MATH5210 Analysis
Assignment 6
Limits of Functions pt. 2
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Claim: Let $f, g, h : (a, b) \to \mathbf{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ for all $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = M$ and $L = M$, then $\lim_{x \to x_0} h(x)$ exists and equals L.

Proof: Let $\epsilon > 0$ be given. I will show that $\exists\ \delta > 0$ s.t. $|g(x) - L| < \epsilon$ for $0 < |x - x_0| < \delta$.
We know $\exists\ \delta_1$ s.t.

$$|f(x) - L| < \epsilon$$

$$\Rightarrow -\epsilon < f(x) - L < \epsilon$$

$$\Rightarrow L - \epsilon < f(x) < L + \epsilon, \text{ for } 0 < |x - x_0| < \delta_1$$

We also know $\exists\ \delta_2$ s.t.

$$|h(x) - M| < \epsilon$$

$$\Rightarrow -\epsilon < h(x) - M < \epsilon$$

$$\Rightarrow M - \epsilon < h(x) < M + \epsilon, \text{ for } 0 < |x - x_0| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then for any x s.t. $0 < |x - x_0| < \delta$

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < M + \epsilon$$

and since $L = M$

$$\Rightarrow L - \epsilon < g(x) < L + \epsilon$$

$$\Rightarrow |g(x) - L| < \epsilon, \text{ for } 0 < |x - x_0| < \delta$$

Therefore, if $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = M$ and $L = M$, then $\lim_{x \to x_0} h(x)$ exists and equals L.

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Claim: Let $f, g : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0} f(x)g(x) = LM$.

Proof: I will use sequential characterization of limits to prove the claim. For **all** sequences a_n s.t.

$$\lim_{n \to \infty} a_n = x_0$$

then

$$\lim_{n \to \infty} f(a_n) = L$$

and

$$\lim_{n \to \infty} g(a_n) = M$$

Let $b_n = f(a_n)$ and $c_n = g(a_n)$ then

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

and

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} g(a_n) = M$$

Observe $\lim_{n \to \infty} b_n c_n$. We know $\lim_{n \to \infty} b_n$ exists and $\lim_{n \to \infty} c_n$ exists, so by the product property of sequences

$$\lim_{n \to \infty} b_n c_n = \lim_{n \to \infty} b_n \cdot \lim_{n \to \infty} c_n = LM$$

Therefore, since $\lim_{n \to \infty} b_n c_n$ is the sequential characterization of $\lim_{n \to \infty} f(x)g(x)$, if $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0} f(x)g(x) = LM$.

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Claim: If $\lim_{x \to x_0} f(x) = L$ and $\lim_{y \to L} g(y) = M$, then $\lim_{x \to x_0} g \circ f(x) = M$

Proof:
Remark: For all sequences a_n s.t. $\lim_{n \to \infty} a_n = x_0$ then

$$\lim_{n \to \infty} f(a_n) = L$$

Let $b_n = f(a_n)$ so

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

Remark: For all sequences c_n s.t. $\lim_{n \to \infty} c_n = L$ then

$$\lim_{n \to \infty} g(c_n) = M$$

Note that b_n is a sequence s.t. $\lim_{n \to \infty} b_n = L$, so b_n satisfies the sequential characterization of $g(x)$ therefore

$$\lim_{n \to \infty} g(b_n) = M$$

Since we defined $b_n = f(a_n)$ then

$$\lim_{n \to \infty} g(b_n) = \lim_{n \to \infty} g(f(a_n)) = M$$

So, since $\lim_{n \to \infty} g(f(a_n)) = M$ then

$$\lim_{n \to \infty} g(f(a_n)) = \lim_{n \to \infty} g \circ f(x) = M$$

Therefore if $\lim_{x \to x_0} f(x) = L$ and $\lim_{y \to L} g(y) = M$, then $\lim_{x \to x_0} g \circ f(x) = M$.

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9 $\epsilon - \delta$

Claim: $\lim_{x \to 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$

Proof: For all $\epsilon > 0$ there exists $\delta = \min\{1, \epsilon\}$ s.t. for all x, where $|x - 1| < \delta$, $|f(x) - L| < \epsilon$. Then

$$|f(x) - L| = \left| \frac{x^2 + 4}{x^2 - 4} + \frac{5}{3} \right|$$

$$= \left| \frac{8x^2 - 8}{3(x^2 - 4)} \right|$$

$$= \frac{8}{3}\left|\frac{(x+1)(x-1)}{x^2+4)}\right|$$

$$< \frac{8}{3}\delta\left|\frac{(x+1)}{(x^2+4)}\right|$$

From our choice of delta, we know that $|x-1| < \delta = \min\{1, \epsilon\}$ so

$$|x-1| < \delta < \epsilon < 1$$

$$\Rightarrow |x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1$$

$$\Rightarrow 0 < x < 2$$

This implies

$$x^2+4 < 8 \text{ and } x+1 < 3$$

so

$$< \frac{8}{3}\delta\left|\frac{(x+1)}{x^2+4)}\right| < \frac{8}{3}\delta \cdot \frac{3}{8}$$

$$\Rightarrow \delta < \epsilon$$

Therefore $\lim_{x\to 1}\frac{x^2+4}{x^2-4} = -\frac{5}{3}$.

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10 $\epsilon - \delta$

Claim: $\lim_{x\to 0}\sqrt{x^2+1} = 1$

Proof: For all $\epsilon > 0$ there exists $\delta = \epsilon$ s.t. for all x, where $|x = x_0| < \delta$, $|f(x) - L| < \epsilon$. Then

$$|f(x) - L| = \left|\sqrt{x^2+1} - 1\right|$$

$$= \left|\sqrt{x^2+1} - 1\right|$$

$$= \frac{x^2+1-1}{\sqrt{x^2+1}+1}$$

$$= \frac{x^2}{\sqrt{x^2+1}+1}$$

$$< \frac{x^2}{\sqrt{x^2}}$$

$$= \frac{x^2}{x}$$

$$= x = \delta = \epsilon$$

Therefore $\lim_{x \to 0} \sqrt{x^2+1} = 1$.

$\blacksquare$

11 $\epsilon - \delta$

Claim: $\lim_{x \to -1} x^3 + x + 2 = 0$

Proof: For all $\epsilon > 0$ there exists $\delta = \min\{1, \frac{\epsilon}{4}\}$ s.t. for all x, where $|x - x_0| < \delta$, $|f(x) - L| < \epsilon$. Then

$$|f(x) - L| = |x^3 + x + 2 - 0|$$

$$= |(x+1)(x^2 - x + 2)|$$

$$= |\delta(x^2 - x + 2)|$$

$$= \delta(|x|^2 - |x| + 2)$$

From our choice of delta, we know that $|x+1| < \delta = \min\{1, \frac{\epsilon}{4}\}$ so

$$|x+1| < \delta < \frac{\epsilon}{4} < 1$$

$$\Rightarrow -1 < x + 1 < 1$$

$$\Rightarrow -2 < x < 0$$

$$\Rightarrow 0 < |x| < 2$$

This implies

$$= \delta(|x|^2 - |x| + 2) < \delta(4 - 2 + 2)$$

$$= 4\delta = \epsilon$$

Therefore $\lim_{x \to -1} x^3 + x + 2 = 0$.

12 - 7 limit properties

Claim: $\lim_{x\to 2} x^2 + x - 5 = 1$

Proof: As discussed in class, the function $f(x) = x$ and $f(x) = x^2$ are continuous functions. This means that $\lim_{x\to x_0} f(x) = f(x_0)$ by the definition of continuous functions. This also means that we know

$$\lim_{x\to 2} x = 2 \text{ and } \lim_{x\to 2} x^2 = 2^2 = 4$$

Because we know these limits exist we can use the sum property of limits to say

$$\lim_{x\to 2} x^2 + x - 5 = \lim_{x\to 2} x^2 + \lim_{x\to 2} x - \lim_{x\to 2} 5$$

and we know the values of the limits because they are continuous functions so

$$\lim_{x\to 2} x^2 + \lim_{x\to 2} x - \lim_{x\to 2} = 4 + 2 - 5 = 1$$

Therefore $\lim_{x\to 2} x^2 + x - 5 = 1$.

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12 - 8 limit properties

Claim: $\lim_{x\to 1} \frac{x}{x^2+4} = \frac{1}{5}$

Proof: As discussed in class, the function $f(x) = x$ and $f(x) = x^2$ are continuous functions. This means that $\lim_{x\to x_0} f(x) = f(x_0)$ by the definition of continuous functions. This also means that we know

$$\lim_{x\to 1} x = 1 \text{ and } \lim_{x\to 1} x^2 = 1^2 = 1$$

Because we know these limits exist we can use the sum property of limits to say

$$\lim_{x\to 1} x^2 + 4 = \lim_{x\to 1} x^2 + \lim_{x\to 1} 4$$

and we know the values of the limits because they are continuous functions so

$$\lim_{x\to 1} x^2 + \lim_{x\to 1} 4 = 1 + 4 = 5$$

Now since we know that the limit of the numerator and the denominator exist we can say

$$\lim_{x\to 1} \frac{x}{x^2+4} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1} x^2 + 4}$$

$$= \frac{\lim_{x\to 1} x}{\lim_{x\to 1} x^2 + \lim_{x\to 1} 4}$$

$$= \frac{1}{1+4} = \frac{1}{5}$$

Therefore $\lim_{x\to 1} \frac{x}{x^2+4} = \frac{1}{5}$.

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13 - 9 sequential characterization

Claim: $\lim_{x\to 1} \frac{x^2+4}{x^2-4} = -\frac{5}{3}$

Proof: Let $f(x) = \frac{x^2+4}{x^2-4}$. Let a_n be any sequence s.t. $\lim_{n\to\infty} a_n = 1$ and let $b_n = f(a_n)$. Then observe

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{a_n^2+4}{a_n^2-4}$$

We know that $\lim_{n\to\infty} a_n = 1$ so by the product property of sequences

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} a_n \cdot a_n = 1 \cdot 1 = 1$$

Now we know $\lim_{n\to\infty} a_n^2$ exists so we can use the sum property of limits to say

$$\lim_{n\to\infty} a_n^2 - 4 = \lim_{n\to\infty} a_n^2 - \lim_{n\to\infty} 4$$

We know the values of these limits so

$$\lim_{n\to\infty} a_n^2 - \lim_{n\to\infty} 4 = 1 - 4 = -3$$

By similar logic we can use the sum property of limits to say

$$\lim_{n\to\infty} a_n^2 + 4 = \lim_{n\to\infty} a_n^2 + \lim_{n\to\infty} 4$$

We know the values of these limits so

$$\lim_{n\to\infty} a_n^2 + \lim_{n\to\infty} 4 = 1 + 4 = 5$$

Now we know that the limit of the numerator and denominator exist so we can use the quotient property of limits to say

$$\lim_{n\to\infty} \frac{a_n^2+4}{a_n^2-4} = \frac{\lim_{n\to\infty} a_n^2 + 4}{\lim_{n\to\infty} a_n^2 - 4}$$

We know the values of these limits so

$$\frac{\lim_{n\to\infty} a_n^2 + 4}{\lim_{n\to\infty} a_n^2 - 4} = \frac{5}{-3} = -\frac{5}{3}$$

Since b_n is the sequential characterization of $f(x)$ and $\lim_{n\to\infty} b_n = -\frac{5}{3}$ then by sequential characterization of limits of functions $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2+4}{x^2-4} = -\frac{5}{3}$.

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13 - 10 sequential characterization

Claim: $\lim_{x\to 0} \sqrt{x^2+1} = 1$

Proof: Let $f(x) = \sqrt{x^2+1}$. Let a_n be any sequence s.t. $\lim_{n\to\infty} a_n = 0$ and let $b_n = f(a_n)$. Then observe

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sqrt{a_n^2+1}$$

We know that $\lim_{n\to\infty} a_n = 0$ so by the product property of sequences

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} a_n \cdot a_n = 0 \cdot 0 = 0$$

Now we know $\lim_{n\to\infty} a_n^2$ exists so we can use the sum property of limits to say

$$\lim_{n\to\infty} a_n^2 + 1 = \lim_{n\to\infty} a_n^2 + \lim_{n\to\infty} 1$$

We know the values of these limits so

$$\lim_{n\to\infty} a_n^2 + \lim_{n\to\infty} 1 = 0 + 1 = 1$$

Now that we know that the limit inside the square root, we can use the square root property to say that

$$\lim_{n\to\infty} \sqrt{a_n^2+1} = \sqrt{\lim_{n\to\infty} a_n + 1}$$

We know the values of these limits so

$$\sqrt{\lim_{n\to\infty} a_n + 1} = \sqrt{1} = 1$$

Since b_n is the sequential characterization of $f(x)$ and $\lim_{n\to\infty} b_n = 1$ then by sequential characterization of limits of functions $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt{x^2+1} = 1$.

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