

MATH5210 ANALYSIS
Assignment 6
Limits of Functions pt. 2
Philip Nelson

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Claim: Let $f, g, h : (a, b) \rightarrow \mathbf{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ for all $x_0 \in (a, b)$.
If $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} h(x) = M$ and $L = M$, then $\lim_{x \rightarrow x_0} g(x)$ exists and equals L .

Proof: Let $\epsilon > 0$ be given. I will show that $\exists \delta > 0$ s.t. $|g(x) - L| < \epsilon$ for $0 < |x - x_0| < \delta$.
We know $\exists \delta_1$ s.t.

$$|f(x) - L| < \epsilon$$

$$\Rightarrow -\epsilon < f(x) - L < \epsilon$$

$$\Rightarrow L - \epsilon < f(x) < L + \epsilon, \text{ for } 0 < |x - x_0| < \delta_1$$

We also know $\exists \delta_2$ s.t.

$$|h(x) - M| < \epsilon$$

$$\Rightarrow -\epsilon < h(x) - M < \epsilon$$

$$\Rightarrow M - \epsilon < h(x) < M + \epsilon, \text{ for } 0 < |x - x_0| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then for any x s.t. $0 < |x - x_0| < \delta$

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < M + \epsilon$$

and since $L = M$

$$\Rightarrow L - \epsilon < g(x) < L + \epsilon$$

$$\Rightarrow |g(x) - L| < \epsilon, \text{ for } 0 < |x - x_0| < \delta$$

Therefore, if $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} h(x) = M$ and $L = M$, then $\lim_{x \rightarrow x_0} h(x)$ exists and equals L . ■

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Claim: Let $f, g : (a, b) \rightarrow \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$, then $\lim_{x \rightarrow x_0} f(x)g(x) = LM$.

Proof: I will use sequential characterization of limits to prove the claim. For all sequences a_n s.t.

$$\lim_{n \rightarrow \infty} a_n = x_0$$

then

$$\lim_{n \rightarrow \infty} f(a_n) = L$$

and

$$\lim_{n \rightarrow \infty} g(a_n) = M$$

Let $b_n = f(a_n)$ and $c_n = g(a_n)$ then

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} f(a_n) = L$$

and

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} g(a_n) = M$$

Observe $\lim_{n \rightarrow \infty} b_n c_n$. We know $\lim_{n \rightarrow \infty} b_n$ exists and $\lim_{n \rightarrow \infty} c_n$ exists, so by the product property of sequences

$$\lim_{n \rightarrow \infty} b_n c_n = \lim_{n \rightarrow \infty} b_n \cdot \lim_{n \rightarrow \infty} c_n = LM$$

Therefore, since $\lim_{n \rightarrow \infty} b_n c_n$ is the sequential characterization of $\lim_{x \rightarrow x_0} f(x)g(x)$, if $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$, then $\lim_{x \rightarrow x_0} f(x)g(x) = LM$. ■

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Claim: If $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{y \rightarrow L} g(y) = M$, then $\lim_{x \rightarrow x_0} g \circ f(x) = M$

Proof:

Remark: For all sequences a_n s.t. $\lim_{n \rightarrow \infty} a_n = x_0$ then

$$\lim_{n \rightarrow \infty} f(a_n) = L$$

Let $b_n = f(a_n)$ so

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} f(a_n) = L$$

Remark: For all sequences c_n s.t. $\lim_{n \rightarrow \infty} c_n = L$ then

$$\lim_{n \rightarrow \infty} g(c_n) = M$$

Note that b_n is a sequence s.t. $\lim_{n \rightarrow \infty} b_n = L$, so b_n satisfies the sequential characterization of $g(x)$ therefore

$$\lim_{n \rightarrow \infty} g(b_n) = M$$

Since we defined $b_n = f(a_n)$ then

$$\lim_{n \rightarrow \infty} g(b_n) = \lim_{n \rightarrow \infty} g(f(a_n)) = M$$

So, since $\lim_{n \rightarrow \infty} g(f(a_n)) = M$ then

$$\lim_{n \rightarrow \infty} g(f(a_n)) = \lim_{n \rightarrow \infty} g \circ f(x) = M$$

Therefore if $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{y \rightarrow L} g(y) = M$, then $\lim_{x \rightarrow x_0} g \circ f(x) = M$. ■

9 $\epsilon - \delta$

Claim: $\lim_{x \rightarrow 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$

Proof: For all $\epsilon > 0$ there exists $\delta = \min\{1, \epsilon\}$ s.t. for all x , where $|x - 1| < \delta$, $|f(x) - L| < \epsilon$. Then

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 + 4}{x^2 - 4} + \frac{5}{3} \right| \\ &= \left| \frac{8x^2 - 8}{3(x^2 - 4)} \right| \end{aligned}$$

$$\begin{aligned}
&= \frac{8}{3} \left| \frac{(x+1)(x-1)}{x^2+4} \right| \\
&< \frac{8}{3} \delta \left| \frac{(x+1)}{(x^2+4)} \right|
\end{aligned}$$

From our choice of delta, we know that $|x-1| < \delta = \min\{1, \epsilon\}$ so

$$|x-1| < \delta < \epsilon < 1$$

$$\Rightarrow |x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1$$

$$\Rightarrow 0 < x < 2$$

This implies

$$x^2 + 4 < 8 \text{ and } x+1 < 3$$

so

$$\begin{aligned}
&< \frac{8}{3} \delta \left| \frac{(x+1)}{x^2+4} \right| < \frac{8}{3} \delta \cdot \frac{3}{8} \\
&\Rightarrow \delta < \epsilon
\end{aligned}$$

Therefore $\lim_{x \rightarrow 1} \frac{x^2+4}{x^2-4} = -\frac{5}{3}$.

■

10 $\epsilon - \delta$

Claim: $\lim_{x \rightarrow 0} \sqrt{x^2+1} = 1$

Proof: For all $\epsilon > 0$ there exists $\delta = \epsilon$ s.t. for all x , where $|x - x_0| < \delta$, $|f(x) - L| < \epsilon$. Then

$$\begin{aligned}
|f(x) - L| &= \left| \sqrt{x^2+1} - 1 \right| \\
&= \left| \sqrt{x^2+1} - 1 \right| \\
&= \frac{x^2+1-1}{\sqrt{x^2+1}+1} \\
&= \frac{x^2}{\sqrt{x^2+1}+1}
\end{aligned}$$

$$\begin{aligned}
&< \frac{x^2}{\sqrt{x^2}} \\
&= \frac{x^2}{x} \\
&= x = \delta = \epsilon
\end{aligned}$$

Therefore $\lim_{x \rightarrow 0} \sqrt{x^2 + 1} = 1$. ■

11 $\epsilon - \delta$

Claim: $\lim_{x \rightarrow -1} x^3 + x + 2 = 0$

Proof: For all $\epsilon > 0$ there exists $\delta = \min\{1, \frac{\epsilon}{8}\}$ s.t. for all x , where $|x - x_0| < \delta$, $|f(x) - L| < \epsilon$. Then

$$\begin{aligned}
|f(x) - L| &= |x^3 + x + 2 - 0| \\
&= |(x + 1)(x^2 - x + 2)| \\
&= |\delta(x^2 - x + 2)| \\
&= \delta(|x|^2 - |x| + 2)
\end{aligned}$$

From our choice of delta, we know that $|x + 1| < \delta = \min\{1, \frac{\epsilon}{8}\}$ so

$$\begin{aligned}
|x + 1| &< \delta < \frac{\epsilon}{8} < 1 \\
\Rightarrow -1 &< x + 1 < 1 \\
\Rightarrow -2 &< x < 0 \\
\Rightarrow 0 &< |x| < 2
\end{aligned}$$

This implies by the triangle inequality that

$$\begin{aligned}
&= \delta(|x|^2 + |x| + 2) < \delta(4 + 2 + 2) \\
&= 8\delta = \epsilon
\end{aligned}$$

Therefore $\lim_{x \rightarrow -1} x^3 + x + 2 = 0$.

12 - 7 limit properties

Claim: $\lim_{x \rightarrow 2} x^2 + x - 5 = 1$

Proof: As discussed in class, the function $f(x) = x$ and $f(x) = x^2$ are continuous functions. This means that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ by the definition of continuous functions. This also means that we know

$$\lim_{x \rightarrow 2} x = 2 \text{ and } \lim_{x \rightarrow 2} x^2 = 2^2 = 4$$

Because we know these limits exist we can use the sum property of limits to say

$$\lim_{x \rightarrow 2} x^2 + x - 5 = \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 5$$

and we know the values of the limits because they are continuous functions so

$$\lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 5 = 4 + 2 - 5 = 1$$

Therefore $\lim_{x \rightarrow 2} x^2 + x - 5 = 1$. ■

12 - 8 limit properties

Claim: $\lim_{x \rightarrow 1} \frac{x}{x^2 + 4} = \frac{1}{5}$

Proof: As discussed in class, the function $f(x) = x$ and $f(x) = x^2$ are continuous functions. This means that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ by the definition of continuous functions. This also means that we know

$$\lim_{x \rightarrow 1} x = 1 \text{ and } \lim_{x \rightarrow 1} x^2 = 1^2 = 1$$

Because we know these limits exist we can use the sum property of limits to say

$$\lim_{x \rightarrow 1} x^2 + 4 = \lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} 4$$

and we know the values of the limits because they are continuous functions so

$$\lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} 4 = 1 + 4 = 5$$

Now since we know that the limit of the numerator and the denominator exist we can say

$$\lim_{x \rightarrow 1} \frac{x}{x^2 + 4} = \frac{\lim_{x \rightarrow 1} x}{\lim_{x \rightarrow 1} x^2 + 4}$$

$$\begin{aligned}
&= \frac{\lim_{x \rightarrow 1} x}{\lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} 4} \\
&= \frac{1}{1 + 4} = \frac{1}{5}
\end{aligned}$$

Therefore $\lim_{x \rightarrow 1} \frac{x}{x^2 + 4} = \frac{1}{5}$. ■

13 - 9 sequential characterization

Claim: $\lim_{x \rightarrow 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$

Proof: Let $f(x) = \frac{x^2 + 4}{x^2 - 4}$. Let a_n be any sequence s.t. $\lim_{n \rightarrow \infty} a_n = 1$ and let $b_n = f(a_n)$. Then observe

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} \frac{a_n^2 + 4}{a_n^2 - 4}$$

We know that $\lim_{n \rightarrow \infty} a_n = 1$ so by the product property of sequences

$$\lim_{n \rightarrow \infty} a_n^2 = \lim_{n \rightarrow \infty} a_n \cdot a_n = 1 \cdot 1 = 1$$

Now we know $\lim_{n \rightarrow \infty} a_n^2$ exists so we can use the sum property of limits to say

$$\lim_{n \rightarrow \infty} a_n^2 - 4 = \lim_{n \rightarrow \infty} a_n^2 - \lim_{n \rightarrow \infty} 4$$

We know the values of these limits so

$$\lim_{n \rightarrow \infty} a_n^2 - \lim_{n \rightarrow \infty} 4 = 1 - 4 = -3$$

By similar logic we can use the sum property of limits to say

$$\lim_{n \rightarrow \infty} a_n^2 + 4 = \lim_{n \rightarrow \infty} a_n^2 + \lim_{n \rightarrow \infty} 4$$

We know the values of these limits so

$$\lim_{n \rightarrow \infty} a_n^2 + \lim_{n \rightarrow \infty} 4 = 1 + 4 = 5$$

Now we know that the limit of the numerator and denominator exist so we can use the quotient property of limits to say

$$\lim_{n \rightarrow \infty} \frac{a_n^2 + 4}{a_n^2 - 4} = \frac{\lim_{n \rightarrow \infty} a_n^2 + 4}{\lim_{n \rightarrow \infty} a_n^2 - 4}$$

We know the values of these limits so

$$\frac{\lim_{n \rightarrow \infty} a_n^2 + 4}{\lim_{n \rightarrow \infty} a_n^2 - 4} = \frac{5}{-3} = -\frac{5}{3}$$

Since b_n is the sequential characterization of $f(x)$ and $\lim_{n \rightarrow \infty} b_n = -\frac{5}{3}$ then by sequential characterization of limits of functions $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$. ■

13 - 10 sequential characterization

Claim: $\lim_{x \rightarrow 0} \sqrt{x^2 + 1} = 1$

Proof: Let $f(x) = \sqrt{x^2 + 1}$. Let a_n be any sequence s.t. $\lim_{n \rightarrow \infty} a_n = 0$ and let $b_n = f(a_n)$. Then observe

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} \sqrt{a_n^2 + 1}$$

We know that $\lim_{n \rightarrow \infty} a_n = 0$ so by the product property of sequences

$$\lim_{n \rightarrow \infty} a_n^2 = \lim_{n \rightarrow \infty} a_n \cdot a_n = 0 \cdot 0 = 0$$

Now we know $\lim_{n \rightarrow \infty} a_n^2$ exists so we can use the sum property of limits to say

$$\lim_{n \rightarrow \infty} a_n^2 + 1 = \lim_{n \rightarrow \infty} a_n^2 + \lim_{n \rightarrow \infty} 1$$

We know the values of these limits so

$$\lim_{n \rightarrow \infty} a_n^2 + \lim_{n \rightarrow \infty} 1 = 0 + 1 = 1$$

Now that we know that the limit inside the square root, we can use the square root property to say that

$$\lim_{n \rightarrow \infty} \sqrt{a_n^2 + 1} = \sqrt{\lim_{n \rightarrow \infty} a_n^2 + 1}$$

We know the values of these limits so

$$\sqrt{\lim_{n \rightarrow \infty} a_n^2 + 1} = \sqrt{1} = 1$$

Since b_n is the sequential characterization of $f(x)$ and $\lim_{n \rightarrow \infty} b_n = 1$ then by sequential characterization of limits of functions $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sqrt{x^2 + 1} = 1$. ■