MATH5210 Analysis
Assignment 7
Uniform Continuity
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For Problems 1 to 6, give $\mathcal{E} - \delta$ proofs of uniform continuity.

1

Claim: $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[2, 4]$.

Proof: Let $\mathcal{E} > 0$ be given. Then there exists $\delta > 0$ s.t. for any $x, y \in [2, 4]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \mathcal{E}$$

Choose $\delta = \frac{\mathcal{E}}{10}$. Then

$$|f(x) - f(y)| = |x^2 + 2x - 3 - y^2 - 2y + 3|$$

$$= |x^2 - y^2 + 2x - 2y|$$

$$= |(x - y)(x + y) + 2(x - y)|$$

Since $|x - y| < \delta$

$$< |\delta(x + y) + 2\delta|$$

$$= \delta|x + y + 2|$$

We need to bound $x + y + 2$ above and since it is an increasing function, it will achieve its maximum at the upper bound of it's domain when $x, y = 4$. Therefore

$$< \delta|4 + 4 + 2|$$

$$= 10\delta = \mathcal{E}$$

Therefore $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[2, 4]$.

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2

Claim: $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[0, 10]$

Proof:

3

Claim: $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, 5]$.

Proof: Let $\mathcal{E} > 0$ be given. There exists $\delta > 0$ s.t. for any $x, y \in [0, 5]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \mathcal{E}$$

Choose $\delta = \mathcal{E}$ then

$$\begin{aligned}|f(x) - f(y)| &= \left|\frac{1}{x+1} - \frac{1}{y+1}\right| \\ &= \left|\frac{y+1-x-1}{(x+1)(y+1)}\right| \\ &= \left|\frac{x-y}{(x+1)(y+1)}\right|\end{aligned}$$

Since $|x - y| < \delta$

$$< \delta \left|\frac{1}{(x+1)(y+1)}\right|$$

We need to bound $\frac{1}{(x+1)(y+1)}$ above and since it is a decreasing function, it will achieve it's maximum at the lower bound of it's domain when $x, y = 0$. Therefore

$$\begin{aligned}&< \delta \left|\frac{1}{(0+1)(0+1)}\right| \\ &= \delta \left|\frac{1}{1}\right| = \mathcal{E}\end{aligned}$$

Therefore $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, 5]$.

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4

Claim: $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, \infty)$

Proof:

5

Claim: $h(x) = \frac{x}{x+1}$ is uniformly continuous on the interval $[0, \infty)$.

Proof: Let $\mathcal{E} > 0$ be given. Then there exists $\delta > 0$ s.t. for all $x, y \in [0, \infty]$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \mathcal{E}$. Choose $\delta = \mathcal{E}$ then

$$f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right|$$

$$= \left| \frac{x - y + xy - xy}{(x+1)(y+1)} \right|$$

Since $|x - y| < \delta$

$$< \delta \left| \frac{1}{(x+1)(y+1)} \right|$$

We need to bound $\frac{1}{(x+1)(y+1)}$ above and since it is a decreasing function it achieves a maximum at the lower bound of it's domain when $x, y = 0$. So

$$< \delta \left| \frac{1}{(0+1)(0+1)} \right|$$

$$= \delta \left| \frac{1}{1} \right| = \delta = \mathcal{E}$$

Therefore $h(x) = \frac{x}{x+1}$ is uniformly continuous on the interval $[0, \infty)$.

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6

Claim: $h(x) = \frac{x}{x^2+1}$ is uniformly continuous on the interval $(-\infty, \infty)$.

Proof: Let $\mathcal{E} > 0$ be given. Then there exists $\delta > 0$ s.t. for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \mathcal{E}$. Choose $\delta = \min\{1, \mathcal{E}\}$ then

$$|f(x) - f(y)| = \left| \frac{1}{x^2+1} - \frac{1}{y^2+1} \right|$$

$$= \left| \frac{(x-y)(x+y) + 1 - 1}{(x^2+1)(y^2+1)} \right|$$

Since $|x - y| < \delta$

$$< \delta \left| \frac{(x+y)}{(x^2+1)(y^2+1)} \right|$$

then we can leverage the triangle inequality

$$< \delta \left(\frac{|x|}{|x^2+1||y^2+1|} + \frac{|y|}{|x^2+1||y^2+1|} \right)$$

Aside:

Let $f(x) = \dfrac{x}{x^2+1} \Rightarrow f'(x) = \dfrac{1-x^2}{(x^2+1)^2} \Rightarrow$ critical points $x = 1, -1$

Let $g(y) = \dfrac{1}{y^2+1} \Rightarrow g'(y) = \dfrac{-2y}{(y^2+1)^2} \Rightarrow$ critical points $y = 0$

Let $i(y) = \dfrac{y}{y^2+1} \Rightarrow i'(y) = \dfrac{1-y^2}{(y^2+1)^2} \Rightarrow$ critical points $y = 1, -1$

Let $j(x) = \dfrac{1}{x^2+1} \Rightarrow j'(x) = \dfrac{-2x}{(x^2+1)^2} \Rightarrow$ critical points $x = 0$

We see from the aside that f is maximized when $x = 1, y = 0$ and g is maximized when $x = 0, y = 1$ and the same is true for i and j respectively. So

$$\delta \left(\frac{|x|}{|x^2+1||y^2+1|} + \frac{|y|}{|x^2+1||y^2+1|} \right) < \delta \left(\frac{|1|}{|2||1|} + \frac{|1|}{|1||2|} \right)$$

$$= \delta \left(\frac{1}{2} + \frac{1}{2} \right) = \delta = \mathcal{E}$$

Therefore $h(x) = \frac{x}{x^2+1}$ is uniformly continuous on the interval $(-\infty, \infty)$

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For problems 7 and 8 use the sequential characterization of uniform continuity to show that the function is not uniformly continuous.

7

Claim: $f(x) = x^2 + 2x - 3$ is not uniformly continuous on the interval $[0, \infty)$

Proof: If $f(x)$ is uniformly continuous, then given any two sequences a_n, b_n s.t.

$$\lim_{n \to \infty} a_n - b_n = 0$$

then

$$\lim_{n \to \infty} g(a_n) - g(b_n) = 0$$

Observe the following two sequences

$$a_n = n + \frac{1}{n} \text{ and } b_n = n$$

We will begin by observing

$$\lim_{n \to \infty} a_n - b_n$$

$$= \lim_{n \to \infty} n + \frac{1}{n} - n$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

and by previous proofs we know $\lim_{n\to\infty} \frac{1}{n} = 0$ therefore the first condition is satisfied

$$\lim_{n\to\infty} a_n - b_n = 0$$

Great! Now we will consider the second condition.

$$\lim_{n\to\infty} f(a_n) - f(b_n)$$

$$= \lim_{n\to\infty} \left(n + \frac{1}{n}\right)^2 + 2\left(n + \frac{1}{n}\right) - 3 - n^2 - 2n + 3$$

$$= \lim_{n\to\infty} n^2 + \frac{1}{n^2} + 2 + 2n + \frac{2}{n} - 3 - n^2 - 2n + 3$$

$$= \lim_{n\to\infty} \frac{1}{n^2} + \frac{2}{n} + 2$$

By previous proofs we know $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{2}{n} = 0$ and $\lim_{n\to\infty} 2 = 2$ so we can use the sum property of limits to say

$$\lim_{n\to\infty} \frac{1}{n^2} + \frac{2}{n} + 2 = \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 2$$

$$= 0 + 0 + 2 = 2$$

Since $\lim_{n\to\infty} f(a_n) - f(b_n)$ clearly does not equal 0 then $f(x)$ is not uniformly continuous on $[0, \infty]$.

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8

Claim: $g(x) = \frac{1}{x+1}$ is not uniformly continuous on the interval $(-1, 1)$

Proof: If $g(x)$ is uniformly continuous, then given any two sequences a_n, b_n s.t.

$$\lim_{n\to\infty} a_n - b_n = 0$$

then

$$\lim_{n\to\infty} g(a_n) - g(b_n) = 0$$

Observe the following two sequences

$$a_n = \frac{-n+1}{n} \text{ and } b_n = \frac{-n+3}{n}$$

We can see that for any constant c,

$$\lim_{n\to\infty} \frac{-n+c}{n} = \lim_{n\to\infty} -\frac{n}{n} + \frac{c}{n} = \lim_{n\to\infty} -1 + \frac{c}{n}$$

By previous proofs we know that $\lim_{n\to\infty} \frac{c}{n} = 0$ and the limit of a constant function is the constant, so we can use the sum property of limits to say

$$\lim_{n\to\infty} -1 + \frac{c}{n} = \lim_{n\to\infty} -1 + \lim_{n\to\infty} \frac{c}{n} = -1 + 0 = -1$$

So $\lim_{n\to\infty} a_n = -1$ and $\lim_{n\to\infty} b_n = -1$ so we can use the sum property of limits to show that a_n and b_n satisfy the first condition

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = -1 - (-1) = 0$$

Perfect! Now let us consider the second condition.

$$\lim_{n\to\infty} g(a_n) - g(b_n)$$

$$= \lim_{n\to\infty} \frac{1}{a_n + 1} - \frac{1}{b_n + 1}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{-n+1}{n} + 1} - \frac{1}{\frac{-n+3}{n} + 1}$$

$$= \lim_{n\to\infty} \frac{1}{-1 + \frac{1}{n} + 1} - \frac{1}{-1 + \frac{3}{n} + 1}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}} - \frac{1}{\frac{3}{n}}$$

$$= \lim_{n\to\infty} n - \frac{n}{3}$$

$$= \lim_{n\to\infty} \frac{2}{3} n$$

Aside:
A set s is bounded if there exists M s.t. for all $x \in s$, $x \leq M$
For $n > \dfrac{3M}{2} \Rightarrow \dfrac{2}{3}n > M$ therefore $\frac{2}{3}n$ is not bounded.

From the aside we know that $\lim_{n\to\infty} \frac{2}{3} n$ does not exist, therefore

$$\lim_{n\to\infty} g(a_n) - g(b_n) \neq 0$$

so $g(x)$ is not uniformly continuous on $(-1, 1)$.

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