

MATH5210 ANALYSIS  
Assignment 4  
Monotone Convergence Theorem  
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**1**

Let  $\{a_n\}$  be a sequence of non-negative real numbers. Show that the partial sums for the infinite series  $\sum_{n=1}^{\infty} a_n$  form a monotone series.

**2**

What is the approximation property for the infimum of a set  $A$ ? If  $m = \inf(A)$ , prove that there is a sequence  $a_n \in A$  with  $\lim a_n = m$ . Follow the proof given in class.

**3**

Give a detailed proof of the MCT in the case of a monotone decreasing sequence which is bounded below. Follow the proof given in class.

**4**

If  $0 < r < 1$ , show that  $\lim_{n \rightarrow \infty} r^n = 0$ . Note that with  $a_n = r^n$ ,  $a_{n+1} = ra_n$ . What happens if  $0 \leq r \leq 1$ ?

**5**

Let  $a_1 \in (0, 1)$ . Prove that  $a_{n+1} = \sqrt{a_n + 1} - 1$  is an increasing sequence which converges to 0. What happens if  $a_0 = 0$ , if  $a_0 = -1$ ?

## 6

Suppose  $a_1 \geq 2$  and  $a_{n+1} = 2 + \sqrt{a_n - 2}$ . Show that  $a_n$  converges to 2 or 3. How does the limit depend upon the value of  $a_1$

## 7

(Bonus) Let  $0 < b_1 < a_1$  and let

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

### a

Show that  $0 < b_n < a_n$

### b

Show that  $b_n$  is increasing and bounded above.

### c

Show that  $a_n$  is decreasing and bounded below.

### d

Show that  $0 < a_{n+1} - b_{n+1} < (a_1 - b_1)/2^n$

### e

Conclude that the sequences  $a_n$  and  $b_n$  converge to a common limit.