MATH5210 Analysis
Assignment 4
Monotone Convergence Theorem
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1

Let $\{a_n\}$ be a sequence of non-negative real numbers. Show that the partial sums for the infinite series $\sum_{n=1}^{\infty} a_n$ form a monotone series.

2

What is the approximation property for the infimum of a set $\mathbf{A}$? If $m = \inf(\mathbf{A})$, prove that there is a sequence $a_n \in \mathbf{A}$ with $\lim a_n = m$.

The approximation property for the infimum of a set $\mathbf{A}$ states
Suppose that $m = \inf(\mathbf{A})$
Then for any $\epsilon > 0, \exists x \in \mathbf{A}$ s.t. $m \leq x \leq m + \epsilon$

Since $m = \inf(\mathbf{A})$, $\{a_n\}$ is bounded below and has a greatest upper bound (gub) s.t. $\text{gub}(\{a_n\}) = m$. We will prove will prove that $\lim_{n\to\infty} a_n = m$.
Let $\epsilon > 0$ be given. We have to find N s.t. $|a_n - m| < \epsilon$ and $a_n - m$ is always positive because $a_n > m$. So $a_n - m < \epsilon$ for all $n > N$.
Then by the approximation property of the infimum of $\mathbf{A}$, there exists an $x \in \{a_n\}$ s.t. $m \leq x < m + \epsilon$ where $x = a_N$. So

$$a_N \leq m + \epsilon \tag{1}$$

Finally if $n > N$, then because of monotone decreasing

$$a_N < a_n \tag{2}$$

Combining 1 and 2 gives us $m + \epsilon > a_N > a_n$. Thus $a_n - m < \epsilon$ for all $n > N$. ■

3

Give a detailed proof of the MCT in the case of a monotone decreasing sequence which is bounded below. Follow the proof given in class.

4

If $0 < r < 1$, show that $\lim_{n\to\infty} r^n = 0$. Note that with $a_n = r^n$, $a_{n+1} = ra_n$. What happens if $0 \leq r \leq 1$?

5

Let $a_1 \in (0, 1)$ Prove that $a_{n+1} = \sqrt{a_n + 1} - 1$ is a increasing sequence which converges to 0. What happens if $a_0 = 0$, if $a_0 = -1$?

6

Suppose $a_1 \geq 2$ and $a_{n+1} = 2 + \sqrt{a_n - 2}$. Show that a_n converges to 2 or 3. How does the limit depend upon the value of a_1

7

(Bonus) Let $0 < b_1 < a_1$ and let

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

a

Show that $0 < b_n < a_n$

b

Show that b_n is increasing and bounded above.

c

Show that a_n is decreasing and bounded below.

d

Show that $0 < a_{n+1} - b_{n+1} < (a_1 - b_1)/2^n$

e

Conclude that the sequences a_n and b_n converge to a common limit.