MATH5210 Analysis
Assignment 2: Limit Problems pt. 2
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Prove $\lim_{n\to\infty} \frac{2n+1}{2n-4} = 1$

Lemma 1: $2n-4$ is greater than n
We will show that $2n-4$ is greater than n by the following:
$2n-4 = n+n-4$ and $n-4 > 1$ for all $n > 4$ thus $2n-4 > n$ for all $n > 4$.

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{2n+1}{2n-4}$, converges to 1 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > 5\cdot\epsilon$. Then $|a_n - A| = |\frac{2n+1}{2n-4} - 1| < \epsilon$. Next we change 1 into $\frac{2n-4}{2n-4}$ and combine both fractions leaving us with $|\frac{2n+1-2n+4}{2n-4}| < \epsilon$. By lemma 1 we will make an estimation that makes the denominator smaller. We will also simplify the numerator. This gives us $|\frac{2n+1-2n+4}{2n-4}| < |\frac{5}{n}|$. Since we know that $n > N$ from the beginning, we can say that $|\frac{5}{n}| < \frac{5}{N}$. Finally, because we chose $N > 5\cdot\epsilon$, then $\frac{5}{N} < \epsilon$. This proves that $\lim_{n\to\infty} \frac{2n+1}{2n-4}$, converges to 1 for all $N > 5\cdot\epsilon$. ∎

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Prove $\lim_{n\to\infty} \frac{n}{2^n} = 0$

Lemma 2: 2^n is greater than n^3
We will show that 2^n is greater than n^3 by the following:
$2^n > n^3 \to \ln(2^n) > \ln(n^3) \to n\ln(2) > 3\ln(n) \to \frac{n}{\ln(n)} > \frac{3}{\ln(2)}$.
Thus $\frac{n}{\ln(n)} > \frac{3}{\ln(2)}$ is true for $n > 10$ and so 2^n is greater than n^3 for $n > 10$.

Lemma 3: n^2 is greater than n
We will show that n^2 is greater than n by the following:
$n^2 > n \to n > 1$
Thus $n > 1$ is true for $n > 1$ and so n^2 is greater than n for $n > 1$.

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{n}{2^n}$, converges to 0 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > \frac{1}{\epsilon}$. Then $|a_n - A| = |\frac{n}{2^n} - 0| < \epsilon$. By lemma 2 we will make an estimation that makes the denominator smaller. This gives us $|\frac{n}{2^n}| < |\frac{n}{n^3}|$. We will continue by making another estimation using lemma 3 that makes the numerator larger. This gives us that $|\frac{n}{n^3}| < |\frac{n^2}{n^3}|$. Then we can continue to transform this by $|\frac{n^2}{n^3}| < |\frac{1}{n}|$ and since we know that $n > N$ from the beginning, we can say that $|\frac{1}{n}| < \frac{1}{N}$. Finally, because we chose $N > \frac{1}{\epsilon}$, then $\frac{1}{N} < \epsilon$. This proves that $\lim_{n\to\infty} \frac{n}{2^n}$, converges to 0.

$\blacksquare$

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Prove $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

Lemma 4: $e^{\frac{2}{\epsilon}}$ is greater than $e^{\frac{1}{\epsilon}}$
We will show that $e^{\frac{2}{\epsilon}}$ is greater than $e^{\frac{1}{\epsilon}}$ by the following:
$e^{\frac{2}{\epsilon}} > e^{\frac{1}{\epsilon}} \to \frac{2}{\epsilon} > \frac{1}{\epsilon} \to 2 > 1$
Thus $2 > 1$ is true for all ϵ and so $e^{\frac{2}{\epsilon}}$ is greater than $e^{\frac{1}{\epsilon}}$.

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{1}{\ln n}$, converges to 0 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > e^{\frac{1}{\epsilon}}$. Then $|a_n - A| = |\frac{1}{\ln n} - 0| < \epsilon$. Since $\ln n > 0$ for all $n > 1$ we have $|\frac{1}{\ln n}| = \frac{1}{\ln n} < \epsilon$. As stated above, $n > N$, thus $\frac{1}{\ln n} < \frac{1}{\ln N} < \epsilon$. Now since we chose $N > e^{\frac{1}{\epsilon}}$ we can use lemma 4 and replace $\frac{1}{\ln N} < \epsilon$ with $\frac{1}{\ln e^{\frac{2}{\epsilon}}}$ which is equal to $\frac{\epsilon}{2} < \epsilon$ which is true for all ϵ. Thus we see that $|a_n - A| < \epsilon$ for all $n > N$ and $\lim_{n\to\infty} \frac{1}{\ln n}$ converges to 0. $\blacksquare$

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Prove $\lim_{n\to\infty} \frac{1}{\ln(\ln n)} = 0$

Lemma 5: $e^{e^{\frac{2}{\epsilon}}}$ is greater than $e^{e^{\frac{1}{\epsilon}}}$
We will show that $e^{e^{\frac{2}{\epsilon}}}$ is greater than $e^{e^{\frac{1}{\epsilon}}}$ by the following:
$e^{e^{\frac{2}{\epsilon}}} > e^{e^{\frac{1}{\epsilon}}} \to e^{\frac{2}{\epsilon}} > e^{\frac{1}{\epsilon}} \to \frac{2}{\epsilon} > \frac{1}{\epsilon} \to 2 > 1$
Thus $2 > 1$ is true for all ϵ and so $e^{e^{\frac{2}{\epsilon}}}$ is greater than $e^{e^{\frac{1}{\epsilon}}}$.

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{1}{\ln(\ln n)}$, converges to 0 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > e^{e^{\frac{1}{\epsilon}}}$. Then $|a_n - A| = |\frac{1}{\ln(\ln n)} - 0| < \epsilon$. Since $\ln(\ln n) > 0$ for all $n > 3$ we have $|\frac{1}{\ln(\ln n)}| = \frac{1}{\ln(\ln n)} < \epsilon$. As stated above, $n > N$, thus $\frac{1}{\ln(\ln n)} < \frac{1}{\ln(\ln N)} < \epsilon$. Now since we chose $N > e^{e^{\frac{1}{\epsilon}}}$ we can use lemma 5 and replace $\frac{1}{\ln(\ln N)} < \epsilon$ with $\frac{1}{\ln(\ln e^{e^{\frac{2}{\epsilon}}})}$ which is equal to $\frac{1}{\frac{2}{\epsilon}} = \frac{\epsilon}{2} < \epsilon$ which is true for all ϵ. Thus we see that $|a_n - A| < \epsilon$ for all $n > N$ and $\lim_{n\to\infty} \frac{1}{\ln(\ln n)}$ converges to 0. $\blacksquare$

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Prove $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$

Lemma 6: $\sqrt{n}$ is unbounded
We will show that $\sqrt{n}$ is unbound by the following:
Recall that a set s is bounded if $\exists M$ s.t. $\forall x \in s$ we have $x \leq M$. Then observe, let M be given, then $n > M^2$ and $\sqrt{n} > M$. So we see that $\sqrt{n}$ is unbounded.

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{n}{\ln n}$, does not converge by showing that $\frac{n}{\ln n}$ is unbounded. Our first step will be to make an estimate which is less than $\frac{n}{\ln n}$. From discussion in class we can say that $\sqrt{n} < \frac{n}{\ln n}$. Then using lemma 6 we know that $\sqrt{n}$ is not bounded. Since $\sqrt{n}$ is less than $\frac{n}{\ln n}$, $\frac{n}{\ln n}$ is also unbounded and therefor the limit does not converge. ■

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Prove $\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n} = 0$

Lemma 7: $\sqrt{n+2} + \sqrt{n}$ is greater than $\sqrt{n}$
We will show that $\sqrt{n+2} + \sqrt{n}$ is greater than $\sqrt{n}$ by the following:
$\sqrt{n+2} + \sqrt{n} > \sqrt{n} \to \sqrt{n+2} + \sqrt{n} - \sqrt{n} > 0 \to \sqrt{n+2} > 0$
Thus $\sqrt{n+2} > 0$ is true for all $n \geq 0$ and so $\sqrt{n+2} + \sqrt{n}$ is greater than $\sqrt{n}$

Lemma 8: $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$
We will show that $\sqrt{a} - \sqrt{b}$ is equal to $\frac{a-b}{\sqrt{a}+\sqrt{b}}$ by the following:
$\sqrt{a} - \sqrt{b} = \sqrt{a} - \sqrt{b} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n}$, converges to 0 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > \frac{4}{\epsilon^2}$. Then $|a_n - A| = |\sqrt{n+2} - \sqrt{n} - 0| < \epsilon$. By using lemma 8 we can transform the equation into $|\sqrt{n+2} - \sqrt{n}| = |\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}|}$. By lemma 7 we can make an estimation that makes the denominator smaller, and simplify the numerator, $|\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}}| < |\frac{2}{\sqrt{n}}|$. Since $\sqrt{n} > 0$ for all $n \geq 0$ and as stated above, $n > N$, we have $\frac{2}{\sqrt{n}} < \frac{2}{\sqrt{N}} < \epsilon$. Since we chose $N > \frac{4}{\epsilon^2}$, thus we see that $|a_n - A| < \epsilon$ for all $n > N$ and $\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n}$ converges to 0.

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Prove $\lim_{n\to\infty} \frac{\sqrt{n+2}}{\sqrt{n-3}} = 1$

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \frac{\sqrt{n+2}}{\sqrt{n-3}}$, converges to 1 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > 3 + \frac{5}{\epsilon}$. Then $|a_n - A| = |\frac{\sqrt{n+2}}{\sqrt{n-3}} - 1| < \epsilon$. We will continue by using a series of algebraic techniques to simplify the equation as follows: $|\frac{\sqrt{n+2}}{n-3} - 1| = |\frac{\sqrt{n+2}-\sqrt{n-3}}{\sqrt{n-3}}|$. Then we will multiply the top and bottom by the conjugate of the numerator $|\frac{n+2-n+3}{n-3+\sqrt{(n+2)(n-3)}}|$. Then we can make an estimate and make the denominator smaller giving us $|\frac{5}{n-3}|$ which is positive for $n > 3$. Since $n > N$, we have $\frac{5}{n-3} < \frac{5}{N-3}$. Then, since we chose $N > 3 + \frac{5}{\epsilon}$, we see that $|a_n - A| < \epsilon$ for all $n > N$ and $\lim_{n\to\infty} \frac{\sqrt{n+2}}{\sqrt{n-3}}$ converges to 1.

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Prove $\lim_{n\to\infty} \sqrt{n^2+2} - \sqrt{n^2+1} = 0$

Proof: Let there be given an $\epsilon > 0$. n is larger than N. We will show that the above limit, $\lim_{n\to\infty} \sqrt{n^2+2} - \sqrt{n^2+1}$, converges to 0 by showing that $|a_n - A| < \epsilon$ for all $n > N$. Begin by choosing $N > \frac{1}{\epsilon}$. Then $|a_n - A| = |\lim_{n\to\infty} \sqrt{n^2+2} - \sqrt{n^2+1} - 0| < \epsilon$. Using lemma 8 we rewrite as $|\frac{n^2+2-n^2-1}{\sqrt{n^2+2}+\sqrt{n^2+1}}|$. Then we simplify the numerator and drop positive terms from the denominator to make a smaller estimate which leaves us $\frac{1}{\sqrt{n^2}\sqrt{n^2}} = \frac{1}{n}$. Since $n > N$ $\frac{1}{n} < \frac{1}{N} < \epsilon$ Finally, since we chose $N > \frac{1}{\epsilon}$, we see that $|a_n - A| < \epsilon$ for all $n > N$ and $\lim_{n\to\infty} \sqrt{n^2+2} - \sqrt{n^2+1}$ converges to 0. ■