MATH5210 Analysis
Assignment 4
Monotone Convergence Theorem
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1

Let $\{a_n\}$ be a sequence of non-negative real numbers. Show that the partial sums for the infinite series $\sum_{n=1}^{\infty} a_n$ form a monotone series.

Proof: Let us review the definition of a monotone increasing series. Let b_m be a series, b_m is monotone increasing iff $b_m \leq b_{m+1}$ for all $m > 0$. Let b_m be the partial sums of $\sum_{n=1}^{\infty} a_n$ such that $b_1 = a_1$, $b_2 = b_1 + a_2$, and $b_{m+1} = b_m + a_n$. From the definition of the problem, $\{a_n\}$ is a sequence of non-negative real numbers so $a_n \geq 0 \Rightarrow b_m + a_n \geq b_m \Rightarrow b_{m+1} \geq b_m$. Therefore b_m is a monotone series. ■

2

What is the approximation property for the infimum of a set $\mathbf{A}$? If $m = \inf(\mathbf{A})$, prove that there is a sequence $a_n \in \mathbf{A}$ with $\lim a_n = m$.

The approximation property for the infimum of a set $\mathbf{A}$ states
Suppose that $m = \inf(\mathbf{A})$
Then for any $\epsilon > 0, \exists x \in \mathbf{A}$ s.t. $m \leq x \leq m + \epsilon$

Since $m = \inf(\mathbf{A})$, $\{a_n\}$ is bounded below and has a greatest upper bound (gub) s.t. $\text{gub}(\{a_n\}) = m$. We will prove will prove that $\lim_{n\to\infty} a_n = m$.
Let $\epsilon > 0$ be given. We have to find N s.t. $|a_n - m| < \epsilon$ and $a_n - m$ is always positive because $a_n > m$. So $a_n - m < \epsilon$ for all $n > N$.
Then by the approximation property of the infimum of $\mathbf{A}$, there exists an $x \in \{a_n\}$ s.t. $m \leq x < m + \epsilon$ where $x = a_N$. So

$$a_N \leq m + \epsilon \tag{1}$$

Finally if $n > N$, then because of monotone decreasing

$$a_N < a_n \tag{2}$$

Combining 1 and 2 gives us $m + \epsilon > a_N > a_n$. Thus $a_n - m < \epsilon$ for all $n > N$. ■

3

Give a detailed proof of the MCT in the case of a monotone decreasing sequence which is bounded below. Follow the proof given in class.

4

If $0 < r < 1$, show that $\lim_{n\to\infty} r^n = 0$. Note that with $a_n = r^n$, $a_{n+1} = ra_n$. What happens if $0 \leq r \leq 1$?

Lemma 3.1: $r^n > 0$ for all $n > 0$ and $0 < r < 1$.
Let us observe $a_1 = r^1$ is positive because $0 < r < 1$ and $a_2 = r^2 = r \cdot r$ is positive because a positive number times a positive number is positive. Assume $a_n > 0$, then $a_{n+1} = r \cdot a_n$ which is positive because as previously stated, a positive number times a positive number is positive. Therefore $r^n > 0$ for all $n > 0$ and $0 < r < 1$.

Lemma 3.2: a_n is monotone decreasing.
From the bounds of r we know that $r < 1$. By lemma 3.1 we know that $r^n > 0$ so we can multiply both sides by r^n while preserving inequality so $r \cdot r^n < r^n$. Then we can use the definition $a_n = r^n$ so we have $r \cdot a_n < a_n$. Then we can use the definition $a_{n+1} = r \cdot r^n$ so we have $a_{n+1} < a_n$. Therefore the sequence is monotone decreasing.

Claim: $\lim_{n\to\infty} r^n = 0$ for $0 < r < 1$.

Proof: Through lemma 3.1 and 3.2, we know that a_n is bounded below by zero and monotone decreasing so by the Monotone Convergence Theorem we know that the limit of a_n exists. Let $A = \lim_{n\to\infty} r^n$. Then starting with $a_{n+1} = r \cdot a_n$ we take the limit of both sides giving us $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} r \cdot r^n$. By the subsequence property of limits we have $A = \lim_{n\to\infty} r \cdot r^n$. Then by the scalar product property of limits, we have $A = r \cdot A$ which is $r \cdot A - A = 0$ and so $A(r-1) = 0$. This can be satisfied by only $A = 0$ therefore $\lim_{n\to\infty} r^n = 0$. ■

5

Let $a_1 \in (-1,0)$ Prove that $a_{n+1} = \sqrt{a_n+1}-1$ is a increasing sequence which converges to 0. What happens if $a_0 = 0$, if $a_0 = -1$?

Lemma 5.1 $a_{n+1} = \sqrt{a_n+1}-1 < 0$ for $a_1 \in (-1,0)$
From the problem statement we know that $-1 < a_n < 0$. Then if $-1 < a_n < 0$ then $0 < a_n+1 < 1$. Since the square root is an increasing function, we can apply it whilst preserving inequality so $0 < \sqrt{a_n+1} < 1$ then $-1 < \sqrt{a_n+1}-1 < 0$ which is $-1 < a_{n+1} < 0$. Therefore $a_{n+1} = \sqrt{a_n+1}-1 < 0$ for $a_1 \in (-1,0)$.

Lemma 5.2 $a_{n+1} = \sqrt{a_n+1}-1$ is monotone increasing for $a_1 \in (-1,0)$.
If $a_1 = -0.5$ then $a_{n+1} = \sqrt{-0.5+1}-1 \approx -0.29$ therefore $a_1 < a_2$. If $a_n < a_{n+1}$ then $a_n+1 < a_{n+1}+1$. Since the square root function is an increasing function, we can apply it whilst preserving inequality so $\sqrt{a_n+1} < \sqrt{a_{n+1}+1}$ then $\sqrt{a_n+1}-1 < \sqrt{a_{n+1}+1}-1$ which is $a_{n+1} < a_{n+2}$. Therefore $a_{n+1} = \sqrt{a_n+1}-1$ is monotone increasing for $a_1 \in (-1,0)$.

Claim: $\lim\limits_{n\to\infty} a_n = 0$ for $a_1 \in (-1.0)$

Proof: Through lemma 5.1 and 5.2 we know that a_n is bounded above by 0 and is an increasing sequence. Therefore we know by the Monotone Convergence Theorem that $\lim\limits_{n\to\infty} a_n$ exists. Let $A = \lim\limits_{n\to\infty} a_n$. Then starting with the definition $a_{n+1} = \sqrt{a_n+1}-1$ we take the limit of both sides so $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{a_n+1}-1$. Let $b_n = 1$ and remark that $\sqrt{a_n+1}-1 = \sqrt{a_n+b_n}-b_n$. Let $c_n = \sqrt{a_n-b_n}$. Now see that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 1 = 1$ since b_n is a constant sequence and $\lim\limits_{n\to\infty} a_n = A$ as previously stated. Then because the limit exists we see that $\lim\limits_{n\to\infty} a_n + 1 = A+1$ by the summation property. So by the summation property and square root property we can express the original limit as $\lim\limits_{n\to\infty} a_{n+1} = \sqrt{\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} 1} - \lim\limits_{n\to\infty} 1$. Then by the subsequence property and evaluation of limits as previously discussed $A = \sqrt{A+1}-1$. Then $A+1 = \sqrt{A+1}$ and then $A^2+2A+1 = A+1$ and then $A^2+A = 0$ so $A = -1, 0$. Returning to the bounds set on a_1 we know that $-1 < a_1 0$ and since a_n is an increasing sequence $A \neq -1$. Therefore $\lim\limits_{n\to\infty} a_n = 0$ for $a_1 \in (-1.0)$. ■

Claim: $\lim\limits_{n\to\infty} a_n = 0$ for $a_1 = 0$

Proof: If $a_1 = 0$ then $a_2 = \sqrt{1}-1 = 0$, therefore $a_1 = a_2$. If $a_n = a_{n+1}$ then $a_n+1 = a_{n+1}+1$ Since the square root is an increasing function we can apply whilst preserving inequality so $\sqrt{a_n+1} = \sqrt{a_{n+1}+1}$ then $\sqrt{a_n+1}-1 = \sqrt{a_{n+1}+1}-1$ which is $a_{n+1} = a_{n+2}$. Therefore $\lim\limits_{n\to\infty} a_n = 0$ for $a_1 = 0$. ■

Claim: $\lim\limits_{n\to\infty} a_n = 1$ for $a_1 = 1$

Proof: If $a_1 = -1$ then $a_2 = \sqrt{0}-1 = -1$, therefore $a_1 = a_2$. If $a_n = a_{n+1}$ then $a_n+1 = a_{n+1}+1$ Since the square root is an increasing function we can apply whilst preserving inequality so $\sqrt{a_n+1} = \sqrt{a_{n+1}+1}$ then $\sqrt{a_n+1}-1 = \sqrt{a_{n+1}+1}-1$ which is $a_{n+1} = a_{n+2}$. Therefore $\lim_{n\to\infty} a_n = -1$ for $a_1 = 0$. ∎

6

Suppose $a_1 \geq 2$ and $a_{n+1} = 2 + \sqrt{a_n - 2}$. Show that a_n converges to 2 or 3. How does the limit depend upon the value of a_1

Lemma 6.1: a_n is bounded above by 3 for all $n > 0$ and $2 < a_n < 3$.
Assuming $2 < a_n < 3$ then $0 < a_n - 2 < 1$. Since the square root function is an increasing function, we can apply it whilst preserving inequality so $0 < \sqrt{a_n - 2} < 1$ then $2 < \sqrt{a_n - 2} + 2 < 3$. Since $a_{n+1} = 2 + \sqrt{a_n - 2}$ we can see $2 < a_{n+1} < 3$. Therefore a_n is bounded above by 3 for all $n > 0$ and $2 < a_n < 3$.

Lemma 6.2: $a_{n+1} = 2 + \sqrt{a_n - 2}$ is monotone increasing for $2 < a_n < 3$.
Observe $a_1 = 2.1$ then $a_2 \approx 2.3$ therefore $a_1 < a_2$. If $a_n < a_{n+1}$ then $a_n - 2 < a_{n+1} - 2$. Since the square root is a positive function, we can apply it to both sides and maintain inequality. So $\sqrt{a_n - 2} < \sqrt{a_{n+1} - 2}$. Then $2 + \sqrt{a_n - 2} < 2 + \sqrt{a_{n+1} - 2}$ which means that $a_{n+1} < a_{n+2}$. Therefore $a_{n+1} = 2 + \sqrt{a_n - 2}$ is monotone increasing for $2 < a_n < 3$.

Lemma 6.3: $\lim_{n\to\infty} a_n = 3$ for $2 < a_n < 3$
By lemma 6.1 and 6.2 we know that a_n is bounded above by 3 and is monotone increasing for $2 < a_n < 3$ so by the Monotone Convergence Theorem we know $\lim_{n\to\infty} a_n$ exists on these bounds. Let $A = \lim_{n\to\infty} a_n$. Starting with $a_{n+1} = 2 + \sqrt{a_n - 2}$ we take the limit of both sides so $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 2 + \sqrt{a_n - 2}$. Let $b_n = 2$ and remark that $2 + \sqrt{a_n - 2} = b_n + \sqrt{a_n - b_n}$. Let $c_n = \sqrt{a_n - b_n}$. Now see that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 2 = 2$ since b_n is a constant sequence and $\lim_{n\to\infty} a_n = A$ as previously stated. Then because the limit exists we see that $\lim_{n\to\infty} a_n - 2 = A - 2$ by the summation property. So by the summation property and the square root property we can express the original limit as $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 2 + \sqrt{\lim_{n\to\infty} a_n - \lim_{n\to\infty} 2}$. Then by the subsequence property of limits and evaluation of limits as previously discussed $A = 2 + \sqrt{A - 2}$ Then $A - 2 = \sqrt{A - 2}$ and then $A^2 - 4A + 2 = A - 2$ and then $A^2 - 5A + 6 = 0$ and then $(A-3)(A-2) = 0$ so $A = 2, 3$ are fixed points. Returning to our bounds on a_1 we know that $A \neq 2$ since $2 < a_1 < 3$ and a_n is monotone increasing for $2 < a_n < 3$. Therefore $\lim_{n\to\infty} a_n = 3$ for $2 < a_n < 3$.

Lemma 6.4 a_n is bounded below by 3 for all $n > 0$ and $a_n > 3$
Assuming $a_n > 3$ then $a_n - 2 > 1$. Since the square root function is an increasing, we can apply it whilst preserving inequality so $\sqrt{a_n - 2} > 1$. Then it follows that $2 + \sqrt{a_n - 2} > 3$. Therefore a_n is bounded below by 3 for all $n > 0$ and $a_n > 3$

Lemma 6.5: $a_{n+1} = 2 + \sqrt{a_n - 2}$ is monotone decreasing for $a_n > 3$.

Observe $a_1 = 4$ then $a_2 \approx 3.4$ therefore $a_1 > a_2$. If $a_n > a_{n+1}$ then $a_n - 2 > a_{n+1} - 2$. Since the square root is a positive function, we can apply it to both sides and maintain inequality. So $\sqrt{a_n - 2} > \sqrt{a_{n+1} - 2}$. Then $2 + \sqrt{a_n - 2} > 2 + \sqrt{a_{n+1} - 2}$ which means that $a_{n+1} > a_{n+2}$. Therefore $a_{n+1} = 2 + \sqrt{a_n - 2}$ is monotone decreasing for $a_n > 3$.

Lemma 6.6: $\lim\limits_{n\to\infty} a_n = 3$ for $a_n > 3$

By lemma 6.4 and 6.5 we know that a_n is bounded below by 3 and is monotone decreasing for $a_n > 3$ so by the Monotone Convergence Theorem we know $\lim\limits_{n\to\infty} a_n$ exists on these bounds. Let $B = \lim\limits_{n\to\infty} a_n$. Starting with $a_{n+1} = 2+\sqrt{a_n - 2}$ we take the limit of both sides so $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} 2+\sqrt{a_n - 2}$. Let $b_n = 2$ and remark that $2+\sqrt{a_n - 2} = b_n + \sqrt{a_n - b_n}$. Let $c_n = \sqrt{a_n - b_n}$. Now see that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 2 = 2$ since b_n is a constant sequence and $\lim\limits_{n\to\infty} a_n = B$ as previously stated. Then because the limit exists we see that $\lim\limits_{n\to\infty} a_n - 2 = B - 2$ by the summation property. So by the summation property and the square root property we can express the original limit as $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} 2 + \sqrt{\lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} 2}$. Then by the subsequence property of limits and evaluation of limits as previously discussed $B = 2 + \sqrt{B-2}$ Then $B - 2 = \sqrt{B-2}$ and then $B^2 - 4B + 2 = B - 2$ and then $B^2 - 5B + 6 = 0$ and then $(B-3)(B-2) = 0$ so $B = 2, 3$ are fixed points. Returning to our bounds on a_1 we know that $B \neq 2$ since $a_1 > 3$ and a_n is monotone decreasing for $a_n > 3$. Therefore $\lim\limits_{n\to\infty} a_n = 3$ for $a_n > 3$.

Lemma 6.7 $\lim\limits_{n\to\infty} a_n = 3$ for $a_1 = 3$

If $a_1 = 3$ then $a_2 = 2 + \sqrt{3-2} = 3$, therefore $a_1 = a_2$. If $a_n = a_{n+1}$ then $a_n - 2 = a_{n+1} - 2$. Since the square root is an increasing function we can apply whilst preserving inequality so $\sqrt{a_n - 2} = \sqrt{a_{n+1} - 2}$ then $2+\sqrt{a_n - 2} = 2 + \sqrt{a_{n+1} - 2}$ which is $a_{n+1} = a_{n+2}$. Therefore $\lim\limits_{n\to\infty} a_n = 3$ for $a_1 = 3$.

Lemma 6.8 $\lim\limits_{n\to\infty} a_n = 2$ for $a_1 = 2$

If $a_1 = 2$ then $a_2 = 2 + \sqrt{2-2} = 2$, therefore $a_1 = a_2$. If $a_n = a_{n+1}$ then $a_n - 2 = a_{n+1} - 2$. Since the square root is an increasing function we can apply whilst preserving inequality so $\sqrt{a_n - 2} = \sqrt{a_{n+1} - 2}$ then $2+\sqrt{a_n - 2} = 2 + \sqrt{a_{n+1} - 2}$ which is $a_{n+1} = a_{n+2}$. Therefore $\lim\limits_{n\to\infty} a_n = 3$ for $a_1 = 2$.

Proof: For $a_1 \geq 2$ and $a_{n+1} = 2 + \sqrt{a_n - 2}$. $\lim\limits_{n\to\infty} a_n$ converges to 2 or 3.
From lemmas 6.3, 6.6, 6.7 and 6.8 we know that for $a_1 \geq 2$, $\lim\limits_{n\to\infty} a_n$ can only converge to 2 or 3. It depends on a_1 which value $\lim\limits_{n\to\infty} a_n$ converges to. When $a_1 = 2$, then $\lim\limits_{n\to\infty} a_n = 2$, lemma 6.8. When $a_1 > 2$, then $\lim\limits_{n\to\infty} a_n = 3$. ■

7

(Bonus) Let $0 < b_1 < a_1$ and let

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

a

Show that $0 < b_n < a_n$

b

Show that b_n is increasing and bounded above.

c

Show that a_n is decreasing and bounded below.

d

Show that $0 < a_{n+1} - b_{n+1} < (a_1 - b_1)/2^n$

e

Conclude that the sequences a_n and b_n converge to a common limit.