MATH5210 Analysis
Assignment 5
Limits of Functions

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Lemma 1.1: If α is a number s.t. $0 \leq \alpha < \epsilon$, then $\alpha = 0$

Proof: Assume that $\alpha > 0$, let $\epsilon = \frac{\alpha}{2}$.
Then
$$\alpha < \frac{\alpha}{2}$$
This conclusion is not possible, then the assumption must be incorrect. Therefore $\alpha = 0$.

Claim: Let $f : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x)$ exists, then the limit is unique.

Proof: Let $\epsilon > 0$ be given.
Then $\exists\ \delta_1$ s.t.
$$|f(x) - L| < \frac{\epsilon}{2}$$
for all x s.t.
$$0 < |x - x_0| < \delta_1$$
Likewise $\exists\ \delta_2$ s.t.
$$|f(x) - M| < \frac{\epsilon}{2}$$
for all x s.t.
$$0 < |x - x_0| < \delta_2$$
Then for $\delta = \min\{\delta_1, \delta_2\}$,
$$|L - M| = |L - f(x) + f(x) - B|$$
and then by the triangle inequality
$$|L - f(x) + f(x) - B| < |L - f(x)| + |f(x) - B|$$

which is equal to

$$= |\frac{\epsilon}{2}| + |\frac{\epsilon}{2}|$$

Since $\epsilon > 0$ was given

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So $0 \leq A - B < \epsilon$ and by lemma 1.1,

$$A - B = 0$$

$$\Rightarrow A = B$$

Therefore if $\lim_{x \to x_0} f(x)$ exists, then the limit is unique. ■

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Claim: Let $f : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $L > 0$, then $\exists\ \alpha, \delta > 0$ s.t. $f(x) > \alpha\ \forall\ x \in (x_0 - \delta, x_0 + \delta)$.

Proof: Since the limit exists, this is true for $\epsilon = \frac{L}{3}$, $\exists\ \delta_1$ s.t. $0 < |x - x_0| < \delta_1$. Using the definition of convergence,

$$|f(x) - L| < \frac{L}{3}$$

$$\Rightarrow -\frac{L}{3} < f(x) - L < \frac{L}{3}$$

$$\Rightarrow L - \frac{L}{3} < f(x) < L + \frac{L}{3}$$

$$\Rightarrow \frac{2}{3}L < f(x) < \frac{4}{3}L$$

So let $\alpha = \frac{2}{3}L$, then for all x s.t. $0 < |x - x_0| < \delta_1$

$$0 < \alpha = \frac{2}{3}L < f(x)$$

Therefore, if $\lim_{x \to x_0} f(x) = L$ and $L > 0$, then $\exists\ \alpha, \delta > 0$ s.t. $f(x) > \alpha\ \forall\ x \in (x_0 - \delta, x_0 + \delta)$.

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Claim: Let $f, g, h : (a, b) \to \mathbf{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ for all $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = M$ and $L = M$, then $\lim_{x \to x_0} h(x)$ exists and equals L.

Proof: Let $\epsilon > 0$ be given. I will show that $\exists\ \delta > 0$ s.t. $|g(x) - L| < \epsilon$ for $0 < |x - x_0| < \delta$.
We know $\exists\ \delta_1$ s.t.

$$|f(x) - L| < \epsilon$$

$$\Rightarrow -\epsilon < f(x) - L < \epsilon$$

$$\Rightarrow L - \epsilon < f(x) < L + \epsilon, \text{ for } 0 < |x - x_0| < \delta_1$$

We also know $\exists\ \delta_2$ s.t.

$$|h(x) - M| < \epsilon$$

$$\Rightarrow -\epsilon < h(x) - M < \epsilon$$

$$\Rightarrow M - \epsilon < h(x) < M + \epsilon, \text{ for } 0 < |x - x_0| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then for any x s.t. $0 < |x - x_0| < \delta$

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < M + \epsilon$$

and since $L = M$

$$\Rightarrow L - \epsilon < g(x) < L + \epsilon$$

$$\Rightarrow |g(x) - L| < \epsilon, \text{ for } 0 < |x - x_0| < \delta$$

Therefore, if $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = M$ and $L = M$, then $\lim_{x \to x_0} h(x)$ exists and equals L.

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Claim: Let $f : (a,b) \to \mathbf{R}$ and let $x_0 \in (a,b)$. If $f(x) \geq 0$ for all $x \in (a,b)$ and $\lim_{x \to x_0} f(x) = L$ exists, then $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

Proof: I will use sequential characterization of limits to prove the claim. For **all** sequences a_n s.t.

$$\lim_{n \to \infty} a_n = x_0$$

then

$$\lim_{n \to \infty} f(a_n) = L$$

Let $b_n = f(a_n)$ then

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

Observe $\lim_{n \to \infty} \sqrt{b_n}$. We know $\lim_{n \to \infty} b_n$ exists, so by the square root property of sequences

$$\lim_{n \to \infty} \sqrt{b_n} = \sqrt{\lim_{n \to \infty} b_n} = \sqrt{L}$$

Therefore, if $f(x) \geq 0$ for all $x \in (a,b)$ and $\lim_{x \to x_0} f(x) = L$ exists, then $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

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Claim: Let $f, g : (a,b) \to \mathbf{R}$ and let $x_0 \in (a,b)$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0} f(x)g(x) = LM$.

Proof: I will use sequential characterization of limits to prove the claim. For **all** sequences a_n s.t.

$$\lim_{n \to \infty} a_n = x_0$$

then

$$\lim_{n \to \infty} f(a_n) = L$$

and

$$\lim_{n \to \infty} g(a_n) = M$$

Let $b_n = f(a_b)$ and $c_n = g(a_n)$ then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = L$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = L$$

Observe $\lim_{n\to\infty} b_n c_n$. We know $\lim_{n\to\infty} b_n$ exists and $\lim_{n\to\infty} c_n$ exists, so by the product property of sequences

$$\lim_{n\to\infty} b_n c_n = \lim_{n\to\infty} b_n \cdot \lim_{n\to\infty} c_n = LM$$

Therefore if $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then $\lim_{x\to x_0} f(x)g(x) = LM$.

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