Chap. 3: Single Stage Network Comparisons

- one way to compare: ability of one network to do connections of another
  - — SIMD mode - all N PEs active
  - — must simulate connection for all PEs
  - — count number of inter-PE transfers (dominant factor)

- Lower bound on simulation transfers
  - — cannot do simulation in fewer transfers
  - — shown by mathematical proof

- Upper bound on simulation transfers
  - — can do simulation in this number of transfers (or less)
  - — shown by algorithm to do simulation

- Ground rules
  - — to simulate i.f. f: DTR of P $\rightarrow$ DTR of f(P), $0 \leq P < N$
  - — i.f. of network being simulated that takes most time to simulate determines bounds

Lower and Upper Bounds on Network Simulation Times

		PM2I	*Cube*	*Illiac*	*Shuffle-Exchange*
PM2I	lower	—	2	1	m
	upper	—	2	1	$m + 1$
Cube	lower	m	—	m	m
	upper	m	—	m	m
Illiac	lower	$n/2$	$(n/2) + 1$	—	$2n - 4$
	upper	$n/2$	$(n/2) + 1$	—	$2n - 1$
Shuffle-Exchange	lower	$2m - 1$	$m + 1$	$2m - 1$	—
	upper	$2m$	$m + 1$	$2m$	—

Note: The entries in row a and column b are lower and upper bounds on the time required for network a to simulate network b.

PM2I → Cube

cube_{m-1}:

$$\text{cube}_{m-1}(p_{m-1}\cdots p_1 p_0) = \bar{p}_{m-1} p_{m-2}\cdots p_1 p_0$$
$$= \text{PM2}_{\pm(m-1)}(p_{m-1}\cdots p_1 p_0)$$

$$\therefore\ \text{cube}_{m-1} = \text{PM2}_{\pm(m-1)}$$

Lower bound = upper bound = 1

$$\text{cube}_i(p_{m-1}\cdots p_1 p_0) = p_{m-1}\cdots p_{i+1}\bar{p}_i p_{i-1}\cdots p_0 \qquad 0 \le i < m$$

$$\text{PM2}_{+i}(P) = P+2^i \bmod N \qquad 0 \le i < m$$

$$\text{PM2}_{-i}(P) = P-2^i \bmod N \qquad 0 \le i < m$$

PM2I → Cube

Lower bound ($0 \le i < m\text{-}1$):

$\text{cube}_i \neq \text{PM2}_{\pm j}$ for any i,j

$\therefore$ need 2 transfers → lower bound $= 2$

$$\text{cube}_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \bar{p}_i p_{i-1} \cdots p_0 \qquad 0 \le i < m$$

$$\text{PM2}_{+i}(P) = P + 2^i \bmod N \qquad 0 \le i < m$$

$$\text{PM2}_{-i}(P) = P - 2^i \bmod N \qquad 0 \le i < m$$

PM2I → Cube

cube_i

Upper bound ($0 \leq i < m-1$):

(S1) $PM2_{+i}$ $[X^m]$

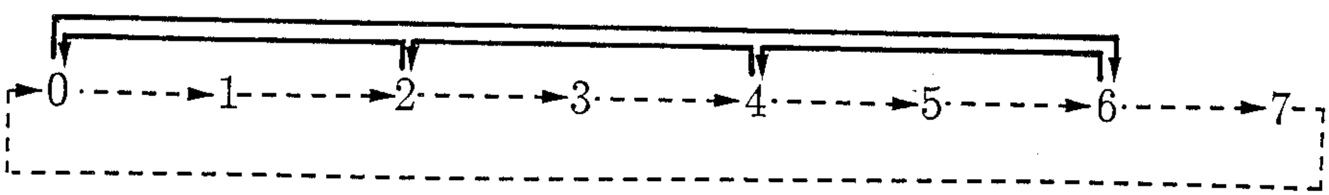
(S2) $PM2_{-(i+1)}$ $[X^{m-(i+1)}0X^i]$

Example: $i=0$, $N=8$.

to perform cube_0:

(S1) $PM2_{+0}$ [XXX] (dashed line)

(S2) $PM2_{-1}$ [XXO] (solid line)



$$\text{cube}_0\,(p_2p_1p_0) = p_2p_1\bar{p}_0$$

For $p_0 = 0$: $\text{cube}_0(p_2p_1 0) = PM2_{+0}(p_2p_1 0) = p_2p_1 1$

For $p_0 = 1$: $\text{cube}_0(p_2p_1 1) =$

$$PM2_{-1}(PM2_{+0}(p_2p_1 1)) = PM2_{-1}(p_2p_1 0{+}2)$$

$$= p_2p_1 0$$

$$\text{cube}_i(p_{m-1}\cdots p_1p_0) = p_{m-1}\cdots p_{i+1}\bar{p}_ip_{i-1}\cdots p_0 \qquad 0 \leq i < m$$

$$PM2_{+i}(P) = P+2^i \bmod N \qquad 0 \leq i < m$$

$$PM2_{-i}(P) = P-2^i \bmod N \qquad 0 \leq i < m$$

PM2I → Cube

cube_i

Upper bound $(0 \le i < m-1)$:

(S1) $PM2_{+i}\ [X^m]$

(S2) $PM2_{-(i+1)}\ [X^{m-(i+1)}0X^i]$

Ex. $i = 2$, $N = 16$, PE $5 \rightarrow \text{cube}_2(5) = 1$

$PM2_{+2}$ [XXXX] $5 \rightarrow 9$

$PM2_{-3}$ [X0XX] $9 \rightarrow 1$

Correctness proof:

Case 1: PE address of form $X^{m-(i+1)}0X^i$

S1: $X^{m-(i+1)}0X^i \rightarrow \underset{(\text{cube}_i)}{X^{m-(i+1)}1X^i}$

S2: no match

Case 2: PE address of form $X^{m-(i+1)}1X^i$

S1: $X^{m-(i+1)}1X^i \rightarrow X^{m-(i+1)}0X^i + 2^{i+1}$

S2: $X^{m-(i+1)}0X^i + 2^{i+1} \rightarrow \underset{(\text{cube}_i)}{X^{m-(i+1)}0X^i}$

Upper bound = 2

$$\text{cube}_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \bar{p}_i p_{i-1} \cdots p_0 \qquad 0 \le i < m$$

$$PM2_{+i}(P) = P + 2^i \bmod N \qquad 0 \le i < m$$

$$PM2_{-i}(P) = P - 2^i \bmod N \qquad 0 \le i < m$$

PM2I $\rightarrow$ Illiac

Illiac - 4 i.f.s (where $n = \sqrt{N}$):

$$\begin{aligned}
\text{Illiac}_{+1}(P) &= P + 1 \bmod N \\
\text{Illiac}_{-1}(P) &= P - 1 \bmod N \\
\text{Illiac}_{+n}(P) &= P + n \bmod N \\
\text{Illiac}_{-n}(P) &= P - n \bmod N
\end{aligned}$$

PM2I - Plus Minus 2^i - 2m i.f.s

$$\begin{aligned}
\text{PM2}_{+i}(P) &= P + 2^i \bmod N \\
\text{PM2}_{-1}(P) &= P - 2^i \bmod N
\end{aligned}$$

$$0 \leq i < m$$

$$n = 2^{m/2}$$

$$\pm 2^{m/2} = \pm n$$

$$\begin{aligned}
\text{Illiac}_{\pm 1} &= \text{PM2}_{\pm 0} \\
\text{Illiac}_{\pm n} &= \text{PM2}_{\pm m/2}
\end{aligned}$$

lower bound = upper bound = 1

PM2I $\rightarrow$ Shuffle-Exchange

Exchange:

$$\text{exchange}\,(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_1 \bar{p}_0$$

$$= \text{cube}_0(p_{m-1} \cdots p_1 p_0)$$

because exchange $=$ cube$_0$ can use PM2I $\rightarrow$ cube$_0$

From the PM2I $\rightarrow$ Cube analysis:

$$\text{lower bound} = \text{upper bound} = 2$$

PM2I $\rightarrow$ Shuffle

Lower bound:

metric -

# of distances $\left\{ \begin{array}{l} \# \text{ distinct integers added to } \{0,1,\ldots,N\text{-}1\} \text{ by} \\ \text{execution of i.f.} \end{array} \right.$

$0 \leq P < N/2 \quad P = 0p_{m-2}\cdots p_1 p_0$

$\text{shuffle}(P) = p_{m-2}\cdots p_1 p_0 0 = 2P$

$\text{shuffle}(P) - P = P$ (adds $0,1,\ldots,(N/2)-1$)

$N/2 \leq P < N \quad P = 1p_{m-2}\cdots p_1 p_0$

$\text{shuffle}(P) = p_{m-2}\cdots p_1 p_0 1 = 2P+1$

$\text{shuffle}(P) - P = P+1$

$(N/2)+1, (N/2)+2,\ldots,N-1, N(=0 \bmod N)$

Total of $N-1$ distinct integers added

$\text{shuffle}\ (p_{m-1}\cdots p_1 p_0) = p_{m-2}p_{m-3}\cdots p_1 p_0 p_{m-1}$

PM2I → Shuffle

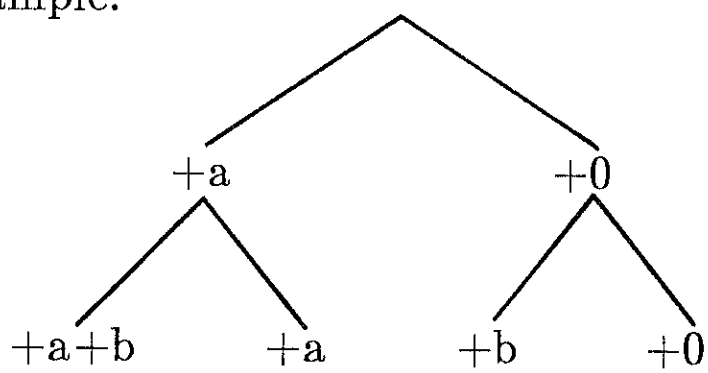
Lower bound:

For PM2I (all arithmetic mod N)

each i.f. executed adds 2^i (or $N-2^i$) if PE active
adds 0 if PE inactive

after y i.f.s executed, at most 2^y different integers added

Example:



For PM2I to add N−1 distinct integers: $\lceil \log_2(N-1) \rceil = m$ transfers

Lower bound = m

$PM2_{+i}(P) = P+2^i \bmod N \qquad 0 \leq i < m$

$PM2_{-i}(P) = P-2^i \bmod N \qquad 0 \leq i < m$

PM2I → Shuffle

Upper bound:

The Concept Underlying the PM2I → Shuffle Algorithm *(N = 8)*

Origin PE Number	*Distance Moved by Shuffle*	*Distance Moved by PM2I*				*Total*
0 = 000	+0	-	-	-	-	+0
1 = 001	+1	+1	-	-	-	+1
2 = 010	+2	-	+2	-	-	+2
3 = 011	+3	+1	+2	-	-	+3
4 = 100	+5	-	-	+4	+1	+5
5 = 101	+6	+1	-	+4	+1	+6
6 = 110	+7	-	+2	+4	+1	+7
7 = 111	+0	+1	+2	+4	+1	+0
		$PM2_{+0}$	$PM2_{+1}$	$PM2_{+2}$	$PM2_{+0}$	

Note: The distance the data item in the DTR of each PE is moved by the shuffle and by the PM2I network simulating the shuffle is shown.

1. Data originally in DTR of PE P moved

   by $PM2_{+i}$ if $p_i = 1$, $i = 0,1,\ldots,m-1$

2. Data originally in DTR of PE P where $p_{m-1} = 1$

   moved by $PM2_{+0}$

$$PM2_{+i}(P) = P+2^i \bmod N \qquad 0 \leq i < m$$

$$\text{shuffle}\ (p_{m-1}\cdots p_1 p_0) = p_{m-2}p_{m-3}\cdots p_1 p_0 p_{m-1}$$

PM2I → Shuffle

Upper bound:

(S1) $A \leftarrow$ DTR $[X^{m-1}0]$
(S2) $PM2_{+0}$ $[X^{m-1}1]$
(S3) for j = 1 until m−1 do
(S4) $\quad A \xleftrightarrow{swap}$ DTR $[X^{m-j-1}1X^{j-1}0]$
(S5) $\quad PM2_{+j}$ $[X^{m-1}0]$
(S6) $PM2_{+0}$ $[X^{m-1}0]$
(S7) DTR $\leftarrow A$ $[X^{m-1}0]$

PE (short hand)

Ex. N = 8, m = 3, DTR PE 3 → DTR 6

(S1) [XX0] {no match}
(S2) $PM2_{+0}$ [XX1] {DTR 3 → DTR 4}
(S4) (j = 1) [X10] {no match}
(S5) (j = 1) $PM2_{+1}$ [XX0] {DTR 4 → DTR 6}
(S4) (j = 2) A ⟷ DTR [1X0] {DTR 6 → A 6}
(S5) (j = 2) [XX0] {data in A register}
(S6) [XX0] {data in A register}
(S7) DTR ← A [XX0] {A 6 → DTR 6}

Algorithm uses m+1 inter-PE transfers and
m+1 register-to-register moves

Upper bound = m+1

$PM2_{+i}(P) = P + 2^i \bmod N \qquad 0 \leq i < m$

$\text{shuffle}\ (p_{m-1} \cdots p_1 p_0) = p_{m-2} p_{m-3} \cdots p_1 p_0 p_{m-1}$

PM2I → Shuffle

Upper bound:

(S1) $A \leftarrow DTR\ [X^{m-1}0]$
(S2) $PM2_{+0}\ [X^{m-1}1]$
(S3) for $j = 1$ until $m-1$ do
(S4) $\quad A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$
(S5) $\quad PM2_{+j}\ [X^{m-1}0]$
(S6) $PM2_{+0}\ [X^{m-1}0]$
(S7) $DTR \leftarrow A\ [X^{m-1}0]$

PM2I → Shuffle Simulation
(N = 8)

PE	Initial DTR Contents	[XX0] S1 A Contents	[XX1] S2 DTR Contents	[X10] S4 j = 1 A Contents	[X10] S4 j = 1 DTR Contents	[XX0] S5 j = 1 DTR Contents	[1X0] S4 j = 2 A Contents	[1X0] S4 j = 2 DTR Contents	[XX0] S5 j = 2 DTR Contents	[XX0] S6 DTR Contents	[XX0] S7 DTR Contents
000	000	000	111	000	111	110	000	110	100	100	000
001	001	—	—	—	—	—	—	—	—	100	100
010	010	010	001	001	010	111	001	111	101	101	001
011	011	—	—	—	—	—	—	—	—	101	101
100	100	100	011	100	011	010	010	100	110	110	010
101	101	—	—	—	—	—	—	—	—	110	110
110	110	110	101	101	110	011	011	101	111	111	011
111	111	—	—	—	—	—	—	—	—	111	111

Note: It is assumed that initially the DTR of PE P contains the integer P, $0 \leq P < 8$.

Proof by induction:

Induction hypothesis:

Sum of the w consecutive numbers beginning with 1:
$\text{Sum}(w) = w(w+1)/2$

Ex. $\text{Sum}(4) =$
$1 + 2 + 3 + 4 = 10$
$4(4+1)/2 = 10$

Induction variable: w

Basis: $w = 1$.
$$w(w+1)/2 = 1(1+1)/2 = 1 = \text{Sum}(1)$$

Induction step:

Assume true for $w = y$, $\quad 1 \leq y$.

Show true for $w = y+1$.

$$w(w+1)/2 = (y+1)((y+1)+1)/2 =$$

$$(y((y+1)+1) + 1((y+1)+1))/2 =$$

$$(y^2+2y+y+2)/2 =$$

$$((y^2+y)/2) + (2y+2)/2 =$$

$$(y(y+1)/2) + (y+1) =$$

$$\text{Sum}(y) + (y+1) = \text{Sum}(y+1)$$

Notation used in algorithm correctness proof and throughout rest of course:

$$q_{a/b} = q_a q_{a-1} \ldots q_{b+1} q_b \quad a \geq b$$

$$q_{m-1/j+1} * 2^{j+1} + q_{j/0} = q_{m-1/j+1} 0^{j+1} + q_{j/0} = q_{m-1/0}$$

$$q_{m-1/j+1} * 2^{j+1} + q_{j/0} * 2 = q_{m-1/j+1} 0^{j+1} + q_{j/0} 0$$

When j = m−1:

$$q_{m-1/j+1} * 2^{j+1} + q_{j/0} * 2 = q_{m-1/0} * 2$$

Ex. $p_3 p_2 p_1 p_0 = 0111$, $m = 4$, $j = 1$

$$p_{m-1/j+1} * 2^{j+1} + p_{j/0} * 2 =$$

$$p_{3/2} * 2^2 + p_{1/0} * 2 =$$

$$p_3 p_2 00 + p_1 p_0 0 =$$

$$0100 + 110 =$$

$$1010$$

PM2I → Shuffle

$Q: q_2q_1q_0 \rightarrow q_2q_1 0 + q_0{*}2$

$\rightarrow q_2 0 0 + q_1q_0{*}2$

$\rightarrow q_2q_1q_0{*}2 = 2Q$

Upper bound:

(S1) A ← DTR $[X^{m-1}0]$
(S2) $PM2_{+0}$ $[X^{m-1}1]$
(S3) for j = 1 until m−1 do
(S4) A ←→ DTR $[X^{m-j-1}1X^{j-1}0]$
(S5) $PM2_{+j}$ $[X^{m-1}0]$
(S6) $PM2_{+0}$ $[X^{m-1}0]$
(S7) DTR ← A $[X^{m-1}0]$

Recall: $PM2_{+i}(P) = P + 2^i \bmod N \quad 0 \le i < m$

Correctness proof: Assume mod N arithmetic

Induction hypothesis: after S2 ($j = 0$) or S5 ($1 \le j < m$)
DTR of $Q = q_{m-1/0} \rightarrow q_{m-1/j+1}{*}2^{j+1} + q_{j/0}{*}2 = P$
A reg. if $q_j = 0$, DTR if $q_j = 1$

Recall: $q_{m-1/j+1} = q_{m-1} \cdots q_{j+2}q_{j+1}$ and

$$q_{m-1/j+1} * 2^{j+1} = q_{m-1/j+1}0^{j+1}$$

Basis:
$j = 0$.

Case 1:
DTR of $q_{m-1/1}0$ (q_0=0). S1: to A reg.
$q_{m-1/1}0 = q_{m-1/1}{*}2^1 + q_0{*}2$.

Case 2:
DTR of $q_{m-1/1}1$ (q_0=1). S2: to DTR of $q_{m-1/1}1 + 1$
$= q_{m-1/1}{*}2 + 2 = q_{m-1/1}{*}2 + q_0{*}2$.

PM2I $\rightarrow$ Shuffle

Upper bound:

(S1) $A \leftarrow DTR\ [X^{m-1}0]$
(S2) $PM2_{+0}\ [X^{m-1}1]$
(S3) for $j = 1$ until $m-1$ do
(S4) $\quad A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$
(S5) $\quad PM2_{+j}\ [X^{m-1}0]$
(S6) $PM2_{+0}\ [X^{m-1}0]$
(S7) $DTR \leftarrow A\ [X^{m-1}0]$

Induction hypothesis: after S2 ($j = 0$) or S5 ($1 \leq j < m$)
DTR of $Q = q_{m-1/0} \rightarrow q_{m-1/j+1}{}^{*}2^{j+1} + q_{j/0}{}^{*}2 = P$
A reg. if $q_j = 0$, DTR if $q_j = 1$

Induction step: Assume true for $j = (k-1)$, show true for $j = k$

Case 1: DTR of $q_{m-1/0}$ with $q_{k-1}=0$. From induction hypothesis, when $j = (k-1)$, in A reg. of $P = q_{m-1/k}{}^{*}2^k + q_{k-1/0}{}^{*}2$. Now consider $j=k$.

(a) $p_k=1$. S4: to DTR of P. S5: to DTR of $P+2^k$.
Since $q_{k-1}=0$ and $p_k=1$, $q_k=1$. ($p_k = q_k + q_{k-1}$)
$P+2^k=(q_{m-1/k+1}1)^{*}2^k + q_{k-1/0}{}^{*}2+2^k$
$=q_{m-1/k+1}{}^{*}2^{k+1} + q_{k-1/0}{}^{*}2+2^{*}2^k$
$=q_{m-1/k+1}{}^{*}2^{k+1} + q_{k/0}{}^{*}2$

(b) $p_k=0$. S4 and S5: no change.
Since $q_{k-1}=0$ and $p_k=0$, $q_k=0$. ($p_k = q_k + q_{k-1}$)
$P= (q_{m-1/k+1}0)^{*}2^k + q_{k-1/0}{}^{*}2$
$= q_{m-1/k+1}{}^{*}2^{k+1} + q_{k/0}{}^{*}2$

PM2I → Shuffle

Upper bound:

(S1) $A \leftarrow DTR\ [X^{m-1}0]$
(S2) $PM2_{+0}\ [X^{m-1}1]$
(S3) for j = 1 until m−1 do
(S4) $\quad A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$
(S5) $\quad PM2_{+j}\ [X^{m-1}0]$
(S6) $PM2_{+0}\ [X^{m-1}0]$
(S7) $DTR \leftarrow A\ [X^{m-1}0]$

Induction hypothesis: after S2 ($j = 0$) or S5 ($1 \leq j \leq m$)
DTR of $Q = q_{m-1/0} \rightarrow q_{m-1/j+1}{}^{*}2^{j+1} + q_{j/0}{}^{*}2 = P$
A reg. if $q_j = 0$, DTR if $q_j = 1$

Case 2:
DTR of $q_{m-1/0}$ with q_{k-1}=1.
Proof technique similar to Case 1.

Upper bound:

(S1) $A \leftarrow \mathrm{DTR}\ [X^{m-1}0]$
(S2) $\mathrm{PM2}_{+0}\ [X^{m-1}1]$
(S3) for $j = 1$ until $m-1$ do
(S4) $\quad A \longleftrightarrow \mathrm{DTR}\ [X^{m-j-1}1X^{j-1}0]$
(S5) $\quad \mathrm{PM2}_{+j}\ [X^{m-1}0]$
(S6) $\mathrm{PM2}_{+0}\ [X^{m-1}0]$
(S7) $\mathrm{DTR} \leftarrow A\ [X^{m-1}0]$

Induction hypothesis: after S2 ($j = 0$) or S5 ($1 \leq j \leq m$)
$\quad$ DTR of $Q = q_{m-1/0} \rightarrow q_{m-1/j+1}*2^{j+1} + q_{j/0}*2 = P$
$\quad$ A reg. if $q_j = 0$, $\quad$ DTR if $q_j = 1$

When $j = m-1$, DTR of $Q \rightarrow Q*2$, A if $q_{m-1}=0$, DTR if $q_{m-1}=1$.

Case 1: DTR data ($q_{m-1}=1$)
S6:DTR of $Q*2 \rightarrow Q*2 + 1$
$\quad q_{m-1}=1 \rightarrow \mathrm{shuffle}(Q) = 2*Q + 1$
S7:no change

Case 2: A data ($q_{m-1}=0$)
S6:no change
S7:A reg. of $Q*2 \rightarrow$ DTR of $Q*2$
$\quad q_{m-1} = 0 \rightarrow \mathrm{shuffle}(Q) = 2*Q$

Q.E.D.

$\mathrm{PM2}_{+i}(P) = P+2^i \bmod N \qquad 0 \leq i < m$

$\mathrm{shuffle}\ (p_{m-1}\cdots p_1 p_0) = p_{m-2}p_{m-3}\cdots p_1 p_0 p_{m-1}$

Cube → PM2I

$PM2_{\pm(m-1)}$:

$$PM2_{\pm(m-1)}(p_{m-1/0}) = \overline{p}_{m-1}p_{m-2/0} = cube_{m-1}(p_{m-1/0})$$

$\therefore$ $PM2_{\pm(m-1)} = cube_{m-1}$ (did this for PM2I → Cube)
lower bound = upper bound = 1

$$PM2_{+i}(P) = P+2^i \bmod N \qquad 0 \le i < m$$

$$PM2_{-i}(P) = P-2^i \bmod N \qquad 0 \le i < m$$

$$cube_i(p_{m-1}\cdots p_1 p_0) = p_{m-1}\cdots p_{i+1}\overline{p}_i p_{i-1}\cdots p_0 \qquad 0\le i<m$$

Cube → PM2I

Lower bound ($0 \leq i < m-1$):

Hamming distance $H(a,b) = \#$ bit positions a and b differ

$H(a, cube_i(a)) = 1$

$H(1^m, PM2_{+i}(1^m)) = H(1^m, 0^{m-i}1^i) = m-i$

$H(0^m, PM2_{-i}(0^m)) = H(0^m, 1^{m-i}0^i) = m-i$

Cube needs $m-i$ transfers to do $PM2_{\pm i}$

When $i = 0$, need m transfers

$\therefore$ lower bound = m

$PM2_{+i}(P) = P+2^i \bmod N \qquad 0 \leq i < m$

$PM2_{-i}(P) = P-2^i \bmod N \qquad 0 \leq i < m$

$cube_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \bar{p}_i p_{i-1} \cdots p_0 \qquad 0 \leq i < m$

Cube → PM2I

Upper bound $(0 \leq i < m-1)$:

For $PM2_{+i}$ ($PM2_{-i}$ is similar)

(S1) for j = m−1 step −1 to i do
cube$_j$ $[X^{m-j}1^{j-i}X^i]$

Ex. i=1, N=16, m=4, DTR PE 3 to $PM2_{+i}(3) = 5$

j=3:	$cube_3$	[X11X]	no change	(3 = 0011)
j=2:	$cube_2$	[XX1X]	3 → 7	(0011 → 0111)
j=1:	$cube_1$	[XXXX]	7 → 5	(0111 → 0101)

Algorithm uses m−i inter-PE transfers,
m when i=0.

Upper bound = m

$$cube_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \overline{p}_i p_{i-1} \cdots p_0 \quad 0 \leq i < m$$

$$PM2_{+i}(P) = P + 2^i \bmod N \qquad 0 \leq i < m$$

$$PM2_{-i}(P) = P - 2^i \bmod N \qquad 0 \leq i < m$$

Cube $\rightarrow$ PM2I

Upper bound $(0 \leq i < m-1)$:

for $j = m-1$ step -1 to i do

  cube$_j$ $[X^{m-j} 1 j-i X^i]$

superscript

Correctness proof: $P = p_{m-1/0}$ where $p_{k+i-1/i} = \underline{1^k}$

k 1's

1111 etc.

Case 1: $p_i = 0$ $(k = 0)$.

No match for $m-1 \geq j > i$.

When $j=i$, S1: $P = p_{m-1/i+1} 0 p_{i-1/0} \rightarrow p_{m-1/i+1} 1 p_{i-1/0}$
$= PM2_{+i}(P)$

Case 2: $p_i = 1$ $(0 < k \leq m-i)$.

$$P = p_{m-1/i+k+1} 0\, 1^k P_{i-1/0}$$

lowercase

k 1's

No match for $m-1 \geq j \geq i+k+1$

When $i+k \geq j \geq i$, match, execute cube$_j$.

$$P = p_{m-1/i+k+1} 0\, 1^k p_{i-1/0} \rightarrow p_{m-1/i+k+1} \bar{0}\, \bar{1}^k p_{i-1/0}$$
$$= p_{m-1/i+k+1} 1\, 0^k p_{i-1/0} = PM2_{+i}(P)$$

$\text{cube}_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \bar{p}_i p_{i-1} \cdots p_0 \quad 0 \leq i < m$

$PM2_{+i}(P) = P + 2^i \bmod N \qquad 0 \leq i < m$

$PM2_{-i}(P) = P - 2^i \bmod N \qquad 0 \leq i < m$

Cube $\rightarrow$ Illiac

Follows from Cube $\rightarrow$ PM2I and PM2I $\rightarrow$ Illiac.

lower bound = upper bound = m

(to simulate $\text{Illiac}_{\pm 1}$)

Cube $\rightarrow$ Shuffle-Exchange

Exchange:

$$\text{exchange}\,(p_{m-1/0}) = p_{m-1/1}\,\overline{p}_0 = \text{cube}_0(p_{m-1/0})$$

$$\text{lower bound} = \text{upper bound} = 1$$

Cube → Shuffle

Lower bound:

$$\text{shuffle}(0^{m-(j+1)}1\,0^{j}) = 0^{m-(j+2)}1\,0^{j+1}$$

Value of bit position j changes.

Occurs $\forall$ j when shuffle executed.

The shuffle changes all m bit positions in set of PE addresses.

Cube can only change one at a time.

$\therefore$ Must take Cube at least m inter-PE transfers.

Lower bound = m

$$\text{shuffle}(p_{m-1}\cdots p_1 p_0) = p_{m-2}p_{m-3}\cdots p_1 p_0 p_{m-1}$$

$$\text{cube}_i(p_{m-1}\cdots p_1 p_0) = p_{m-1}\cdots p_{i+1}\bar{p}_i p_{i-1}\cdots p_0 \quad 0 \leq i < m$$

Cube $\rightarrow$ Shuffle

(S1) where ADDR(m−1) = ADDR(0)
    do A $\leftarrow$ DTR $[X^m]$
    elsewhere cube_0 $[X^m]$
(S2) for j = 1 to m−1 do
(S3)   where ADDR(j) $\neq$ ADDR(j−1)
        do A $\longleftrightarrow$ DTR $[X^m]$
(S4) cube_j $[X^m]$

Correctness proof: After S1 DTR from PE $P = p_{m-1/0} \rightarrow p_{m-1/1}p_{m-1}$; in A if $p_{m-1} = p_0$, in DTR if $p_{m-1} \neq p_0$. All data in PEs where ADDR(m−1) = ADDR(0).

Induction hypothesis: after cube_j in S4, DTR $P \rightarrow p_{m-1/j+1}p_{j-1/0}p_{m-1}$ (lo-order j+1 bits shuffled); in A if $p_j = p_{j-1}$, in DTR if $p_j \neq p_{j-1}$. , (ADDR (m−1) = ADDR (0))

Basis: j = 1.

Case 1: A and DTR of PE where ADDR(1) = ADDR(0). $(P1 = P_{m-1})$
(a) A: from P, where $p_{m-1} = p_0$ (S1). $p_{m-1/1}p_{m-1} = p_{m-1/2}p_0p_{m-1}$ since p_1 = ADDR(1) = ADDR(0) = $p_{m-1} = p_0$.
(b) DTR: from P, where $p_{m-1} \neq p_0$ (S1).
(S4) DTR $p_{m-1/1}p_{m-1} \rightarrow$
DTR $p_{m-1/2}\overline{p}_1p_{m-1} = p_{m-1/2}p_0p_{m-1}$ since
$\overline{p}_1 = \overline{\text{ADDR}(1)} = \overline{\text{ADDR}(0)} = \overline{p}_{m-1} = p_0$.

Case 2: A and DTR of PE where ADDR(1) $\neq$ ADDR(0). Proof technique similar to Case 1.

Cube → Shuffle

(S2) for j = 1 to m−1 do
(S3) where ADDR(j) $\neq$ ADDR(j−1)
do A $\longleftrightarrow$ DTR $[X^m]$
(S4) cube_j $[X^m]$

Induction hypothesis: after cube_j in S4, DTR P → $p_{m-1/j+1}p_{j-1/0}p_{m-1}$ (lo-order j+1 bits shuffled); in A if $p_j = p_{j-1}$, in DTR if $p_j \neq p_{j-1}$. ADDR(m−1) = ADDR(1)

$i \leq j \leq m-1$

Induction step: Assume true for j=i, show true for j = i+1

Case 2: A and DTR of PE where ADDR(i+1) $\neq$ ADDR(i).

(a) A from P, where $p_i = p_{i-1}$, in A of $p_{m-1/i+1}p_{i-1/0}p_{m-1}$ = P′ (I.h.). When j = i+1, (S3) → DTR of P′, (S4) → DTR of $p_{m-1/i+2}\overline{p}_{i+1}p_{i-1/0}p_{m-1}$ = $p_{m-1/i+2}p_{i/0}p_{m-1}$ since $\overline{p}_{i+1} = \overline{\text{ADDR(i+1)}}$ = ADDR(i) = $p_{i-1} = p_i$.

(b) DTR from P, where $p_i \neq p_{i-1}$, in DTR of $p_{m-1/i+1}p_{i-1/0}p_{m-1}$ = P′ (I.h.). When j = i+1, (S3) → A of P′ = $p_{m-1/i+1}p_{i-1/0}p_{m-1} = p_{m-1/i+2}p_{i/0}p_{m-1}$

since p_{i+1} = ADDR(i+1) = $\overline{\text{ADDR(i)}}$ = $\overline{p}_{i-1} = p_i$.

Case 1: A and DTR of PE where ADDR(i+1) = ADDR(i). Proof technique similar to Case 2.

Cube $\rightarrow$ Shuffle

(S2) for j = 1 to m−1 do
(S3) where ADDR(j) $\neq$ ADDR(j−1)
do A $\longleftrightarrow$ DTR $[X^m]$
(S4) cube_j $[X^m]$
(S5) where ADDR(m−1) = ADDR(0)
do DTR $\leftarrow$ A $[X^m]$

Induction hypothesis: after cube_j in S4, DTR P $\rightarrow$ $p_{m-1/j+1}p_{j-1/0}p_{m-1}$ (lo-order j+1 bits shuffled); in A if $p_j = p_{j-1}$, in DTR if $p_j \neq p_{j-1}$; for $1 \le j \le m-2$

When j = m−2: DTR PE P = $p_{m-1/0}$ $\rightarrow p_{m-1}p_{m-3/0}p_{m-1}$; in A if $p_{m-2} = p_{m-3}$, in DTR if $p_{m-2} \neq p_{m-3}$.

j = m−1:

Case 1: A and DTR of PE where ADDR(m−1) = ADDR(m−2). (still considering addr(m−1) = addr(0))

(a) A: from P, where $p_{m-2} = p_{m-3}$, in A of $p_{m-1}p_{m-3/0}p_{m-1} = P'$. (S5) $\rightarrow$ DTR of P' = $p_{m-1}p_{m-3/0}p_{m-1} = p_{m-2/0}p_{m-1}$ since p_{m-1} = ADDR(m−1) = ADDR(m−2) = $p_{m-3} = p_{m-2}$.

(b) DTR: from P, where $p_{m-2} \neq p_{m-3}$, in DTR of $p_{m-1}p_{m-3/0}p_{m-1}$. (S4) $\rightarrow \overline{p}_{m-1}p_{m-3/0}p_{m-1}$ = $p_{m-2/0}p_{m-1}$ since $\overline{p}_{m-1} = \overline{\text{ADDR(m−1)}}$ = $\overline{\text{ADDR(m−2)}} = \overline{p}_{m-3} = p_{m-2}$.

Cube $\rightarrow$ Shuffle

(S2) for j = 1 to m−1 do
(S3) where ADDR(j) $\neq$ ADDR(j−1)
do A $\longleftrightarrow$ DTR [X^m]
(S4) cube_j [X^m]
(S5) where ADDR(m−1) = ADDR(0)
do DTR $\leftarrow$ A [X^m]

Induction hypothesis: after cube_j in S4, DTR P $\rightarrow$ $p_{m-1/j+1} p_{j-1/0} p_{m-1}$ (lo-order j+1 bits shuffled); in A if $p_j = p_{j-1}$, in DTR if $p_j \neq p_{j-1}$.

When j = m−2: DTR PE P = $p_{m-1/0}$ $\rightarrow p_{m-1} p_{m-3/0} p_{m-1}$; in A if $p_{m-2} = p_{m-3}$, in DTR if $p_{m-2} \neq p_{m-3}$.

j = m−1:

Case 2: A and DTR of PE where ADDR(m−1) $\neq$ ADDR(m−2).
Proof technique similar to Case 1.

Cube → Shuffle

Upper bound:

(S1) where ADDR(m−1) = ADDR(0)
    do A ← DTR $[X^m]$
    elsewhere $cube_0$ $[X^m]$
(S2) for j = 1 to m−1 do
(S3)   where ADDR(j) ≠ ADDR(j−1)
        do A ←→ DTR $[X^m]$
(S4) $cube_j$ $[X^m]$
(S5) where ADDR(m−1) = ADDR(0)
    do DTR ← A $[X^m]$

Ex. N = 8, m = 3, DTR PE 3 → DTR 6

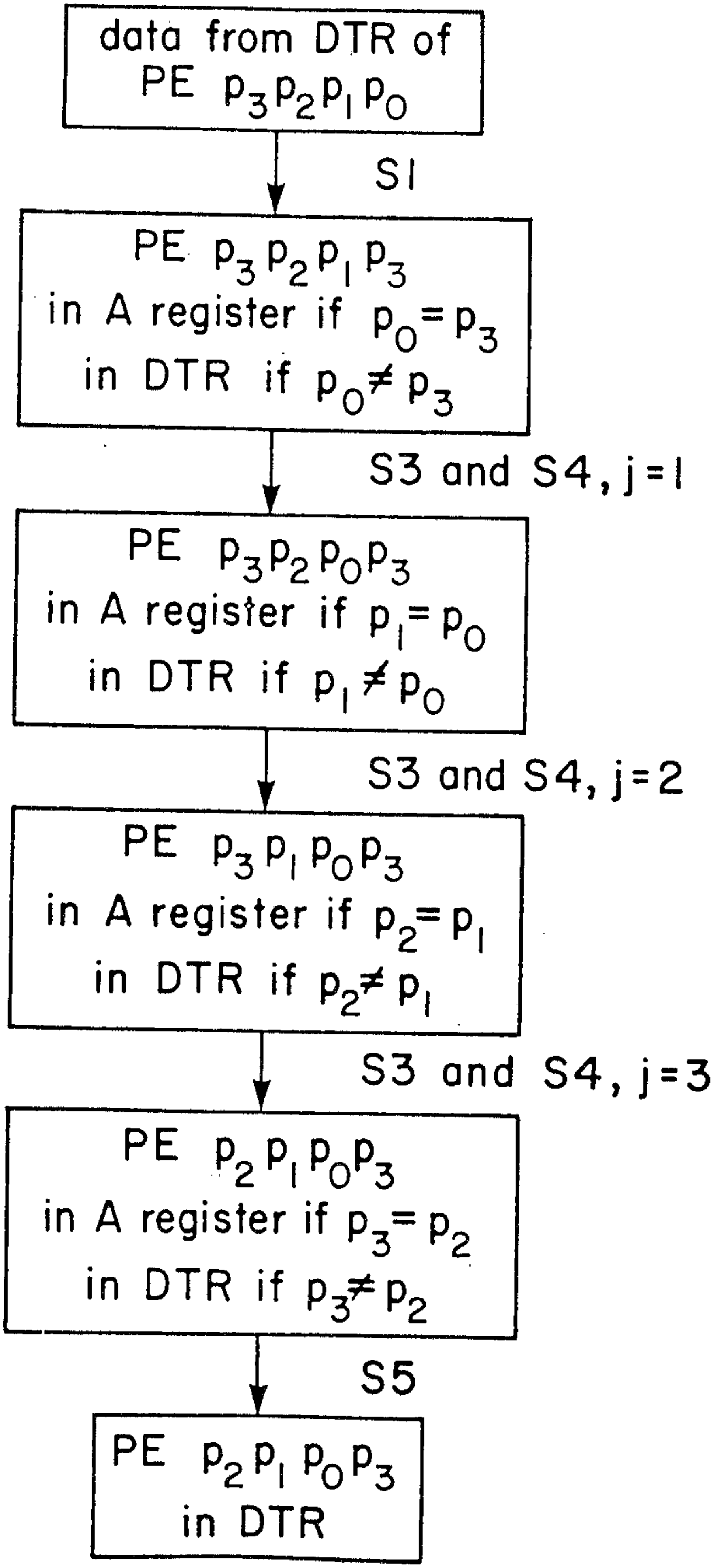
(S1) elsewhere $cube_0$ [XXX] {DTR 3 → DTR 2}
(S3) (j = 1) A ←→ DTR [XXX] {DTR 2 → A 2}
(S4) (j = 1) $cube_1$ [XXX] {data in A register}
(S3) (j = 2) A ←→ DTR [XXX] {A 2 → DTR 2}
(S4) (j = 2) $cube_2$ [XXX] {DTR 2 → DTR 6}

Algorithm uses m inter-PE transfers,
m+1 "where" statements, and
m+1 register-to-register moves

Upper bound = m

$$\text{shuffle}(p_{m-1} \cdots p_1 p_0) = p_{m-2} p_{m-3} \cdots p_1 p_0 p_{m-1}$$

$$\text{cube}_i(p_{m-1} \cdots p_1 p_0) = p_{m-1} \cdots p_{i+1} \overline{p}_i p_{i-1} \cdots p_0 \quad 0 \leq i < m$$



data from DTR of PE $p_3 p_2 p_1 p_0$
S1
PE $p_3 p_2 p_1 p_3$ in A register if $p_0 = p_3$ in DTR if $p_0 \neq p_3$
S3 and S4, j=1
PE $p_3 p_2 p_0 p_3$ in A register if $p_1 = p_0$ in DTR if $p_1 \neq p_0$
S3 and S4, j=2
PE $p_3 p_1 p_0 p_3$ in A register if $p_2 = p_1$ in DTR if $p_2 \neq p_1$
S3 and S4, j=3
PE $p_2 p_1 p_0 p_3$ in A register if $p_3 = p_2$ in DTR if $p_3 \neq p_2$
S5
PE $p_2 p_1 p_0 p_3$ in DTR

Cube $\rightarrow$ Shuffle

Upper bound: (ADDR(i) = bit i of PE address)

(S1) where ADDR(m−1) = ADDR(0)
    do A $\leftarrow$ DTR $[X^m]$
    elsewhere cube_0 $[X^m]$
(S2) for j = 1 to m−1 do
(S3)   where ADDR(j) $\neq$ ADDR(j−1)
        do A $\longleftrightarrow$ DTR $[X^m]$
(S4) cube_j $[X^m]$
(S5) where ADDR(m−1) = ADDR(0)
    do DTR $\leftarrow$ A $[X^m]$

Cube → Shuffle Simulation
(N = 8)

PE	Initial DTR Contents	S1 A Contents	S1 DTR Contents	S3 j = 1 A Contents	S3 j = 1 DTR Contents	S4 j = 1 DTR Contents	S3 j = 2 A Contents	S3 j = 2 DTR Contents	S4 j = 2 DTR Contents	S5 DTR Contents
000	000	000	001	000	001	010	000	010	—	000
001	001	—	001	001	—	011	001	011	100	100
010	010	010	011	011	010	001	001	011	100	001
011	011	—	011	—	011	—	—	—	101	101
100	100	—	100	—	100	—	—	—	010	010
101	101	101	100	100	101	110	110	100	011	110
110	110	—	110	110	—	100	110	100	011	011
111	111	111	110	111	110	101	111	101	—	111

Note: It is assumed that initially the DTR of PE P contains the integer P, $0 \leq P < 8$.

Illiac $\rightarrow$ PM2I

Lower bound:

Let $d(a,b) = |a-b|$ and $j = (m/2)-1$.

$$2^j = n/2 \text{ and } d(0, PM2_{+j}(0)) = n/2.$$

$d(a, Illiac_{\pm n}(a)) = n$, so to move $n/2$ with $Illiac_{\pm n}$ need $1 + (n/2)$ steps

$d(a, Illiac_{\pm 1}(a)) = 1$, so $n/2$ steps required

Illiac $\rightarrow$ PM2I

Upper bound:

$PM2_{+i}$ for $0 \leq i < m/2$ ($PM2_{-i}$ similar):

(S1) for j = 1 to 2^i do Illiac_{+1} $[X^m]$

Ex. N = 16, m = 4, i = 1, DTR PE 0 $\rightarrow$ DTR 2

(S1) j = 1 [XXXX] {DTR 0 $\rightarrow$ DTR 1}

(S1) j = 2 [XXXX] {DTR 1 $\rightarrow$ DTR 2}

Algorithm uses 2^i transfers

n/2 transfers when i = (m/2)−1

Correctness proof:

Let $(\text{Illiac}_{+1})^j$ mean execute Illiac_{+1} j times

$(\text{Illiac}_{+1})^{2^i}(a) = a + 2^i \bmod N = PM2_{+i}(a)$

$PM2_{+(m/2)-1} = +2^{(m/2)-1} = +n/2$

Illiac → PM2I

Upper bound:

$PM2_{+i}$ for $m/2 \leq i < m$ ($PM2_{-i}$ similar):

(S1) for $j = 1$ to $2^i/n$ do Illiac_{+n} [X^m]

Ex. $N = 16$, $m = 4$, $n = 4$, $i = 3$,

DTR PE 0 → DTR 8

(S1) $j = 1$ [XXXXX] {DTR 0 → DTR 4}

S1
(~~S2~~) $j = 2$ [XXXXX] {DTR 4 → DTR 8}

Algorithm uses $2^i/n$ transfers

$n/2$ transfers when $i = m-1$

Correctness proof:

$$(\text{Illiac}_{+n})^{2^i/n}(a) = a + (n * (2^i/n)) \bmod N$$

$$= a + 2^i \bmod N = PM2_{+i}(a)$$

$PM2_{+m/2} = +2^{m/2} = +n$

Illiac $\rightarrow$ Cube

Lower bound:

Let $d(a,b) = |a-b|$ and $j = (m/2)-1$.

$2^j = 2^{(m/2)-1} = +n/2$

$d(0,\text{cube}_j(0)) = n/2$

Need at least $(n/2) + 1$ Illiac transfers

From Illiac $\rightarrow$ PM2I, only way $0 \rightarrow n/2$
in $< (n/2)+1$ steps is $(\text{Illiac}_{+1})^{n/2}$

$\text{cube}_j\ (n/2) = 0$

No subsequence of $(\text{Illiac}_{+1})^{n/2}$ can do.

$\therefore$ To do $\text{cube}_j(0)$ and $\text{cube}_j(n/2)$ need
at least $(n/2)+1$

Illiac $\rightarrow$ Cube

Upper bound:

For cube$_i$ $0 \le i \le (m/2)-2$

(S1) $A \leftarrow DTR\ [X^{m-(i+1)}1X^i]$

(S2) for $j = 1$ to 2^i do $\text{Illiac}_{+1}\ [X^m]$

(S3) $A \longleftrightarrow DTR\ [X^{m-(i+1)}1X^i]$

(S4) for $j = 1$ to 2^i do $\text{Illiac}_{-1}\ [X^m]$

(S5) $DTR \leftarrow A\ [X^{m-(i+1)}1X^i]$

Ex. $N = 16$, $m = 4$, $i = 0$, DTR PE 3 $\rightarrow$ DTR 2

(S1) $A \leftarrow DTR$ [XXX1] {DTR 3 $\rightarrow$ A 3}

(S2) {data in A register}

(S3) $DTR \leftarrow A$ [XXX1] {A 3 $\rightarrow$ DTR 3}

(S4) $(\text{Illiac}_{-1})^1$ {DTR 3 $\rightarrow$ DTR 2}

(S5) [XXX1] {no match}

Algorithm uses	$2*2^i$	inter-PE transfers
	$n/2$	inter-PE transfers when $i = (m/2)-2$
	3	register-to-register moves

For cube_i $0 \leq i \leq (m/2)-2$

(S1) $A \leftarrow DTR\ [X^{m-(i+1)}1X^i]$

(S2) for $j = 1$ to 2^i do $\text{Illiac}_{+1}\ [X^m]$

(S3) $A \longleftrightarrow DTR\ [X^{m-(i+1)}1X^i]$

(S4) for $j = 1$ to 2^i do $\text{Illiac}_{-1}\ [X^m]$

(S5) $DTR \leftarrow A\ [X^{m-(i+1)}1X^i]$

Correctness proof:

Case 1: DTR data originally in PE P, $p_i = 0$.

(S1) no match.
(S2) $P \longrightarrow P+2^i = \text{cube}_i(P)$
(S3) DTR of $\text{cube}_i(P) \longrightarrow$ A of $\text{cube}_i(P)$
(S4) no change
(S5) A of $\text{cube}_i(P) \longrightarrow$ DTR of $\text{cube}_i(P)$

Case 2: DTR data originally in PE P, $p_i = 1$

(S1) DTR of P $\longrightarrow$ A of P

(S2) no change

(S3) A of P $\longrightarrow$ DTR of P

(S4) $P \longrightarrow P-2^i = \text{cube}_i(P)$

(S5) no change

Illiac $\rightarrow$ Cube

Upper bound:

$\text{cube}_{(m/2)-1}$

(S1) for j = 1 to n/2 do Illiac_{+1} $[X^m]$

(S2) Illiac_{-n} $[X^{m/2}0X^{(m/2)-1}]$

Ex. N = 16, m = 4, n = 4, cube_1

DTR PE 3 to DTR 1

(S1) $(\text{Illiac}_{+1})^{4/2}$ [XXXX] {DTR 3 $\rightarrow$ DTR 5}

(S2) Illiac_{-4} [XX0X] {DTR 5 $\rightarrow$ DTR 1}

Algorithm uses (n/2)+1 inter-PE transfers

Correctness proof:

Follows from PM2I $\rightarrow$ Cube and Illiac $\rightarrow$ PM2I

Illiac $\rightarrow$ Cube

Upper bound:

For cube_i $m/2 \leq i \leq m-2$

(S1) $A \leftarrow DTR\ [X^{m-(i+1)}1X^i]$

(S2) for $j = 1$ to $2^i/n$ do $\text{Illiac}_{+n}\ [X^m]$

(S3) $A \longleftrightarrow DTR\ [X^{m-(i+1)}1X^i]$

(S4) for $j = 1$ to $2^i/n$ do $\text{Illiac}_{-n}\ [X^m]$

(S5) $DTR \leftarrow A\ [X^{m-(i+1)}1X^i]$

Ex. $N = 16$, $m = 4$, $i = 2$, DTR PE 3 $\rightarrow$ DTR 7

(S1) [X1XX] {no match}

(S2) $(\text{Illiac}_{+4})^1$ {DTR 3 $\rightarrow$ DTR 7}

(S3) A $\leftarrow$ DTR [X1XX] {DTR 7 $\rightarrow$ A 7}

(S4) {data in A register}

(S5) DTR $\leftarrow$ A [X1XX] {A7 $\rightarrow$ DTR 7}

Algorithm uses	$2*(2^i/n)$	inter-PE transfers
	$n/2$	inter-PE transfers when $i = m-2$
	3	register-to-register moves

Illiac $\rightarrow$ Cube

Upper bound:

For cube_{m-1}:

Use Illiac $\rightarrow$ PM2I since $\text{cube}_{m-1} \equiv \text{PM2}_{\pm(m-1)}$

Algorithm uses n/2 inter-PE transfers

$(\text{Illiac}_{+n})^{n/2} = +n*(n/2) = +N/2 = 2^{m-1}$

Illiac $\rightarrow$ Shuffle-Exchange

Exchange: exchange $=$ cube_0 so use Illiac $\rightarrow$ cube_0

lower bound $=$ upper bound $= 2$

Illiac $\rightarrow$ Shuffle

Lower bound: $2(n-2)$

Upper bound: use PM2I $\rightarrow$ shuffle and Illiac $\rightarrow$ PM2I

(S1$'$) $A \leftarrow DTR\ [X^{m-1}0]$

(S2$'$) $Illiac_{+1}\ [X^{m-1}1]\ \ \{PM2_{+0}\ [X^{m-1}1]\}$

(S3$'$) for $j = 1$ to $(m/2)-1$ do

(S4$'$) $\quad A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$

(S5$'$) $\quad$ for $i = 1$ to 2^j do $Illiac_{+1}\ [X^m]$

$$\{PM2_{+j}\ [X^{m-1}0]\ 1 \leq j < m/2\}$$

(S3$''$) for $j = m/2$ to $m-1$ do

(S4$''$) $\quad A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$

(S5$''$) $\quad$ for $i = 1$ to $2^j/n$ do $Illiac_{+n}\ [X^m]$

$$\{PM2_{+j}\ [X^{m-1}0]\ \ m/2 \leq j < m\}$$

(S6$'$) $Illiac_{+1}\ [X^{m-1}0]\ \{PM2_{+0}\ [X^{m-1}0]\}$

(S7$'$) $DTR \leftarrow A\ [X^{m-1}0]$

Illiac → Shuffle

(S2$'$) Illiac_{+1} $[X^{m-1}1]$ $\{PM2_{+0}\ [X^{m-1}1]\}$

(S3$'$) for j = 1 to (m/2)−1 do

(S4$'$) $A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$

(S5$'$) for i = 1 to 2^j do Illiac_{+1} $[X^m]$

$\{PM2_{+j}\ [X^{m-1}0]\ 0 \le j < m/2\}$

(S3$''$) for j = m/2 to m−1 do

(S4$''$) $A \longleftrightarrow DTR\ [X^{m-j-1}1X^{j-1}0]$

(S5$''$) for i = 1 to $2^j/n$ do Illiac_{+n} $[X^m]$

$\{PM2_{+j}\ [X^{m-1}0]\ \ m/2 \le j < m\}$

(S6$'$) Illiac_{+1} $[X^{m-1}0]$ $\{PM2_{+0}\ [X^{m-1}0]\}$

Algorithm complexity:

inter-PE transfers – S2$'$: 1, and S6$'$: 1.

S5$'$: $\sum_{j=1}^{(m/2)-1} 2^j = 2^{m/2}-2 = n-2$

S5$''$: $\sum_{j=m/2}^{m-1} 2^j/n = \sum_{j=m/2}^{m-1} 2^{j-(m/2)} = \sum_{i=0}^{m/2-1} 2^i = n-1$

2n−1 inter-PE data transfers (total)

n+1 register-to-register moves

Shuffle-Exchange $\rightarrow$ PM2I

Lower bound:

$PM2_{+i}(1^m) = 0^{m-i}1^i \qquad 0 \leq i < m$

$1^m \rightarrow 0^{m-i}1^i$

1 in or rotated to 0^{th} position, mapped to 0 (by exchange), then shuffled to $(m-1)^{st}$ position; need at least m−1 shuffles.

to change m−i 1's to 0's need at least m−i exchanges

need at least 2m−1−i transfers

for i = 0, need at least 2m−1 transfers neat!

$PM2_{-i}$ is similar

Shuffle-Exchange $\rightarrow$ PM2I

Upper bound:

$PM2_{+i}$ $0 \leq i < m$ ($PM2_{-i}$ similar)

(S1) for $j = m-1$ step -1 to i do

(S2) shuffle $[X^m]$

(S3) exchange $[1^{j-i}X^{m-(j-i)}]$

(S4) for $j = i-1$ step -1 to 0 do shuffle $[X^m]$

Ex. $N = 8$, $m = 3$, $i = 1$, DTR PE 3 $\rightarrow$ DTR 5

(S2) (j=2) shuffle [XXX] {DTR 3 $\rightarrow$ DTR 6}

(S3) (j=2) exchange [1XX] {DTR 6 $\rightarrow$ DTR 7}

(S2) (j=1) shuffle [XXX] {DTR 7 $\rightarrow$ DTR 7}

(S3) (j=1) exchange [XXX] {DTR 7 $\rightarrow$ DTR 6}

(S4) (j=0) shuffle [XXX] {DTR 6 $\rightarrow$ DTR 5}

Algorithm uses m shuffles

m$-$i exchanges

Total of 2m inter-PE transfers when $i = 0$

Shuffle-Exchange $\rightarrow$ PM2I

Upper bound:

$PM2_{+i}$ $0 \leq i < m$ ($PM2_{-i}$ similar)

(S1) for $j = m-1$ step -1 to i do

(S2) shuffle $[X^m]$

(S3) exchange $[1^{j-i}X^{m-(j-i)}]$

(S4) for $j = i-1$ step -1 to 0 do shuffle $[X^m]$

Correctness proof:

Induction hypothesis: after S2 and S3 executed,

$P = p_{m-1/0}$ is mapped to

$p_{j-1/i}p'_{i-1/0}p'_{m-1/j}$ where $p'_{m-1/0} = P + 2^i \bmod N$

Prove by induction on j

When $j = i$, $P \rightarrow p'_{i-1/0}p'_{m-1/i}$

S4: $p'_{i-1/0}p'_{m-1/i} \rightarrow p'_{m-1/0} = PM2_{+i}(P)$

Shuffle-Exchange → Cube

Lower bound:

$\text{cube}_{m-1}(10^{m-3}11) = 0^{m-2}11$

- need at least one exchange
- need at least one shuffle to move 1 out of p_{m-1} position
- must have at least m−1 shuffles or else 1 in p_1 position will move into $p_{m-1/2}$ position
- 1 exchange and m−1 shuffles not enough
  - $\text{shuffle}^{m-1}(\text{exchange}(10^{m-3}11)) = 010^{m-3}1$
  - exchange not first, 1 in p_0 position moved to p_{m-1} position by shuffles
- ∴ need at least m+1 transfers

Shuffle-Exchange $\rightarrow$ Cube

Upper bound:

$\text{cube}_0 = \text{exchange}$

$\text{cube}_i \; 0 < i < m$:

(S1) for $j = 1$ to $m-i$ do shuffle $[X^m]$

(S2) exchange $[X^m]$

(S3) for $j = 1$ to i do shuffle $[X^m]$

Ex. $N = 8$, $m = 3$, $i = 1$, DTR PE 3 $\rightarrow$ DTR 1

(S1) (j=1) shuffle [XXX] {DTR 3 $\rightarrow$ DTR 6}

(S1) (j=2) shuffle [XXX] {DTR 6 $\rightarrow$ DTR 5}

(S2) exchange [XXX] {DTR 5 $\rightarrow$DTR 4}

(S3) (j=1) shuffle [XXX] {DTR 4 $\rightarrow$ DTR 1}

Algorithm uses m+1 inter-PE transfers

Shuffle-Exchange $\rightarrow$ Cube

Upper bound:

cube_i $0 < i < m$:

(S1) for $j = 1$ to $m-i$ do shuffle $[X^m]$

(S2) exchange $[X^m]$

(S3) for $j = 1$ to i do shuffle $[X^m]$

Correctness proof:

$$\text{shuffle}^{m-i}(p_{m-1/0}) = p_{i-1/0}p_{m-1/i}$$

$$\text{exchange}(p_{i-1/0}p_{m-1/i}) = p_{i-1/0}p_{m-1/i+1}\overline{p}_i$$

$$\begin{aligned}\text{shuffle}^{i}(p_{i-1/0}p_{m-1/i+1}\overline{p}_i) &= p_{m-1/i+1}\overline{p}_i p_{i-1/0} \\ &= \text{cube}_i(p_{m-1/0})\end{aligned}$$

Shuffle-Exchange $\rightarrow$ Illiac

Follows from Shuffle-Exchange $\rightarrow$ PM2I and

PM2I $\rightarrow$ Illiac

Lower bound $= 2m-1$

Upper bound $= 2m$