MATH5210 Analysis
Assignment 6
Limits of Functions pt. 2

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Claim: Let $f, g, h : (a, b) \to \mathbf{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ for all $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = M$ and $L = M$, then $\lim_{x \to x_0} h(x)$ exists and equals L.

Proof: Let $\epsilon > 0$ be given. I will show that $\exists\ \delta > 0$ s.t. $|g(x) - L| < \epsilon$ for $0 < |x - x_0| < \delta$.
We know $\exists\ \delta_1$ s.t.

$$|f(x) - L| < \epsilon$$

$$\Rightarrow -\epsilon < f(x) - L < \epsilon$$

$$\Rightarrow L - \epsilon < f(x) < L + \epsilon, \text{ for } 0 < |x - x_0| < \delta_1$$

We also know $\exists\ \delta_2$ s.t.

$$|h(x) - M| < \epsilon$$

$$\Rightarrow -\epsilon < h(x) - M < \epsilon$$

$$\Rightarrow M - \epsilon < h(x) < M + \epsilon, \text{ for } 0 < |x - x_0| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then for any x s.t. $0 < |x - x_0| < \delta$

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < M + \epsilon$$

and since $L = M$

$$\Rightarrow L - \epsilon < g(x) < L + \epsilon$$

$$\Rightarrow |g(x) - L| < \epsilon, \text{ for } 0 < |x - x_0| < \delta$$

Therefore, if $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} h(x) = M$ and $L = M$, then $\lim\limits_{x \to x_0} h(x)$ exists and equals L.

$\blacksquare$

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Claim: Let $f, g : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} g(x) = M$, then $\lim\limits_{x \to x_0} f(x)g(x) = LM$.

Proof: I will use sequential characterization of limits to prove the claim. For **all** sequences a_n s.t.

$$\lim_{n \to \infty} a_n = x_0$$

then

$$\lim_{n \to \infty} f(a_n) = L$$

and

$$\lim_{n \to \infty} g(a_n) = M$$

Let $b_n = f(a_b)$ and $c_n = g(a_n)$ then

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

and

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

Observe $\lim\limits_{n \to \infty} b_n c_n$. We know $\lim\limits_{n \to \infty} b_n$ exists and $\lim\limits_{n \to \infty} c_n$ exists, so by the product property of sequences

$$\lim_{n \to \infty} b_n c_n = \lim_{n \to \infty} b_n \cdot \lim_{n \to \infty} c_n = LM$$

Therefore if $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} g(x) = M$, then $\lim\limits_{x \to x_0} f(x)g(x) = LM$.

$\blacksquare$

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Claim: If $\lim_{x \to x_0} f(x) = L$ and $\lim_{y \to L} g(y) = M$, then $\lim_{x \to x_0} g \circ f(x) = M$

Proof:
Remark: For all sequences a_n s.t. $\lim_{n \to \infty} a_n = x_0$ then

$$\lim_{n \to \infty} f(a_n) = L$$

Let $b_n = f(a_n)$
Remark: For all sequences c_n s.t. $\lim_{n \to \infty} c_n = L$ then

$$\lim_{n \to \infty} g(c_n) = M$$

Let $b_n = f(a_n)$
Note that b_n is a sequence s.t. $\lim_{n \to \infty} b_n = L$. b_n satisfies the sequential characterization of $g(x)$ therefore

$$\lim_{n \to \infty} g(b_n) = M$$

Since we defined $b_n = f(a_n)$ then

$$\begin{aligned} \lim_{n \to \infty} g(b_n) &= \lim_{n \to \infty} g(f(a_n)) \\ &= \lim_{n \to \infty} g(L) = M \end{aligned}$$

So, since $\lim_{n \to \infty} g(f(a_n)) = M$ then

$$\lim_{n \to \infty} g(f(a_n)) = \lim_{n \to \infty} g \circ f(x) = M$$

Therefore if $\lim_{x \to x_0} f(x) = L$ and $\lim_{y \to L} g(y) = M$, then $\lim_{x \to x_0} g \circ f(x) = M$

$\blacksquare$

9 $\epsilon - \delta$

Claim: $\lim_{x \to 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$

Proof: For all $\epsilon > 0$ there exists $\delta = \min\{1, \frac{\epsilon}{2}\}$ s.t. for all x, where $|x - 1| < \delta$, $|f(x) - L| < \epsilon$. Then

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 + 4}{x^2 - 4} + \frac{5}{3} \right| \\ &= \left| \frac{8x^2 - 8}{3(x^2 - 4)} \right| \\ &= \frac{8}{3} \left| \frac{(x+1)(x-1)}{(x+2)(x-2)} \right| \\ &< \frac{8}{3} \delta \left| \frac{(x+1)}{(x+2)(x-2)} \right| \end{aligned}$$

10 $\epsilon - \delta$

Claim: $\lim_{x \to 0} \sqrt{x^2 + 1} = 1$

Proof:

11 $\epsilon - \delta$

Claim: $\lim_{x \to -1} x^3 + x + 2 = 0$

Proof:

12 - 7 limit properties

Claim: $\lim_{x \to 2} x^2 + x - 5 = 1$

Proof:

12 - 8 limit properties

Claim: $\lim_{x \to 1} \frac{x}{x^2 + 4} = \frac{1}{5}$

Proof:

13 - 9 limit properties

Claim: $\lim_{x \to 1} \frac{x^2 + 4}{x^2 - 4} = -\frac{5}{3}$

Proof:

13 - 10 limit properties

Claim: $\lim_{x \to 0} \sqrt{x^2 + 1} = 1$

Proof: