based largely on Sect. 8.5 in [96].

Especially for larger problems, the 'brute force' analysis as in Table 10.4 is rather cumbersome. A different and more convenient way is to use the *top trading cycle procedure*. In a given house exchange problem a *top trading cycle* is a sequence $i_1, i_2, \ldots, i_k$ of players, with $k \geq 1$, such that the favorite house of player i_1 is house h_{i_2}, the favorite house of player i_2 is house h_{i_3}, ..., and the favorite house of player i_k is house h_{i_1}. If $k = 1$, then this simply means that player i_1 already owns his favorite house. In the top trading cycle procedure, we look for a top trading cycle, assign houses within the cycle, and next the involved players and their houses leave the scene. Then we repeat the procedure for the remaining players, etc., until no player is left.

In the example in Table 10.3 there is only one top trading cycle, namely $1,3,2$, resulting in the allocation $1 : h_3$, $3 : h_2$, $2 : h_1$, a core allocation: in fact, each player obtains his top house. In general, it is true that *for strict preferences the top trading cycle procedure results in a core allocation.* The reader should check the validity of this claim (Problem 10.8).

What about the other core allocation found in Table 10.4? In this allocation, the grand coalition could *weakly improve*: by the allocation $1 : h_3, 3 : h_2, 2 : h_1$ players 1 and 3 would be strictly better off, while player 2 would not be worse off. We define the *strong core* as consisting of those allocations on which no coalition could even weakly improve, that is, make all its members at least as good off and at least one member strictly better off. In the example, only the allocation $1 : h_3$, $3 : h_2$, $2 : h_1$ is in the strong core. In general, one can show that *the strong core of a house exchange problem with strict preferences consists of the unique allocation produced by the top trading cycle procedure.*

Problems

10.1. *A Division Problem (1)*

Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Player 1 has utility function $u_1(\alpha) = \alpha$ and player 2 has utility function $u_2(\beta) = 1 - (1-\beta)^2$ for amounts $\alpha, \beta \in [0,1]$ of the good.

(a) Determine the set of feasible utility pairs. Make a picture.

(b) Determine the Nash bargaining solution outcome, in terms of utilities as well as of the physical distribution of the good.

(c) Suppose the players' utilities are discounted by a factor $\delta \in [0,1)$. Calculate the Rubinstein bargaining outcome, i.e., the subgame perfect equilibrium outcome of the infinite horizon alternating offers bargaining game.

(d) Determine the limit of the Rubinstein bargaining outcome, for δ approaching 1, in two ways: by using the result of (b) and by using the result of (c).

10.2. *A Division Problem (2)*

Suppose that two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Assume that player 1 has utility function $u(\alpha)$ $(0 \leq \alpha \leq 1)$ and player 2 has utility function $v(\alpha) = 2u(\alpha)$ $(0 \leq \alpha \leq 1)$.

Determine the physical distribution of the good according to the Nash bargaining solution. Can you say something about the resulting utilities? (Hint: use the relevant properties of the Nash bargaining solution.)

10.3. *A Division Problem (3)*

Suppose that two players (bargainers) bargain over the division of two units of a perfectly divisible good. Assume that player 1 has a utility function $u(\alpha) = \frac{\alpha}{2}$ $(0 \leq \alpha \leq 2)$ and player 2 has utility function $v(\alpha) = \sqrt[3]{\alpha}$ $(0 \leq \alpha \leq 2)$.

(a) Determine the physical distribution of the good according to the Rubinstein bargaining procedure, for any discount factor $0 < \delta < 1$.

(b) Use the result to determine the Nash bargaining solution distribution.

(c) Suppose player 1's utility function changes to $w(\alpha) = \alpha$ for $0 \leq \alpha \leq 1.6$ and $w(\alpha) = 1.6$ for $1.6 \leq \alpha \leq 2$. Determine the Nash bargaining solution outcome, both in utilities and in physical distribution, for this new situation.

10.4. *An Exchange Economy*

Consider an exchange economy with two agents A and B and two goods. The agents are endowed with initial bundles $\mathbf{e}^A = (3,1)$ and $\mathbf{e}^B = (1,3)$. Their preferences are represented by the utility functions $u^A(x_1,x_2) = \ln(x_1+1) + \ln(x_2+2)$ and $u^B(x_1,x_2) = 3\ln(x_1+1) + \ln(x_2+1)$.

(a) Compute the demand functions of the agents.

(b) Compute Walrasian equilibrium prices and the equilibrium allocation.

(c) Compute the contract curve and the core. Sketch the Edgeworth box.

(d) Show that the Walrasian equilibrium allocation is in the core.

(e) How would you set up a two-person bargaining problem associated with this economy? Would it make sense to consider the Nash bargaining solution in order to compute an allocation? Why or why not?

10.5. *The Matching Problem of Table 10.1 Continued*[9]

(a) Apply the deferred acceptance procedure to the matching problem of Table 10.1 with the women proposing.

(b) Are there any other stable matchings in this example?

[9] Problems 10.5–10.7 are taken from [96].

Table 10.5 The matching problem of Problem 10.6

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_1	m_1	m_1	m_1
w_2	w_2	w_3	m_2	m_3	m_2
w_3	w_3	w_2	m_3	m_2	m_3

Table 10.6 The matching problem of Problem 10.7

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

10.6. *Another Matching Problem*

Consider the matching problem with three men, three women, and preferences as in Table 10.5.

(a) Compute the two matchings produced by the deferred acceptance procedure with the men and with the women proposing.

(b) Are there any other stable matchings?

(c) Verify the claim made in the text about the optimality of the matchings in (a) for the men and the women, respectively.

10.7. *Yet Another Matching Problem: Strategic Behavior*

Consider the matching problem with three men, three women, and preferences as in Table 10.6.

(a) Compute the two matchings produced by the deferred acceptance procedure with the men and with the women proposing.

(b) Are there any other stable matchings?

Now consider the following noncooperative game. The players are w_1, w_2, and w_3. The strategy set of a player is simply the set of all possible preferences over the men. (Thus, each player has 16 different strategies.) The outcomes of the game are the matchings produced by the deferred acceptance procedure with the men proposing, assuming that each man uses his true preference given in Table 10.6.

(c) Show that the following preferences form a Nash equilibrium: w_2 and w_3 use their true preferences, as given in Table 10.6; w_1 uses the preference (m_1, m_2, m_3). Conclude that sometimes it may pay off to lie about one's true preference. (Hint: in a Nash equilibrium, no player can gain by deviating.)

10.8. *Core Property of Top Trading Cycle Procedure*

Show that for strict preferences the top trading cycle results in a core allocation.

Table 10.7 The house exchange problem of Problem 10.10

Player 1	Player 2	Player 3	Player 4
h_3	h_4	h_1	h_3
h_2	h_1	h_4	h_2
h_4	h_2	h_3	h_1
h_1	h_3	h_2	h_4

10.9. *House Exchange with Identical Preferences*

Consider the n-player house exchange problem where all players have identical strict preferences. Find the house allocation(s) in the core.

10.10. *A House Exchange Problem*[10]

Consider the house exchange problem with four players in Table 10.7.
Compute all core allocations and all strong core allocations.

10.11. *Cooperative Oligopoly*

Consider the Cournot oligopoly game with n firms with different costs $c_1, c_2, \ldots, c_n$. (This is the game of Problem 6.2 with heterogenous costs.) As before, each firm i offers $q_i \geq 0$, and the price-demand function is $p = \max\{0, a - \sum_{i=1}^{n} q_i\}$, where $0 < c_i < a$ for all i.

(a) Show that the reaction function of player i is

$$q_i = \max\{0, \frac{a - c_i - \sum_{j \neq i} q_i}{2}\}.$$

(b) Show that the unique Nash equilibrium of the game is $\mathbf{q}^* = (q_1^*, \ldots, q_n^*)$ with

$$q_i^* = \frac{a - nc_i + \sum_{j \neq i} c_j}{n+1},$$

for each i, assuming that this quantity is positive.

(c) Derive that the corresponding profits are

$$\frac{(a - nc_i + \sum_{j \neq i} c_j)^2}{(n+1)^2}$$

for each player i.

Let the firms now be the players in a cooperative TU-game with player set $N = \{1, 2, \ldots, n\}$, and consider a coalition $S \subseteq N$. What is the total profit that S can make on its own? This depends on the assumptions that we make on the behavior of the

[10] Taken from [83].

players outside S. Very pessimistically, one could solve the problem

$$\max_{q_i : i \in S} \min_{q_j : j \notin S} \sum_{i \in S} P_i(q_1, \ldots, q_n),$$

which is the profit that S can guarantee independent of the players outside S. This view is very pessimistic because it presumes maximal resistance of the outside players, even if this means that these outside players hurt themselves. In the present case it is not hard to see that this results in zero profit for S.

Two alternative scenarios are: S plays a Cournot–Nash equilibrium in the $(n - |S| + 1)$-player oligopoly game against the outside firms as separate firms, or S plays a Cournot–Nash equilibrium in the duopoly game against $N \setminus S$.

In the first case we in fact have an oligopoly game with costs c_j for every player $j \notin S$ and with cost $c_S := \min\{c_i : i \in S\}$ for 'player' (coalition) S.

(d) By using the results of (a)–(c) show that coalition S obtains a profit of

$$v_1(S) = \frac{[a - (n - |S| + 1)c_S + \sum_{j \notin S} c_j]^2}{(n - |S| + 2)^2}$$

in this scenario. Thus, this scenario results in a cooperative TU-game (N, v_1).

(e) Assume $n = 3$, $a = 7$, and $c_i = i$ for $i = 1, 2, 3$. Compute the core, the Shapley value, and the nucleolus for the TU-game (N, v_1).

(f) Show that in the second scenario, coalition S obtains a profit of

$$v_2(S) = \frac{(a - 2c_S + c_{N-S})^2}{9},$$

resulting in a cooperative game (N, v_2).

(g) Assume $n = 3$, $a = 7$, and $c_i = i$ for $i = 1, 2, 3$. Compute the core, the Shapley value, and the nucleolus for the TU-game (N, v_2).

Chapter 11
Social Choice

Social choice theory studies the aggregation of individual preferences into a common or social preference. It overlaps with several social science disciplines, such as political theory (e.g., voting for Parliament, or for a president) and game theory (e.g., voters may vote strategically, or candidates may choose positions strategically). For a general overview see [3] and [4].

In the classical model of social choice, there is a finite number of agents who have preferences over a finite number of alternatives. These preferences are either aggregated into a social preference according to a so-called social welfare function, or result in a common alternative according to a so-called social choice function.

The main purpose of this chapter is to review two classical results, namely Arrow's [2] Theorem and Gibbard's [44] and Satterthwaite's [112] Theorem. The first theorem applies to social welfare functions and says that, if the social preference between any two alternatives should only depend on the individual preferences between these alternatives and, thus, not on individual preferences involving other alternatives, then the social welfare function must be dictatorial. The second theorem applies to social choice functions and says that the only social choice functions that are invulnerable to strategic manipulation are the dictatorial ones. These results are often referred to as 'impossibility theorems' since dictatorships are generally regarded undesirable.

Both results are closely related: indeed, the proof of the Theorem of Gibbard and Satterthwaite in [44] uses Arrow's Theorem. The presentation in this chapter closely follows that in Reny [108], which is both simple and elegant, and which shows the close relation between the two results. The approach in [108] is itself based on a few other sources: see [108] for the references.

Section 11.1 is introductory. Section 11.2 discusses Arrow's Theorem and Sect. 11.3 the Gibbard–Satterthwaite Theorem.

11.1 Introduction and Preliminaries

11.1.1 An Example

Suppose there are three agents (individuals, voters) who have strict preferences over a set of five alternatives ($a_1,\ldots,a_5$), as given in Table 11.1.

In this table the preferences of the players are represented by the *Borda scores*: the best alternative of an agent obtains 5 points, the second best 4 points, etc., until the worst alternative which obtains 1 point. So, for instance, agent 1 has the preference $a_1P_1a_5P_1a_3P_1a_4P_1a_2$ in the notation to be introduced below. We use the Borda scores (Borda [25]) as a convenient way to represent these preferences and, more importantly, to obtain an example of a social welfare as well as a social choice function.

First, suppose that we want to extract a common social ranking of the alternatives from the individual preferences. One way to do this is to add the Borda scores per alternative. In the example this results in $9,7,11,8,10$ for a_1,a_2,a_3,a_4,a_5, respectively, resulting in the social ranking $a_3Pa_5Pa_1Pa_4Pa_2$. If we just want to single out one alternative, then we could take the one with the maximal Borda score, in this case alternative a_3. In the terminology to be introduced formally below, Borda scores give rise to a social welfare as well as a social choice function.[1]

One potential drawback of using Borda scores to obtain a social ranking is, that the ranking between two alternatives may not just depend on the individual preferences between these two alternatives. For instance, suppose that agent 1's preference would change to $a_1P_1a_4P_1a_5P_1a_3P_1a_2$. Then the Borda scores would change to $9,7,8,10,9$ for a_1,a_2,a_3,a_4,a_5, respectively, resulting in the social ranking $a_4Pa_1Ia_5Pa_3Pa_2$ (where I denotes indifference). Observe that no agent's preference between a_1 and a_4 has changed, but that socially this preference is reversed. This is not a peculiarity of using Borda scores: Arrow's Theorem, to be discussed in Sect. 11.2, states that the only way to avoid this kind of preference reversal is to make one agent the dictator, i.e., to have the social preference coincide with the preference of one fixed agent.

A potential drawback of using the Borda scores in order to single out a unique alternative is that this method is vulnerable to strategic manipulation. For instance, suppose that agent 1 would lie about his true preference given in Table 11.1 and

Table 11.1 Borda scores

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
2	1	2	3	4	5
3	3	4	5	2	1

[1] Note that ties may occur, but this need not bother us here.

claim that his preference is $a_1 P_1 a_5 P_1 a_2 P_1 a_4 P_1 a_3$ instead. Then the Borda scores would change to $9, 9, 9, 8, 10$ for a_1, a_2, a_3, a_4, a_5, respectively, resulting in the chosen alternative a_5 instead of a_3. Since agent 1 prefers a_5 over a_3 according to his *true* preference, he gains by this strategic manipulation. Again, this phenomenon is not a peculiarity of the Borda method: the Gibbard–Satterthwaite Theorem in Sect. 11.3 shows that the only way to avoid it is (again) to make one fixed agent a dictator.

11.1.2 Preliminaries

Let $A = \{a_1, \ldots, a_m\}$ be the set of *alternatives*. To keep things interesting we assume $m \geq 3$.[2] The set of *agents* is denoted by $N = \{1, \ldots, n\}$. We assume $n \geq 2$.

A *binary relation* on A is a subset of $A \times A$. In our context, for a binary relation R on A we usually write aRb instead of $(a, b) \in R$ and interpret this as an agent or society (weakly) preferring a over b. Well-known conditions for a binary relation R on A are:

(a) *Reflexivity*: aRa for all $a \in A$.
(b) *Completeness*: aRb or bRa for all $a, b \in A$ with $a \neq b$.
(c) *Antisymmetry*: For all $a, b \in A$, if aRb and bRa, then $a = b$.
(d) *Transitivity*: For all $a, b, c \in A$, aRb and bRc imply aRc.

A *preference* on A is a reflexive, complete and transitive binary relation on A. For a preference R on A we write aPb if aRb and not bRa; and aIb if aRb and bRa. The binary relations P and I are called the *asymmetric* and *symmetric parts* of R, respectively, and interpreted as strict preference and indifference. Check (see Problem 11.1) that P is antisymmetric and transitive but not reflexive and not necessarily complete, and that I is reflexive and transitive but not necessarily antisymmetric and not necessarily complete. By $\mathscr{L}^*$ we denote the set of all preferences on A, and by $\mathscr{L} \subseteq \mathscr{L}^*$ the set of all antisymmetric (i.e., *strict*) preferences on A. In plain words, elements of $\mathscr{L}^*$ order the elements of A but allow for indifferences, while elements of $\mathscr{L}$ order the elements of A strictly.[3]

In what follows, it is assumed that agents have strict preferences while social preferences may have indifferences. A strict preference *profile* is a list $(R_1, \ldots, R_i, \ldots, R_n)$, where R_i is the strict preference of agent i. Hence, $\mathscr{L}^N$ denotes the set of all strict preference profiles. A *social choice function* is a map $f : \mathscr{L}^N \to A$, i.e., it assigns a unique alternative to every profile of strict preferences. A *social welfare function* is a map $F : \mathscr{L}^N \to \mathscr{L}^*$, i.e., it assigns a (possibly non-strict) preference to every profile of strict preferences.

[2] See Problem 11.7 for the case $m = 2$.

[3] Elements of $\mathscr{L}$ are usually called *linear orders* and those of $\mathscr{L}^*$ *weak orders*.

11.2 Arrow's Theorem

In this section the focus is on social welfare functions. We formulate three properties for a social welfare function $F : \mathscr{L}^N \to \mathscr{L}^*$. Call F:

(1) *Pareto Efficient* (PE) if for each profile $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and all $a, b \in A$, if aR_ib for all $i \in N$, then aPb, where P is the asymmetric part of $R = F(R_1, \ldots, R_n)$.
(2) *Independent of Irrelevant Alternatives* (IIA) if for all $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and $(R'_1, \ldots, R'_n) \in \mathscr{L}^N$ and all $a, b \in A$, if $aR_ib \Leftrightarrow aR'_ib$ for all $i \in N$, then $aRb \Leftrightarrow aR'b$, where $R = F(R_1, \ldots, R_n)$ and $R' = F(R'_1, \ldots, R'_n)$.
(3) *Dictatorial* (D) if there is an $i \in N$ such that $F(R_1, \ldots, R_n) = R_i$ for all $(R_1, \ldots, R_n) \in \mathscr{L}^N$.

Pareto Efficiency requires that, if all agents prefer an alternative a over an alternative b, then the social ranking should also put a above b. Independence of Irrelevant Alternatives says that the social preference between two alternatives should only depend on the agents' preferences between these two alternatives and not on the position of any other alternative.[4] Dictatoriality says that the social ranking is always equal to the preference of a fixed agent, the *dictator*. Clearly, there are exactly n dictatorial social welfare functions.

The first two conditions are regarded as desirable but the third clearly not. Unfortunately, Arrow's Theorem implies that the first two conditions imply the third one.[5]

Theorem 11.1 (Arrow's Theorem). *Let F be a Pareto Efficient and IIA social welfare function. Then F is dictatorial.*

Proof.

Step 1. Consider a profile in $\mathscr{L}^N$ and two distinct alternatives $a, b \in A$ such that every agent ranks a on top and b at bottom. By Pareto Efficiency, the social ranking assigned by F must also rank a on top and b at bottom.

Now change agent 1's ranking by raising b in it one position at a time. By IIA, a is ranked socially (by F) on top as long as b is still below a in the preference of agent 1. In the end, if agent 1 ranks b first and a second, we have a or b on top of the social ranking by Pareto efficiency of F. If a is still on top in the social ranking, then continue the same process with agents 2,3, etc., until we reach some agent k such that b is on top of the social ranking after moving b above a in agent k's preference. Tables 11.2 and 11.3 give the situations just before and just after b is placed above a in agent k's preference.[6]

[4] Although there is some similarity in spirit, this condition is not in any formal sense related to the IIA condition in bargaining, see Sect. 10.1 or Chap. 21.

[5] For this reason the theorem is often referred to as Arrow's Impossibility Theorem.

[6] In these tables and also the ones below, we generically denote all preferences by $R_1, \ldots, R_n$. The last column in every table will be used in Sect. 11.3.

Table 11.2

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
b	$\cdots$	b	a	a	$\cdots$	a	a	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	b	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b	$\cdot$	

Table 11.3

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
b	$\cdots$	b	b	a	$\cdots$	a	b	b
a	$\cdots$	a	a	$\cdot$	$\cdots$	$\cdot$	a	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b	$\cdot$	

Table 11.4

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$	a	a
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$	b	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a	$\cdot$	
a	$\cdots$	a	$\cdot$	b	$\cdots$	b	$\cdot$	

Table 11.5

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
b	$\cdots$	b	b	$\cdot$	$\cdots$	$\cdot$	b	b
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	a	
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a	$\cdot$	
a	$\cdots$	a	$\cdot$	b	$\cdots$	b	$\cdot$	

Step 2. Now consider Tables 11.4 and 11.5. The profile in Table 11.4 arises from the one in Table 11.2 by moving a to the last position for agents $i < k$ and to the second last position for agents $i > k$. In exactly the same way, the profile in Table 11.5 arises from the one in Table 11.3. Then IIA applied to Tables 11.3 and 11.5 implies that b is socially top-ranked in Table 11.5. Next, IIA applied to the transition from Table 11.5 to Table 11.4 implies that in Table 11.4 b must still be socially ranked above every alternative except perhaps a. But IIA applied to the transition from Table 11.2 to Table 11.4 implies that in Table 11.4 a must still be socially ranked above every alternative. This proves that the social rankings in Tables 11.4 and 11.5 are correct.

Table 11.6

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$	a	a
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
c	$\cdots$	c	$\cdot$	c	$\cdots$	c	$\cdot$	
b	$\cdots$	b	$\cdot$	a	$\cdots$	a	$\cdot$	
a	$\cdots$	a	$\cdot$	b	$\cdots$	b	$\cdot$	

Table 11.7

R_1	$\cdots$	R_{k-1}	R_k	R_{k+1}	$\cdots$	R_n	F	f
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$	a	a
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$	$\cdot$	
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$	c	
c	$\cdots$	c	$\cdot$	c	$\cdots$	c	$\cdot$	
b	$\cdots$	b	$\cdot$	b	$\cdots$	b	b	
a	$\cdots$	a	$\cdot$	a	$\cdots$	a	$\cdot$	

Step 3. Consider a third alternative c distinct from a and b. The social ranking in Table 11.6 is obtained by from Table 11.4 by applying IIA.

Step 4. Consider the profile in Table 11.7, obtained from the profile in Table 11.6 by switching a and b for agents $i > k$. By IIA applied to the transition from Table 11.6 to Table 11.7, we have that a must still be socially ranked above every alternative except possibly b. However, b must be ranked below c by Pareto efficiency, which shows that the social ranking in Table 11.7 is correct.

Step 5. Consider any arbitrary profile in which agent k prefers a to b. Change the profile by moving c between a and b for agent k and to the top of every other agent's preference (if this is not already the case). By IIA this does not affect the social ranking of a vs. b. Since the preference of every agent concerning a and c is now as in Table 11.7, IIA implies that a is socially ranked above c, which itself is socially ranked above b by Pareto Efficiency. Hence, by transitivity of the social ranking we may conclude that a is socially ranked above b whenever it is preferred by agent k over b. By repeating the argument with the roles of b and c reversed, and recalling that c was an arbitrary alternative distinct from a and b, we may conclude that the social ranking of a is above some alternative whenever agent k prefers a to that alternative: k is a 'dictator' for a. Since a was arbitrary, we can repeat the whole argument to conclude that there must be a dictator for every alternative. Since there cannot be distinct dictators for distinct alternatives, there must be a single dictator for all alternatives. □

11.3 The Gibbard–Satterthwaite Theorem

The Gibbard–Satterthwaite Theorem applies to social choice functions. We start by listing the following possible properties of a social choice function $f : \mathscr{L}^N \to A$. Call f:

(1) *Unanimous* (UN) if for each profile $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and each $a \in A$, if aR_ib for all $i \in N$ and all $b \in A \setminus \{a\}$, then $f(R_1, \ldots, R_n) = a$.
(2) *Monotonic* (MON) if for all profiles $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and $(R'_1, \ldots, R'_n) \in \mathscr{L}^N$ and all $a \in A$, if $f(R_1, \ldots, R_n) = a$ and $aR_ib \Rightarrow aR'_ib$ for all $b \in A \setminus \{a\}$ and $i \in N$, then $f(R'_1, \ldots, R'_n) = a$.
(3) *Dictatorial* (D) if there is an $i \in N$ such that $f(R_1, \ldots, R_n) = a$ where aR_ib for all $b \in A \setminus \{a\}$, for all $(R_1, \ldots, R_n) \in \mathscr{L}^N$.
(4) *Strategy-Proof* (SP) if for all profiles $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and $(R'_1, \ldots, R'_n) \in \mathscr{L}^N$ and all $i \in N$, if $R'_j = R_j$ for all $j \in N \setminus \{i\}$, then $f(R_1, \ldots, R_n)\ R_i\ f(R'_1, \ldots, R'_n)$.

Unanimity requires that, if all agents have the same top alternative, then this alternative should be chosen. Monotonicity says that, if some alternative a is chosen and the profile changes in such a way that a is still preferred by every agent over all alternatives over which it was originally preferred, then a should remain to be chosen. Dictatoriality means that there is a fixed agent whose top element is always chosen. Strategy-Proofness says that no agent can obtain a better chosen alternative by lying about his true preference.

In accordance with mathematical parlance, call a social choice function $f : \mathscr{L}^N \to A$ *surjective* if for every $a \in A$ there is some profile $(R_1, \ldots, R_n) \in \mathscr{L}^N$ such that $f(R_1, \ldots, R_n) = a$. Hence, each a is chosen at least once.[7] The Gibbard–Satterthwaite Theorem is as follows.

Theorem 11.2 (Gibbard–Satterthwaite Theorem). *Let $f : \mathscr{L}^N \to A$ be a surjective and strategy-proof social choice function. Then f is dictatorial.*

We will prove the Gibbard–Satterthwaite Theorem by using the next theorem, which is a variant of the Muller–Satterthwaite Theorem [84].

Theorem 11.3 (Muller–Satterthwaite). *Let $f : \mathscr{L}^N \to A$ be a unanimous and monotonic social choice function. Then f is dictatorial.*

Proof of Theorem 11.2. We prove that f is unanimous and monotonic. The result then follows from Theorem 11.3.

Suppose that $f(R_1, \ldots, R_n) = a$ for some profile $(R_1, \ldots, R_n) \in \mathscr{L}^N$ and some alternative $a \in A$. Let $i \in N$ and let $(R'_1, \ldots, R'_n) \in \mathscr{L}^N$ be a profile such that for all $j \in N \setminus \{i\}$ we have $R'_j = R_j$ and for all $b \in A \setminus \{a\}$ we have aR'_ib if aR_ib. We wish to show that $f(R'_1, \ldots, R'_n) = a$. Suppose, to the contrary, that $f(R'_1, \ldots, R'_n) = b \neq a$. Then SP implies aR_ib, and hence aR'_ib. By SP, however, bR'_ia, hence, by antisymmetry of R'_i, $a = b$, a contradiction. This proves $f(R'_1, \ldots, R'_n) = a$.

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[7] In the social choice literature this property is sometimes called *citizen-sovereignty*.

Now suppose that $(R'_1,\ldots,R'_n) \in \mathscr{L}^N$ is a profile such that for all $i \in N$ and all $b \in A \setminus \{a\}$ we have aR'_ib if aR_ib. By applying the argument in the preceding paragraph n times, it follows that $f(R'_1,\ldots,R'_n) = a$. Hence, f is monotonic.

To prove unanimity, suppose that $(R_1,\ldots,R_n) \in \mathscr{L}^N$ and $a \in A$ such that aR_ib for all $i \in N$ and $b \in A \setminus \{a\}$. By surjectivity there is $(R'_1,\ldots,R'_n) \in \mathscr{L}^N$ with $f(R'_1,\ldots,R'_n) = a$. By monotonicity we may move a to the top of each agent's preference and still have a chosen. Next, again by monotonicity, we may change each agent i's preference to R_i without changing the chosen alternative, i.e., $f(R_1,\ldots,R_n) = a$. Hence, f is unanimous. □

Proof of Theorem 11.3. The proof parallels the proof of Theorem 11.1 and uses analogous steps and the same tables.

Step 1. Consider a profile in $\mathscr{L}^N$ and two distinct alternatives $a,b \in A$ such that every agent ranks a on top and b at bottom. By unanimity, f chooses a.

Now change agent 1's ranking by raising b in it one position at a time. By MON, a is chosen by f as long as b is still below a in the preference of agent 1. In the end, if agent 1 ranks b first and a second, we have a or b chosen by f, again by MON. If a is still chosen, then continue the same process with agents 2,3, etc., until we reach some agent k such that b is chosen after moving b above a in agent k's preference. Tables 11.2 and 11.3 give the situations just before and just after b is placed above a in agent k's preference.

Step 2. Now consider Tables 11.4 and 11.5. The profile in Table 11.4 arises from the one in Table 11.2 by moving a to the last position for agents $i < k$ and to the second last position for agents $i > k$. In exactly the same way, the profile in Table 11.5 arises from the one in Table 11.3.

Then MON applied to Tables 11.3 and 11.5 implies that b is chosen in Table 11.5. Next, MON applied to the transition from Table 11.5 to Table 11.4 implies that in Table 11.4 the choice must be either b or a. Suppose b would be chosen. Then MON applied to the transition from Table 11.4 to Table 11.2 implies that in Table 11.2 b must be chosen as well, a contradiction. Hence, a is chosen in Table 11.4. This proves that the choices by f in Tables 11.4 and 11.5 are correct.

Step 3. Consider a third alternative c distinct from a and b. The choice in Table 11.6 is obtained by from Table 11.4 by applying MON.

Step 4. Consider the profile in Table 11.7, obtained from the profile in Table 11.6 by switching a and b for agents $i > k$. If the choice in Table 11.7 were some d unequal to a or b, then by MON it would also be d in Table 11.6, a contradiction. If it were b, then by MON it would remain b even if c would be moved to the top of every agent's preference, contradicting unanimity. Hence, it must be a.

Step 5. Consider any arbitrary profile with a at the top of agent k's preference. Such a profile can always be obtained from the profile in Table 11.7 without worsening the position of a with respect to any other alternative in any agent's preference. By MON therefore, a must be chosen whenever it is at the top of agent k's preference, so k is a 'dictator' for a. Since a was arbitrary, we can find a dictator for every other alternative but, clearly, these must be one and the same agent. Hence, this agent is the dictator. □

There is a large literature that tries to escape the rather negative conclusions of Theorems 11.1–11.3 by adapting the model and/or restricting the domain. Examples of this are provided in Problems 6.22 and 6.23.

Problems

11.1. *Preferences*

Let R be a preference on A, with symmetric part I and asymmetric part P.

(a) Prove that P is antisymmetric and transitive but not reflexive and not necessarily complete.

(b) Prove that I is reflexive and transitive but not necessarily complete and not necessarily antisymmetric.

11.2. *Pairwise Comparison*

For a profile $r = (R_1, \ldots, R_n) \in \mathscr{L}^N$ and $a, b \in A$ define

$$N(a,b,r) = \{i \in N \mid aR_ib\} ,$$

i.e., $N(a,b,r)$ is the set of agents who (strictly) prefer a to b in the profile r. With r we can associate a binary relation $C(r)$ on A by defining $aC(r)b :\Leftrightarrow |N(a,b,r)| \geq |N(b,a,r)|$ for all $a, b \in A$. If $aC(r)b$ we say that 'a beats b by pairwise majority'.

(a) Is $C(r)$ reflexive? Complete? Antisymmetric?

(b) Show that $C(r)$ is not transitive, by considering the famous Condorcet profile[8] for $N = \{1,2,3\}$ and $A = \{a,b,c\}$: aR_1bR_1c, bR_2cR_2a, cR_3aR_3b.

(c) Call a a *Condorcet winner* if $|N(a,b,r)| > |N(b,a,r)|$ for all $b \in A \setminus \{a\}$. Is there a Condorcet winner in the example in Sect. 11.1?

11.3. *Independence of the Conditions in Theorem 11.1*

Show that the conditions in Theorem 11.1 are independent. That is, exhibit a social welfare function that is Pareto efficient and does not satisfy IIA or dictatoriality, and one that satisfies IIA and is not dicatorial nor Pareto efficient.

11.4. *Independence of the Conditions in Theorem 11.2*

Show that the conditions in Theorem 11.2 are independent.

11.5. *Independence of the Conditions in Theorem 11.3*

Show that the conditions in Theorem 11.3 are independent.

[8] See [27], and see [43] for a comprehensive study of this so-called *Condorcet paradox*.

11.6. *Copeland Score and Kramer Score*

The *Copeland score* of an alternative $a \in A$ at a profile $r = (R_1, \ldots, R_n) \in \mathscr{L}^N$ is defined by

$$c(a,r) = |\{b \in A \mid N(a,b,r) \geq N(b,a,r)\}| ,$$

i.e., the number of alternatives that a beats (cf. Problem 11.2). The *Copeland ranking* is obtained by ranking the alternatives according to their Copeland scores.

(a) Is the Copeland ranking a preference? Is it antisymmetric? Does the derived social welfare function satisfy IIA? Pareto efficiency?

The *Kramer score* of an alternative $a \in A$ at a profile $r = (R_1, \ldots, R_n) \in \mathscr{L}^N$ is defined by

$$k(a,r) = \min_{b \in A \setminus \{a\}} |N(a,b,r)| ,$$

i.e., the worst score among all pairwise comparisons. The *Kramer ranking* is obtained by ranking the alternatives according to their Kramer scores.

(b) Is the Kramer ranking a preference? Is it antisymmetric? Does the derived social welfare function satisfy IIA? Pareto efficiency?

11.7. *Two Alternatives*

Show that Theorems 11.1–11.3 no longer hold if there are just two alternatives, i.e., if $m = 2$.

Chapter 12
Matrix Games

In this chapter we study finite two-person zerosum games – matrix games – more rigorously. In particular, von Neumann's Minimax Theorem [140] is proved. The chapter builds on Chap. 2 in Part I, and the reader is advised to read this chapter and in particular Definition 2.1 before continuing.

Section 12.1 presents a proof of the Minimax Theorem, and Sect. 12.2 shows how a matrix game can be solved – optimal strategies and the value of the game can be found – by solving an associated linear programming problem.

12.1 The Minimax Theorem

Let A be an $m \times n$ matrix game. For any strategy $\mathbf{p} \in \Delta^m$ of player 1, let $v_1(\mathbf{p}) = \min_{\mathbf{q} \in \Delta^n} \mathbf{p}A\mathbf{q}$. It is easy to see that $v_1(\mathbf{p}) = \min_{j \in \{1,\ldots,n\}} \mathbf{p}A\mathbf{e}^j$, since $\mathbf{p}A\mathbf{q}$ is a convex combination of the numbers $\mathbf{p}A\mathbf{e}^j$. In the matrix game A player 1 can guarantee a payoff of at least

$$v_1(A) := \max_{\mathbf{p} \in \Delta^m} v_1(\mathbf{p}).$$

Similarly, for any strategy $\mathbf{q} \in \Delta^n$ of player 2 let $v_2(\mathbf{q}) = \max_{\mathbf{p} \in \Delta^m} \mathbf{p}A\mathbf{q} = \max_{i \in \{1,\ldots,m\}} \mathbf{e}^i A\mathbf{q}$, then player 2 can guarantee to have to pay at most

$$v_2(A) := \min_{\mathbf{q} \in \Delta^n} v_2(\mathbf{q}).$$

Intuitively, player 1 should not be able to guarantee to obtain more than what player 2 can guarantee to pay maximally. Indeed, we have the following lemma.

Lemma 12.1. *For any $m \times n$ matrix game, $v_1(A) \leq v_2(A)$.*

Proof. Problem 12.1. □

The following theorem is due to von Neumann [140].

Theorem 12.2 (Minimax Theorem for Matrix Games). *For any $m \times n$ matrix game A, $v_1(A) = v_2(A)$.*

Proof. Suppose that A is any $m \times n$ matrix game. Then either (1) or (2) in Lemma 22.3 has to hold.

First suppose that (1) holds. Then there are $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^m$ with $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$ and $A\mathbf{y} + \mathbf{z} = 0$. It cannot be the case that $\mathbf{y} = \mathbf{0}$, since that would imply that also $\mathbf{z} = 0$, a contradiction. Hence $\sum_{k=1}^{n} y_k > 0$. Define $\mathbf{q} \in \Delta^n$ by $q_j = y_j / \sum_{k=1}^{n} y_k$ for every $j = 1, \ldots, n$. Then $A\mathbf{q} = -\mathbf{z} / \sum_{k=1}^{n} y_k \leq \mathbf{0}$. Hence $v_2(\mathbf{q}) \leq 0$, and therefore $v_2(A) \leq 0$.

Suppose instead that (2) holds. Then there is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}A > \mathbf{0}$. Define $\mathbf{p} \in \Delta^m$ by $\mathbf{p} = \mathbf{x} / \sum_{i=1}^{m} x_i$, then $v_1(\mathbf{p}) > 0$ and therefore $v_1(A) > 0$.

We conclude that, for any matrix game A, it is not possible to have $v_1(A) \leq 0 < v_2(A)$.

Fix some matrix game B, and suppose that $v_1(B) < v_2(B)$. We derive a contradiction, and by Lemma 12.1 this completes the proof of the theorem.

Let A be the matrix game arising by subtracting the number $v_1(B)$ from all entries of B. Then, clearly, $v_1(A) = v_1(B) - v_1(B) = 0$ and $v_2(A) = v_2(B) - v_1(B) > 0$. Hence, $v_1(A) \leq 0 < v_2(A)$, which is the desired contradiction. □

In view of Theorem 12.2 we can define the *value* of the game A by $v(A) = v_1(A) = v_2(A)$. An *optimal strategy* of player 1 is a strategy $\mathbf{p}$ such that $v_1(\mathbf{p}) \geq v(A)$. Similarly, an *optimal strategy* of player 2 is a strategy $\mathbf{q}$ such that $v_2(\mathbf{q}) \leq v(A)$. Theorem 12.2 implies that $v_1(\mathbf{p}) = v_2(\mathbf{q}) = v(A)$ for such optimal strategies. In particular, the optimal strategies for player 1 are exactly the maximin strategies, and the optimal strategies for player 2 are exactly the minimax strategies (cf. Definition 2.2).

For computation of optimal strategies and the value of matrix games in some special cases, see Chap. 2 and Problems 12.2 and 12.3. In general, matrix games can be solved by linear programming. This is demonstrated in the next section.

12.2 A Linear Programming Formulation

Let A be an $m \times n$ matrix game. Adding the same number to all entries of A changes the value by that same number but not the optimal strategies of the players. So we may assume without loss of generality that all entries of A are positive. We define the $(m+1) \times (n+1)$ matrix B as follows:

$$B = \begin{pmatrix} & & & & -1 \\ & & & & -1 \\ & & A & & \vdots \\ & & & & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix}.$$

Let $\mathbf{b} = (0,\ldots,0,-1) \in \mathbb{R}^{n+1}$ and $\mathbf{c} = (0,\ldots,0,-1) \in \mathbb{R}^{m+1}$. Define $V := \{\mathbf{x} \in \mathbb{R}^{m+1} \mid \mathbf{x}B \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and $W := \{\mathbf{y} \in \mathbb{R}^{n+1} \mid B\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$. It is easy to check that $V, W \neq \emptyset$. The Duality Theorem of Linear Programming (Theorem 22.5) therefore implies:

Corollary 12.3. $\min\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x} \in V\} = \max\{\mathbf{b} \cdot \mathbf{y} \mid \mathbf{y} \in W\}$.

The minimum and maximum problems in this corollary are so-called linear programming (LP) problems. If we call the minimization problem the *primal* problem, then the maximization problem is the *dual* problem – or vice versa. The common minimum/maximum is called the *value* of the LP, and $\mathbf{x}$ and $\mathbf{y}$ achieving the value are called *optimal solutions*. Denote the sets of optimal solutions by $O_{\min}$ and $O_{\max}$, respectively.

We shall prove the following result.

Theorem 12.4. *Let A be an $m \times n$ matrix game with all entries positive.*

(1) *If $\mathbf{p} \in \Delta^m$ is an optimal strategy for player 1 and $\mathbf{q} \in \Delta^n$ is an optimal strategy for player 2 in A, then $(\mathbf{p}, v(A)) \in O_{min}$ and $(\mathbf{q}, v(A)) \in O_{max}$. The value of the LP is $-v(A)$.*
(2) *If $\mathbf{x} = (x_1,\ldots,x_m,x_{m+1}) \in O_{min}$ and $\mathbf{y} = (y_1,\ldots,y_n,y_{n+1}) \in O_{max}$, then $(x_1,\ldots,x_m)$ is an optimal strategy for player 1 in A, $(y_1,\ldots,y_n)$ is an optimal strategy for player 2 in A, and $v(A) = x_{m+1} = y_{n+1}$.*

Proof. (1) Let $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$ be optimal strategies in the matrix game A. Then $\mathbf{p}A\mathbf{e}^j \geq v(A)$ and $\mathbf{e}^i A\mathbf{q} \leq v(A)$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Since all entries of A are positive and therefore $v(A) > 0$, this implies $(\mathbf{p}, v(A)) \in V$ and $(\mathbf{q}, v(A)) \in W$. Since $(\mathbf{p}, v(A)) \cdot \mathbf{c} = -v(A)$ and $(\mathbf{q}, v(A)) \cdot \mathbf{b} = -v(A)$, Lemma 22.8 implies that the value of the LP is $-v(A)$, $(\mathbf{p}, v(A)) \in O_{\min}$ and $(\mathbf{q}, v(A)) \in O_{\max}$.

(2) Let $\mathbf{x} = (x_1,\ldots,x_{m+1}) \in O_{\min}$. Since $\mathbf{x} \cdot \mathbf{c} = -v(A)$ by (1), we have $x_{m+1} = v(A)$. Since $\mathbf{x}B \geq \mathbf{b}$, we have $(x_1,\ldots,x_m)A\mathbf{e}^j \geq v(A)$ for all $j = 1,\ldots,n$, $x_i \geq 0$ for all $i = 1,\ldots,m$, and $\sum_{i=1}^m x_i \leq 1$. Suppose that $\sum_{i=1}^m x_i < 1$. Obviously, $\sum_{i=1}^m x_i > 0$, otherwise $\mathbf{x} = (0,\ldots,0,v(A)) \notin V$ since $v(A) > 0$. Then, letting $t = (\sum_{i=1}^m x_i)^{-1} > 1$, we have $t\mathbf{x} \in V$ and $t\mathbf{x} \cdot \mathbf{c} = -t\,v(A) < -v(A)$, contradicting $\mathbf{x} \in O_{\min}$. Hence, $\sum_{i=1}^m x_i = 1$, and $(x_1,\ldots,x_m)$ is an optimal strategy of player 1 in A.

The proof of the second part of (2) is analogous. □

The interest of this theorem derives from the fact that since the invention of the simplex algorithm by George Dantzig in 1947 (see any textbook on linear programming or operations research) solving linear programming problems is a well established area. Thus, one can apply any (computer) method for solving LPs to find the value and optimal strategies of a matrix game.

Observe that, by slightly modifying part (2) of the proof of Theorem 12.4, we can in fact derive the Minimax Theorem from the Duality Theorem (Problem 12.4). Conversely, with each LP we can associate a matrix game and thereby derive the Duality Theorem from the Minimax Theorem (see, e.g., [98]). This confirms the close relationship between linear programming (Duality Theorem) and the theory of matrix games (Minimax Theorem).

Problems

12.1. *Proof of Lemma 12.1*

Prove Lemma 12.1.

12.2. 2×2 *Games*

Consider the 2×2 matrix game

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Assume that A has no saddlepoints (cf. Definition 2.3).

(a) Show that we may assume, without loss of generality,

$$a_{11} > a_{12}, \quad a_{12} < a_{22}, \quad a_{21} < a_{22}, \quad a_{11} > a_{21}.$$

(b) Show that the unique optimal strategies $\mathbf{p}$ and $\mathbf{q}$ and the value of the game are given by:

$$\mathbf{p} = \frac{\mathbf{J}A^*}{\mathbf{J}A^*\mathbf{J}^T}, \quad \mathbf{q} = \frac{A^*\mathbf{J}^T}{\mathbf{J}A^*\mathbf{J}^T}, \quad v(A) = \frac{|A|}{\mathbf{J}A^*\mathbf{J}^T},$$

where A^* is the adjoint matrix of A, i.e.,

$$A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$

$|A|$ is the determinant of A, and $\mathbf{J} := (1,1)$.[1]

12.3. *Symmetric Games*

An $m \times n$ matrix game $A = (a_{ij})$ is called *symmetric* if $m = n$ and $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, m$.

Prove that the value of a symmetric game is zero and that the sets of optimal strategies of players 1 and 2 coincide.

12.4. *The Duality Theorem Implies the Minimax Theorem*

Modify the proof of part (2) of Theorem 12.4 in order to derive the Minimax Theorem from the Duality Theorem. (Hint: first show that the value of the LP must be negative.)

[1] $\mathbf{J}$ denotes the row vector and $\mathbf{J}^T$ the transpose, i.e., the column vector. In general, we omit the transpose notation if confusion is unlikely.

12.5. *Infinite Matrix Games*

Consider the following two-player game. Each player mentions a natural number. The player with the highest number receives one Euro from the player with the lowest number. If the numbers are equal then no player receives anything.

(a) Write this game in the form of an infinite matrix game A.

(b) Compute $\sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathbf{p}A\mathbf{q}$ and $\inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathbf{p}A\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are probability distributions over the rows and the columns of A, respectively. (Conclude that this game has no 'value'.)

12.6. *Equalizer Theorem*

Let v be the value of the $m \times n$-matrix game A, and suppose that $\mathbf{p}A\mathbf{e}^n = v$ for every optimal strategy $\mathbf{p}$ of player 1. Show that player 2 has an optimal strategy $\mathbf{q}$ with $q_n > 0$ (cf. Chap. 20 in [6]).

Chapter 13
Finite Games

This chapter builds on Chap. 3, where we studied finite two person games – bimatrix games. The reader is advised to (re)read this chapter before continuing. The present chapter offers a more rigorous treatment of finite games, i.e., games with finitely many players – often two – who have finitely many pure strategies over which they can randomize.[1]

In Sect. 13.1 a proof of Nash's existence theorem is provided. Section 13.2 goes deeper into bimatrix games. In Sect. 13.3 the notion of iterated dominance is studied, and its relation with rationalizability indicated. Sections 13.4–13.6 present some basics about refinements of Nash equilibrium. Section 13.7 is on correlated equilibrium in bimatrix games, and Sect. 13.8 concludes with an axiomatic characterization of Nash equilibrium based on a reduced game (consistency) condition.

13.1 Existence of Nash Equilibrium

A *finite game* is a $2n+1$-tuple

$$G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n),$$

where

- $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, $n \geq 1$, is the set of *players*.
- for every $i \in N$, S_i is the finite *pure strategy set* of player i.
- for every $i \in N$, $u_i : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ is the *payoff function* of player i; i.e., for every pure strategy combination $(s_1, \ldots, s_n) \in S$ where $s_1 \in S_1, \ldots, s_n \in S_n$, $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ is player i's payoff.

This definition is identical to the definition of an n-person game in Chap. 6, except that the pure strategy sets are now finite. The elements of S_i are the pure strategies

[1] We only discuss games with complete information. In the terminology of Chap. 5, each player has only one type.

of player i. A (mixed) *strategy* of player i is a probability distribution over S_i. The set of (mixed) strategies of player i is denoted by $\Delta(S_i)$. Observe that, whenever we talk about a strategy, we mean a mixed strategy (which may of course be pure).

Let $(\sigma_1,\ldots,\sigma_n) \in \Delta(S_1)\times\ldots\times\Delta(S_n)$ be a strategy combination. Player i's *payoff* from this strategy combination is defined to be his expected payoff. With some abuse of notation this is also denoted by $u_i(\sigma_1,\ldots,\sigma_n)$. Formally,

$$u_i(\sigma_1,\ldots,\sigma_n) = \sum_{(s_1,\ldots,s_n)\in S} \left(\prod_{i\in N} \sigma_i(s_i)\right) u_i(s_1,\ldots,s_n).$$

For a strategy combination σ and a player $i \in N$ we denote by (σ_i', σ_{-i}) the strategy combination in which player i plays $\sigma_i' \in \Delta(S_i)$ and each player $j \neq i$ plays σ_j.

A *best reply* of player i to the strategy combination σ_{-i} of the other players is a strategy $\sigma_i \in \Delta(S_i)$ such that $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Delta(S_i)$.

A *Nash equilibrium* of G is a strategy combination $\sigma^* \in \prod_{i\in N}\Delta(S_i)$ such that for each player i, σ_i^* is a best reply to σ_{-i}^*.

As in Chaps. 3 and 6, β_i denotes player i's *best reply correspondence*. That is, $\beta_i : \prod_{j\in N,\, j\neq i}\Delta(S_j) \to \Delta(S_i)$ assigns to each strategy combination of the other players the set of all best replies of player i.

Nash [91] proved that every finite game has a Nash equilibrium in mixed strategies. Formally:

Theorem 13.1 (Existence of Nash equilibrium). *Every finite game* $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ *has a Nash equilibrium.*

The proof of this theorem is based on Kakutani's fixed point theorem. Readers not familiar with this theorem should consult Sect. 22.5 first.

Proof of Theorem 13.1. Consider the correspondence

$$\beta : \prod_{i\in N}\Delta(S_i) \to \prod_{i\in N}\Delta(S_i),\ (\sigma_1,\ldots,\sigma_n) \mapsto \prod_{i\in N}\beta_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n).$$

This correspondence is convex-valued and upper semi-continuous (Problem 13.1). By Kakutani's fixed point theorem (Theorem 22.10) it has a fixed point σ^*. By definition of β, any fixed point is a Nash equilibrium of G. □

13.2 Bimatrix Games

Two-person finite games – bimatrix games – were studied in Chap. 3. Here we present some extensions. In Sect. 13.2.1 we give some formal relations between pure and mixed strategies in a Nash equilibrium. In Sect. 13.2.2 we extend the graphical method for computing Nash equilibria (cf. Sect. 3.2.2). In Sect. 13.2.3 a general mathematical programming method is described by which equilibria of bimatrix games can be found. Section 13.2.4 reconsiders matrix games as a special kind of bimatrix games. Section 13.2.5 is about Zermelo's theorem on the game of chess.

13.2.1 Pure and Mixed Strategies in Nash Equilibrium

Let (A,B) be an $m \times n$ bimatrix game (Definition 3.1). The first lemma says that to determine whether a strategy pair is a Nash equilibrium it is sufficient to compare the expected payoff of a (mixed) strategy with the payoffs of pure strategies.

Lemma 13.2. *Strategy combination* $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ *is a Nash equilibrium of* (A,B) *if and only if* $\mathbf{p}^* A \mathbf{q}^* \geq \mathbf{e}^i A \mathbf{q}^*$ *for all* $i = 1, \ldots, m$ *and* $\mathbf{p}^* A \mathbf{q}^* \geq \mathbf{p}^* B \mathbf{e}^j$ *for all* $j = 1, \ldots, n$.

Proof. Problem 13.2. □

The next lemma says that a player always has a pure best reply against any strategy of the opponent.

Lemma 13.3. *Let* $\mathbf{p} \in \Delta^m$ *and* $\mathbf{q} \in \Delta^n$. *Then there is an* $i \in \{1, \ldots, m\}$ *with* $\mathbf{e}^i \in \beta_1(\mathbf{q})$ *and a* $j \in \{1, \ldots, n\}$ *with* $\mathbf{e}^j \in \beta_2(\mathbf{p})$.

Proof. Problem 13.3. □

In light of these lemmas it makes sense to introduce the pure best reply correspondences.

Definition 13.4. Let (A,B) be an $m \times n$ bimatrix game and let $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$. Then

$$PB_1(\mathbf{q}) = \{i \in \{1, \ldots, m\} \mid \mathbf{e}^i A \mathbf{q} = \max_k \mathbf{e}^k A \mathbf{q}\}$$

is the set of *pure best replies* of player 1 to $\mathbf{q}$ and

$$PB_2(\mathbf{p}) = \{j \in \{1, \ldots, n\} \mid \mathbf{p} A \mathbf{e}^j = \max_k \mathbf{p} A \mathbf{e}^k\}$$

is the set of *pure best replies* of player 2 to $\mathbf{p}$.

Observe that, with some abuse of notation, the pure best replies in this definition are labelled by the row and column numbers.

The *carrier* $C(\mathbf{p})$ of a mixed strategy $\mathbf{p} \in \Delta^k$, where $k \in \mathbb{N}$, is the set of coordinates that are positive, i.e.,

$$C(\mathbf{p}) = \{i \in \{1, \ldots, k\} \mid p_i > 0\}.$$

The next lemma formalizes the observation used already in Chap. 3, namely that in a best reply a player puts positive probability only on those pure strategies that maximize his expected payoff (cf. Problem 3.8).

Lemma 13.5. *Let (A,B) be an $m \times n$ bimatrix game, $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$. Then*

$$\mathbf{p} \in \beta_1(\mathbf{q}) \Leftrightarrow C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$$

and

$$\mathbf{q} \in \beta_2(\mathbf{p}) \Leftrightarrow C(\mathbf{q}) \subseteq PB_2(\mathbf{p}).$$

Proof. We only show the first equivalence.

First let $\mathbf{p} \in \beta_1(\mathbf{q})$, and assume $i \in C(\mathbf{p})$ and, contrary to what we want to prove, that $\mathbf{e}^i A\mathbf{q} < \max_k \mathbf{e}^k A\mathbf{q}$. Then

$$\mathbf{p}A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q} = \sum_{k=1}^m p_k \max_k \mathbf{e}^k A\mathbf{q} > \sum_{k=1}^m p_k \mathbf{e}^k A\mathbf{q} = \mathbf{p}A\mathbf{q},$$

where the first equality follows from Lemma 13.3. This is a contradiction, hence $\mathbf{e}^i A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}$ and $i \in PB_1(\mathbf{q})$.

Next, assume that $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$. Then

$$\mathbf{p}A\mathbf{q} = \sum_i p_i \mathbf{e}^i A\mathbf{q} = \sum_{i \in C(\mathbf{p})} p_i \mathbf{e}^i A\mathbf{q} = \sum_{i \in C(\mathbf{p})} p_i \max_k \mathbf{e}^k A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}.$$

So $\mathbf{p}A\mathbf{q} \geq \mathbf{e}^i A\mathbf{q}$ for all $i = 1, \ldots, m$, which by Lemma 13.2 implies $\mathbf{p} \in \beta_1(\mathbf{q})$. $\square$

The following corollary is an immediate consequence of Lemma 13.5. It is, in principle, helpful to find Nash equilibria or to determine whether a given strategy combination is a Nash equilibrium. See Example 13.7.

Corollary 13.6. *A strategy pair $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium in a bimatrix game (A,B) if and only if $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$ and $C(\mathbf{q}) \subseteq PB_2(\mathbf{p})$.*

Example 13.7. Consider the bimatrix game

$$(A,B) = \begin{pmatrix} 1,1 & 0,1 & 0,1 & 0,1 \\ 1,1 & 1,1 & 0,1 & 0,1 \\ 1,1 & 1,1 & 1,1 & 0,1 \\ 1,1 & 1,1 & 1,1 & 1,1 \end{pmatrix}$$

and the strategies $\mathbf{p} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{q} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Since

$$A\mathbf{q} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{p}B = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix},$$

$PB_1(\mathbf{q}) = \{2,3,4\}$ and $PB_2(\mathbf{p}) = \{1,2,3,4\}$. Since $C(\mathbf{p}) = \{2,3,4\}$ and $C(\mathbf{q}) = \{1,2\}$, we have $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$ and $C(\mathbf{q}) \subseteq PB_2(\mathbf{p})$. So Corollary 13.6 implies that $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium.

13.2.2 Extension of the Graphical Method

In Sect. 3.2.2 we learnt how to solve 2×2 bimatrix games graphically. We now extend this method to 2×3 and 3×2 games. For larger games it becomes impractical or impossible to use this graphical method.

As an example consider the 2×3 bimatrix game

$$(A,B) = \begin{pmatrix} 2,1 & 1,0 & 1,1 \\ 2,0 & 1,1 & 0,0 \end{pmatrix}.$$

The Nash equilibria of this game are elements of the set $\Delta^2 \times \Delta^3$ of all possible strategy combinations. This set can be represented as in Fig. 13.1.
Here player 2 chooses a point in the triangle with vertices $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, while player 1 chooses a point of the horizontal line segment with vertices $\mathbf{e}_1$ and $\mathbf{e}_2$.

In order to determine the best replies of player 1 note that

$$A\mathbf{q} = \begin{pmatrix} 2q_1 + q_2 + q_3 \\ 2q_1 + q_2 \end{pmatrix}.$$

As $\mathbf{e}_1 A\mathbf{q} = \mathbf{e}_2 A\mathbf{q} \Leftrightarrow q_3 = 0$, it follows that

$$\beta_1(\mathbf{q}) = \begin{cases} \{\mathbf{e}^1\} & \text{if } q_3 > 0 \\ \Delta^2 & \text{if } q_3 = 0. \end{cases}$$

This yields the best reply correspondence represented in Fig. 13.2.
Similarly,

$$pB = \begin{pmatrix} p_1 & p_2 & p_1 \end{pmatrix}$$

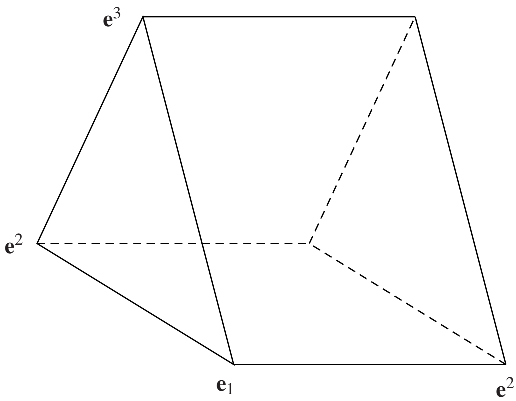

Fig. 13.1 The set $\Delta^2 \times \Delta^3$

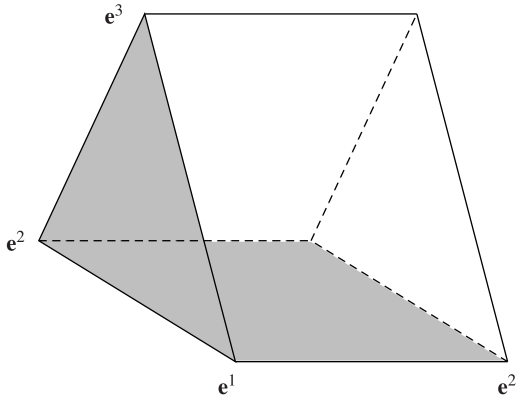

Fig. 13.2 The best reply correspondence of player 1 (*shaded*)

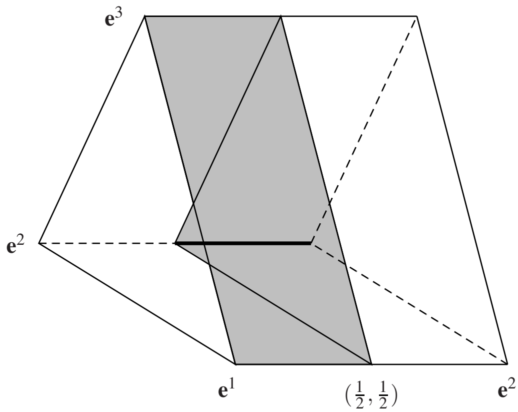

Fig. 13.3 The best reply correspondence of player 2 (*shaded/thick*)

implies

$$\beta_2(\mathbf{p}) = \begin{cases} \{\mathbf{e}^2\} & \text{if } p_1 < p_2 \\ \Delta^3 & \text{if } p_1 = p_2 \\ \{\mathbf{q} \in \Delta^3 \mid q_2 = 0\} & \text{if } p_1 > p_2. \end{cases}$$

This yields the best reply correspondence represented in Fig. 13.3.

Figure 13.4 represents the intersection of the two best reply correspondences and, thus, the set of Nash equilibria.

13.2.3 A Mathematical Programming Approach

In Sect. 12.2 we have seen that matrix games can be solved by linear programming. Nash equilibria of an $m \times n$ bimatrix game (A, B) can be found by considering the

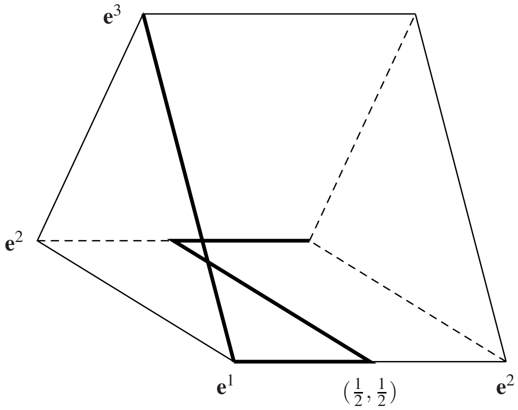

Fig. 13.4 The set of Nash equilibria

following quadratic programming problem (cf. Mangasarian and Stone [76]):

$$\begin{aligned} \max_{\mathbf{p}\in\Delta^m,\ \mathbf{q}\in\Delta^n,\ a,b\in\mathbb{R}} \quad & f(\mathbf{p},\mathbf{q},a,b) := \mathbf{p}A\mathbf{q} + \mathbf{p}B\mathbf{q} - a - b \\ \text{subject to} \quad & \mathbf{e}^i A\mathbf{q} \le a \quad \text{for all } i = 1,2,\ldots,m \\ & \mathbf{p}B\mathbf{e}^j \le b \quad \text{for all } j = 1,2,\ldots,n. \end{aligned} \tag{13.1}$$

Theorem 13.8. $(\mathbf{p},\mathbf{q},a,b)$ *is a solution of* (13.1) *if and only if* $(\mathbf{p},\mathbf{q})$ *is a Nash equilibrium of* (A,B)*,* $a = \mathbf{p}A\mathbf{q}$*,* $b = \mathbf{p}B\mathbf{q}$*.*

Proof. Problem 13.7. □

If (A,B) is a zero-sum game, i.e., if $B = -A$, then (13.1) reduces to

$$\begin{aligned} \max_{\mathbf{p}\in\Delta^m,\ \mathbf{q}\in\Delta^n,\ a,b\in\mathbb{R}} \quad & -a-b \\ \text{subject to} \quad & \mathbf{e}^i A\mathbf{q} \le a \quad \text{for all } i = 1,2,\ldots,m \\ & -\mathbf{p}A\mathbf{e}^j \le b \quad \text{for all } j = 1,2,\ldots,n. \end{aligned} \tag{13.2}$$

Program (13.2) can be split up into two independent programs

$$\begin{aligned} \max_{\mathbf{q}\in\Delta^n,\ a\in\mathbb{R}} \quad & -a \\ \text{subject to} \quad & \mathbf{e}^i A\mathbf{q} \le a \quad \text{for all } i = 1,2,\ldots,m \end{aligned} \tag{13.3}$$

and

$$\begin{aligned} \min_{\mathbf{p}\in\Delta^m,\ b\in\mathbb{R}} \quad & b \\ \text{subject to} \quad & \mathbf{p}A\mathbf{e}^j \ge -b \quad \text{for all } j = 1,2,\ldots,n. \end{aligned} \tag{13.4}$$

One can check that these problems are equivalent to the LP and its dual for matrix games in Sect. 12.2, see Problem 13.8.

Lemke and Howson [70] provide an algorithm to find at least one Nash equilibrium of a bimatrix game. See [143] for an overview.

13.2.4 Matrix Games

Since matrix games are also bimatrix games, everything that we know about bimatrix games is also true for matrix games. In fact, the Minimax Theorem (Theorem 12.2) can be derived directly from the existence theorem for Nash equilibrium (Theorem 13.1). Moreover, each Nash equilibrium in a matrix game consists of a pair of optimal (maximin and minimax) strategies, and each such pair is a Nash equilibrium. As a consequence, in a matrix game, Nash equilibrium strategies are exchangeable – there is no coordination problem, and all Nash equilibria result in the same payoffs.

All these facts are collected in the following theorem. For terminology concerning matrix games see Chap. 12. The 'new' contribution of this theorem is part (2), part (1) is just added to provide an alternative proof of the Minimax Theorem.

Theorem 13.9. *Let A be an $m \times n$ matrix game. Then:*

(1) $v_1(A) = v_2(A)$.
(2) *A pair $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ is a Nash equilibrium of $(A, -A)$ if and only if $\mathbf{p}^*$ is an optimal strategy for player 1 in A and $\mathbf{q}^*$ is an optimal strategy for player 2 in A.*

Proof. (1) In view of Lemma 12.1 it is sufficient to prove that $v_1(A) \geq v_2(A)$. Choose $(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \in \Delta^m \times \Delta^n$ to be a Nash equilibrium of $(A, -A)$ – this is possible by Theorem 13.1. Then

$$\mathbf{p}A\hat{\mathbf{q}} \leq \hat{\mathbf{p}}A\hat{\mathbf{q}} \leq \hat{\mathbf{p}}A\mathbf{q} \quad \text{for all } \mathbf{p} \in \Delta^m,\ \mathbf{q} \in \Delta^n.$$

This implies $\max_{\mathbf{p}} \mathbf{p}A\hat{\mathbf{q}} \leq \hat{\mathbf{p}}A\mathbf{q}$ for all $\mathbf{q}$, hence $v_2(A) = \min_{\mathbf{q}'} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}' \leq \hat{\mathbf{p}}A\mathbf{q}$ for all $\mathbf{q}$. So

$$v_2(A) \leq \min_{\mathbf{q}} \hat{\mathbf{p}}A\mathbf{q} \leq \max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q} = v_1(A).$$

(2) First, suppose that $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ is a Nash equilibrium of $(A, -A)$. Then

$$\mathbf{p}^*A\mathbf{q}^* = \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}^* = v_2(\mathbf{q}^*) \geq \min_{\mathbf{q}} v_2(\mathbf{q}) = v_2(A) = v(A).$$

If $\mathbf{p}^*$ were not optimal, then $\mathbf{p}^*A\mathbf{q} < v(A)$ for some $\mathbf{q} \in \Delta^n$, so $\mathbf{p}^*A\mathbf{q}^* \leq \mathbf{p}^*A\mathbf{q} < v(A)$, a contradiction. Similarly, $\mathbf{q}^*$ must be optimal.

Conversely, suppose that $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies. Since $\mathbf{p}A\mathbf{q}^* \leq v(A)$ for all $\mathbf{p} \in \Delta^m$ and $\mathbf{p}^*A\mathbf{q} \geq v(A)$ for all $\mathbf{q} \in \Delta^n$, it follows that $\mathbf{p}^*$ and $\mathbf{q}^*$ are mutual best replies and, thus, $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium in $(A, -A)$. □

13.2.5 The Game of Chess: Zermelo's Theorem

On of the earliest formal results in game theory is Zermelo's Theorem on the game of chess, published in [150]. In this subsection we provide a simple proof of this theorem, based on Theorem 13.9.

The game of chess is a classical example of a zero-sum game. There are three possible outcomes: a win for White, a win for Black, and a draw. Identifying player 1 with White and player 2 with Black, we can associate with these outcomes the payoffs $(1,-1)$, $(-1,1)$, and $(0,0)$, respectively. In order to guarantee that the (extensive form) game stops after finitely many moves, we assume the following stopping rule: if the same configuration on the chess board has occurred more than twice, the game ends in a draw. Since there are only finitely many configurations on the chess board, the game must stop after finitely many moves. Note that the chess game is a finite extensive form game of perfect information and therefore has a Nash equilibrium in pure strategies – see Sect. 4.3. To be precise, this is a pure strategy Nash equilibrium in the associated matrix game, where mixed strategies are allowed as well.

Theorem 13.10 (Zermelo's Theorem). *In the game of chess, either White has a pure strategy that guarantees a win, or Black has a pure strategy that guarantees a win, or both players have pure strategies that guarantee at least a draw.*

Proof. Let $A = (a_{ij})$ denote the associated matrix game, and let row i^* and column j^* constitute a pure strategy Nash equilibrium. We distinguish three cases.

Case 1. $a_{i^*j^*} = 1$, i.e., White wins. By Theorem 13.9, $v(A) = 1$. I.e., White has a pure strategy that guarantees a win, namely play row i^*.

Case 2. $a_{i^*j^*} = -1$, i.e., Black wins. By Theorem 13.9, $v(A) = -1$. I.e., Black has a pure strategy that guarantees a win, namely play column j^*.

Case 3. $a_{i^*j^*} = 0$, i.e., the game ends in a draw. By Theorem 13.9, $v(A) = 0$. Hence, both White and Black can guarantee at least a draw by playing row i^* and column j^*, respectively. $\square$

13.3 Iterated Dominance and Best Reply

A pure strategy of a player in a finite game is strictly dominated if there is another (mixed or pure) strategy that yields always – whatever the other players do – a strictly higher payoff. Such a strategy is not played in a Nash equilibrium, and can therefore be eliminated. In the smaller game there may be another pure strategy of the same or of another player that is strictly dominated and again may be eliminated. This way a game may be reduced to a smaller game for which it is easier to compute the Nash equilibria. If the procedure results in a unique surviving strategy combination then the game is called *dominance solvable*, but this is a rare exception.

We applied these ideas before, in Chaps. 2 and 3. In this section we show, formally, that by this procedure of iterated elimination of strictly dominated strategies no Nash equilibria of the original game are lost, and no Nash equilibria are added.

For iterated elimination of weakly dominated strategies the situation is different: Nash equilibria may be lost, and the final result may depend on the order of elimination. See Problem 3.6.

We start with repeating the definition of a strictly dominated strategy for an arbitrary finite game.

Definition 13.11. Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game, $i \in N$, $s_i \in S_i$. Strategy s_i is *strictly dominated by strategy* $\sigma_i \in \Delta(S_i)$ if $u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$ for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. Strategy $s_i \in S_i$ is *strictly dominated* if it is strictly dominated by some strategy $\sigma_i \in \Delta(S_i)$.

The fact that iterated elimination of strictly dominated strategies does not essentially change the set of Nash equilibria of a game is a straightforward consequence of the following lemma.

Lemma 13.12. *Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game, $i \in N$, and let $s_i \in S_i$ be strictly dominated. Let $G' = (N, S_1, \ldots, S_{i-1}, S_i \setminus \{s_i\}, S_{i+1}, \ldots, S_n, u'_1, \ldots, u'_n)$ be the game arising from G be eliminating s_i from S_i and restricting the utility functions accordingly. Then:*

(1) *If σ is a Nash equilibrium in G, then $\sigma_i(s_i) = 0$ (where $\sigma_i(s_i)$ is the probability assigned by σ_i to pure strategy $s_i \in S_i$) and σ' is a Nash equilibrium in G', where $\sigma'_j = \sigma_j$ for each $j \in N \setminus \{i\}$ and σ'_i is the restriction of σ_i to $S_i \setminus \{s_i\}$.*
(2) *If σ' is a Nash equilibrium in G', then σ is a Nash equilibrium in G, where $\sigma_j = \sigma'_j$ for each $j \in N \setminus \{i\}$ and $\sigma_i(t_i) = \sigma'_i(t_i)$ for all $t_i \in S_i \setminus \{s_i\}$.*

Proof. (1) Let σ be a Nash equilibrium in G, and let $\tau_i \in \Delta(S_i)$ strictly dominate s_i. If $\sigma_i(s_i) > 0$, then

$$u_i(\hat{\sigma}_i + \sigma_i(s_i)\tau_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}),$$

where $\hat{\sigma}_i : S_i \to \mathbb{R}$ is defined by $\hat{\sigma}_i(t_i) = \sigma_i(t_i)$ for all $t_i \in S_i \setminus \{s_i\}$ and $\hat{\sigma}_i(s_i) = 0$. This contradicts the assumption that σ is a Nash equilibrium in G. Therefore, $\sigma_i(s_i) = 0$. With σ' and G' as above, we have

$$u'_i(\sigma'_1, \ldots, \sigma'_{i-1}, \tau'_i, \sigma'_{i+1}, \ldots, \sigma'_n) = u_i(\sigma_1, \ldots, \sigma_{i-1}, \bar{\tau}'_i, \sigma_{i+1}, \ldots, \sigma_n),$$

for every $\tau'_i \in \Delta(S_i \setminus \{s_i\})$, where $\bar{\tau}'_i \in \Delta(S_i)$ assigns 0 to s_i and is equal to τ'_i otherwise. From this it follows that σ'_i is still a best reply to σ'_{-i}. It is straightforward that also for each $j \neq i$, σ'_j is still a best reply to σ'_{-j}. Hence σ' is a Nash equilibrium in G'.

(2) Let σ' and σ be as above in (2). Obviously, for every player $j \neq i$, σ_j is still a best reply in σ since $\sigma_i(s_i) = 0$, i.e., player i puts zero probability on the new pure strategy s_i. For player i, σ_i is certainly a best reply among all strategies that put zero

probability on s_i. But then, σ_i is a best reply among all strategies, since strategies that put nonzero probability on s_i can never be best replies by the first argument in the proof of (1). Hence, σ is a Nash equilibrium in G. $\square$

Obviously, a strictly dominated pure strategy is not only never played in a Nash equilibrium, but, a forteriori, is never (part of) a best reply. Formally, we say that a pure strategy s_i of player i in the finite game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ is *never a best reply* if for all $(\sigma_j)_{j \neq i}$ and all $\sigma_i \in \beta_i((\sigma_j)_{j \neq i})$, we have $\sigma_i(s_i) = 0$. The following result (cf. [99]) shows that for two-player games also the converse holds.

Theorem 13.13. *In a finite two-person game every pure strategy that is never a best reply, is strictly dominated.*

Proof. Let (A, B) be an $m \times n$ bimatrix game and suppose without loss of generality that pure strategy $\mathbf{e}^1 \in \Delta^m$ of player 1 is never a best reply. Let $\mathbf{b} = (-1, -1, \ldots, -1) \in \mathbb{R}^n$.

Let $\tilde{A}$ be the $(m-1) \times n$ matrix with i-th row equal to the first row of A minus the $i+1$-th row of A, i.e., $\tilde{a}_{ij} = a_{1j} - a_{i+1,j}$ for every $i = 1, \ldots, m-1$ and $j = 1, \ldots, n$. The assumption that the pure strategy $\mathbf{e}^1$ of player 1 is never a best reply is equivalent to the statement that the system

$$\tilde{A}\mathbf{q} \geq \mathbf{0}, \quad \mathbf{q} \in \Delta^n$$

has no solution. This, in turn, is equivalent to the statement that the system

$$\tilde{A}\mathbf{q} \geq \mathbf{0}, \quad \mathbf{q} \geq \mathbf{0}, \quad \mathbf{q} \cdot \mathbf{b} < 0$$

has no solution. This means that the system in (2) of Lemma 22.6 (with $\tilde{A}$ instead of A there) has no solution. Hence, this lemma implies that the system

$$\mathbf{x} \in \mathbb{R}^{m-1}, \quad \mathbf{x}\tilde{A} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}$$

has a solution. By definition of $\mathbf{b}$ and $\tilde{A}$ we have for such a solution $\mathbf{x} = (x_2, \ldots, x_m)$:

$$\mathbf{x} \geq \mathbf{0} \ \text{ and } \ \sum_{i=2}^{m} x_i \mathbf{e}^i A \geq \sum_{i=2}^{m} x_i \mathbf{e}^1 A + (1, \ldots, 1).$$

This implies that $\mathbf{x} \neq \mathbf{0}$ and therefore that $\mathbf{e}^1$ is strictly dominated by the strategy

$$(0, x_2 / \sum_{i=2}^{m} x_i, \ldots, x_m / \sum_{i=2}^{m} x_i) \in \Delta^m.$$

Hence, $\mathbf{e}^1$ is strictly dominated. $\square$

For games with more than two players Theorem 13.13 does not hold, see Problem 13.11 for a counterexample.

The concept of 'never a best reply' is closely related to the concept of *rationalizability* (Bernheim [10], Pearce [99]). Roughly, rationalizable strategies are strategies that survive a process of iterated elimination of strategies that are never a best reply. Just like the strategies surviving iterated elimination of strictly dominated strategies, rationalizable strategies constitute a set-valued solution concept. The above theorem implies that for two-player games the two solution concepts coincide. In general, the set of rationalizable strategies is a subset of the set of strategies that survive iterated elimination of strictly dominated strategies.

The implicit assumption justifying iterated elimination of strategies that are dominated or never a best reply is quite demanding. Not only should a player believe that some other player will not play a such a strategy, but he should also believe that the other player believes that he (the first player) believes this and, in turn, will not use such a strategy in the reduced game, etc.[2]

13.4 Perfect Equilibrium

Since a game may have many, quite different Nash equilibria, literature has focused since a long time on so-called *refinements* of Nash equilibrium. We have seen examples of this in extensive form games, such as subgame perfect equilibrium and perfect Bayesian equilibrium (Chaps. 4, 5). One of the earliest and best known refinements of Nash equilibrium in strategic form games is the concept of 'trembling hand perfection', introduced by Selten [118]. This refinement excludes Nash equilibria that are not robust against 'trembles' in the players' strategies.

Formally, let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game and let μ be an *error function*, assigning a number $\mu_{ih} \in (0,1)$ to every $i \in N$ and $h \in S_i$, such that $\sum_{h \in S_i} \mu_{ih} < 1$ for every player i. The number μ_{ih} is the minimum probability with which player i is going to play pure strategy h, perhaps by 'mistake' ('trembling hand'). Let, for each $i \in N$, $\Delta(S_i, \mu) = \{\sigma_i \in \Delta(S_i) \mid \sigma_i(h) \geq \mu_{ih}$ for all $h \in S_i\}$, and let $G(\mu)$ denote the game derived from G by assuming that each player i may only choose strategies from $\Delta(S_i, \mu)$. The game $G(\mu)$ is called the *μ-perturbed game*. Denote the set of Nash equilibria of G by $NE(G)$ and of $G(\mu)$ by $NE(G(\mu))$.

Lemma 13.14. *For every error function μ, $NE(G(\mu)) \neq \emptyset$.*

Proof. Analogous to the proof of Theorem 13.1. □

A perfect Nash equilibrium is a Nash equilibrium that is the limit of *some* sequence of Nash equilibria of perturbed games. Formally:

Definition 13.15. A strategy combination $\sigma \in NE(G)$ is *perfect* if there is a sequence $G(\mu^t)$, $t \in \mathbb{N}$ of perturbed games with $\mu^t \to \mathbf{0}$ for $t \to \infty$ and a sequence of Nash equilibria $\sigma^t \in G(\mu^t)$ such that $\sigma^t \to \sigma$ for $t \to \infty$.

[2] See [132] or [102].

As follows from the proof of Theorem 13.17 below, a perfect strategy combination is indeed a Nash equilibrium.

Call a strategy combination σ in G *completely mixed* if $\sigma_i(h) > 0$ for all $i \in N$ and $h \in S_i$.

Lemma 13.16. *A completely mixed Nash equilibrium of G is perfect.*

Proof. Problem 13.12. □

Also if a game has no completely mixed Nash equilibrium, it still has a perfect Nash equilibrium.

Theorem 13.17. *Every finite game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ has a perfect Nash equilibrium.*

Proof. Take any sequence $(G(\mu^t))_{\mu^t \to \mathbf{0}}$ of perturbed games and $\sigma^t \in NE(G(\mu^t))$ for each $t \in \mathbb{N}$. Since $\prod_{i \in N} \Delta(S_i)$ is a compact set we may assume without loss of generality that the sequence $(\sigma^t)_{t \in \mathbb{N}}$ converges to some $\sigma \in \prod_{i \in N} \Delta(S_i)$. It is easy to verify that $\sigma \in NE(G)$, hence, σ is a perfect Nash equilibrium. □

The following lemma relates the Nash equilibria of a perturbed game to pure best replies in such a game.

Lemma 13.18. *Let $G(\mu)$ be a perturbed game and σ a strategy combination in $G(\mu)$.*

(1) *If $\sigma \in NE(G(\mu))$, $i \in N$, $h \in S_i$, and $\sigma_i(h) > \mu_{ih}$, then $h \in \beta_i(\sigma_{-i})$.*
(2) *If for all $i \in N$ and $h \in S_i$, $\sigma_i(h) > \mu_{ih}$ implies $h \in \beta_i(\sigma_{-i})$, then $\sigma \in NE(G(\mu))$.*

Proof. (1) Let $\sigma \in NE(G(\mu))$, $i \in N$, $h \in S_i$, and $\sigma_i(h) > \mu_{ih}$. Suppose, contrary to what we wish to prove, that $h \notin \beta_i(\sigma_{-i})$. Take $h' \in S_i$ with $h' \in \beta_i(\sigma_{-i})$. (Such an h' exists by an argument similar to the proof of Lemma 13.3.) Consider the strategy σ_i' defined by $\sigma_i'(h) = \mu_{ih}$, $\sigma_i'(h') = \sigma_i(h') + \sigma_i(h) - \mu_{ih}$, and $\sigma_i'(k) = \sigma_i(k)$ for all $k \in S_i \setminus \{h, h'\}$. Then $\sigma_i' \in \Delta(S_i, \mu)$ and $u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma)$, contradicting the assumption $\sigma \in NE(G(\mu))$.

(2) Let $i \in N$. The condition in (2) implies that, if $h \in S_i$ and $h \notin \beta_i(\sigma_{-i})$, then $\sigma_i(h) = \mu_{ih}$. This implies that σ_i is a best reply to $(\sigma_j)_{j \neq i}$. Thus, $\sigma \in NE(G(\mu))$. □

Below we present two characterizations of perfect Nash equilibrium that both avoid sequences of perturbed games. The first one is based on the notion of ε-equilibrium, introduced in [87].

Let $\varepsilon > 0$. A strategy combination $\sigma \in \prod_{i \in N} \Delta(S_i)$ is an *ε-perfect equilibrium* of G if it is completely mixed and $\sigma_i(h) \leq \varepsilon$ for all $i \in N$ and all $h \in S_i$ with $h \notin \beta_i(\sigma_{-i})$.

An ε-perfect equilibrium of G need not be a Nash equilibrium of G, but it puts probabilities of at most ε on pure strategies that are not best replies.

The announced characterizations are collected in the following theorem.[3]

[3] This theorem is based on [118] and [87] and appears as Theorem 2.2.5 in [138].

Theorem 13.19. *Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ and $\sigma \in \prod_{i \in N} \Delta(S_i)$. The following statements are equivalent:*

(1) *σ is a perfect Nash equilibrium of G;*
(2) *σ is a limit of a sequence of ε-perfect equilibria σ^{ε} of G for $\varepsilon \to 0$;*
(3) *σ is a limit of a sequence of completely mixed strategy combinations σ^{ε} for $\varepsilon \to 0$, where $\sigma_i \in \beta_i(\sigma^{\varepsilon}_{-i})$ for each $i \in N$ and each σ^{ε} in this sequence.*

Proof. (1) $\Rightarrow$ (2): Take a sequence of perturbed games $G(\mu^t)$, $t \in \mathbb{N}$ with $\mu^t \to \mathbf{0}$ and a sequence $\sigma^t \in NE(G(\mu^t))$ with $\sigma^t \to \sigma$. For each t define $\varepsilon^t \in \mathbb{R}$ by $\varepsilon^t = \max\{\mu_{ih} \mid i \in N,\ h \in S_i\}$. Then, by Lemma 13.18(1), σ^t is an ε^t-equilibrium for every t. So (2) follows.

(2) $\Rightarrow$ (3): Take a sequence of ε-equilibria σ^{ε} as in (2) converging to σ for $\varepsilon \to 0$. Let $i \in N$. By the definition of ε-perfect equilibrium, if $\sigma_i(h) > 0$ for some $h \in S_i$, then for ε sufficiently small we have $h \in \beta_i(\sigma_{-i})$. This implies $\sigma_i \in \beta_i(\sigma_{-i})$. So (3) follows.

(3) $\Rightarrow$ (1): Let σ^{ε^t} ($t \in \mathbb{N}$) be a sequence as in (3) with $\varepsilon^t \to 0$ and $\sigma^{\varepsilon^t} \to \sigma$ as $t \to \infty$. For each $t \in \mathbb{N}$, $i \in N$ and $h \in S_i$ define $\mu^t_{ih} = \sigma^{\varepsilon^t}_i(h)$ if $\sigma_i(h) = 0$ and $\mu^t_{ih} = \varepsilon^t$ otherwise. Then, for ε^t sufficiently small, μ^t is an error function, $G(\mu^t)$ is a perturbed game, and σ^{ε^t} is a strategy combination in $G(\mu^t)$. By Lemma 13.18(2), $\sigma^{\varepsilon^t} \in NE(G(\mu^t))$. So σ is a perfect Nash equilibrium of G. □

There is a close relation between the concept of domination and the concept of perfection. We first extend the concept of (weak) domination to mixed strategies. In the game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$, call a strategy $\sigma_i \in \Delta(S_i)$ *(weakly) dominated by* $\sigma'_i \in \Delta(S_i)$ if $u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$, with at least one inequality strict. (Observe that it is actually sufficient to check this for combinations $s_{-i} \in \prod_{j \neq i} S_i$.) Call σ_i *undominated* if there is no σ'_i by which it is dominated, and call a strategy combination σ *undominated* if σ_i is undominated for every $i \in N$. We now have:

Theorem 13.20. *Every perfect Nash equilibrium in G is undominated.*

Proof. Let σ be a perfect Nash equilibrium and suppose that (say) σ_1 is dominated. Then there is a $\sigma'_1 \in \Delta(S_1)$ such that $u_1(\sigma_1, s_{-1}) \leq u_1(\sigma'_1, s_{-1})$ for all $s_{-1} \in \prod_{i=2}^{n} S_i$, with at least one inequality strict. Take a sequence $(\sigma^t)_{t \in \mathbb{N}}$ of strategy combinations as in (3) of Theorem 13.19, converging to σ. Then, since every σ^t is completely mixed, we have $u_1(\sigma_1, \sigma^t_{-1}) < u_1(\sigma'_1, \sigma^t_{-1})$ for every t. This contradicts the fact that σ_1 is a best reply to σ^t_{-1}. □

The converse of Theorem 13.20 is only true for two-person games. For a counterexample involving three players, see Problem 13.13.

For proving the converse of the theorem for bimatrix games, we use the following auxiliary lemma. In this lemma, for a matrix game $\tilde{A}$, $C_2(\tilde{A})$ denotes the set of all columns of $\tilde{A}$ that are in the carrier of some optimal strategy of player 2 in $\tilde{A}$.

Lemma 13.21. *Let $G = (A,B)$ be an $m \times n$ bimatrix game and let $\mathbf{p} \in \Delta^m$. Define the $m \times n$ matrix $\tilde{A} = (\tilde{a}_{ij})$ by $\tilde{a}_{ij} = a_{ij} - \mathbf{p}A\mathbf{e}^j$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Then $\mathbf{p}$ is undominated in G if and only if $v(\tilde{A}) = 0$ and $C_2(\tilde{A}) = \{1,\ldots,n\}$.*

Proof. First note $\mathbf{p}\tilde{A} = \mathbf{0}$ and therefore $v(\tilde{A}) \geq 0$.

For the if-direction, suppose that $\mathbf{p}$ is dominated in G, say by $\mathbf{p}'$. Then $\mathbf{p}'A \gneqq \mathbf{p}A$, hence $\mathbf{p}'\tilde{A} \gneqq \mathbf{0}$. Therefore, if $v(\tilde{A}) = 0$, then $\mathbf{p}'$ is an optimal strategy in $\tilde{A}$ and, thus, $j \notin C_2(\tilde{A})$ whenever $\mathbf{p}'\tilde{A}\mathbf{e}^j > 0$. This proves the if-direction.

For the only-if direction suppose that $\mathbf{p}$ is undominated in G. Then clearly $v(\tilde{A}) = 0$ otherwise $\mathbf{p}$ would be dominated in G by any strategy that is optimal for player 1 in $\tilde{A}$. Suppose there is a column j that is not an element of $C_2(\tilde{a})$. By Problem 12.6 there must be an optimal strategy $\mathbf{p}'$ of player 1 in $\tilde{A}$ such that $\mathbf{p}'\tilde{A}\mathbf{e}^j > 0$, so that $\mathbf{p}'\tilde{A} \gneqq \mathbf{0}$, hence $\mathbf{p}'A \gneqq \mathbf{p}A$. So $\mathbf{p}'$ dominates $\mathbf{p}$ in G, a contradiction. This proves the only-if direction. □

Theorem 13.22. *Let $G = (A,B)$ be a bimatrix game, and let $(\mathbf{p},\mathbf{q})$ be an undominated Nash equilibrium. Then $(\mathbf{p},\mathbf{q})$ is perfect.*

Proof. Let $\tilde{A}$ as in Lemma 13.21, then $\mathbf{p}$ is an optimal strategy for player 1 in $\tilde{A}$ since $\mathbf{p}\tilde{A} = \mathbf{0}$ and $v(\tilde{A}) = 0$. By Lemma 13.21 we can find a completely mixed optimal strategy $\mathbf{q}'$ for player 2 in $\tilde{A}$. So $\mathbf{p}$ is a best reply to $\mathbf{q}'$ in $\tilde{A}$, i.e., $\mathbf{p}\tilde{A}\mathbf{q}' \geq \tilde{\mathbf{p}}\tilde{A}\mathbf{q}'$ for all $\tilde{\mathbf{p}}$, and thus $\tilde{\mathbf{p}}A\mathbf{q}' - \mathbf{p}A\mathbf{q}' \leq 0$ for all $\tilde{\mathbf{p}}$. So $\mathbf{p}$ is also a best reply to $\mathbf{q}'$ in G. For $1 > \varepsilon > 0$ define $\mathbf{q}^\varepsilon = (1-\varepsilon)\mathbf{q} + \varepsilon\mathbf{q}'$. Then $\mathbf{q}^\varepsilon$ is completely mixed, $\mathbf{p}$ is a best reply to $\mathbf{q}^\varepsilon$, and $\mathbf{q}^\varepsilon \to \mathbf{q}$ for $\varepsilon \to 0$. In the same way we can construct a sequence $\mathbf{p}^\varepsilon$ with analogous properties, converging to $\mathbf{p}$. Then implication (3) $\Rightarrow$ (1) in Theorem 13.19 implies that $(\mathbf{p},\mathbf{q})$ is perfect. □

The following example shows an advantage but at the same time a drawback of perfect Nash equilibrium.

Example 13.23. Consider the bimatrix game

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 1,1 & 10,0 \\ 0,10 & 10,10 \end{pmatrix} \end{array},$$

which has two Nash equilibria, both pure, namely (U,L) and (D,R). Only (U,L) is perfect, as can be seen by direct inspection or by applying Theorems 13.20 and 13.22. At the equilibrium (D,R), each player has an incentive to deviate to the other pure strategy since the opponent may deviate by mistake. This equilibrium is excluded by perfection. On the other hand, the unique perfect equilibrium (U,L) is payoff-dominated by the equilibrium (D,R).

Another drawback of perfect equilibrium is the fact that adding dominated strategies may result in adding perfect Nash equilibria, as the following example shows.

Example 13.24. In the game

$$\begin{array}{c} \\ U \\ M \\ D \end{array}\begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 & -1,-2 \\ 0,0 & 0,0 & 0,-2 \\ -2,-1 & -2,0 & -2,-2 \end{pmatrix} \end{array},$$

there are two perfect Nash equilibria, namely (U,L) and (M,C). If we reduce the game by deleting the pure strategies D and R, the only perfect equilibrium that remains is (U,L).

This motivated the introduction of a further refinement called *proper Nash equilibrium* in [87]. See the next section.

13.5 Proper Equilibrium

A perfect equilibrium is required to be robust only against *some* 'trembles' and, moreover, there are no further conditions on these trembles. Myerson [87] proposes the additional restriction that trembles be less probable if they are more 'costly'.

Given some $\varepsilon > 0$, call a strategy combination σ in the game $G = (N, S_1, \dots, S_n, u_1, \dots, u_n)$ an *ε-proper equilibrium* if σ is completely mixed and for all $i \in N$ and $h,k \in S_i$ we have

$$u_i(h, \sigma_{-i}) < u_i(k, \sigma_{-i}) \Rightarrow \sigma_i(h) \le \varepsilon \sigma_i(k).$$

Observe that an ε-proper equilibrium does not have to be a Nash equilibrium.

Definition 13.25. A strategy combination σ in G is *proper* if, for some sequence $\varepsilon^t \to 0$, there exist ε^t-proper equilibria $\sigma(\varepsilon^t)$ such that $\sigma(\varepsilon^t) \to \sigma$.

Since, in a proper strategy combination σ, a pure strategy h of a player i that is not a best reply to σ_{-i} is played with probability 0, it follows that a proper strategy combination is a Nash equilibrium.[4] Moreover, since it is straightforward by the definitions that an ε-proper equilibrium is also an ε-perfect equilibrium, it follows from Theorem 13.19 that a proper equilibrium is perfect. Hence, properness is a refinement of perfection. Example 13.24 shows that this refinement is strict: the Nash equilibrium (M,C) is perfect but not proper. To see this, note that in an ε-proper equilibrium $(\mathbf{p}, \mathbf{q})$ we must have at least $q_3 \le \varepsilon q_1$. But then U is a best reply of player 1, hence it is not possible that a sequence of such ε-proper equilibria converges to (M,C).

[4] Note that, by replacing the word 'proper' by 'perfect' in Definition 13.25, we obtain an alternative definition of perfect (Nash) equilibrium. This follows from Theorem 13.19.

A proper Nash equilibrium always exists:

Theorem 13.26. *Let* $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ *be a finite game. Then* G *has a proper Nash equilibrium.*

Proof. It is sufficient to show that for $\varepsilon > 0$ close to 0 there exists an ε-proper equilibrium of G. Let $0 < \varepsilon < 1$ and define the error function μ by $\mu_{ik} = \varepsilon^{|S_i|}/|S_i|$ for all $i \in N$ and $k \in S_i$. For every $i \in N$ and $\sigma \in \prod_{j \in N} \Delta(S_j, \mu)$ define

$$F_i(\sigma) = \{\tau_i \in \Delta(S_i, \mu) \mid \forall k, l \in S_i \, [u_i(k, \sigma_{-i}) < u_i(l, \sigma_{-i}) \Rightarrow \tau_i(k) \leq \varepsilon \tau_i(l)]\}.$$

Then $F_i(\sigma) \neq \emptyset$ as can be seen as follows. Define

$$v_i(\sigma, k) = |\{l \in S_i \mid u_i(k, \sigma_{-i}) < u_i(l, \sigma_{-i})\}|,$$

for all $k \in S_i$, and $\tau_i(k) = \varepsilon^{v_i(\sigma,k)} / \sum_l \varepsilon^{v_i(\sigma,l)}$ for all $k \in S_i$. Then $\tau_i \in F_i(\sigma)$. Consider the correspondence

$$F : \prod_{j \in N} \Delta(S_j, \mu) \to \prod_{j \in N} \Delta(S_j, \mu), \; \sigma \mapsto \prod_{i \in N} F_i(\sigma).$$

Then F satisfies the conditions of the Kakutani fixed point theorem (Theorem 22.10) – see Problem 13.14. Hence, F has a fixed point, and each fixed point of F is an ε-proper equilibrium of G. $\square$

In spite of the original motivation for introducing properness, this concept suffers from the same deficit as perfect equilibrium: adding strictly dominated strategies may enlarge the set of proper Nash equilibria. See Problem 13.15 for an example of this.

13.6 Strictly Perfect Equilibrium

Another refinement of perfect equilibrium is obtained by requiring robustness of a Nash equilibrium with respect to *all* 'trembles'. This results in the concept of strictly perfect equilibrium (Okada [95]).

Definition 13.27. A strategy combination σ in the game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ is *strictly perfect* if, for every sequence $\{G(\mu^t)\}_{\mu^t \to 0}$ of perturbed games there exist profiles $\sigma^t \in NE(G(\mu^t))$ such that $\sigma^t \to \sigma$.

Clearly, a strictly perfect strategy combination is a perfect Nash equilibrium. For some further observations concerning strictly perfect equilibrium see Problem 13.16.

13.7 Correlated Equilibrium

In the preceding sections we studied several refinements of Nash equilibrium. In this section the set of Nash equilibria is extended in a way to become clear below. It is, however, not the intention to enlarge the set of Nash equilibria but rather to enable the players to reach better payoffs by allowing some communication device. This is going to result in the concept of *correlated equilibrium*, introduced in [5]. Attention in this section is restricted to bimatrix games, and we closely follow the presentation in [98].

In order to fix ideas, consider the situation where two car drivers approach a road crossing. Each driver has two pure strategies: 'stop' (s) or 'cross'(c). The preferences for the resulting combinations are as expressed by the following table:

$$(A,B) \;=\; \begin{array}{c} \\ c \\ s \end{array} \begin{array}{c} \begin{array}{cc} c & \quad s \end{array} \\ \begin{pmatrix} -10,-10 & 5,0 \\ 0,5 & -1,-1 \end{pmatrix} \end{array}.$$

This bimatrix game has two asymmetric and seemingly unfair pure Nash equilibria, and one symmetric mixed Nash equilibrium $((3/8,5/8),(3/8,5/8))$, resulting in an expected payoff of $-5/8$ for both, and therefore also not quite satisfying.

Now suppose that traffic lights are installed that indicate c ('green') or s ('red') according to the probabilities in the following table:

$$\begin{array}{c} \\ c \\ s \end{array} \begin{array}{c} \begin{array}{cc} c & \quad s \end{array} \\ \begin{pmatrix} 0.00 & 0.55 \\ 0.40 & 0.05 \end{pmatrix} \end{array}.$$

E.g., with probability 0.55 (55% of the time) the light is green for driver 1 and red for driver 2. Assume that the players (drivers) are not forced to obey the traffic lights but know the probabilities as given in the table. We argue that it is in each player's own interest to obey the lights if the other player does so.

If the light is green for player 1 then player 1 knows with certainty that the light is red for player 2. So if player 2 obeys the lights and stops, it is indeed optimal for player 1 to cross. If the light is red for player 1, then the conditional probability that player 2 crosses (if he obeys the lights) is equal to $0.4/0.45 \approx 0.89$ and the conditional probability that player 2 stops is $0.05/0.45 \approx 0.11$. So if player 1 stops, his expected payoff is $0.89 \cdot 0 + 0.11 \cdot -1 = -0.11$, and if he crosses his expected payoff is $0.89 \cdot -10 + 0.11 \cdot 5 = -8.35$. Clearly, it is optimal for player 1 to obey the light and stop. For player 2 the argument is similar, so that we can indeed talk of an equilibrium: such an equilibrium is called a *correlated equilibrium*. Note that there is no mixed strategy combination in the game (A,B) that induces these probabilities. In terms of the situation in the example, this particular equilibrium cannot be reached without traffic lights serving as a communication device between the players. The overall expected payoffs of the players are $0.55 \cdot 5 + 0.05 \cdot -1 = 2.7$ for player 1 and

$0.40 \cdot 5 + 0.05 \cdot -1 = 1.95$ for player 2, which is considerably better for both than the payoffs in the mixed Nash equilibrium.

In general, let (A,B) be an $m \times n$ bimatrix game. A *correlated strategy* is an $m \times n$ matrix $P = (p_{ij})$ with $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$ and $p_{ij} \geq 0$ for all $i = 1,\ldots,m$, $j = 1,\ldots,n$. A correlated strategy P can be thought of as a communication device: the pair (i,j) is chosen with probability p_{ij}, and if that happens, player 1 receives the signal i and player 2 the signal j. Suppose player 2 obeys the signal. If player 1 receives signal i and indeed plays i, his expected payoff is

$$\sum_{j=1}^{n} p_{ij} a_{ij} / \sum_{j=1}^{n} p_{ij},$$

and if he plays row k instead, his expected payoff is

$$\sum_{j=1}^{n} p_{ij} a_{kj} / \sum_{j=1}^{n} p_{ij}.$$

So to keep player 1 from disobeying the received signal, we should have

$$\sum_{j=1}^{n} (a_{ij} - a_{kj}) p_{ij} \geq 0 \quad \text{for all } i,k = 1,\ldots,m. \tag{13.5}$$

The analogous condition for player 2 is

$$\sum_{i=1}^{m} (b_{ij} - b_{il}) p_{ij} \geq 0 \quad \text{for all } j,l = 1,\ldots,n. \tag{13.6}$$

Definition 13.28. A *correlated equilibrium* in the bimatrix game (A,B) is a correlated strategy $P = (p_{ij})$ satisfying (13.5) and (13.6).

For the two-driver example conditions (13.5) and (13.6) result in four inequalities, which are not difficult to solve (Problem 13.17). In general, any Nash equilibrium of a bimatrix game results in a correlated equilibrium (Problem 13.18), so existence of a correlated equilibrium is not really an issue.

The set of correlated equilibria is convex (Problem 13.19), so the convex hull of all payoff pairs corresponding to the Nash equilibria of a bimatrix game consists of payoff pairs attainable in correlated equilibria. Problem 13.20 presents an example of a game in which some payoff pairs can be reached in correlated equilibria but not as convex combinations of payoff pairs of Nash equilibria.

In general, correlated equilibria can be computed using linear programming. Specifically, let (A,B) be an $m \times n$ bimatrix game. We associate with (A,B) an $mn \times (m(m-1) + n(n-1))$ matrix C as follows. For each pair $(i,j)$of a row and a column in (A,B)

we have a row in C, and for each pair (h,k) of two different rows in (A,B) or two different columns in (A,B) we have a column in C. We define

$$c_{(i,j)(h,k)} = \begin{cases} a_{ij} - a_{kj} & \text{if } i = h \in \{1,\ldots,m\} \text{ and } k \in \{1,\ldots,m\} \\ b_{ij} - b_{ih} & \text{if } j = k \in \{1,\ldots,n\} \text{ and } h \in \{1,\ldots,n\} \\ 0 & \text{otherwise.} \end{cases}$$

Consider C as a matrix game. By construction of C and (13.5) and (13.6), any correlated equilibrium $P = (p_{ij})$ of (A,B) viewed as a strategy $\mathbf{p}$ of player 1 in C (not to be confused with player 1 in (A,B)) has an expected payoff vector $\mathbf{p}C \geq \mathbf{0}$, and therefore $v(C)$, the value of the matrix game C, is nonnegative. In particular, this implies that any optimal strategy of player 1 in C is a correlated equilibrium in (A,B). If $v(C) = 0$ then any correlated equilibrium in (A,B) is an optimal strategy of player 1 in C, but if $v(C) > 0$ then there may be correlated equilibria in (A,B) that are not optimal strategies for player 1 in C – they may only guarantee zero. The latter is the case in the two-drivers example (Problem 13.21).

Matrix games can be solved by linear programming, see Sect. 12.2.

We conclude with an example[5] in which the described technique is applied.

Example 13.29. Consider the bimatrix game

$$(A,B) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 3,1 & 2,5 & 6,0 \\ 1,4 & 3,3 & 2,6 \end{pmatrix} \end{array}.$$

The associated matrix game C is as follows.

$$\begin{array}{c} \\ (1,1') \\ (1,2') \\ (1,3') \\ (2,1') \\ (2,2') \\ (2,3') \end{array} \begin{array}{c} \begin{array}{cccccccc} (1,2) & (2,1) & (1',2') & (1',3') & (2',1') & (2',3') & (3',1') & (3',2') \end{array} \\ \begin{pmatrix} 2 & 0 & -4 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 5 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & -2 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -3 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix} \end{array}.$$

It can be checked that this game has value 0, by using the optimal strategies

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 0 & 0.3 & 0.075 \\ 0 & 0.5 & 0.125 \end{pmatrix} \end{array}$$

for player 1 in C and $(0,0,1/2,1/2,0,0,0,0)$ for player 2 in C. The optimal strategy for player 1 in C is unique, and since $v(C) = 0$ this implies that the game (A,B) has a unique correlated equilibrium. Consequently, this must correspond to the unique

[5] This is Example VIII.4.4 in [98].

Nash equilibrium of the game. Indeed, $((3/8,5/8),(0,4/5,1/5))$ is the unique Nash equilibrium of (A,B) and it results in the probabilities given by P.

13.8 A Characterization of Nash Equilibrium

The concept of Nash equilibrium requires strong behavioral assumptions about the players. Each player should be able to guess what other players will do, assume that other players know this and make similar conjectures, etc., and all this should be in equilibrium. The basic difficulty is that Nash equilibrium is a circular concept. Not surprisingly, theories of repeated play or learning or, more generally, dynamic models that aim to explain how players in a game come to play a Nash equilibrium, have in common that they change the strategic decision into a collection of single-player decision problems.[6]

In this section (based on [101]) we review a different approach, which is axiomatic in nature. The Nash equilibrium concept is viewed as a solution concept: a correspondence which assigns to any finite game a set of strategy combinations. One of the conditions (axioms) put on this correspondence is a condition of consistency with respect to changes in the number of players: if a player leaves the game, leaving his strategy as an input behind, then the other players should not want to change their strategies. This is certainly true for Nash equilibrium, but can be put as an abstract condition on a solution correspondence. By assuming that players in single-player games – hence, in 'simple' maximization problems – behave rationally, and by adding a converse consistency condition, it follows that the solution correspondence must be the Nash equilibrium correspondence. We proceed with a formal treatment of this axiomatic characterization.

Let Γ be a collection of finite games of the form $G=(N,S_1,\ldots,S_n,u_1,\ldots,u_n)$. (It is implicit that also the set of players N may vary in Γ.) A *solution* on Γ is a function φ that assigns to each $G\in\Gamma$ a set of strategy combinations $\varphi(G)\subseteq\prod_{i\in N}\Delta(S_i)$.[7] A particular solution is the Nash correspondence NE, assigning to each $G\in\Gamma$ the set $NE(G)$ of all Nash equilibria in G.

Definition 13.30. The solution φ satisfies *one-person rationality* (OPR) if for every one-person game $G=(\{i\},S_i,u_i)$ in Γ

$$\varphi(G)=\{\sigma_i\in\Delta(S_i)\mid u_i(\sigma_i)\geq u_i(\tau_i)\text{ for all }\tau_i\in\Delta(S_i)\}.$$

The interpretation of OPR is clear and needs no further comments.

Let $G=(N,S_1,\ldots,S_n,u_1,\ldots,u_n)$ be a game, $\emptyset\neq M\subseteq N$, and let σ be a strategy combination in G. The *reduced game* of G with respect to M and σ is the game $G^{M,\sigma}=(M,(S_i)_{i\in M},(u_i^\sigma)_{i\in M})$, where $u_i^\sigma(\tau)=u_i(\tau,\sigma_{N\setminus M})$ for all $\tau\in\prod_{j\in M}\Delta(S_j)$.[8]

[6] See, e.g., [149].

[7] The presentation in [101] is for more general games.

[8] For a subset $T\subseteq N$, denote $(\sigma_j)_{j\in T}$ by σ_T.

The interpretation of such a reduced game is straightforward: if the players of $N \setminus M$ leave the game, leaving their strategy combination $\sigma_{N\setminus M}$ behind, then the remaining players are faced with the game $G^{M,\sigma}$. Alternatively, if it is common knowledge among the players in M that the players outside M play according to σ, then they are faced with the game $G^{M,\sigma}$. Call a collection of games Γ *closed* if it is closed under taking reduced games.

Definition 13.31. Let Γ be a closed collection of games and let φ be a solution on Γ. Then φ is *consistent* (CONS) if for every game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$, every $\emptyset \neq M \subseteq N$, and every strategy combination $\sigma \in \varphi(G)$, we have $\sigma_M \in \varphi(G^{M,\sigma})$.

The interpretation of consistency is: if the players outside M have left the game while leaving the strategy combination $\sigma_{N\setminus M}$ behind, then there should be no need for the remaining players to revise their strategies.

The consequence of imposing OPR and CONS on a solution correspondence is that it can contain only Nash equilibria:

Proposition 13.32. *Let Γ be a closed collection of games and let φ be a solution on Γ satisfying* OPR *and* CONS. *Then $\varphi(G) \subseteq NE(G)$ for every $G \in \Gamma$.*

Proof. Let $G = (N, S_1, \ldots, S_n,\ u_1, \ldots, u_n) \in \Gamma$ and $\sigma \in \varphi(G)$. By CONS, $\sigma_i \in \varphi(G^{\{i\},\sigma})$ for every $i \in N$. By OPR, $u_i^\sigma(\sigma_i) \geq u_i^\sigma(\tau_i)$ for every $\tau_i \in \Delta(S_i)$ and $i \in N$. Hence

$$u_i(\sigma_i, \sigma_{N\setminus\{i\}}) \geq u_i(\tau_i, \sigma_{N\setminus\{i\}}) \text{ for every } \tau_i \in \Delta(S_i) \text{ and } i \in N.$$

Thus, $\sigma \in NE(G)$. □

Proposition 13.32 says that *NE* is the maximal solution (with respect to set-inclusion) satisfying OPR and CONS. (It it trivial to see that NE satisfies these conditions.) To derive a similar minimal set-inclusion result we use another condition.

Let Γ be a closed collection of games and let φ be a solution on Γ. For a game $G = (N, S_1, \ldots, S_n,\ u_1, \ldots, u_n) \in \Gamma$ with $|N| \geq 2$ denote

$$\tilde{\varphi}(G) = \{\sigma \in \prod_{i \in N} \Delta(S_i) \mid \text{ for all } \emptyset \neq M \subsetneq N,\ \sigma_M \in \varphi(G^{M,\sigma})\}.$$

Definition 13.33. A solution φ on a closed collection of games satisfies *converse consistency* (COCONS) if for every game G with at least two players, $\tilde{\varphi}(G) \subseteq \varphi(G)$.

Converse consistency says that strategy combinations of which the restrictions belong to the solution in smaller reduced games should also belong to the solution in the game itself. Note that consistency can be defined by the converse inclusion $\varphi(G) \subseteq \tilde{\varphi}(G)$ for every $G \in \Gamma$, which explains the expression 'converse consistency'. Obviously, the Nash equilibrium correspondence satisfies COCONS.

Proposition 13.34. *Let Γ be a closed collection of games and let φ be a solution on Γ satisfying* OPR *and* COCONS. *Then $\varphi(G) \supseteq NE(G)$ for every $G \in \Gamma$.*

Proof. The proof is by induction on the number of players. For one-person games the inclusion follows (with equality) from OPR. Assume that $NE(G) \subseteq \varphi(G)$ for all t-person games in Γ where $t \leq k$ and $k \geq 1$. Let G_0 be a $k+1$-person game in Γ. Note that $NE(G_0) \subseteq \widetilde{NE}(G_0)$ by CONS of NE. By the induction hypothesis, $\widetilde{NE}(G_0) \subseteq \tilde{\varphi}(G_0)$ and by COCONS, $\tilde{\varphi}(G_0) \subseteq \varphi(G_0)$. Thus, $NE(G_0) \subseteq \varphi(G_0)$. $\square$

Corollary 13.35. *Let Γ be a closed family of games. The Nash equilibrium correspondence is the unique solution on Γ satisfying* OPR, CONS, *and* COCONS.

It can be shown that the axioms in Corollary 13.35 are independent (Problem 13.23).

The consistency approach fails when applied to refinements of Nash equilibrium, see [93] for details. For instance, Problem 13.24 shows that the correspondence of perfect equilibria is not consistent.

Problems

13.1. *Existence of Nash Equilibrium*

Prove that the correspondence β in the proof of Theorem 13.1 is upper semi-continuous and convex-valued. Also check that every fixed point of β is a Nash equilibrium of G.

13.2. *Lemma 13.2*

Prove Lemma 13.2.

13.3. *Lemma 13.3*

Prove Lemma 13.3.

13.4. *Dominated Strategies*

Let (A,B) be an $m \times n$ bimatrix game. Suppose there exists a $\mathbf{q} \in \Delta^n$ such that $q_n = 0$ and $B\mathbf{q} > B\mathbf{e}^n$ (i.e., there exists a mixture of the first $n-1$ columns of B that is strictly better than playing the n-th column).

(a) Prove that $q_n^* = 0$ for every Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$.

Let (A', B') be the bimatrix game obtained from (A,B) be deleting the last column.

(b) Prove that $(\mathbf{p}*, \mathbf{q}')$ is a Nash equilibrium of (A', B') if and only if $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium of (A,B), where $\mathbf{q}'$ is the strategy obtained from $\mathbf{q}^*$ by deleting the last coordinate.

13.5. *A* 3×3 *Bimatrix Game*

Consider the 3×3 bimatrix game

$$(A,B) = \begin{pmatrix} 0,4 & 4,0 & 5,3 \\ 4,0 & 0,4 & 5,3 \\ 3,5 & 3,5 & 6,6 \end{pmatrix}.$$

Let $(\mathbf{p},\mathbf{q})$ be a Nash equilibrium in (A,B).

(a) Prove that $\{1,2\} \nsubseteq C(\mathbf{p})$.

(b) Prove that $C(\mathbf{p}) \neq \{2,3\}$.

(c) Find all Nash equilibria of this game.

13.6. *A* 3×2 *Bimatrix Game*

Use the graphical method to compute the Nash equilibria of the bimatrix game

$$(A,B) = \begin{pmatrix} 0,0 & 2,1 \\ 2,2 & 0,2 \\ 2,2 & 0,2 \end{pmatrix}.$$

13.7. *Proof of Theorem 13.8*

Prove Theorem 13.8.

13.8. *Matrix Games*

Show that the pair of linear programs (13.3) and (13.4) is equivalent to the LP and its dual in Sect. 12.2 for solving matrix games.

13.9. *Tic-Tac-Toe*

The two-player game of Tic-Tac-Toe is played on a 3×3 board. Player 1 starts by putting a cross on one of the nine fields. Next, player 2 puts a circle on one of the eight remaining fields. Then player 1 puts a cross on one of the remaining seven fields, etc. If player 1 achieves three crosses or player 2 achieves three circles in a row (either vertically or horizontally or diagonally) then that player wins. If this does not happen and the board is full, then the game ends in a draw.

(a) Design a pure maximin strategy for player 1. Show that this maximin strategy guarantees at least a draw to him.

(b) Show that player 1 cannot guarantee a win.

(c) What is the value of this game?

13.10. *Iterated Elimination in a Three-Player Game*

Solve the following three-player game (from [145]), where player 1 chooses rows, player 2 columns, and player 3 one of the two games L and R:

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 14,24,32 & 8,30,27 \\ 30,16,24 & 13,12,50 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 16,24,30 & 30,16,24 \\ 30,23,14 & 14,24,32 \end{pmatrix} \end{array}.$$

13.11. *Never a Best Reply and Domination*

In the following game (taken from [39]) player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. The diagram gives the payoffs of player 3. Show that Y is never a best reply for player 3, and that Y is not strictly dominated.

$$V: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad W: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0 & 9 \\ 9 & 0 \end{pmatrix} \end{array}$$

$$X: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix} \end{array} \quad Y: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

13.12. *Completely Mixed Nash Equilibria are Perfect*

Prove Lemma 13.16.

13.13. *A Three-Player Game with an Undominated but not Perfect Equilibrium*

Consider the following three-player game, where player 1 chooses rows, player 2 columns, and player 3 matrices (taken from [138]):

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,1 & 1,0,1 \\ 1,1,1 & 0,0,1 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,0 & 0,0,0 \\ 0,1,0 & 1,0,0 \end{pmatrix} \end{array}.$$

(a) Show that (U,l,L) is the only perfect Nash equilibrium of this game.

(b) Show that (D,l,L) is an undominated Nash equilibrium.

13.14. *Existence of Proper Equilibrium*

Prove that the correspondence F in the proof of Theorem 13.26 satisfies the conditions of the Kakutani fixed point theorem.

13.15. *Strictly Dominated Strategies and Proper Equilibrium*

Consider the three-person game (cf. [138], p. 31)

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,1 & 0,0,1 \\ 0,0,1 & 0,0,1 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 0,0,0 & 0,0,0 \\ 0,0,0 & 1,1,0 \end{pmatrix} \end{array},$$

where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices.

(a) First assume that player 3 is a dummy and has only one strategy, namely L. Compute the perfect and proper Nash equilibrium or equilibria of the game.

(b) Now suppose that player 3 has two pure strategies. Compute the perfect and proper Nash equilibrium or equilibria of the game. Conclude that adding a strictly dominated strategy (namely, R) has resulted in an additional proper equilibrium.

13.16. *Strictly Perfect Equilibrium*

(a) Show that a completely mixed Nash equilibrium in a finite game G is strictly perfect.

(b) Show that a strict Nash equilibrium in a game G is strictly perfect. (A Nash equilibrium is *strict* if any unilateral deviation of a player leads to a strictly lower payoff for that player.)

(c) Compute all Nash equilibria, perfect equilibria, proper equilibria, and strictly perfect equilibria in the following game, where $\alpha, \beta > 0$. (Conclude that strictly perfect equilibria may fail to exist.)

$$(A,B) = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} 0,\beta & \alpha,0 & 0,0 \\ 0,\beta & 0,0 & \alpha,0 \end{pmatrix} \end{array}.$$

13.17. *Correlated Equilibria in the Two-Driver Example (1)*

Compute all correlated equilibria in the game

$$(A,B) = \begin{array}{c} \\ c \\ s \end{array} \begin{array}{c} \begin{array}{cc} c & s \end{array} \\ \begin{pmatrix} -10,-10 & 5,0 \\ 0,5 & -1,-1 \end{pmatrix} \end{array},$$

by using the definition of correlated equilibrium.

13.18. *Nash Equilibria are Correlated*

Let $(\mathbf{p},\mathbf{q})$ be a Nash equilibrium in the $m \times n$ bimatrix game (A,B). Let $P = (p_{ij})$ be the $m \times n$ matrix defined by $p_{ij} = p_i q_j$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Show that P is a correlated equilibrium.

13.19. *The Set of Correlated Equilibria is Convex*

Show that the set of correlated equilibria in a bimatrix game (A,B) is convex.

13.20. *Correlated vs. Nash Equilibrium*

Consider the bimatrix game (cf. [5])

$$(A,B) = \begin{pmatrix} 6,6 & 2,7 \\ 7,2 & 0,0 \end{pmatrix}$$

and the correlated strategy

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

(a) Compute all Nash equilibria of (A,B).

(b) Show that P is a correlated equilibrium and that the associated payoffs fall outside the convex hull of the payoff pairs associated with the Nash equilibria of (A,B).

13.21. *Correlated Equilibria in the Two-Driver Example (2)*

Consider again the game of Problem 13.17 and set up the associated matrix C as in Sect. 13.7. Show that the value of the matrix game C is equal to 3, and that player 1 in C has a unique optimal strategy. (Hence, this method gives one particular correlated equilibrium.)

13.22. *Finding Correlated Equilibria*

Compute (the) correlated equilibria in the following game directly, and by using the associated matrix game.

$$(A,B) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 5,2 & 1,3 \\ 2,3 & 4,1 \end{pmatrix} \end{array}.$$

13.23. *Independence of the Axioms in Corollary 13.35*

Show that the three conditions in Corollary 13.35 are independent: for each pair of conditions, exhibit a solution that satisfies these two conditions but not the third one.

13.24. *Inconsistency of Perfect Equilibria*

Consider the three-person game G_0

$$D: \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1,1,1 & 1,0,1 \\ 1,1,1 & 0,0,1 \end{pmatrix} \end{array} \quad U: \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0,1,0 & 0,0,0 \\ 1,1,0 & 0,0,0 \end{pmatrix} \end{array}$$

where player 1 chooses rows, player 2 columns, and player 3 matrices. Let Γ consist of this game and all its reduced games. Use this collection to show that the perfect Nash equilibrium correspondence is not consistent.

Chapter 14
Extensive Form Games

A game in extensive form specifies when each player in the game is to move, what his information is about the sequence of previous moves, which chance moves occur, and what the final payoffs are. Such games are discussed in Chaps. 4 and 5, and also occur in Chaps. 6 and 7. The present chapter extends the material introduced in the first two mentioned chapters, and the reader is advised to (re)read these chapters before continuing.

Section 14.1 formally introduces extensive form structures and games, and Sect. 14.2 introduces behavioral strategies and studies the relation between behavioral and mixed strategies. Section 14.3 is on Nash equilibrium and its main refinements, namely subgame perfect equilibrium and sequential equilibrium. For more about refinements and some relations with refinements of Nash equilibrium in strategic form games see [138] and [102].

14.1 Extensive Form Structures and Games

An extensive form game[1] is based on a directed rooted tree. A *directed rooted tree* is a pair $T = (X, E)$, where:

- X is a finite set with $|X| \geq 2$. The elements of X are called *nodes*.
- E is a subset of $X \times X$. The elements of E are called *edges*. An edge $e = (x, y) \in E$ is called an *outgoing* edge of x and an *ingoing* edge of y.
- There is an $x_0 \in X$, called the *root*, such that for each $x \in X \setminus \{x_0\}$ there is a unique path from x_0 to x. Here, a *path* from x_0 to x is a series of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, x)$ for some $k \geq 0$.
- x_0 has no ingoing edges.

These conditions imply that each node which is not the root, has exactly one ingoing edge. Moreover, there are nodes which have no outgoing edges. These nodes are called *end nodes*. The set of end nodes is generally denoted by $Z (\subseteq X)$.

[1] The presentation in this section is based on Perea [102].

An *extensive form structure* is a tuple $\mathscr{S} = (T, N, P, \mathscr{H}, \mathscr{A}, \tau)$, where:

- $T = (X, E)$ is a directed rooted tree with root x_0 and set of end nodes Z.
- $N = \{1, \ldots, n\}$ with $n \geq 1$ is the set of *players*.
- $P : X \setminus Z \to N \cup \{C\}$ is a function assigning to each non-end node either a player or Chance C. If $P(x)$ is a player, then node x is a *decision node* of player $P(x)$, otherwise x is a *chance node*.
- $\mathscr{H} = (H_i)_{i \in N}$ where for each $i \in N$, H_i is a partition of the set $P^{-1}(i)$ of decision nodes of player i. The sets $h \in H_i$ are called *information sets* of player i. Each $h \in H_i$ is assumed to satisfy (1) every path in T intersects H_i at most once and (2) every node in h has the same number of outgoing edges.
- $\mathscr{A} = (A(h))_{h \in H}$, where $H = \cup_{i \in N} H_i$, and for each $h \in H$, $A(h)$ is a partition of the set of edges outgoing from nodes $x \in h$. The partition $A(h)$ is such that for each $x \in h$ and each $a \in A(h)$, a contains exactly one edge outgoing from x. Every set $a \in A(h)$ is called an *action* at h. It is assumed that $|A(h)| \geq 2$ for each $h \in H$.
- τ assigns to each chance node a probability distribution over set of the outgoing edges, where it is assumed that all these probabilities are positive.

In Fig. 14.1 – which is a partial reproduction of Fig. 4.1 – these concepts are illustrated.

In this extensive form structure, the directed rooted tree has 14 nodes $x_0, x_1, \ldots, x_{13}$ and, consequently, 13 edges. The set of end nodes is $Z = \{x_6, \ldots, x_{13}\}$ and the player set is $N = \{1, 2\}$. The function $P : \{x_0, \ldots, x_5\} \to \{1, 2\} \cup \{C\}$ is defined by $P(x) = 1$ for $x \in \{x_0, x_5\}$, $P(x) = 2$ for $x \in \{x_2, x_3, x_4\}$, and $P(x_1) = C$. The information sets are $\{x_0\}, \{x_5\} \in H_1$ and $\{x_2, x_3\}, \{x_4\} \in H_2$. The actions for player 1 are: $\{(x_0, x_2)\}$, $\{(x_0, x_3)\}$, $\{(x_0, x_1)\} \in A(\{x_0\})$ and $\{(x_5, x_{12})\}$, $\{(x_5, x_{13})\} \in A(\{x_5\})$. The actions for player 2 are: $\{(x_2, x_6), (x_3, x_8)\}$, $\{(x_2, x_7), (x_3, x_9)\} \in A(\{x_2, x_3\})$ and $\{(x_4, x_{10})\}$, $\{(x_4, x_{11})\} \in A(\{x_4\})$. Finally, $\tau(x_1) = (1/4, 3/4)$, where $1/4$ is the probability of (x_1, x_4) and $3/4$ is the probability of (x_1, x_5).

Clearly, this formal notation is quite cumbersome and we try to avoid its use as much as possible. It is only needed to give precise definitions and proofs.

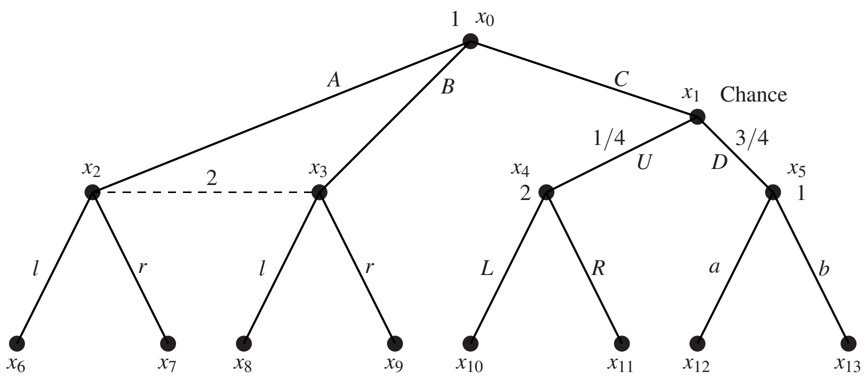

Fig. 14.1 An extensive form structure

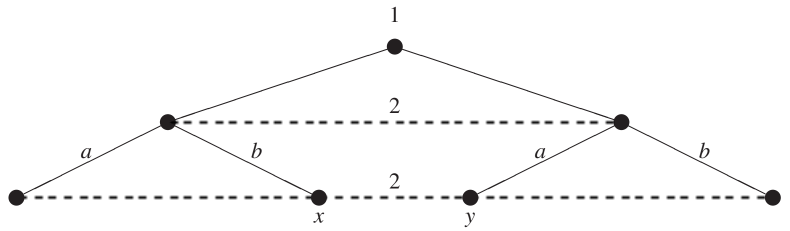

Fig. 14.2 Part of an extensive form structure without perfect recall

It is usually assumed that an extensive form structure $\mathscr{S}$ satisfies *perfect recall*: this means that each player always remembers what he did in the past. The formal definition is as follows.

Definition 14.1. An extensive form structure $\mathscr{S}$ satisfies *perfect recall for player* $i \in N$ if for every information set $h \in H_i$ and each pair of nodes $x, y \in h$, player i's outgoing edges on the path from the root to x belong to the same player i actions as player i's outgoing edges on the path from the root to y. We say that $\mathscr{S}$ satisfies *perfect recall* if it satisfies perfect recall for every player.

Figure 14.2 shows part of an extensive form structure *without* perfect recall.

The condition of perfect recall, introduced by Kuhn [67], plays an important role for the relation between mixed and behavioral strategies (see Sect. 14.2).

We also repeat the definitions of perfect and imperfect information (cf. Chap. 4).

Definition 14.2. An extensive form structure $\mathscr{S}$ has *perfect information* if for every $i \in N$ and $h \in H_i$, $|h| = 1$. Otherwise, $\mathscr{S}$ has *imperfect information*.

We conclude this section with the formal definition of an extensive form game. An *extensive form game* Γ is an $n+1$ tuple $\Gamma = (\mathscr{S}, u_1, \ldots, u_n)$, where $\mathscr{S}$ is an extensive form structure and for each player $i \in N$, $u_i : Z \to \mathbb{R}$. The function u_i is player i's *payoff function*.

A game $\Gamma = (\mathscr{S}, u_1, \ldots, u_n)$ has *(im)perfect information* if $\mathscr{S}$ has (im)perfect information.

14.2 Pure, Mixed and Behavioral Strategies

Let $\mathscr{S} = (T, N, P, \mathscr{H}, \mathscr{A}, \tau)$ be an extensive form structure. A *pure strategy* s_i of player $i \in N$ is a map assigning an action $a \in A(h)$ to every information set $h \in H_i$. By S_i we denote the (finite) set of pure strategies of player i.

Any strategy combination $(s_1, \ldots, s_n)$, when played, results in an end node of the game tree. Therefore, with each extensive form game $\Gamma = (\mathscr{S}, u_1, \ldots, u_n)$ we can associate a strategic form game $G(\Gamma) = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ in the obvious way. A (mixed) *strategy* of player $i \in N$ is an element of $\Delta(S_i)$, i.e., a probability distribution over the elements of S_i.

When considering an extensive form game (structure) it seems more natural to consider, instead of mixed strategies, so-called *behavioral strategies*. A behavioral strategy of a player assigns to each information set of that player a probability distribution over the actions at that information set. Formally, we have the following definition.

Definition 14.3. Let $\mathscr{S} = (T, N, P, \mathscr{H}, \mathscr{A}, \tau)$ be an extensive form structure. A *behavioral strategy* of player $i \in N$ is a map b_i assigning to each information set $h \in H_i$ a probability distribution over the set of actions $A(h)$.

Given a behavioral strategy there is an obvious way to define an associated mixed strategy: for each pure strategy, simply multiply all probabilities assigned by the behavioral strategy to the actions occurring in the pure strategy. Consider for instance the extensive form structure in Fig. 14.3. In this diagram a behavioral strategy b_1 of player 1 is indicated. The associated mixed strategy σ_1 assigns the probabilities $\sigma_1(A,l) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, $\sigma_1(A,r) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, $\sigma_1(B,l) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, and $\sigma_1(B,r) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. Strategy σ_1 is the 'right' mixed strategy associated with b_1 in the following sense. Suppose player 2 plays the mixed or behavioral strategy – there is no difference in this case – which puts probability α on L and $1-\alpha$ on R. If player 1 plays the behavioral strategy b_1 then the probability distribution generated over the end nodes of the game is $x_5 \mapsto \frac{1}{6} \cdot \alpha$, $x_6 \mapsto \frac{1}{3} \cdot \alpha$, $x_7 \mapsto \frac{1}{6} \cdot (1-\alpha)$, $x_8 \mapsto \frac{1}{3} \cdot (1-\alpha)$. The same distribution is generated by the mixed strategy σ_1. E.g., the probability that x_5 is reached equals $\sigma_1(A,l) \cdot \alpha = \frac{1}{6} \cdot \alpha$, etc. We call b_1 and σ_1 *outcome equivalent*. Obviously, it would have been sufficient to check this for the two pure strategies L and R for player 2.

We summarize these considerations in a definition and a proposition, which is presented without a formal proof.

Definition 14.4. Two (behavioral or mixed) strategies of player i in $\mathscr{S}$ are *outcome equivalent* if for each pure strategy combination s_{-i} of the other players the probability distributions generated by the two strategies over the end nodes are equal.

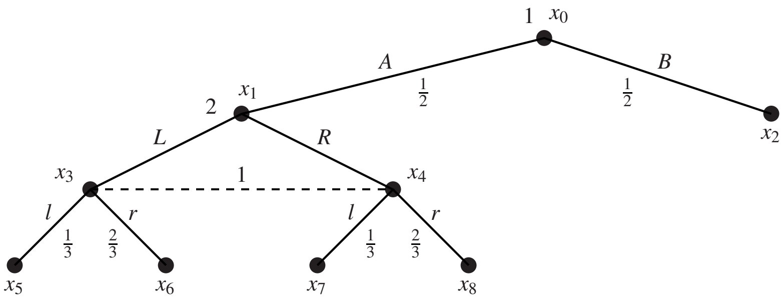

Fig. 14.3 From behavioral to mixed strategies

Proposition 14.5. *Let b_i be a behavioral strategy of player i in $\mathscr{S}$. Then there is a mixed strategy σ_i of player i that is outcome equivalent to b_i. Such a strategy σ_i is obtained by assigning to each pure strategy s_i of player i the product of the probabilities assigned by b_i to the actions chosen by s_i.*

It should be noted that there is not necessarily a unique mixed strategy that is outcome equivalent to a given behavioral strategy. For instance, in the example above, if we change the behavioral strategy of player 1 such that it assigns probability 0 to action A and probability 1 to action B at information set $\{x_0\}$, then any mixed strategy which puts zero probability on (A,l) and (A,r) is outcome equivalent, resulting in each end node other than x_2 with zero probability.

Also for the converse question it is not hard to figure out a procedure. Suppose that σ_1 is a mixed strategy of player i, h is an information set of player i, and a is an action in h. First, let $S_i(h)$ denote the set of pure strategies of player i such that the play of the game possibly reaches h, in other words, such that there exists a path through h containing the actions prescribed by the pure strategy under consideration. Then $\sigma_i(S_i(h))$ is the total probability assigned by σ_i to this set of pure strategies. Within this set, consider those pure strategies that assign a to h and divide their total probability by $\sigma_i(S_i(h))$ if $\sigma_i(S_i(h)) > 0$: the result is defined to be $b_i(h)(a)$. If $\sigma_i(S_i(h)) = 0$ then we can choose $b_i(h)$ arbitrary. This way, we construct a behavioral strategy b_i that is outcome equivalent to the mixed strategy σ_i.

As an illustration consider the extensive form structure in Fig. 14.4, which is the same as the one in Fig. 14.3. Consider the mixed strategy σ_1 of player 1 defined by: $(A,l) \mapsto \frac{1}{5}$, $(A,r) \mapsto \frac{1}{10}$, $(B,l) \mapsto \frac{2}{5}$, $(B,r) \mapsto \frac{3}{10}$. Following the above procedure we obtain

$$\begin{aligned}
b_1(A) &= \tfrac{\sigma_1(A,l)+\sigma_1(A,r)}{1} = \tfrac{1}{5} + \tfrac{1}{10} = \tfrac{3}{10} \\
b_1(B) &= \tfrac{\sigma_1(B,l)+\sigma_1(B,r)}{1} = \tfrac{2}{5} + \tfrac{3}{10} = \tfrac{7}{10} \\
b_1(l) &= \tfrac{\sigma_1(A,l)}{\sigma_1(A,l)+\sigma_1(A,r)} = \tfrac{1/5}{1/5+1/10} = \tfrac{2}{3} \\
b_1(r) &= \tfrac{\sigma_1(A,r)}{\sigma_1(A,l)+\sigma_1(A,r)} = \tfrac{1/10}{1/5+1/10} = \tfrac{1}{3} .
\end{aligned}$$

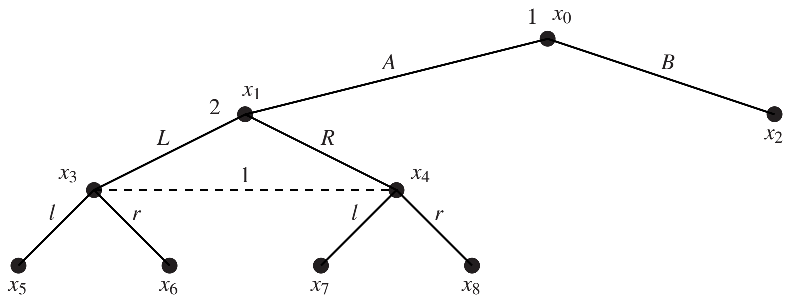

Fig. 14.4 From mixed to behavioral strategies

It is straightforward to verify that b_1 and σ_1 are outcome equivalent.

Outcome equivalence is not guaranteed without perfect recall: see Problem 14.2 for an example. With perfect recall, we have the following theorem.[2]

Theorem 14.6 (Kuhn). *Let the extensive form structure $\mathscr{S}$ satisfy perfect recall. Then, for every player i and every mixed strategy σ_i there is a behavioral strategy b_i that is outcome equivalent to σ_i.*[3]

14.3 Nash Equilibrium and Refinements

Let $\Gamma = (\mathscr{S}, (u_i)_{i \in N})$ be an extensive form game with associated strategic form game $G(\Gamma) = ((S_i)_{i \in N}, (u_i)_{i \in N})$. We assume that $\mathscr{S}$ satisfies perfect recall.

A *pure strategy Nash equilibrium* of Γ is defined to be a pure strategy Nash equilibrium of $G(\Gamma)$. Note that, if Γ has perfect information, then a pure strategy Nash equilibrium exists (cf. Chap. 4).

A *mixed strategy Nash equilibrium* of Γ is defined to be a (mixed strategy) Nash equilibrium of $G(\Gamma)$. By Theorem 13.1 such an equilibrium always exists.

Consider, now, a behavioral strategy combination $b = (b_i)_{i \in N}$ in Γ. Such a strategy combination generates a probability distribution over the end nodes and, thus, an expected payoff for each player. We call b_i a *best reply* of player $i \in N$ to the strategy combination b_{-i} if there is no other behavioral strategy b'_i of player i such that (b'_i, b_{-i}) generates a higher expected payoff for player i. We call b a *Nash equilibrium* (in behavioral strategies) of Γ if b_i is a best reply to b_{-i} for every player $i \in N$.

Let σ be a mixed strategy Nash equilibrium of Γ. By Theorem 14.6 there is a behavioral strategy combination b that is outcome equivalent to σ. We claim that b is a Nash equilibrium of Γ. Suppose not, then there is a player $i \in N$ and a behavioral strategy b'_i that gives player i a higher expected payoff against b_{-i}. By Proposition 14.5 there is a mixed strategy σ'_i that is outcome equivalent to b'_i. Consequently, σ'_i gives player i a higher expected payoff against σ_{-i}, a contradiction. We have thus proved:

Theorem 14.7. *Every extensive form game has a Nash equilibrium.*

In fact, a similar argument as the one leading to this theorem can be applied to show that every Nash equilibrium (in behavioral strategies) results in a Nash equilibrium in mixed strategies. Hence, one way to find the (behavioral strategy) Nash equilibria of an extensive form game is to determine all (mixed strategy) Nash equilibria of the associated strategic form game. Which way is most convenient depends on the game at hand. In particular for refinements it is often easier to compute behavioral equilibrium strategies directly, without first computing the mixed strategy Nash equilibria. Before discussing these refinements we first consider an example.

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[2] For a proof see [67] or [102], Theorem 2.4.4.

[3] The behavioral strategy b_i can be constructed as described in the text.

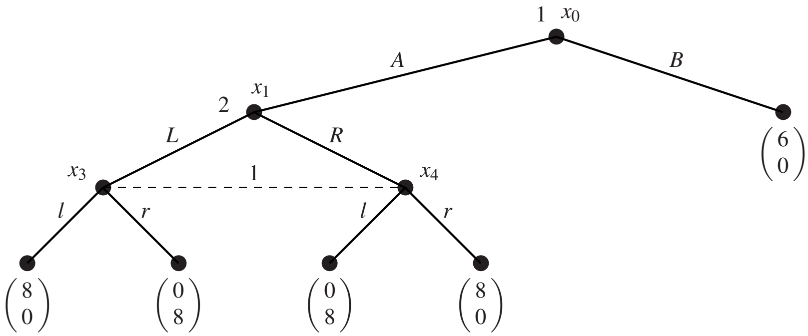

Fig. 14.5 The game Γ_1

Table 14.1 The strategic form $G(\Gamma_1)$ of Γ_1

$$\begin{array}{c c}
 & \begin{array}{cc} L & R \end{array} \\
\begin{array}{c} Al \\ Ar \\ Bl \\ Br \end{array} & \begin{pmatrix} 8,0 & 0,8 \\ 0,8 & 8,0 \\ 6,0 & 6,0 \\ 6,0 & 6,0 \end{pmatrix}
\end{array}$$

Example 14.8. Consider the extensive form game Γ_1 in Fig. 14.5. This game is based on the extensive form structure of Fig. 14.4, in which the symbols for the end nodes are replaced by payoffs for player 1 (upper number) and player 2 (lower number). The associated strategic form of this game is given in Table 14.1.

(Note that in $G(\Gamma)$ there is no essential difference between Bl and Br.[4]) To find the Nash equilibria in $G(\Gamma_1)$, first note that player 2 will never play pure in a Nash equilibrium. Suppose, in equilibrium, that player 2 plays $(\alpha, 1-\alpha)$ with $0 < \alpha < 1$, and player 1 plays $\mathbf{p} = (p_1, p_2, p_3)$, where p_1 is the probability on Al, p_2 the probability on Ar, and p_3 the joint probability on Bl and Br. Since $0 < \alpha < 1$, player 2 is indifferent between L and R, which implies that $p_1 = p_2$. Suppose $p_1 = p_2 > 0$. Then we must have $8\alpha \geq 6$ and $8(1-\alpha) \geq 6$, which is impossible. Hence $p_3 = 1$ and both $6 \geq 8\alpha$ and $6 \geq 8(1-\alpha)$, so $1/4 \leq \alpha \leq 3/4$. This implies that $b = (b_1, b_2)$ is a (behavioral strategy) Nash equilibrium of Γ if and only if

$$b_1(A) = 0, \quad b_1(B) = 1, \quad 1/4 \leq b_2(L) \leq 3/4.$$

So $b_1(l)$ may take any arbitrary value.

In the remainder of this section we consider refinements of Nash equilibrium.

[4] Indeed, some authors do not distinguish between these strategies, e.g., [102].

14.3.1 Subgame Perfect Equilibrium

Let x be a non-end node in an extensive form structure $\mathscr{S}$ and let $T^x = (V^x, E^x)$ be the subtree starting from x – i.e., V^x is the subset of V consisting of all nodes of V that can be reached by a path starting from x, and E^x is the subset of E of all edges with endpoints in V^x. Assume that every information set of $\mathscr{S}$ is contained either in V^x or in $V \setminus V^x$. (This implies, in particular, that $\{x\}$ is a singleton information set.) Let $\mathscr{S}^x$ denote the substructure obtained by restricting $\mathscr{S}$ to T^x. Then, for the extensive form game $\Gamma = (\mathscr{S}, (u_i)_{i\in N})$, the game $\Gamma^x = (\mathscr{S}^x, (u_i^x)_{i\in N})$ is defined by restricting the payoff functions to the end nodes still available in V^x. We call Γ^x a *subgame* of Γ. For a behavioral strategy combination $b = (b_i)_{i\in N}$ we denote by $b^x = (b_i^x)_{i\in N}$ the restriction to Γ^x.

Definition 14.9. A behavioral strategy combination b in Γ is a *subgame perfect equilibrium* if b^x is a Nash equilibrium for every subgame Γ^x.

Clearly, this definition extends the definition of subgame perfection for pure strategy combinations given in Chap. 4.

Since the whole game Γ is a subgame ($\Gamma = \Gamma^{x_0}$, where x_0 is the root of the game tree), every subgame perfect equilibrium is a Nash equilibrium. By carrying out a backward induction procedure as in Sect. 4.3, it can be seen that a subgame perfect equilibrium exists in any extensive form game.[5]

Subgame perfection often implies a considerable reduction of the set of Nash equilibria. The following example is the continuation of Example 14.8.

Example 14.10. To find the subgame perfect equilibria in Γ_1, we only have to analyze the subgame $\Gamma_1^{x_1}$. It is easy to see that this subgame has a unique Nash equilibrium, namely player 2 playing L and R each with probability $1/2$, and player 1 playing l and r each with probability $1/2$. This results in a unique subgame perfect equilibrium $b = (b_1, b_2)$ given by

$$b_1(B) = 1, \quad b_1(l) = 1/2, \quad b_2(L) = 1/2\,.$$

14.3.2 Perfect Bayesian and Sequential Equilibrium

In games without proper subgames and in games of imperfect information the subgame perfection requirement may not have much bite (see the examples in Chaps. 4 and 5). The concept of sequential equilibrium allows to distinguish between Nash equilibria by considering beliefs of players on information sets.

[5] This is intuitive but not trivial. See, e.g., [102].

Consider an extensive form structure $\mathscr{S} = (T, N, P, \mathscr{H}, \mathscr{A}, \tau)$. A *belief system* β assigns to every information set $h \in H = \cup_{i \in N} H_i$ a probability distribution β_h over the nodes in h. An *assessment* is a pair (b, β) of a behavioral strategy combination $b = (b_i)_{i \in N}$ and a belief system β.

For any node x in the game tree T, let $\mathbb{P}_b(x)$ denote the probability that x is reached given b, that is, $\mathbb{P}_b(x)$ is the product of the probabilities of all edges on the unique path form the root x_0 to x: these probabilities are given by b, or by τ in case of a chance node on the path. For $h \in H$, $\mathbb{P}_b(h) = \sum_{x \in h} \mathbb{P}_b(x)$ is the probability that the information set h is reached, given b.

The next definition is the formal version of the consistency requirement introduced already in Chap. 4.

Definition 14.11. An assessment (b, β) in $\mathscr{S}$ is *Bayesian consistent* if $\beta_h(x) = \mathbb{P}_b(x)/\mathbb{P}_b(h)$ for all $h \in H$ for which $\mathbb{P}_b(h) > 0$ and all $x \in h$.

By Bayesian consistency the players' beliefs are determined by the behavioral strategies on all information sets that are reached with positive probability. In general, this requirement is quite weak and does not imply, for instance, that the beliefs of one and the same player are internally consistent. Consider the extensive form structure in Fig. 14.6. Suppose that player 1 plays a and player 2 plays e. Then the beliefs of player 2 at the information sets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are not restricted by Bayesian consistency. Moreover, a belief $\beta(x_1) = \beta(x_4) = 1$ is allowed, which means that player 2's beliefs are not internally consistent.

In many applications this drawback does not occur, and Bayesian consistency is strong enough. For instance, in signaling games (see Chap. 5) Bayesian consistency implies the usual stronger versions like consistency in the sense of Kreps and Wilson [66] below. See Problem 14.4.

In the literature[6] the condition of *updating consistency* has been proposed to remedy the indicated defect. In the example above, with player 1 playing a and

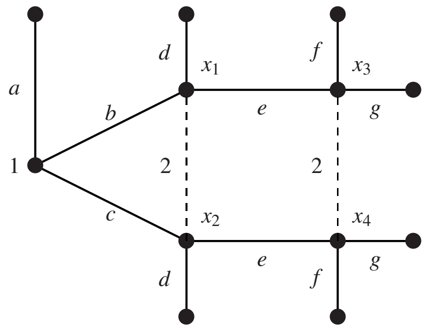

Fig. 14.6 If player 1 plays a, then player 2's beliefs at $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are independent under Bayesian consistency

[6] See [102] and the references therein.

player 2 playing e, updating consistency would imply that $\beta(x_3) = \beta(x_1)$, as seems natural.

A much stronger requirement is the original consistency condition in [66]. Call a behavioral strategy b_i of player i *completely mixed* if $b_i(h)(x) > 0$ for each $h \in H_i$ and $x \in h$. A behavioral strategy combination $b = (b_i)_{i \in N}$ is *completely mixed* if b_i is completely mixed for every $i \in N$. Observe that, if b is completely mixed and the assessment (b, β) is Bayesian consistent, then β is uniquely determined by b.

Definition 14.12. An assessment (b, β) in $\mathscr{S}$ is *consistent* if there exists a sequence $(b^m, \beta^m)_{m \in \mathbb{N}}$ of Bayesian consistent assessments with each b^m completely mixed and $\lim_{m \to \infty}(b^m, \beta^m) = (b, \beta)$.

Consistency implies Bayesian consistency (Problem 14.3). Consistency is clearly stronger than Bayesian consistency. For instance, in the extensive form structure of Fig. 14.6, it is easily seen that consistency requires player 2 to have identical beliefs on his two information sets, i.e., $\beta(x_1) = \beta(x_3)$. This is true even if on his right information set player 2 is replaced by some other player 3, as in Fig. 14.7. It shows that (this strong form of) consistency is stronger than Bayesian consistency combined with 'updating consistency', since the latter condition would only require a player to 'update' his own earlier beliefs.

Consider now an extensive form game $\Gamma = (\mathscr{S}, (u_i)_{i \in N})$. Let (b, β) be an assessment. Let $i \in N$, $h \in H_i$, $a \in A(h)$, and $x \in h$. Suppose player i is at node x and takes action a. This corresponds to an edge (x, y) in the game tree. Then each end node on a path starting from x and passing though y is reached with a probability that is equal to the product of the probabilities of all edges on this path following y, given by b, and the probabilities of eventual chance nodes on this path. This way, we can compute the expected payoff to player i from playing a, conditional on being at node x: denote this payoff by $u_i(a|b, x)$. Player i's expected payoff from action a, given information set h, is then equal to $\sum_{x \in h} \beta(x) u_i(a|b, x)$.

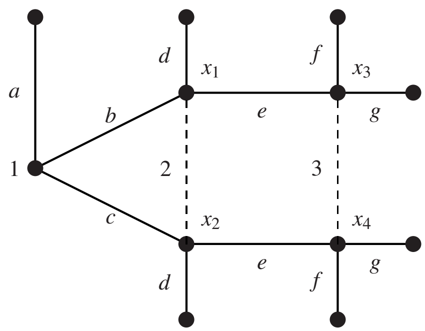

Fig. 14.7 Under consistency, the beliefs of players 2 and 3 are identical: $\beta(x_1) = \beta(x_3)$

Definition 14.13. An assessment (b,β) in $\Gamma = (\mathscr{S}, (u_i)_{i\in N})$ is *sequentially rational* if for every $i \in N$, $h \in H_i$, and $a \in A(h)$ we have

$$b_i(h)(a) > 0 \Rightarrow \sum_{x\in h} \beta(x)\, u_i(a|b,x) = \max_{a'\in A(h)} \sum_{x\in h} \beta(x)\, u_i(a'|b,x).$$

Sequential rationality of an assessment (b,β) means that a player puts only positive probability on those actions at an information set h that maximize his expected payoff, given h and his belief $\{\beta(x)|x \in h\}$.

Definition 14.14. An assessment (b,β) in $\Gamma = (\mathscr{S}, (u_i)_{i\in N})$ is a *sequential equilibrium* if it is sequentially rational and consistent.

Sequential equilibria were introduced in [66].[7]

Theorem 14.15. *Every sequential equilibrium is subgame perfect and, in particular, a Nash equilibrium.*

For a proof of this theorem (which is somewhat cumbersome in terms of notation), see [102], Lemma 4.2.6, or [138], Theorem 6.3.2.

In Chap. 4 we introduced, somewhat informally, the notion of *perfect Bayesian equilibrium*. An assessment (b,β) is a perfect Bayesian equilibrium if the beliefs β satisfy Bayesian updating in the extensive form structure 'as much as possible', given the behavioral strategy combination b, and if it is sequentially rational. There is no unified definition in the literature of the expression 'as much as possible'. It implies at least Bayesian consistency, but it should also imply, for instance, that the beliefs on the information set $\{x_3,x_4\}$ are identical to the beliefs on the information set $\{x_1,x_2\}$ in Figs. 14.6 and 14.7. It should imply Bayesian consistency of beliefs on subgames (cf. the game in Fig. 4.7 in Chap. 4), and it should imply subgame perfection. The main motivation for considering perfect Bayesian equilibrium is to avoid the condition involving limiting assessments in the definition of consistency.[8] In this chapter we stick to sequential equilibrium but keep in mind that, in order to compute sequential equilibria for a given game, the method of Bayesian updating 'as much as possible' is a good heuristic.

There is hardly any general method available to compute sequential equilibria: it depends very much on the game at hand what the best way is. We consider one example and refer to the problem section for other examples.

Example 14.16. Consider the three-player game in Fig. 14.8. To find the sequential equilibria of this game, first observe that consistency requires the beliefs of players 2

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[7] Actually, the definition of sequential rationality in Definition 14.13 is called 'local' sequential rationality. Together with consistency, it implies the sequential rationality condition in [66]. It is, however, easier to apply. See also [102], Chap. 4, or [138], Chap. 6. See the same sources for a proof that sequential equilibria always exist.

[8] For instance, Fudenberg and Tirole [40] provide two definitions of perfect Bayesian equilibrium for different classes of games and show that their concepts coincide with sequential equilibrium.

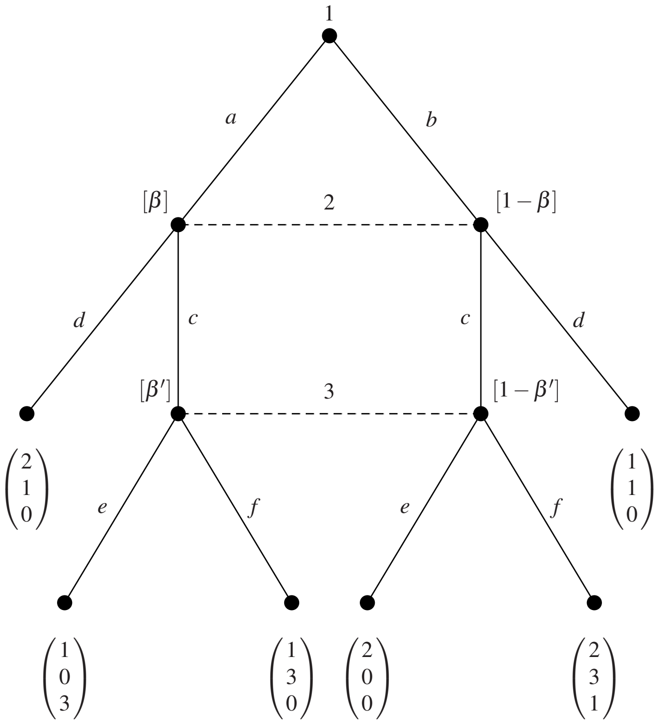


Fig. 14.8 Sequential equilibrium analysis in a three-player game

and 3 to be the same, so $\beta = \beta'$. Denote behavioral strategies of the players by $b_1(a)$ and $b_1(b) = 1 - b_1(a)$, $b_2(c)$ and $b_2(d) = 1 - b_2(c)$, and $b_3(e)$ and $b_3(f) = 1 - b_3(e)$.

Starting with player 3, sequential rationality requires $b_3(e) = 1$ if $\beta > \frac{1}{4}$, $b_3(e) = 0$ if $\beta < \frac{1}{4}$, and $0 \leq b_3(e) \leq 1$ if $\beta = \frac{1}{4}$.

Using this, if $\beta > \frac{1}{4}$ then playing c yields 0 for player 2 and playing d yields 1. Therefore $b_2(c) = 0$. Similarly, $b_2(c) = 1$ if $\beta < \frac{1}{4}$. If $\beta = \frac{1}{4}$ then d yields 1 whereas c yields $3b_3(f)$. Hence, $b_2(c) = 0$ if $\beta = \frac{1}{4}$ and $b_3(f) < \frac{1}{3}$; $b_2(c) = 1$ if $\beta = \frac{1}{4}$ and $b_3(f) > \frac{1}{3}$; and $0 \leq b_2(c) \leq 1$ if $\beta = \frac{1}{4}$ and $b_3(f) = \frac{1}{3}$.

We finally consider player 1. If $b_1(a) > \frac{1}{4}$ then consistency requires $\beta = b_1(a) > \frac{1}{4}$ and therefore $b_2(c) = 0$. So player 1 obtains $2b_1(a) + 1(1 - b_1(a)) = 1 + b_1(a)$, which is maximal for $b_1(a) = 1$. Obviously, player 1 cannot improve on this. So we have the following sequential equilibrium:

$$b_1(a) = 1, \quad b_2(c) = 0, \quad b_3(e) = 1, \quad \beta = \beta' = 1.$$

If $b_1(a) < \frac{1}{4}$ then $\beta = b_1(a) < \frac{1}{4}$ and therefore $b_2(c) = 1$ and $b_3(e) = 0$. So player 1 obtains $1b_1(a) + 2(1 - b_1(a)) = 2 - b_1(a)$, which is maximal for $b_1(a) = 0$. Obviously again, player 1 cannot improve on this. So we have a second sequential equilibrium:

$$b_1(a) = 0, \quad b_2(c) = 1, \quad b_3(e) = 0, \quad \beta = \beta' = 0\,.$$

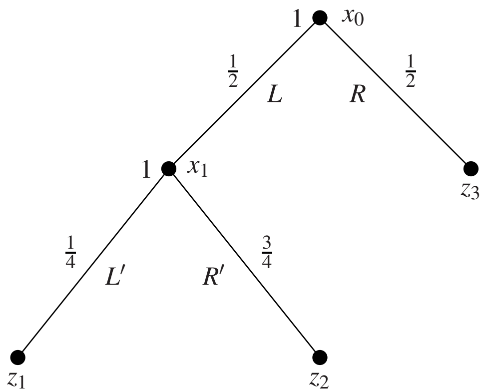
If $b_1(a) = \frac{1}{4}$ then player 1 must be indifferent between a and b. This implies that the expected payoff from a should be equal to the expected payoff from b, hence that $2(1-b_2(c)) + 1b_2(c) = 2b_2(c) + 1(1-b_2(c))$ which is true for $b_2(c) = \frac{1}{2}$. The preceding analysis for player 2 shows that for player 2 to play completely mixed we need $b_3(e) = \frac{2}{3}$. So we have a third sequential equilibrium

$$b_1(a) = \frac{1}{4}, \quad b_2(c) = \frac{1}{2}, \quad b_3(e) = \frac{2}{3}, \quad \beta = \beta' = \frac{1}{4}\,.$$

Problems

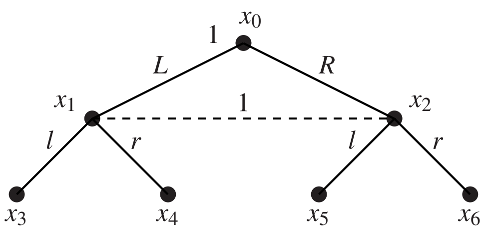
14.1. *Mixed and Behavioral Strategies*

Determine all mixed strategies that are outcome equivalent with the behavioral strategy represented in the following one-player extensive form structure.



14.2. *An Extensive form Structure Without Perfect Recall*

Consider the following extensive form structure:



(a) Show that this one-player extensive form structure has no perfect recall.

(b) Consider the mixed strategy σ_1 that assigns probability $1/2$ to both (L,l) and (R,r). Show that there is no behavioral strategy that generates the same probability distribution over the end nodes as σ_1 does.

14.3. *Consistency Implies Bayesian Consistency*

Let (b,β) be a consistent assessment in an extensive form structure $\mathscr{S}$. Show that (b,β) is Bayesian consistent.

14.4. *(Bayesian) Consistency in Signaling Games*

Prove that Bayesian consistency implies consistency in a signaling game.

[A general definition of a signaling game (cf. also Chap. 5) is as follows. The set of players is $N=\{1,2\}$ and for the extensive form structure $\mathscr{S}$ we have:

(1) The directed rooted tree is $T=(X,E)$ with root x_0,

$$X=\{x_0,x_1,\ldots,x_k,x_{i1},\ldots,x_{il},x_{ij1},\ldots,x_{ijm}\mid i=1,\ldots,k,\ j=1,\ldots,l\},$$

where $k,l,m\geq 2$,

$$E=\{(x_0,x_i),(x_i,x_{ij}),(x_{ij},x_{ijj'})\mid i=1,\ldots,k,\ j=1,\ldots,l,\ j'=1,\ldots,m\}.$$

Hence

$$Z=\{x_{ijj'}\mid i=1,\ldots,k,\ j=1,\ldots,l,\ j'=1,\ldots,m\}.$$

(2) The chance and player assignment P is defined by $P(x_0)=C$, $P(x_i)=1$ for all $i=1,\ldots,k$, $P(x_{ij})=2$ for all $i=1,\ldots,k$, $j=1,\ldots,l$.

(3) The information sets are

$$H_1=\{\{x_1\},\ldots,\{x_k\}\},\ H_2=\{\{x_{1j},\ldots,x_{kj}\}\mid j=1,\ldots,l\}.$$

(4) The action sets are

$$A(\{x_i\})=\{\{(x_i,x_{ij})\}\mid j=1,\ldots,l\}\qquad \text{for every } i=1,\ldots,k$$

for player 1 and

$$A(\{x_{1j},\ldots,x_{kj}\})=\{\{(x_{1j},x_{1jj'}),\ldots,(x_{kj},x_{kjj'})\}\mid j'=1,\ldots,m\}$$

for every $j=1,\ldots,l$, for player 2.

(5) The map τ assigns a positive probability to each edge in the set $\{(x_0,x_1),\ldots,(x_0,x_k)\}$. (Player 1 has k 'types'.)

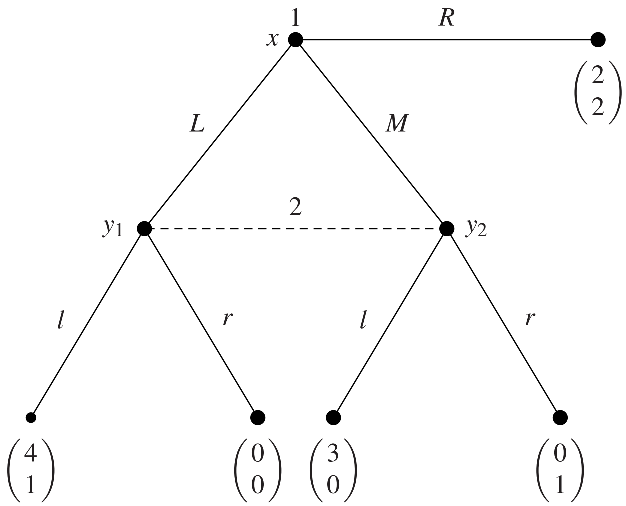
Finally, the players have payoff functions $u_1,u_2:Z\to\mathbb{R}$.]

14.5. *Computation of Sequential Equilibrium (1)*

Compute the sequential equilibrium or equilibria in the game Γ_1 in Fig. 14.5.

14.6. *Computation of Sequential Equilibrium (2)*

Consider the following extensive form game.



(a) Determine the strategic form of this game.

(b) Determine all Nash and subgame perfect Nash equilibria of this game.

(c) Determine all sequential equilibria of this game.

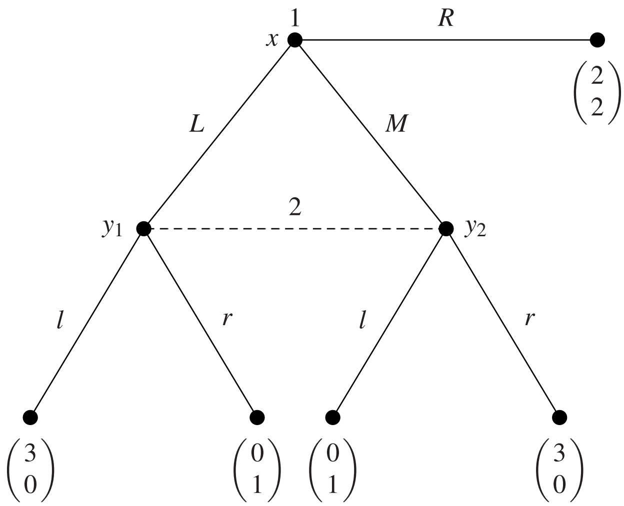
14.7. *Computation of Sequential Equilibrium (3)*

Consider the following extensive form game below.

(a) Determine the strategic form of this game.

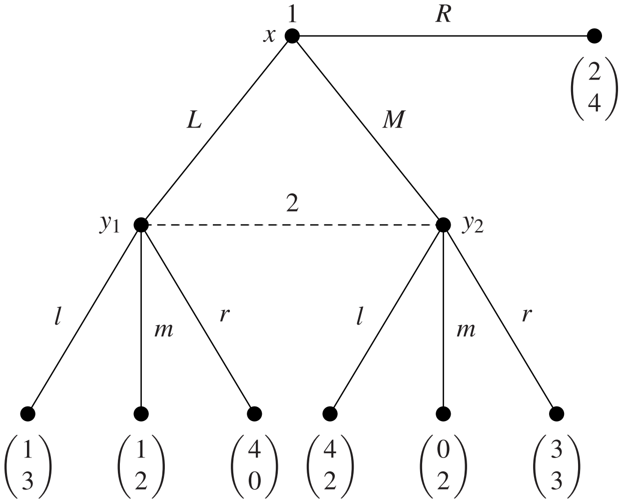
(b) Compute the Nash equilibria and subgame perfect Nash equilibria.

(c) Compute the sequential equilibria.



14.8. *Computation of Sequential Equilibrium (4)*

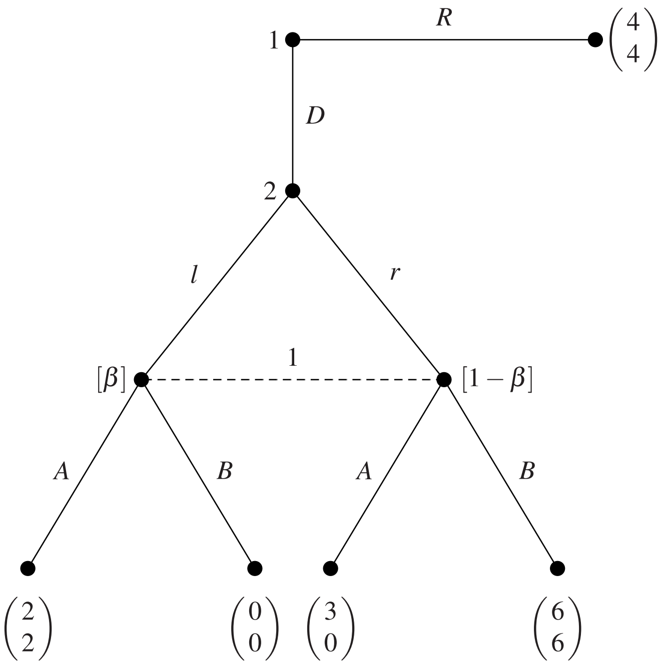
Consider the following extensive form game.



Compute all sequential equilibria of this game.

14.9. *Computation of Sequential Equilibrium (5)*

Compute all Nash, subgame perfect and sequential equilibria in the following game.



Chapter 15
Evolutionary Games

In this chapter we go somewhat deeper into evolutionary game theory. The concepts of evolutionary stable strategy and replicator dynamics, introduced in Chap. 8, are further explored, and proofs of results mentioned in that chapter are provided. We advise the reader to study Chap. 8 first, although the present chapter is largely self-contained.

This chapter is based mainly on [147]. In Sect. 15.1 we briefly review symmetric two-player games. Section 15.2 discusses evolutionary stable strategies and Sect. 15.3 replicator dynamics.

15.1 Symmetric Two-Player Games

Much of evolutionary game theory is concerned with symmetric two-player games. A (finite) symmetric two-player game is completely determined by a pair of $m \times m$ payoff matrices (A,B) such that $B = A^T$, i.e., $B = (b_{ij})_{i,j=1}^m$ is the transpose of $A = (a_{ij})_{i,j=1}^m$. In other words, for all $i, j \in \{1,2,\ldots,m\}$, we have $b_{ij} = a_{ji}$.

In such a game we are particularly interested in symmetric (pure and mixed strategy) Nash equilibria. A Nash equilibrium (σ_1,σ_2) is *symmetric* if $\sigma_1 = \sigma_2$. We denote by $NE(A,A^T)$ the set of all Nash equilibria of (A,A^T) and by

$$NE(A) = \{\mathbf{x} \in \Delta^m \mid (\mathbf{x},\mathbf{x}) \in NE(A,A^T)\}$$

the set of all strategies that occur in a symmetric Nash equilibrium. By a standard application of Kakutani's fixed point theorem we prove that this set is nonempty.

Proposition 15.1. *For any $m \times m$-matrix A, $NE(A) \neq \emptyset$.*

Proof. For each $\mathbf{x} \in \Delta^m$, viewed as a strategy of player 2 in (A,A^T), let $\beta_1(\mathbf{x})$ be the set of best replies of player 1 in (A,A^T). Then the correspondence $\mathbf{x} \mapsto \beta_1(\mathbf{x})$ is upper semi-continuous and convex-valued (check this), so that by the Kakutani Fixed Point Theorem 22.10 there is an $\mathbf{x}^* \in \Delta^m$ with $\mathbf{x}^* \in \beta_1(\mathbf{x}^*)$. Since player 2's payoff matrix is the transpose of A, it follows that also $\mathbf{x}^* \in \beta_2(\mathbf{x}^*)$. Hence, $(\mathbf{x}^*, \mathbf{x}^*) \in NE(A, A^T)$, so $\mathbf{x}^* \in NE(A)$. □

15.1.1 Symmetric 2 × 2 Games

For later reference it is convenient to have a classification of symmetric 2×2 games with respect to their symmetric Nash equilibria. Such a game is described by the payoff matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

For the purpose of Nash equilibrium analysis, we may consider without loss of generality (check!) the matrix

$$A' = \begin{pmatrix} a_{11} - a_{21} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

where $a_1 := a_{11} - a_{21}$ and $a_2 := a_{22} - a_{12}$. For a generic matrix A, implying $a_1, a_2 \neq 0$, there are essentially three different cases:

(1) $a_1 < 0, a_2 > 0$. In this case, $NE(A') = \{\mathbf{e}^2\}$, i.e., each player playing the second strategy is the unique symmetric Nash equilibrium.
(2) $a_1, a_2 > 0$. In this case, $NE(A') = \{\mathbf{e}^1, \mathbf{e}^2, \hat{\mathbf{x}}\}$, where $\hat{\mathbf{x}} = (a_2/(a_1 + a_2), a_1/(a_1 + a_2))$.
(3) $a_1, a_2 < 0$. In this case, $NE(A') = \{\hat{\mathbf{x}}\}$ with $\hat{\mathbf{x}}$ as in (2).

15.2 Evolutionary Stability

15.2.1 Evolutionary Stable Strategies

In evolutionary game theory the interpretation of a symmetric two-person game is that players in a possibly large population randomly meet in pairs. Let such a game be described by A, then a mixed strategy $\mathbf{x} \in \Delta^m$ is interpreted as a vector of population shares: for each k, x_k is the share of the population that 'plays' pure strategy k. Such a strategy is called *evolutionary stable* if it performs better against a small 'mutation' than that mutation itself. For convenience the formal definition is reproduced here (cf. [77] and Definition 8.3).

Definition 15.2. A strategy $\mathbf{x} \in \Delta^m$ is an *evolutionary stable strategy* (ESS) in A if for every strategy $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$, there exists some $\varepsilon_{\mathbf{y}} \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_{\mathbf{y}})$ we have

$$\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}). \tag{15.1}$$

The set of all ESS is denoted by $ESS(A)$.

Evolutionary stable strategies can be characterized as follows.

Proposition 15.3. *Let A be a symmetric $m \times m$ game. Then*

$$ESS(A) = \{\mathbf{x} \in NE(A) \mid \forall \mathbf{y} \in \Delta^m, \mathbf{y} \neq \mathbf{x}\ [\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \Rightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}]\} \,.$$

This proposition follows from Propositions 8.4 and 8.5.

From Problems 15.1–15.3 it follows that the concept of ESS does not 'solve' the prisoners' dilemma nor the 'coordination problem'. Also, an ESS may be completely mixed (Hawk–Dove), or fail to exist (Rock–Paper–Scissors).

15.2.2 The Structure of the Set ESS(A)

Let $\mathbf{x}$ be an ESS and let $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$ such that the carrier[1] of $\mathbf{y}$ is contained in the carrier of $\mathbf{x}$, i.e., $C(\mathbf{y}) \subseteq C(\mathbf{x})$. Since $\mathbf{x} \in NE(A)$ by Proposition 15.3, this implies $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ and hence, again by Proposition 15.3, $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$. We have established:

Proposition 15.4. *If $\mathbf{x} \in ESS(A)$ and $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ and $C(\mathbf{y}) \subseteq C(\mathbf{x})$, then $\mathbf{y} \notin NE(A)$.*

This implies the following corollary (check!):

Corollary 15.5. *The set $ESS(A)$ is finite. If $\mathbf{x} \in ESS(A)$ is completely mixed, then $ESS(A) = \{\mathbf{x}\}$.*

15.2.3 Relations with Other Refinements

If $\mathbf{x} \in NE(A)$ is weakly dominated by $\mathbf{y} \in \Delta^m$, then $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ and $\mathbf{y}A\mathbf{y} \geq \mathbf{x}A\mathbf{y}$; so by Proposition 15.3, $\mathbf{x} \notin ESS(A)$. Therefore, if $\mathbf{x} \in ESS(A)$, then $(\mathbf{x},\mathbf{x})$ is an undominated equilibrium and hence perfect by Theorem 13.22. It can even be shown that $(\mathbf{x},\mathbf{x})$ is proper (see [138]). The next proposition summarizes these facts.

Proposition 15.6. *If $\mathbf{x} \in ESS(A)$, then $(\mathbf{x},\mathbf{x}) \in NE(A,A^T)$ is undominated, perfect and proper.*

The unique (symmetric) equilibrium in the Rock–Paper–Scissors game in Problem 15.3 is proper (why?), but the associated equilibrium strategy is not ESS, so the converse of Proposition 15.6 does not hold.

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[1] Recall – see Chap. 13 – that the carrier of $\mathbf{y}$, $C(\mathbf{y})$, is the set $\{i \in \{1,\ldots,m\} \mid y_i > 0\}$.

15.2.4 Other Characterizations of ESS

Uniform Invasion Barriers

The number $\varepsilon_{\mathbf{y}}$ in the definition of an ESS can be interpreted as an 'invasion barrier': if the share of the mutant strategy $\mathbf{y}$ is smaller than $\varepsilon_{\mathbf{y}}$, then the 'incumbent' strategy $\mathbf{x}$ fares better against the mutated population than the mutant $\mathbf{y}$ itself does, so that the mutant strategy becomes extinct. In a large but finite population, it would not make sense if this invasion barrier could become arbitrarily small since then the 'mutant' population would sometimes have to consist of less than one individual to guarantee survival of the strategy $\mathbf{x}$ under consideration. This gives rise to the following definition (from [139]).

Definition 15.7. A strategy $\mathbf{x} \in \Delta^m$ has a *uniform invasion barrier* if there exists an $\bar{\varepsilon} \in (0,1)$ such that (15.1) holds for all strategies $\mathbf{y} \neq \mathbf{x}$ and every $\varepsilon \in (0, \bar{\varepsilon})$.

It turns out that possessing a uniform invasion barrier characterizes an evolutionary stable strategy.

Proposition 15.8. *For each* $\mathbf{x} \in \Delta^m$*,* $\mathbf{x} \in ESS(A)$ *if and only if* $\mathbf{x}$ *has a uniform invasion barrier.*

Proof. Let $\mathbf{x} \in \Delta^m$. If $\mathbf{x}$ has a uniform invasion barrier $\bar{\varepsilon}$, then clearly $\mathbf{x}$ is an ESS by choosing, in (15.1), $\varepsilon_{\mathbf{y}} = \bar{\varepsilon}$ for each $\mathbf{y} \in \Delta^m$.

Conversely, let $\mathbf{x}$ be an ESS. Define the function $b : \Delta^m \setminus \{\mathbf{x}\} \to [0,1]$ by

$$b(\mathbf{y}) = \sup\{\delta \in [0,1] \mid \forall \varepsilon \in (0,\delta)\ [(\mathbf{x}-\mathbf{y})A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > 0]\}$$

for all $\mathbf{y} \in \Delta^m \setminus \{\mathbf{x}\}$. We first consider the function b on the compact set $Z = \{\mathbf{z} \in \Delta^m \mid z_i = 0 \text{ for some } i \in C(\mathbf{x})\}$. Consider $\mathbf{y} \in Z$. Since $\mathbf{x}$ is an ESS, we have that $(\mathbf{x} - \mathbf{y})A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$ is positive for small positive values of ε. Since this expression depends linearly on ε, this implies that there can be at most one value of ε, which we denote by $\varepsilon_{\mathbf{y}}$, such that $(\mathbf{x}-\mathbf{y})A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) = 0$. If $\varepsilon_{\mathbf{y}} \in (0,1)$, then $(\mathbf{x} - \mathbf{y})A(\varepsilon_{\mathbf{y}}\mathbf{y} + (1-\varepsilon_{\mathbf{y}})\mathbf{x}) = 0$ implies that $(\mathbf{x}-\mathbf{y})A(\mathbf{x}-\mathbf{y}) \neq 0$, since otherwise

$$0 = (\mathbf{x}-\mathbf{y})A(\varepsilon_{\mathbf{y}}\mathbf{y} + (1-\varepsilon_{\mathbf{y}})\mathbf{x}) = (\mathbf{x}-\mathbf{y})A\mathbf{x}$$

and, thus, $(\mathbf{x}-\mathbf{y})A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) = 0$ for all ε, a contradiction. Hence, in that case, $b(\mathbf{y}) = \varepsilon_{\mathbf{y}} = (\mathbf{x}-\mathbf{y})A\mathbf{x}/(\mathbf{x}-\mathbf{y})A(\mathbf{x}-\mathbf{y})$; otherwise, $b(\mathbf{y}) = 1$. It is not hard to see this implies that b is a continuous function. Since b is positive and Z is compact, $\min_{\mathbf{y}\in Z} b(\mathbf{y}) > 0$. Hence, $\mathbf{x}$ has a uniform invasion barrier, namely this minimum value, on the set Z.

Now suppose that $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$. We claim that there is a $\lambda \in (0,1]$ such that $\mathbf{y} = \lambda\mathbf{z} + (1-\lambda)\mathbf{x}$ for some $\mathbf{z} \in Z$. To see this, first note that we can take $\lambda = 1$ if $\mathbf{y} \in Z$. If $\mathbf{y} \notin Z$ then consider, for each $\mu \geq 0$, the point $\mathbf{z}(\mu) = (1-\mu)\mathbf{x} + \mu\mathbf{y}$, and let $\hat{\mu} \geq 1$ be the largest value of μ such that $\mathbf{z}(\mu) \in \Delta^m$. Then there is a coordinate $i \in \{1,\ldots,m\}$ with $z_i(\hat{\mu}) = 0$, $z_i(\mu) > 0$ for all $\mu < \hat{\mu}$, and $z_i(\mu) < 0$ for all $\mu > \hat{\mu}$.

Clearly, this implies $x_i > y_i$, hence $i \in C(\mathbf{x})$, and thus $\mathbf{z}(\hat{\mu}) \in Z$. Then, for $\mathbf{z} = \mathbf{z}(\hat{\mu})$ and $\lambda = 1/\hat{\mu}$, we have $\mathbf{y} = \lambda\mathbf{z} + (1-\lambda)\mathbf{x}$.

By straightforward computation we have

$$(\mathbf{x}-\mathbf{y})A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) = \lambda(\mathbf{x}-\mathbf{z})A(\varepsilon\lambda\mathbf{z} + (1-\varepsilon\lambda)\mathbf{x})$$

for each $\varepsilon > 0$, so that $b(\mathbf{y}) = \min\{b(\mathbf{z})/\lambda, 1\} \geq b(\mathbf{z})$.

We conclude that $\bar{\varepsilon} = \min_{\mathbf{y}\in Z} b(\mathbf{y})$ is a uniform invasion barrier for $\mathbf{x}$. □

Local Superiority

By Proposition 15.3, a completely mixed ESS earns a higher payoff against any mutant than such a mutant earns against itself. This global superiority property can be generalized to the following local version [56].

Definition 15.9. The strategy $\mathbf{x} \in \Delta^m$ is *locally superior* if it has an open neighborhood U such that $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$ for all $\mathbf{y} \in U \setminus \{\mathbf{x}\}$.

The local superiority condition provides another characterization of ESS.

Proposition 15.10. *For each* $\mathbf{x} \in \Delta^m$, $\mathbf{x} \in ESS(A)$ *if and only if* $\mathbf{x}$ *is locally superior.*

Proof. Let $\mathbf{x} \in \Delta^m$.

First suppose that $\mathbf{x}$ is locally superior, and let U be as in Definition 15.9. Let $\mathbf{z} \in \Delta^m \setminus \{\mathbf{x}\}$ and define for each $0 < \varepsilon < 1$ the point $\mathbf{w}(\varepsilon)$ by $\mathbf{w}(\varepsilon) = \varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}$. Then there is $\varepsilon_{\mathbf{z}} > 0$ such that $\mathbf{w}(\varepsilon) \in U$ for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$, hence $\mathbf{x}A\mathbf{w}(\varepsilon) > \mathbf{w}(\varepsilon)A\mathbf{w}(\varepsilon)$. This implies $\mathbf{x}A\mathbf{w}(\varepsilon) > \mathbf{z}A\mathbf{w}(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$. In particular, we have

$$\varepsilon\mathbf{x}A\mathbf{z} + (1-\varepsilon)\mathbf{x}A\mathbf{x} > \varepsilon\mathbf{z}A\mathbf{z} + (1-\varepsilon)\mathbf{z}A\mathbf{x}$$

for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$, hence $\mathbf{x}A\mathbf{x} \geq \mathbf{z}A\mathbf{x}$. So $\mathbf{x} \in NE(A)$. Suppose now that $\mathbf{z}A\mathbf{x} = \mathbf{x}A\mathbf{x}$. Then, for $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$,

$$\begin{aligned}
\varepsilon\mathbf{x}A\mathbf{z} &= \mathbf{x}A\mathbf{w}(\varepsilon) - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&> \mathbf{z}A\mathbf{w}(\varepsilon) - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&= \varepsilon\mathbf{z}A\mathbf{z} + (1-\varepsilon)\mathbf{z}A\mathbf{x} - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&= \varepsilon\mathbf{z}A\mathbf{z}\,,
\end{aligned}$$

so that $\mathbf{x}$ is an ESS.

Conversely, let $\mathbf{x}$ be an ESS with uniform invasion barrier (cf. Proposition 15.8) $\bar{\varepsilon} \in (0,1)$, and let Z be as in the proof of Proposition 15.8. Let

$$V = \{\mathbf{y} \in \Delta^m \mid \mathbf{y} = \varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x} \text{ for some } \mathbf{z} \in Z \text{ and } \varepsilon \in [0, \bar{\varepsilon})\}.$$

Since Z is closed and $\mathbf{x} \notin Z$, there is an open neighborhood U of $\mathbf{x}$ such that $U \cap \Delta^m \subseteq V$. Suppose that $\mathbf{y} \neq \mathbf{x}$, $\mathbf{y} \in U \cap \Delta^m$. Then $\mathbf{y} \in V$, and by Proposition 15.8,

$\mathbf{z}A\mathbf{y} = \mathbf{z}A(\varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}) < \mathbf{x}A(\varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}) = \mathbf{x}A\mathbf{y}$, with $\mathbf{z}$ as in the definition of V. This implies $\mathbf{y}A\mathbf{y} = \varepsilon\mathbf{z}A\mathbf{y} + (1-\varepsilon)\mathbf{x}A\mathbf{y} < \mathbf{x}A\mathbf{y}$. □

Local Strict Efficiency

Consider the special case of a symmetric game (A,B) with $A^T = A$, hence A is itself symmetric and $B = A$. Call such a game *doubly symmetric*.

Definition 15.11. A strategy $\mathbf{x} \in \Delta^m$ is *locally strictly efficient* if it has an open neighborhood U such that $\mathbf{x}A\mathbf{x} > \mathbf{y}A\mathbf{y}$ for all $\mathbf{y} \in U \setminus \{\mathbf{x}\}$.

For doubly symmetric games, local strict efficiency characterizes ESS ([57]).

Proposition 15.12. *Let $A = A^T$. Then $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is locally strictly efficient.*

Proof. Let $\mathbf{x} \in \Delta^m$. For any $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$, we have

$$\mathbf{y}A\mathbf{y} = \mathbf{x}A\mathbf{x} - 2\mathbf{x}A\mathbf{z} - 2\mathbf{z}A\mathbf{x} + 4\mathbf{z}A\mathbf{z}\,.$$

Hence, using the symmetry of A,

$$\mathbf{x}A\mathbf{x} - \mathbf{y}A\mathbf{y} = 4\left[\mathbf{x}A\mathbf{z} - \mathbf{z}A\mathbf{z}\right]\,.$$

If $\mathbf{x}$ is locally strictly efficient, then this identity implies that $\mathbf{x}$ is locally superior, and conversely. By Proposition 15.10, it follows that $\mathbf{x}$ is an ESS if and only if $\mathbf{x}$ is locally strictly efficient. □

15.3 Replicator Dynamics and ESS

The concept of an evolutionary stable strategy is based on the idea of *mutation*. Incorporation of the evolutionary concept of *selection* calls for a more explicitly dynamic approach.

15.3.1 Replicator Dynamics

As before, consider a symmetric game described by the $m \times m$ matrix A. A mixed strategy $\mathbf{x} \in \Delta^m$ can be interpreted as a vector of population shares (a *state*) over the pure strategies, evolving over time. To express time dependence, we write $\mathbf{x} = \mathbf{x}(t)$. For each pure strategy i, the expected payoff of playing i when the population is in state $\mathbf{x}$ is equal to $\mathbf{e}^iA\mathbf{x}$, hence the average population payoff is equal to $\sum_{i=1}^m x_i\mathbf{e}^iA\mathbf{x} =$

$\mathbf{x}A\mathbf{x}$. In the *replicator dynamics* ([133]) it is assumed that population shares develop according to the differential equation

$$\dot{x}_i = dx_i(t)/dt = \left[\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x}\right] x_i \tag{15.2}$$

for each pure strategy $i = 1,2,\ldots,m$, where dependence on t is (partly) suppressed from the notation. In other words, the share of the population playing strategy i changes with rate proportional to the difference between the expected payoff of i (individual fitness) and the average population payoff (average fitness).

To study the replicator dynamics in (15.2) one needs to apply the theory of differential equations and dynamical systems (e.g., [55]). For a first analysis we can restrict attention to a few basic concepts and facts.

For each *initial state* $\mathbf{x}(0) = \mathbf{x}^0 \in \Delta^m$, the system (15.2) induces a *solution* or *trajectory* $\xi(t,\mathbf{x}^0)$ in Δ^m. Call state $\mathbf{x}$ a *stationary point* of the dynamics (15.2) if $\dot{\mathbf{x}} = (\dot{x}_1,\ldots,\dot{x}_m) = (0,\ldots,0)$. If $m = 2$ then $\dot{x}_1 = 0$ or $\dot{x}_2 = 0$ is sufficient for $\mathbf{x}$ to be a stationary point, since (15.2) implies the natural condition $\sum_{i=1}^m \dot{x}_i = 0$. Note that any $\mathbf{e}^i$ is a stationary point – this is a more or less artificial property of the replicator dynamics. A state $\mathbf{x}$ is *Lyapunov stable* if every open neighborhood B of $\mathbf{x}$ contains an open neighborhood B^0 of $\mathbf{x}$ such that $\xi(t,\mathbf{x}^0) \in B$ for all $\mathbf{x}^0 \in B^0$ and $t \geq 0$. A state $\mathbf{x}$ *asymptotically stable* if it is Lyapunov stable and it has an open neighborhood B^* such that $\lim_{t\to\infty} \xi(t,\mathbf{x}^0) = \mathbf{x}$ for all $\mathbf{x}^0 \in B^*$. It is not hard to show that Lyapunov stability implies stationarity.

15.3.2 Symmetric 2×2 Games

In order to analyze the replicator dynamics for symmetric 2×2 games corresponding to A, we can without loss of generality restrict attention again to the normalized game

$$A' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Now (15.2) reduces to

$$\dot{x}_1 = [a_1x_1 - a_2x_2]x_1x_2 \tag{15.3}$$

(and $\dot{x}_2 = -\dot{x}_1$). For case I ($a_1 < 0$ and $a_2 > 0$) the stationary points of the dynamics are $\mathbf{x} = \mathbf{e}^1$ and $\mathbf{x} = \mathbf{e}^2$. For all other $\mathbf{x}$, $\dot{x}_1 < 0$, which implies that the system then converges to $\mathbf{e}^2$, the unique ESS. Hence, the (unique) ESS is also the (unique) asymptotically stable state.

From the answers to Problems 15.5 and 15.6 we have:

Proposition 15.13. *Let A be a generic 2×2 matrix and let $\mathbf{x} \in \Delta^2$. Then $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is an asymptotically stable state of the replicator dynamics.*

Note that this proposition implies Proposition 8.6(2). Part (1) of Proposition 8.6 follows from Problem 15.2.

15.3.3 Dominated Strategies

Does the replicator dynamics discard of dominated strategies? One answer to this question is provided by the following proposition, which states that if we start from a completely mixed strategy eventually all strictly dominated pure strategies vanish, i.e., their population shares converge to zero.

By Δ_0^m we denote the (relative) interior of the set Δ^m, i.e., $\Delta_0^m = \{\mathbf{x} \in \Delta^m \mid \mathbf{x} > \mathbf{0}\}$ is the set of completely mixed strategies or states.

Proposition 15.14. *Let* $\mathbf{x}^0 \in \Delta^m$ *be completely mixed and let pure strategy* i *be strictly dominated. Then* $\lim_{t\to\infty} \xi_i(t,\mathbf{x}^0) = 0$.

Proof. Let i be strictly dominated by $\mathbf{y} \in \Delta^m$ and let

$$\varepsilon = \min_{\mathbf{x}\in\Delta^m} \mathbf{y}A\mathbf{x} - \mathbf{e}^i A\mathbf{x} .$$

By continuity of the expected payoff function and compactness of Δ^m, $\varepsilon > 0$. Define $v_i : \Delta_0^m \to \mathbb{R}$ by $v_i(\mathbf{x}) = \ln x_i - \sum_{j=1}^m y_j \ln(x_j)$. The function v_i is differentiable, with time derivative at any point $\mathbf{x} = \xi(t,\mathbf{x}^0)$ equal to

$$\begin{aligned}
\dot{v}_i(\mathbf{x}) &= \left[\frac{dv_i(\xi(t,\mathbf{x}^0))}{dt}\right]_{\xi(t,\mathbf{x}^0)=\mathbf{x}} \\
&= \sum_{j=1}^m \frac{\partial v_i(\mathbf{x})}{\partial x_j}\dot{x}_j \\
&= \frac{\dot{x}_i}{x_i} - \sum_{j=1}^m \frac{y_j\dot{x}_j}{x_j} \\
&= (\mathbf{e}^i - \mathbf{x})A\mathbf{x} - \sum_{j=1}^m y_j(\mathbf{e}^j - \mathbf{x})A\mathbf{x} \\
&= (\mathbf{e}^i - \mathbf{y})A\mathbf{x} \le -\varepsilon < 0 .
\end{aligned}$$

Hence, $v_i(\xi(t,\mathbf{x}^0))$ decreases to minus infinity as $t \to \infty$. This implies $\xi_i(t,\mathbf{x}^0) \to 0$. □

Proposition 15.14 remains true for pure strategies i that are iteratively strictly dominated (see [111]). For weakly dominated strategies several things may happen, see Problem 15.7.

15.3.4 Nash Equilibrium Strategies

Consider again the finite symmetric two-player game with payoff matrix A. What is the relation between the replicator dynamics and Nash equilibrium strategies?

The answer is given by the following proposition, where $ST(A)$ denotes the set of stationary states, hence (check!):

$$ST(A) = \{\mathbf{x} \in \Delta^m \mid \forall i \in C(\mathbf{x})\ [\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}]\}\ . \tag{15.4}$$

Proposition 15.15. *For any finite symmetric two-player game with payoff matrix A we have:*

(1) $\{\mathbf{e}^1, \ldots, \mathbf{e}^m\} \cup NE(A) \subseteq ST(A)$.
(2) $ST(A) \cap \Delta_0^m = NE(A) \cap \Delta_0^m$.
(3) $ST(A) \cap \Delta_0^m$ *is a convex set and if* $\mathbf{z} \in \Delta^m$ *is a linear combination of states in this set, then* $\mathbf{z} \in NE(A)$.

Proof. It is straightforward from (15.2) that $\mathbf{e}^i \in ST(A)$ for every pure strategy i. If $\mathbf{x} \in NE(A)$, then every $i \in C(\mathbf{x})$ is a pure best reply, hence $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$; for $i \notin C(\mathbf{x})$, $x_i = 0$. Hence, $\mathbf{x} \in ST(A)$. This proves (1). Also (2) is immediate since $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$ for every $\mathbf{x} \in ST(A) \cap \Delta_0^m$ and every $\mathbf{x} \in NE(A) \cap \Delta_0^m$.

It remains to prove the last claim. Let $\mathbf{x}$ and $\mathbf{y}$ be completely mixed stationary points, and let $\alpha, \beta \in \mathbb{R}$ and $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y} \in \Delta^m$. For any pure strategy i we have

$$\mathbf{e}^i A\mathbf{z} = \alpha\mathbf{e}^i A\mathbf{x} + \beta\mathbf{e}^i A\mathbf{y} = \alpha\mathbf{x}A\mathbf{x} + \beta\mathbf{y}A\mathbf{y}$$

since $\mathbf{x}, \mathbf{y} \in ST(A) \cap \Delta_0^m$. This implies that actually $\mathbf{e}^i A\mathbf{z} = \mathbf{z}A\mathbf{z}$ for all pure strategies i, hence $\mathbf{z}$ is stationary. If $\mathbf{z}$ is completely mixed, then we are done by part (2). Otherwise, $\mathbf{z}$ is a boundary point of $ST(A) \cap \Delta_0^m$ and hence of $NE(A) \cap \Delta_0^m$, so $\mathbf{z} \in NE(A)$ since $NE(A)$ is a closed set. Finally, since Δ^m is convex and $\mathbf{z} \in ST(A) \cap \Delta_0^m$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $ST(A) \cap \Delta_0^m$ is a convex set. □

Proposition 15.15 implies that every (symmetric) Nash equilibrium is stationary. The weakest form of dynamical stability, Lyapunov stability, leads to a refinement of Nash equilibrium, as the next result shows.

Proposition 15.16. *Let* $\mathbf{x} \in \Delta^m$ *be a Lyapunov stable stationary state. Then* $\mathbf{x} \in NE(A)$.

Proof. Suppose $\mathbf{x} \notin NE(A)$. Then $\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x} > 0$ for some $i \notin C(\mathbf{x})$. By continuity, there is a $\delta > 0$ and an open neighborhood U of $\mathbf{x}$ such that $\mathbf{e}^i A\mathbf{y} - \mathbf{y}A\mathbf{y} \geq \delta$ for all $\mathbf{y} \in U \cap \Delta^m$. But then $\xi_i(t, \mathbf{x}^0) \geq x_i^0 \exp(\delta t)$ for all $\mathbf{x}^0 \in U \cap \Delta^m$ and $t \geq 0$ such that $\xi(t, \mathbf{x}^0) \in U \cap \Delta^m$.[2] So $\xi_i(t, \mathbf{x}^0)$ increases exponentially from any $\mathbf{x}^0 \in U \cap \Delta_0^m$ with $x_i^0 > 0$ whereas $x_i = 0$. This contradicts Lyapunov stability of $\mathbf{x}$. □

The final result in this section says that if a trajectory of the replicator dynamics starts from an interior (completely mixed) state and converges, then the limit state is a Nash equilibrium strategy.

Proposition 15.17. *Let* $\mathbf{x}^0 \in \Delta_0^m$ *and* $\mathbf{x} \in \Delta^m$ *such that* $\mathbf{x} = \lim_{t\to\infty} \xi(t, \mathbf{x}^0)$. *Then* $\mathbf{x} \in NE(A)$.

---

[2] This follows from the fact that the system $\dot{y} = \delta y$ with initial condition $y(0) = y^0$ has solution $y(t) = y^0 \exp(\delta t)$.

Proof. Suppose that $\mathbf{x} \notin NE(A)$. Then there is a pure strategy i and an $\varepsilon > 0$ such that $\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x} = \varepsilon$. Hence, there is a $T \in \mathbb{R}$ such that $\mathbf{e}^i A\xi(t,\mathbf{x}^0) - \xi(t,\mathbf{x}^0)A\xi(t,\mathbf{x}^0) > \varepsilon/2$ for all $t \geq T$. By (15.2), $\dot{x}_i > x_i\varepsilon/2$ for all $t \geq T$, and hence $\xi_i(t,\mathbf{x}^0) > \xi_i(T,\mathbf{x}^0)\exp(\varepsilon(t-T)/2)$ for all $t \geq T$. Since $\xi_i(T,\mathbf{x}^0) > 0$, this implies $\xi_i(t,\mathbf{x}^0) \to \infty$ as $t \to \infty$, a contradiction. □

15.3.5 Perfect Equilibrium Strategies

In the preceding subsection we have seen that Lyapunov stability implies Nash equilibrium. What are the implications of asymptotic stability?

First, asymptotic stability implies Lyapunov stability and therefore also Nash equilibrium. Since Nash equilibrium implies stationarity, however, it must be the case that an asymptotically stable Nash equilibrium strategy is *isolated*, meaning that it has an open neighborhood in which there are no other Nash equilibrium strategies. If not, there would be arbitrarily close stationary states, which conflicts with asymptotic stability.

Second, asymptotic stability also implies perfection ([15]).

Proposition 15.18. *Let $\mathbf{x} \in \Delta^m$ be asymptotically stable. Then $(\mathbf{x},\mathbf{x}) \in NE(A,A^T)$ is isolated and perfect.*

Proof. We still have to prove that $(\mathbf{x},\mathbf{x}) \in NE(A)$ is a perfect equilibrium. Suppose not. Then $\mathbf{x}$ is weakly dominated by some $\mathbf{y} \in \Delta^m \setminus \{\mathbf{x}\}$, see Theorem 13.22. Hence $\mathbf{y}A\mathbf{z} \geq \mathbf{x}A\mathbf{z}$ for all $\mathbf{z} \in \Delta^m$. Define $v : \Delta^m \to \mathbb{R}$ by

$$v(\mathbf{z}) = \sum_{i \in C(\mathbf{z})} (y_i - x_i)\ln(z_i)$$

for all $\mathbf{z} \in \Delta^m$. Similarly as in the proof of Proposition 15.14, we obtain that v is nondecreasing along all interior solution trajectories of (15.2), i.e., at any $\mathbf{z} \in \Delta_0^m$,

$$\dot{v}(\mathbf{z}) = \sum_{i \in C(\mathbf{z})} (y_i - x_i)\frac{\dot{z}_i}{z_i} = \sum_{i=1}^{m} (y_i - x_i)[\mathbf{e}^i A\mathbf{z} - \mathbf{z}A\mathbf{z}] = (\mathbf{y}-\mathbf{x})A\mathbf{z} \geq 0.$$

Since $\mathbf{x}$ is asymptotically stable, it has an open neighborhood U such that $\xi(t,\mathbf{x}^0) \to \mathbf{x}$ for all $\mathbf{x}^0 \in U \cap \Delta^m$. By nondecreasingness of v along all interior solution trajectories this implies $v(\mathbf{x}) \geq v(\mathbf{z})$ for all $\mathbf{z} \in U \cap \Delta_0^m$. We will construct, however, a $\mathbf{z}$ in $U \cap \Delta_0^m$ with $v(\mathbf{z}) > v(\mathbf{x})$. This is a contradiction and, hence, $\mathbf{x}$ must be perfect.

To construct $\mathbf{z}$, define for $\delta \in (0,1)$, $\mathbf{w} \in \Delta_0^m$, and $\varepsilon > 0$,

$$\mathbf{z} = (1-\varepsilon)[(1-\delta)\mathbf{x} + \delta\mathbf{y}] + \varepsilon\mathbf{w}\,.$$

For ε sufficiently small, we have $y_i > x_i \Rightarrow z_i > x_i$ and $y_i < x_i \Rightarrow z_i < x_i$. Moreover

$$\begin{aligned} v(\mathbf{z}) - v(\mathbf{x}) &= \sum_{i=1}^{m} (y_i - x_i) \ln(z_i) - \sum_{i \in C(\mathbf{x})} (y_i - x_i) \ln(x_i) \\ &= \sum_{i \in C(\mathbf{x})} (y_i - x_i)[\ln(z_i) - \ln(x_i)] + \sum_{i \notin C(\mathbf{x})} y_i \ln(z_i). \end{aligned}$$

The second term in the expression after the last equality sign is zero. To see this, it is sufficient to show that $C(\mathbf{y}) \subseteq C(\mathbf{x})$. Suppose that $j \in C(\mathbf{y})$ and $j \notin C(\mathbf{x})$. By asymptotic stability of $\mathbf{x}$, $\xi(t, \mathbf{x}^0) \to \mathbf{x}$ for all $\mathbf{x}^0 \in U \cap \Delta^0$. In particular, $v(\xi(t, \mathbf{x}^0))$ is nondecreasing with t. However, $\xi_j(t, \mathbf{x}^0) \to x_j = 0$ whereas $y_j > 0$, so for some constant γ we have $v(\xi(t, \mathbf{x}^0)) \le \gamma + y_j \ln(\xi_j(t, \mathbf{x}^0)) \to -\infty$, a contradiction.

So we have $v(\mathbf{z}) > v(\mathbf{x})$, which completes the proof. □

Problems

15.1. *Computing ESS in* 2×2 *Games (1)*

Compute $ESS(A)$ for the following payoff matrices A.

(a) $A = \begin{pmatrix} 4 & 0 \\ 5 & 3 \end{pmatrix}$ (Prisoners' Dilemma)

(b) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (Coordination game)

(c) $A = \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix}$ (Hawk–Dove game)

15.2. *Computing ESS in* 2×2 *Games (2)*

Compute $ESS(A')$ for each of the cases (1), (2), and (3) in Sect. 15.1.1. Compare with your answers to Problem 15.1.

15.3. *Rock–Paper–Scissors (1)*

Show that the Rock–Paper–Scissors game

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

has no ESS.

15.4. *Uniform Invasion Barriers*

Find the maximal value of the uniform invasion barrier for the ESS's in each of the cases (1), (2), and (3) in Sect. 15.1.1.

15.5. *Replicator Dynamics in Normalized Game (1)*

Show that A and A' (see Sect. 15.3.2) result in the same replicator dynamics.

15.6. *Replicator Dynamics in Normalized Game (2)*

(a) Simplify the dynamics (15.3) for case (1) in Sect. 15.1.1 by substituting $x_2 = 1 - x_1$ and plot $\dot{x}_1$ as a function of $x_1 \in [0,1]$.

(b) Carry out this analysis also for cases (2) and (3). What is your conclusion?

15.7. *Weakly Dominated Strategies and Replicator Dynamics*

(a) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Investigate the trajectories of the replicator dynamics.

(b) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Investigate the trajectories of the replicator dynamics.

(Cf. Proposition 15.14 and for more on weakly dominated strategies see [147], p. 83.)

15.8. *Stationary Points and Nash Equilibria*

Consider the two-person symmetric game with payoff matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

(a) Compute $NE(A)$.

(b) Compute $ST(A)$.

15.9. *Lyapunov Stable States in* 2×2 *Games*

Consider the normalized two-player symmetric 2×2 game A'. Compute the Lyapunov stable states for cases (1), (2), and (3).

15.10. *Nash Equilibrium and Lyapunov Stability*

Consider the symmetric game with payoff matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute $NE(A)$. Show that the unique element in this set is not Lyapunov stable (see [147], Sect. 3.3.2 for more on this).

15.11. *Rock–Paper–Scissors (2)*

Consider the generalized Rock–Paper–Scissors game with payoff matrix

$$A = \begin{pmatrix} 1 & 2+a & 0 \\ 0 & 1 & 2+a \\ 2+a & 0 & 1 \end{pmatrix}$$

where $a \in \mathbb{R}$.

(a) Write down the three equations of the replicator dynamics.

(b) Define $h(\mathbf{x}) = \ln(x_1 x_2 x_3)$ for $\mathbf{x}$ positive and show that $\dot{h}(\mathbf{x}) = 3 + a - 3\mathbf{x}A\mathbf{x}$.

(c) Show that the average payoff is equal to

$$\mathbf{x}A\mathbf{x} = 1 + \frac{a}{2}(1 - ||\mathbf{x}||^2)$$

for each $\mathbf{x} \in \Delta^3$, where $||\mathbf{x}||$ is the Euclidean norm of $\mathbf{x}$. Conclude that $\dot{h}(\mathbf{x}) = \frac{a}{2}\left(3||\mathbf{x}||^2 - 1\right)$.

(d) Show that $\dot{h}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 0$ and $\dot{h}(\mathbf{x})$ has the same sign as a for other $\mathbf{x} \in \Delta^m$.

(e) Show that the unique Nash equilibrium in this game is (1) asymptotically stable for $a > 0$; (2) Lyapunov but not asymptotically stable for $a = 0$; (3) not Lyapunov stable for $a < 0$.

Chapter 16
TU-Games: Domination, Stable Sets, and the Core

In a game with transferable utility (TU-game) each coalition (subset of players) is characterized by its worth, i.e., a real number representing the payoff or utility that the coalition can achieve if it forms. It is assumed that this payoff can be freely distributed among the members of the coalition in any way desired.

For some examples the reader is referred to Chap. 1. Chapter 9 presents a first acquaintance with transferable utility games. Although the present chapter and the following ones are self-contained, it may be helpful to study the relevant parts of Chaps. 1 and 9 first.

In this chapter the focus is on the core of a transferable utility game. Section 16.1 starts with a weaker concept, the imputation set, and introduces the concept of domination. Section 16.2 introduces the domination core and the core. Section 16.3 studies these solution concepts for a special class of TU-games called simple games. In Sect. 16.4 we briefly review von Neumann and Morgenstern's stable sets, which are also based on the concept of domination. Section 16.5, finally, presents a characterization of games with non-empty cores in terms of balancedness.

16.1 Imputations and Domination

We start with repeating the definition of a game with transferable utility (cf. Definition 9.1).

Definition 16.1. A *cooperative game with transferable utility* or *TU-game* is a pair (N,v), where $N = \{1,2,\ldots,n\}$ with $n \in \mathbb{N}$ is the set of *players*, and v is a function assigning to each *coalition* S, i.e., to each subset $S \subseteq N$ a real number $v(S)$, such that $v(\emptyset) = 0$. The function v is called the *characteristic function* and $v(S)$ is called the *worth* of S. The coalition N is called the *grand coalition*. A *payoff distribution for* coalition S is a vector of real numbers $(x_i)_{i\in S}$.

The set of coalitions is also denoted by 2^N, so that a TU-game is a pair (N, v) with $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The game (N, v) is often denoted by v if no confusion about the set of players is likely to arise. Also, for a coalition $\{i, j, \ldots, k\}$ we sometimes write $v(ij\ldots k)$ instead of $v(\{i, j, \ldots, k\})$. By $\mathcal{G}^N$ the set of all TU-games with player set N is denoted.

We frequently use the notation $x(S) := \sum_{i \in S} x_i$ for a payoff distribution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^N$ and a coalition $S \subseteq N$.

Let (N, v) be a TU-game. A vector $\mathbf{x} \in \mathbb{R}^N$ is called an *imputation* if

(1) $\mathbf{x}$ is *individually rational*, i.e.

$$x_i \geq v(i) \quad \text{for all } i \in N,$$

(2) $\mathbf{x}$ is *efficient* i.e.

$$x(N) = v(N).$$

The set of imputations of (N, v) is denoted by $I(v)$. An element $\mathbf{x} \in I(v)$ is a payoff distribution of the worth $v(N)$ of the grand coalition N which gives each player i a payoff x_i which is at least as much as he can obtain when he operates alone.

Example 16.2. A game v is called *additive* if $v(S \cup T) = v(S) + v(T)$ for all disjoint coalitions S and T. Such a game is completely determined by the worths of the one-person coalitions $v(i)$ $(i \in N)$, since $v(S) = \sum_{i \in S} v(i)$ for every coalition S. For an additive game v, $I(v)$ consists of one point: $I(v) = \{(v(1), v(2), \ldots, v(n))\}$.

Note that for a game v

$$I(v) \neq \emptyset \quad \text{if and only if } v(N) \geq \sum_{i=1}^{n} v(i).$$

For an *essential* game v, that is, a game with $v(N) \geq \sum_{i=1}^{n} v(i)$, $I(v)$ is the convex hull of the points: $\mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^n$ where $f_k^i := v(k)$ if $k \neq i$ and $f_i^i := v(N) - \sum_{k \in N \setminus \{i\}} v(k)$ (see Problem 16.1).

Example 16.3. Let (N, v) be a three-person game with $v(1) = v(3) = 0$, $v(2) = 3$, $v(1,2,3) = 5$. Then $I(v)$ is the triangle with vertices $\mathbf{f}^1 = (2,3,0)$, $\mathbf{f}^2 = (0,5,0)$ and $\mathbf{f}^3 = (0,3,2)$ (see Fig. 16.1).

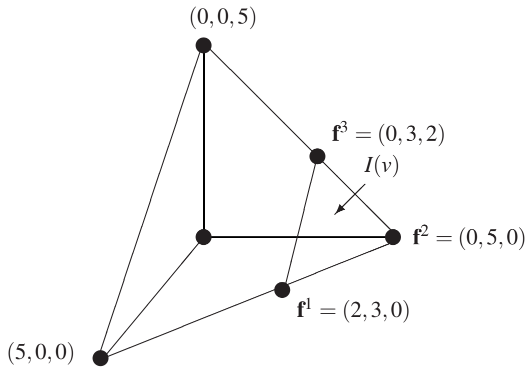
The following concept was already introduced in [141].

Definition 16.4. Let (N, v) be a game. Let $\mathbf{y}$, $\mathbf{z} \in I(v)$, $S \in 2^N \setminus \{\emptyset\}$. Then $\mathbf{y}$ *dominates* $\mathbf{z}$ *via coalition* S, denoted by $\mathbf{y} \operatorname{dom}_S \mathbf{z}$, if

(1) $y_i > z_i$ for all $i \in S$,
(2) $y(S) \leq v(S)$.

For $\mathbf{y}, \mathbf{z} \in I(v)$, $\mathbf{y}$ is said to *dominate* $\mathbf{z}$ (notation: $\mathbf{y} \operatorname{dom} \mathbf{z}$) if there is an $S \in 2^N \setminus \{\emptyset\}$ such that $\mathbf{y} \operatorname{dom}_S \mathbf{z}$.

Fig. 16.1 Example 16.3



Interpretation: the payoff distribution $\mathbf{y}$ is better than $\mathbf{z}$ for all members $i \in S$ if (1) holds and the payoffs $(y_i)_{i\in S}$ are attainable for the members of S by cooperation if (2) holds. Against each $\mathbf{z}$ in

$$D(S) := \{\mathbf{z} \in I(v) \mid \text{ there exists } \mathbf{y} \in I(v) \text{ with } \mathbf{y}\,\mathrm{dom}_S\,\mathbf{z}\}$$

the players of S can protest successfully. The set $D(S)$ consists of the imputations which are dominated via S. Note that always $D(N) = \emptyset$ (see Problem 16.3). We call $\mathbf{x} \in I(v)$ *undominated* if $\mathbf{x} \in I(v) \setminus \bigcup_{S\in 2^N\setminus\{\emptyset\}} D(S)$.

Example 16.5. Let (N,v) be the three-person game with $v(1,2) = 2$, $v(N) = 1$ and $v(S) = 0$ if $S \neq \{1,2\}, N$. Then $D(S) = \emptyset$ if $S \neq \{1,2\}$ and $D(\{1,2\}) = \{\mathbf{x} \in I(v) \mid x_3 > 0\}$. The elements $\mathbf{x}$ in $I(v)$ which are undominated satisfy $x_3 = 0$.

16.2 The Core and the Domination-Core

The concept of domination defined in the preceding section gives rise to the following definitions.

Definition 16.6. The *domination core* (*D-core*) of a game (N,v) is the set

$$DC(v) := I(v) \setminus \bigcup_{S\in 2^N\setminus\{\emptyset\}} D(S),$$

i.e., the set of all undominated elements in $I(v)$. The *core* of a game (N,v) is the set

$$C(v) := \{\mathbf{x} \in I(v) \mid x(S) \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}.$$

If $\mathbf{x} \in C(v)$, then no coalition $S \neq N$ has an incentive to split off if $\mathbf{x}$ is the proposed payoff distribution in N, because the total amount $x(S)$ allocated to S is not smaller than the amount $v(S)$ which the players in S can obtain by forming the coalition S.

For the game in Example 16.5 the D-core is nonempty and the core is empty. In general the following holds.

Theorem 16.7. *The core is a subset of the D-core for each TU-game.*

Proof. Let (N,v) be a game and $\mathbf{x} \in I(v)$, $\mathbf{x} \notin DC(v)$. Then there is a $\mathbf{y} \in I(v)$ and a coalition S such that $\mathbf{y}\,\mathrm{dom}_S\,\mathbf{x}$. Thus, $v(S) \geq y(S) > x(S)$, which implies that $\mathbf{x} \notin C(v)$. □

Elements of $C(v)$ can easily be obtained because the core is defined with the aid of linear inequalities. The core is a polytope. Also the D-core is a convex set: see Problem 16.2.

A natural question that arises is: for which games is the core equal to the D-core? Consider the following condition on a game (N,v):

$$v(N) \geq v(S) + \sum_{i \in N \setminus S} v(i) \text{ for all } S \in 2^N \setminus \{\emptyset\}. \tag{16.1}$$

It turns out that this condition is sufficient for the equality of core and D-core.

Theorem 16.8. *Let (N,v) be a game satisfying (16.1). Then $DC(v) = C(v)$.*

Proof. In view of Theorem 16.7 it is sufficient to show that $DC(v) \subseteq C(v)$.
Claim. Let $\mathbf{x} \in I(v)$ with $x(S) < v(S)$ for some S, then there is a $\mathbf{y} \in I(v)$ such that $\mathbf{y}\,\mathrm{dom}_S\,\mathbf{x}$.

To prove this claim, define $\mathbf{y}$ as follows. If $i \in S$, then $y_i := x_i + |S|^{-1}(v(S) - x(S))$. If $i \notin S$, then $y_i := v(i) + (v(N) - v(S) - \sum_{i \in N \setminus S} v(i))|N \setminus S|^{-1}$. Then $\mathbf{y} \in I(v)$, where $y_i \geq v(i)$ for $i \in N \setminus S$ follows from (16.1). Furthermore, $\mathbf{y}\,\mathrm{dom}_S\,\mathbf{x}$. This proves the claim.

To prove $DC(v) \subseteq C(v)$, suppose $\mathbf{x} \in DC(v)$. Then there is no $\mathbf{y} \in I(v)$ with $\mathbf{y}\,\mathrm{dom}\,\mathbf{x}$. In view of the Claim it follows that $x(S) \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Hence, $\mathbf{x} \in C(v)$. □

Many games v derived from practical situations have the following property:

$$v(S \cup T) \geq v(S) + v(T) \text{ for all disjoint } S,T \subseteq N. \tag{16.2}$$

A game satisfying (16.2) is called *super-additive*. Observe that (16.2) implies (16.1), so that Theorem 16.8 holds for super-additive games in particular.

16.3 Simple Games

In this section we study the core and D-core of simple games. Simple games arise in particular in political situations, see for instance the United Nations Security Council example in Chap. 1.

Definition 16.9. A *simple game* (N,v) is a game where every coalition has either worth 0 or worth 1, and the grand coalition N has worth 1. Coalitions with worth 1 are called *winning*, the other coalitions are called *losing*. A *minimal* winning coalition is a winning coalition for which every proper subset is losing. A player i is called a *dictator* in a simple game (N,v) if $v(S)=1$ if and only if $i \in S$. A player i is called a *veto player* in a simple game (N,v) if i belongs to all winning coalitions. The *set of veto players* of v is denoted by $\text{veto}(v)$. Hence,

$$\text{veto}(v) = \bigcap \{S \in 2^N \mid v(S) = 1\}.$$

The next example suggests that non-emptiness of the core has something to do with the existence of veto players.

For each $i \in N$ let $\mathbf{e}^i \in \mathbb{R}^n$ denote the vector with i-th coordinate equal to 1 and all other coordinates equal to 0.

Example 16.10. (1) For the *dictator game* δ_i, which is the simple game with $\delta_i(S) = 1$ if and only if $i \in S$ one has $I(\delta_i) = \{\mathbf{e}^i\}$, $\text{veto}(\delta_i) = \{i\}$ and $C(\delta_i) = DC(\delta_i) = \{\mathbf{e}^i\}$.
(2) For the three-person *majority game* with $v(S) = 1$ if $|S| \in \{2,3\}$ and $v(S) = 0$ if $|S| \in \{0,1\}$ one has:

$$\{1,2\} \cap \{1,3\} \cap \{2,3\} \cap \{1,2,3\} = \emptyset = \text{veto}(v)$$

and

$$C(v) = DC(v) = \emptyset.$$

(3) For the T*-unanimity game* u_T, which is the simple game with $u_T(S) = 1$ if and only if $T \subseteq S$, $\text{veto}(u_T) = T$ and

$$C(u_T) = DC(u_T) = \text{conv}\{\mathbf{e}^i \mid i \in T\}.$$

The following theorem shows that the core of a simple game is nonempty if and only if the game has veto players. Furthermore, core elements divide the total amount $v(N) = 1$ of the grand coalition among the veto players. The D-core is equal to the core for simple games except in one case where there is exactly one $k \in N$ with $v(k) = 1$ and k is not a veto player. See also Example 16.12 below.

Theorem 16.11. *Let* (N,v) *be a simple game. Then:*

(1) $C(v) = \text{conv}\{\mathbf{e}^i \in \mathbb{R}^n \mid i \in \text{veto}(v)\}$.
(2) *If* $\text{veto}(v) = \emptyset$ *and* $\{i \in N \mid v(i) = 1\} = \{k\}$, *then* $C(v) = \emptyset$ *and* $DC(v) = \{\mathbf{e}^k\}$. *Otherwise,* $DC(v) = C(v)$.

Proof. (a) Suppose $i \in \text{veto}(v)$. Let $S \in 2^N \setminus \{\emptyset\}$. If $i \in S$ then $e^i(S) = 1 \geq v(S)$, otherwise $e^i(S) = 0 = v(S)$. Obviously, $e^i(N) = 1 = v(N)$. So $\mathbf{e}^i \in C(v)$. This proves the inclusion $\supseteq$ in (1) because $C(v)$ is a convex set.

(b) To prove the inclusion $\subseteq$ in (1), let $\mathbf{x} \in C(v)$. It is sufficient to prove: $i \notin \text{veto}(v) \Rightarrow x_i = 0$. Suppose, to the contrary, that $x_i > 0$ for some non-veto player i. Take S with $v(S) = 1$ and $i \notin S$ (such an S exists otherwise i would be a veto player). Then $x(S) = x(N) - x(N \setminus S) \leq 1 - x_i < 1$, contradicting the fact that $\mathbf{x}$ is a core element. This concludes the proof of (1).

(c) If $\text{veto}(v) = \emptyset$ and k is the only player in the set $\{i \in N \mid v(i) = 1\}$, then $C(v) = \emptyset$ by part (1), whereas $I(v) = \{\mathbf{e}^k\}$, hence $DC(v) = \{\mathbf{e}^k\}$. If $\text{veto}(v) = \emptyset$ and $\{i \in N \mid v(i) = 1\} = \emptyset$ then (16.1) is satisfied, so that core and D-core are equal by Theorem 16.8. If $\text{veto}(v) = \emptyset$ and $|\{i \in N \mid v(i) = 1\}| \geq 2$ then $I(v) = \emptyset$ so that $C(v) = DC(v) = \emptyset$.

(d) To complete the proof of (2), suppose $\text{veto}(v) \neq \emptyset$. Then $|\{i \in N \mid v(i) = 1\}| \leq 1$. If $\{i \in N \mid v(i) = 1\} = \{k\}$ then $\text{veto}(v) = \{k\}$ and $I(v) = \{\mathbf{e}^k\}$, so that $C(v) = \{\mathbf{e}^k\} = DC(v)$. If, finally, $\{i \in N \mid v(i) = 1\} = \emptyset$, then (16.1) is satisfied and the core equals the D-core by Theorem 16.8. □

Example 16.12. Let $N = \{1,2,3\}$, $v(1) = v(2,3) = v(1,2,3) = 1$ and $v(S) = 0$ for the other coalitions. Then $\text{veto}(v) = \emptyset$, $C(v) = \emptyset$, $DC(v) = \{\mathbf{e}^1\}$. Note that this simple game is not super-additive, and does not satisfy (16.1).

16.4 Stable Sets

Stable sets were already introduced by von Neumann and Morgenstern [141] – the core was introduced by Gillies [46]. The definition is again based on the concept of domination. By way of example, let v be the three-person game with all worths equal to 1 except for the one-person coalitions, which have worth equal to 0. Observe that the three vectors $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},0,\frac{1}{2})$, and $(0,\frac{1}{2},\frac{1}{2})$ are imputations that do not dominate each other. Moreover, each imputation other than one of these three is dominated by one of these three (see Problem 16.5). For this reason, von Neumann and Morgenstern called the set of these three imputations a 'solution' of the game.

Definition 16.13. Let v be a game and let $A \subseteq I(v)$. The set A is called a *stable set* if

(1) if $\mathbf{x}, \mathbf{y} \in A$ then $\mathbf{x}$ does not dominate $\mathbf{y}$.
(2) if $\mathbf{x} \in I(v) \setminus A$ then there is a $\mathbf{y} \in A$ that dominates $\mathbf{x}$.

The first property in Definition 16.13 is called *internal stability* and the second one *external stability*.

The three-person game described above has many stable sets: see Problem 16.5. But even if a game has only one stable set then still a selection would have to be made, for practical purposes; stability, however, is a property of sets, not of single payoff distributions. The core does not suffer from this problem and, moreover, in that case there exist some plausible choices (like the nucleolus, see Chap. 19). Moreover, games with non-empty cores have been exactly characterized (see Sect. 16.5), whereas the problem of existence of stable sets is only partially solved. Lucas [72] gives an example of a(n essential) game that does not have a stable set; see also [98], p. 253.

Some partial existence results are given now. First, essential simple games always have stable sets:

Theorem 16.14. *Let v be a simple game and let S be a minimal winning coalition. Let Δ^S be the set of those imputations $\mathbf{x}$ with $x_i = 0$ for every $i \notin S$. Then, if $\Delta^S \neq \emptyset$, it is a stable set.*

Proof. Problem 16.8. □

A game (N,v) is called a *zero-one game* if all one-person coalitions have worth 0 and the grand coalition N has worth 1. In the following example symmetric three-person zero-one games are considered.

Example 16.15. Let (N,v) be a game with $N = \{1,2,3\}$ and $v(i) = 0$ for all $i \in N$, $v(N) = 1$, and $v(S) = \alpha$ for every two-person coalition S, where $0 \leq \alpha \leq 1$. Then:

(a) Let $\alpha \geq \frac{2}{3}$. Then

$$\{(x,x,1-2x),(x,1-2x,x),(1-2x,x,x) \mid \frac{\alpha}{2} \leq x \leq \frac{1}{2}\} \tag{16.3}$$

is a stable set.
(b) For $\alpha < \frac{2}{3}$, the set in (16.3) is internally but not externally stable. The union of this set with the core of the game is a stable set.
(c) For $\alpha \leq \frac{1}{2}$ the core is a (the unique) stable set.

For the proofs of these statements see Problem 16.9.

The next theorem gives the relation between the domination core and stable sets.

Theorem 16.16. *Let (N,v) be a game. Then:*

(a) The D-core of v is a subset of any stable set.
(b) Suppose the D-core of v is a stable set. Then it is the unique stable set of the game.

Proof. Problem 16.10. □

16.5 Balanced Games and the Core

In this section we derive the Bondareva–Shapley ([16], [122]) Theorem which characterizes games with non-empty cores in terms of balancedness. First, the concepts of balanced maps, collections, and games are introduced.

Let $N := \{1,2,\ldots,n\}$. A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$ is called a *balanced map* if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\mathbf{e}^S = \mathbf{e}^N.$$

Here $\mathbf{e}^S$ is the *characteristic vector* for coalition S with

$$e_i^S = 1 \quad \text{if } i \in S \quad \text{and} \quad e_i^S = 0 \quad \text{if } i \in N \setminus S.$$

A collection B of coalitions is called a *balanced collection* if there is a balanced map λ such that

$$B = \{S \in 2^N \mid \lambda(S) > 0\}.$$

Example 16.17. (1) Let $N_1, N_2, \ldots, N_k$ be a partition of N, i.e., $N = \bigcup_{r=1}^{k} N_r$, $N_s \cap N_t = \emptyset$ if $s \neq t$. Then $\{N_1, N_2, \ldots, N_k\}$ is a balanced collection, corresponding to the balanced map λ with $\lambda(S) = 1$ if $S \in \{N_1, N_2, \ldots, N_k\}$ and $\lambda(S) = 0$, otherwise.

(2) For $N = \{1,2,3\}$ the set $B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ is balanced and corresponds to the balanced map λ with

$$\lambda(S) = 0 \quad \text{if } |S| \in \{1,3\} \quad \text{and} \quad \lambda(S) = \frac{1}{2} \quad \text{if } |S| = 2.$$

In order to have an interpretation of a balanced map, one can think of each player having one unit of time (or perhaps energy, labor) to spend. Each player can distribute his time over the various coalitions of which he is a member. Such a distribution is 'balanced' if it corresponds to a balanced map λ, where $\lambda(S)$ is interpreted as the length of time that the coalition S exists ('cooperates'); balancedness of λ means that each player spends exactly his one unit of time over the various coalitions.

Definition 16.18. A game (N, v) is called a *balanced game* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have

$$\sum_S \lambda(S) v(S) \leq v(N). \tag{16.4}$$

Extending the interpretation of a balanced map in terms of a distribution of time to a game, balancedness of a game could be interpreted as saying that it is at least as productive to have the grand coalition operate during one unit of time as to have a balanced distribution of time over various smaller coalitions – worths of coalitions being interpreted as productivities. Thus, in a balanced game, it seems advantageous to form the grand coalition. Indeed, technically the importance of the notion of balancedness follows from Theorem 16.21, proved by Bondareva [16] and Shapley [122]; this theorem characterizes games with a nonempty core. Its proof is based on the following duality theorem.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the usual inner product: $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$.

Theorem 16.19. *Let A be an $n \times p$-matrix, $\mathbf{b} \in \mathbb{R}^p$ and $\mathbf{c} \in \mathbb{R}^n$, and let $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}A \geq \mathbf{b}\} \neq \emptyset$ and $\{\mathbf{y} \in \mathbb{R}^p \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} \neq \emptyset$. Then*

$$\min\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x}A \geq \mathbf{b}\} = \max\{\mathbf{b} \cdot \mathbf{y} \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}.$$

Proof. Problem 16.13. □

Remark 16.20. In Theorem 16.19 also the following holds: if one of the programs is infeasible (i.e., one of the two sets in the theorem is empty), then both programs do not have an optimal solution (i.e., neither the minimum nor the maximum are attained). See Problem 16.14 for a proof.

Theorem 16.21. *Let (N, v) be a TU-game. Then the following two assertions are equivalent:*

(1) $C(v) \neq \emptyset$.
(2) (N, v) *is a balanced game.*

Proof. First note that $C(v) \neq \emptyset$ if and only if

$$v(N) = \min\{\sum_{i=1}^{n} x_i \mid \mathbf{x} \in \mathbb{R}^N,\ x(S) \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}. \qquad (16.5)$$

By the duality theorem, Theorem 16.19, equality (16.5) holds if and only if

$$v(N) = \max\{\sum \lambda(S)v(S) \mid \sum \lambda(S)\mathbf{e}^S = \mathbf{e}^N, \lambda \geq \mathbf{0}\}. \qquad (16.6)$$

(Take for A the matrix with the characteristic vectors $\mathbf{e}^S$ as columns, let $\mathbf{c} := \mathbf{e}^N$ and let $\mathbf{b}$ be the vector of coalitional worths. Obviously, the non-emptiness conditions in Theorem 16.19 are satisfied.) Now (16.6) holds if and only if (16.4) holds. Hence (1) and (2) are equivalent. □

Problems

16.1. *Imputation Set of an Essential Game*

Prove that for an essential game v, $I(v)$ is the convex hull of the points $\mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^n$, as claimed in Sect. 16.1.

16.2. *Convexity of the Domination Core*

Prove that for each game the domination core is a convex set.

16.3. *Dominated Sets of Imputations*

(1) Prove that for each game (N, v), $D(S) = \emptyset$ if $|S| \in \{1, n\}$.

(2) Determine for each S the set $D(S)$ for the cost savings game (three communities game) in Chap. 1. Answer the same questions for the glove game in Chap. 1.

16.4. *The Domination Relation*

(1) Prove that dom and dom_S are irreflexive relations and that dom_S is transitive and antisymmetric.[1]

(2) Construct a game (N, v) and imputations $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$ and $\mathbf{y}\,\mathrm{dom}\,\mathbf{x}$.

(3) Construct a game (N, v) and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in I(v)$ with $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$ and $\mathbf{y}\,\mathrm{dom}\,\mathbf{z}$ and not $\mathbf{x}\,\mathrm{dom}\,\mathbf{z}$.

[1] See Sect. 11.1 for definitions.

16.5. *Stable Sets in a Three-Person Game*

Let $(\{1,2,3\},v)$ be the game with all worths equal to 1 except for the one-person and the empty coalitions, which have worth equal to 0.

(1) Prove that each element of the imputation set of this game is dominated by another element.

(2) Prove that in this game each $\mathbf{x} \in I(v) \setminus A$ is dominated by an element of $A := \{(\frac{1}{2},\frac{1}{2},0),\ (\frac{1}{2},0,\frac{1}{2}),\ (0,\frac{1}{2},\frac{1}{2})\}$.

(3) If $c \in [0,\frac{1}{2})$ and $B := \{\mathbf{x} \in I(v) \mid x_3 = c\}$, then each element of $I(v) \setminus B$ is dominated by an element of B. Show this.

16.6. *Singleton Stable Set*

Prove that if a game (N,v) has a one-element stable set then $v(N) = \sum_{i \in N} v(i)$ (from [81]).

16.7. *A Glove Game*

Consider the three-person simple game v defined by

$$v(S) := \begin{cases} 1 & \text{if } S = \{1,2\} \text{ or } \{2,3\} \text{ or } \{1,2,3\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that any imputation (x_1,x_2,x_3) that is not equal to $\mathbf{e}^2$ is dominated by another imputation.

(b) Compute the core and the domination core.

(c) Show that the domination core is not a stable set.

(d) Show that

$$B := \{(\lambda, 1-2\lambda, \lambda) \mid 0 \le \lambda \le \frac{1}{2}\}$$

is a stable set.

16.8. *Proof of Theorem 16.14*

Prove Theorem 16.14.

16.9. *Example 16.15*

Prove the statements in Example 16.15.

16.10. *Proof of Theorem 16.16*

Prove Theorem 16.16. Does this theorem also hold for the core instead of the D-core?

16.11. *Core and D-core*

Is (16.1) also a necessary condition for equality of the core and the D-core? (cf. Theorem 16.8).

16.12. *Strategic Equivalence*

Let (N, w) be *strategically equivalent* to (N, v), that is, there are $k \in \mathbb{R}$, $k > 0$ and $\mathbf{a} \in \mathbb{R}^N$ such that for each coalition S: $w(S) = kv(S) + a(S)$. Show that

(1) $C(w) = kC(v) + \mathbf{a}$ $(:= \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} = k\mathbf{y} + \mathbf{a}$ for some $\mathbf{y} \in C(v)\})$
(2) $DC(w) = kDC(v) + \mathbf{a}$.

(The equalities (1) and (2) express that the core and the D-core are covariant w.r.t. strategic equivalence.)

16.13. *Proof of Theorem 16.19*

Prove Theorem 16.19. (Hint: use Theorem 22.5.)

16.14. *Infeasible Programs in Theorem 16.19*

Prove the claim made in Remark 16.20. Hint: Suppose, say, that there is no $\mathbf{y} \geq \mathbf{0}$ with $A\mathbf{y} = \mathbf{c}$. Then, certainly, the max-program does not have an optimal solution. Use Farkas' Lemma (Lemma 22.4) to conclude that there exists a vector $\mathbf{z}$ with $\mathbf{z}A \geq \mathbf{0}$ and $\mathbf{z} \cdot \mathbf{c} < 0$. Suppose the min-program is feasible, i.e., there is an $\mathbf{x}$ with $\mathbf{x}A \geq \mathbf{b}$. Then, show that the min-program does not have an optimal solution by considering the vectors $\mathbf{x} + t\mathbf{z}$ for $t \in \mathbb{R}$, $t > 0$.

16.15. *Balanced Maps and Collections*

(1) Show that for any balanced map λ one has $\sum_S \lambda(S) \geq 1$, with equality if and only if the corresponding balanced collection equals $\{N\}$.

(2) If B is a balanced collection unequal to $\{N\}$, then

$$B^c := \{S \in 2^N \setminus \{\emptyset\} \mid N \setminus S \in B\}$$

is also a balanced collection. Give the corresponding balanced map.

(3) Let $S \in 2^N \setminus \{\emptyset, N\}$. Prove that $\{S, (N \setminus \{i\})_{i \in S}\}$ is balanced collection.

(4) Prove that the balanced maps form a convex set Λ^n.

16.16. *Minimum of Balanced Games*

Show that the minimum of two balanced games is again balanced.

16.17. *Balanced Simple Games*

A simple game has a non-empty core if and only if it has veto players, cf. Theorem 16.11(1). Derive this result from Theorem 16.21.

Chapter 17
The Shapley Value

In Chap. 16 set-valued solution concepts for games with transferable utilities were studied: the imputation set, core, domination core, and stable sets. In this chapter, a one-point (single-valued) solution concept is discussed: the Shapley value. It may again be helpful to first study the relevant parts of Chaps. 1 and 9.

Section 17.1 introduces the Shapley value by several formulas and presents (a variation on) Shapley's axiomatic characterization using additivity. In Sect. 17.2 we present three other characterizations of the Shapley value: a description in terms of Harsanyi dividends; an axiomatic characterization of Young based on strong monotonicity; and Owen's formula for the Shapley value based on a multilinear extension of games. Section 11.3 discusses Hart and Mas-Colell's approach to the Shapley value based on potential and reduced games.

17.1 Definition and Shapley's Characterization

The Shapley value is one of the most interesting solution concepts in cooperative game theory. The seminal paper [121] is the starting point of a large literature on this solution concept and related concepts. For overviews see [109] and Chaps. 53 and 54 in [6].

Let (N, v) be a TU-game and let $\sigma : N \to N$ be a permutation of the player set. Imagine that the players enter a room one by one in the ordering $\sigma(1)$, $\sigma(2), \ldots,$ $\sigma(n)$ and give each player the marginal contribution he creates in the game. To be more specific, let the *set of predecessors of i in σ* be the coalition

$$P_\sigma(i) := \{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\}.$$

For example, if $N = \{1,2,3,4,5\}$ and $\sigma(1) = 2$, $\sigma(2) = 5$, $\sigma(3) = 4$, $\sigma(4) = 1$, and $\sigma(5) = 3$, player 2 enters first, next players 5, 4, 1, and 3. So $P_\sigma(1) = \{2,5,4\}$.

Define the *marginal vector* m^σ by

$$m_i^\sigma = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)). \qquad (17.1)$$

Thus, the marginal vector m^σ gives each player his marginal contribution to the coalition formed by his entrance, according to the ordering σ.[1]

Definition 17.1. The *Shapley value* $\Phi(v)$ of a game (N,v) is the average of the marginal vectors of the game, i.e.

$$\Phi(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma. \tag{17.2}$$

(Here $\Pi(N)$ denotes the set of permutations of N.)

Example 17.2. (1) For a two-person game (N,v) the Shapley value is

$$\Phi(v) = \left(v(1) + \frac{v(N) - v(1) - v(2)}{2}, v(2) + \frac{v(N) - v(1) - v(2)}{2} \right).$$

(2) Let (N,v) be the three-person game with $v(1) = v(2) = v(3) = 0$, $v(1,2) = 4$, $v(1,3) = 7$, $v(2,3) = 15$, $v(1,2,3) = 20$. Then the marginal vectors are given in the Table 17.1. The Shapley value of this game is equal to $\frac{1}{6}(21,45,54)$, as one easily obtains from this table.

(3) The Shapley value $\Phi(v)$ for an additive game is equal to $(v(1), v(2), \ldots, v(n))$.

Based on (17.2), a probabilistic interpretation of the Shapley value is as follows. Suppose we draw from an urn, containing the elements of $\Pi(N)$, a permutation σ (probability $(n!)^{-1}$). Then let the players enter a room one by one in the order σ and give each player the marginal contribution created by him. Then the i-th coordinate $\Phi_i(v)$ of $\Phi(v)$ is the expected payoff to player i according to this random procedure.

Using (17.1) formula (17.2) can be rewritten as

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)). \tag{17.3}$$

The terms at the right hand side of the summation sign are of the form $v(S \cup \{i\}) - v(S)$, where S is a subset of N not containing i. For how many orderings does

Table 17.1 Example 17.2(2)

$(\sigma(1), \sigma(2), \sigma(3))$	m_1^σ	m_2^σ	m_3^σ
$(1,2,3)$	0	4	16
$(1,3,2)$	0	13	7
$(2,1,3)$	4	0	16
$(2,3,1)$	5	0	15
$(3,1,2)$	7	13	0
$(3,2,1)$	5	15	0
Σ	21	45	54

[1] Of course, m^σ depends on the game v.

one have $P_\sigma(i) = S$? The answer is $|S|!(n-1-|S|)!$, where the first factor $|S|!$ corresponds to the number of orderings of S and the second factor $(n-1-|S|)!$ to the number of orderings of $N \setminus (S \cup \{i\})$. Hence, (17.3) can be rewritten to obtain

$$\Phi_i(v) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!}(v(S \cup \{i\}) - v(S)). \tag{17.4}$$

Note that

$$\frac{|S|!(n-1-|S|)!}{n!} = \frac{1}{n}\binom{n-1}{|S|}^{-1}.$$

This gives rise to a second probabilistic interpretation of the Shapley value. Construct a subset S to which i does not belong, as follows. First, draw at random a number from the urn containing the numbers (possible sizes) $0, 1, 2, \ldots, n-1$, where each number has probability n^{-1} to be drawn. If size s is chosen, draw a set from the urn containing the subsets of $N \setminus \{i\}$ of size s, where each set has the same probability $\binom{n-1}{s}^{-1}$ to be drawn. If S is drawn with $|S| = s$, then pay player i the amount $v(S \cup \{i\}) - v(S)$. Then, in view of (17.4), the expected payoff for player i in this random procedure is the Shapley value for player i of the game (N, v).

Shapley [121] gave an axiomatic characterization of the Shapley value. That is, he formulated a number of properties that a one-point solution should (or might) have and then showed that the Shapley value is the only solution with these properties. This characterization – in a somewhat different form – is the next subject of this section.

Definition 17.3. A *value* on $\mathcal{G}^N$ is a map $\psi : \mathcal{G}^N \to \mathbb{R}^N$.[2]

The following axioms for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are used in the announced characterization of the Shapley value.

Efficiency (EFF): $\sum_{i=1}^n \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.

The efficiency (sometimes called Pareto optimality or Pareto efficiency) axiom needs no further explanation.

Call a player i in a game (N, v) a *null-player* if $v(S \cup i) - v(S) = 0$ for every coalition $S \in 2^N$. Such a player does not contribute anything to any coalition, in particular also $v(i) = 0$. So it seems reasonable that such a player obtains zero according to the value. This is what the following axiom requires.

Null-player Property (NP): $\psi_i(v) = 0$ for all $v \in \mathcal{G}^N$ and all a null-players i in v.

Call players i and j *symmetric* in the game (N, v) if $v(S \cup i) = v(S \cup j)$ for every coalition $S \subseteq N \setminus \{i, j\}$. Symmetric players have the same contribution to any coalition, and therefore it seems reasonable that they should obtain the same payoff according to the value. That is the content of the following axiom.

Symmetry (SYM): $\psi_i(v) = \psi_j(v)$ for all $v \in \mathcal{G}^N$ and all symmetric players i and j in v.

[2] Occasionally, also the word *solution* will be used.

The last axiom needed in the announced characterization can be interpreted as follows. Suppose the game (N,v) is played today and the game (N,w) tomorrow. If the value ψ is applied then player i obtains in total: $\psi_i(N,v)+\psi_i(N,w)$. One may also argue that, in total, the game $(N,v+w)$ is played and that, accordingly, player i should obtain $\psi_i(N,v+w)$. The following axiom expresses the possible point of view that these two evaluations should not make a difference.

Additivity (ADD): $\psi(v+w)=\psi(v)+\psi(w)$ for all $v,w\in\mathscr{G}^N$.

The announced characterization is the following theorem.

Theorem 17.4. *Let $\psi:\mathscr{G}^N\to\mathbb{R}^N$ be a value. Then ψ satisfies EFF, NP, SYM, and ADD, if, and only if, ψ is the Shapley value Φ.*

The proof of Theorem 17.4 uses, through the additivity axiom, the fact that $\mathscr{G}^N$ is a linear space, with addition defined by $(v+w)(S)=v(S)+w(S)$ and scalar multiplication $(\alpha v)(S)=\alpha v(S)$ for all $v,w\in\mathscr{G}^N$, $S\in 2^N$, and $\alpha\in\mathbb{R}$. An obvious basis for $\mathscr{G}^N$ is the set $\{1_T\in\mathscr{G}^N\mid T\in 2^N\setminus\{\emptyset\}\}$, where 1_T is the game defined by $1_T(T)=1$ and $1_T(S)=0$ for all $S\neq T$ (cf. Problem 17.1). This basis is not very well suited for the present purpose because the Shapley value $\Phi(1_T)$ $(T\in 2^N\setminus\{\emptyset\})$ cannot easily be determined from the axioms in Theorem 17.4: for example, there is no null-player in the game 1_T (cf. Problem 17.1).

Another basis is the collection of unanimity games $\{u_T\in\mathscr{G}^N\mid T\in 2^N\setminus\{\emptyset\}\}$, see Example 16.10(3) for the definition, and Problem 17.2. This basis is used in the following proof.

Proof of Theorem 17.4. That the Shapley value satisfies the four axioms in the theorem is the subject of Problem 17.3.

Conversely, suppose ψ satisfies the four axioms. It has to be proved that $\psi=\Phi$. Take $v\in\mathscr{G}^N$. Then there are unique numbers c_T $(T\neq\emptyset)$ such that $v=\sum_{T\neq\emptyset}c_Tu_T$ (cf. Problem 17.2). By ADD of ψ and Φ it follows that

$$\psi(v)=\sum_{T\neq\emptyset}\psi(c_Tu_T),\quad \Phi(v)=\sum_{T\neq\emptyset}\Phi(c_Tu_T).$$

So it is sufficient to show that for all $T\neq\emptyset$ and $c\in\mathbb{R}$:

$$\psi(cu_T)=\Phi(cu_T). \tag{17.5}$$

Take $T\neq\emptyset$ and $c\in\mathbb{R}$. Note first that for all $i\in N\setminus T$:

$$cu_T(S\cup\{i\})-cu_T(S)=0 \quad \text{for all } S,$$

implying that i is a null-player in cu_T. So, by NP of ψ and Φ:

$$\psi_i(cu_T)=\Phi_i(cu_T)=0 \quad \text{for all } i\in N\setminus T. \tag{17.6}$$

Now suppose that $i,j\in T$, $i\neq j$. Then, for every coalition $S\subseteq N\setminus\{i,j\}$, $cu_T(S\cup i)=cu_T(S\cup j)=0$, which implies that i and j are symmetric in cu_T. Hence, by SYM of ψ and Φ:

$$\Phi_i(cu_T) = \Phi_j(cu_T) \quad \text{for all } i, j \in T \tag{17.7}$$

and similarly

$$\psi_i(cu_T) = \psi_j(cu_T) \quad \text{for all } i, j \in T. \tag{17.8}$$

Then EFF, (17.6), (17.7), and (17.8) imply that

$$\psi_i(cu_T) = \Phi_i(cu_T) = |T|^{-1}c \quad \text{for all } i \in T. \tag{17.9}$$

Now (17.6) and (17.9) imply (17.5). □

The following two axioms (for a value ψ) are stronger versions of the null-player property and symmetry, respectively. Call a player i in a game (N, v) a *dummy player* if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus \{i\}$.

Dummy player Property (DUM): $\psi_i(v) = v(i)$ for all $v \in \mathcal{G}^N$ and all a dummy players i in v.

A dummy player only contributes his own worth to every coalition, and that is what he should be payed according to the dummy property.

For a permutation $\sigma \in \Pi(N)$ define the game v^σ by

$$\begin{aligned} v^\sigma(\sigma(U)) &:= v(U) \quad \text{for all } U \in 2^N \text{ or} \\ v^\sigma(S) &= v(\sigma^{-1}(S)) \quad \text{for all } S \in 2^N \end{aligned}$$

and define $\sigma^* : \mathbb{R}^N \to \mathbb{R}^N$ by $(\sigma^*(x))_{\sigma(k)} := x_k$ for all $x \in \mathbb{R}^N$ and $k \in N$.

Anonymity (AN): $\psi(v^\sigma) = \sigma^*(\psi(v))$ for all $v \in \mathcal{G}^N$ and all $\sigma \in \Pi(N)$.

Anonymity implies that a value does not discriminate between the players solely on the basis of their 'names' (i.e., numbers).

The dummy player property implies the null-player property, and anonymity implies symmetry. The Shapley value has the dummy player property. See Problem 17.4 for these claims.

The Shapley value is also anonymous.

Lemma 17.5. *The Shapley value Φ is anonymous.*

Proof. (1) First we show that

$$\rho^*(m^\sigma(v)) = m^{\rho\sigma}(v^\rho) \quad \text{for all } v \in \mathcal{G}^N, \rho, \sigma \in \pi(N).$$

This follows because for all $i \in N$:

$$\begin{aligned} m^{\rho\sigma}(v^\rho)_{\rho\sigma(i)} &= v^\rho(\{\rho\sigma(1), \ldots, \rho\sigma(i)\} - v^\rho(\{\rho\sigma(1), \ldots, \rho\sigma(i-1)\}) \\ &= v(\{\sigma(1), \ldots, \sigma(i)\}) - v(\{\sigma(1), \ldots, \sigma(i-1)\}) \\ &= (m^\sigma(v))_{\sigma(i)} = (\rho^*(m^\sigma(v)))_{\rho\sigma(i)}. \end{aligned}$$

(2) Take $v \in \mathscr{G}^N$ and $\rho \in \Pi(N)$. Then (1), the fact that $\rho \mapsto \rho\sigma$ is a surjection on $\Pi(N)$ and the linearity of ρ^* imply

$$\begin{aligned}\Phi(v^\rho) &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma(v^\sigma) = \frac{1}{n!}\sum_\sigma m^{\rho\sigma}(v^\rho)\\ &= \frac{1}{n!}\sum_\sigma \rho^*(m^\sigma(v)) = \rho^*(\frac{1}{n!}\sum_\sigma m^\sigma) = \rho^*(\Phi(v)).\end{aligned}$$

This proves the anonymity of Φ. □

As is clear by now the Shapley value has many appealing properties. In the following sections more properties and characterizations of the Shapley value are considered. However, it also has some drawbacks. In a balanced game the Shapley value does not necessarily assign a core element. Also, it does not have to be individually rational (cf. Problem 17.5).

Variations of the Shapley value, obtained by omitting the symmetry (or anonymity) requirement in particular, are discussed in Chap. 18.

17.2 Other Characterizations

In this section three other characterizations of the Shapley value are discussed: the dividend approach, the axiomatization of Young [148] based on strong monotonicity, and the multilinear approach of Owen [97].

17.2.1 Dividends

Harsanyi [48] has introduced the following concept.

Definition 17.6. Let (N,v) be game. For each coalition T the *dividend* $\Delta_v(T)$ is defined, recursively, as follows.

$$\begin{aligned}\Delta_v(\emptyset) &:= 0\\ \Delta_v(T) &:= v(T) - \sum_{S:S\subsetneq T} \Delta_v(S) \text{ if } |T| \geq 1.\end{aligned}$$

The relation between dividends and the Shapley value is described in the next theorem. The Shapley value of a player in a game turns out to be the sum of all equally distributed dividends of coalitions to which the player belongs.

Theorem 17.7. *Let* $v = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T u_T$ *(as in Problem 17.2). Then:*

(1) $\Delta_v(T) = c_T$ *for all* $T \neq \emptyset$.

(2) The Shapley value $\Phi_i(v)$ for player i is equal to the sum of the equally distributed dividends of the coalitions to which player i belongs i.e.,

$$\Phi_i(v) = \sum_{T:i\in T} \frac{\Delta_v(T)}{|T|}.$$

Proof. In the proof of Theorem 17.4 it was shown that $\Phi(c_T u_T) = |T|^{-1} c_T \mathbf{e}^T$ for each T, so by ADD, $\Phi(v) = \sum_{T\neq\emptyset} c_T |T|^{-1}\mathbf{e}^T$.

Hence, $\Phi_i(v) = \sum_{T:i\in T} c_T |T|^{-1}$. The only thing left to show is that

$$c_T = \Delta_v(T) \quad \text{for all } T \neq \emptyset. \tag{17.10}$$

The proof of this is done by induction. If $|T| = 1$, say $T = \{i\}$, then $c_T = v(i) = \Delta_v(T)$. Suppose (17.10) holds for all $S \subsetneqq T$. Then $\Delta_v(T) = v(T) - \sum_{S\subsetneqq T} \Delta_v(S) = v(T) - \sum_{S\subsetneqq T} c_S = c_T$ because $v(T) = \sum_{S\subseteq T} c_S$. □

The concept of coalitional dividend is important not only for the Shapley value but for a wider range of solution concepts. See [30].

17.2.2 Strong Monotonicity

The Shapley value obviously has the property that if a player contributes at least as much to any coalition in a game v than in a game w, then his payoff from the Shapley value in v is at least as large as that in w. Formally, the Shapley value satisfies the following axiom for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$; a proof is immediate from (17.4).

Strong Monotonicity (SMON): $\psi_i(v) \geq \psi_i(w)$ for all $v, w \in \mathcal{G}^N$ that satisfy

$$v(S\cup\{i\}) - v(S) \geq w(S\cup\{i\}) - w(S) \quad \text{for all } S \in 2^N.$$

Young [148] has proved that together with efficiency and symmetry this axiom characterizes the Shapley value.

Theorem 17.8. *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. Then ψ satisfies EFF, SYM, and SMON, if and only if ψ is the Shapley value Φ.*

Proof. Obviously, Φ satisfies the three axioms. Conversely, suppose ψ satisfies the three axioms:

(1) Let z be the game that is identically zero. In this game, all players are symmetric, so SYM and EFF together imply $\psi(z) = 0$.
(2) Let i be a null-player in a game v. Then the condition in SMON applies to z and v with all inequalities being equalities. So SMON yields $\psi_i(v) \geq \psi_i(z)$ and $\psi_i(z) \geq \psi_i(v)$. Hence by (1), $\psi_i(v) = \mathbf{0}$.
(3) Let $c \in \mathbb{R}$ and $T \in 2^N \setminus \{\emptyset\}$. Then (2) implies $\psi_i(cu_T) = 0$ for every $i \in N\setminus T$. This, SYM, and EFF imply $\psi_i(cu_T) = c|T|^{-1}$ for every $i \in T$. Hence, $\psi(cu_T) = c|T|^{-1}\mathbf{e}^T$.

(4) Each $v \in \mathscr{G}^N$ can be written in a unique way as a linear combination of $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ (see Problem 17.2). So v is of the form $\sum c_T u_T$. The proof of $\psi(v) = \Phi(v)$ will be completed by induction on the number $\alpha(v)$ of terms in $\sum c_T u_T$ with $c_T \neq 0$.

From (1), $\psi(v) = \Phi(v) = \mathbf{0}$ if $\alpha(v) = 0$, and from (3), $\psi(v) = \Phi(v)$ if $\alpha(v) = 1$ because $\Phi(cu_T) = c|T|^{-1}\mathbf{e}^T$. Suppose $\psi(w) = \Phi(w)$ for all $w \in \mathscr{G}^N$ with $\alpha(w) < k$, where $k \geq 2$. Let v be a game with $\alpha(v) = k$. Then there are coalitions $T_1, T_2, \ldots,$ T_k and real numbers $c_1, c_2, \ldots, c_k$, unequal to zero, such that $v = \sum_{r=1}^k c_r u_{T_r}$. Let $D := \cap_{r=1}^k T_r$.

For $i \in N \setminus D$, define $w^i := \sum_{r:i \in T_r} c_r u_{T_r}$. Because $\alpha(w^i) < k$, the induction hypothesis implies: $\psi_i(w^i) = \Phi_i(w^i)$. Further, for every $S \in 2^N$:

$$\begin{aligned} v(S \cup i) - v(S) &= \sum_{r=1}^k c_r u_{T_r}(S \cup i) - \sum_{r=1}^k c_r u_{T_r}(S) \\ &= \sum_{r:i \in T_r} c_r u_{T_r}(S \cup i) - \sum_{r:i \in T_r} c_r u_{T_r}(S) \\ &= w^i(S \cup i) - w^i(S), \end{aligned}$$

so that, by SMON of ψ and Φ, it follows that $\psi_i(v) = \psi_i(w^i) = \Phi_i(w^i) = \Phi_i(v)$. So

$$\psi_i(v) = \Phi_i(v) \quad \text{for all } i \in N \setminus D. \tag{17.11}$$

(17.11) and EFF for ψ and Φ yield

$$\sum_{i \in D} \psi_i(v) = \sum_{i \in D} \Phi_i(v). \tag{17.12}$$

Let $i, j \in D$, then for every $S \subseteq N \setminus \{i, j\}$:

$$(0 =)\ v(S \cup i) = \sum_{r=1}^k c_r u_{T_r}(S \cup i) = \sum_{r=1}^k c_r u_{T_r}(S \cup j) = v(S \cup j),$$

so i and j are symmetric. Hence, by SYM of ψ and Φ:

$$\psi_i(v) = \psi_j(v), \quad \Phi_i(v) = \Phi_j(v). \tag{17.13}$$

Now $\psi(v) = \Phi(v)$ follows from (17.11), (17.12) and (17.13). □

17.2.3 Multilinear Extension

The Shapley value of a game may also be described by means of the multilinear extension of a game (cf. [97, 98]). Let (N, v) be game. Consider the function $f : [0,1]^N \to \mathbb{R}$ on the hypercube $[0,1]^N$, defined by

$$f(x_1,x_2,\ldots,x_n) = \sum_{S\in 2^N}\left(\prod_{i\in S} x_i \prod_{i\in N\setminus S}(1-x_i)\right) v(S). \tag{17.14}$$

Observe that the set of extreme points of $[0,1]^N$, $\text{ext}([0,1]^N)$, is equal to $\{\mathbf{e}^S \mid S \in 2^N\}$. By Problem 17.8(1),

$$f(\mathbf{e}^S) = v(S) \quad \text{for each } S \in 2^N. \tag{17.15}$$

So f can be seen as an extension of $\tilde{v}$: $\text{ext}([0,1]^N) \to \mathbb{R}$ with $\tilde{v}(\mathbf{e}^S) := v(S)$. In view of Problem 17.8, f is called the *multilinear extension* of ($\tilde{v}$ or) v.

One can give a probabilistic interpretation of $f(\mathbf{x})$. Suppose that each of the players $i \in N$, independently, decides whether to cooperate (probability x_i) or not (probability $1-x_i$). So with probability $\prod_{i\in S} x_i \prod_{i\in N\setminus S}(1-x_i)$ the coalition S forms, which has worth $v(S)$. Then $f(\mathbf{x})$ as given in (17.14) can be seen as the expectation of the worth of the formed coalition.

Another interpretation is to see $\mathbf{x} \in [0,1]^N$ as a fuzzy set, where x_i is the intensity of availability of player i and to see f as a characteristic function, defined for fuzzy coalitions in N.

Denote by $D_k f(\mathbf{x})$ the derivative of f w.r.t. the k-th coordinate in $\mathbf{x}$. The following result [97] provides another description of the Shapley value, as the integral along the main diagonal of $[0,1]^N$ of $D_k f$.

Theorem 17.9. *$\Phi_k(v) = \int_0^1 (D_k f)(t,t,\ldots,t)dt$ for each $k \in N$.*

Proof.

$$\begin{aligned} D_k f(\mathbf{x}) &= \sum_{T:k\in T}\left[\prod_{i\in T\setminus\{k\}} x_i \prod_{i\in N\setminus T}(1-x_i)\right] v(T) \\ &\quad - \sum_{S:k\notin S}\left[\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})}(1-x_i)\right] v(S) \\ &= \sum_{S:k\notin S}\left[\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})}(1-x_i)\right](v(S\cup\{k\})-v(S)). \end{aligned}$$

Hence, $\int_0^1 (D_k f)(t,t,\ldots,t)dt = \sum_{S:k\notin S}\left(\int_0^1 t^{|S|}(1-t)^{n-|S|-1}dt\right)(v(S\cup\{k\})-v(S))$.

Using the well-known (beta-)integral formula

$$\int_0^1 t^{|S|}(1-t)^{n-|S|-1}dt = \frac{|S|!(n-|S|-1)!}{n!}$$

it follows that $\int_0^1 (D_k f)(t,t,\ldots,t)dt = \sum_{S:k\notin S}\frac{|S|!(n-|S|-1)!}{n!}(v(S\cup\{k\})-v(S)) = \Phi_k(v)$ by (17.4). $\square$

Example 17.10. Let (N,v) be the three-person game with $v(1) = v(2) = v(3) = v(1,2) = 0$, $v(1,3) = 1$, $v(2,3) = 2$, $v(N) = v(1,2,3) = 4$. Then $f(x_1,x_2,x_3) = x_1(1-x_2)x_3 + 2(1-x_1)\,x_2x_3 + 4x_1x_2x_3 = x_1x_3 + 2x_2x_3 + x_1x_2x_3$ for all $x \in [0,1]^N$.

So $D_1 f(\mathbf{x}) = x_3 + x_2x_3$, $D_2 f(\mathbf{x}) = 2x_3 + x_1x_3$, $D_3 f(\mathbf{x}) = x_1 + 2x_2 + x_1x_2$. Theorem 17.9 implies

$$\Phi_1(v) = \int_0^1 D_1 f(t,t,t)dt = \int_0^1 (t+t^2)dt = \frac{5}{6},$$
$$\Phi_2(v) = \int_0^1 (2t+t^2)dt = 1\frac{1}{3},\ \Phi_3(v) = \int_0^1 (3t+t^2)dt = 1\frac{5}{6}.$$

17.3 Potential and Reduced Game

This section starts off with discussing the potential approach to the Shapley value. The potential is, in a sense, dual to the concept of dividends. Next, reduced games are considered, which leads to another axiomatic characterization of the Shapley value. This section is based on [52].

17.3.1 *The Potential Approach to the Shapley Value*

Denote by $\mathscr{G}$ is the family of all games (N,v) with an arbitrary finite (player) set N (not necessarily the set of the first n natural numbers). It is convenient to include also the game $(\emptyset, v)$, with empty player set. Thus,

$$\mathscr{G} = \bigcup_{N \subseteq \mathbb{N},\ |N|<\infty} \mathscr{G}^N.$$

Definition 17.11. A *potential* (Hart and Mas-Colell [52]) is a function $P : \mathscr{G} \to \mathbb{R}$ satisfying

$$P(\emptyset, v) = 0 \tag{17.16}$$
$$\sum_{i\in N} D_i P(N,v) = v(N) \quad \text{for all } (N,v) \in \mathscr{G}. \tag{17.17}$$

Here $D_iP(N,v) := P(N,v) - P(N \setminus \{i\}, v)$ with v the restriction to $N \setminus \{i\}$ in the last expression.

If P is a potential then (17.17) says that the 'gradient' $\operatorname{grad} P(N,v) := (D_iP(N,v))_{i\in N}$ is an efficient payoff vector for the game (N,v).

Note that

$$P(\{i\},v) = v(\{i\})$$
$$P(\{i,j\},v) = \frac{1}{2}(v(\{i,j\}) + v(\{i\}) + v(\{j\}))$$

if P is a potential. More generally, it follows from (17.17) that

$$P(N,v) = |N|^{-1}(v(N) + \sum_{i \in N} P(N \setminus \{i\}, v)). \tag{17.18}$$

So the potential of (N,v) is uniquely determined by the potential of subgames of (N,v), which implies with (17.16) the first assertion in Theorem 17.12 below. The second assertion in this theorem connects the potential of a game to the Harsanyi dividends (see Sect. 17.2.1). Further, the gradient of P in (N,v) is equal to the Shapley value of (N,v). It also provides an algorithm to calculate the Shapley value, by calculating with (17.16) and (17.18) the potentials of the game and its subgames and then using Theorem 17.12(3).

Theorem 17.12. (1) *There is a unique potential* $P : \mathscr{G} \to \mathbb{R}$.
(2) $P(N,v) = \sum_{\emptyset \neq T \subseteq N} c_T |T|^{-1}$ *for* $v = \sum_{\emptyset \neq T \subseteq N} c_T u_T$.
(3) grad $P(N,v) = \Phi(N,v)$.

Proof. (1) follows immediately from (17.16) and (17.18).
(2) and (3). Let $Q : \mathscr{G} \to \mathbb{R}$ be defined by $Q(\emptyset, v) := 0$ and $Q(N,v) := \sum_{T \in 2^N \setminus \{\emptyset\}} c_T |T|^{-1}$ for all (N,v), where $v = \sum c_T u_T$, if $N \neq \emptyset$. Further, for each (N,v) and $i \in N$

$$D_i Q(N,v) = \sum_{\emptyset \neq T \subseteq N} c_T |T|^{-1} - \sum_{\emptyset \neq T \subseteq N \setminus \{i\}} c'_T |T|^{-1} \tag{17.19}$$

if $v = \sum_{T \subseteq N} c_T u_T$ and $v' = \sum_{T \subseteq N \setminus \{i\}} c'_T u_T$ where v' is the restriction of v to $2^{N \setminus \{i\}}$. Since $v'(S) = \sum_{T \subseteq N \setminus \{i\}} c'_T u_T(S)$ for all $S \subseteq N \setminus \{i\}$ and $\{u_T \mid T \subseteq N \setminus \{i\}\}$ is a basis of the linear space of games with player set $N \setminus \{i\}$, we obtain $c_T = c'_T$ for all $T \subseteq N \setminus \{i\}$. But then (17.19) and Theorem 17.7 imply

$$D_i Q(N,v) = \sum_{T:\, i \in T} c_T |T|^{-1} = \Phi_i(N,v).$$

Further, by efficiency (EFF) of Φ:

$$\sum_{i \in N} D_i Q(N,v) = \sum_{i \in N} \Phi_i(N,v) = v(N).$$

So Q is a potential. From (1) it follows that $Q = P$ and then (3) holds. □

The potential of a game can also be expressed directly in terms of the coalitional worths, as in the following theorem.

Theorem 17.13. *For each game* $(N,v) \in \mathscr{G}$*:*

$$P(N,v) = \sum_{S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} v(S).$$

Proof. Let for each $(N,v) \in \mathscr{G}$,

$$Q(N,v) := \sum_{S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} v(S).$$

Then $Q(\emptyset, v) = 0$. It is sufficient for the proof of $Q(N,v) = P(N,v)$, to show that $D_i Q(N,v) = \Phi_i(N,v)$ for all $i \in N$. Now

$$\begin{aligned}
D_i Q(N,v) &= Q(N,v) - Q(N \setminus \{i\}, v) \\
&= \sum_{T \subseteq N} \frac{(|T|-1)!(|N|-|T|)!}{|N|!} v(T) \\
&\quad - \sum_{S \subseteq N \setminus \{i\}} \frac{(|S|-1)!(|N|-1-|S|)!}{(|N|-1)!} v(S) \\
&= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} v(S \cup \{i\}) \\
&\quad + \sum_{S \subseteq N \setminus \{i\}} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} v(S) \\
&\quad - \sum_{S \subseteq N \setminus \{i\}} \frac{(|S|-1)!(|N|-1-|S|)!}{(|N|-1)!} v(S) \\
&= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S \cup \{i\}) - v(S)) = \Phi_i(N,v)
\end{aligned}$$

where the last equality follows from (17.4). □

A probabilistic interpretation of Theorem 17.13 is the following. The number $|N|^{-1} P(N,v)$ is the expectation of the normalized worth $|S|^{-1} v(S)$ of the formed coalition $S \subseteq N$ if the probability that S forms is $|N|^{-1} \binom{|N|}{|S|}^{-1}$ (corresponding to drawing first a size $s \in \{1, 2, \ldots, |N|\}$ and then a set S with $|S| = s$).

Theorem 17.12 can be used to calculate the Shapley value, as the following example shows.

Example 17.14. Consider the three-person game (N,v) given in Table 17.2. The dividends of the subcoalitions and the potential of the subgames are given in lines 3 and 4 of this table, respectively. It follows that

Table 17.2 Example 17.14

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1	2	3	5	6	9	15
$\Delta_v(S)$	0	1	2	3	2	2	4	1
$\Delta_v(S)/\|S\|$	0	1	2	3	1	1	2	$\frac{1}{3}$
$P(S,v)$	0	1	2	3	4	5	7	$10\frac{1}{3}$

$$\Phi(N,v) = (3\tfrac{1}{3}, 5\tfrac{1}{3}, 6\tfrac{1}{3})$$

$$\begin{aligned}\Phi(N,v) &= (D_1P(N,v), D_2P(N,v), D_3P(N,v)) \\ &= (10\frac{1}{3}-7, 10\frac{1}{3}-5, 10\frac{1}{3}-4) = (3\frac{1}{3}, 5\frac{1}{3}, 6\frac{1}{3}) \\ \Phi(\{2,3\},v) &= (7-3, 7-2) = (4,5).\end{aligned}$$

17.3.2 Reduced Games

To introduce the concept of a reduced game, consider the game in Example 17.14. Suppose that the players agree on using the Shapley value, and consider the coalition $\{1,2\}$. If players 1 and 2 pool their Shapley value payoffs then together they have $3\frac{1}{3}+5\frac{1}{3}=8\frac{2}{3}$. Another way to obtain this amount is to take the worth of the grand coalition, 15, and to subtract player 3's payoff, $6\frac{1}{3}$. Consider $\{1\}$ as a subcoalition of $\{1,2\}$. Player 1 could form a coalition with player 3 and obtain the worth 6, but he would have to pay player 3 according to the Shapley value of the two-player game $(\{1,3\},v)$, which is the vector $(2,4)$; recall that the players agree on using the Shapley value. So player 1 is left with $6-4=2$. Similarly, player 2 could form a coalition with player 3 and obtain $v(2,3)=9$ minus the Shapley value payoff for player 3 in the game $(\{2,3\},v)$, which is 5. So player 2 is left with $9-5=4$. So a 'reduced' game $(\{1,2\},\tilde{v})$ has been constructed with $\tilde{v}(1)=2$, $\tilde{v}(2)=4$, and $\tilde{v}(1,2)=8\frac{2}{3}$. The Shapley value of this game is the vector $(3\frac{1}{3},5\frac{1}{3})$. Observe that these payoffs are equal to the Shapley value payoffs in the original game. This is not a coincidence; the particular way of constructing a reduced game as illustrated here leaves the Shapley value invariant.

A general game theoretic principle is the following. Suppose in a game a subset of the players consider the game arising among themselves; then, if they apply the same 'solution rule' as in the original game, their payoffs should not change – they should have no reason to renegotiate. Of course, the formulation of this principle leaves open many ways to define 'the game arising among themselves'. Different definitions correspond to different solution rules. Put differently, there are many ways to define reduced games, leading to many different 'reduced game properties' as specific instances of the above general game theoretic principle. Instead of 'reduced game property' also the term 'consistency' is used. This concept has been very fruitful over the past decades – in cooperative as well as noncooperative game theory.[3]

For the Shapley value the following reduced game turns out to be relevant. It is the reduced game applied in the example above.

Definition 17.15. Let ψ be a value, assigning to each element (N,v) of $\mathscr{G}$ a vector $\psi(N,v)\in\mathbb{R}^N$. For $(N,v)\in\mathscr{G}$ and $U\subseteq N$, $U\neq\emptyset$, the game $(N\setminus U, v_{U,\psi})$ is defined by $v_{U,\psi}(\emptyset)=0$ and

$$v_{U,\psi}(S) := v(S\cup U) - \sum_{k\in U}\psi_k(S\cup U, v) \quad \text{for all } S\in 2^{N\setminus U}\setminus\{\emptyset\}.$$

$v_{U,\psi}$ is called the (U,ψ)-*reduced game* of v.

[3] Cf. Sect. 13.8.

Thus, the worth of coalition S in the game $v_{U,\psi}$ is obtained by subtracting from the worth of $S \cup U$ in the game $(S \cup U, v)$ the payoffs of the players in U according to the value ψ. The reduced game property or consistency property for a value ψ on $\mathscr{G}$ based on this reduced game is the following.

HM-consistency (HMC)[4]: for all games (N,v) and all $U \subseteq N$, $U \neq \emptyset$

$$\psi_i(N \setminus U, v_{U,\psi}) = \psi_i(N,v) \quad \text{for all } i \in N \setminus U. \tag{17.20}$$

The following lemma is used in the proof of HM-consistency of the Shapley value.

Lemma 17.16. *Let $(N,v) \in \mathscr{G}$. Suppose $Q : 2^N \to \mathbb{R}$ satisfies*

$$\sum_{i \in S} (Q(S) - Q(S \setminus \{i\})) = v(S) \quad \textit{for each } S \in 2^N \setminus \{\emptyset\}.$$

Then for each $S \in 2^N$:

$$Q(S) = P(S,v) + Q(\emptyset). \tag{17.21}$$

Proof. The proof of (17.21) is by induction on $|S|$. Obviously, (17.21) holds if $|S| = 0$. Take T with $|T| > 0$ and suppose (17.21) holds for all S with $|S| < |T|$. Then

$$\begin{aligned}
Q(T) &= |T|^{-1}(v(T) + \sum_{i \in T} Q(T \setminus \{i\})) \\
&= |T|^{-1}(v(T) + |T|Q(\emptyset) + \sum_{i \in T} P(T \setminus \{i\}, v)) \\
&= Q(\emptyset) + |T|^{-1}(v(T) + \sum_{i \in T} P(T \setminus \{i\}, v)) \\
&= Q(\emptyset) + P(T,v).
\end{aligned}$$

□

Lemma 17.17. *The Shapley value Φ is HM-consistent.*

Proof. Take (N,v) in $\mathscr{G}$, and $\emptyset \neq U \subseteq N$. We have to prove that

$$\Phi_i(N \setminus U, v_{U,\Phi}) = \Phi_i(N,v) \quad \text{for all } i \in N \setminus U. \tag{17.22}$$

Note that, in view of the definition of $v_{U,\Phi}$, efficiency of Φ, and Theorem 17.12(3), we have for all $S \subseteq N \setminus U$:

$$\begin{aligned}
v_{U,\Phi}(S) &= v(S \cup U) - \sum_{i \in U} \Phi_i(S \cup U, v) = \sum_{i \in S} \Phi_i(S \cup U, v) \\
&= \sum_{i \in S} P(S \cup U, v) - P((S \cup U) \setminus \{i\}, v).
\end{aligned} \tag{17.23}$$

[4] Named after Hart and Mas-Colell.

For $S \in 2^{N \setminus U}$ define $Q(S) := P(S \cup U, v)$. Then, by Lemma 17.16, with $N \setminus U$ and $v_{U,\Phi}$ in the roles of N and v, (17.23) implies

$$Q(S) = P(S, v_{U,\Phi}) + Q(\emptyset) = P(S, v_{U,\Phi}) + P(U, v)$$

for all $S \subseteq N \setminus U$. Hence, by the definition of Q:

$$P(S \cup U, v) = P(S, v_{U,\Phi}) + P(U, v). \tag{17.24}$$

(17.24) and Theorem 17.12(3) imply

$$\begin{aligned} \Phi_i(N \setminus U, v_{U,\Phi}) &= P(N \setminus U, v_{U,\Phi}) - P((N \setminus U) \setminus \{i\}, v_{U,\Phi}) \\ &= P(N, v) - P(N \setminus \{i\}, v) = \Phi_i(N, v). \end{aligned}$$

So (17.22) holds. □

Call a value ψ *standard for two-person games* if for all $(\{i, j\}, v)$

$$\begin{aligned} \psi_i(\{i, j\}, v) &= \frac{1}{2}(v(i, j) + v(i) - v(j)), \\ \psi_j(\{i, j\}, v) &= \frac{1}{2}(v(i, j) - v(i) + v(j)). \end{aligned}$$

Note that the Shapley value is standard for two-person games. It turns out that this property, together with HM-consistency, characterizes the Shapley value.

Theorem 17.18. *Let ψ be a value on $\mathcal{G}$. Then ψ is standard for two-person games and HM-consistent if, and only if, ψ is the Shapley value.*

Proof. (a) By Lemma 17.17, Φ is HM-consistent. Further, Φ is standard for two-person games.

(b) For the proof of the converse, assume ψ has the two properties in the theorem. It will first be proved that ψ is efficient. For two-person games this is true by standardness. Let $(\{i\}, v)$ be a one-person game. To prove that $\psi_i(\{i\}, v) = v(i)$, construct a two-person game $(\{i, j\}, v^*)$ (with $j \neq i$) by defining

$$v^*(i) := v(i), \quad v^*(j) := 0, \quad v^*(i, j) := v(i).$$

From the standardness of ψ it follows that

$$\psi_i((i, j), v^*) = v(i), \psi_j((i, j), v^*) = 0. \tag{17.25}$$

The definition of $v^*_{j,\psi}$ and (17.25) imply: $v^*_{j,\psi}(i) = v^*(i, j) - \psi_j((i, j),\, v^*) = v(i) - 0 = v(i)$. So

$$(\{i\}, v^*_{j,\psi}) = (\{i\}, v). \tag{17.26}$$

Then, by (17.26), the consistency of ψ and (17.25):

$$\psi_i(\{i\}, v) = \psi_i(\{i\}, v^*_{j,\psi}) = \psi_i(\{i, j\}, v^*) = v(i).$$

So for one- and two-person games the rule ψ is efficient. Take a game (S,v) with $|S| \geq 3$ and suppose that for all games (T,v) with $|T| < |S|$ the rule ψ is efficient. Take $k \in S$. Then the consistency of ψ implies

$$\begin{aligned}
\sum_{i\in S} \psi_i(S,v) &= \psi_k(S,v) + \sum_{i\in S\setminus\{k\}} \psi_i(S,v) \\
&= \psi_k(S,v) + \sum_{i\in S\setminus\{k\}} \psi_i(S\setminus\{k\}, v_{k,\psi}) \\
&= \psi_k(S,v) + v_{k,\psi}(S\setminus\{k\}) \\
&= v(S).
\end{aligned}$$

Hence, ψ is an efficient rule.

(c) If a $Q : \mathscr{G} \to \mathbb{R}$ can be constructed with $Q(\emptyset, v) = 0$ and

$$\psi_i(N,v) = Q(N,v) - Q(N\setminus\{i\}, v) \tag{17.27}$$

for all (N,v) in $\mathscr{G}$ and $i \in N$, then by (b) and Theorem 17.12(1), $Q = P$ and then $\psi_i(N,v) = \Phi_i(N,v)$ by Theorem 17.12(3).

Hence, the only thing to do is to construct such a Q. Start with $Q(\emptyset,v) := 0$, $Q(\{i\},v) := v(i)$, $Q(\{i,j\},v) := \frac{1}{2}(v(i,j) + v(i) + v(j))$ and continue in a recursive way as follows. Let $(N,v) \in \mathscr{G}$, $|N| \geq 3$ and suppose Q with property (17.27) has been defined already for games with less than $|N|$ players. Then one can define $Q(N,v) := \alpha$ if, and only if

$$\alpha - Q(N\setminus\{i\},v) = \psi_i(N,v) \quad \text{for all } i \in N.$$

This implies that the proof is complete if it can be shown that

$$\psi_i(N,v) + Q(N\setminus\{i\},v) = \psi_j(N,v) + Q(N\setminus\{j\},v) \quad \text{for all } i,j \in N. \tag{17.28}$$

To prove (17.28) take $k \in N\setminus\{i,j\} (|N| \geq 3)$. Then

$$\begin{aligned}
\psi_i(N,v) - \psi_j(N,v) &= \psi_i(N\setminus\{k\}, v_{k,\psi}) - \psi_j(N\setminus\{k\}, v_{k,\psi}) \\
&= Q(N\setminus\{k\}, v_{k,\psi}) - Q(N\setminus\{k,i\}, v_{k,\psi}) \\
&\quad - Q(N\setminus\{k\}, v_{k,\psi}) + Q(N\setminus\{k,j\}, v_{k,\psi}) \\
&= (-Q(N\setminus\{k,i\}, v_{k,\psi}) + Q(N\setminus\{i,j,k\}, v_{k,\psi})) \\
&\quad + Q(N\setminus\{k,j\}, v_{k,\psi}) - Q(N\setminus\{i,j,k\}, v_{k,\psi}) \\
&= -\psi_j(N\setminus\{k,i\}, v_{k,\psi}) + \psi_i(N\setminus\{k,j\}, v_{k,\psi}) \\
&= -\psi_j(N\setminus\{i\}, v) + \psi_i(N\setminus\{j\}, v) \\
&= -Q(N\setminus\{i\}, v) + Q(N\setminus\{i,j\}, v) \\
&\quad + Q(N\setminus\{j\}, v) - Q(N\setminus\{i,j\}, v) \\
&= Q(N\setminus\{j\}, v) - Q(N\setminus\{i\}, v),
\end{aligned}$$

where HM-consistency of ψ is used in the first and fifth equality and (17.27) for subgames in the second, fourth and sixth equality.

This proves (17.28). □

Problems

17.1. *The Games* 1_T

(a) Show that $\{1_T \in \mathcal{G}^N \mid T \in 2^N \setminus \{\emptyset\}\}$ is a basis for $\mathcal{G}^N$.

(b) Show that there is no null-player in a game $(N, 1_T)$ $(T \neq \emptyset)$.

(c) Determine the Shapley value $\Phi(N, 1_T)$.

17.2. *Unanimity Games*

(a) Prove that the collection of unanimity games $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ is a basis for $\mathcal{G}^N$. (Hint: In view of Problem 17.1(a) it is sufficient to show linear independence.)

(b) Prove that for each game $v \in \mathcal{G}^N$:

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T \qquad \text{with} \qquad c_T = \sum_{S : S \subseteq T} (-1)^{|T|-|S|} v(S).$$

17.3. *Necessity of the Axioms in Theorem 17.4*

Show that the Shapley value satisfies EFF, NP, SYM, and ADD.

17.4. *Dummy Player Property and Anonymity*

Show that DUM implies NP, and that AN implies SYM, but that the converses of these implications do not hold. Show that the Shapley value has the dummy player property.

17.5. *Shapley Value, Core, and Imputation Set*

Show that the Shapley value of a game does not have to be an element of the core or of the imputation set, even if these sets are non-empty. How about the case of two players?

17.6. *Shapley Value as a Projection*

The Shapley value $\Phi : \mathcal{G}^N \to \mathbb{R}^N$ can be seen as a map from $\mathcal{G}^N$ to the space A^N of additive games by identifying $\mathbb{R}^N$ with A^N.

Prove that $\Phi : \mathcal{G}^N \to A^N$ is a projection i.e. $\Phi \circ \Phi = \Phi$.

17.7. *Shapley Value of Dual Game*

The dual game (N, v^*) of a game (N, v) is defined by $v^*(S) = v(N) - v(N \setminus S)$ for every $S \subseteq N$. Prove that the Shapley value of v^* is equal to the Shapley value of v. (Hint: If $v = \sum \alpha_T u_T$, then $v^* = \sum \alpha_T u_T^*$.)

17.8. *Multilinear Extension*

(1) Prove (17.15).

(2) Show that f in (17.14) is a multilinear function. (A function $g : \mathbb{R}^n \to \mathbb{R}$ is called multilinear if g is of the form $g(\mathbf{x}) = \sum_{T \subseteq N} c_T \left(\prod_{i \in T} x_i\right)$ for arbitrary real numbers c_T.)

(3) Prove that f is the unique multilinear extension of $\tilde{v}$ to $[0,1]^N$.

17.9. *The Beta-Integral Formula*

Prove the beta-integral formula used in the proof of Theorem 17.9.

17.10. *Path Independence of* Φ

Let $(N,v) \in \mathscr{G}^N$ with $N = \{1,2,\ldots,n\}$. Prove that for each permutation $\tau : N \to N$ we have

$$\sum_{k=1}^{n} \Phi_{\tau(k)}(\{\tau(1),\tau(2),\ldots,\tau(k)\},v) = \sum_{k=1}^{n} \Phi_k(\{1,2,\ldots,k\},v).$$

17.11. *An Alternative Characterization of the Shapley Value*

Let ψ be a value on $\mathscr{G}$. Prove: $\psi = \Phi$ if and only if ψ has the following four properties:

(1) ψ is HM-consistent.

(2) ψ is efficient for two-person games.

(3) ψ is anonymous for two-person games.

(4) ψ is relative invariant w.r.t. strategic equivalence for two-person games. (This means: $(\psi_i(\tilde{v}),\psi_j(\tilde{v})) = \alpha(\psi_i(v),\psi_j(v)) + (\beta_i,\beta_j)$ whenever $\tilde{v}(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for every coalition S, where $\alpha > 0$ and $\beta_i,\beta_j \in \mathbb{R}$.)

Chapter 18
Core, Shapley Value, and Weber Set

In Chap. 17 we have seen that the Shapley value of a game does not have to be in the core of the game, nor even an imputation (Problem 17.5). In this chapter we introduce a set-valued extension of the Shapley value, the Weber set, and show that it always contains the core (Sect. 18.1). Next, we study so-called convex games and show that these are exactly those games for which the core and the Weber set coincide. Hence, for such games the Shapley value is an attractive core selection (Sect. 18.2). Finally, we study random order values (Sect. 18.3), which fill out the Weber set, and the subset of weighted Shapley values, which still cover the core (Sect. 18.4).

18.1 The Weber Set

Let $(N,v) \in \mathscr{G}^N$. Recall the definition of a marginal vector from Sect. 17.1.

Definition 18.1. The *Weber set* of a game $(N,v) \in \mathscr{G}^N$ is the convex hull of its marginal vectors:

$$W(v) := \mathrm{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\} .$$

Example 18.2. Consider the three-person game $(\{1,2,3\},v)$ defined by $v(12) = v(13) = 1$, $v(23) = -1$, $v(123) = 3$, and $v(i) = 0$ for every $i \in \{1,2,3\}$. The marginal vectors of this game, the core and the Weber set are given in Fig. 18.1.

We show now that the core is always a subset of the Weber set (Weber [146]; the proof is from Derks [29]).

Theorem 18.3. *Let $(N,v) \in \mathscr{G}^N$. Then $C(v) \subseteq W(v)$.*

Proof. Suppose there is an $\mathbf{x} \in C(v) \setminus W(v)$. By a separation theorem (Theorem 22.1), there exists a vector $\mathbf{y} \in \mathbb{R}^N$ such that $\mathbf{w} \cdot \mathbf{y} > \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{w} \in W(v)$. In particular,

$$m^\sigma \cdot \mathbf{y} > \mathbf{x} \cdot \mathbf{y} \quad \text{for every } \sigma \in \Pi(N) . \tag{18.1}$$

$(\sigma(1),\sigma(2),\sigma(3))$	m_1^σ	m_2^σ	m_3^σ
$(1,2,3)$	0	1	2
$(1,3,2)$	0	2	1
$(2,1,3)$	1	0	2
$(2,3,1)$	4	0	-1
$(3,1,2)$	1	2	0
$(3,2,1)$	4	-1	0

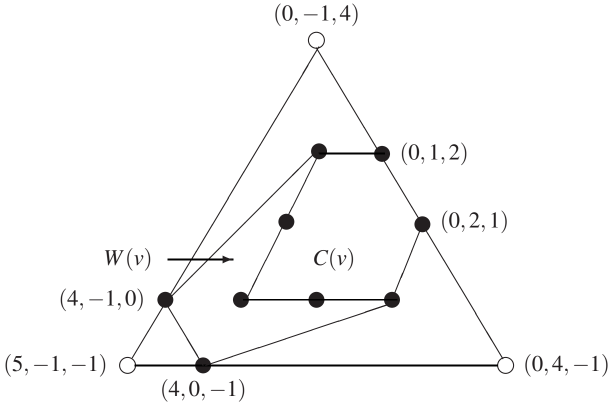


Fig. 18.1 Example 18.2. The core is the convex hull of the vectors $(3,0,0)$, $(1,2,0)$, $(0,2,1)$, $(0,1,2)$, and $(1,0,2)$. The Weber set is the convex hull of the six marginal vectors

Let $\pi \in \Pi(N)$ with $y_{\pi(1)} \geq y_{\pi(2)} \geq \cdots \geq y_{\pi(n)}$. Since $\mathbf{x} \in C(v)$,

$$\begin{aligned}
m^\pi \cdot \mathbf{y} &= \sum_{i=1}^{n} y_{\pi(i)}(v(\pi(1),\pi(2),\ldots,\pi(i)) - v(\pi(1),\pi(2),\ldots,\pi(i-1))) \\
&= y_{\pi(n)}v(N) - y_{\pi(1)}v(\emptyset) + \sum_{i=1}^{n-1}(y_{\pi(i)} - y_{\pi(i+1)})v(\pi(1),\pi(2),\ldots,\pi(i)) \\
&\leq y_{\pi(n)}\sum_{j=1}^{n} x_{\pi(j)} + \sum_{i=1}^{n-1}(y_{\pi(i)} - y_{\pi(i+1)})\sum_{j=1}^{i} x_{\pi(j)} \\
&= \sum_{i=1}^{n} y_{\pi(i)}\sum_{j=1}^{i} x_{\pi(j)} - \sum_{i=2}^{n} y_{\pi(i)}\sum_{j=1}^{i-1} x_{\pi(j)} \\
&= \sum_{i=1}^{n} y_{\pi(i)}x_{\pi(i)} = \mathbf{x}\cdot\mathbf{y}
\end{aligned}$$

contradicting (18.1). □

18.2 Convex Games

For the coincidence of core and Weber set the following possible property of a game plays a crucial role.

Definition 18.4. A TU-game (N,v) is *convex* if the following condition holds for all $S,T \subseteq N$:

$$v(S)+v(T) \leq v(S \cup T)+v(S \cap T)\ . \tag{18.2}$$

Observe that convexity of a game implies super-additivity (cf. (16.2)): v is super-additive if (18.2) holds whenever S and T have empty intersection. The intuition is similar: larger coalitions have a relatively larger worth. This intuition is also apparent in the following condition:

$$\text{For all } i \in N \text{ and } S \subseteq T \subseteq N \setminus \{i\}\text{:}\ \ v(S \cup i)-v(S) \leq v(T \cup i)-v(T)\ . \tag{18.3}$$

Lemma 18.5. *A game* $(N,v) \in \mathcal{G}^N$ *is convex if and only if it satisfies* (18.3).

Proof. Let $v \in \mathcal{G}^N$. Obviously, (18.3) follows from (18.2) by taking, for S and T in (18.2), $S \cup i$ and T from (18.3), respectively.

In order to derive (18.2) from (18.3), first take $S_0 \subseteq T_0 \subseteq N$ and $R \subseteq N \setminus T_0$, say $R=\{i_1,\ldots,i_k\}$. By repeated application of (18.3) one obtains

$$\begin{aligned}
v(S_0 \cup i_1)-v(S_0) &\leq v(T_0 \cup i_1)-v(T_0)\\
v(S_0 \cup i_1 i_2)-v(S_0 \cup i_1) &\leq v(T_0 \cup i_1 i_2)-v(T_0 \cup i_1)\\
&\ \ \vdots\\
v(S_0 \cup i_1 \cdots i_k)-v(S_0 \cup i_1 \cdots i_{k-1}) &\leq v(T_0 \cup i_1 \cdots i_k)-v(T_0 \cup i_1 \cdots i_{k-1})\ .
\end{aligned}$$

Adding these inequalities yields

$$v(S_0 \cup R)-v(S_0) \leq v(T_0 \cup R)-v(T_0)$$

for all $R \subseteq N \setminus T_0$. Applying this inequality to arbitrary S,T by setting $S_0=S \cap T$, $T_0=T$, and $R=S \setminus T$, yields (18.2). $\square$

The importance of convex games for the relation between the core and the Weber set follows from the following theorem (Shapley [124]; Ichiishi [58]).

Theorem 18.6. *Let* $v \in \mathcal{G}^N$. *Then* v *is convex if and only if* $C(v)=W(v)$.

Proof. (a) Suppose v is convex. For the 'only if' part it is, in view of Theorem 18.3 and convexity of the core, sufficient to prove that each marginal vector $m^\pi(v)$ of v is in the core. In order to show this, assume for notational simplicity that π is identity. Let $S \subseteq N$ be an arbitrary coalition, say $S=\{i_1,\ldots,i_s\}$. Then, for $1 \leq k \leq s$, by (18.3):

$$v(i_1,\ldots,i_k)-v(i_1,\ldots,i_{k-1}) \leq v(1,2,\ldots,i_k)-v(1,2,\ldots,i_k-1)=m^\pi_{i_k}(v)\ .$$

Summing these inequalities from $k = 1$ to $k = s$ yields

$$v(S) = v(i_1, \ldots, i_s) \leq \sum_{k=1}^{s} m^{\pi}_{i_k}(v) = \sum_{i \in S} m^{\pi}_i(v) \;,$$

which shows $m^{\pi}(v) \in C(v)$.

(b) For the converse, suppose that all marginal vectors of v are in the core. Let $S, T \subseteq N$ be arbitrary. Order the players of N as follows:

$$N = \{\underbrace{i_1, \ldots, i_k}_{S \cap T}, \underbrace{i_{k+1}, \ldots, i_\ell}_{T \setminus S}, \underbrace{i_{\ell+1}, \ldots, i_s}_{S \setminus T}, \underbrace{i_{s+1}, \ldots, i_n}_{N \setminus (S \cup T)}\} \;.$$

This defines an ordering or permutation π with corresponding marginal vector $m(v) = m^{\pi}(v)$. Since $m(v) \in C(v)$,

$$\begin{aligned}
v(S) &\leq \sum_{i \in S} m_i(v) \\
&= \sum_{j=1}^{k} m_{i_j}(v) + \sum_{j=\ell+1}^{s} m_{i_j}(v) \\
&= v(i_1, \ldots, i_k) \\
&\quad + [v(i_1, \ldots, i_{\ell+1}) - v(i_1, \ldots, i_\ell)] \\
&\quad + [v(i_1, \ldots, i_{\ell+2}) - v(i_1, \ldots, i_{\ell+1}] \\
&\quad + \cdots [v(i_1, \ldots, i_s) - v(i_1, \ldots, i_{s-1}] \\
&= v(S \cap T) + v(S \cup T) - v(T) \;,
\end{aligned}$$

which implies (18.2). So v is a convex game. □

An immediate consequence of Theorem 18.6 and the definition of the Shapley value (Definition 17.1) is the following corollary.

Corollary 18.7. *Let $v \in \mathcal{G}^N$ be convex. Then $\Phi(v) \in C(v)$, i.e., the Shapley value is in the core.*

18.3 Random Order Values

A value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is called a *random order value* if there is a probability distribution p over the set of permutations $\Pi(N)$ of N such that

$$\psi(N, v) = \sum_{\pi \in \Pi(N)} p(\pi) m^{\pi}(N, v)$$

for every $(N, v) \in \mathcal{G}^N$. In that case, we denote ψ by Φ^p. Observe that Φ^p is the Shapley value Φ if $p(\pi) = 1/n!$ for every $\pi \in \Pi(N)$. Obviously,

$$W(v) = \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} = \Phi^p(v) \text{ for some } p\}.$$

Random order values satisfy the following two conditions.

Monotonicity (MON): $\psi(v) \geq \mathbf{0}$ for all monotonic games $v \in \mathcal{G}^N$. [The game v is *monotonic* if $S \subseteq T$ implies $v(S) \leq v(T)$ for all $S, T \subseteq N$.]

Linearity (LIN): $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$ for all $v, w \in \mathcal{G}^N$, and $\alpha, \beta \in \mathbb{R}$ (where, for each S, $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$).

Monotonicity says that in a monotonic game, where larger coalitions have higher worths, i.e., all marginal contributions are nonnegative, every player should receive a nonnegative payoff. Linearity is a strengthening of additivity. The main result in this section is the following characterization of random order values (see Problem 18.7 for a strengthening of the 'only if' part of this theorem).

Theorem 18.8. *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. Then ψ satisfies LIN, DUM, EFF, and MON if and only if it is a random order value.*

The proof of Theorem 18.8 is based on a sequence of propositions and lemmas, which are of independent interest.

Proposition 18.9. *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a linear value. Then there is a collection of constants $\{a^i_T \in \mathbb{R} \mid i \in N,\ \emptyset \neq T \subseteq N\}$ such that $\psi_i(v) = \sum_{\emptyset \neq T \subseteq N} a^i_T v(T)$ for every $v \in \mathcal{G}^N$ and $i \in N$.*

Proof. Let $a^i_T := \psi_i(1_T)$ for all $i \in N$ and $\emptyset \neq T \subseteq N$ (cf. Problem 17.1). For every $v \in \mathcal{G}^N$ we have $v = \sum_{T \neq \emptyset} v(T)1_T$. The desired conclusion follows from linearity of ψ. $\square$

Proposition 18.10. *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a linear value satisfying DUM. Then there is a collection of constants $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$[1] with $\sum_{T \subseteq N \setminus i} p^i_T = 1$ for all $i \in N$, such that for every $v \in \mathcal{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] \ .$$

Proof. Let $v \in \mathcal{G}^N$ and $i \in N$. By Proposition 18.9 there are a^i_T such that $\psi_i(v) = \sum_{\emptyset \neq T \subseteq N} a^i_T v(T)$. Then $0 = \psi_i(u_{N \setminus i}) = a^i_N + a^i_{N \setminus i}$, where the first equality follows from DUM. Assume now as induction hypothesis that $a^i_{T \cup i} + a^i_T = 0$ for all $T \subseteq N \setminus i$ with $|T| \geq k \geq 2$ (we have just established this for $k = n-1$), and let $S \subseteq N \setminus i$ with $|S| = k-1$. Then

$$\begin{aligned} 0 &= \psi_i(u_S) \\ &= \sum_{T : S \subseteq T} a^i_T \\ &= \sum_{T : S \subsetneq T \subseteq N \setminus i} (a^i_{T \cup i} + a^i_T) + a^i_{S \cup i} + a^i_S \\ &= a^i_{S \cup i} + a^i_S \ , \end{aligned}$$

[1] As before, we write $N \setminus i$ instead of $N \setminus \{i\}$ (etc.) for brevity.

where the last equality follows by induction and the first one by DUM. Hence, we have proved that $a^i_{T\cup i} + a^i_T = 0$ for all $T \subseteq N \setminus i$ with $0 < |T| \le n-1$. Now define, for all $i \in N$ and all such T, $p^i_T := a^i_{T\cup i} = -a^i_T$, and define $p^i_\emptyset := a^i_{\{i\}}$. Then for every $v \in \mathscr{G}^N$ and $i \in N$:

$$\psi_i(v) = \sum_{\emptyset \neq T \subseteq N} a^i_T v(T) = \sum_{T \subseteq N\setminus i} p^i_T [v(T\cup i) - v(T)] \,.$$

Finally, by DUM,

$$1 = u_{\{i\}}(i) = \psi_i(u_{\{i\}}) = \sum_{T\subseteq N\setminus i} p^i_T \,,$$

which completes the proof of the proposition. □

Proposition 18.11. *Let $\psi : \mathscr{G}^N \to \mathbb{R}^N$ be a linear value satisfying DUM and MON. Then there is a collection of constants $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ with $\sum_{T\subseteq N\setminus i} p^i_T = 1$ and $p^i_S \ge 0$ for all $S \subseteq N \setminus i$ and $i \in N$, such that for every $v \in \mathscr{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T\subseteq N\setminus i} p^i_T [v(T\cup i) - v(T)] \,.$$

Proof. In view of Proposition 18.10 we only have to prove that the weights p^i_T are nonnegative. Let $i \in N$ and $T \subseteq N \setminus i$ and consider the game $\hat{u}_T$ assigning worth 1 to all strict supersets of T and 0 otherwise. Then $\psi_i(\hat{u}_T) = p^i_T$ by Proposition 18.10. Since $\hat{u}_T$ is a monotonic game, this implies $p^i_T \ge 0$. □

Lemma 18.12. *Let $\psi : \mathscr{G}^N \to \mathbb{R}^N$ be a value and $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ be a collection of constants such that for every $v \in \mathscr{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T\subseteq N\setminus i} p^i_T [v(T\cup i) - v(T)] \,.$$

Then ψ is efficient if and only if $\sum_{i\in N} p^i_{N\setminus i} = 1$ and $\sum_{i\in T} p^i_{T\setminus i} = \sum_{j\in N\setminus T} p^j_T$ for all $\emptyset \neq T \neq N$.

Proof. Let $v \in \mathscr{G}^N$. Then

$$\begin{aligned}\psi(v)(N) &= \sum_{i\in N}\sum_{T\subseteq N\setminus i} p^i_T [v(T\cup i) - v(T)] \\ &= \sum_{T\subseteq N} v(T) \left(\sum_{i\in T} p^i_{T\setminus i} - \sum_{j\in N\setminus T} p^j_T \right) .\end{aligned}$$

Clearly, this implies efficiency of ψ if the relations in the lemma hold. Conversely, suppose that ψ is efficient. Let $\emptyset \neq T \subseteq N$ and consider the games u_T and $\hat{u}_T$. Then the preceding equation implies that

$$\psi(u_T)(N) - \psi(\hat{u}_T)(N) = \sum_{i\in T} p^i_{T\setminus i} - \sum_{j\in N\setminus T} p^j_T \,.$$

The relations in the lemma now follow by efficiency of ψ, since $u_T(N) - \hat{u}_T(N)$ is equal to 1 if $T = N$ and equal to 0 otherwise. □

We are now sufficiently equipped to prove Theorem 18.8.

Proof of Theorem 18.8. We leave it to the reader to verify that random order values satisfy the four axioms in the theorem. Conversely, let ψ satisfy these four axioms. By Proposition 18.11 there is a collection of constants $\{p_T^i \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ with $\sum_{T \subseteq N \setminus i} p_T^i = 1$ and $p_S^i \geq 0$ for all $S \subseteq N \setminus i$ and $i \in N$, such that for every $v \in \mathscr{G}^N$ and every $i \in N$:

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p_T^i [v(T \cup i) - v(T)] \ .$$

For all $i \in N$ and $T \subseteq N \setminus i$ define $A(T) := \sum_{j \in N \setminus T} p_T^j$ and $A(i;T) := p_T^i / A(T)$ if $A(T) \neq 0$, $A(i;T) := 0$ if $A(T) = 0$. For any permutation $\pi \in \Pi(N)$ write $\pi = (i_1, \ldots, i_n)$ (that is, $\pi(k) = i_k$ for all $k \in N$). Define

$$p(\pi) = p_\emptyset^{i_1} A(i_2; \{i_1\}) A(i_3; \{i_1, i_2\}) \cdots A(i_n; \{i_1, \ldots, i_{n-1}\}) \ .$$

Then it can be verified (Problem 18.8) that

$$\sum_{\pi \in \Pi(N)} p(\pi) = \sum_{i \in N} p_\emptyset^i \ . \tag{18.4}$$

We claim that for every $0 \leq t \leq n-1$ we have

$$\sum_{T : |T| = t} \ \sum_{i \in N \setminus T} p_T^i = 1 \ . \tag{18.5}$$

To prove this, first let $t = n-1$. Then the sum on the left-hand side of (18.5) is equal to $\sum_{i \in N} p_{N \setminus i}^i$, which is equal to 1 by Lemma 18.12. Now as induction hypothesis assume that (18.5) holds for $t+1$. Then

$$\begin{aligned} \sum_{T : |T| = t} \ \sum_{i \in N \setminus T} p_T^i &= \sum_{T : |T| = t+1} \ \sum_{i \in T} p_{T \setminus i}^i \\ &= \sum_{T : |T| = t+1} \ \sum_{i \in N \setminus T} p_T^i \\ &= 1 \ , \end{aligned}$$

where the second equality follows by Lemma 18.12 and the last equality by induction. This proves (18.5). In particular, for $t = 0$, we have $\sum_{i \in N} p_\emptyset^i = 1$. Together with (18.4) this implies that $p(\cdot)$ as defined above is a probability distribution on $\Pi(N)$.

In order to complete the proof, it is sufficient to show that $\psi = \Phi^p$. For every game $v \in \mathscr{G}^N$ and $i \in N$ we can write

$$\Phi_i^p(v) = \sum_{\pi \in \Pi(N)} p(\pi) [v(P_\pi(i) \cup i) - v(P_\pi(i))] \ ,$$

where $P_\pi(i)$ denotes the set of predecessors of player i under the permutation π (cf. Sect. 17.1). Hence, it is sufficient to prove that for all $i \in N$ and $T \subseteq N \setminus i$ we have

$$p_T^i = \sum_{\pi \in \Pi(N):\ T = P_\pi(i)} p(\pi) \,. \tag{18.6}$$

This is left as an exercise (Problem 18.9) to the reader. □

18.4 Weighted Shapley Values

The Shapley value is a random order value that distributes the dividend of each coalition equally among all the members of that coalition (see Theorem 17.7). In this sense, it treats players consistently over coalitions. This is not necessarily the case for every random order value. To be specific, consider Example 18.2. The payoff vector $(2.5, -0.5, 1)$ is a point of the Weber set that can be obtained uniquely as $\Phi^p(v)$, where the probability distribution p assigns weights $1/2$ to the permutations $(3,2,1)$ and $(2,1,3)$. If we look at the distribution of dividends resulting from the associated marginal vectors (cf. Problem 18.1), we obtain that $\Delta_v(123)$ is split equally between players 1 and 3 (they are the only ones that occur last in the two permutations), whereas $\Delta_v(23)$ is split equally between players 2 and 3. ($\Delta_v(12)$ goes to player 1 and $\Delta_v(13)$ is split evenly between players 1 and 3.) Hence, whereas player 2 has zero power compared to player 3 in distributing $\Delta_v(123)$, they have equal power in distributing $\Delta_v(23)$. In this respect, players 2 and 3 are not treated consistently by Φ^p.

In order to formalize the idea of consistent treatment, we first define positively weighted Shapley values. Let $\omega \in \mathbb{R}^N$ with $\omega > \mathbf{0}$. The *positively weighted Shapley value* Φ^ω is defined as the unique linear value which assigns to each unanimity game (N, u_S):

$$\Phi_i^\omega(u_S) = \begin{cases} \omega_i/\omega(S) & \text{for } i \in S \\ 0 & \text{for } i \in N \setminus S \,. \end{cases} \tag{18.7}$$

Owen [97] has shown that these positively weighted values are random order values. More precisely, define independently distributed random variables X_i $(i \in N)$ on $[0,1]$ by their cumulative distribution functions $[0,1] \ni t \mapsto t^{\omega^i}$ (that is, $X_i \le t$ with probability t^{ω^i}). Then, define the probability distribution p^ω by

$$p^\omega(\pi) = \int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} dt_1^{\omega_1} \cdots dt_{n-2}^{\omega_{n-2}} dt_{n-1}^{\omega_{n-1}} dt_n^{\omega_n} \tag{18.8}$$

for every permutation $\pi = (i_1, i_2, \ldots, i_n)$. That is, $p^\omega(\pi)$ is defined as the probability that i_1 comes before i_2, i_2 before i_3, etc., evaluated according to the independent random variables X_i. Then we have

Theorem 18.13. *For every $\omega \in \mathbb{R}^N$ with $\omega > \mathbf{0}$, $\Phi^\omega = \Phi^{p^\omega}$.*

Proof. Let S be a nonempty coalition and $i \in S$. It is sufficient to prove that $\Phi_i^{p^\omega}(u_S) = \omega_i/\omega(S)$. Note that

$$\Phi_i^{p^\omega}(u_S) = \sum_{\pi : S \setminus i \subseteq P_\pi(i)} p^\omega(\pi) \, ,$$

and, the right-hand side of this identity is equal to

$$\int_0^1 \int_0^{t_i} dt^{\omega(S\setminus i)} dt_i^{\omega_i} \, ,$$

which, in turn, is equal to $\omega_i/\omega(S)$. □

Next, we extend the concept of weighted Shapley value to include zero weights. Consider for instance, the three-person random order value that puts weight $1/2$ on the permutations $(1,2,3)$ and $(1,3,2)$. Then (cf. again Problem 18.1) the dividend $\Delta_v(12)$ goes to player 2, the dividend $\Delta_v(13)$ to player 3, and the dividends $\Delta_v(23)$ and $\Delta_v(123)$ are split equally between players 2 and 3. Thus, this random order value treats players consistently but we cannot just formalize this by giving player 1 weight 0 since player 1 does obtain $\Delta_v(1)$.

To accommodate this kind of random order values we introduce the concept of a weight system. A *weight system* w is an ordered partition $(S_1, \ldots, S_k)$ of N together with a vector $\omega \in \mathbb{R}^N$ such that $\omega > \mathbf{0}$. The *weighted Shapley value* Φ^w is defined as the unique linear value which assigns to each unanimity game $u_S \in \mathcal{G}^N$:

$$\Phi_i^w(u_S) = \begin{cases} \omega_i/\omega(S \cap S_m) & \text{for } i \in S \cap S_m \text{ and } m = \max\{h : S_h \cap S \neq \emptyset\} \\ 0 & \text{otherwise.} \end{cases} \tag{18.9}$$

Hence, S_h is more powerful as h is larger; for each coalition S we consider the subset of the most powerful players $S \cap S_m$, where m is the largest index h such that the intersection of S_h with S is nonempty, and they distribute the dividend of coalition S according to their (relative) weights $\omega_i/\omega(S \cap S_m)$. Clearly, for $k = 1$ we obtain a positively weighted Shapley value as defined above.

Weighted Shapley values are again random order values. For a weight system w with ordered partition $(S_1, \ldots, S_k)$ we only assign positive probability to those permutations in which all players of S_1 enter before all players of S_2, all players of S_2 enter before all players of S_3, etc. Given such a permutation we can assign probability $p_1(\pi)$ to the order induced by π on S_1 in the same way as we did above in (18.8); similarly, we assign probabilities $p_2(\pi), \ldots, p_k(\pi)$ to the orders induced on $S_2, \ldots, S_k$, respectively. Then we define

$$p^w(\pi) = \prod_{h=1}^{k} p_h(\pi) \, .$$

It can be shown again that $\Phi^w = \Phi^{p^w}$.

An axiomatic characterization of weighted Shapley values is provided in Kalai and Samet [62], see also Derks et al. [30], and Problem 18.10.

By Theorem 18.3 we know that the core of any game is included in the Weber set and, thus, in any game any core element corresponds to at least one random order value. The following theorem states that, in fact, the core is always covered by the set of weighted Shapley values.

Theorem 18.14. *Let $v \in \mathscr{G}^N$ and $\mathbf{x} \in C(v)$. Then there is a weight system w such that $\mathbf{x} = \Phi^w(v)$.*

For a proof of this theorem we refer to Monderer et al. [79].

Problems

18.1. *Marginal Vectors and Dividends*

Let $(N,v) \in \mathscr{G}^N$.

(1) Show that

$$v(S) = \sum_{T \subseteq S} \Delta_v(T), \tag{18.10}$$

where $\Delta_v(T)$ are the dividends defined in Sect. 17.1.

(2) Express each marginal vector m^π in terms of dividends.

18.2. *Convexity and Marginal Vectors*

Prove that a game (N,v) is convex if and only if for all $T \in 2^N \setminus \{\emptyset\}$:

$$v(T) = \min_{\pi \in \Pi(N)} \sum_{i \in T} m_i^\pi(v) \ .$$

18.3. *Strictly Convex Games*

Call a game (N,v) *strictly convex* if all inequalities in (18.3) hold strictly. Show that in a strictly convex game all marginal vectors are different.

18.4. *Sharing Profits*

Consider the following situation with $n+1$ players. Player 0 (the landlord) owns the land and players $1,2,\ldots,n$ are n identical workers who own their labor only. The production $f : \{0,1,\ldots,n\} \to \mathbb{R}$ describes how much is produced by the workers. Assume that f is nondecreasing and that $f(0) = 0$. We associate with this situation a TU-game that reflects the production possibilities of coalitions. Without agent 0 a coalition has zero worth, otherwise the worth depends on the number of workers. More precisely,

$$v(S) := \begin{cases} 0 & \text{if } 0 \notin S \\ f(|S|-1) & \text{if } 0 \in S \end{cases}$$

for every coalition $S \subseteq \{0, 1, \ldots, n\}$.

(a) Compute the marginal vectors and the Shapley value of this game.

(b) Compute the core of this game.

(c) Give a necessary and sufficient condition on f such that the game is convex. (So in that case, the core and the Weber set coincide and the Shapley value is in the core.)

18.5. *Sharing Costs*

(Cf. [71].) Suppose that n airlines share the cost of a runway. To serve the planes of company i, the length of the runway must be c_i, which is also the cost of a runway of that length. Assume $0 \le c_1 \le c_2 \le \cdots \le c_n$. The cost of coalition S is defined as $c_S = \max_{i \in S} c_i$ for every nonempty coalition S.

(a) Model this situation as a cost savings game (cf. the three communities game in Chap. 1).

(b) Show that the resulting game is convex, and compute the marginal vectors, the Shapley value, and the core.

18.6. *Independence of the Axioms in Theorem 18.8*

Show that the axioms in Theorem 18.8 are independent.

18.7. *Null-Player in Theorem 18.8*

Show that Theorem 18.8 still holds if DUM is replaced by NP.

18.8. *Equation (18.4)*

Prove (18.4).

18.9. *Equation (18.6)*

Prove (18.6).

18.10. *Characterization of Weighted Shapley Values*

Say that a value $\psi : \mathscr{G}^N \to \mathbb{R}^N$ satisfies the *Partnership* axiom if $\psi_i(\psi(u_T)(S)u_S) = \psi_i(u_T)$ for all $S \subseteq T \subseteq N$ and all $i \in S$. Prove that a value ψ satisfies LIN, DUM, EFF, MON, and Partnership, if and only if it is a weighted Shapley value.

18.11. *Core and Weighted Shapley Values in Example 18.2*

In Example 18.2, determine for each $\mathbf{x} \in C(v)$ a weight system w such that $\mathbf{x} = \Phi^w(v)$.

Chapter 19
The Nucleolus

The core of a game with transferable utility can be a large set, but it can also be empty. The Shapley value assigns to each game a unique point, which, however, does not have to be in the core.

The *nucleolus* (Schmeidler [116]) assigns to each game with a nonempty imputation set a unique element of that imputation set; moreover, this element is in the core if the core of the game is nonempty. The *pre-nucleolus* always exists (and does not have to be an imputation, even if this set is nonempty), but for balanced games it coincides with the nucleolus.

In this chapter, which is partially based on the treatment of the subject in [100] and [98], we consider both the nucleolus and the pre-nucleolus. The reader is advised to read the relevant part of Chap. 9 first.

In Sect. 19.1 we start with an example illustrating the (pre-)nucleolus and Kohlberg's balancedness criterion (Kohlberg [65]). Section 19.2 introduces the lexicographic order, on which the definition of the (pre-)nucleolus in Sect. 19.3 is based. Section 19.4 presents the Kohlberg criterion, which is a characterization of the (pre-) nucleolus in terms of balanced collections of coalitions. Computational aspects are discussed in Sect. 19.5, while Sect. 19.6 presents Sobolev's [127] characterization of the pre-nucleolus based on a reduced game property.

19.1 An Example

Consider the three-person TU-game given by Table 19.1. It is easy to see that this game has a nonempty core; for instance, $(8,8,8)$ is a core element. Let $\mathbf{x} = (x_1,x_2,x_3)$ be an arbitrary efficient payoff distribution. For a nonempty coalition S, define the *excess of S at* $\mathbf{x}$ as $e(S,\mathbf{x}) := v(S) - x(S)$. For an efficient vector $\mathbf{x}$ the excess of the grand coalition N is always zero, and is therefore omitted from consideration. The idea underlying the nucleolus is as follows. For an arbitrary efficient vector consider the corresponding vector of excesses. Among all imputations (and for the pre-nucleolus: among all efficient vectors) find those where the maximal excess is minimal. If this set consists of one point, then this is the (pre-)nucleolus. Otherwise, continue by minimizing the second largest excess, etc. Note that if a

Table 19.1 The example of Sect. 19.1

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	4	4	8	12	16	24
$e(S,(8,8,8))$		-4	-4	-4	-8	-4	0	
$e(S,(6,9,9))$		-2	-5	-5	-7	-3	-2	
$e(S,(6,8,10))$		-2	-4	-6	-6	-4	-2	

game has a nonempty core then every core element has by definition only non-positive excesses, whereas efficient payoff vectors outside the core have at least one positive excess. This implies that for balanced games the successive minimization of excesses can be restricted to core elements, and the pre-nucleolus and the nucleolus coincide.

In order to illustrate these ideas, consider again Table 19.1, where the excesses of some core vectors for this example are calculated. The highest excess for the core vector $(8,8,8)$ is equal to zero, attained for the coalition $\{2,3\}$. Obviously, this excess can be decreased by increasing the payoff for players 2 and 3 together, at the expense of player 1, who has an excess of -4. Thus, a next 'try' is the payoff vector $(6,9,9)$, which indeed has maximal excess -2 reached for coalitions $\{1\}$ and $\{2,3\}$. It is then obvious that this is indeed the minimal maximal excess, because the excess for coalition $\{1\}$ can only be decreased by increasing the excess for $\{2,3\}$, and vice versa. Observe that the collection $\{\{1\},\{2,3\}\}$ is balanced (in particular, it is a partition).[1] At $(6,9,9)$ the second maximal excess is equal to -3, reached by the coalition $\{1,3\}$. Again, this might be decreased by improving the payoff for players 1 and 3 together at the expense of player 2. Because the payoff for player 1 has already been fixed at 6, this means that the payoff for player 3 has to be increased and that of player 2 has to be decreased. These observations lead to the next 'try' $(6,8,10)$, where the maximal excess is still equal to -2, and the second maximal excess equals -4, reached by the coalitions $\{2\}$ and $\{1,3\}$. It is obvious that this second maximal excess, as well as the third maximal excess of -6, cannot be decreased any further. Observe that also the collections $\{\{1\},\{2,3\},\{2\},\{1,3\}\}$ and $\{\{1\},\{2,3\},\{2\},\{1,3\},\{3\},\{1,2\}\}$ are all balanced.

It follows that $(6,8,10)$ is the (pre-)nucleolus of this game. Moreover, the excesses are closely related to balanced collections of coalitions; this will appear to be a more general phenomenon, known as the Kohlberg criterion.

19.2 The Lexicographic Order

The definition of the nucleolus is based on a comparison of vectors by means of the lexicographic order. We briefly discuss this order and examine some of its properties. Let $\mathbb{R}^k$ be the real vector space of dimension k. A binary relation $\succeq$ on $\mathbb{R}^k$ that satisfies:

[1] See Chap. 16.

(1) *Reflexivity*: $\mathbf{x} \succeq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^k$

(2) *Transitivity*: $\mathbf{x} \succeq \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$ with $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$

is called a *partial order*. For a partial order $\succeq$, we write $\mathbf{x} \succ \mathbf{y}$ to indicate that $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \not\succeq \mathbf{x}$. A partial order $\succeq$ is called a *weak order* if it satisfies

(3) *Completeness*: $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ with $\mathbf{x} \neq \mathbf{y}$

If also

(4) *Antisymmetry*: $\mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ with $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$

the relation is called a *linear order*.[2]

On the vector space $\mathbb{R}^k$ we define the linear order $\succeq_{\text{lex}}$ as follows. For any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^K$, $\mathbf{x}$ is *lexicographically* larger than or equal to $\mathbf{y}$ – notation: $\mathbf{x} \succeq_{\text{lex}} \mathbf{y}$ – if either $\mathbf{x} = \mathbf{y}$, or $\mathbf{x} \neq \mathbf{y}$ and for

$$i = \min\{j \in \{1, \ldots, k\} \mid x_j \neq y_j\}$$

we have that $x_i > y_i$. In other words, $\mathbf{x}$ should assign a higher value than $\mathbf{y}$ to the *first* coordinate on which $\mathbf{x}$ and $\mathbf{y}$ are different. For obvious reasons the order $\succeq_{\text{lex}}$ is called the *lexicographic order* on $\mathbb{R}^k$.

The lexicographic order cannot be represented by a continuous utility function (cf. Problem 19.5). In fact, it can be shown that the lexicographic order on $\mathbb{R}^k$ cannot be represented by *any* utility function (cf. Problem 19.6).

19.3 The (Pre-)Nucleolus

Let (N, v) be a TU-game and let $X \subseteq \mathbb{R}^N$ be some set of payoff distributions. For every non-empty coalition $S \subseteq N$ and every $\mathbf{x} \in X$ the *excess of S at $\mathbf{x}$* is the number

$$e(S, \mathbf{x}, v) := v(S) - x(S).$$

This number can be interpreted as the dissatisfaction (complaint, regret) of the coalition S if $\mathbf{x}$ is the payoff vector. For every $\mathbf{x} \in X$ let $\theta(\mathbf{x})$ denote the vector of excesses at $\mathbf{x}$ arranged in non-increasing order, hence

$$\theta(\mathbf{x}) = (e(S_1, \mathbf{x}, v), \ldots, e(S_{2^n-1}, \mathbf{x}, v))$$

such that $e(S_t, \mathbf{x}, v) \geq e(S_p, \mathbf{x}, v)$ for all $1 \leq t \leq p \leq 2^n - 1$. Let $\succeq_{\text{lex}}$ be the lexicographic order on $\mathbb{R}^{2^n-1}$, as defined in Sect. 19.2. The *nucleolus of (N, v) with respect to X* is the set

$$\mathcal{N}(N, v, X) := \{\mathbf{x} \in X \mid \theta(\mathbf{y}) \succeq_{\text{lex}} \theta(\mathbf{x}) \text{ for all } \mathbf{y} \in X\}.$$

[2] See also Chap. 11. Here, we repeat some of the definitions for convenience.

So the nucleolus consists of all payoff vectors in X at which the excess vectors are lexicographically minimized. The motivation for this is that we first try to minimize the dissatisfaction of those coalitions for which this is maximal, next for those coalitions that have second maximal excess, etc.

We first establish conditions under which the nucleolus with respect to a set X is non-empty.

Theorem 19.1. *Let X be non-empty and compact. Then $\mathcal{N}(N,v,X) \neq \emptyset$ for every game v.*

Proof. First observe that all excess functions $e(S,\cdot,v)$ are continuous and therefore $\theta(\cdot)$ is continuous. Define $X_0 := X$ and, recursively,

$$X_t := \{\mathbf{x} \in X_{t-1} \mid \theta_t(\mathbf{y}) \geq \theta_t(\mathbf{x}) \text{ for all } \mathbf{y} \in X_{t-1}\}$$

for all $t = 1,2,\ldots,2^n-1$. Since $\theta(\cdot)$ is continuous, Weierstrass' Theorem implies that every X_t is a non-empty compact subset of X_{t-1}. This holds in particular for $t = 2^n - 1$ and, clearly, $X_{2^n-1} = \mathcal{N}(N,v,X)$. □

We will show that, if X is, moreover, convex, then the nucleolus with respect to X consists of a single point. We start with the following lemma.

Lemma 19.2. *Let X be convex, $\mathbf{x},\mathbf{y} \in X$, and $0 \leq \alpha \leq 1$. Then*

$$\alpha\theta(\mathbf{x}) + (1-\alpha)\theta(\mathbf{y}) \succeq_{\text{lex}} \theta(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}). \tag{19.1}$$

Proof. Let $S_1,\ldots,S_{2^n-1}$ be an ordering of the non-empty coalitions such that

$$\theta(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) = (e(S_1,\alpha\mathbf{x} + (1-\alpha)\mathbf{y},v),\ldots,e(S_{2^n-1},\alpha\mathbf{x} + (1-\alpha)\mathbf{y},v)).$$

The right-hand side of this equation is equal to $\alpha\mathbf{a} + (1-\alpha)\mathbf{b}$, where $\mathbf{a} = (e(S_1,\mathbf{x},v),\ldots,e(S_{2^n-1},\mathbf{x},v))$ and $\mathbf{b} = (e(S_1,\mathbf{y},v),\ldots,e(S_{2^n-1},\mathbf{y},v))$. Since $\theta(\mathbf{x}) \succeq_{\text{lex}} \mathbf{a}$ and $\theta(\mathbf{y}) \succeq_{\text{lex}} \mathbf{b}$ it follows that

$$\alpha\theta(\mathbf{x}) + (1-\alpha)\theta(\mathbf{y}) \succeq_{\text{lex}} \alpha\mathbf{a} + (1-\alpha)\mathbf{b} = \theta(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}),$$

hence (19.1) holds. □

Remark 19.3. Formula (19.1) shows that $\theta(\cdot)$ is *quasi-convex* with respect to the ordering $\succeq_{\text{lex}}$.

Theorem 19.4. *Let X be non-empty, compact, and convex. Then, for every game (N,v), the nucleolus with respect to X consists of a single point.*

Proof. By Theorem 19.1 the nucleolus is non-empty. Let $\mathbf{x},\mathbf{y} \in \mathcal{N}(N,v,X)$ and $0 < \alpha < 1$. Then $\theta(\mathbf{x}) = \theta(\mathbf{y})$. By Lemma 19.2,

$$\theta(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) = \alpha\theta(\mathbf{x}) + (1-\alpha)\theta(\mathbf{y}).$$

Hence $\theta(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$ and therefore $\theta(\mathbf{x}) = \theta(\mathbf{y}) = \alpha\mathbf{a} + (1-\alpha)\mathbf{b}$, with $\mathbf{a}$ and $\mathbf{b}$ as in the proof of Lemma 19.2. Since $\theta(\mathbf{x}) \succeq_{\text{lex}} \mathbf{a}$ and $\theta(\mathbf{y}) \succeq_{\text{lex}} \mathbf{b}$ it follows that $\mathbf{a} = \theta(\mathbf{x})$ and $\mathbf{b} = \theta(\mathbf{y})$. In $\mathbf{a}$ and $\mathbf{b}$ the coalitions are ordered in the same way and therefore this is also the case in $\theta(\mathbf{x})$ and $\theta(\mathbf{y})$. Hence $\mathbf{x}$ and $\mathbf{y}$ have all excesses equal, and thus $\mathbf{x} = \mathbf{y}$. □

Well-known choices for the set X are the imputation set $I(N,v)$ of a game (N,v) and the set of efficient payoff distributions $I^*(N,v) = \{\mathbf{x} \in \mathbb{R}^N \mid x(N) = v(N)\}$. For an essential game (N,v) the set $I(N,v)$ is non-empty, compact and convex, and therefore Theorem 19.4 implies that the nucleolus with respect to $I(N,v)$ consists of a single point, called the *nucleolus of* (N,v), and denoted by $\nu(N,v)$. Although the set $I^*(N,v)$ is not compact, the nucleolus of (N,v) with respect to this set exists and is also single-valued (see Problem 19.7): its unique member is called the *pre-nucleolus of* (N,v), denoted by $\nu^*(N,v)$. In Problem 19.8 the reader is asked to show that both points are in the core of the game if this set is non-empty, and then coincide.

19.4 The Kohlberg Criterion

In this section we derive the so-called Kohlberg criterion for the pre-nucleolus, which characterizes this solution in terms of balanced sets (cf. Chap. 16). A similar result can be derived for the nucleolus, see Problem 19.9, but the formulation for the pre-nucleolus is slightly simpler.

We start, however, with Kohlberg's characterization in terms of side-payments. A *side-payment* is a vector $\mathbf{y} \in \mathbb{R}^N$ satisfying $y(N) = 0$. Let (N,v) be a game and for every $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$ denote by

$$\mathscr{D}(\alpha,\mathbf{x},v) = \{S \subseteq N \setminus \{\emptyset\} \mid e(S,\mathbf{x},v) \geq \alpha\}$$

the set of coalitions with excess at least α at $\mathbf{x}$.

Theorem 19.5. *Let (N,v) be a game and $\mathbf{x} \in I^*(N,v)$. Then the following two statements are equivalent.*

(1) $\mathbf{x} = \nu^*(N,v)$.
(2) *For every α such that $\mathscr{D}(\alpha,\mathbf{x},v) \neq \emptyset$ and for every side-payment $\mathbf{y}$ with $y(S) \geq 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$ we have $y(S) = 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$.*

Proof. Assume that $\mathbf{x} = \nu^*(N,v)$ and that the conditions in (2) are fulfilled for $\mathbf{x}$, α, and $\mathbf{y}$. Define $\mathbf{z}_\varepsilon = \mathbf{x} + \varepsilon\mathbf{y}$ for every $\varepsilon > 0$. Then $\mathbf{z}_\varepsilon \in I^*(N,v)$. Choose $\varepsilon^* > 0$ such that, for all $S \in \mathscr{D}(\alpha,\mathbf{x},v)$ and non-empty $T \notin \mathscr{D}(\alpha,\mathbf{x},v)$,

$$e(S,\mathbf{z}_{\varepsilon^*},v) > e(T,\mathbf{z}_{\varepsilon^*},v). \tag{19.2}$$

For every $S \in \mathscr{D}(\alpha, \mathbf{x}, v)$,

$$\begin{aligned} e(S, \mathbf{z}_{\varepsilon^*}, v) &= v(S) - (x(S) + \varepsilon^* y(S)) \\ &= e(S, \mathbf{x}, v) - \varepsilon^* y(S) \\ &\leq e(S, \mathbf{x}, v). \end{aligned} \tag{19.3}$$

Assume, contrary to what we want to prove, that $y(S) > 0$ for some $S \in \mathscr{D}(\alpha, \mathbf{x}, v)$. Then, by (19.2) and (19.3), $\theta(\mathbf{x}) \succeq_{\text{lex}} \theta(\mathbf{z}_{\varepsilon^*})$, a contradiction.

Next, let $\mathbf{x} \in I^*(N, v)$ satisfy (2). Let $\mathbf{z} = \nu^*(N, v)$. Denote

$$\{e(S, \mathbf{x}, v) \mid S \in 2^N \setminus \{\emptyset\}\} = \{\alpha_1, \ldots, \alpha_p\}$$

with $\alpha_1 > \cdots > \alpha_p$. Define $\mathbf{y} = \mathbf{z} - \mathbf{x}$. Hence, $\mathbf{y}$ is a side-payment. Since $\theta(\mathbf{x}) \succeq_{\text{lex}} \theta(\mathbf{z})$, we have $e(S, \mathbf{x}, v) = \alpha_1 \geq e(S, \mathbf{z}, v)$ for all $S \in \mathscr{D}(\alpha_1, \mathbf{x}, v)$ and thus

$$e(S, \mathbf{x}, v) - e(S, \mathbf{z}, v) = (z - x)(S) = y(S) \geq 0.$$

Therefore, by (2), $y(S) = 0$ for all $S \in \mathscr{D}(\alpha_1, \mathbf{x}, v)$.
Assume now that $y(S) = 0$ for all $S \in \mathscr{D}(\alpha_t, \mathbf{x}, v)$ for some $1 \leq t \leq p$. Then, since $\theta(\mathbf{x}) \succeq_{\text{lex}} \theta(\mathbf{z})$,

$$e(S, \mathbf{x}, v) = \alpha_{t+1} \geq e(S, \mathbf{z}, v) \text{ for all } S \in \mathscr{D}(\alpha_{t+1}, \mathbf{x}, v) \setminus \mathscr{D}(\alpha_t, \mathbf{x}, v).$$

Hence $y(S) \geq 0$ and thus, by (2), $y(S) = 0$ for all $y \in \mathscr{D}(\alpha_{t+1}, \mathbf{x}, v)$. We conclude that $y(S) = 0$ for all $S \in 2^N \setminus \{\emptyset\}$, so $\mathbf{y} = \mathbf{0}$ and $\mathbf{x} = \mathbf{z}$. □

We can now prove Kohlberg's characterization of the pre-nucleolus by balanced collections.

Theorem 19.6 (Kohlberg). *Let (N, v) be a game and $\mathbf{x} \in I^*(N, v)$. Then the following two statements are equivalent.*

(1) $\mathbf{x} = \nu^*(N, v)$.
(2) *For every α, $\mathscr{D}(\alpha, \mathbf{x}, v) \neq \emptyset$ implies that $\mathscr{D}(\alpha, \mathbf{x}, v)$ is a balanced collection.*

Proof. Assume that $\mathbf{x}$ satisfies (2). Let $\alpha \in \mathbb{R}$ such that $\mathscr{D}(\alpha, \mathbf{x}, v) \neq \emptyset$, and let $\mathbf{y}$ be a side-payment with $y(S) \geq 0$ for all $S \in \mathscr{D}(\alpha, \mathbf{x}, v)$. Since, by (2), $\mathscr{D}(\alpha, \mathbf{x}, v)$ is balanced there are numbers $\lambda(S) > 0$, $S \in \mathscr{D}(\alpha, \mathbf{x}, v)$, such that

$$\sum_{S \in \mathscr{D}(\alpha, \mathbf{x}, v)} \lambda(S) \mathbf{e}^S = \mathbf{e}^N.$$

By taking the product on both sides with $\mathbf{y}$ this implies

$$\sum_{S \in \mathscr{D}(\alpha, \mathbf{x}, v)} \lambda(S) y(S) = y(N) = 0.$$

Therefore $y(S) = 0$ for every $S \in \mathscr{D}(\alpha, \mathbf{x}, v)$. Thus, Theorem 19.5 implies $\mathbf{x} = \nu^*(N, v)$.

Assume next that $\mathbf{x} = \nu^*(N,v)$. Let $\alpha \in \mathbb{R}$ such that $\mathscr{D}(\alpha,\mathbf{x},v) \neq \emptyset$. Consider the linear program

$$\max \sum_{S\in\mathscr{D}(\alpha,\mathbf{x},v)} y(S) \text{ subject to } -y(S) \leq 0,\ S \in \mathscr{D}(\alpha,\mathbf{x},v), \text{ and } y(N) = 0. \tag{19.4}$$

This program is feasible and, by Theorem 19.5, its value is 0. Hence (see Problem 19.10) its dual is feasible, that is, there are $\lambda(S) \geq 0$, $S \in \mathscr{D}(\alpha,\mathbf{x},v)$, and $\lambda(N) \in \mathbb{R}$ such that

$$-\sum_{S\in\mathscr{D}(\alpha,\mathbf{x},v)} \lambda(S)\mathbf{e}^S + \lambda(N)\mathbf{e}^N = \sum_{S\in\mathscr{D}(\alpha,\mathbf{x},v)} \mathbf{e}^S.$$

Hence $\lambda(N)\mathbf{e}^N = \sum_{S\in\mathscr{D}(\alpha,\mathbf{x},v)}(1+\lambda(S))\mathbf{e}^S$. Since $1+\lambda(S) > 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$, we have $\lambda(N) > 0$ and thus $\mathscr{D}(\alpha,\mathbf{x},v)$ is balanced. □

19.5 Computation of the Nucleolus

For two-person games, the (pre-)nucleolus is easy to compute (see Problem 19.13). In general, the computation of the nucleolus can be based on the subsequent determination of the sets $X_0, X_1, X_2, \ldots$ in Theorem 19.1, but this may not be easy, as the following example shows.

Example 19.7. Consider the TU-game v with player set $N = \{1,2,3,4\}$ defined by[3]

$$v(S) = \begin{cases} 20 & \text{if } S = N \\ 8 & \text{if } S = \{1,2\} \\ 8 & \text{if } S = \{3,4\} \\ 4 & \text{if } S = \{1\} \\ 2 & \text{if } S = \{3\} \\ 0 & \text{otherwise.} \end{cases}$$

First observe that it is easy to find some imputation (e.g., $\mathbf{x} = (6,4,5,5)$) such that the excesses of $\{1,2\}$ and $\{3,4\}$ are both equal to -2 and all other excesses are at most -2. Clearly, this must be the minimal (over all imputations) maximal excess attainable, since decreasing the excess of $\{1,2\}$ implies increasing the excess of $\{3,4\}$ by efficiency, and vice versa. Thus,

$$X_1 = \{\mathbf{x} \in I(v) \mid \theta(\mathbf{x})_1 = -2\},$$

and $X_2 = X_1$ since the excess of -2 is reached at the two coalitions $\{1,2)\}$ and $\{3,4\}$. Consistently with the Kohlberg criterion, these coalitions form a balanced collection. Next, observe that the remaining excesses are always at most as large as at least one of the excesses of the four one-person coalitions. So we can find the

[3] Observe that this game has a non-empty core and therefore the nucleolus and pre-nucleolus coincide and are elements of the core. Cf. Problem 19.8.

second-highest excess by minimizing α subject to the constraints[4]

$$\begin{cases} 8-x_1-x_2 = -2 \\ 8-x_3-x_4 = -2 \\ 4-x_1 \leq \alpha \\ -x_2 \leq \alpha \\ 2-x_3 \leq \alpha \\ -x_4 \leq \alpha \,. \end{cases}$$

This can be rewritten as the system

$$\begin{cases} x_1+x_2 = 10 \\ x_3+x_4 = 10 \\ x_1 \geq 4-\alpha \\ x_2 \geq -\alpha \\ x_3 \geq 2-\alpha \\ x_4 \geq -\alpha \end{cases}$$

with an obvious minimum value $\alpha = -3$. So the next two coalitions of which the excesses become fixed are $\{1\}$ and $\{2\}$, and, thus, the nucleolus allocates $x_1 = 7$ to player 1 and $x_2 = 3$ to player 2. The third step in the computation is to decrease α even further subject to the constraints

$$\begin{cases} x_3+x_4 = 10 \\ x_3 \geq 2-\alpha \\ x_4 \geq -\alpha. \end{cases}$$

(Note that the constraints that only refer to x_1 and x_2 have become superfluous.) This linear program has an obvious solution $\alpha = -4$, which yields $x_3 = 6$ and $x_4 = 4$. Thus, the (pre-)nucleolus of this game is

$$\nu(v) = \nu^*(v) = (7,3,6,4).$$

It is interesting to see that, even though at first glance player 4 does not seem to have any noticeable advantage over player 2, he is still doing better in the nucleolus. This is due to the fact that early on in the process player 4 was grouped together with player 3, who has a lower individual value than player 1, with whom player 2 becomes partnered. Thus, player 4 obtains a bigger slice of the cake of size 10 that he has to share with player 3, than player 2 does in a similar situation.

This example raises the question how the nucleolus of a given TU-game can be computed in a systematic way. More generally, let (N,v) be a game. In order to compute the nucleolus $\mathscr{N}(N,v,X)$ for $X \subseteq \mathbb{R}^N$ a convex polyhedral set, determined by a system of linear (in)equalities, we can start by solving the linear program[5]

$$\text{Minimize } \alpha \text{ subject to } x(S)+\alpha \geq v(S), \quad \forall \emptyset \neq S \in 2^N, \mathbf{x} \in X.$$

[4] The game v is omitted from the notation.

[5] Under appropriate restrictions this program is feasible and bounded.

Let α_1 denote the minimum of this program and let $X_1 \subseteq X$ be the set of points where the minimum is obtained. If $|X_1| = 1$ then $\mathcal{N}(N,v,X) = X_1$. Otherwise, let $\mathcal{B}_1$ be the set of coalitions S such that $e(S,\mathbf{x},v) = \alpha_1$ for all $\mathbf{x} \in X_1$, and solve the linear program

$$\text{Minimize } \alpha \text{ subject to } x(S) + \alpha \geq v(S), \quad \forall \emptyset \neq S \in 2^N \setminus \mathcal{B}_1, \mathbf{x} \in X_1.$$

Continuing in this way, we eventually reach a unique point which constitutes the nucleolus of v with respect to X (cf. Theorem 19.4). The following example (taken from [98]) illustrates this.

Example 19.8. Let $N = \{1,2,3,4\}$ and $v(N) = 100$, $v(123) = 95$, $v(124) = 85$, $v(134) = 80$, $v(234) = 55$, $v(ij) = 50$ for all $i \neq j$, $v(i) = 0$ for all i. We compute the pre-nucleolus (which will turn out to be an imputation and therefore equal to the nucleolus). We start with the linear program

Minimize α

subject to

$$\begin{array}{rrrrrcl}
x_1 & + x_2 & + x_3 & & + \alpha & \geq & 95 \\
x_1 & + x_2 & & + x_4 & + \alpha & \geq & 85 \\
x_1 & & + x_3 & + x_4 & + \alpha & \geq & 80 \\
 & x_2 & + x_3 & + x_4 & + \alpha & \geq & 55 \\
 & & x_i & + x_j & + \alpha & \geq & 50 \\
 & & x_i & & + \alpha & \geq & 0 \\
x_1 & + x_2 & + x_3 & + x_4 & & = & 100.
\end{array}$$

Solving this program results in $\alpha_1 = 10$, obtained over the set X_1 given by

$$x_1 + x_2 = 60, \quad x_1 \geq 30, \quad x_2 \geq 25, \quad x_3 = 25, \quad x_4 = 15,$$

and

$$\mathcal{B}_1 = \{123, 124, 34\},$$

which is a balanced set, as was to be expected (cf. Theorem 19.6).

The second linear program is now

Minimize α

subject to

$$\begin{array}{rrrrrcl}
x_1 & & + x_3 & + x_4 & + \alpha & \geq & 80 \\
 & x_2 & + x_3 & + x_4 & + \alpha & \geq & 55 \\
x_1 & & + x_3 & & + \alpha & \geq & 50 \\
x_1 & & & + x_4 & + \alpha & \geq & 50 \\
 & x_2 & + x_3 & & + \alpha & \geq & 50 \\
 & x_2 & & + x_4 & + \alpha & \geq & 50 \\
 & & & x_i & + \alpha & \geq & 0 \\
 & & & & \mathbf{x} & \in & X_1.
\end{array}$$

By some simplifications this program reduces to

Minimize α

subject to

$$\begin{aligned} x_1 + \alpha &\geq 40 \\ x_2 + \alpha &\geq 35 \\ x_1 + x_2 &= 60 \\ x_1 &\geq 30 \\ x_2 &\geq 25, \end{aligned}$$

with solution $\alpha_2 = 7.5$, $x_1 = 32.5$, $x_2 = 27.5$. Hence, the (pre-)nucleolus of this game is

$$\nu(N,v) = \nu^*(N,v) = (32.5, 27.5, 25, 15),$$

and $\mathscr{B}_1 = \{123, 124, 34\}$, $\mathscr{B}_2 = \{134, 234\}$. By Theorem 19.6 it can be verified that we have indeed found the pre-nucleolus (Problem 19.15).

19.6 A Characterization of the Pre-Nucleolus

The pre-nucleolus is a single-valued solution concept, defined for any game (N,v). Hence, it is an example of a value on $\mathscr{G}$ (see Chap. 17). In this section we provide a characterization based on a reduced game property. This characterization was provided by Sobolev [127]; here, we follow the presentation of [100]. We will not give a complete proof of this characterization but, instead, refer the reader to the cited literature.

Let ψ be a value on $\mathscr{G}$, and let $\mathbf{x} \in \mathbb{R}^N$. Let S be a non-empty coalition. The *DM reduced game for S at* $\mathbf{x}$ is the game $(S, v_{S,\mathbf{x}}) \in \mathscr{G}^S$ defined by

$$v_{S,\mathbf{x}}(T) := \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \setminus S) & \text{if } T = S \\ \max_{Q \subseteq N \setminus S} v(T \cup Q) - x(Q) & \text{otherwise.} \end{cases}$$

This reduced game was introduced by Davis and Maschler [24]. Its interpretation is as follows. Suppose $\mathbf{x}$ is the payoff vector for the grand coalition. The coalition S could renegotiate these payoffs among themselves. Assume that the outside players are happy with $\mathbf{x}$. Hence, S has $v(N) - x(N \setminus S)$ to redistribute. Any smaller coalition T, however, could cooperate with zero or more outside players and pay them according to $\mathbf{x}$: then $v_{S,\mathbf{x}}(T)$ as defined above is the maximum they could get. Hence, the redistribution game takes the form $v_{S,\mathbf{x}}$.

The following axiom for a value ψ on $\mathscr{G}$ requires the outcome of the redistribution game for S to be equal to the original outcome.

Davis–Maschler consistency (DMC) $\psi_i(S, v_{S,\mathbf{x}}) = \psi_i(N,v)$ for every $(N,v) \in \mathscr{G}$, $\emptyset \neq S \subseteq N$, $\mathbf{x} = \psi(N,v)$ and $i \in S$.

The announced characterization is based on two other axioms, namely Anonymity (AN, see Sect. 17.1) and the following axiom.

Covariance (COV) $\psi(N, \alpha v + \mathbf{b}) = \alpha\psi(N, v) + \mathbf{b}$ for all $(N, v) \in \mathscr{G}$, every $\alpha \in \mathbb{R}$, $\alpha > 0$, and every $\mathbf{b} \in \mathbb{R}^N$, where $(\alpha v + \mathbf{b})(S) := \alpha v(S) + b(S)$ for every non-empty coalition $S \subseteq N$.

Remark 19.9. Covariance requires that the value ψ respects *strategic equivalence* of games, cf. Problem 16.12.

We first prove that the pre-nucleolus is Davis–Maschler consistent.

Lemma 19.10. *The pre-nucleolus, as a value on $\mathscr{G}$, is Davis–Maschler consistent.*

Proof. Let $(N, v) \in \mathscr{G}$, $\mathbf{x} = v^*(N, v)$ and $\emptyset \neq S \subseteq N$. Let $\mathbf{x}_S \in \mathbb{R}^S$ be the restriction of $\mathbf{x}$ to S, then we have to prove that $\mathbf{x}_S = v^*(S, v_{S,\mathbf{x}})$. Let $\alpha \in \mathbb{R}$ with $\mathscr{D}(\alpha, \mathbf{x}_S, v_{S,\mathbf{x}}) \neq \emptyset$ and let $\mathbf{y}_S \in \mathbb{R}^S$ be a side-payment with $y_S(Q) \geq 0$ for every $Q \in \mathscr{D}(\alpha, \mathbf{x}_S, v_{S,\mathbf{x}})$ then, in view of Theorem 19.5 it is sufficient to prove that $y_S(Q) = 0$ for every $Q \in \mathscr{D}(\alpha, \mathbf{x}_S, v_{S,\mathbf{x}})$. Note that

$$\{T \cap S \mid T \in \mathscr{D}(\alpha, \mathbf{x}, v),\ \emptyset \neq T \cap S \neq S\} = \mathscr{D}(\alpha, \mathbf{x}_S, v_{S,\mathbf{x}}) \setminus \{S\}. \tag{19.5}$$

Extend $\mathbf{y}_S$ to a vector $\mathbf{y} \in \mathbb{R}^N$ by setting $y_i = 0$ for all $i \in N \setminus S$. Then $y(N) = 0$ and, by (19.5), $y(Q) \geq 0$ for all $Q \in \mathscr{D}(\alpha, \mathbf{x}, v)$. By Theorem 19.5, it follows that $y(Q) = 0$ for all $Q \in \mathscr{D}(\alpha, \mathbf{x}, v)$. Hence, by (19.5), $y_S(Q) = 0$ for all $Q \in \mathscr{D}(\alpha, \mathbf{x}_S, v_{S,\mathbf{x}})$, which completes the proof. □

As an additional result, we prove that COV and DMC imply Efficiency (EFF).

Lemma 19.11. *Let ψ be a value on $\mathscr{G}$ satisfying COV and DMC. Then ψ satisfies EFF.*

Proof. Let $(\{i\}, v)$ be a one-person game. If $v(i) = 0$ then, by COV, $\psi(\{i\}, 0) = \psi(\{i\}, 2 \cdot 0) = 2\psi(\{i\}, 0)$, hence $\psi(\{i\}, 0) = 0$. Again by COV,

$$\psi(\{i\}, v) = \psi(\{i\}, 0 + v) = \psi(\{i\}, 0) + v(i) = v(i),$$

so EFF is satisfied. Now let $(N, v) \in \mathscr{G}$ with at least two players. Let $\mathbf{x} = \psi(N, v)$ and $i \in N$. By DMC, $x_i = \psi(\{i\}, v_{\{i\},\mathbf{x}})$. Hence, $x_i = v_{\{i\},\mathbf{x}}(i) = v(N) - x(N \setminus \{i\})$, where the second equality follows by definition of the reduced game. Thus, $x(N) = v(N)$ and the proof is complete. □

The announced characterization of the pre-nucleolus by Sobolev [127] is as follows.

Theorem 19.12. *A value ψ on $\mathscr{G}$ satisfies COV, AN, and DMC if and only if it is the pre-nucleolus.*

Proof. COV and AN of the pre-nucleolus follow from Problem 19.17. DMC follows from Lemma 19.10. For the (quite involved) proof of the only-if statement we refer the reader to [127] or [100]. □

Snijders [126] provides a characterization of the nucleolus on the class of all games with non-empty imputation set by modifying the Davis–Maschler consistency condition.

Problems

19.1. *Binary Relations*

Give an example of a relation that satisfies (1)–(3), but not (4) in Sect. 19.2. Also find an example that only violates (3), and one that only violates (2). What about (1)? Give an example of a partial order that is neither antisymmetric nor complete.

19.2. *Linear Orders*

Let $\succeq$ be a linear order. Show that $\mathbf{x} \succ \mathbf{y}$ holds if and only if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

19.3. *The Lexicographic Order (1)*

Show that $\succeq_{\text{lex}}$ is indeed a linear order.

19.4. *The Lexicographic Order (2)*

Find the set of points (x_1,x_2) in $\mathbb{R}^2$ for which $(x_1,x_2) \succeq_{\text{lex}} (3,1)$. Draw this set in the Cartesian plane. Is this set closed?

19.5. *Representability of Lexicographic Order (1)*

Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Define $\succeq_u$ by $\mathbf{x} \succeq_u \mathbf{y}$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$. Use Problem 19.4 to show that $\succeq_u \neq \succeq_{\text{lex}}$.

19.6. *Representability of Lexicographic Order (2)*

Show that the lexicographic order cannot be represented by any utility function. [Hint: take the lexicographic order on $\mathbb{R}^2$ and argue that representability implies that for each real number t we can find a rational number $q(t)$ such that $q(t) \neq q(s)$ whenever $t \neq s$. Hence, we have uncountably many different rational numbers, a contradiction.]

19.7. *Single-Valuedness of the Pre-Nucleolus*

Prove that the nucleolus of any game (N,v) with respect to $I^*(N,v)$ is single-valued.

19.8. *(Pre-)nucleolus and Core*

Let (N,v) be a game with $C(N,v) \neq \emptyset$. Prove that $\nu(N,v) = \nu^*(N,v) \in C(N,v)$.

19.9. *Kohlberg Criterion for the Nucleolus*

Let (N,v) be a game satisfying $I(N,v) \neq \emptyset$, and let $\mathbf{x} \in I(N,v)$. Prove that $\mathbf{x} = \nu(N,v)$ if and only if for every $\alpha \in \mathbb{R}$: if $\mathscr{D}(\alpha,v,\mathbf{x}) \neq \emptyset$ then there exists a set $\mathscr{E}(\alpha,\mathbf{x},v) \subseteq \{\{j\} \mid j \in N,\ x_j = v(j)\}$ such that $\mathscr{D}(\alpha,\mathbf{x},v) \cup \mathscr{E}(\alpha,\mathbf{x},v)$ is balanced.

19.10. *Proof of Theorem 19.6*

In the proof of Theorem 19.6, determine the dual program and conclude that it is feasible. Hint: use Theorem 16.19 and Remark 16.20.

19.11. *Nucleolus of a Three-Person Game (1)*

Compute the nucleolus of the three-person game v defined by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	4	3	2	4	3	2	12

19.12. *Nucleolus of a Three-Person Game (2)*

(a) Compute the nucleolus of the three-person TU-game defined by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	1	7	5	3	10

(b) Make a graphical representation of the sets $X_0, X_1, X_2, \ldots$

19.13. *Nucleolus of a Two-Person Game*

Compute the pre-nucleolus of a two-person game and the nucleolus of an essential two-person game.

19.14. *Individual Rationality Restrictions for the Nucleolus*

Compute the nucleolus and the pre-nucleolus of the three-person TU-game defined by $v(12) = v(13) = 2$, $v(123) = 1$ and $v(S) = 0$ for all other coalitions.

19.15. *Example 19.8*

Verify that $(32.5, 27.5, 25, 15)$ is indeed the pre-nucleolus of the game in Example 19.8, by applying Theorem 19.6.

19.16. *(Pre-)nucleolus of a Symmetric Game*

Let v be an essential game. Suppose that v is symmetric (meaning that there exists a function $f: \mathbb{R} \to \mathbb{R}$ such that $v(S) = f(|S|)$ for every coalition S.)

(a) Prove that the (pre-)nucleolus is symmetric, that is, $\nu(v)_i = \nu(v)_j$ and $\nu^*(v)_i = \nu^*(v)_j$ for all players $i, j \in N$. Give a formula for the (pre-)nucleolus.

(b) Prove that X_1 (cf. Theorem 19.1) is a singleton set. Which coalitions have maximal excess in the (pre-)nucleolus?

19.17. *COV and AN of the Pre-Nucleolus*

Prove that the pre-nucleolus satisfies COV and AN.

19.18. *Apex Game*

Consider the five-person *apex game* (N, v) with $N = \{1,2,3,4,5\}$ and $v(S) = 1$ if $1 \in S$ and $|S| \geq 2$ or if $|S| \geq 4$, and $v(S) = 0$ otherwise. Compute the (pre-)nucleolus of this game.

19.19. *Landlord Game*

Consider the landlord game in Problem 18.4. (See also [82].)

(a) Assume that for all $i = 1,\ldots,n-1$ we have $f(i) - f(i-1) \geq f(i+1) - f(i)$. Show that the (pre-)nucleolus of this game assigns $f(n) - \frac{n}{2}[f(n) - f(n-1)]$ to the landlord. Compare with the Shapley value.

(b) Assume that for all $i = 1,\ldots,n-1$ we have $f(i) - f(i-1) \leq f(i+1) - f(i)$, and that $\frac{1}{n+1} f(n) \leq \frac{1}{2}[f(n) - f(n-1)]$. Show that the (pre-)nucleolus of this game treats all players (including the landlord) equally.

19.20. *Game in Sect. 19.1*

Use the algorithm of solving successive linear programs to find the (pre-)nucleolus of the game discussed in Sect. 19.1. Use Theorem 19.6 to verify that the (pre-) nucleolus has been found.

19.21. *The Prekernel*

For a game (N,v) define the *pre-kernel* $\mathcal{K}^*(N,v) \subseteq I^*(N,v)$ by

$$\begin{aligned}\mathcal{K}^*(N,v) = \{x \in I^*(N,v) \mid & \max_{S \subseteq N \setminus \{j\},\; i \in S} e(S,\mathbf{x},v) \\ &= \max_{S \subseteq N \setminus \{i\},\; j \in S} e(S,\mathbf{x},v) \text{ for all } i,j \in N\}.\end{aligned}$$

Prove that $\nu^*(N,v) \in \mathcal{K}^*(N,v)$.

Chapter 20
Special Transferable Utility Games

In this chapter we consider several classes of games with transferable utility which are derived from specific economic (or political) models or combinatorial problems. In particular, we study assignment and permutation games, flow games, and voting games.[1]

20.1 Assignment and Permutation Games

An example of a permutation game is the 'dentist game' described in Sect. 1.3.4. An example of an assignment game is the following (from [22]).

Example 20.1. Vladimir (player 1), Wanda (player 2), and Xavier (player 3) each own a house that they want to sell. Yolanda (player 4) and Zarik (player 5) each want to buy a house. Vladimir, Wanda, and Xavier value their houses at 1, 1.5, and 2, respectively (each unit is 100,000 Euros). The worths of their houses to Yolanda and Zarik, respectively, are 0.8 and 1.5 for Vladimir's house, 2 and 1.2 for Wanda's house, and 2.2 and 2.3 for Xavier's house.

This situation gives rise to a five-player TU-game, where the worth of each coalition is defined to be the maximal surplus that can be generated by buying and selling within the coalition. For instance, in the coalition $\{2,3,5\}$ the maximum surplus is generated if Zarik buys the house of Xavier, namely $2.3-2=0.3$, which is greater than the $1.2-1.5=-0.3$ that results if Zarik buys Wanda's house. Each coalition can generate a payoff of at least 0 because it can refrain from trading at all. The complete game is described in Table 20.1, where coalitions with only buyers or only sellers are left out. A game like this is called an assignment game.

We will examine such games in detail,[2] starting with the basic definitions.

[1] Both our choice of topics and our treatment of these are limited. There is a large literature on combinatorial games and on voting games.

[2] The presentation is mainly based on [22], Chap. 3.

Table 20.1 An assignment game

S	$v(S)$	S	$v(S)$	S	$v(S)$
14	0	125	0.5	345	0.3
15	0.5	134	0.2	1,234	0.5
24	0.5	135	0.5	1,235	0.5
25	0	145	0.5	1,245	1
34	0.2	234	0.5	1,345	0.7
35	0.3	235	0.3	2,345	0.8
124	0.5	245	0.5	12,345	1

Let M and P be two finite, disjoint sets. For each pair $(i, j) \in M \times P$ the number $a_{ij} \geq 0$ is interpreted as the value of the matching between i and j. With this situation a cooperative game (N, v) can be associated, as follows. The player set N is the set $M \cup P$. For each coalition $S \subseteq N$ the worth $v(S)$ is the maximum that S can achieve by making pairs among its own members. Formally, if $S \subseteq M$ or $S \subseteq P$ then $v(S) := 0$, because no pairs can be formed at all. Otherwise, $v(S)$ is equal to the value of the following integer programming problem.

$$\begin{array}{rll} \max & \sum_{i \in M} \sum_{j \in P} a_{ij} x_{ij} & \\ \text{subject to} & \sum_{j \in P} x_{ij} \leq 1_S(i) & \text{for all } i \in M \\ & \sum_{i \in M} x_{ij} \leq 1_S(j) & \text{for all } j \in P \\ & x_{ij} \in \{0, 1\} & \text{for all } i \in M,\ j \in P. \end{array} \tag{20.1}$$

Here, $1_S(i) := 1$ if $i \in S$ and equal to zero otherwise. Games defined by (20.1) are called *assignment games*. These games were introduced by Shapley and Shubik [125]. Verify that in Example 20.1 the numbers a_{ij} are given by $a_{ij} = \max\{h_{ij} - c_i, 0\}$, where h_{ij} is the value of the house of player i to player j and c_i is the value of the house of player i for himself.

As will become clear below, a more general situation is the following. For each $i \in N = \{1, 2, \ldots, n\}$ let $k_{i\pi(i)}$ be the value placed by player i on the permutation $\pi \in \Pi(N)$. (The implicit assumption is that $k_{i\pi(i)} = k_{i\sigma(i)}$ whenever $\pi(i) = \sigma(i)$.) Each coalition $S \subseteq N$ may achieve a permutation π involving only the players of S, that is, $\pi(i) = i$ for all $i \notin S$. Let $\Pi(S)$ denote the set of all such permutations. Then a game v can be defined by letting, for each nonempty coalition S, the worth

$$v(S) := \max_{\pi \in \Pi(S)} \sum_{i \in S} k_{i\pi(i)}. \tag{20.2}$$

The game thus obtained is called a *permutation game*, introduced in [136]. Alternatively, the worth $v(S)$ in such a game can be defined by the following integer programming problem.

$$\begin{array}{rll} \max & \sum_{i \in N} \sum_{j \in N} k_{ij} x_{ij} & \\ \text{subject to} & \sum_{j \in N} x_{ij} = 1_S(i) & \text{for all } i \in N \end{array}$$

$$\begin{aligned} &\textstyle\sum_{i\in N} x_{ij} = 1_S(j) &&\text{for all } j \in N \\ &x_{ij} \in \{0,1\} &&\text{for all } i,j \in N. \end{aligned} \tag{20.3}$$

The two definitions are equivalent, and both can be used to verify that the 'dentist game' of Sect. 1.3.4 is indeed a permutation game (Problem 20.1).

The relation between the class of assignment games and the class of permutation games is a simple one. The former class is contained in the latter, as the following theorem shows.

Theorem 20.2. *Every assignment game is a permutation game.*

Proof. Let v be an assignment game with player set $N = M \cup P$. For all $i, j \in N$ define

$$k_{ij} := \begin{cases} a_{ij} & \text{if } i \in M,\ j \in P \\ 0 & \text{otherwise.} \end{cases}$$

Let w be the permutation game defined by (20.3) with k_{ij} as above. Note that the number of variables in the integer programming problem defining $v(S)$ is $|M| \times |P|$, while the number of variables in the integer programming problem defining $w(S)$ is $(|M| + |P|)^2$. For $S \subseteq M$ or $S \subseteq P$, $w(S) = 0 = v(S)$. Let now $S \subseteq N$ with $S \not\subseteq M$ and $S \not\subseteq P$. Let $x \in \{0,1\}^{|M|\times|P|}$ be an optimal solution for (20.1). Define $\hat{x} \in \{0,1\}^{(|M|+|P|)^2}$ by

$$\begin{aligned} \hat{x}_{ij} &:= x_{ij} \quad \text{if } i \in M,\ j \in P \\ \hat{x}_{ij} &:= x_{ji} \quad \text{if } i \in P,\ j \in M \\ \hat{x}_{ii} &:= 1_S(i) - \sum_{j\in P} x_{ij} \quad \text{if } i \in M \\ \hat{x}_{jj} &:= 1_S(j) - \sum_{i\in M} x_{ij} \quad \text{if } j \in P \\ \hat{x}_{ij} &:= 0 \quad \text{in all other cases.} \end{aligned}$$

Then $\hat{x}$ satisfies the conditions in problem (20.3). Hence, for every S,

$$w(S) \geq \sum_{i\in N}\sum_{j\in N} k_{ij}\hat{x}_{ij} = \sum_{i\in M}\sum_{j\in P} a_{ij}x_{ij} = v(S).$$

On the other hand, let $z \in \{0,1\}^{(|M|+|P|)^2}$ be an optimal solution for (20.3). Define $\hat{z} \in \{0,1\}^{|M|\times|P|}$ by

$$\hat{z}_{ij} := z_{ij} \quad \text{for } i \in M,\ j \in P.$$

Then $\hat{z}$ satisfies the conditions in problem (20.1). Hence, for every S,

$$v(S) \geq \sum_{i\in M}\sum_{j\in P} a_{ij}\hat{z}_{ij} = \sum_{i\in M}\sum_{j\in P} k_{ij}z_{ij} = w(S).$$

Consequently, $v = w$. □

The converse of Theorem 20.2 is not true, as the following example shows. As a matter of fact, a necessary condition for a permutation game to be an assignment game is the existence of a partition of the player set N of the permutation game into two subsets N_1 and N_2, such that the value of a coalition S is 0 whenever $S \subseteq N_1$ or $S \subseteq N_2$. The example shows that this is not a sufficient condition.

Example 20.3. Let $N = \{1,2,3\}$ and let v be the permutation game with the numbers k_{ij} given in the following matrix:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Then $v(i) = 0$ for every $i \in N$, $v(1,2) = v(1,3) = 3$, $v(2,3) = 0$, and $v(N) = 4$. Note that this game satisfies the condition formulated above with $N_1 = \{1\}$ and $N_2 = \{2,3\}$, but it is not an assignment game (Problem 20.2).

The main purpose of this section is to show that permutation games and, hence, assignment games are balanced and, in fact, totally balanced. A TU-game (N,v) is *totally balanced* if the *subgame* (M,v) – where v is the restriction to M – is balanced for every $M \subseteq N$. Balanced games are exactly those games that have a non-empty core, see Chap. 16.

Theorem 20.4. *Assignment games and permutation games are totally balanced.*

Proof. In view of Theorem 20.2, it is sufficient to prove that permutation games are totally balanced. Because any subgame of a permutation game is again a permutation game (see Problem 20.3), it is sufficient to prove that any permutation game is balanced.

Let (N,v) be a permutation game, defined by (20.3). By the Birkhoff–von Neumann Theorem (Theorem 22.11) the integer restriction can be dropped so that each $v(S)$ is also defined by the following program:

$$\begin{array}{rll} \max & \sum_{i\in N}\sum_{j\in N} k_{ij}x_{ij} & \\ \text{subject to} & \sum_{j\in N} x_{ij} = 1_S(i) & \text{for all } i \in N \\ & \sum_{i\in N} x_{ij} = 1_S(j) & \text{for all } j \in N \\ & x_{ij} \geq 0 & \text{for all } i,j \in N. \end{array} \tag{20.4}$$

Note that this is a linear programming problem of the same format as the maximization problem in Theorem 16.19. Namely, with notations as there, take

$$\begin{aligned} \mathbf{y} &= (x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{n1},\ldots,x_{nn}) \\ \mathbf{b} &= (k_{11},\ldots,k_{1n},k_{21},\ldots,k_{2n},\ldots,k_{n1},\ldots,k_{nn}) \\ \mathbf{c} &= (1_S,1_S). \end{aligned}$$

Further, let A be the $2n \times n^2$-matrix with row $k \in \{1,\ldots,n\}$ containing a 1 at columns $k, k+n, k+2n, \ldots, k+(n-1)n$ and zeros otherwise; and with row $k+n$ ($k \in \{1,\ldots,n\}$) containing a 1 at columns $(k-1)n+1,\ldots,kn$ and zeros otherwise.

The corresponding dual problem, the minimization problem in Theorem 16.19, then has the form:

$$\begin{aligned} &\min \sum_{i\in N} 1_S(i)y_i + \sum_{j\in N} 1_S(j)z_j \\ \text{subject to} \quad & y_i + z_j \geq k_{ij} \quad \text{for all } i,j \in N. \end{aligned} \tag{20.5}$$

Let $(\hat{\mathbf{y}},\hat{\mathbf{z}})$ be an optimal solution of problem (20.5) for $S = N$. Then, by Theorem 16.19 and the fact that the maximum in problem (20.4) for $S = N$ is equal to $v(N)$ by definition, it follows that

$$\sum_{i\in N}(\hat{y}_i + \hat{z}_i) = v(N).$$

Since $(\hat{\mathbf{y}},\hat{\mathbf{z}})$ satisfies the restrictions in problem (20.5) for every $S \subseteq N$, it furthermore holds that for every $S \subseteq N$,

$$\sum_{i\in S}(\hat{y}_i + \hat{z}_i) = \sum_{i\in N} 1_S(i)\hat{y}_i + \sum_{i\in N} 1_S(i)\hat{z}_i \geq v(S).$$

Therefore, $\mathbf{u} \in \mathbb{R}^N$ defined by $u_i := \hat{y}_i + \hat{z}_i$ is in the core of v. □

20.2 Flow Games

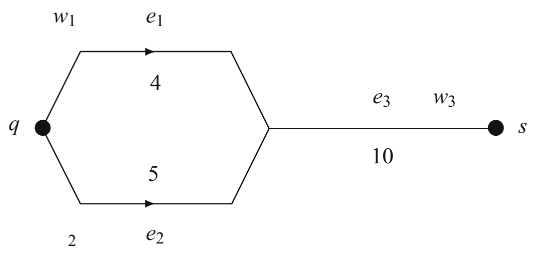
In this section another class of balanced games is considered. These games are derived from the following kind of situation. There is a given capacitated network, the edges of which are controlled by subsets of players. These coalitions can send a flow through the network. The flow is maximal if all players cooperate, and then the question arises how to distribute the profits. One can think of an almost literal example, where the edges represent oil pipelines, and the players are in power in different countries through which these pipelines cross. Alternatively, one can think of rail networks between cities, or information channels between different users.

Capacitated networks are treated in Sect. 22.7, which should be read before continuing.

Consider a capacitated network (V,E,k) and a set of players $N := \{1,2,\ldots,n\}$. Suppose that with each edge in E a simple game is associated.[3] The winning coalitions in this simple game are supposed to control the corresponding edge; the capacitated network is called a *controlled* capacitated network. For any coalition $S \subseteq N$ consider the capacitated network arising from the given network by deleting the edges that are *not* controlled by S. A game can be defined by letting the worth of S be equal to the value of a maximal flow through this restricted network. The game thus arising is called a *flow game*.

[3] See Sect. 16.3.

Fig. 20.1 Example 20.5



Example 20.5. Consider the capacitated network in Fig. 20.1. This network has three edges denoted e_1, e_2, and e_3 with capacities 4,5 and 10, respectively. The control games are w_1, w_2, w_3 with $N = \{1,2,3\}$ and

$$\begin{array}{ll} w_1(S) = 1 & \text{if } S \in \{\{1,2\},N\} \text{ and } w_1(S) = 0 \text{ otherwise} \\ w_2(S) = 1 & \text{if } S \in \{\{1,3\},N\} \text{ and } w_2(S) = 0 \text{ otherwise} \\ w_3(S) = 1 & \text{if, and only if, } 1 \in S. \end{array}$$

The coalition $\{1,2\}$ can only use the edges e_1 and e_3, so the maximal flow (per time unit) for $\{1,2\}$ is 4. This results in $v(\{1,2\}) = 4$ for the corresponding flow game (N,v). This game is given by $v(i) = 0$ for all $i \in N$, $v(\{1,2\}) = 4$, $v(\{1,3\}) = 5$, $v(\{2,3\}) = 0$ and $v(N) = 9$.

A minimum cut in this network corresponding to the grand coalition is $(\{q\}, N \setminus \{q\})$. By the Max-Flow Min-Cut Theorem of Ford and Fulkerson [36], Theorem 22.15, the sum of the capacities of e_1 and e_2 $(4+5)$ is equal to $v(N)$. Divide $v(N)$ as follows. Divide 4 equally among the veto players of w_1, and 5 equally among the veto players of w_2. The result for the players is the payoff vector $(4\frac{1}{2}, 2, 2\frac{1}{2})$. Note that this vector is in $C(v)$.

The next theorem shows that the non-emptiness of the core of the control games is inherited by the flow game.

Theorem 20.6 (cf. [23]). *Suppose all control games in a controlled capacitated network have veto players. Then the corresponding flow game is balanced.*

Proof. Take a maximal flow for the grand coalition and a minimum cut in the network for the grand coalition, consisting of the edges

$$e_1, e_2, \ldots, e_p \text{ with capacities } k_1, k_2, \ldots, k_p$$

and control games w_1, w_2, ..., w_p, respectively. Then Theorem 22.15 implies that $v(N) = \sum_{r=1}^{p} k_r$. For each r take $x^r \in C(w_r)$ and divide k_r according to the division key x^r (i.e., $k_r x_i^r$ is the amount for player i). Note that non-veto players get nothing. Then $\sum_{r=1}^{p} k_r x^r \in C(v)$ since:

(1) $\sum_{i=1}^{n} \sum_{r=1}^{p} k_r x_i^r = \sum_{r=1}^{p} k_r \sum_{i=1}^{n} x_i^r = \sum_{r=1}^{p} k_r = v(N)$.

(2) For each coalition S, the set

$$E_S := \{e_r : r \in \{1, \dots p\}, w_r(S) = 1\}$$

is associated with a cut of the network, governed by the coalition S.

Hence, $\sum_{i \in S} \left(\sum_{r=1}^{p} k_r x_i^r \right) = \sum_{r=1}^{p} k_r \sum_{i \in S} x_i^r \geq \sum_{r=1}^{p} k_r w_r(S) = \sum_{e_r \in E_S} k_r =$ capacity$(E_S) \geq v(S)$, where the last inequality follows from Theorem 22.15. □

The next theorem is a kind of converse to Theorem 20.6.

Theorem 20.7. *Each nonnegative balanced game arises from a controlled capacitated network where all control games possess veto players.*

Proof. See Problem 20.5. □

20.3 Voting Games: The Banzhaf Value

Voting games constitute another special class of TU-games. Voting games are simple games which reflect the distribution of voting power within, for instance, political systems. There is a large body of work on voting games within the political science literature. In this section we restrict ourselves to a brief discussion of a well-known example of a power index, to so-called Banzhaf–Coleman index and the associated value, the Banzhaf value.

A *power index* is a value applied to voting (simple) games. The payoff vector assigned to a game is interpreted as reflecting power distribution – e.g., the probability of having a decisive vote – rather than utility.

We start with an illustrating example.

Example 20.8. Consider a parliament with three parties 1, 2, and 3. The numbers of votes are, respectively, 50, 30, and 20. To pass any law, a two-third majority is needed. This leads to a simple game with winning coalitions $\{1,2\}$, $\{1,3\}$, and $\{1,2,3\}$. The Shapley value[4] of this game is $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, as can easily be checked. By definition of the Shapley value this means that in four of the six permutations player 1 makes the coalition of his predecessors winning by joining them, whereas for players 2 and 3 this is only the case with one permutation for each. The coalitions that are made winning by player 1 if he joins, are $\{2\}$, $\{3\}$, and $\{2,3\}$. In the Shapley value the last coalition is counted double. It might be more natural to count this coalition only once. This would lead to an outcome $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, instead of the Shapley value. The associated value is called the *normalized Banzhaf–Coleman index*.

For a simple game (N, v), the *normalized Banzhaf–Coleman index* can be defined as follows. Define a *swing* for player i as a coalition $S \subseteq N$ with $i \in S$, S wins, and $S \setminus \{i\}$ loses. Let θ_i be the number of swings for player i, then define the numbers

[4] Also called the *Shapley–Shubik power index* in this context.

$$\beta_i(N,v) := \frac{\theta_i}{\sum_{j=1}^n \theta_j}.$$

The vector $\beta(N,v)$ is the normalized Banzhaf–Coleman index of the simple game (N,v).

For a general game (N,v) write

$$\theta_i(v) := \sum_{S\subset N:\ i\notin S} [v(S\cup i) - v(S)].$$

For a simple game v this number $\theta_i(v)$ coincides with the number θ_i above.

Next, define the value $\Psi : \mathcal{G}^N \to \mathbb{R}^N$ by

$$\Psi_i(v) := \frac{\theta_i(v)}{2^{|N|-1}} = \sum_{S\subset N:\ i\notin S} \frac{1}{2^{|N|-1}}[v(S\cup i) - v(S)]. \tag{20.6}$$

The value Ψ is called the *Banzhaf value*. The remainder of this section is devoted to an axiomatic characterization of this value. In the literature many characterizations are available. The one presented below is based on Nowak [94]. The characterization uses the axioms SYM (Symmetry), SMON (Strong Monotonicity), and DUM (Dummy Property), which were all introduced in Chap. 17. Besides, it uses a kind of 'reduced game' or 'amalgamation' property, as follows.

For a game (N,v) (with at least two players) and different players i,j put $p = \{i,j\}$ and define the game $((N\setminus p)\cup\{p\}, v_p)$ by

$$v_p(S) = v(S) \quad \text{and} \quad v_p(S\cup\{p\}) = v(S\cup p), \quad \text{for any } S\subseteq N\setminus p. \tag{20.7}$$

Thus, v_p is an $(n-1)$-person game obtained by amalgamating players i and j in v into one player p in v_p.

Let ψ be an arbitrary value (on the class $\mathcal{G}$ of all games with arbitrary player set). The announced axiom is as follows.

2-Efficiency (2-EFF): $\psi_i(v) + \psi_j(v) = \psi_p(v_p)$ for all v,i,j,p,v_p as above.

The following theorem gives a characterization of the Banzhaf value.

Theorem 20.9. *The value ψ on $\mathcal{G}$ satisfies 2-EFF, SYM, DUM, and SMON, if and only if ψ is the Banzhaf value Ψ.*

Proof. That the Banzhaf value satisfies the four axioms in the theorem is the subject of Problem 20.9. For the converse, let ψ be a value satisfying the four axioms. We prove that $\psi = \Psi$.

Step 1: Let u_T be a unanimity game. We first show that

$$\psi_i(u_T) = 1/2^{|T|-1} \quad \text{if } i\in T \quad \text{and} \quad \psi_i(u_T) = 0 \quad \text{if } i\notin T. \tag{20.8}$$

If $|T| = 1$ then every player is a dummy, so that (20.8) follows from DUM. Suppose (20.8) holds whenever $|T| \le k$ or $|N| \le m$, and consider a unanimity game u_T where now the number of players is $m+1$, and T contains $k+1$ players. Let $i,j\in T$, put

$p = \{i, j\}$ and consider the game $(u_T)_p$. Then $(u_T)_p$ is the m-person unanimity game of the coalition $T' = (T \setminus p) \cup \{p\}$, and $|T'| = k$. By the induction hypothesis

$$\psi_p((u_T)_p) = 1/2^{|T'|-1} = 1/2^{k-1}.$$

By 2-EFF this implies

$$\psi_i(u_T) + \psi_j(u_T) = 1/2^{k-1}.$$

From this and SYM it follows that

$$\psi_i(u_T) = 1/2^k = 1/2^{|T|-1}, \quad i \in T,$$

and by DUM, $\psi_j(u_T) = 0$ when $j \notin T$. Thus, ψ is the Banzhaf value on unanimity games for any finite set of players. In the same way, one shows that this is true for any real multiple cu_T of a unanimity game.

Step 2: For an arbitrary game v write $v = \sum_{\emptyset \neq T} c_T u_T$, and let $\alpha(v)$ denote the number of nonzero coefficients in this representation. The proof will be completed by induction on the number $\alpha(v)$ and the number of players. For $\alpha(v) = 1$ Step 1 implies $\psi(v) = \Psi(v)$ independent of the number of players. Assume that $\psi(v) = \Psi(v)$ on any game v with at most n players, and also any game v with $\alpha(v) \leq k$ for some natural number k and with $n+1$ players, and let v be a game with $n+1$ players and with $\alpha(v) = k+1$. There are $k+1$ different nonempty coalitions $T_1, \ldots, T_{k+1}$ with

$$v = \sum_{r=1}^{k+1} c_{T_r} u_{T_r},$$

where all coefficients are nonzero. Let $T := T_1 \cap \ldots \cap T_{k+1}$. Because $k+1 \geq 2$, it holds that $N \setminus T \neq \emptyset$. Assume $i \notin T$. Define the game w by

$$w = \sum_{r:\ i \in T_r} c_{T_r} u_{T_r}.$$

Then $\alpha(w) \leq k$ and $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for every coalition S not containing player i. By SMON and the induction hypothesis it follows that $\psi_i(v) = \psi_i(w) = \Psi_i(w) = \Psi_i(w)$. Hence,

$$\psi_i(v) = \Psi_i(v) \quad \text{for every } i \in N \setminus T. \tag{20.9}$$

Let $j \in T$ and $i \in N \setminus T$, put $p = \{i, j\}$, and consider the game v_p. Because the game v_p has n players the induction hypothesis implies

$$\psi_p(v_p) = \Psi_p(v_p). \tag{20.10}$$

Applying axiom 2-EFF to both ψ and Ψ yields

$$\psi_p(v_p) = \psi_i(v) + \psi_j(v) \quad \text{and} \quad \Psi_p(v_p) = \Psi_i(v) + \Psi_j(v). \tag{20.11}$$

Combining (20.9), (20.10), and (20.11) implies $\psi_j(v) = \Psi_j(v)$ for every $j \in T$. Together with (20.9) this completes the induction argument, and therefore the proof. □

Problems

20.1. *The Dentist Game*

Show that (20.2) and (20.3) are equivalent, and use each of these to verify that the 'dentist game' of Sect. 1.3.4 is a permutation game.

20.2. *Example 20.3*

Show that the game in Example 20.3 is not an assignment game.

20.3. *Subgames of Permutation Games*

Prove that subgames of permutation games are again permutation games. Is this also true for assignment games?

20.4. *A Flow Game*

Consider the network in Fig. 20.2. Suppose that this is a controlled capacitated network with player set $N = \{1,2,3,4\}$, suppose that all edges have capacity 1 and that $w_1 = \delta_1$, $w_2 = \delta_2$, $w_3 = \delta_3$ and $w_4(S) = 1$ iff $S \in \{\{3,4\},N\}$. (Here, δ_i is the simple game where a coalition is winning if, and only if, it contains player i.)

(1) Calculate the corresponding flow game (N,v).

(2) Calculate $C(v)$.

(3) The proof of Theorem 20.6 describes a way to find core elements by looking at minimum cuts and dividing the capacities of edges in the minimum cut in some way among the veto players of the corresponding control game. Which elements of $C(v)$ can be obtained in this way?

20.5. *Every Nonnegative Balanced Game is a Flow Game*

Prove that every nonnegative balanced game is a flow game. (Hint. Use the following result (cf. [28]): every nonnegative balanced game can be written as a positive linear combination of balanced simple games.)

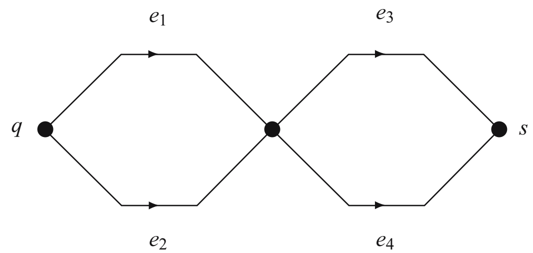


Fig. 20.2 The network of Problem 20.4

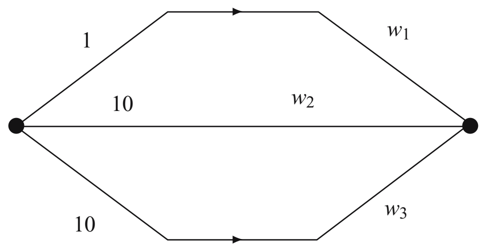


Fig. 20.3 The network of Problem 20.6

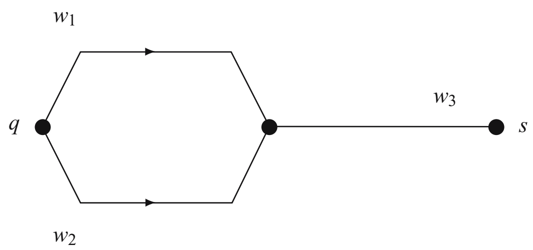


Fig. 20.4 The network of Problem 20.7

20.6. *On Theorem 20.6 (1)*

(1) Consider a controlled capacitated network with a minimum cut, where all control games corresponding to the edges in this minimum cut (connecting vertices between the two sets in the cut) have veto players. Prove that the corresponding flow game is balanced.

(2) Show that the flow game, corresponding to Fig. 20.3, where the winning coalitions of w_1 are $\{1,3\}$, $\{2,4\}$ and $N = \{1,2,3,4\}$, where the winning coalitions of w_2 are $\{1,2\}$ and N and of w_3 $\{3,4\}$ and N and where the capacities are 1, 10, 10 respectively, has a nonempty core. Note that there is no minimum cut where all control games have veto players.

20.7. *On Theorem 20.6 (2)*

Prove that the two-person flow game corresponding to the controlled capacitated network of Fig. 20.4 has an empty core, where $w_1 = \delta_1$, $w_2 = \delta_2$, $w_3(S) = 1$ if $S \neq \emptyset$, and where the capacities of the edges are equal to 1.

20.8. *Totally Balanced Flow Games*

(Cf. [64].) Let (N, v) be the flow game corresponding to a controlled capacitated network where all control games are dictatorial games (games of the form δ_i, see Problem 20.4). Prove that each subgame (S, v_S) (where v_S is the restriction of v to 2^S) has a nonempty core, i.e., that the game (N, v) is totally balanced.

20.9. *If-Part of Theorem 20.9*

Prove that the Banzhaf value satisfies 2-EFF, SYM, DUM, and SMON. Is it possible to weaken DUM to NP (the null-player property) in Theorem 20.9? Give an example showing that the Banzhaf value is not efficient.

Chapter 21
Bargaining Problems

The game-theoretic literature on bargaining can be divided in two strands: the cooperative and the noncooperative approach. Here, the focus is on the cooperative approach, which was initiated by Nash [90] and which is axiomatic in nature.[1] A seminal article on noncooperative bargaining is Rubinstein [110]. The basic idea of that paper is briefly repeated below, see Sect. 6.7 for a more elaborate discussion. We conclude the chapter with a few remarks on games with non-transferable utility (NTU-games).

21.1 The Bargaining Problem

Bargaining problems[2] were introduced by Nash [90]. A two-*person bargaining problem* is a pair $(S,\mathbf{d})$ where S is a compact convex nonempty subset of $\mathbb{R}^2$ and $\mathbf{d}$ is an element of S such that $\mathbf{x} > \mathbf{d}$ for some $\mathbf{x} \in S$. The elements of S are called *outcomes* and $\mathbf{d}$ is the *disagreement outcome*. The interpretation of such a problem $(S,\mathbf{d})$ is as follows. Two bargainers, 1 and 2, have to agree on some outcome $\mathbf{x} \in S$, yielding utility x_i to bargainer i. If they fail to reach such an agreement, they end up with the disagreement utilities $\mathbf{d} = (d_1, d_2)$. B denotes the family of all two-person bargaining problems.

A (*bargaining*) *solution* is a map $F : B \to \mathbb{R}^2$ such that $F(S,\mathbf{d}) \in S$ for all $(S,\mathbf{d}) \in B$. Nash [90] proposed to characterize such a solution by requiring it to satisfy certain axioms. More precisely, he proposed the following axioms.[3]

Weak Pareto Optimality (WPO): $F(S,\mathbf{d}) \in W(S)$ for all $(S,\mathbf{d}) \in B$, where $W(S) := \{\mathbf{x} \in S \mid \forall \mathbf{y} \in \mathbb{R}^2 : \mathbf{y} > \mathbf{x} \Rightarrow \mathbf{y} \notin S\}$ is the *weakly Pareto optimal subset of* S.

[1] For comprehensive surveys see [103] or [135].

[2] See Sect. 10.1 for a first discussion.

[3] See Fig. 10.2 for an illustration of these axioms. In Sect. 10.1 the stronger Pareto Optimality is imposed instead of Weak Pareto Optimality. In the diagram – panel (a) – that does not make a difference.

Symmetry (SYM): $F_1(S,\mathbf{d}) = F_2(S,\mathbf{d})$ for all $(S,\mathbf{d}) \in B$ that are *symmetric*, i.e., $d_1 = d_2$ and $S = \{(x_2,x_1) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\}$.

Scale Covariance (SC): $F(\mathbf{a}S+\mathbf{b},\mathbf{ad}+\mathbf{b}) = \mathbf{a}F(S,\mathbf{d})+\mathbf{b}$ for all $(S,\mathbf{d}) \in B$, where $\mathbf{b} \in \mathbb{R}^2$, $\mathbf{a} \in \mathbb{R}^2_{++}$, $\mathbf{ax} := (a_1x_1,a_2x_2)$ for all $\mathbf{x} \in \mathbb{R}^2$, and $\mathbf{a}S := \{\mathbf{ax} \mid \mathbf{x} \in S\}$.

Independence of Irrelevant Alternatives (IIA): $F(S,\mathbf{d}) = F(T,\mathbf{e})$ for all $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B$ with $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, and $F(T,\mathbf{e}) \in S$.

Weak Pareto Optimality says that it should not be possible for both bargainers to gain with respect to the solution outcome. If a game is symmetric, then there is no way to distinguish between the bargainers, and a solution should not do that either: that is what Symmetry requires. Scale Covariance requires the solution to be covariant under positive affine transformations: the underlying motivation is that the *utility functions* of the bargainers are usually assumed to be of the von Neumann–Morgenstern type, which implies that they are representations of preferences unique only up to positive affine transformations (details are omitted here). Independence of Irrelevant Alternatives requires the solution outcome not to change when the set of possible outcomes shrinks, the original solution outcome still remaining feasible.

The *Nash (bargaining) solution* $N : B \to \mathbb{R}^2$ is defined as follows. For every $(S,\mathbf{d}) \in B$,

$$N(S,\mathbf{d}) = \operatorname{argmax}\{(x_1-d_1)(x_2-d_2) \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{d}\}.$$

That the Nash bargaining solution is well defined, follows from Problem 21.3.

Nash [90] proved the following theorem.

Theorem 21.1. *Let $F : B \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(1) $F = N$.
(2) *F satisfies WPO, SYM, SC, IIA.*

Proof. The implication (1)⇒(2) is the subject of Problem 21.4. For the implication (2)⇒(1), assume F satisfies WPO, SYM, SC, and IIA. Let $(S,\mathbf{d}) \in B$, and $\mathbf{z} := N(S,\mathbf{d})$. Note that $\mathbf{z} > \mathbf{d}$. Let $T := \{((z_1-d_1)^{-1}, (z_2-d_2)^{-1})(\mathbf{x}-\mathbf{d}) \mid \mathbf{x} \in S\}$. By SC,

$$F(T,\mathbf{0}) = ((z_1-d_1)^{-1},(z_2-d_2)^{-1})F(S,\mathbf{d}) - ((z_1-d_1)^{-1},(z_2-d_2)^{-1})\mathbf{d} \quad (21.1)$$

and

$$N(T,\mathbf{0}) = ((z_1-d_1)^{-1},(z_2-d_2)^{-1})(\mathbf{z}-\mathbf{d}) = (1,1). \quad (21.2)$$

Hence, in order to prove $F(S,\mathbf{d}) = N(S,\mathbf{d})$, it is, in view of (21.1) and (21.2), sufficient to show that $F(T,\mathbf{0}) = (1,1)$. By (21.2) and Problem 21.5, there is a supporting line of T at $(1,1)$ with slope -1. So the equation of this supporting line is $x_1 + x_2 = 2$. Choose $\alpha > 0$ so large that $T \subseteq D := \operatorname{conv}\{(-\alpha,-\alpha), (-\alpha,2+\alpha), (2+\alpha,-\alpha)\}$. Cf. Fig. 21.1.

Then $(D,\mathbf{0}) \in B$, $(D,\mathbf{0})$ is symmetric, and $W(D) = \operatorname{conv}\{(-\alpha,2+\alpha), (2+\alpha,-\alpha)\}$. Hence by SYM and WPO of F:

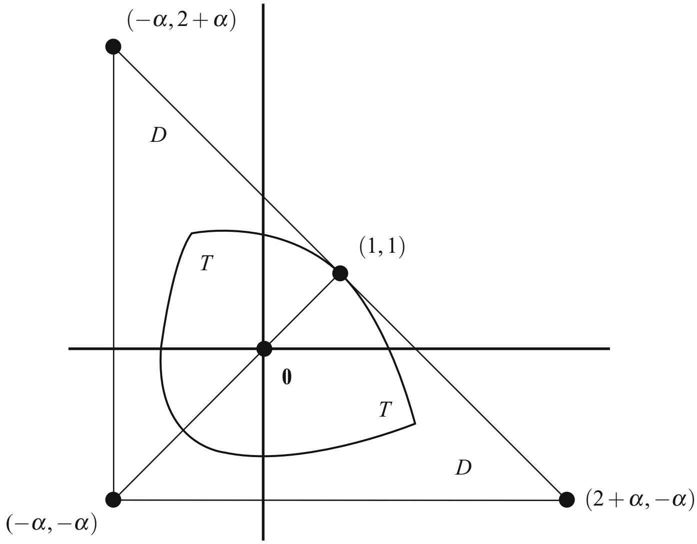


Fig. 21.1 Proof of Theorem 21.1

$$F(D,0) = (1,1). \tag{21.3}$$

Since $T \subseteq D$ and $(1,1) \in T$, we have by IIA and (21.3): $F(T,0) = (1,1)$. This completes the proof. □

21.2 The Raiffa–Kalai–Smorodinsky Solution

Kalai and Smorodinsky [63] replaced Nash's IIA (the most controversial axiom in Theorem 21.1) by the following condition. For a problem $(S,\mathbf{d}) \in B$,

$$u(S,\mathbf{d}) := (\max\{x_1 \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{d}\}, \max\{x_2 \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{d}\})$$

is called the *utopia point of* $(S,\mathbf{d})$.

Individual Monotonicity (IM): $F_j(S,\mathbf{d}) \leq F_j(T,\mathbf{e})$ for all $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B$ and $i, j \in \{1,2\}$ with $i \neq j$, $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, and $u_i(S,\mathbf{d}) = u_i(T,\mathbf{e})$.

The *Raiffa–Kalai–Smorodinsky solution* (Raiffa [105]) $R : B \to \mathbb{R}^2$ is defined as follows. For every $(S,\mathbf{d}) \in B$, $R(S,\mathbf{d})$ is the point of intersection of $W(S)$ with the straight line joining $\mathbf{d}$ and $u(S,d)$.

The following theorem is a modified version of the characterization theorem obtained by Kalai and Smorodinsky [63]. In order to understand the proof it is advisable to draw pictures, just as in the proof of the characterization of the Nash bargaining solution.

Theorem 21.2. *Let $F : B \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(1) *$F = R$.*
(2) *F satisfies WPO, SYM, SC, and IM.*

Proof. The implication (1)⇒(2) is the subject of Problem 21.7. For the converse implication, assume F has the four properties stated. Let $(S,\mathbf{d}) \in B$ and let $T := \{\mathbf{ax}+\mathbf{b} \mid \mathbf{x} \in S\}$ with $\mathbf{a} := ((u_1(S,\mathbf{d})-d_1)^{-1}, (u_2(S,\mathbf{d})-d_2)^{-1})$, $\mathbf{b} := -\mathbf{ad}$. By SC of R and F, $R(T,\mathbf{0}) = \mathbf{a}R(S,\mathbf{d})+\mathbf{b}$ and $F(T,\mathbf{0}) = \mathbf{a}F(S,\mathbf{d})+\mathbf{b}$. Hence, for $F(S,\mathbf{d}) = R(S,\mathbf{d})$, it is sufficient to prove that $R(T,\mathbf{0}) = F(T,\mathbf{0})$.

Since $u(T,\mathbf{0}) = (1,1)$, $R(T,\mathbf{0})$ is the point of $W(T)$ with equal coordinates, so $R_1(T,\mathbf{0}) = R_2(T,\mathbf{0})$. If $R(T,\mathbf{0}) = (1,1) = u(T,\mathbf{0})$, then let $L :=$ conv$\{(0,0),(1,1)\}$. Then by WPO, $F(L,\mathbf{0}) = (1,1)$, so by IM, $F(T,\mathbf{0}) \geq F(L,\mathbf{0})$, hence $F(T,\mathbf{0}) = F(L,\mathbf{0}) = R(T,\mathbf{0})$.

Next assume $R(T,\mathbf{0}) < (1,1)$. Let $\widetilde{T} := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{y} \leq \mathbf{x} \leq \mathbf{z}$ for some $\mathbf{y},\mathbf{z} \in T\}$. Clearly $T \subseteq \widetilde{T}$ and $u(\widetilde{T},0) = u(T,\mathbf{0}) = (1,1)$ so by IM:

$$F(\widetilde{T},\mathbf{0}) \geq F(T,\mathbf{0}), \tag{21.4}$$

and further, since $R(T,\mathbf{0}) \in W(T)$ and $R_1(\widetilde{T},\mathbf{0}) = R_2(\widetilde{T},\mathbf{0})$,

$$R(\widetilde{T},\mathbf{0}) = R(T,\mathbf{0}). \tag{21.5}$$

Let $V :=$ conv$\{\mathbf{0},R(T,\mathbf{0}),(1,0),(0,1)\}$. By WPO and SYM, $F(V,\mathbf{0}) = R(T,\mathbf{0})$. By $V \subseteq \widetilde{T}$, $u(V,\mathbf{0}) = u(\widetilde{T},\mathbf{0}) = (1,1)$, and IM, we have $F(\widetilde{T},\mathbf{0}) \geq F(V,\mathbf{0}) = R(T,\mathbf{0})$, hence $F(\widetilde{T},\mathbf{0}) = R(T,\mathbf{0})$. Combined with (21.4), this implies $R(T,\mathbf{0}) \geq F(T,\mathbf{0})$, hence $R(T,\mathbf{0}) = F(T,\mathbf{0})$ by WPO and the fact $R(T,\mathbf{0}) < (1,1)$. This completes the proof. □

21.3 The Egalitarian Solution

Consider the following two properties for a bargaining solution F.

Pareto Optimality (PO): $F(S,\mathbf{d}) \in P(S)$ for all $(S,\mathbf{d}) \in B$, where $P(S) := \{\mathbf{x} \in S \mid \forall \mathbf{y} \in S : \mathbf{y} \geq \mathbf{x} \Rightarrow \mathbf{y} = \mathbf{x}\}$ is the *Pareto optimal subset of S*.

Monotonicity (MON): $F(S,\mathbf{d}) \leq F(T,\mathbf{e})$ for all $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B$ with $S \subseteq T$ and $\mathbf{d} = \mathbf{e}$.

Clearly, $P(S) \subseteq W(S)$ for every $(S,\mathbf{d}) \in B$, and Pareto optimality is a stronger requirement than Weak Pareto Optimality. The Nash and Raiffa solutions are Pareto optimal, and therefore WPO can be replaced by PO in Theorems 21.1 and 21.2. Monotonicity is much stronger than Individual Monotonicity or Restricted Monotonicity (see Problem 21.8 for the definition of the last axiom) and in fact it is inconsistent with Weak Pareto Optimality (see Problem 21.10).

Call a problem $(S,\mathbf{d}) \in B$ *comprehensive* if $\mathbf{z} \le \mathbf{y} \le \mathbf{x}$ implies $\mathbf{y} \in S$ for all $\mathbf{z}, \mathbf{x} \in S$, $\mathbf{y} \in \mathbb{R}^2$. By B^c we denote the subclass of comprehensive problems.

The *egalitarian solution* $E : B^c \to \mathbb{R}^2$ assigns to each problem $(S,\mathbf{d}) \in B^c$ the point $E(S,\mathbf{d}) \in W(S)$ with $E_1(S,\mathbf{d}) - d_1 = E_2(S,\mathbf{d}) - d_2$.

The following axiom is a weakening of Scale Covariance.

Translation Covariance (TC): $F(S+\mathbf{e},\mathbf{d}+\mathbf{e}) = F(S,\mathbf{d}) + \mathbf{e}$ for all problems $(S,\mathbf{d})$ and all $\mathbf{e} \in \mathbb{R}^2$.

The following theorem gives a characterization of the egalitarian solution based on Monotonicity.

Theorem 21.3. *Let $F : B^c \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(1) *$F = E$.*
(2) *F satisfies WPO, MON, SYM, and TC.*

Proof. The implication (1)⇒(2) is the subject of Problem 21.11. For the converse implication, let $(S,\mathbf{d}) \in B^c$. We want to show $F(S,\mathbf{d}) = E(S,\mathbf{d})$.

In view of TC of F and E, we may assume $\mathbf{d} = \mathbf{0}$. Let $V := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{0} \le \mathbf{x} \le E(S,\mathbf{0})\}$. Clearly, $(V,\mathbf{0}) \in B^c$ is a symmetric problem, so $F(V,\mathbf{0}) = E(S,\mathbf{0})$ by SYM and WPO of F. By MON,

$$F(S,\mathbf{0}) \ge F(V,\mathbf{0}) = E(S,\mathbf{0}). \tag{21.6}$$

If $E(S,\mathbf{0}) \in P(S)$, then (21.6) implies $F(S,\mathbf{0}) = E(S,\mathbf{0})$, so we are done. Now suppose $E(S,\mathbf{0}) \in W(S) \setminus P(S)$. Without loss of generality, assume $E_1(S,\mathbf{0}) = u_1(S,\mathbf{0})$, i.e., $E_1(S,\mathbf{0}) = \max\{x_1 \mid \mathbf{x} \in S,\, \mathbf{x} \ge \mathbf{0}\}$. Hence, $E_1(S,\mathbf{0}) = F_1(S,\mathbf{0})$ by (21.6).

Suppose $F_2(S,\mathbf{0}) > E_2(S,\mathbf{0})$. The proof will be finished by contradiction. Let $\alpha > 0$ with $E_2(S,\mathbf{0}) < \alpha < F_2(S,\mathbf{0})$. Let $T := \mathrm{conv}(S \cup \{(\alpha,0),\,(\alpha,\alpha)\})$. Then $(T,\mathbf{0}) \in B^c$ and $E(T,\mathbf{0}) = (\alpha,\alpha) \in P(T)$, so $F(T,\mathbf{0}) = (\alpha,\alpha)$ by our earlier argument (see the line below (21.6)). On the other hand, by MON, $F_2(T,\mathbf{0}) \ge F_2(S,\mathbf{0}) > \alpha$, a contradiction. □

An alternative characterization of the egalitarian solution can be obtained by using the following axioms.

Super-Additivity (SA): $F(S+T,\mathbf{d}+\mathbf{e}) \ge F(S,\mathbf{d}) + F(T,\mathbf{e})$ for all $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B^c$. Here, $S+T := \{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in S,\, \mathbf{y} \in T\}$.

Individual Rationality (IR): $F(S,\mathbf{d}) \ge \mathbf{d}$ for all $(S,\mathbf{d}) \in B$.

Theorem 21.4. *Let $F : B^c \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(1) *$F = E$.*
(2) *F satisfies WPO, SA, SYM, IR, and TC.*

Proof. (1)⇒(2) follows from Theorem 21.3 and Problem 21.12. For the converse implication, let $(S,\mathbf{d}) \in B^c$. We wish to show $F(S,\mathbf{d}) = E(S,\mathbf{d})$. In view of TC of F and E we may assume $\mathbf{d} = \mathbf{0}$. For every $1 > \varepsilon > 0$ let $V^\varepsilon := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{0} \leq \mathbf{x} \leq (1-\varepsilon)E(S,\mathbf{0})\}$. Then $(V^\varepsilon,\mathbf{0}) \in B^c$ and $F(V^\varepsilon,\mathbf{0}) = E(V^\varepsilon,\mathbf{0}) = (1-\varepsilon)E(S,\mathbf{0})$ by WPO and SYM of F and E. Since $S = V^\varepsilon + (S - V^\varepsilon)$, we have by *SA*:

$$F(S,\mathbf{0}) \geq (1-\varepsilon)E(S,\mathbf{0}) + F(S - V^\varepsilon,\mathbf{0}). \tag{21.7}$$

Letting ε decrease to 0, we obtain by (21.7) and IR:

$$F(S,\mathbf{0}) \geq E(S,\mathbf{0}). \tag{21.8}$$

If $E(S,\mathbf{0}) \in P(S)$, then (21.8) implies $F(S,\mathbf{0}) = E(S,\mathbf{0})$ and we are done. Otherwise, suppose without loss of generality that $E_1(S,\mathbf{0}) = \max\{x_1 \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{0}\}$. Let $\mathbf{z}$ be the point of $P(S)$ with $z_1 = E_1(S,\mathbf{0})$, hence $\alpha := E_2(S,\mathbf{0}) - z_2 < 0$ since, by assumption, $E(S,\mathbf{0}) \notin P(S)$. For $\varepsilon > 0$, let $R_\varepsilon := \text{conv}\{(0,\varepsilon), (0,\alpha), (\varepsilon,\alpha)\}$. Then $(R_\varepsilon,\mathbf{0}) \in B^c$. Further, let $T_\varepsilon := S + R_\varepsilon$. By construction, $E(T_\varepsilon,\mathbf{0}) \in P(T_\varepsilon)$, hence, as before, $F(T_\varepsilon,\mathbf{0}) = E(T_\varepsilon,\mathbf{0})$. If ε approaches 0, $F(T_\varepsilon,\mathbf{0})$ converges to $E(S,\mathbf{0})$ and by SA and IR, $F(T_\varepsilon,\mathbf{0}) \geq F(S,\mathbf{0})$. So $E(S,\mathbf{0}) \geq F(S,\mathbf{0})$. Combined with (21.8), this gives $F(S,\mathbf{0}) = E(S,\mathbf{0})$. □

21.4 Noncooperative Bargaining

A different approach to bargaining is obtained by studying it a as strategic process. In this section we discuss the basics of the model of Rubinstein [110] in an informal manner.[4]

Point of departure is a bargaining problem $(S,\mathbf{d}) \in B$. Assume $\mathbf{d} = \mathbf{0}$ and write S instead of $(S,\mathbf{d})$. Suppose bargaining takes place over time, at moments $t = 0,1,2,\ldots$ At even moments, player 1 makes some proposal $\mathbf{x} = (x_1,x_2) \in P(S)$ and player 2 accepts or rejects it. At odd moments, player 2 makes some proposal $\mathbf{x} = (x_1,x_2) \in P(S)$ and player 1 accepts or rejects it. The game ends as soon as a proposal is accepted. If a proposal $\mathbf{x} = (x_1,x_2)$ is accepted at time t, then the players receive payoffs $(\delta^t x_1, \delta^t x_2)$. Here $0 < \delta < 1$ is a so called discount factor; it reflects impatience of the players, for instance because of foregone interest payments ('shrinking cake'). If no proposal is ever accepted, then the game ends with the disagreement payoffs of $(0,0)$.

Suppose player 1 has in mind to make some proposal $\mathbf{y} = (y_1,y_2) \in P(S)$, and that player 2 has in mind to make some proposal $\mathbf{z} = (z_1,z_2) \in P(S)$. So player 1 offers to player 2 the amount y_2. Player 2 expects to get z_2 if he rejects $\mathbf{y}$, but he will get z_2 one round later. So player 1's proposal y will be rejected by player 2 if $y_2 < \delta z_2$; on the other hand, there is no need to offer strictly more than δz_2. This leads to the equation

$$y_2 = \delta z_2. \tag{21.9}$$

[4] See also Sect. 6.7.

By reversing in this argument the roles of players 1 and 2 one obtains

$$z_1 = \delta y_1. \tag{21.10}$$

These two equations define unique points $\mathbf{y}$ and $\mathbf{z}$ in $P(S)$. The result of the Rubinstein bargaining approach is that player 1 starts by offering $\mathbf{y}$, player 2 accepts, and the game ends with the payoffs $\mathbf{y} = (y_1, y_2)$.

This description is informal. Formally, one defines a dynamic noncooperative game and looks for the (in this case) subgame perfect Nash equilibria of this game. It can be shown that all such equilibria result in the payoffs $\mathbf{y}$ (or in $\mathbf{z}$ if player 2 would start instead of player 1).[5]

The surprising fact is that, although at first sight the Rubinstein approach is quite different from the axiomatic approach by Nash (Theorem 21.1) the resulting outcomes turn out to be closely related. From (21.9), (21.10) one derives easily that $y_1y_2 = z_1z_2$, i.e., the points $\mathbf{y}$ and $\mathbf{z}$ are on the same level curve of the function $\mathbf{x} = (x_1, x_2) \mapsto x_1x_2$, which appears in the definition of the Nash bargaining solution. Moreover, if the discount factor δ approaches 1, the points $\mathbf{y}$ and $\mathbf{z}$ converge to one another on the curve $P(S)$, and hence to the Nash bargaining solution outcome. In words, as the players become more patient, the outcome of the Rubinstein model converges to the Nash bargaining solution outcome.[6]

21.5 Games with Non-Transferable Utility

Both TU-games and bargaining problems are special cases of NTU-games, games with non-transferable utility. In an NTU-game, the possibilities from cooperation for each coalition are described by a set, rather than a single number. For a TU-game (N, v) those sets can be defined as

$$V(S) = \{\mathbf{x} \in \mathbb{R}^S \mid x(S) \le v(S)\}$$

for every coalition S. For a two-person bargaining problem (S, d) the set of feasible payoffs is S for the grand coalition $\{1, 2\}$ and $(-\infty, d_i]$ for each player i.

The core concept can be extended to NTU-games (for two- or n-person bargaining problems it is just the part of the Pareto optimal set weakly dominating the disagreement outcome). Also the balancedness concept can be extended; the main result here is that balanced games have a nonempty core, but the converse is not true (Scarf [114]).

Most other solution concepts for NTU-games (in particular the Harsanyi [49] and Shapley [123] NTU-values, and the consistent value of Hart and Mas-Colell [53]) extend the Nash bargaining solution as well as the Shapley value for TU-games. An exception are the monotonic solutions of Kalai and Samet [61], which extend

[5] For a proof of this fact, see Rubinstein [110] or Sutton [131]. For a relatively recent and elaborate discussion of noncooperative bargaining models see Muthoo [85].

[6] See also Binmore et al. [13].

the egalitarian solution of the bargaining problem. See de Clippel et al. [26] for an overview of various axiomatic characterizations of values for NTU-games, and see Peters [104] for an overview of NTU-games in general.

Most (though not all) results of this chapter on bargaining can be extended to the n-person case without too much difficulty. This is not true for the Rubinstein approach, the extension of which is not obvious. One possibility is presented by Hart and Mas-Colell [53].

Problems

21.1. *Anonymity and Symmetry*

Call a two-person bargaining solution *anonymous* if $F_1(S',\mathbf{d}') = F_2(S,\mathbf{d})$ and $F_2(S',\mathbf{d}') = F_1(S,\mathbf{d})$ whenever $(S,\mathbf{d})$, $(S',\mathbf{d}') \in B$ with $S' = \{(x_2,x_1) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\}$ and $(d'_1,d'_2) = (d_2,d_1)$. Prove that Anonymity implies Symmetry but not vice versa.

21.2. *Revealed Preference*

Let $B_0 = \{(S,\mathbf{d}) \in B \mid \mathbf{d} = (0,0),\ S \text{ is zero-comprehensive}\}$ where S is *zero-comprehensive* if for all $\mathbf{x} \in S$ and $\mathbf{y} \in \mathbb{R}^2_+$ with $\mathbf{y} \le \mathbf{x}$ we have $\mathbf{y} \in S$. Write S instead of $(S,\mathbf{0})$. Let $\succeq$ be a binary relation on $\mathbb{R}^2$ and $F : B_0 \to \mathbb{R}^2$ a solution. Say that $\succeq$ *represents* F if for every $S \in B_0$:

$$\{F(S)\} = \{\mathbf{x} \in S \mid \mathbf{x} \succeq \mathbf{y} \text{ for every } \mathbf{y} \in S\},$$

i.e., if F uniquely maximizes $\succeq$ on S. Prove: F satisfies IIA if and only if F can be represented by a binary relation $\succeq$.

21.3. *The Nash Solution is Well-defined*

Show that N is well defined, i.e., that the function $(x_1 - d_1)(x_2 - d_2)$ takes its maximum on $\{\mathbf{x} \in S \mid \mathbf{x} \ge \mathbf{d}\}$ at a unique point.

21.4. *(1) $\Rightarrow$ (2) in Theorem 21.1*

Show that N satisfies the properties WPO, SYM, SC, and IIA.

21.5. *Geometric Characterization of the Nash Bargaining solution*

Show that, for every $(S,\mathbf{d}) \in B$, $N(S,\mathbf{d}) = \mathbf{z} > \mathbf{d}$ if and only if there is a supporting line of S at $\mathbf{z}$ with slope the negative of the slope of the straight line through $\mathbf{d}$ and $\mathbf{z}$.

21.6. *Strong Individual Rationality*

Call a solution F *strongly individually rational* (SIR) if $F(S,\mathbf{d}) > \mathbf{d}$ for all $(S,\mathbf{d}) \in B$. The *disagreement* solution D is defined by $D(S,\mathbf{d}) := \mathbf{d}$ for every $(S,\mathbf{d}) \in B$. Show that the following two statements for a solution F are equivalent:

(1) $F = N$ or $F = D$.

(2) F satisfies IR, SYM, SC, and IIA.

Derive from this that N is the unique solution with the properties SIR, SYM, SC, and IIA. (Hint: For the implication (2)⇒(1), show that, for every $(S,\mathbf{d}) \in B$, either $F(S,\mathbf{d}) = \mathbf{d}$ or $F(S,\mathbf{d}) \in W(S)$. Also show that, if $F(S,\mathbf{d}) = \mathbf{d}$ for *some* $(S,\mathbf{d}) \in B$, then $F(S,\mathbf{d}) = \mathbf{d}$ for *all* $(S,\mathbf{d}) \in B$.)

21.7. *(1) ⇒ (2) in Theorem 21.2*

Show that the Raiffa–Kalai–Smorodinsky solution has the properties WPO, SYM, SC, and IM.

21.8. *Restricted Monotonicity*

Call a solution $F : B \to \mathbb{R}^2$ *restrictedly monotonic* (RM) if $F(S,\mathbf{d}) \leq F(T,\mathbf{e})$ whenever $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B$, $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, $u(S,\mathbf{d}) = u(T,\mathbf{e})$.

(1) Prove that IM implies RM.

(2) Show that RM does not imply IM.

21.9. *Global Individual Monotonicity*

For a problem $(S,\mathbf{d}) \in B$, $g(S) := (\max\{x_1 \mid \mathbf{x} \in S\}, \max\{x_2 \mid \mathbf{x} \in S\})$ is called the *global utopia point of* S. *Global Individual Monotonicity* (GIM) is defined in the same way as IM, with the condition "$u_i(S,\mathbf{d}) = u_i(T,\mathbf{e})$" replaced by: $g_i(S) = g_i(T)$. The solution $G : B \to \mathbb{R}^2$ assigns to each $(S,\mathbf{d}) \in B$ the point of intersection of $W(S)$ with the straight line joining $\mathbf{d}$ and $g(S)$. Show that G is the unique solution with the properties WPO, SYM, SC, and GIM.

21.10. *Monotonicity and (weak) Pareto Optimality*

(1) Show that there is no solution satisfying MON and WPO.

(2) Show that, on the subclass B_0 introduced in Problem 21.2, there is no solution satisfying MON and PO. Can you find a solution on this class with the properties MON and WPO?

21.11. *The Egalitarian Solution (1)*

(1) Show that E satisfies MON, SYM, and WPO (on B^c).

(2) Show that E is translation covariant on B^c.

21.12. *The Egalitarian Solution (2)*

Show that the egalitarian solution is super-additive.

21.13. *Independence of Axioms*

In the characterization Theorems 21.1–21.4, show that none of the axioms used can be dispensed with.

21.14. *Nash and Rubinstein*

Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Player 1 has utility function $u_1(\alpha) = \alpha$ and player 2 has utility function $u_2(\alpha) = 1 - (1 - \alpha)^2$ for amounts $\alpha \in [0, 1]$ of the good. If they do not reach an agreement on the division of the good they both receive nothing.

(a) Determine the set of feasible utility pairs. Make a picture.

(b) Determine the Nash bargaining solution outcome, in terms of utilities as well as of the physical distribution of the good.

(c) Suppose the players' utilities are discounted by a factor $\delta \in (0, 1)$. Calculate the Rubinstein bargaining outcome.

(d) Determine the limit of the Rubinstein bargaining outcome, for δ approaching 1, in two ways: by using the result of (b) and by using the result of (c).

Chapter 22
Tools

This chapter collects some mathematical tools used in this book: (direct) convex separation results in Sects. 22.2 and 22.6; Lemmas of the Alternative, in particular Farkas' Lemma in Sect. 22.3; the Linear Duality Theorem in Sect. 22.4; the Brouwer and Kakutani Fixed Point Theorems in Sect. 22.5; the Krein–Milman Theorem and the Birkhoff–von Neumann Theorem in Sect. 22.6; and the Max-Flow Min-Cut Theorem of Ford and Fulkerson in Sect. 22.7.

22.1 Some Definitions

A subset $Z \subseteq \mathbb{R}^n$ is *convex* if with any two points $\mathbf{x}, \mathbf{y} \in Z$, also the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ is contained in Z. Formally:

$$\forall\, \mathbf{x}, \mathbf{y} \in Z\, \forall\, 0 \leq \lambda \leq 1 : \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in Z.$$

If Z is a closed set[1] then for convexity it is sufficient to check this condition for $\lambda = 1/2$ (see Problem 22.1). It is easy to see that a set $Z \subseteq \mathbb{R}^n$ is convex if and only if $\sum_{j=1}^{k} \lambda_j \mathbf{x}^j \in Z$ for all $\mathbf{x}^1, \ldots, \mathbf{x}^k \in Z$ and all nonnegative $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $\sum_{j=1}^{k} \lambda_j = 1$. Such a sum $\sum_{j=1}^{k} \lambda_j \mathbf{x}^j$ is called a *convex combination* of the $\mathbf{x}^j$.

For vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{n} x_i y_i$$

denotes the *inner product* of $\mathbf{x}$ and $\mathbf{y}$, and

$$||\mathbf{x} - \mathbf{y}|| := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

[1] A set $Z \subseteq \mathbb{R}^n$ is *closed* if it contains the limit of every converging sequence in Z.

is the *Euclidean distance* between $\mathbf{x}$ and $\mathbf{y}$. A set $C \subseteq \mathbb{R}^n$ is a (convex) *cone* if, with each $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$, also $\lambda \mathbf{x} \in C$ and $\mathbf{x} + \mathbf{y} \in C$.

22.2 A Separation Theorem

In this section we derive the simplest version of a separation result, namely separating a point from a convex set.

Theorem 22.1. *Let $Z \subseteq \mathbb{R}^n$ be a closed convex set and let $\mathbf{x} \in \mathbb{R}^n \setminus Z$. Then there is a $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \cdot \mathbf{z} > \mathbf{y} \cdot \mathbf{x}$ for every $\mathbf{z} \in Z$.*

Thus, this theorem states the geometrically obvious fact that a closed convex set and a point not in that set can be *separated* by a hyperplane (with normal $\mathbf{y}$).

Proof of Theorem 22.1. Let $\mathbf{z}' \in Z$ such that $0 < ||\mathbf{x} - \mathbf{z}'|| \leq ||\mathbf{x} - \mathbf{z}||$ for all $\mathbf{z} \in Z$. Such a $\mathbf{z}'$ exists by the Theorem of Weierstrass, since the Euclidean distance from $\mathbf{x}$ is a continuous function on the set Z, and for the minimum of $\mathbf{z} \to ||\mathbf{x} - \mathbf{z}||$ on Z attention can be restricted to a compact subset of Z. Let $\mathbf{y} = \mathbf{z}' - \mathbf{x}$. Let $\mathbf{z} \in Z$. For any α, $0 \leq \alpha \leq 1$, convexity of Z implies $\mathbf{z}' + \alpha(\mathbf{z} - \mathbf{z}') \in Z$, and thus

$$||\mathbf{z}' + \alpha(\mathbf{z} - \mathbf{z}') - \mathbf{x}||^2 \geq ||\mathbf{z}' - \mathbf{x}||^2.$$

Hence,

$$2\alpha(\mathbf{z}' - \mathbf{x}) \cdot (\mathbf{z} - \mathbf{z}') + \alpha^2 ||\mathbf{z} - \mathbf{z}'||^2 \geq 0.$$

Thus, letting $\alpha \downarrow 0$, it follows that $(\mathbf{z}' - \mathbf{x}) \cdot (\mathbf{z} - \mathbf{z}') \geq 0$. From this, $(\mathbf{z}' - \mathbf{x}) \cdot \mathbf{z} \geq (\mathbf{z}' - \mathbf{x}) \cdot \mathbf{z}' = (\mathbf{z}' - \mathbf{x}) \cdot \mathbf{x} + (\mathbf{z}' - \mathbf{x}) \cdot (\mathbf{z}' - \mathbf{x}) > (\mathbf{z}' - \mathbf{x}) \cdot \mathbf{x}$.

Because $\mathbf{z}$ was arbitrary, it follows that $\mathbf{y} \cdot \mathbf{z} > \mathbf{y} \cdot \mathbf{x}$ for every $\mathbf{z} \in Z$. □

Remark 22.2. A consequence of Theorem 22.1 is that there are real numbers α and β satisfying $\mathbf{y} \cdot \mathbf{z} > \alpha$ and $\mathbf{y} \cdot \mathbf{x} < \alpha$, and $\mathbf{y} \cdot \mathbf{z} > \beta$ and $\mathbf{y} \cdot \mathbf{x} = \beta$, for all $\mathbf{z} \in Z$ (notations as in the Lemma).

22.3 Lemmas of the Alternative

Theorem 22.1 can be used to derive several *lemmas of the alternative*. These lemmas have in common that they describe two systems of linear inequalities and equations, exactly one of which has a solution.

Lemma 22.3 (Theorem of the Alternative for Matrices). *Let A be an $m \times n$ matrix. Exactly one of the following two statements is true:*

(1) *There are $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^m$ with $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$ and $A\mathbf{y} + \mathbf{z} = 0$.*
(2) *There is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}A > \mathbf{0}$.*

Proof. We leave it to the reader to prove that at most one of the systems in (1) and (2) has a solution (Problem 22.2).

Now suppose that (1) is not true. It is sufficient to prove that the system in (2) must have a solution. Observe that (1) implies that $\mathbf{0}$ is a convex combination of the columns of A and the set $\{\mathbf{e}^j \in \mathbb{R}^m \mid j = 1, \ldots, m\}$. This follows from dividing both sides of the equation $A\mathbf{y} + \mathbf{z} = 0$ by $\sum_{j=1}^n y_j + \sum_{i=1}^m z_i$. Hence, the assumption that (1) is not true means that $\mathbf{0} \notin Z$, where $Z \subseteq \mathbb{R}^m$ is the convex hull of the columns of A and the set $\{\mathbf{e}^j \in \mathbb{R}^m \mid j = 1, \ldots, m\}$. By Theorem 22.1 and Remark 22.2 there is an $\mathbf{x} \in \mathbb{R}^m$ and a number $\beta \in \mathbb{R}$ such that $\mathbf{x} \cdot \mathbf{z} > \beta$ for all $z \in Z$ and $\mathbf{x} \cdot \mathbf{0} = \beta$. Hence, $\beta = 0$ and, in particular, $\mathbf{x}A > \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$ since the columns of A and all $\mathbf{e}^j$ for $j = 1, \ldots, m$ are elements of Z. Thus, (2) is true. □

Another lemma of the alternative is Farkas's Lemma.

Lemma 22.4 (Farkas' Lemma). *Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Exactly one of the following two statements is true:*

(1) *There is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}A = \mathbf{b}$.*
(2) *There is a $\mathbf{y} \in \mathbb{R}^n$ with $A\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b} \cdot \mathbf{y} < \mathbf{0}$.*

Proof. We leave it to the reader to show that at most one of the two systems in (1) and (2) can have a solution (Problem 22.3). Assume that the system in (1) does not have a solution. It is sufficient to prove that the system in (2) must have a solution.

The assumption that the system in (1) does not have a solution is equivalent to the statement $\mathbf{b} \notin Z$ where

$$Z = \{\mathbf{z} \in \mathbb{R}^n \mid \text{ there exists an } \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{z} = \mathbf{x}A\}.$$

Observe that Z is a closed convex set. By Theorem 22.1 and Remark 22.2 it follows that there is a $\mathbf{y} \in \mathbb{R}^n$ and an $\alpha \in \mathbb{R}$ with $\mathbf{y} \cdot \mathbf{b} < \alpha$ and $\mathbf{y} \cdot \mathbf{z} > \alpha$ for all $\mathbf{z} \in Z$. Because $\mathbf{0} \in Z$ it follows that $\alpha < \mathbf{y} \cdot \mathbf{0} = 0$, hence $\mathbf{y} \cdot \mathbf{b} < \alpha < 0$. To prove that the system in (2) has a solution, it is sufficient to prove that $A\mathbf{y} \geq \mathbf{0}$. Suppose not, i.e., there is an i with $(A\mathbf{y})_i < 0$. Then $\mathbf{e}^i A\mathbf{y} < 0$, so $(M\mathbf{e}^i)A\mathbf{y} \to -\infty$ as $\mathbb{R} \ni M \to \infty$. Observe, however, that $(M\mathbf{e}^i)A \in Z$ for every $M > 0$, so that $(M\mathbf{e}^i)A\mathbf{y} > \alpha$ for every such M. This contradiction completes the proof of the lemma. □

These lemmas can be interpreted geometrically. We show this for Farkas' Lemma in Fig. 22.1. Consider the row vectors $\mathbf{r}_i$ of A as points in $\mathbb{R}^n$. The set of all nonnegative linear combinations of the $\mathbf{r}_i$ forms a cone C. The statement that the system in (1) in Lemma 22.4 has no nonnegative solution means that the vector $\mathbf{b}$ does not lie in C. In this case, the lemma asserts the existence of a vector $\mathbf{y}$ which makes an obtuse angle with $\mathbf{b}$ and a nonobtuse angle with each of the vectors $\mathbf{r}_i$. This means that the hyperplane L orthogonal to $\mathbf{y}$ has the cone C on one side and the point $\mathbf{b}$ on the other.

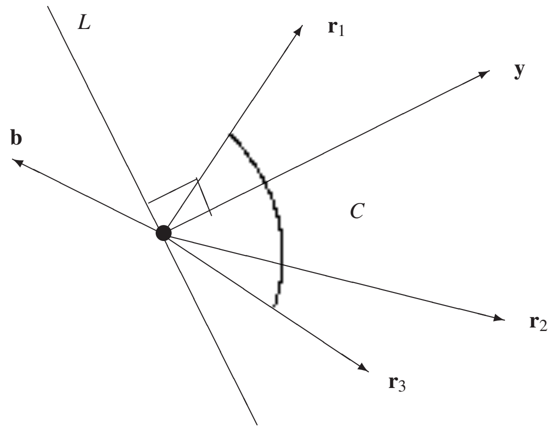


Fig. 22.1 Geometric interpretation of Farkas' Lemma

22.4 The Duality Theorem of Linear Programming

In this section we prove the following theorem.

Theorem 22.5 (Duality Theorem of Linear Programming). *Let A be an $n \times p$ matrix, $\mathbf{b} \in \mathbb{R}^p$, and $\mathbf{c} \in \mathbb{R}^n$. Suppose $V := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}A \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\} \neq \emptyset$ and $W := \{\mathbf{y} \in \mathbb{R}^p \mid A\mathbf{y} \leq \mathbf{c},\ \mathbf{y} \geq \mathbf{0}\} \neq \emptyset$. Then* $\min\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x} \in V\} = \max\{\mathbf{b} \cdot \mathbf{y} \mid \mathbf{y} \in W\}$.

To prove this theorem, we first prove the following variant of Farkas' Lemma.

Lemma 22.6. *Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Exactly one of the following two statements is true:*

(1) *There is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x}A \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*
(2) *There is a $\mathbf{y} \in \mathbb{R}^n$ with $A\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$, and $\mathbf{y} \geq \mathbf{0}$.*

Proof. Problem 22.4. □

The following three lemmas are further preparations for the proof of the Duality Theorem.

Lemma 22.7. *Let $\mathbf{x} \in V$ and $\mathbf{y} \in W$ (cf. Theorem 22.5). Then $\mathbf{x} \cdot \mathbf{c} \geq \mathbf{b} \cdot \mathbf{y}$.*

Proof. $\mathbf{x} \cdot \mathbf{c} \geq \mathbf{x}A\mathbf{y} \geq \mathbf{b} \cdot \mathbf{y}$. □

Lemma 22.8. *Let $\hat{\mathbf{x}} \in V$, $\hat{\mathbf{y}} \in W$ with $\hat{\mathbf{x}} \cdot \mathbf{c} = \mathbf{b} \cdot \hat{\mathbf{y}}$. Then $\hat{\mathbf{x}} \cdot \mathbf{c} = \min\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x} \in V\}$ and $\mathbf{b} \cdot \hat{\mathbf{y}} = \max\{\mathbf{b} \cdot \mathbf{y} \mid \mathbf{y} \in W\}$.*

Proof. By Lemma 22.7, for every $\mathbf{x} \in V$: $\mathbf{x} \cdot \mathbf{c} \geq \mathbf{b} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \mathbf{c}$. Similarly, $\mathbf{b} \cdot \mathbf{y} \leq \hat{\mathbf{x}} \cdot \mathbf{c} = \mathbf{b} \cdot \hat{\mathbf{y}}$ for every $\mathbf{y} \in W$. □

Proof of Theorem 22.5. In view of Lemmas 22.7 and 22.8, it is sufficient to show the existence of $\hat{\mathbf{x}} \in V$ and $\hat{\mathbf{y}} \in W$ with $\hat{\mathbf{x}} \cdot \mathbf{c} \leq \mathbf{b} \cdot \hat{\mathbf{y}}$. So it is sufficient to show that the system

$$(\mathbf{x},\mathbf{y})\begin{pmatrix} -A & \mathbf{0} & \mathbf{c} \\ \mathbf{0} & A^t & -\mathbf{b} \end{pmatrix} \le (-\mathbf{b},\mathbf{c},\mathbf{0}),\ \mathbf{x}\ge\mathbf{0},\mathbf{y}\ge\mathbf{0}$$

has a solution. Suppose this is not the case. By Lemma 22.6, there exists a vector $(\mathbf{z},\mathbf{w},t)\in\mathbb{R}^p\times\mathbb{R}^n\times\mathbb{R}$ with

$$\begin{pmatrix} -A & \mathbf{0} & \mathbf{c} \\ \mathbf{0} & A^t & -\mathbf{b} \end{pmatrix}\begin{pmatrix} \mathbf{z} \\ \mathbf{w} \\ t \end{pmatrix} \ge \mathbf{0},\ \ (-\mathbf{b},\mathbf{c},0)\cdot(\mathbf{z},\mathbf{w},t) < \mathbf{0},\ \ \mathbf{z}\ge\mathbf{0},\ \ \mathbf{w}\ge\mathbf{0},\ \ t\ge 0.$$

Hence

$$A\mathbf{z} \le t\mathbf{c} \tag{22.1}$$

$$\mathbf{w}A \ge t\mathbf{b} \tag{22.2}$$

$$\mathbf{c}\cdot\mathbf{w} < \mathbf{b}\cdot\mathbf{z}. \tag{22.3}$$

If $t=0$, then $A\mathbf{z}\le\mathbf{0}$, $\mathbf{w}A\ge\mathbf{0}$, hence, for $\mathbf{x}\in V$ and $\mathbf{y}\in W$:

$$\mathbf{b}\cdot\mathbf{z} \le \mathbf{x}A\mathbf{z} \le 0 \le \mathbf{w}A\mathbf{y} \le \mathbf{w}\cdot\mathbf{c}$$

contradicting (22.3). If $t>0$, then by (22.1) and (22.2), $t^{-1}\mathbf{z}\in W$ and $t^{-1}\mathbf{w}\in V$. By (22.3), $\mathbf{b}\cdot(t^{-1}\mathbf{z}) > (t^{-1}\mathbf{w})\cdot\mathbf{c}$, which contradicts Lemma 22.7. Hence, the first system above must have a solution. □

22.5 Some Fixed Point Theorems

Let $Z\subseteq\mathbb{R}^n$ be a nonempty convex and compact set.[2] Let $f: Z\to Z$ be a continuous function. A point $\mathbf{x}^*\in Z$ is a *fixed point* of f if $f(\mathbf{x}^*)=\mathbf{x}^*$.

If $n=1$, then Z is a closed interval of the form $[a,b]\subseteq\mathbb{R}$, and then it is clear from intuition that f must have a fixed point: formally, this is a straightforward implication of the intermediate-value theorem.

More generally, Brouwer [18] proved the following theorem. For a proof, see e.g. [113].

Theorem 22.9 (Brouwer Fixed Point Theorem). *Let $Z\subseteq\mathbb{R}^n$ be a nonempty compact and convex set and let $f: Z\to Z$ be a continuous function. Then f has a fixed point.*

A generalization of Brouwer's fixed point theorem is Kakutani's fixed point theorem [60]. Let $F: Z\to Z$ be a *correspondence*, i.e., $F(\mathbf{x})$ is a nonempty subset of Z for every $\mathbf{x}\in Z$. Call F *convex-valued* if $F(\mathbf{x})$ is a convex set for every $\mathbf{x}\in Z$. Call F *upper semi-continuous* if the following holds: for every sequence $(\mathbf{x}^k)_{k\in\mathbb{N}}$ in Z converging to $\mathbf{x}\in Z$ and for every sequence $(\mathbf{y}^k)_{k\in\mathbb{N}}$ in Z converging to $\mathbf{y}\in Z$, if

[2] A set $Z\subseteq\mathbb{R}^n$ is *compact* if it is closed and bounded. A set $Z\subseteq\mathbb{R}^n$ is bounded if there is an $M>0$ such that $||\mathbf{x}||<M$ for all $\mathbf{x}\in Z$.

$\mathbf{y}^k \in F(\mathbf{x}^k)$ for every $k \in \mathbb{N}$, then $\mathbf{y} \in F(\mathbf{x})$. A point $\mathbf{x}^* \in Z$ is a *fixed point* of Z if $\mathbf{x}^* \in F(\mathbf{x}^*)$.

Theorem 22.10 (Kakutani Fixed Point Theorem). *Let $Z \subseteq \mathbb{R}^n$ be a nonempty compact and convex set and let $F : Z \to Z$ be an upper semi-continuous and convex-valued correspondence. Then F has a fixed point.*

One way to prove this theorem is to derive it from the Brouwer fixed point Theorem: see, e.g., [54].

22.6 The Birkhoff–von Neumann Theorem

Let C be a convex set in some linear space V. An element $e \in C$ is called an *extreme point* of C if for all $x, y \in C$ with $e = \frac{1}{2}(x+y)$ it holds that $x = y$ $(= e)$. By $\text{ext}(C)$ the set of extreme points of C is denoted. See Problem 22.5 for alternative characterizations of extreme points.

An $n \times n$-matrix D is called *doubly stochastic* if $0 \leq d_{ij} \leq 1$ for all $i, j = 1, \ldots, n$, $\sum_{j=1}^{n} d_{ij} = 1$ for all i, and $\sum_{i=1}^{n} d_{ij} = 1$ for all j. If moreover $d_{ij} \in \{0,1\}$ for all $i, j = 1, \ldots, n$, then D is called a *permutation matrix*. Let $D_{n\times n}$ denote the set of all $n \times n$ doubly stochastic matrices, and let $P_{n\times n}$ denote the set of all $n \times n$ permutation matrices. Note that $D_{n\times n}$ is a convex compact set, and that $P_{n\times n}$ is a finite subset of $D_{n\times n}$. The following theorem gives the exact relation.

Theorem 22.11 (Birkhoff–von Neumann).

(1) $\text{ext}(D_{n\times n}) = P_{n\times n}$
(2) $D_{n\times n} = \text{conv}(P_{n\times n})$.

Part (2) of Theorem 22.11 follows from the Theorem of Krein–Milman (Theorem 22.13 below). In the proof of the latter theorem the dimension of a subset of a linear space V plays a role. A subset of V of the form $a + L$ where $a \in V$ and L is a linear subspace of V, is called an *affine subspace*. Check that a subset A of V is affine if, and only if, with any two different elements x and y of A, also the straight line through x and y is contained in A (Problem 22.6). For an affine subspace $a + L$ of V the *dimension* is defined to be the dimension of the linear subspace L. For an arbitrary subset A of V, its *dimension* $\dim(A)$ is defined to be the dimension of the smallest affine subspace of V containing the set A.

The following separation lemma is used in the proof of the Theorem of Krein–Milman.

Lemma 22.12. *Let C be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n \setminus C$. Then there exists a $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $\mathbf{p} \cdot \mathbf{a} \leq \mathbf{p} \cdot \mathbf{c}$ for every $\mathbf{c} \in C$.*

Proof. We distinguish two cases: $\mathbf{a} \in \text{clo}(C)$ and $\mathbf{a} \notin \text{clo}(C)$ (clo denotes the topological closure).

(1) Suppose $\mathbf{a} \notin \mathrm{clo}(C)$. Then the result follows from Theorem 22.1, with $\mathrm{clo}(C)$ in the role of the set Z there.

(2) Suppose $\mathbf{a} \in \mathrm{clo}(C)$. Because $\mathbf{a} \notin C$ it follows that $\mathbf{a}$ is not in the interior of C. Hence, there is a sequence $\mathbf{a}^1, \mathbf{a}^2, \ldots \in \mathbb{R}^n \setminus \mathrm{clo}(C)$ converging to $\mathbf{a}$. By Theorem 22.1 again, for each k there is a $\mathbf{p}^k \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $\mathbf{p}^k \cdot \mathbf{a}^k \leq \mathbf{p}^k \cdot \mathbf{c}$ for all $\mathbf{c} \in \mathrm{clo}(C)$, and we can take these vectors $\mathbf{p}^k$ such that $||\mathbf{p}^k|| = 1$ for every k ($||\cdot||$ denotes the Euclidean norm). Because $\{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| = 1\}$ is a compact set, there exists a converging subsequence $\mathbf{p}^{k(1)}, \mathbf{p}^{k(2)}, \ldots$ of $\mathbf{p}^1, \mathbf{p}^2, \ldots$ with limit, say, $\hat{\mathbf{p}}$. Then $\hat{\mathbf{p}} \cdot \mathbf{a} = \lim_{\ell \to \infty} \mathbf{p}^{k(\ell)} \cdot \mathbf{a}^{k(\ell)} \leq \lim_{\ell \to \infty} \mathbf{p}^{k(\ell)} \cdot \mathbf{c} = \hat{\mathbf{p}} \cdot \mathbf{c}$ for all $\mathbf{c} \in \mathrm{clo}(C)$. □

Theorem 22.13 (Krein–Milman). *Let C be a nonempty compact and convex subset of $\mathbb{R}^n$. Then* $\mathrm{ext}(C) \neq \emptyset$ *and* $C = \mathrm{conv}(\mathrm{ext}(C))$.

Proof. (1) Because C is compact and $\mathbf{x} \mapsto ||\mathbf{x}||$ (where $||\cdot||$ denotes the Euclidean norm) is continuous, there exists by the Theorem of Weierstrass an $\mathbf{e} \in C$ with $||\mathbf{e}|| = \max_{\mathbf{x} \in C} ||\mathbf{x}||$. Then $\mathbf{e} \in \mathrm{ext}(C)$, which can be proved as follows. Suppose that $\mathbf{e} = \frac{1}{2}(\mathbf{x}^1 + \mathbf{x}^2)$ for some $\mathbf{x}^1, \mathbf{x}^2 \in C$. Then

$$||\mathbf{e}|| = ||\frac{1}{2}(\mathbf{x}^1 + \mathbf{x}^2)|| \leq \frac{1}{2}||\mathbf{x}^1|| + \frac{1}{2}||\mathbf{x}^2|| \leq \frac{1}{2}||\mathbf{e}|| + \frac{1}{2}||\mathbf{e}||$$

implies $||\mathbf{x}^1|| = ||\mathbf{x}^2|| = ||\frac{1}{2}(\mathbf{x}^1 + \mathbf{x}^2)||$. By definition of the Euclidean norm this is only possible if $\mathbf{x}^1 = \mathbf{x}^2 = \mathbf{e}$. This shows $\mathbf{e} \in \mathrm{ext}(C)$. Hence, $\mathrm{ext}(C) \neq \emptyset$.

(2) The second statement in the theorem will be proved by induction on $\dim(C)$.

(a) If $\dim(C) = 0$, then $C = \{\mathbf{a}\}$ for some $\mathbf{a} \in \mathbb{R}^n$, so $\mathrm{ext}(C) = \{\mathbf{a}\}$ and $\mathrm{conv}(\mathrm{ext}(C)) = \{\mathbf{a}\} = C$.

(b) Let $k \in \mathbb{N}$, and suppose that $\mathrm{conv}(\mathrm{ext}(D)) = D$ for every nonempty compact and convex subset D of $\mathbb{R}^n$ with $\dim(D) < k$. Let C be a k-dimensional compact convex subset of $\mathbb{R}^n$. Obviously, $\mathrm{conv}(\mathrm{ext}(C)) \subseteq C$. So to prove is still: $C \subseteq \mathrm{conv}(\mathrm{ext}(C))$. Without loss of generality assume $\mathbf{0} \in C$ (otherwise, shift the whole set C). Let W be the smallest affine (hence, linear) subset of $\mathbb{R}^n$ containing C. Hence, $\dim(W) = k$. From part (1) of the proof there is an $\mathbf{e} \in \mathrm{ext}(C)$. Let $\mathbf{x} \in C$. If $\mathbf{x} = \mathbf{e}$ then $\mathbf{x} \in \mathrm{conv}(\mathrm{ext}(C))$. If $\mathbf{x} \neq \mathbf{e}$ then the intersection of the straight line through $\mathbf{x}$ and $\mathbf{e}$ with C is a line segment of which one of the endpoints is $\mathbf{e}$. Let $\mathbf{b}$ be the other endpoint. Then $\mathbf{b}$ is a boundary point of C. Then, by Lemma 22.12, there is a linear function $f : W \to \mathbb{R}$ with $f(\mathbf{b}) = \min\{f(\mathbf{c}) \mid \mathbf{c} \in C\}$ and $f \neq 0$ (check this).

Let $D := \{\mathbf{y} \in C \mid f(\mathbf{y}) = f(\mathbf{b})\}$. Then D is a compact and convex subset of C. Because $f \neq 0$ it follows that $\dim(D) < k$. By the induction hypothesis, $D = \mathrm{conv}(\mathrm{ext}(D))$. Also, $\mathrm{ext}(D) \subseteq \mathrm{ext}(C)$, see Problem 22.7. Hence, $\mathbf{b} \in D = \mathrm{conv}(\mathrm{ext}(D)) \subseteq \mathrm{conv}(\mathrm{ext}(C))$. Further, $\mathbf{e} \in \mathrm{ext}(C)$. Because $\mathbf{x} \in \mathrm{conv}\{\mathbf{b}, \mathbf{e}\}$ it follows that $\mathbf{x} \in \mathrm{conv}(\mathrm{ext}(C))$. So $C \subseteq \mathrm{conv}(\mathrm{ext}(C))$. □

Proof of Theorem 22.11. Because $D_{n\times n}$ is compact and convex, part (2) follows from part (1) and Theorem 22.13. So only (1) still has to be proved.

(a) We first prove that $P_{n\times n} \subseteq \mathrm{ext}(D_{n\times n})$. Let $P = [p_{ij}]_{i,j=1}^n$ be a permutation matrix with $P = \frac{1}{2}(A + B)$ for some $A, B \in D_{n\times n}$. Then $p_{ij} = \frac{1}{2}(a_{ij} + b_{ij})$ and $p_{ij} \in \{0, 1\}$

for all $i, j \in \{1,2,\ldots,n\}$. If $p_{ij} = 0$ then $a_{ij} = b_{ij} = 0$ because $a_{ij}, b_{ij} \geq 0$. If $p_{ij} = 1$ then $a_{ij} = b_{ij} = 1$ because $a_{ij}, b_{ij} \leq 1$. Hence, $A = B$, so that $P \in \text{ext}(D_{n\times n})$.

(b) Let now $D = [d_{ij}] \in D_{n\times n}$ such that D is not a permutation matrix. The proof is complete if we show that D is not an extreme point. For this, it is sufficient to show that there exists an $n \times n$-matrix $C \neq [0]$ with:

(1) $c_{ij} = 0$ whenever $d_{ij} = 0$ or $d_{ij} = 1$.
(2) $\sum_{i=1}^{n} c_{ij} = 0$ for all $j \in \{1,2,\ldots,n\}$ with $d_{ij} \neq 1$ for every i.
(3) $\sum_{j=1}^{n} c_{ij} = 0$ for all $i \in \{1,2,\ldots,n\}$ with $d_{ij} \neq 1$ for every j.

For in that case, for $\varepsilon > 0$ sufficiently small, the matrices $D + \varepsilon C$ and $D - \varepsilon C$ are two different doubly stochastic matrices with $D = \frac{1}{2}((D + \varepsilon C) + (D - \varepsilon C))$, implying that $D \notin \text{ext}(D_{n\times n})$.

We are left to construct C. In order to satisfy (1), for those rows or columns of D that contain a 1 the corresponding rows or columns of C contain only zeros. Suppose there are k rows (and hence columns) of D that do not contain a 1. Because D is not a permutation matrix, $2 \leq k \leq n$. In these k rows there are at least $2k$ elements unequal to 0 and 1. The corresponding $2k$ or more elements of C are to be chosen such that they satisfy the system of $2k$ homogeneous linear equations described in (2) and (3). This system is dependent ($\sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}$) so that it has a nontrivial solution (because there are more variables than independent equalities). This gives the required $C \neq [0]$. $\square$

22.7 The Max-Flow Min-Cut Theorem

A *capacitated network* is a triple (V, E, k), where V is a finite set containing at least two distinguished elements $q, s \in V$ called *source* (q) and *sink* (s); E is a subset of $V \times V$ such that $v \neq w$, $v \neq s$, and $w \neq q$ for all $(v, w) \in E$; and $k : E \to \mathbb{R}_+$. Elements of V are called *vertices* and elements of E are called *edges*. The number $k(e)$ is the *capacity* of the edge e; if $e = (v, w)$ then $k(e)$ is interpreted as the maximal amount that can flow from v to w through edge e. The source has only outgoing and the sink only incoming edges. See Fig. 22.2 for an example.

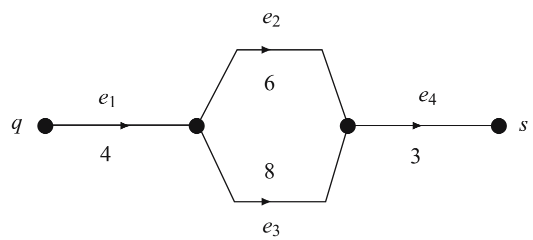


Fig. 22.2 A capacitated network

A *flow* in this network is a map $f : E \to \mathbb{R}_+$ with $f(e) \le k(e)$ and such that for all $v \in V \setminus \{q,s\}$

$$\sum_{w \in V:\ (w,v) \in E} f(w,v) = \sum_{w \in V:\ (v,w) \in E} f(v,w).$$

In other words, a flow satisfies the capacity constraints and for all vertices (except source and sink) the 'inflow' equals the 'outflow'.

The *value* of a flow f is defined as the inflow in the sink, i.e., as the number

$$\sum_{v \in V:\ (v,s) \in E} f(v,s).$$

A flow is called *maximal* if it has maximal value among all possible flows. Intuitively, the value of a maximal flow is determined by the 'bottlenecks' in the network. In order to formalize this, define a *cut* in the network to be a partition of V into two subsets V_1 and V_2 such that $q \in V_1$ and $s \in V_2$. Such a cut is denoted by (V_1,V_2). The *capacity* of a cut is the number

$$k(V_1,V_2) := \sum_{v \in V_1, w \in V_2 : (v,w) \in E} k(v,w),$$

i.e., the total capacity along edges going from V_1 to V_2. A cut is called *minimal* if it has minimal capacity among all possible cuts.

In the example in Fig. 22.2, a minimal cut has only the sink in V_2, and its capacity is equal to 3. Obviously, this is also the value of a maximal flow, but such a flow is not unique.

Flows and cuts are, first of all, related as described in the following lemma.

Lemma 22.14. *Let f be a flow in the capacitated network (V,E,k), and let $\varphi : E \to \mathbb{R}$ be an arbitrary function. Then:*

(1) $\sum_{v \in V} \sum_{(w,v):\ (w,v) \in E} \varphi(w,v) = \sum_{v \in V} \sum_{(v,w):\ (v,w) \in E} \varphi(v,w)$.

(2) $\sum_{(q,v):\ (q,v) \in E} f(q,v) = \sum_{(v,s):\ (v,s) \in E} f(v,s)$.

(3) *For every cut (V_1,V_2) the value of the flow f is equal to*

$$\sum_{(v,w):\ (v,w) \in E, v \in V_1, w \in V_2} f(v,w) - \sum_{(v,w):\ (v,w) \in E, v \in V_2, w \in V_1} f(v,w).$$

Proof. (1) follows because summation at both sides is taken over the same sets. Part (1) moreover implies

$$\sum_{(q,v):\ (q,v) \in E} f(q,v) + \sum_{(v,w) \in E:\ v \ne q} f(v,w) = \sum_{(v,s):\ (v,s) \in E} f(v,s) \\ + \sum_{(v,w) \in E:\ w \ne s} f(v,w)$$

which implies (2) because $\sum_{(v,w)\in E:\ v\neq q} f(v,w) = \sum_{(v,w)\in E:\ w\neq s} f(v,w)$ by definition of a flow ('inflow' equals 'outflow' at every vertex that is not the source and not the sink). For part (3), let (V_1, V_2) be a cut of the network. Then

$$\begin{aligned}
\sum_{(v,w)\in E:\ v\in V_1, w\in V_2} f(v,w) &= \sum_{(v,w)\in E:\ v\in V_1} f(v,w) - \sum_{(v,w)\in E:\ v,w\in V_1} f(v,w) \\
&= \sum_{(v,w)\in E:\ v=q} f(v,w) + \sum_{(v,w)\in E:\ w\in V_1} f(v,w) \\
&\quad - \sum_{(v,w)\in E:\ v,w\in V_1} f(v,w) \\
&= \sum_{(v,w)\in E:\ v=q} f(v,w) + \sum_{(v,w)\in E:\ v\in V_2, w\in V_1} f(v,w) \\
&= \sum_{(v,w)\in E:\ w=s} f(v,w) + \sum_{(v,w)\in E:\ v\in V_2, w\in V_1} f(v,w).
\end{aligned}$$

This implies part (3) of the lemma. □

The following theorem is the famous Max-Flow Min-Cut Theorem of Ford and Fulkerson [36].

Theorem 22.15. *Let (V, E, k) be a capacitated network. Then the value of a maximal flow is equal to the capacity of a minimal cut.*

Proof. Let f be a maximal flow. (Note that f is an optimal solution of a feasible bounded linear program, so that existence of f is guaranteed.) Part (3) of Lemma 22.14 implies that the value of *any* flow is smaller than or equal to the capacity of *any* cut, so that it is sufficient to find a cut of which the capacity is equal to the value of f.

For points v, w in the network define a *path* as a sequence of different non-directed edges starting in v and ending in w; 'non-directed' means that for any edge $(x, y) \in E$, (x, y) as well as (y, x) may be used in this path. Such a path may be described by a sequence $v = x_1, x_2, \ldots, x_m = w$ with $(x_i, x_{i+1}) \in E$ or $(x_{i+1}, x_i) \in E$ for every $i = 1, \ldots, m-1$. Call such a path *non-satiated* if for every $i = 1, \ldots, m-1$ it holds that $f(x_i, x_{i+1}) < k(x_i, x_{i+1})$ if $(x_i, x_{i+1}) \in E$, and $f(x_{i+1}, x_i) > 0$ if $(x_{i+1}, x_i) \in E$. In other words, the flow is below capacity in edges that are traversed in the 'right' way, and positive in edges that are traversed in the 'wrong' way.

Define V_1 to be the set of vertices x for which there is a non-satiated path from q to x, together with the vertex q, and let V_2 be the complement of V_1 in V. Then $s \in V_2$ because otherwise there would be a non-satiated path from q to s, implying that f would not be maximal; the flow f could be increased by increasing it in edges on this path that are traversed in the right way and decreasing it in edges along the path that are traversed in the wrong way, without violating the capacity constraints or the inflow–outflow equalities. Hence (V_1, V_2) is a cut in the network.

Let $(x, y) \in E$ with $x \in V_1$ and $y \in V_2$. Then $f(x, y) = k(x, y)$ because otherwise there would be a non-satiated path from q to a vertex in V_2. Similarly, $f(x', y') = 0$ whenever $(x', y') \in E$ with $x' \in V_2$ and $y' \in V_1$. By Lemma 22.14, part (3), the value of the flow f is equal to

$$\sum_{(v,w):\ (v,w)\in E, v\in V_1, w\in V_2} f(v,w) - \sum_{(v,w):\ (v,w)\in E, v\in V_2, w\in V_1} f(v,w)$$

hence to $\sum_{(v,w):\ (v,w)\in E, v\in V_1, w\in V_2} k(v,w)$, which is by definition the capacity of the cut (V_1, V_2). This completes the proof. $\square$

Observe that the proof of Theorem 22.15 suggests an algorithm to determine a maximal flow, by starting with an arbitrary flow, looking for a non-satiated path, and improving this path. By finding an appropriate cut, maximality of a flow can be checked. Theorem 22.15 is actually a (linear) duality result, but the above proof is elementary.

Problems

22.1. *Convex Sets*

Prove that a closed set $Z \subseteq \mathbb{R}^n$ is convex if and only if $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} \in Z$ for all $\mathbf{x}, \mathbf{y} \in Z$.

22.2. *Proof of Lemma 22.3*

Prove that at most one of the systems in Lemma 22.3 has a solution.

22.3. *Proof of Lemma 22.4*

Prove that at most one of the systems in Lemma 22.4 has a solution.

22.4. *Proof of Lemma 22.6*

Prove Lemma 22.6.

22.5. *Extreme Points*

Let C be a convex set in a linear space V and let $e \in C$. Prove that the following three statements are equivalent:

(1) $e \in \text{ext}(C)$.
(2) For all $0 < \alpha < 1$ and all $x, y \in C$, if $x \neq y$ then $e \neq \alpha x + (1 - \alpha)y$.
(3) $C \setminus \{e\}$ is a convex set.

22.6. *Affine Subspaces*

Prove that a subset A of a linear space V is affine if, and only if, with any two different elements x and y of A, also the straight line through x and y is contained in A.

22.7. *The Set of Sup-Points of a Linear Function on a Convex Set*

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Let C be a convex subset of $\mathbb{R}^n$ and $\alpha := \sup\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$, $D := \{\mathbf{x} \in C \mid f(\mathbf{x}) = \alpha\}$. Show that D is convex and that $\text{ext}(D) \subseteq \text{ext}(C)$.

Hints, Answers and Solutions

Problems of Chap. 1

1.2 *Variant of Matching Pennies*
There are saddlepoint(s) if and only if $x \leq -1$.

1.3 *Mixed Strategies*
(b) $(3/4, 1/4)$.

(c) $(1/2, 1/2)$.

(d) By playing $(3/4, 1/4)$ player 1 obtains $10/4 = 2.5$ for sure (independent of what player 2 does). Similarly, by playing $(1/2, 1/2)$, player 2 is sure to pay 2.5. So 2.5 is the value of this game. Given a rational opponent, no player can hope to do better by playing differently.

1.5 *Glove Game*
(a) $(0, 0, 1)$ is the unique vector in the core of the glove game.

1.6 *Dentist Appointments*
The Shapley value $(9\frac{1}{2}, 6\frac{1}{2}, 8)$ is *not* in the core of this game. The nucleolus is in the core of the game.

1.7 *Nash Bargaining*
The problem to solve is $\max_{0 \leq \alpha \leq 1} \alpha\sqrt{1-\alpha}$. Obviously, the solution must be interior: $0 < \alpha < 1$. Set the first derivative equal to 0, solve, and check that the second derivative is negative.

1.8 *Variant of Glove Game*
The worth of a coalition S depends on the minimum of the numbers of right-hand and left-hand players in the coalition. Write $N = L \cup R$ and find a formal expression for $v(S)$.

Problems of Chap. 2

2.1 *Solving Matrix Games*
(a) The optimal strategies are $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2. The value of the game is $30/11$. In the original game the optimal strategies are $(5/11, 6/11, 0)$ for player 1 and $(5/11, 6/11, 0)$ for player 2.

(b) The value of the game is 0. The unique maximin strategy is $(0, 1, 0)$. The minimax strategies are $(0, q, 1-q, 0)$ for any $0 \le q \le 1$.

(c) The value of the game is 1, the unique minimax strategy is $(1/2, 0, 1/2)$, and the maximin strategies are: $(p, (1-p)/2, (1-p)/2)$ for $0 \le p \le 1$.

(d) The value of the game is 9 and player 1's maximin strategy is $(1/2, 1/2, 0, 0)$. The set of all minimax strategies is $\{(\alpha, \frac{7-14\alpha}{10}, \frac{3+4\alpha}{10}) \in \mathbb{R}^3 \mid 0 \le \alpha \le 1/2\}$.

(e) The value is $8/5$. The unique maximin strategy is $(2/5, 3/5)$ and the unique minimax strategy is $(0, 4/5, 1/5, 0)$.

(f) The value is equal to 1, player 2 has a unique minimax strategy namely $(0, 1, 0)$, and the set of maximin strategies is $\{(0, p, 1-p) \mid 0 \le p \le 1\}$.

2.2 *Saddlepoints*
(b) There are saddlepoints at $(1, 4)$ and at $(4, 1)$.

2.3 *Rock–Paper–Scissors*
The associated matrix game is

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

Problems of Chap. 3

3.1 *Some Applications*
(a) Let Smith be the row player and Brown the column player, then the bimatrix game is

$$\begin{array}{c} \\ L \\ S \end{array} \begin{array}{c} \begin{array}{cc} L & S \end{array} \\ \begin{pmatrix} 2,2 & -1,-1 \\ -1,-1 & 1,1 \end{pmatrix} \end{array}.$$

(b) Let the government be the row player and the pauper the column player. The bimatrix game is

$$\begin{array}{c} \\ \text{aid} \\ \text{not} \end{array} \begin{array}{c} \begin{array}{cc} \text{work} & \text{not} \end{array} \\ \begin{pmatrix} 3,2 & -1,3 \\ -1,1 & 0,0 \end{pmatrix} \end{array}.$$

(c) This game has two pure strategy Nash equilibria and one other (mixed strategy) Nash equilibrium.

(e) This situation can be modeled as a 3×3 bimatrix game.

3.2 *Matrix Games*
(a) You should find the same solution, namely $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2, as the unique Nash equilibrium.

(b) If player 2 plays a minimax strategy then 2's payoff is at least $-v$, where v is the value of the game. Hence, any strategy that gives player 1 at least v is a best reply. So a maximin strategy is a best reply. Similarly, a minimax strategy is a best reply against a maximin strategy, so any pair consisting of a maximin and a minimax strategy is a Nash equilibrium.

Conversely, in a Nash equilibrium the payoffs must be $(v, -v)$ otherwise one of the players could improve by playing an optimal (maximin or minimax) strategy. But then player 1's strategy must be a maximin strategy since otherwise player 2 would have a better reply, and player 2's strategy must be a minimax strategy since otherwise player 1 would have a better reply.

(c) The appropriate definition for player 2 would be: a maximin strategy *for player 2* in B, since now B represents the payoffs to player 2, and not what player 2 has to pay to player 1.

The Nash equilibrium of Problem 3.1(b), for instance, does not consist of maximin strategies of the players. The maximin strategy of player 1 in A is $(1/5, 4/5)$, which is not part of a (the) Nash equilibrium. The maximin strategy of player 2 (!) in B is $(1, 0)$, which is not part of a (the) Nash equilibrium.

3.3 *Strict Domination*
(c) There are three Nash equilibria: $((1,0),\ (1,0,0,0))$, $((0,1),(0,0,1,0))$, and $((3/7,4/7),(1/3,0,2/3,0))$.

3.4 *Iterated Elimination (1)*
(b) The unique equilibrium is (B, Y).

3.5 *Iterated Elimination (2)*
The Nash equilibria are $((1/3,2/3,0),(2/3,0,1/3))$, $((0,1,0),(1,0,0))$, and $((1,0,0),(0,0,1))$.

3.6 *Weakly Dominated Strategies*

(b) Consecutive deletion of Z, C, A results in the Nash equilibria (B, X) and (B, Y). Consecutive deletion of C, Y, B, Z results in the Nash equilibrium (A, X).

3.7 *A Parameter Game*
Distinguish three cases: $a > 2$, $a = 2$, and $a < 2$.

3.8 *Equalizing Property of Mixed Equilibrium Strategies*
(a) Check by substitution.
(b) Suppose the expected payoff (computed by using $\mathbf{q}^*$) of row i played with positive probability (p_i^*) in a Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$, hence the number $\mathbf{e}^i A \mathbf{q}^*$,

would not be maximal. Then player 1 would improve by adding the probability p_i^* to some row j with higher expected payoff $\mathbf{e}^j A\mathbf{q}^* > \mathbf{e}^i A\mathbf{q}^*$, and in this way increase his payoff, a contradiction. A similar argument can be made for player 2 and the columns.

Problems of Chap. 4

4.1 *Counting Strategies*
White has 20 possible opening moves, and therefore also 20 possible strategies. Black has many more strategies.

4.2 *Extensive vs. Strategic Form*
One possibility is the following extensive form game. Player 1 starts and has two possible actions, say U and D (corresponding to the two rows of player 1). These actions lead to the unique information set of player 2. At each of the two nodes in this information set, player 2 has four actions, namely C_1, C_2, C_3, and C_4 (corresponding to the four columns of player 2). The payoffs following U are (a_1,a_2), (b_1,b_2), (e_1,e_2), and (f_1,f_2), respectively. The payoffs following D are (c_1,c_2), (d_1,d_2), (g_1,g_2), and (h_1,h_2), respectively.

Another possibility is to switch the roles of the players. Player 2 starts and has four actions C_1, C_2, C_3, and C_4. Player 1 has one information set with four nodes and at each node the actions U and D. The payoffs (following U and D) are: (a_1,a_2) and (c_1,c_2); (b_1,b_2) and (d_1,d_2); (e_1,e_2) and (g_1,g_2); and (f_1,f_2) and (h_1,h_2), respectively.

If, for instance, $b_i = a_i$, $f_i = e_i$, $g_i = c_i$, and $h_i = d_i$ for $i = 1,2$, then the following extensive form game of perfect information is possible. Player 1 starts and has two actions, U and D. After U player 2 has two actions, say L and R, and after D player 2 has again two actions, say l and r. The path U,L is followed by the payoff pair (a_1,a_2), the path U,R by (e_1,e_2), the path D,l by (c_1,c_2), and the path D,r by (d_1,d_2).

Still another possibility is to let player 2 start with two possible actions, leading to one information set of player 1 with two actions at each of the two nodes. Finally, all the actions of player 1 result in one information set of player 2 at which this player has again two actions. Formally, this extensive form is not allowed since player 2 has imperfect recall.

4.4 *Choosing Objects*
(c) In any subgame perfect equilibrium the game is played as follows: player 1 picks O_3, then player 2 picks O_2 or O_1, and finally player 1 picks O_4. These are the (two) subgame perfect equilibrium outcomes of the game. Due to ties (of player 2) there is more than one subgame perfect equilibrium, namely eight in total. All subgame perfect equilibria result in the same distribution of the objects.

4.5 *An Extensive Form Game*
There is a unique pure strategy Nash equilibrium, which is also subgame perfect. This equilibrium is perfect Bayesian for an appropriate choice of player 2's belief.

4.6 *Another Extensive Form Game*
There is a unique Nash equilibrium (in pure strategies). This equilibrium is not perfect Bayesian.

4.7 *A Centipede Game*
(b) Consider any strategy combination. The last player that has continued when playing his strategy could have improved by stopping if possible. Hence, in equilibrium the play of the game must have stopped immediately.

To exhibit a non-subgame perfect Nash equilibrium, assume that player 1 always stops, and that player 2 also always stops except at his second decision node. Check that this is a Nash equilibrium. (One can also write down the strategic form, which is an 8×8 bimatrix game.)

4.8 *Finitely Repeated Games*
(c) The unique subgame perfect Nash equilibrium is where player 1 always plays B and player 2 always R. This is true for any finite repetition of the game.

(e) Player 1: play T at the first stage. Player 2: play L at the first stage. Second stage play is given by the following diagram:

$$\begin{array}{c c} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} T \\ C \\ B \end{array} & \begin{pmatrix} B,R & C,R & C,R \\ B,M & B,R & B,R \\ B,M & B,R & B,R \end{pmatrix} \end{array}.$$

For instance, if first stage play results in (C, L), then player 1 plays B and player 2 plays M at stage 2. Verify that this defines a subgame perfect equilibrium in which (T, L) is played at the second stage. (Other solutions are possible, as long as players 1 and 2 are punished for unilateral deviations at stage 1.)

Problems of Chap. 5

5.1 *Battle-of-the-Sexes*
The strategic form is a 4×4 bimatrix game. List the strategies of the players as in the text. We can then compute the expected payoffs. E.g., if the first row corresponds to strategy SS of player 1 and strategies SS, SB, BS, and BB of player 2, then the payoffs are, respectively, $1/6$ times $(8,3)$, $(6,9)$, $(6,0)$, and $(4,6)$. The (pure strategy) Nash equilibria are (SS,SB) and (BS,BB).

5.2 *A Static Game of Incomplete Information*
There are three pure Nash equilibria: (TT,L), (TB,R), and (BB,R). (The first letter in a strategy of player 1 applies to Game 1, the second letter to Game 2.)

5.3 *Another Static Game of Incomplete Information*
(b) The unique pure strategy Nash equilibrium is: t_1 and t'_1 play B, t_2 and t'_2 play R.

5.4 *Job-Market Signaling*
(b) There is a pooling equilibrium where both worker types take no education, and wages are always low. This equilibrium is sustained by the firm's belief that an educated worker is of the high type with probability at least $1/2$. It survives the intuitive criterion if the firm believes that an educated worker is of the high type with probability 1.

There is also a separating equilibrium in which the high type takes education and the low type doesn't. Both types get offered the low wage.

5.5 *A Joint Venture*
(c) There is a unique Nash equilibrium (even in mixed strategies). This is also subgame perfect and perfect Bayesian.

5.6 *Entry Deterrence*
For $x \leq 100$ the strategy combination where the entrant always enters and the incumbent colludes is a perfect Bayesian equilibrium. For $x \geq 50$, the combination where the entrant always stays out and the incumbent fights is a perfect Bayesian equilibrium if the incumbent believes that, if the entrant enters, then fighting yields 0 with probability at most $1 - \frac{50}{x}$. Both equilibria satisfy the intuitive criterion.

5.7 *The Beer–Quiche Game*
(b) There are two perfect Bayesian equilibria, both of which are pooling. In the first one, player 1 always eats quiche. This equilibrium does not survive the intuitive criterion. In the second one, player 1 always drinks beer. This equilibrium does survive the intuitive criterion.

5.8 *Issuing Stock*
(b) There is a pooling equilibrium in which the manager never proposes to issue new stock, and such a proposal would not be approved of by the existing shareholders since they believe that this proposal signals a good state with high enough probability. (The background of this is that a new stock issue would dilute the value of the stock of the existing shareholders in a good state of the world, see the original article for details.) This equilibrium (just about) survives the intuitive criterion.

There is also a separating equilibrium in which a stock issue is proposed in the bad state but not in the good state. If a stock issue is proposed, then it is approved of.

Finally, there is a separating equilibrium in which a stock issue is proposed in the good state but not in the bad state. If a stock issue is proposed, then it is not approved of.

(c) In this case, a stock issue proposal would always be approved of, so the 'bad news effect' of a stock issue vanishes. The reason is that the investment opportunity is now much more attractive.

5.9 *More Signaling Games*
(a) There is a unique, pooling perfect Bayesian equilibrium. This equilibrium does not survive the intuitive criterion.

(b) There are two strategy combinations that are perfect Bayesian. Only one of them survives the intuitive criterion.

Problems of Chap. 6

6.1 *Cournot with Asymmetric Costs*
The Nash equilibrium is $q_1 = (a-2c_1+c_2)/3$ and $q_2 = (a-2c_2+c_1)/3$, given that these amounts are nonnegative.

6.2 *Cournot Oligopoly*
(b) The reaction function of player i is: $\beta_i(q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_n) = (a-c-\sum_{j\neq i} q_j)/2$ if $\sum_{j\neq i} q_j \leq a-c$, and $\beta_i(q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_n) = 0$ otherwise.

(c) One should compute the point of intersection of the n reaction functions. This amounts to solving a system of n linear equations in n unknowns $q_1,\ldots,q_n$. Alternatively, one may guess that there is a solution $q_1 = q_2 = \ldots = q_n$. Then $q_1 = (a-c-(n-1)q_1)/2$, resulting in $q_1 = \frac{a-c}{n+1}$. Hence, each firm producing $\frac{a-c}{n+1}$ is a Nash equilibrium. If the number of firms becomes large then this amount converges to 0, which is no surprise since demand is bounded by a.

(d) To show that this equilibrium is unique, one may first argue that it is unique given that all equilibrium outputs are positive (e.g., by showing that the given system of n linear equations has a unique solution by computing the determinant of the coefficient matrix). Moreover, in equilibrium there is at least one firm producing a positive output and having positive profit. If there would be some other firm with zero output, then that firm could improve by producing a small amount and making positive profit.

6.3 *Quantity Competition with Heterogenous Goods*
(a) $\Pi_i(q_1,q_2) = q_i p_i(q_1,q_2) - cq_i$ for $i = 1,2$.

(b) The equilibrium is: $q_1 = (21-4c)/33$, $q_2 = (13-3c)/22$, $p_1 = (21+7c)/11$, $p_2 = (39+13c)/22$.

(c) $q_1 = (57-10c)/95$, $q_2 = (38-10c)/95$, $p_1 = (228+50c)/95$, $p_2 = (228+45c)/95$.

(d) $q_1 = \max\{1-\frac{1}{2}p_1+\frac{1}{3}p_2, 0\}$, $q_2 = \max\{1-\frac{1}{2}p_2+\frac{1}{4}p_1\}$. The profit functions are now $\Pi_1(p_1,p_2) = p_1q_1 - cq_1$ and $\Pi_2(p_1,p_2) = p_2q_2 - cq_2$, with q_1 and q_2 as given.

(e) The equilibrium is $p_1 = \frac{16+8c}{11}$, $p_2 = \frac{30+15c}{22}$. Note that these prices are different from the ones in (c).

(f) These are the same prices and quantities as under (c).
(g) See the answers to (e) and (f).

6.4 *A Numerical Example of Cournot with Incomplete Information*
$q_1 = 18/48, q_H = 9/48, q_L = 15/48$. In the complete information case with low cost we have $q_1 = q_2 = 16/48$, with high cost it is $q_1 = 20/48$ and $q_2 = 8/48$. Note that the low cost firm 'suffers' from incomplete information since firm 1 attaches some positive probability to firm 2 having high cost and therefore has higher supply. For the high cost firm the situation is reversed: it 'benefits' from incomplete information.

6.5 *Cournot with Two-sided Incomplete Information*
Similar to (6.3) we derive

$$q_\ell = q_\ell(q_H, q_L) = \frac{a - c_\ell - \vartheta q_H - (1-\vartheta)q_L}{2},$$

$$q_h = q_h(q_H, q_L) = \frac{a - c_h - \vartheta q_H - (1-\vartheta)q_L}{2},$$

$$q_L = q_L(q_h, q_\ell) = \frac{a - c_L - \pi q_h - (1-\pi)q_\ell}{2},$$

$$q_H = q_H(q_h, q_\ell) = \frac{a - c_H - \pi q_h - (1-\pi)q_\ell}{2}.$$

Here, q_ℓ and q_h correspond to the low and high cost types of firm 1 and q_L, and q_H correspond to the low and high cost types of firm 2. The (Bayesian) Nash equilibrium follows by solving these four equations in the four unknown quantities.

6.6 *Incomplete Information about Demand*
The equilibrium is: $q_1 = \frac{\vartheta a_H + (1-\vartheta)a_L - c}{3}$, $q_H = \frac{a_H - c}{3} + \frac{1-\vartheta}{6}(a_H - a_L)$, $q_L = \frac{a_L - c}{3} - \frac{\vartheta}{6}(a_H - a_L)$. (Assume that all these quantities are positive.)

6.7 *Variations on Two-person Bertrand*
(a) If $c_1 < c_2$ then there is no Nash equilibrium. (Write down the reaction functions or – easier – consider different cases.)

(b) If $c_1 = c_2 = c < a - 1$ then there are two equilibria, namely $p_1 = p_2 = c$ and $p_1 = p_2 = c + 1$. (If $c = a - 1$ then there are two additional equilibria, namely $p_1 = c, p_2 = c + 1$ and $p_1 = c + 1, p_2 = c$.) Assume $c_1 < c_2$. There are several cases. E.g., if $c_1 < c_2 - 1$ and $a > 2c_2 - c_1$ then there are two equilibria, namely $p_1 = c_2 - 1, p_2 = c_2$ and $p_1 = c_2, p_2 = c_2 + 1$.

6.8 *Bertrand with More than Two Firms*
A strategy combination is a Nash equilibrium if and only if at least two firms charge a price of c and the other firms charge prices higher than c.

6.9 *Variations on Stackelberg*
(a) With firm 1 as a leader we have $q_1 = (1/2)(a - 2c_1 + c_2)$ and $q_2 = (1/4)(a + 2c_1 - 3c_2)$. With firm 2 as a leader we have $q_2 = (1/2)(a - 2c_2 + c_1)$ and $q_1 = (1/4)(a + 2c_2 - 3c_1)$.

(b) The leader in the Stackelberg game can always play the Cournot quantity: since the follower plays the best reply, this results in the Cournot outcome. Hence, the

Stackelberg equilibrium – where the leader maximizes – can only give a higher payoff. (This argument holds for an arbitrary game where one player moves first and the other player moves next, having observed the move of the first player.)

(c) $q_i = (1/2^i)(a-c)$ for $i = 1,2,\ldots,n$.

6.10 *First-price Sealed-bid Auction*
(b) Suppose that in some Nash equilibrium player i wins with valuation $v_i < v_1$. Then the winning bid b_i must be at most v_i otherwise player i makes a negative profit and therefore can improve by bidding (e.g.) v_i. But then player 1 can improve by bidding higher than b_i (and win) but lower than v_1 (and make positive profit). Other Nash equilibria: $(v_1, v_1, 0, 0, \ldots, 0)$, $(b, b, b, \ldots, b)$ with $v_1 \geq b \geq v_2$, etc.

(d) If not, then there would be a Nash equilibrium in which – in view of (c) – all players bid below their valuations. By (b) a player with the highest valuation wins the auction, so this must be player 1 if each player bids below his true valuation. But then player 1 can improve if $b_1 \geq v_2$ and player 2 can improve if $b_1 < v_2$.

6.11 *Second-price Sealed-bid Auction*
(d) Also $(v_1, 0\ldots, 0)$ is a Nash equilibrium.

(e) The equilibria are: $\{(b_1,b_2) \mid b_2 \geq v_1,\ 0 \leq b_1 \leq v_2\} \cup \{(b_1,b_2) \mid b_1 \geq v_2,\ b_2 \leq b_1,\ 0 \leq b_2 \geq v_1\}$.

6.12 *Third-price Sealed-bid Auction*
(b) Suppose $v_1 > v_2 > v_3 > \cdots$, then bidder 2 could improve by bidding higher than v_1.

(c) Everybody bidding the highest valuation v_1 is a Nash equilibrium. Also everybody bidding the second highest valuation v_2 is a Nash equilibrium. (There are many more!)

6.13 *n-Player First-price Sealed-bid with Incomplete Information*
Suppose every player $j \neq i$ plays s_j^*. If player i's type is v_i and he bids b_i (which can be assumed to be at most $1-1/n$ since no other bidder bids higher than this) then the probability of winning the auction is equal to the probability that very bid b_j, $j \neq i$, is at most b_i (including equality since this happens with zero probability). In turn, this is equal to the probability that $v_j \leq n/(n-1)b_i$ for every $j \neq i$. Since the players's valuations are independently drawn from the uniform distribution, the probability that player i wins the auction is equal to $(n/(n-1)b_i)^{n-1}$, hence player i should maximize the expression $(v_i - b_i)(n/(n-1)b_i)^{n-1}$, resulting in $b_i = (1-1/n)v_i$.

6.14 *Mixed Strategies and Objective Uncertainty*
(a) $((1/2,1/2),(2/5,3/5))$.

6.15 *Variations on Finite Horizon Bargaining*
(a) Adapt Table 6.1 for the various cases.

(b) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1-\delta_2+\delta_1\delta_2, \delta_2-\delta_1\delta_2)$ at $t=0$ and player 2 accepts.

(c) The subgame perfect equilibrium *outcome* in shares of the good is: player 1 proposes $(1-\delta_2^2+\delta_1\delta_2^2, \delta_2^2-\delta_1\delta_2^2)$ at $t=0$ and player 2 accepts.

(d) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1-\delta+\delta^2-\ldots+\delta^{T-1}-\delta^T s_1, \delta-\delta^2+\ldots-\delta^{T-1}+\delta^T s_1)$ at $t=0$ and player 2 accepts.

(e) The limits are $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$, independent of $\mathbf{s}$.

6.16 *Variations on Infinite Horizon Bargaining*
(a) Conditions (6.6) are replaced by $x_2^* = \delta_2 y_2^*$ and $y_1^* = \delta_1 x_1^*$. This implies $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$ and $y_1^* = \frac{\delta_1-\delta_1\delta_2}{1-\delta_1\delta_2}$. In the strategies (σ_1^*) and (σ_2^*), replace δ by δ_1 and δ_2, respectively. The equilibrium outcome is that player 1's proposal x^* at $t=0$ is accepted.

(b) Nothing essential changes. Player 2's proposal y^* is accepted at $t=0$.

(c) Nothing changes compared to the situation in the text, since $\mathbf{s}$ is only obtained at $t=\infty$.

(e) Let p denote the probability that the game ends. Then p is also the probability that the game ends given that it does not end at $t=0$. Hence, $p=(1-\delta)+\delta p$, so that $p=1$.

6.17 *A Principal–agent Game*
(a) This is a game of complete information. The employer starts and has an infinite number of actions available, namely any pair (w_L, w_H) of nonnegative wages. After each of these actions, the worker has three possible actions: reject, resulting in 0 for the employer and 2 for the worker; accept and exert high effort, resulting in the (expected) payoffs of $9.2-\frac{8w_H+2w_L}{10}$ for the employer and $\frac{8w_H+2w_L}{10}-3$ for the worker; accept and exert low effort, resulting in the (expected) payoffs of $6.8-\frac{2w_H+8w_L}{10}$ for the employer and $\frac{2w_H+8w_L}{10}$ for the worker. The actions of the employer are also his strategies. A strategy for the worker is a function $(w_L, w_H) \mapsto \{$ reject (r), accept and exert high effort (h), accept and exert low effort (l) $\}$.

(b) Any pair of wages (w_L, w_H) with $\frac{2w_H+8w_L}{10}=2$ and $w_H \le w_L+5$, together with the worker choosing a payoff maximizing action but certainly l if l is payoff maximizing following any wage offer package (w_L, w_H), is a subgame perfect equilibrium. All these equilibria result in the worker accepting and exerting low effort, and final payoffs of 4.8 for the employer and 2 for the worker.

6.18 *The Market for Lemons*
(b) There are many subgame perfect equilibria: the buyer offers $p \le 5{,}000$ and the seller accepts any price of at least 5,000 if the car is bad and of at least 15,000 if the car is good. All these equilibria result in expected payoff of zero for both. There are no other subgame perfect equilibria.

6.19 *Corporate Investment and Capital Structure*
(b) Suppose the investor's belief that $\pi=L$ after observing s is equal to q. Then the investor accepts s if and only if

$$s[qL+(1-q)H+R] \ge I(1+r). \qquad (*)$$

The entrepreneur prefers to receive the investment if and only if

$$s \leq R/(\pi + R), \qquad (**)$$

for $\pi \in \{L, H\}$.

In a pooling equilibrium, $q = p$. Note that $(**)$ is more difficult to satisfy for $\pi = H$ than for $\pi = L$. Thus, $(*)$ and $(**)$ imply that a pooling equilibrium exists only if

$$\frac{I(1+r)}{pL + (1-p)H + R} \leq \frac{R}{H+R}.$$

A separating equilibrium always exists. The low-profit type offers $s = I(1 + r)/(L+R)$, which the investor accepts, and the high-profit type offers $s < I(1 + r)/(H+R)$, which the investor rejects.

6.20 *A Poker Game*

(a) The strategic form of this game is as follows:

$$\begin{array}{cc} & \begin{array}{cccc} \text{aa} & \text{aq} & \text{ka} & \text{kq} \end{array} \\ \begin{array}{c} \text{believe} \\ \text{show} \end{array} & \begin{pmatrix} -1,1 & -1/3,1/3 & -2/3,2/3 & 0,0 \\ 2/3,-2/3 & 1/3,-1/3 & 0,0 & -1/3,1/3 \end{pmatrix} \end{array}.$$

Here, 'believe' and 'show' are the strategies of player I. The first letter in any strategy of player II is what player II says if the dealt card is a King, the second letter is what II says if the dealt card is a Queen – if the dealt card is an Ace player II has no choice.

(b) Player I has a unique optimal (maximin) strategy and player 2 has a unique optimal (minimax) strategy. The value of the game is $-2/9$.

6.21 *A Hotelling Location Problem*

(a) $x_1 = x_2 = \frac{1}{2}$.

(c) $x_1 = x_2 = \frac{1}{2}$.

6.22 *Median Voting*

(a) The strategy set of each player is the interval $[0,30]$. If each player i plays x_i, then the payoff to each player i is $-|\frac{x_1+\ldots+x_n}{n} - t_i|$. A Nash equilibrium always exists.

(b) The payoff to player i is now $-|\operatorname{med}(x_1,\ldots,x_n) - t_i|$, where $\operatorname{med}(\cdot)$ denotes the median. For each player, proposing a temperature different from his true ideal temperature either leaves the median unchanged or moves the median farther away from the ideal temperature, whatever the proposals of the other players. Hence, proposing one's ideal temperature is a weakly dominant strategy.

6.23 *The Uniform Rule*

(b) $M = 4:\ (1,3/2,3/2), M = 5:\ (1,2,2), M = 5.5:\ (1,2,5/2), M = 6:\ (1,2,3)$, $M = 7:\ (2,2,3), M = 8:\ (5/2,5/2,3), M = 9:\ (3,3,3)$.

(c) If player i reports t_i and receives $s_i > t_i$ then, apparently the total reported quantity is above M and thus, player i can only further increase (hence, worsen) his share by reporting a different quantity. If player i reports t_i and receives $s_i < t_i$ then, apparently the total reported quantity is below M and thus, player i can only further decrease (hence, worsen) his share by reporting a different quantity.

There exist other Nash equilibria, but they do not give different outcomes (shares). E.g., if $M > \sum_{j=1}^{n} t_j$, then player 1 could just as well report 0 instead of t_1.

6.24 *Reporting a Crime*
(b) $p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$.

(c) The probability of the crime being reported in this equilibrium is $1-(1-p)^n = 1 - \left(\frac{c}{v}\right)^{\frac{n}{n-1}}$. This converges to $1-(c/v)$ for n going to infinity. Observe that both p and the probability of the crime being reported decrease if n becomes larger.

6.25 *Firm Concentration*
Let, in equilibrium, n firms locate downtown and m firms in the suburbs, with $n = 6$ and $m = 4$.

6.26 *Tragedy of the Commons*
(d) Suppose, to the contrary, $G^* \leq G^{**}$. Then $v(G^*) \geq v(G^{**})$ since $v' < 0$, and $0 > v'(G^*) \geq v'(G^{**})$ since $v'' < 0$. Also, $G^*/n < G^{**}$. Hence

$$v(G^*) + (1/n)G^*v'(G^*) - c > v(G^{**}) + G^{**}v'(G^{**}) - c,$$

a contradiction since both sides should be zero.

Problems of Chap. 7

7.1 *On Discounting and Limiting Average*
(a) See the solution to Problem 6.16(e).

(b) A sequence like $1, 3, 5, 7, \ldots$ has a limiting average of infinity. More interestingly, one may construct a sequence containing only the numbers $+1$ and -1 of which the finite averages ‘oscillate’, e.g., below $-1/2$ and above $+1/2$, so that the limit does not exist.

7.2 *Nash and Subgame Perfect Equilibrium in a Repeated Game*
(a) The unique Nash equilibrium is (T, R); $v(A) = 1$ and the minimax strategy in A is R; $v(-B) = -1$ and the maximin strategy in $-B$ is row B.

(b) Only $(1, 5)$, independent of δ.

(c) $\delta \geq \frac{1}{2}$, to keep player 2 from deviating to R.

7.3 *Nash and Subgame Perfect Equilibrium in Another Repeated Game*
(a) The limiting average payoffs $(2, 1)$, $(1, 2)$, and $(2/3, 2/3)$, resulting from playing, respectively, the Nash equilibria (T, L), (B, R), and $((2/3, 1/3), (1/3, 2/3))$ at every stage; and all payoffs (x_1, x_2) with $x_1, x_2 > 2/3$.

(b) $v(A) = 2/3$ and $-v(-B) = 2/3$. Hence, all payoffs (x_1, x_2) with $x_1, x_2 > 2/3$.

7.4 *Repeated Cournot and Bertrand*
(a) The relevant restriction on δ is given by $\frac{1}{8} \geq \frac{9}{64}(1-\delta) + \frac{1}{9}\delta$, which yields $\delta \geq \frac{9}{17}$.

(b) $\delta \geq \frac{1}{2}$.

Problems of Chap. 8

8.1 *Symmetric Games*
(a) $(0,1)$ is the only ESS.

(b) Both $(1,0)$ and $(0,1)$ are ESS. The (Nash equilibrium) strategy $(1/3, 2/3)$ is not an ESS.

8.2 *More Symmetric Games*
The replicator dynamics in the matrix A is $\dot{p} = p(p-1)(p-\frac{1}{2})$, with rest points $p = 0, 1, \frac{1}{2}$, of which only $p = \frac{1}{2}$ is stable. The game (A, A^T) has a unique symmetric Nash equilibrium, namely $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$. The unique ESS is $(\frac{1}{2}, \frac{1}{2})$.

8.3 *Asymmetric Games*
In (B, B^T), the replicator dynamics is given by the equations $\dot{p} = pq(1-p)$ and $\dot{q} = pq(1-q)$. There is one stable rest point, namely $p = q = 1$, corresponding to the unique strict Nash equilibrium $((1,0),(1,0))$ of the game. The other rest points are all points in the set

$$\{(p,q) \mid \ p = 0 \text{ and } 0 \leq q \leq 1 \text{ or } q = 0 \text{ and } 0 \leq p \leq 1\}.$$

8.4 *Frogs Call for Mates*
Note that for (a) and (b) Proposition 8.6 can be used. Similarly, for (c) we can use Proposition 8.7, by stating the conditions under which each of the four pure strategy combinations is a strict Nash equilibrium: if $z_1 < P + m - 1$ and $z_2 < P + m - 1$ then (Call, Call) is a strict Nash equilibrium, etc.

8.5 *Video Market Game*
There are four rest points, of which only one is stable.

Problems of Chap. 9

9.2 *Computing the Core*
(a) $\{(0,0,1)\}$; (b) polygon with vertices $(15,5,4)$, $(9,5,10)$, $(14,6,4)$, and $(8,6,10)$.

9.4 *'Non-monotonicity' of the Core*
(b) The core of (N, v') is the set $\{(0,0,1,1)\}$ (use the fact that $C(N,v') \subseteq C(N,v)$). Hence, player 1 can only obtain less in the core although the worth of coalition $\{1,3,4\}$ has increased.

9.5 *Efficiency of the Shapley Value*
Consider an order $i_1, i_2, \ldots, i_n$ of the players. The sum of the coordinates of the associated marginal vector is $[v(\{i_1\}) - v(\emptyset)] + [v(\{i_1, i_2\}) - v(\{i_1\})] + [v(\{i_1, i_2, i_3\}) - v(\{i_1, i_2\})] + \cdots + [v(N) - v(N \setminus \{i_n\})] = v(N) - v(\emptyset) = v(N)$. Hence, every marginal vector is efficient, so the Shapley value is efficient since it is the average of the marginal vectors.

9.6 *Computing the Shapley Value*
(a) $\Phi(N,v) = (1/6, 1/6, 2/3) \notin C(N,v)$; (b) $(9\frac{1}{2}, 6\frac{1}{2}, 8)$, not in the core.

9.7 *The Shapley Value and the Core*
(a) $a = 3$ (use Problem 9.3).

(b) $(2.5, 2, 1.5)$.

(c) The Shapley value is $(a/3 + 1/2, a/3, a/3 - 1/2)$. The minimal value of a for which this is in the core is $15/4$.

9.8 *Shapley Value in a Two-player Game*
$\Phi(N,v) = (v(\{1\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}, v(\{2\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2})$.

9.9 *The Nucleolus and the Core*
Use the fact that at a core element all excesses are nonpositive.

9.10 *Computing the Nucleolus*
(a) $(0, 0, 1)$.

(b) $(11.5, 5.5, 7)$.

(c) $(1/5, 1/5, 1/5, 1/5, 1/5, 0, \ldots, 0) \in \mathbb{R}^{15}$.

(d) In (N,v): $(1/2, 1/2, 1/2, 1/2)$; in (N,v): $(0, 0, 1, 1)$.

9.11 *Nucleolus of Two-player Games*
The nucleolus is $(v(\{1\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}, v(\{2\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2})$.

9.12 *Computing the Core, the Shapley Value, and the Nucleolus*
(a) The nucleolus and Shapley value coincide and are equal to $(1.5, 2, 2.5)$.

(c) The maximal value of $v(\{1\})$ is 2. For that value the unique core element is $(2, 2, 2)$.

9.13 *Properties of the Shapley Value*
(a) In $\Phi_i(N,v)$ the term $v(S \cup \{i\}) - v(S)$ occurs the same number of times as the term $v(S \cup \{j\}) - v(S)$ in $\Phi_j(N,v)$, for every coalition $S \subseteq N \setminus \{i, j\}$. Let S be a coalition with $i \in S$ and $j \notin S$. Then $v(S \setminus \{i\} \cup \{j\}) = v(S \setminus \{i\} \cup \{i\})$, so that

$$\begin{aligned} v(S \cup \{j\}) - v(S) &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{i\}) \\ &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{j\}), \end{aligned}$$

and also these expressions occur the same umber of times. Similarly for coalitions S that contain j but not i.

(b) This is obvious from Definition 9.3.

(c) Observe that it is sufficient to show $\sum_{S:i\notin S} \frac{|S|!(n-|S|-1)!}{n!} = 1$. To show this, note that $\frac{|S|!(n-|S|-1)!}{n!} = \frac{1}{n}\begin{pmatrix} n-1 \\ |S| \end{pmatrix}^{-1}$, so that

$$\sum_{S:i\notin S} \frac{|S|!(n-|S|-1)!}{n!} = \frac{1}{n} \sum_{s=0,1,\ldots,n-1} \binom{n-1}{s} \binom{n-1}{s}^{-1}$$
$$= \frac{1}{n} \cdot n = 1.$$

Problems of Chap. 10

10.1 *A Division Problem (1)*
(b) In terms of utilities: $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, in terms of distribution: $(\frac{1}{3}\sqrt{3}, 1-\frac{1}{3}\sqrt{3})$.

(c) The Rubinstein outcome is x^* where $x_1^* = \sqrt{\frac{1}{1+\delta+\delta^2}}$ and $x_2^* = 1 - \frac{1}{1+\delta+\delta^2}$.

(d) $\lim_{\delta\to 1} x_1^* = \frac{1}{3}\sqrt{3}$, consistent with what was found under (a).

10.2 *A Division Problem (2)*
Use symmetry, Pareto optimality and covariance of the Nash bargaining solution.

10.3 *A Division Problem (3)*
(a) The distribution of the good is $\left(2\frac{1-\delta^3}{1-\delta^4}, 2-2\frac{1-\delta^3}{1-\delta^4}\right)$. In utility terms this is $\left(\frac{1-\delta^3}{1-\delta^4}, \sqrt[3]{2-2\frac{1-\delta^3}{1-\delta^4}}\right)$.

(b) By taking the limit for $\delta \to 1$ in (b), we obtain $(1.5, 0.5)$ as the distribution assigned by the Nash bargaining solution. In utilities: $(0.75, \sqrt[3]{0.5})$.

10.4 *An Exchange Economy*
(a) $x_1^A(p_1,p_2) = (3p_2+2p_1)/2p_1$, $x_2^A = (4p_1-p_2)/2p_2$, $x_1^B = (p_1+6p_2)/2p_1$, $x_2^B = p_1/2p_2$.

(b) $(p_1,p_2) = (9,5)$ (or any positive multiple thereof); the equilibrium allocation is $((33/18, 31/10), (39/18, 9/10))$.

(c) The (non-boundary part of the) contract curve is given by the equation $x_2^A = (17x_1^A+5)/(2x_1^A+8)$. The core is the part of this contract curve such that $\ln(x_1^A+1)+\ln(x_2^A+2) \geq \ln 4+\ln 3 = \ln 12$ (individual rationality constraint for A) and $3\ln(5-x_1^A)+\ln(5-x_2^A) \geq 3\ln 2+\ln 4 = \ln 12$ (individual rationality constraint for B).

(d) The point $\mathbf{x}^A = (33/18, 31/10)$ satisfies the equation $x_2^A = (17x_1^A+5)/(2x_1^A+8)$.

(e) For the disagreement point $\mathbf{d}$ one can take the point $(\ln 12, \ln 12)$. The set S contains all points $\mathbf{u} \in \mathbb{R}^2$ that can be obtained as utilities from any distribution of the goods that does not exceed total endowments $\mathbf{e} = (4,4)$. Unlike the Walrasian equilibrium allocation, the allocation obtained by applying the Nash bargaining solution is not independent of arbitrary monotonic transformations of the utility functions. It is a 'cardinal' concept, in contrast to the Walrasian allocation, which is 'ordinal'.

10.5 *The Matching Problem of Table 10.1 Continued*
(a) The resulting matching is (w_1,m_1), (w_2,m_2), w_3 and m_3 remain single.

10.6 *Another Matching Problem*
(a) With the men proposing: (m_1,w_1), (m_2,w_2), (m_3,w_3). With the women proposing: (m_1,w_1), (m_2,w_3), (m_3,w_2).

(b) Since in any stable matching we must have (m_1,w_1), the matchings found in (a) are the only stable ones.

(c) Obvious: every man weakly or strongly prefers the men proposing matching in (a); and vice versa for the women.

10.7 *Yet Another Matching Problem: Strategic Behavior*
(b) There are no other stable matchings.

(c) The resulting matching is (m_1,w_1), (m_2,w_3), (m_3,w_2). This is clearly better for w_1.

10.8 *Core Property of Top Trading Cycle Procedure*
All players in a top trading cycle get their top houses, and thus none of these players can be a member of a blocking coalition, say S. Omitting these players and their houses from the problem, by the same argument none of the players in a top trading cycle in the second round can be a member of S: the only house that such a player may prefer is no longer available in S; etc.

10.9 *House Exchange with Identical Preferences*
Without loss of generality, assume that each player has the same preference $h_1h_2\ldots h_n$. Show that in a core allocation each player keeps his own house.

10.10 *A House Exchange Problem*
There are three core allocations namely: (1) $1:h_3$, $2:h_4$, $3:h_1$, $4:h_2$; (2) $1:h_2$, $2:h_4$, $3:h_1$, $4:h_3$; (3) $1:h_3$, $2:h_1$, $3:h_4$, $4:h_2$. Allocation (1) is in the strong core.

10.11 *Cooperative Oligopoly*
(a)–(c) Analogous to Problems 6.1, 6.2. Parts (d) and (f) follow directly from (c). For parts (e) and (g) use the methods of Chap. 9.

Problems of Chap. 11

11.1 *Preferences*
(a) If $a \neq b$ and aRb and bRa then neither aPb nor bPa, so P is not necessarily complete.

(b) I is not complete unless aRb for all $a,b \in A$. I is only antisymmetric if R is a linear order.

11.2 *Pairwise Comparison*
(a) $C(r)$ is reflexive and complete but not antisymmetric.

(c) There is no Condorcet winner in this example.

11.3 *Independence of the Conditions in Theorem 11.1*
The social welfare function based on the Borda scores is Pareto efficient but does not satisfy IIA and is not dictatorial (cf. Sect. 11.1). The social welfare function that assigns to each profile of preferences the reverse preference of agent 1 satisfies IIA and is not dicatorial but also not Pareto efficient.

11.4 *Independence of the Conditions in Theorem 11.2*
A constant social welfare function (i.e., always assigning the same fixed alternative) is strategy-proof and nondictatorial but not surjective. The social welfare function that always assigns the bottom element of agent 1 is surjective, nondictatorial, and not strategy-proof.

11.5 *Independence of the Conditions in Theorem 11.3*
A constant social welfare function (i.e., always assigning the same fixed alternative) is monotonic and nondictatorial but not unanimous. A social welfare function that assigns the common top alternative to any profile where all agents have the same top alternative, and a fixed constant alternative to any other profile, is unanimous and nondictatorial but not monotonic.

11.6 *Copeland Score and Kramer Score*
(a) The Copeland ranking is a preference. The Copeland ranking is not antisymmetric. It is easy to see that the Copeland ranking is Pareto efficient. By Arrow's Theorem therefore, it does not satisfy IIA.

(b) The Kramer ranking is a preference. The Kramer ranking is not antisymmetric and not Pareto efficient. It violates IIA.

11.7 *Two Alternatives*
Consider the social welfare function based on majority rule, i.e., it assigns aPb if $|N(a,b,r)| > |N(b,a,r)|$; bPa if $|N(a,b,r)| < |N(b,a,r)|$; and aIb if $|N(a,b,r)| = |N(b,a,r)|$.

Problems of Chap. 12

12.2 2×2 *Games*
(a) To have no saddlepoints we need $a_{11} > a_{12}$ or $a_{11} < a_{12}$. Assume the first, then the other inequalities follow.

(b) For optimal strategies $\mathbf{p} = (p, 1-p)$ and $\mathbf{q} = (q, 1-q)$ we must have $0 < p < 1$ and $0 < q < 1$. Then use that p should be such that player 2 is indifferent between the two columns and q such that player 1 is indifferent between the two rows.

12.3 *Symmetric Games*
Let $\mathbf{x}$ be optimal for player 1. Then $\mathbf{x}A\mathbf{y} \geq v(A)$ for all $\mathbf{y}$; hence $\mathbf{y}A\mathbf{x} = -\mathbf{x}A\mathbf{y} \leq -v(A)$ for all $\mathbf{y}$; hence (take $\mathbf{y} = \mathbf{x}$) $v(A) \leq -v(A)$, so $v(A) \leq 0$. Similarly, derive the converse inequality by considering an optimal strategy for player 2.

12.4 *The Duality Theorem Implies the Minimax Theorem*
Let A be an $m \times n$ matrix game. Without loss of generality assume that all entries of A are positive. Consider the associated LP as in Sect. 12.2.

Consider the vector $\bar{\mathbf{x}} = (1/m, \ldots, 1/m, \eta) \in \mathbb{R}^{m+1}$ with $\eta > 0$. Since all entries of A are positive it is straightforward to check that $\bar{\mathbf{x}} \in V$ if $\eta \leq \sum_{i=1}^{m} a_{ij}/m$ for all $j = 1, \ldots, n$. Since $\bar{\mathbf{x}} \cdot c = -\eta < 0$, it follows that the value of the LP must be negative.

Let $\mathbf{x} \in O_{\min}$ and $\mathbf{y} \in O_{\max}$ be optimal solutions of the LP. Then $-x_{m+1} = -y_{n+1} < 0$ is the value of the LP. We have $x_i \geq 0$ for every $i = 1, \ldots, m$, $\sum_{i=1}^{m} x_i \leq 1$, and $(x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ (> 0) for every $j = 1, \ldots, n$. Optimality in particular implies $\sum_{i=1}^{m} x_i = 1$, so that $v_1(A) \geq (x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ for all j, hence $v_1(A) \geq x_{m+1}$. Similarly, it follows that $v_2(A) \leq y_{n+1} = x_{m+1}$, so that $v_2(A) \leq v_1(A)$. The Minimax Theorem now follows.

12.5 *Infinite Matrix Games*
(a) A is an infinite matrix game with for all $i, j \in \mathbb{N}$: $a_{ij} = 1$ if $i > j$, $a_{ij} = 0$ if $i = j$, and $a_{ij} = -1$ if $i < j$.

(b) Fix a mixed strategy $\mathbf{p} = (p_1, p_2, \ldots)$ for player 1 with $p_i \geq 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} p_i = 1$. If player 2 plays pure strategy j, then the expected payoff for player 1 is equal to $-\sum_{i=1}^{j-1} p_i + \sum_{i=j+1}^{\infty} p_i$. Since $\sum_{i=1}^{\infty} p_i = 1$, this expected payoff converges to -1 as j approaches ∞. Hence, $\inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$, so $\sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$. Similarly, one shows $\inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathbf{p}A\mathbf{q} = 1$, hence the game has no 'value'.

12.6 *Equalizer Theorem*
Assume, without loss of generality, $v = 0$. It is sufficient to show that there exists $\mathbf{q} \in \mathbb{R}^n$ with $\mathbf{q} \geq \mathbf{0}$, $A\mathbf{q} \leq \mathbf{0}$, and $q_n = 1$. The required optimal strategy is then obtained by normalization.

This is equivalent to existence of a vector $(\mathbf{q}, w) \in \mathbb{R}^{n+1}$ with $\mathbf{q} \geq \mathbf{0}$, $w \geq 0$, such that

$$\begin{pmatrix} A & I \\ \mathbf{e}^n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ w \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix},$$

where row vector $\mathbf{e}^n \in \mathbb{R}^n$, I is the $m \times m$ identity matrix, $\mathbf{0}$ is an $1 \times m$ vector on the left hand side and an $m \times 1$ vector on the right hand side. Thus, we have to show that the vector $\mathbf{x} := (\mathbf{0}, 1) \in \mathbb{R}^{m+1}$ is in the cone spanned by the columns of the $(m+1) \times (n+m)$ matrix on the left hand side. Call this matrix B and call this cone Z. Assume $\mathbf{x} \notin Z$ and derive a contradiction using Theorem 22.1.

Problems of Chap. 13

13.1 *Existence of Nash Equilibrium*
For upper semi-continuity of β, take a sequence σ^k converging to σ, a sequence $\tau^k \in \beta(\sigma^k)$ converging to τ, and show $\tau \in \beta(\sigma)$.

13.2 *Lemma 13.2*
The only-if direction is straightforward from the definition of Nash equilibrium.

13.3 *Lemma 13.3*
Take i such that $\mathbf{e}^i A\mathbf{q} \geq \mathbf{e}^k A\mathbf{q}$ for all $k = 1, \ldots, m$. Then, clearly, $\mathbf{e}^i A\mathbf{q} \geq \mathbf{p}' A\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$, so $\mathbf{e}^i \in \beta_1(\mathbf{q})$. The second part is analogous.

13.4 *Dominated Strategies*
(b) Denote by $NE(A,B)$ the set of Nash equilibria of (A,B). Then

$$\begin{aligned}
(\mathbf{p}^*, \mathbf{q}^*) \in NE(A,B) &\Leftrightarrow (\mathbf{p}^*, (\mathbf{q}', 0)) \in NE(A,B) \text{ where } (\mathbf{q}', 0) = \mathbf{q}^* \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1} [\mathbf{p}^* A(\mathbf{q}', 0) \geq \mathbf{p}A(\mathbf{q}', 0), \\
&\qquad \mathbf{p}^* B(\mathbf{q}', 0) \geq \mathbf{p}^* B(\mathbf{q}, 0)] \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1} [\mathbf{p}^* A'\mathbf{q}' \geq \mathbf{p}A\mathbf{q}', \\
&\qquad \mathbf{p}^* B'\mathbf{q}' \geq \mathbf{p}^* B\mathbf{q}] \\
&\Leftrightarrow (\mathbf{p}^*, \mathbf{q}') \in NE(A', B').
\end{aligned}$$

Note that the first equivalence follows by part (a).

13.5 *A* 3×3 *Bimatrix Game*
(c) The unique Nash equilibrium is $((0,0,1),(0,0,1))$.

13.6 *A* 3×2 *Bimatrix Game*
The set of Nash equilibria is $\{(\mathbf{p}, \mathbf{q}) \in \Delta^3 \times \Delta^2 \mid p_1 = 0,\ q_1 \geq \frac{1}{2}\} \cup \{((1,0,0),(0,1))\}$.

13.7 *Proof of Theorem 13.8*
'If': conditions (13.1) are satisfied and $f = 0$, which is optimal since $f \leq 0$ always. 'Only-if': clearly we must have $a = \mathbf{p}A\mathbf{q}$ and $b = \mathbf{p}B\mathbf{q}$ (otherwise $f < 0$ which cannot be optimal). From the conditions (13.1) we have $\mathbf{p}'A\mathbf{q} \leq a = \mathbf{p}A\mathbf{q}$ and $\mathbf{p}B\mathbf{q}' \leq b = \mathbf{p}B\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$ and $\mathbf{q}' \in \Delta^n$, which implies that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

13.8 *Matrix Games*
This is a repetition of the proof of Theorem 12.4. Note that the solutions of program (13.3) give exactly the value of the game a and the optimal (minimax) strategies of player 2. The solutions of program (13.4) give exactly the value of the game $-b$ and the optimal (maximin) strategies of player 1.

13.9 *Tic-Tac-Toe*
(a) Start by putting a cross in the center square. Then player 2 has essentially two possibilities for the second move, and it is easy to see that in each of the two cases player 1 has a forcing third move. After this, it is equally easy to see that player 1 can always enforce a draw.

(b) If player 1 does not start at the center, then player 2 can put his first circle at the center and then can place his second circle in such a way that it becomes forcing. If player 1 starts at the center then either a pattern as in (a) is followed, leading to a draw, or player 2's second circle becomes forcing, also resulting in a draw.

13.10 *Iterated Elimination in a Three-player Game*
The resulting strategy combination is (D, l, L).

13.11 *Never a Best Reply and Domination*
First argue that strategy Y is not strictly dominated. Next assume that Y is a best reply to strategies $(p, 1-p)$ of player 1 and $(q, 1-q)$ of player 2, and derive a contradiction.

13.13 *A Three-player Game with an Undominated but not Perfect Equilibrium*
(a) (U, l, L) is the only perfect equilibrium.

(b) The equilibrium (D, l, L) is undominated.

13.14 *Existence of Proper Equilibrium*
Tedious but straightforward.

13.15 *Strictly Dominated Strategies and Proper Equilibrium*
(a) The only Nash equilibria are $(U, l(, L))$ and $(D, r(, L))$. Obviously, only the first one is perfect and proper.

(b) (D, r, L), is a proper Nash equilibrium.

13.16 *Strictly Perfect Equilibrium*
(a) Identical to the proof of Lemma 13.16, see Problem 13.12: note that any sequence of perturbed games converging to the given game must eventually contain any given completely mixed Nash equilibrium σ.

(c) The set of Nash equilibria is $\{((p, 1-p), L) \mid 0 \leq p \leq 1\}$, where p is the probability on U. Every Nash equilibrium of the game (A, B) is perfect and proper. No Nash equilibrium is strictly perfect.

13.17 *Correlated Equilibria in the Two-driver Example (1)*
Use inequalities (13.5) and (13.6) to derive the conditions: $p_{11} + p_{12} + p_{21} + p_{22} = 1$, $p_{ij} \geq 0$ for all $i, j \in \{1, 2\}$, $p_{11} \leq \frac{3}{5} \min\{p_{12}, p_{21}\}$, $p_{22} \leq \frac{5}{3} \min\{p_{12}, p_{21}\}$.

13.18 *Nash Equilibria are Correlated*
Check that (13.5) and (13.6) are satisfied for P.

13.19 *The Set of Correlated Equilibria is Convex*
Let P and Q be correlated equilibria and $0 \leq t \leq 1$. Check that (13.5) and (13.6) are satisfied for $tP + (1-t)Q$.

13.20 *Correlated vs. Nash Equilibrium*
(a) The Nash equilibria are: $((1, 0), (0, 1))$, $((0, 1), (1, 0))$, and $((2/3, 1/3), (2/3, 1/3))$.

13.21 *Correlated Equilibria in the Two-driver Example (2)*
The matrix C is

$$\begin{array}{c} \\ (1,1') \\ (1,2') \\ (2,1') \\ (2,2') \end{array} \begin{array}{c} \begin{array}{cccc} (1,2) & (2,1) & (1',2') & (2',1') \end{array} \\ \begin{pmatrix} -10 & 0 & 0 & -10 \\ 6 & 0 & 10 & 0 \\ 0 & 10 & 0 & 6 \\ 0 & -6 & -6 & 0 \end{pmatrix} \end{array}.$$

13.22 *Finding Correlated Equilibria*
There is a unique correlated equilibrium

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

13.23 *Independence of the Axioms in Corollary 13.35*
Not OPR: take the set of all strategy combinations in every game. Not CONS: in games with maximal player set take all strategy combinations, in other games take the set of Nash equilibria. Not COCONS: drop a Nash equilibrium in some game with maximal player set, but otherwise always take the set of all Nash equilibria.

13.24 *Inconsistency of Perfect Equilibria*
First show that the perfect equilibria in G_0 are all strategy combinations where player 2 plays L, player 3 plays D, and player 1 plays any mixture between T and B. Next consider the reduced game by fixing player 3's strategy at D.

Problems of Chap. 14

14.2 *An Extensive form Structure without Perfect Recall*
(a) The paths $\{(x_0,x_1)\}$ and $\{(x_0,x_2)\}$ contain different player 1 actions.

14.3 *Consistency Implies Bayesian Consistency*
With notations as in Definition 14.12, for $h \in H$ with $\mathbb{P}_b(h) > 0$ and $x \in h$ we have $\beta_h(x) = \lim_{m\to\infty} \beta_h^m(x) = \lim_{m\to\infty} \mathbb{P}_{b^m}(x)/\mathbb{P}_{b^m}(h) = \mathbb{P}_b(x)/\mathbb{P}_b(h)$. Here, the second equality follows from Bayesian consistency of the (b^m, β^m).

14.4 *(Bayesian) Consistency in Signaling Games*
The idea of the proof is as follows. Suppose player 1 puts zero probability on some action a, so that $\mathbb{P}_b(h) = 0$, where h is player 2's information set following a. Let the beliefs of player 2 on h be given by $\beta_1, \ldots, \beta_k$ for the different types of player 1. Consider, for any $\varepsilon > 0$, a behavioral strategy of player 1 that puts probability $\varepsilon\beta_k$ on action a for type k. Do this for any action of player 1 played with zero probability, and complete the behavioral strategies thus obtained by maintaining the original (conditional) probabilities on actions played with positive probabilities in

the original assessment. Letting ε go to zero, we have a sequence of completely mixed Bayesian consistent assessments converging to the original assessment.

14.5 *Computation of Sequential Equilibrium (1)*
The unique sequential equilibrium consists of the behavioral strategies where player 1 plays B with probability 1 and l with probability $1/2$, and player 2 plays L with probability $1/2$; and player 1 believes that x_3 and x_4 are equally likely.

14.6 *Computation of Sequential Equilibrium (2)*
(b) The Nash equilibria are (L,l), and $(R,(\alpha,1-\alpha))$ for all $\alpha \leq 1/2$, where α is the probability with which player 2 plays l.

(c) Let π be the belief player 2 attaches to node y_1. Then the sequential equilibria are: (L,l) with belief $\pi = 1$; (R,r) with belief $\pi \leq 1/2$; and $(R,(\alpha,1-\alpha))$ for any $\alpha \leq 1/2$ with belief $\pi = 1/2$.

14.7 *Computation of Sequential Equilibrium (3)*
(b) The Nash equilibria are $(R,(q,1-q))$ with $1/3 \leq q \leq 2/3$. (The conditions on q keep player 1 from deviating to L or M.)

14.8 *Computation of Sequential Equilibrium (4)*
The Nash equilibria in this game are: $(R,(q_1,q_2,q_3))$ with $q_3 \leq 1/3$ and $q_1 \leq 1/2-(3/4)q_3$, where q_1,q_2,q_3 are the probabilities put on l,m,r, respectively; and $((1/4,3/4,0),(1/4,0,3/4)$ (probabilities on L,M,R and l,m,r, respectively).

Let π be the belief attached by player 2 to y_1. Then with $\pi = 1/4$ the equilibrium $((1/4,3/4,0),(1/4,0,3/4)$ becomes sequential. The first set of equilibria contains no equilibrium that can be extended to a sequential equilibrium.

14.9 *Computation of Sequential Equilibrium (5)*
The Nash equilibria are: (DB,r); $((R,(s,1-s)),(q,1-q))$ with $0 \leq s \leq 1$ and $q \geq 1/3$, where s is the probability on A and q is the probability on l. The subgame perfect equilibria are: (DB,r); (RA,l); $((R,(3/4,1/4)),(3/5,2/5))$. The first one becomes sequential with $\beta = 0$; the second one with $\beta = 1$; and the third one with $\beta = 3/5$.

Problems of Chap. 15

15.1 *Computing ESS in* 2×2 *Games (1)*
$ESS(A)$ can be computed using Proposition 15.3.
(a) $ESS(A) = \{\mathbf{e}^2\}$. (b) $ESS(A) = \{\mathbf{e}^1,\mathbf{e}^2\}$. (c) $ESS(A) = \{(2/3,1/3)\}$.

15.2 *Computing ESS in* 2×2 *Games (2)*
Case (1): $ESS(A') = \{\mathbf{e}^2\}$; case (2): $ESS(A') = \{\mathbf{e}^1,\mathbf{e}^2\}$; case (3): $ESS(A') = \{\hat{\mathbf{x}}\} = \{(a_2/(a_1+a_2),a_1/(a_1+a_2))\}$.

15.3 *Rock–Paper–Scissors (1)*
The unique Nash equilibrium is $((1/3,1/3,1/3),(1/3,1/3,1/3))$, which is symmetric. But $(1/3,1/3,1/3)$ is not an ESS.

15.4 *Uniform Invasion Barriers*
Case (1), $\mathbf{e}^2$: maximal uniform invasion barrier is 1.
Case (2), $\mathbf{e}^1$: maximal uniform invasion barrier is $a_1/(a_1+a_2)$.
Case (2), $\mathbf{e}^2$: maximal uniform invasion barrier is $a_2/(a_1+a_2)$.
Case (3), $\hat{\mathbf{x}}$: maximal uniform invasion barrier is 1.

15.5 *Replicator Dynamics in Normalized Game (1)*
Straightforward computation.

15.6 *Replicator Dynamics in Normalized Game (2)*
The replicator dynamics can be written as $\dot{x} = [x(a_1+a_2) - a_2]x(1-x)$, where $\dot{x} = \dot{x}_1$. So $x = 0$ and $x = 1$ are always stationary points. In case I the graph of $\dot{x}$ on $(0,1)$ is below the horizontal axis. In case II there is another stationary point, namely at $x = a_2/(a_1+a_2)$; on $(0, a_2/(a_1+a_2))$ the function $\dot{x}$ is negative, on $(a_2/(a_1+a_2), 1)$ it is positive. In case III the situation of case II is reversed: the function $\dot{x}$ is positive on $(0, a_2/(a_1+a_2))$ and negative on $((a_2/(a_1+a_2), 1)$.

15.7 *Weakly Dominated Strategies and Replicator Dynamics*
(b) The stationary points are $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and all points with $x_3 = 0$. Except $\mathbf{e}^3$, all stationary points are Lyapunov stable. None of these points is asymptotically stable. Also, $\mathbf{e}^3$ is strictly dominated (by $\mathbf{e}^1$). (One can also derive $d(x_1/x_2)/dt = x_1x_3/x_2 > 0$ at completely mixed strategies, i.e., at the interior of Δ^3. Hence, the share of subpopulation 1 grows faster than that of 2 but this difference goes to zero if x_3 goes to zero [$\mathbf{e}^2$ is weakly dominated by $\mathbf{e}^1$].)

15.8 *Stationary Points and Nash Equilibria*
(a) $NE(A) = \{(\alpha, \alpha, 1-2\alpha) \mid 0 \leq \alpha \leq 1/2\}$.

(b) By Proposition 15.15 and (a) it follows that $\{(\alpha, \alpha, 1-2\alpha) \mid 0 \leq \alpha \leq 1/2\} \cup \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\} \subseteq ST(A)$, and that possibly other stationary points must be boundary points of Δ^3. By considering the replicator dynamics it follows that there are no additional stationary points. All stationary points except $\mathbf{e}^1$ and $\mathbf{e}^2$ are Lyapunov stable, but no point is asymptotically stable.

15.9 *Lyapunov Stable States in* 2×2 *Games*
Case (1): $\mathbf{e}^1$; case (2): $\mathbf{e}^1$ and $\mathbf{e}^2$; case (3): $\hat{\mathbf{x}}$ (cf. Problem 15.6).

15.10 *Nash Equilibrium and Lyapunov Stability*
$NE(A) = \{\mathbf{e}^1\}$. If we start at a completely mixed strategy close to $\mathbf{e}^1$, then first x_3 increases, and we can make the solution trajectory pass $\mathbf{e}^3$ as closely as desired. This shows that $\mathbf{e}^1$ is not Lyapunov stable.

15.11 *Rock–Paper–Scissors (2)*
(e) Follows from (d). If $a > 0$ then any trajectory converges to the maximum point of $x_1x_2x_3$, i.e., to $(1/3, 1/3, 1/3)$. If $a = 0$ then the trajectories are orbits ($x_1x_2x_3$ constant) around $(1/3, 1/3, 1/3)$. If $a < 0$ then the trajectories move outward, away from $(1/3, 1/3, 1/3)$.

Problems of Chap. 16

16.1 *Imputation Set of an Essential Game*
Note that $I(v)$ is a convex set and $\mathbf{f}^i \in I(v)$ for every $i = 1, \ldots, n$. Thus, $I(v)$ contains the convex hull of $\{\mathbf{f}^i \mid i \in N\}$. Now let $\mathbf{x} \in I(v)$, and write $\mathbf{x} = (v(1), \ldots, v(n)) + (\alpha_1, \ldots, \alpha_n)$, where $\sum_{i \in N} \alpha_i = v(N) - \sum_{i \in N} v(i) =: \alpha$.

16.2 *Convexity of the Domination Core*
First prove the following claim: For each $\mathbf{x} \in I(v)$ and $\emptyset \neq S \subseteq N$ we have

$$\exists \mathbf{z} \in I(v) : \mathbf{z}\,\mathrm{dom}_S \mathbf{x} \Leftrightarrow x(S) < v(S) \quad \text{and} \quad x(S) < v(N) - \sum_{i \notin S} v(i).$$

Use this claim to show that $I(v) \setminus D(S)$ is a convex set. Finally, conclude that $DC(v)$ must be convex.

16.3 *Dominated Sets of Imputations*
(2) In both games, $D(ij) = \{\mathbf{x} \in I(v) \mid x_i + x_j < v(ij)\}$, $i, j \in \{1,2,3\}$, $i \neq j$.

16.7 *A Glove Game*
(b) The core and the domination core are both equal to $\{(0,1,0)\}$, cf. Theorem 16.11.

16.11 *Core and D-core*
Condition (16.1) is not a necessary condition for equality of the core and the D-core. To find a counterexample, first note that if $C(v) \neq \emptyset$ then (16.1) must hold. Therefore, a counterexample has to be some game with empty core and D-core.

16.12 *Strategic Equivalence*
Straightforward using the definitions.

16.13 *Proof of Theorem 16.19*

Write $B = \begin{pmatrix} A \\ -A \end{pmatrix}$. Then

$$\begin{aligned}
\max\{\mathbf{b} \cdot \mathbf{y} \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} &= \max\{\mathbf{b} \cdot \mathbf{y} \mid B\mathbf{y} \leq (\mathbf{c}, -\mathbf{c}), \mathbf{y} \geq \mathbf{0}\} \\
&= \min\{(\mathbf{c}, -\mathbf{c}) \cdot (\mathbf{x}, \mathbf{z}) \mid (\mathbf{x}, \mathbf{z})B \geq \mathbf{b}, (\mathbf{x}, \mathbf{z}) \geq \mathbf{0}\} \\
&= \min\{\mathbf{c} \cdot (\mathbf{x} - \mathbf{z}) \mid (\mathbf{x} - \mathbf{z})A \geq \mathbf{b}, (\mathbf{x}, \mathbf{z}) \geq \mathbf{0}\} \\
&= \min\{\mathbf{c} \cdot \mathbf{x}' \mid \mathbf{x}'A \geq \mathbf{b}\}.
\end{aligned}$$

The second equality follows from Theorem 22.5.

16.14 *Infeasible Programs in Theorem 16.19*
Follow the hint.

16.16 *Minimum of Balanced Games*
Follows by using the definition of balancedness or by Theorem 16.21.

16.17 *Balanced Simple Games*
Let (N, v) be a simple game.

Suppose i is a veto player. Let B be a balanced collection with balanced map λ. Then

$$\sum_{S \in B} \lambda(S)v(S) = \sum_{S \in B: i \in S} \lambda(S)v(S) \leq 1 = v(N),$$

since i is a veto player. Hence, v is balanced.

For the converse, suppose v is balanced, and distinguish two cases:
Case 1: There is an i with $v(\{i\}) = 1$. Show that i is a veto player.
Case 2: $v(\{i\}) = 0$ for every $i \in N$. Show that also in this case v has veto players.

Problems of Chap. 17

17.1 *The Games* 1_T
(c) For $i \in T$: $\Phi_i(1_T) = \frac{(|T|-1)!(n-|T|)!}{n!}$.

17.2 *Unanimity Games*
(a) Suppose $\sum_{T \neq \emptyset} \alpha_T u_T = 0$, where 0 means the zero-game, for some $\alpha_T \in \mathbb{R}$. Show that all α_T are zero by induction, starting with one-person coalitions.

(b) Let $W \in 2^N$, then show

$$\sum_{T \neq \emptyset} c_T u_T(W) = v(W) + \sum_{S:\ S \subsetneq W} v(S) \sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|}.$$

It is sufficient to show that the second term of the last expression is equal to 0, hence that $\sum_{T:S \subseteq T \subseteq W} (-1)^{|T|-|S|} = 0$.

17.3 *Necessity of the Axioms in Theorem 17.4*
EFF, NP and ADD are straightforward. SYM needs more attention. Let i, j be symmetric in v. Note that for $S \subseteq N$ with $i \notin S$ and $j \in S$ we have $v((S \cup i) \setminus j) = v(S)$ by symmetry of i and j, since $v((S \cup i) \setminus j) = v((S \setminus j) \cup i)$ and $v(S) = v((S \setminus j) \cup j)$. Use this to show $\Phi_i(v) = \Phi_j(v)$ by collecting terms in a clever way.

17.4 *Dummy Player Property and Anonymity*
That DUM implies NP and the Shapley value satisfies DUM is straightforward. AN implies SYM: Let i and j be symmetric players, and let the value ψ satisfy AN. Then consider the permutation σ with $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ otherwise.

17.5 *Shapley Value, Core, and Imputation Set*
In the case of two players the core and the imputation set coincide. If the core is not empty then the Shapley value is in the core, cf. Example 17.2. In general, consider any game with $v(1) = 2$, $v(N) = 3$, and $v(S) = 0$ otherwise. Then the Shapley value is not even in the imputation set as soon as $n \geq 3$.

17.6 *Shapley Value as a Projection*
If a is an additive game then $\Phi(a) = (a(1), a(2), \ldots, a(n))$. For a general game v let a^v be the additive game generated by $\Phi(v)$. Then $\Phi(a^v) = (a^v(1), \ldots, a^v(n)) = \Phi(v)$.

17.7 *Shapley Value of Dual Game*
Follow the hint, or give a direct proof by using (17.4).

17.8 *Multilinear Extension*
(3) Let g be another multilinear extension of $\tilde{v}$ to $[0,1]^n$, say $g(\mathbf{x}) = \sum_{T \subseteq N} b_T$ $(\prod_{i \in T} x_i)$. Show $b_T = c_T$ for all T by induction, starting with one-player coalitions.

17.9 *The Beta-integral Formula*
Apply partial integration.

17.10 *Path Independence of* Φ
Use Theorem 17.12(3).

17.11 *An Alternative Characterization of the Shapley Value*
The Shapley value satisfies all these conditions. Conversely, (2)–(4) imply standardness for two-person games, so the result follows from Theorem 17.18.

Problems of Chap. 18

18.1 *Marginal Vectors and Dividends*
(2) For each $i \in N$, $m_i^\pi = \sum_{T \subseteq P_\pi(i) \cup i,\ i \in T} \Delta_v(T)$.

18.2 *Convexity and Marginal Vectors*
Use Theorems 18.3 and 18.6.

18.3 *Strictly Convex Games*
Let π and σ be two different permutations and suppose that $k \geq 1$ is the minimal number such that $\pi(k) \neq \sigma(k)$. Then show that $m^\pi_{\pi(k)}(v) < m^\sigma_{\pi(k)}(v)$. Hence, $m^\pi \neq m^\sigma$.

18.4 *Sharing Profits*
(a) For the landlord: $\Phi_0(v) = \frac{1}{n+1}\left[\sum_{s=0}^n f(s)\right]$.

(c) Extend f to a piecewise linear function on $[0,n]$. Then v is convex if and only if f is convex.

18.5 *Sharing Costs*
(a) For every nonempty coalition S, $v(S) = \sum_{i \in S} c_i - \max\{c_i \mid i \in S\}$. If we regard $c = (c_1, \ldots, c_2)$ as an additive game we can write $v = c - c_{\max}$, where $c_{\max}(S) = \max\{c_i \mid i \in S\}$.

18.6 *Independence of the Axioms in Theorem 18.8*
(1) Consider the value which, for every game v, gives each dummy player his individual worth and distributes the rest, $v(N) - \sum_{i \in D} v(i)$ where D is the subset of dummy players, evenly among the players. This value satisfies all axioms except LIN.

(2) Consider the value which, for every game v, distributes $v(N)$ evenly among all players. This value satisfies all axioms except DUM.

(3) The value which gives each player his individual worth satisfies all axioms except EFF.

(4) Consider any set of weights $\{\alpha_\pi \mid \pi \in \Pi(N)\}$ with $\alpha_\pi \in \mathbb{R}$ for all π and $\sum_{\pi\in\Pi(N)} \alpha_\pi = 1$. The value $\sum_{\pi\in\Pi(N)} \alpha_\pi m^\pi$ satisfies LIN, DUM and EFF, but not MON unless all weights are nonnegative.

18.7 *Null-player in Theorem 18.8*
Check that the dummy axiom in the proof of this theorem is only used for unanimity games. In those games, dummy players are also null-players, so it is sufficient to require NP. Alternatively, one can show that DUM is implied by ADD (and, thus, LIN), EFF and NP.

18.8 *Equation (18.4)*
When considering the sum $\sum_{\pi\in\Pi(N)} p(\pi)$ we may leave out any π with $p(\pi) = 0$. This means in particular that in what follows all expressions $A(\cdot;\cdot)$ are positive. Now

$$\begin{aligned}
\sum_{\pi\in\Pi(N)} p(\pi) &= \sum_{i_1\in N} \sum_{i_2\in N\setminus i_1} \cdots \sum_{i_{n-1}\in N\setminus i_1\cdots i_{n-2}} p_\emptyset^{i_1} A(i_2;\{i_1\}) \times A(i_3;\{i_1,i_2\}) \\
&\quad \times \cdots \times A(i_{n-1};\{i_1,\ldots,i_{n-2}\}) \times A(i_n;\{i_1,\ldots,i_{n-1}\}) \\
&= \sum_{i_1\in N} \sum_{i_2\in N\setminus i_1} \cdots \sum_{i_{n-2}\in N\setminus i_1\cdots i_{n-3}} p_\emptyset^{i_1} A(i_2;\{i_1\}) \times A(i_3;\{i_1,i_2\}) \\
&\quad \times \cdots \times A(i_{n-2};\{i_1,\ldots,i_{n-3}\}) \\
&\quad \times \left(\frac{p^\ell_{N\setminus i_1\cdots i_{n-2}}}{p^\ell_{N\setminus i_1\cdots i_{n-2}} + p^k_{N\setminus i_1\cdots i_{n-2}}} + \frac{p^k_{N\setminus i_1\cdots i_{n-2}}}{p^\ell_{N\setminus i_1\cdots i_{n-2}} + p^k_{N\setminus i_1\cdots i_{n-2}}} \right) \\
&= \sum_{i_1\in N} \sum_{i_2\in N\setminus i_1} \cdots \sum_{i_{n-2}\in N\setminus i_1\cdots i_{n-3}} p_\emptyset^{i_1} A(i_2;\{i_1\}) \times A(i_3;\{i_1,i_2\}) \\
&\quad \times \cdots \times A(i_{n-2};\{i_1,\ldots,i_{n-3}\}) \\
&= \sum_{i_1\in N} \sum_{i_2\in N\setminus i_1} \cdots \sum_{i_{n-3}\in N\setminus i_1\cdots i_{n-4}} p_\emptyset^{i_1} A(i_2;\{i_1\}) \times A(i_3;\{i_1,i_2\}) \\
&\quad \times \cdots \times A(i_{n-3};\{i_1,\ldots,i_{n-4}\}) \\
&= \cdots \\
&= \sum_{i_1\in N} p_\emptyset^{i_1},
\end{aligned}$$

where after the first equality sign, $i_n \in N \setminus \{i_1,\ldots,i_{n-1}\}$, and $A(i_n;\{i_1,\ldots,i_{n-1}\}) = 1$ by definition; after the second equality sign $\ell,k \in N \setminus \{i_1,\ldots,i_{n-2}\}$ with $\ell \neq k$; the third equality sign follows since the sum involving ℓ and k is equal to 1; the remaining equality signs follow from repetition of this argument.

18.9 *Equation (18.6)*
Let $|T| = t$. By using a similar argument as for the proof of (18.4) in the solution to Problem 18.8, we can write

$$\sum_{\pi:T=P_\pi(i)} p(\pi) = \sum_{i_1\in T}\sum_{i_2\in T\setminus i_1}\cdots\sum_{i_t\in T\setminus i_1 i_2\cdots i_{t-1}} p_\emptyset^{i_1} A(i_2;\{i_1\})\cdots A(i_t;T\setminus\{i_t\})$$
$$\times A(i;T).$$

Hence,

$$\begin{aligned}
\sum_{\pi:T=P_\pi(i)} p(\pi) &= \frac{p_T^i}{\sum_{j\in N\setminus T} p_T^j} \\
&\quad\times \sum_{i_t\in T}\frac{p_{T\setminus i_t}^{i_t}}{\sum_{j\in(N\setminus T)\cup i_t} p_{T\setminus i_t}^j}\times\sum_{i_{t-1}\in T\setminus i_t}\frac{p_{T\setminus i_t i_{t-1}}^{i_{t-1}}}{\sum_{j\in(N\setminus T)\cup i_t i_{t-1}} p_{T\setminus i_t i_{t-1}}^j} \\
&\quad\times\cdots\times\sum_{i_1\in T\setminus i_t\cdots i_2} p_\emptyset^{i_1} \\
&= \frac{p_T^i}{\sum_{j\in T} p_{T\setminus j}^j} \\
&\quad\times \sum_{i_t\in T}\frac{p_{T\setminus i_t}^{i_t}}{\sum_{j\in T\setminus i_t} p_{(T\setminus i_t)\setminus j}^j}\times\sum_{i_{t-1}\in T\setminus i_t}\frac{p_{T\setminus i_t i_{t-1}}^{i_{t-1}}}{\sum_{j\in T\setminus i_t i_{t-1}} p_{(T\setminus i_t i_{t-1})\setminus j}^j} \\
&\quad\times\cdots\times\sum_{i_1\in T\setminus i_t\cdots i_2} p_\emptyset^{i_1} \\
&= p_T^i.
\end{aligned}$$

Here, the first equality sign follows from rearranging terms and substituting the expressions for $A(\cdot;\cdot)$; the second equality sign follows from Lemma 18.12; the final equality sign follows from reading the preceding expression from right to left, noting that the remaining sum in each enumerator cancels against the preceding denominator.

18.10 *Characterization of Weighted Shapley Values*
Check that every weighted Shapley value satisfies the Partnership axiom. Conversely, let ψ be a value satisfying the Partnership axiom and the four other axioms. Let $S^1 := \{i\in N \mid \psi_i(u_N)>0\}$ and for every $i\in S^1$ let $\omega_i := \psi_i(u_N)$. Define, recursively, $S^k := \{i\in N\setminus(S^1\cup\cdots\cup S^{k-1}) \mid \psi_i(u_{N\setminus(S^1\cup\cdots\cup S^{k-1})})>0\}$ and for every $i\in S^k$ let $\omega_i := \psi_i(u_{N\setminus(S^1\cup\cdots\cup S^{k-1})})$. This results in a partition $(S^1,\ldots,S^m)$ of N. Now define the weight system w by the partition $(S_1,\ldots,S_m)$ with $S_1 := S^m$, $S_2 := S^{m-1}$, …, $S_m := S^1$, and the weights ω. Then it is sufficient to prove that for each coalition S and each player $i\in S$ we have $\psi_i(u_S)=\Phi_i^w(u_S)$. Let $h:=\max\{j\mid S\cap S_j\neq\emptyset\}$, then with $T=N\setminus(S_{h+1}\cup\cdots\cup S_m)$ we have by the Partnership axiom: $\psi_i(u_S)=\frac{1}{\psi(u_T)(S)}\psi_i(u_T)$. If $i\notin S_h$ then $\psi_i(u_T)=0$, hence $\psi_i(u_S)=0=\Phi_i^w(u_S)$. If $i\in S_h$ then $\psi_i(u_S)=\frac{\omega_i}{\sum_{j\in S\cap S_h}\omega_j}=\Phi_i^w(u_S)$.

18.11 *Core and Weighted Shapley Values in Example 18.2*
First write the game as a sum of unanimity games:

$$v = u_{\{1,2\}} + u_{\{1,3\}} - u_{\{2,3\}} + 2u_N.$$

Then consider all possible ordered partitions of N, there are 13 different ones, and associated weight vectors. This results in a description of all payoff vectors associated with weighted Shapley values, including those in the core of the game.

Problems of Chap. 19

19.1 *Binary Relations*
Not (4): $\succeq$ on $\mathbb{R}$ defined by $x \succeq y \Leftrightarrow x^2 \geq y^2$.
Not (3): $\geq$ on $\mathbb{R}^2$.
Not (2): $\succeq$ on $\mathbb{R}$ defined by: for all x, y, $x \geq y$, let $x \succeq y$ if $x - y \geq 1$, and let $y \succeq x$ if $x - y < 1$.
Not (1): $>$ on $\mathbb{R}$.
The ordering on $\mathbb{R}$, defined by $[x \succeq y] \Leftrightarrow [x = y$ or $0 \leq x, y \leq 1]$ is reflexive and transitive but not complete and not anti-symmetric.

19.2 *Linear Orders*
If $\mathbf{x} \succ \mathbf{y}$ then by definition $\mathbf{x} \succeq \mathbf{y}$ and not $\mathbf{y} \succeq \mathbf{x}$: hence $\mathbf{x} \neq \mathbf{y}$ since otherwise $\mathbf{y} \succeq \mathbf{x}$ by reflexivity.
If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then not $\mathbf{y} \succeq \mathbf{x}$ since otherwise $\mathbf{x} = \mathbf{y}$ by anti-symmetry. Hence $\mathbf{x} \succ \mathbf{y}$.

19.3 *The Lexicographic Order (1)*
Check (1)–(4) in Sect. 19.2 for $\succeq_{\text{lex}}$. Straightforward.

19.4 *The Lexicographic Order (2)*
This is the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid [x_1 = 3,\ x_2 \geq 1]$ or $[x_1 > 3]\}$. This set is not closed.

19.5 *Representability of Lexicographic Order (1)*
Consider Problem 19.4. Since $(\alpha, 0) \succeq_{\text{lex}} (3, 1)$ for all $\alpha > 3$, we have $u(\alpha, 0) \geq u(3, 1)$ for all $\alpha > 3$ and hence, by continuity of u, $\lim_{\alpha \downarrow 3} u(\alpha, 0) \geq u(3, 1)$. Therefore $(3, 0) \succeq_{\text{lex}} (3, 1)$, a contradiction.

19.6 *Representability of Lexicographic Order (2)*
Suppose that u represents $\succeq_{\text{lex}}$ on $\mathbb{R}^2$, that is, $\mathbf{x} \succeq_{\text{lex}} \mathbf{y}$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then for any $t \in \mathbb{R}$ let $q(t)$ be a rational number in the interval $[u(t, 0), u(t, 1)]$. Since $(t, \alpha) \succ_{\text{lex}} (s, \beta)$ and hence $u(t, \alpha) > u(s, \beta)$ for all $t > s$ and all $\alpha, \beta \in [0, 1]$, we have $[u(t, 0), u(t, 1)] \cap [u(s, 0), u(s, 1)] = \emptyset$ for all $t \neq s$. Hence, $q(t) \neq q(s)$ for all $t \neq s$. Therefore, we have found uncountably many different rational numbers, a contradiction.

19.7 *Single-valuedness of the Pre-nucleolus*
Consider the pre-nucleolus on a suitable compact convex subset and apply Theorem 19.4.

19.8 *(Pre-)nucleolus and Core*
Use the fact that core elements have all excesses non-positive.

19.9 *Kohlberg Criterion for the Nucleolus*
First observe that the following modification of Theorem 19.5 holds:

Theorem 19.5′ Let (N,v) be a game and $\mathbf{x} \in I(N,v)$. Then the following two statements are equivalent:

(1) $\mathbf{x} = \nu(N,v)$.
(2) *For every α such that $\mathscr{D}(\alpha,\mathbf{x},v) \neq \emptyset$ and for every side-payment $\mathbf{y}$ with $y(S) \geq 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$ and with $y_i \geq 0$ for all $i \in N$ with $x_i = v(i)$ we have $y(S) = 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$.*

The proof of this theorem is almost identical to the proof of Theorem 19.5. In the second sentence of the proof, note that $\mathbf{z}_\varepsilon \in I(N,v)$ for ε small enough. In the second part of the proof, (2) $\Rightarrow$ (1), note that $y_i = z_i - x_i \geq 0$ whenever $x_i = v(i)$.

For the 'if'-part of the statement in this problem, let $\mathbf{x} \in I(N,v)$, $\mathscr{D}(\alpha,\mathbf{x},v) \neq \emptyset$, and $\mathscr{E}(\alpha,\mathbf{x},v)$ such that $\mathscr{D}(\alpha,\mathbf{x},v) \cup \mathscr{E}(\alpha,\mathbf{x},v)$ is balanced. Consider a side-payment $\mathbf{y}$ with $y(S) \geq 0$ for every $S \in \mathscr{D}(\alpha,\mathbf{x},v)$ and $y_i \geq 0$ for every i with $x_i = v(i)$ (hence in particular for every i with $\{i\} \in \mathscr{E}(\alpha,\mathbf{x},v)$). The argument in the first part of the proof of Theorem 19.6 now applies to $\mathscr{D}(\alpha,\mathbf{x},v) \cup \mathscr{E}(\alpha,\mathbf{x},v)$, and Theorem 19.5′ implies $\mathbf{x} = \nu(N,v)$.

For the 'only-if' part, consider the program (19.4) in the second part of the proof of Theorem 19.6 but add the constraints $-y_i \leq 0$ for every $i \in N$ with $x_i = v(i)$. Theorem 19.5′ implies that the dual of this program is feasible, that is, there are $\lambda(S) \geq 0$, $S \in \mathscr{D}(\alpha,\mathbf{x},v)$, $\lambda(\{i\}) \geq 0$, i such that $x_i = v(i)$, and $\lambda(N) \in \mathbb{R}$ such that

$$-\sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}} - \sum_{S \in \mathscr{D}(\alpha,\mathbf{x},v)} \lambda(S)\mathbf{e}^S + \lambda(N)\mathbf{e}^N = \sum_{S \in \mathscr{D}(\alpha,\mathbf{x},v)} \mathbf{e}^S.$$

Hence $\lambda(N)\mathbf{e}^N = \sum_{S \in \mathscr{D}(\alpha,\mathbf{x},v)} (1 + \lambda(S))\mathbf{e}^S + \sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}}$. Let $\mathscr{E}(\alpha,\mathbf{x},v)$ consist of those one-person coalitions $\{i\}$ with $x_i = v(i)$ and $\lambda(\{i\}) > 0$, then $\mathscr{D}(\alpha,\mathbf{x},v) \cup \mathscr{E}(\alpha,\mathbf{x},v)$ is balanced.

19.10 *Proof of Theorem 19.6*
To formulate the dual program, use for instance the formulation in Theorem 16.19. For instance, the primal (19.4) can be converted to the minimization problem in Theorem 16.19; then the dual corresponds to the maximization problem in Theorem 16.19. Feasibility of the dual follows from Problem 16.14.

19.11 *Nucleolus of a Three-person Game (1)*
The nucleolus is $(5,4,3)$.

19.12 *Nucleolus of a Three-person Game (2)*
The (pre-)nucleolus is $(5,3,2)$.

19.14 *Individual Rationality Restrictions for the Nucleolus*
The nucleolus is $(1,0,0)$. The pre-nucleolus is $(5/3,-1/3,-1/3)$.

19.15 *Example 19.8*
The set $\mathscr{B}_1 = \{123,124,34\}$ is balanced with weights all equal to $1/2$. The set $\mathscr{B}_1 \cup \mathscr{B}_2 = \{123,124,34,134,234\}$ is balanced with weights, respectively, equal to $5/12, 5/12, 3/12, 2/12, 2/12$.

19.16 *(Pre-)nucleolus of a Symmetric Game*
(a) $\nu(v) = \nu^*(v) = (v(N)/n)\mathbf{e}^N$.

(b) The maximal excess is reached by all coalitions S for which $f(|S|) - (|S|/n)f(n)$ is maximal.

19.17 *COV and AN of the Pre-nucleolus*
Covariance of the pre-nucleolus follows since applying a transformation as in the definition of this property changes all excesses (only) by the same positive (multiplicative) factor.

Anonymity of the pre-nucleolus follows since a permutation of the players does not change the ordered vectors $\theta(\mathbf{x})$, but only permutes the coalitions to which the excesses correspond.

19.18 *Apex Game*
The (pre-)nucleolus is $(3/7,1/7,1/7,1/7,1/7)$. This can easily be verified using the Kohlberg criterion.

19.19 *Landlord Game*
(a) By anonymity, each worker is assigned $\frac{1}{2}[f(n) - f(n-1)]$. By computing the excesses, it follows that among all coalitions containing the landlord, with this payoff vector the maximal excesses are reached by the coalitions containing $n-1$ workers, and further also by the coalitions consisting of a single worker and not the landlord. By the Kohlberg criterion this immediately implies that the given vector is the (pre-)nucleolus. For the Shapley value, see Problem 18.4.

(b) Compute the excesses for the payoff vector $\frac{f(n)}{n+1}\mathbf{e}^{\{0,1,\dots,n\}}$, and apply the Kohlberg criterion.

19.20 *Game in Sect. 19.1*
The first linear program is: minimize α subject to the constraints $x_i + \alpha \geq 4$ for $i = 1,2,3$, $x_1 + x_2 + \alpha \geq 8$, $x_1 + x_3 + \alpha \geq 12$, $x_2 + x_3 + \alpha \geq 16$, $x_1 + x_2 + x_3 = 24$. The program has optimal value $\alpha = -2$, reached for $x_1 = 6$ and $x_2, x_3 \geq 6$.

In the second program x_1 has been eliminated. This program reduces to: minimize α subject to $x_2 + \alpha \geq 4$, $x_2 \leq 12 + \alpha$, $x_2 + x_3 = 18$. This has optimal value $\alpha = -4$, reached for $x_2 = 8$, $x_3 = 10$.

19.21 *The Prekernel*
For $i, j \in N$ denote by $\mathscr{T}_{ij}$ those coalitions that contain player i and not player j. For a payoff vector $\mathbf{x}$ denote by $s_{ij}(\mathbf{x}, v)$ the maximum of $e(S,\mathbf{x},v)$ over all $S \in \mathscr{T}_{ij}$.

Let now $\mathbf{x}$ be the pre-nucleolus and suppose, contrary to what has to be proved, that there are two distinct players k,ℓ such that $s_{k\ell}(\mathbf{x},v) > s_{\ell k}(\mathbf{x},v)$. Let $\delta = (s_{k\ell}(\mathbf{x},v) - s_{\ell k}(\mathbf{x},v))/2$ and define $\mathbf{y}$ by $y_k = x_k + \delta$, $y_\ell = x_\ell - \delta$, and $y_i = x_i$ for all $i \neq k,\ell$. Denote $\mathscr{S} = \{S \in 2^N \setminus \mathscr{T}_{k\ell} \mid e(S,\mathbf{x},v) \geq s_{kl}(\mathbf{x},v)\}$ and $s = |S|$. Then $\theta_{s+1}(\mathbf{x}) = s_{k\ell}(\mathbf{x},v)$. For $S \in 2^N \setminus (\mathscr{T}_{k\ell} \cup \mathscr{T}_{\ell k})$, we have $e(S,\mathbf{x},v) = e(S,\mathbf{y},v)$. For $S \in \mathscr{T}_{k\ell}$ we have $e(S,\mathbf{y},v) = e(S,\mathbf{x},v) - \delta$. Finally, for $S \in \mathscr{T}_{\ell k}$ we have

$$e(S,\mathbf{y},v) = e(S,\mathbf{x},v) + \delta \leq s_{\ell k}(\mathbf{x},v) + \delta = s_{k\ell}(\mathbf{x},v) - \delta.$$

Thus, $\theta_t(\mathbf{y}) = \theta_t(\mathbf{x})$ for all $t \leq s$ and $\theta_{s+1}(\mathbf{y}) < s_{k\ell}(\mathbf{x},v) = \theta_{s+1}(\mathbf{x})$. Hence $\theta(\mathbf{x}) \succ_{\text{lex}} \theta(\mathbf{y})$, a contradiction.

Problems of Chap. 20

20.2 *Example 20.3*
Argue that $a_{12} = a_{13} = 3$ if v were an assignment game. Use this to derive a contradiction.

20.3 *Subgames of Permutation Games*
That a subgame of a permutation game is again a permutation game follows immediately from the definition: in (20.3) the worth $v(S)$ depends only on the numbers k_{ij} for $i,j \in S$. By a similar argument (consider (20.1)) this also holds for assignment games.

20.4 *A Flow Game*
(3) $(1,1,0,0)$, corresponding to the minimum cut through e_1 and e_2; $\{(0,0,1+\alpha,1-\alpha) \mid 0 \leq \alpha \leq 1\}$, corresponding to the minimum cut through e_3 and e_4.

20.5 *Every Nonnegative Balanced Game is a Flow Game*
Let v be a nonnegative balanced game, and write (following the hint to the problem) $v = \sum_{r=1}^{k} \alpha_r w_r$, where $\alpha_r > 0$ and w_r a balanced simple game for each $r = 1,\ldots,k$. Consider the controlled capacitated network with two vertices, the source and the sink, and k edges connecting them, where each edge e_r has capacity α_r and is controlled by w_r. Then show that the associated flow game is v.

20.6 *On Theorem 20.6 (1)*
(1) This follows straightforwardly from the proof of Theorem 20.6.

(2) E.g., each player receiving $5\frac{1}{4}$ is a core element.

20.7 *On Theorem 20.6 (2)*
In any core element, player should 1 receive at least 1 and player 2 also, but $v(N) = 1$. Hence the game has an empty core.

20.8 *Totally Balanced Flow Games*
This follows immediately from Theorem 20.6, since every dictatorial game is balanced, i.e., has veto players.

20.9 *If-part of Theorem 20.9*

We show that the Banzhaf value satisfies 2-EFF (the other properties are obvious). With notations as in the formulation of 2-EFF, we have

$$\begin{aligned}\psi_p(v_p) &= \sum_{S\subseteq(N\backslash p)\cup\{p\}:\ p\notin S} \frac{1}{2^{|N|-2}}\left[v_p(S\cup\{p\})-v_p(S)\right]\\ &= \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{|N|-2}}\left[v(S\cup\{ij\})-v(S)\right]\\ &= \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{|N|-1}}\left[2v(S\cup\{ij\})-2v(S)\right].\end{aligned}$$

The term in brackets can be written as

$$\begin{aligned}&[v(S\cup\{i,j\})-v(S\cup\{i\})+v(S\cup\{j\})-v(S)]\\ +\ &[v(S\cup\{i,j\})-v(S\cup\{j\})+v(S\cup\{i\})-v(S)],\end{aligned}$$

hence $\psi_p(v_p)=\psi_j(v)+\psi_i(v)$.

Show that DUM cannot be weakened to NP by finding a different value satisfying 2-EFF, SYM, NP, and SMON.

Problems of Chap. 21

21.1 *Anonymity and Symmetry*
An example of a symmetric but not anonymous solution is as follows. To symmetric problems, assign the point in $W(S)$ with equal coordinates; otherwise, assign the point of S that is lexicographically (first player 1, then player 2) maximal.

21.3 *The Nash Solution is Well-defined*
The function $\mathbf{x}\mapsto(x_1-d_1)(x_2-d_2)$ is continuous on the compact set $\{\mathbf{x}\in S\mid \mathbf{x}\geq\mathbf{d}\}$ and hence attains a maximum on this set. We have to show that this maximum is attained at a unique point. In general, consider two points $\mathbf{z},\mathbf{z}'\in\{\mathbf{x}\in S\mid\mathbf{x}\geq\mathbf{d}\}$ with $(z_1-d_1)(z_2-d_2)=(z_1'-d_1)(z_2'-d_2)=\alpha$. Then one can show that at the point $\mathbf{w}=\frac{1}{2}(\mathbf{z}+\mathbf{z}')\in S$ one has $(w_1-d_1)(w_2-d_2)>\alpha$. This implies that the maximum is attained at a unique point.

21.4 *(1) $\Rightarrow$ (2) in Theorem 21.1*
WPO and IIA are straightforward by definition, and SC follows from an easy computation. For SYM, note that if $N(S,\mathbf{d})=\mathbf{z}$ for a symmetric problem $(S,\mathbf{d})$, then also $(z_2,z_1)=N(S,\mathbf{d})$ by definition of the Nash bargaining solution. Hence, $z_1=z_2$ by uniqueness.

21.5 *Geometric Characterization of the Nash Bargaining Solution*
Let $(S,\mathbf{d})\in B$ and $N(S,\mathbf{d})=\mathbf{z}$. The slope of the tangent line ℓ to the graph of the function $x_1\mapsto(z_1-d_1)(z_2-d_2)/(x_1-d_1)+d_2$ (which describes the level set of

$\mathbf{x} \mapsto (x_1 - d_1)(x_2 - d_2)$ through $\mathbf{z}$) at $\mathbf{z}$ is equal to $-(z_2 - d_2)(z_1 - d_1)$, i.e., the negative of the slope of the straight line through d and z. Clearly, ℓ supports S at $\mathbf{z}$: this can be seen by invoking a separating hyperplane theorem, but also as follows. Suppose there were some point $\mathbf{z}'$ of S on the other side of ℓ than $\mathbf{d}$. Then there is a point $\mathbf{w}$ on the line segment connecting $\mathbf{z}'$ and $\mathbf{z}$ (hence, $\mathbf{w} \in S$) with $(w_1 - d_1)(w_2 - d_2) > (z_1 - d_1)(z_2 - d_2)$, contradicting $\mathbf{z} = N(S, \mathbf{d})$. The existence of such a point $\mathbf{w}$ follows since otherwise the straight line through $\mathbf{z}'$ and $\mathbf{z}$ would also be a tangent line to the level curve of the Nash product at $\mathbf{z}$.

For the converse, suppose that at a point $\mathbf{z}$ there is a supporting line of S with slope $-(z_2 - d_2)(z_1 - d_1)$. Clearly, this line is tangent to the graph of the function $x_1 \mapsto (z_1 - d_1)(z_2 - d_2)/(x_1 - d_1) + d_2$ at $\mathbf{z}$. It follows that $\mathbf{z} = N(S, \mathbf{d})$.

21.6 *Strong Individual Rationality*
The implication (1) $\Rightarrow$ (2) is straightforward. For (2) $\Rightarrow$ (1), if F is also weakly Pareto optimal, then $F = N$ by Theorem 21.1. So it is sufficient to show that, if F is not weakly Pareto optimal then $F = D$. Suppose that F is not weakly Pareto optimal. Then there is an $(S, \mathbf{d}) \in B$ with $F(S, \mathbf{d}) \notin W(S)$. By IR, $F(S, \mathbf{d}) \geq \mathbf{d}$. Suppose $F(S, \mathbf{d}) \neq \mathbf{d}$. By SC, we may assume w.l.o.g. $\mathbf{d} = (0,0)$. Let $\alpha > 0$ be such that $F(S,(0,0)) \in W((\alpha,\alpha)S)$. Since $F(S,(0,0)) \notin W(S)$, $\alpha < 1$. So $(\alpha,\alpha)S \subseteq S$. By IIA, $F((\alpha,\alpha)S,(0,0)) = F(S,(0,0))$, so by SC, $F((\alpha,\alpha)S,(0,0)) = (\alpha,\alpha)F(S,(0,0)) = F(S,(0,0))$, contradicting $\alpha < 1$. So $F(S,(0,0)) = (0,0)$. Suppose $F(T,(0,0)) \neq (0,0)$ for some $(T,(0,0)) \in B$. By SC we may assume $(0,0) \neq F(T,(0,0)) \in S$. By IIA applied twice, $(0,0) = F(S \cap T,(0,0)) = F(T,(0,0)) \neq (0,0)$, a contradiction. Hence, $F = D$.

21.7 *(1) $\Rightarrow$ (2) in Theorem 21.2*
Straightforward. Note in particular that in a symmetric game the utopia point is also symmetric, and that the utopia point is 'scale covariant'.

21.8 *Restricted Monotonicity*
(1) Follows from applying IM twice.

(2) For $(S, \mathbf{d})$ with $\mathbf{d} = (0,0)$ and $u(S, \mathbf{d}) = (1,1)$, let $F(S, \mathbf{d})$ be the lexicographically (first player 1, then player 2) maximal point of $S \cap \mathbb{R}^2_+$. Otherwise, let F be equal to R. This F satisfies RM but not IM.

21.9 *Global Individual Monotonicity*
It is straightforward to verify that G satisfies WPO, SYM, SC, and GIM. For the converse, suppose that F satisfies these four axioms, let $(S, \mathbf{d}) \in B$ and $\mathbf{z} := G(S, \mathbf{d})$. By SC, w.l.o.g. $\mathbf{d} = (0,0)$ and $g(S) = (1,1)$. Let $\alpha \leq 0$ such that $S \subseteq \widetilde{S}$ where $\widetilde{S} := \{\mathbf{x} \in \mathbb{R}^2 \mid (\alpha,\alpha) \leq \mathbf{x} \leq \mathbf{y}$ for some $\mathbf{y} \in S\}$. In order to prove $F(S,(0,0)) = G(S,(0,0))$ it is sufficient to prove that $F(\widetilde{S},(0,0)) = G(\widetilde{S},(0,0))$ (in view of GIM and WPO). Let $T = \text{conv}\{\mathbf{z}, (\alpha, g_2(\widetilde{S})), (g_1(\widetilde{S}), \alpha)\} = \text{conv}\{\mathbf{z}, (\alpha, 1), (1, \alpha)\}$. By SYM and WPO, $F(T,(0,0)) = \mathbf{z}$. By GIM, $F(\widetilde{S},(0,0)) \geq F(T,(0,0)) = \mathbf{z} = G(S,(0,0)) = G(\widetilde{S},(0,0))$, so by WPO: $F(\widetilde{S},(0,0)) = G(\widetilde{S},(0,0))$. (Make pictures. Note that this proof is analogous to the proof of Theorem 21.2.)

21.10 *Monotonicity and (weak) Pareto Optimality*
(1) Consider problems of the kind $(\text{conv}\{\mathbf{d},\mathbf{a}\},\mathbf{d})$ for some $\mathbf{a} > \mathbf{d}$.

(2) The egalitarian solution E satisfies MON and WPO on B_0.

21.11 *The Egalitarian Solution (1)*
Straightforward.

21.12 *The Egalitarian Solution (2)*
Let $\mathbf{z} := E(S,\mathbf{d}) + E(T,\mathbf{e})$. Then it is straightforward to derive that $z_1 - (d_1 + e_1) = z_2 - (d_2 + e_2)$. Since $E(S+T, d+e)$ is the maximal point $\mathbf{x}$ such that $x_1 - (d_1 + e_1) = x_2 - (d_2 + e_2)$, it follows that $E(S+T, d+E) \geq \mathbf{z}$.

21.13 *Independence of Axioms*
Theorem 21.1:

WPO, SYM, SC: $F = R$; WPO, SYM, IIA: $F = L$, where $L(S,\mathbf{d})$ is the point of $P(S)$ nearest to the point $\mathbf{z} \geq \mathbf{d}$ with $z_1 - d_1 = z_2 - d_2$ measured along the boundary of S; WPO, SC, IIA: $F = D^1$, where $D^1(S,\mathbf{d})$ is the point of $\{\mathbf{x} \in P(S) \mid \mathbf{x} \geq \mathbf{d}\}$ with maximal first coordinate; SYM, SC, IIA: $F = D$ (disagreement solution).

Theorem 21.2:

WPO, SYM, SC: $F = N$; WPO, SYM, IM: $F = L$; WPO, SC, IM: if $\mathbf{d} = (0,0)$ and $u(S,\mathbf{d}) = (1,1)$, let F assign the point of intersection of $W(S)$ and the line segment connecting $(1/4, 3/4)$ and $(1,1)$ and, otherwise, let F be determined by SC; SYM, SC, IM: $F = D$.

Theorem 21.2:

WPO, MON, SYM: $F(S,\mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$ if $\mathbf{d} = (1,0)$, $F(S,\mathbf{d}) = E(S,\mathbf{d})$ otherwise; WPO, MON, TC: $F(S,\mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$; WPO, SYM, TC: $F = N$; MON, SYM, TC: $F = D$.

21.14 *Nash and Rubinstein*
(b) The Nash bargaining solution outcome is $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, hence $(\frac{1}{3}\sqrt{3}, 1 - \frac{1}{3}\sqrt{3})$ is the resulting distribution of the good.

(c) The Rubinstein bargaining outcome is $\left(\sqrt{\frac{1-\delta}{1-\delta^3}}, \frac{\delta - \delta^3}{1-\delta^3}\right)$.

(d) The outcome in (c) converges to the outcome in (b) if δ converges to 1.

Problems of Chap. 22

22.1 *Convex Sets*
The only-if part is obvious. For the if-part, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in Z the condition implies that $\frac{k}{2^m}\mathbf{x} + \frac{2^m - k}{2^m}\mathbf{y} \in Z$ for every $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^m\}$. By closedness of Z, this implies that $\text{conv}\{\mathbf{x},\mathbf{y}\} \subseteq Z$, hence Z is convex.

22.2 *Proof of Lemma 22.3*
Suppose that both systems have a solution, say $(\mathbf{y},\mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y},\mathbf{z}) \neq \mathbf{0}$, $A\mathbf{y}+\mathbf{z} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. Then $\mathbf{x}A\mathbf{y}+\mathbf{x}\cdot\mathbf{z} = \mathbf{x}(A\mathbf{y}+\mathbf{z}) = 0$, hence $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ since $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. This contradicts $(\mathbf{y},\mathbf{z}) \neq \mathbf{0}$.

22.3 *Proof of Lemma 22.4*
Suppose that both systems have a solution, say $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A = \mathbf{b}$, $A\mathbf{y} \geq \mathbf{0}$, $\mathbf{b}\cdot\mathbf{y} < 0$. Then $\mathbf{x}A\mathbf{y} < 0$, contradicting $\mathbf{x} > \mathbf{0}$ and $A\mathbf{y} \geq \mathbf{0}$.

22.4 *Proof of Lemma 22.6*
(a) If $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}A \leq \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}\cdot\mathbf{y} < 0$ then $\mathbf{x}A\mathbf{y} \leq \mathbf{b}\cdot\mathbf{y} < 0$, so $A\mathbf{y} \not\geq \mathbf{0}$. This shows that at most one of the two systems has a solution.
(b) Suppose the system in (1) has no solution. Then also the system $\mathbf{x}A + \mathbf{z}I = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$ has no solution. Hence, by Farkas' Lemma the system $\begin{pmatrix} A \\ I \end{pmatrix}\mathbf{y} \geq \mathbf{0}$, $\mathbf{b}\cdot\mathbf{y} < 0$ has a solution. Therefore, the system in (2) has a solution.

22.5 *Extreme Points*
The implication (2) $\Rightarrow$ (1) follows by definition of an extreme point.

For the implication (1) $\Rightarrow$ (3), let $x,y \in C\setminus\{e\}$ and $0 < \lambda < 1$. Let $z = \lambda x + (1-\lambda)y$. If $z \neq e$ then $z \in C\setminus\{e\}$ by convexity of C. Suppose now that $z = e$. W.l.o.g. let $\lambda \geq 1/2$. Then $e = \lambda x + (1-\lambda)y = (1/2)x + (1/2)[\mu x + (1-\mu)y]$ for $\mu = 2\lambda - 1$. Since $\mu x + (1-\mu)y \in C$, this implies that e is not an extreme point of C. This proves the implication (1) $\Rightarrow$ (3).

For the implication (3) $\Rightarrow$ (2), let $x,y \in C$, $x \neq y$, and $0 < \alpha < 1$. If $x = e$ or $y = e$ then clearly $\alpha x + (1-\alpha)y \neq e$. If $x \neq e$ and $y \neq e$ then $\alpha x + (1-\alpha)y \in C\setminus\{e\}$ by convexity of $C\setminus\{e\}$, hence $\alpha x + (1-\alpha)y \neq e$ as well.

22.6 *Affine Subspaces*
Let $A = a + L$ be an affine subspace, $x,y \in A$, and $\lambda \in \mathbb{R}$. Write $x = a + \bar{x}$ and $y = a + \bar{y}$ for $\bar{x},\bar{y} \in L$, then $\lambda x + (1-\lambda)y = a + \lambda\bar{x} + (1-\lambda)\bar{y} \in A$ since $\lambda\bar{x} + (1-\lambda)\bar{y} \in L$ (L is a linear subspace).

Conversely, suppose that A contains the straight line through any two of its elements. Let a be an arbitrary element of A and let $L := \{x - a \mid x \in A\}$. Then it follows straightforwardly that L is a linear subspace of V, and thus $A = a + L$ is an affine subspace.

22.7 *The Set of Sup-points of a Linear Function on a Convex Set*
In general, linearity of f implies that, if $f(\mathbf{x}) = f(\mathbf{y})$, then $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = f(\mathbf{x}) = f(\mathbf{y})$ for any two points of C and $0 < \lambda < 1$. It follows, in particular, that the set D is convex.

Let $\mathbf{e} \in \text{ext}(D)$ and suppose $\mathbf{e} = (1/2)\mathbf{x} + (1/2)\mathbf{y}$ for some $\mathbf{x},\mathbf{y} \in C$. Then by linearity of f, $f(\mathbf{e}) = (1/2)f(\mathbf{x}) + (1/2)f(\mathbf{y})$, hence $\mathbf{x},\mathbf{y} \in D$ since $\mathbf{e} \in D$. So $\mathbf{e} = \mathbf{x} = \mathbf{y}$ since $\mathbf{e}$ is an extreme point of D. Thus, $\mathbf{e}$ is also an extreme point of C.

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