Assignment 3
Limit Problems (using limit properties)

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1.

List the properties of limits discussed in class. Give the citation for these properties in the textbook

Properties:

#1. Uniqueness (p 26)

#2. Bounded 2.18 (p 35)

#3. Bounded Away 2.21 (p 36)

#4. Preservation of Inequality 2.9 (p 28)

#5. Bounded and Zero 2.12 (p 29)

#6. Squeeze 2.22 (p 37)

#7. Sum Property 2.10 (p 28)

#8. Polynomial Property 2.17 (p 31)

#9. Product Property 2.13 (p 30)

#10. Quotient Property 2.15 (p 31)

#11. Square Roots 2.17 (p 32)

#12. Convergent Subsequence 2.33 (p 45)

#13. Monotone Convergence 2.25 (p 38)

2.1

Claim: $\lim\limits_{n\to\infty} \frac{5n-6}{6n+7} = \frac{5}{6}$

Proof: $\lim\limits_{n\to\infty} \frac{5n-6}{6n+7} = \lim\limits_{n\to\infty} \frac{5-\frac{6}{n}}{6+\frac{7}{n}}$

Let $a_n = 5 - \frac{6}{n}$
Then the $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 5 - \frac{6}{n}$
Using the sum property of limits, $\lim\limits_{n\to\infty} 5 - \frac{6}{n} = \lim\limits_{n\to\infty} 5 - \lim\limits_{n\to\infty} \frac{6}{n}$
From common knowledge and previous $\epsilon - N$ proofs, we know $\lim\limits_{n\to\infty} 5 = 5$ and $\lim\limits_{n\to\infty} \frac{6}{n} = 0$
Thus $\lim\limits_{n\to\infty} 5 - \frac{6}{n} = 5 + 0 = 5$

Let $b_n = 6 + \frac{7}{n}$
Then the $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 6 + \frac{7}{n}$
Using the sum property of limits, $\lim\limits_{n\to\infty} 6 - \frac{7}{n} = \lim\limits_{n\to\infty} 6 - \lim\limits_{n\to\infty} \frac{7}{n}$
From common knowledge and previous $\epsilon - N$ proofs, we know $\lim\limits_{n\to\infty} 6 = 6$ and $\lim\limits_{n\to\infty} \frac{7}{n} = 0$
Thus $\lim\limits_{n\to\infty} 6 - \frac{7}{n} = 6 + 0 = 6$

Finally, using the quotient property of limits and previous two statements $\lim\limits_{n\to\infty} \frac{5n-6}{6n+7} = \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{5}{6}$ ∎

2.2

Claim: $\lim\limits_{n\to\infty} \frac{n^2+n+1}{n^2-4} = 1$

Proof: $\lim\limits_{n\to\infty} \frac{n^2+n+1}{n^2-4} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{4}{n^2}}$

Let $a_n = 1 + \frac{1}{n} + \frac{1}{n^2}$
Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2}$
Using the sum property of limits, $\lim\limits_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2} = \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n} + \lim\limits_{n\to\infty} \frac{1}{n^2}$
From common knowledge and previous $\epsilon - N$ proofs, we know
$\lim\limits_{n\to\infty} 1 = 1$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, and $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$
Thus $\lim\limits_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2} = 1 + 0 + 0 = 1$

Let $b_n = 1 - \frac{4}{n^2}$
Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} = 1 - \frac{4}{n^2}$
Using the sum property of limits, $\lim\limits_{n\to\infty} = 1 - \frac{4}{n^2} = \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \frac{4}{n^2}$
From common knowledge and previous $\epsilon - N$ proofs, we know
$\lim\limits_{n\to\infty} 1 = 1$ and $\lim\limits_{n\to\infty} \frac{4}{n^2} = 0$
Thus $\lim\limits_{n\to\infty} 1 - \frac{4}{n^2} = 1 + 0 = 1$

Finally, using the quotient property of limits and previous two statements $\lim_{n\to\infty} \frac{n^2+n+1}{n^2-4} = \frac{1}{1} = 1$

$\blacksquare$

2.3

Claim: $\lim_{n\to\infty} \frac{\sqrt{n^3+1}}{n+2} = \infty$

Proof: Begin by multiplying by $\frac{1}{n}/\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sqrt{n^3+1}}{n+2} = \lim_{n\to\infty} \frac{\sqrt{n+\frac{1}{n}}}{1+\frac{2}{n}}$$

By the quotient property of limits

$$= \frac{\lim_{n\to\infty} \sqrt{n+\frac{1}{n}}}{\lim_{n\to\infty} 1+\frac{2}{n}}$$

Then by the polynomial property of limits

$$= \frac{\sqrt{\lim_{n\to\infty} n+\frac{1}{n}}}{\lim_{n\to\infty} 1+\frac{2}{n}}$$

Then by the summation property of limits

$$= \frac{\sqrt{\lim_{n\to\infty} n+\lim_{n\to\infty} \frac{1}{n}}}{\lim_{n\to\infty} 1+\lim_{n\to\infty} \frac{2}{n}}$$

Then by common knowledge and previous proofs

$$= \frac{\sqrt{\lim_{n\to\infty} n+0}}{1+0} = \frac{\sqrt{\lim_{n\to\infty} n}}{1} = \sqrt{\lim_{n\to\infty} n}$$

Then by the polynomial property

$$= \lim_{n\to\infty} \sqrt{n}$$

Then as previously proven

$$= \lim_{n\to\infty} \sqrt{n} = \infty$$

Thus we can see that $\lim_{n\to\infty} \frac{\sqrt{n^3+1}}{n+2} = \infty$

$\blacksquare$

2.4

Claim: $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Proof: We will show by the bounded and zero property of limits that $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Let $a_n = \sin(n)$
As we have previously discussed $|\sin(n)| \leq 1$
Thus $\sin(n)$ is bounded.

Then let $b_n = \frac{1}{n}$.
We know from previous $\epsilon - N$ proofs that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ ∎

2.5

Claim: $\lim_{n\to\infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist

Proof: We will show through the convergent sub sequence property that $\lim_{n\to\infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist

Let $a_n = 1 + 3(-1)^n + 4(-1)^{n+1}$

Let $b_n = a_{2n} = 1 - 3 + 4$
Then the sequence $\{b_n\} = \{0, 0, 0, \ldots\}$
So $\lim_{n\to\infty} b_n = 0$

Let $c_n = a_{2n+1} = 1 + 3 - 4$ Then the sequence $\{c_n\} = \{1, 1, 1, \ldots\}$
So $\lim_{n\to\infty} c_n = 1$

The convergent subsequence property states that if $\lim_{n\to\infty} a_n = A$ then the limit of all subsequence of a_n also converge to A. Here we have shown two subsequence b_n and c_n which do not converge to the same value, hence $\lim_{n\to\infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist.

∎

2.6

Claim: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Proof: We will show by the bounded and zero property of limits that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Let $a_n = (-1)^n$
Then $|a_n| = 1$
So a_n is bounded above by 1.

Then let $b_n = \frac{1}{n}$.
We know from previous $\epsilon - N$ proofs that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ $\blacksquare$

2.7

Claim: $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

Proof: By the squeeze property of limits we will show that $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

Let $a_n = \frac{1}{n}$, $b_n = \frac{1}{\ln(n)}$ and $c_n = \frac{1}{\ln(\ln n)}$

As previously shown $2^n > n^2 > n$ for $n > 1$. Observe then that $e^n > 2^n > n \Rightarrow e^n > n$.
Which $\Rightarrow n > \ln n$
And $\Rightarrow \frac{1}{n} < \frac{1}{\ln n}$

Next, as we just proved $n > \ln n$
Then $\Rightarrow \ln n > \ln(\ln n)$
And $\Rightarrow \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$

Note, from previous proofs we know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{\ln(\ln n)} = 0$
So we have two series, $a_n < b_n < c_n$ and since $\lim_{n\to\infty} a_b = 0$ and $\lim_{n\to\infty} c_n = 0$ then $\lim_{n\to\infty} b_n = 0$

$\blacksquare$

2.8

Claim: $\lim_{n\to\infty} \sin\frac{n\pi}{3}$ does not exist

Proof: By the convergent subsequence property we will show that $\lim_{n\to\infty} \sin\frac{n\pi}{3}$ does not exist

Let $a_n = \sin\frac{n\pi}{3}$

Let $b_n = a_{3n} = \sin(n\pi)$
Then the sequence $\{b_n\} = \{0, 0, 0, \ldots\}$
So $\lim_{n\to\infty} b_n = 0$

Let $c_n = a_{6n+1} = \sin(2\pi + \frac{\pi}{3})$
Then the sequence $\{c_n\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\}$
So $\lim_{n\to\infty} c_n = \frac{1}{2}$

The convergent subsequence property states that if $\lim_{n\to\infty} a_n = A$ then the limit of all subsequence of a_n also converge to A. Here we have shown two subsequence b_n and c_n which do not converge to the same value, hence $\lim_{n\to\infty} \sin\frac{n\pi}{3}$ does not exist.

■

2.9

Claim: $\lim_{n\to\infty} \frac{2\sin(n)+3\cos(n)}{\sqrt{n}} = 0$

Proof: We will show through the bounded and zero property of limits that
$\lim_{n\to\infty} \frac{2\sin(n)+3\cos(n)}{\sqrt{n}} = 0$

First we will show that $|2\sin(n) + 3\cos(n)| \leq M$ for all n and thus is bounded.
By the triangle inequality we know $|2\sin(n) + 3\cos(n)| \leq |2\sin(n)| + |3\cos(n)|$
As previously discussed, $\sin(n)$ and $\cos(n)$ are bounded by ± 1
So $|2\sin(n)| + |3\cos(n)| \leq 2\cdot 1 + 3\cdot 1 = 5$
Thus $|2\sin(n) + 3\cos(n)| \leq 5$ and so it is bounded.

As previously proven, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Finally, by the product of limits property, we have the product of a bounded series and zero which is zero so $\lim_{n\to\infty} \frac{2\sin(n)+3\cos(n)}{\sqrt{n}} = 0$

■

2.10

Claim: $\lim_{n\to\infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = 1$

Proof: $\lim_{n\to\infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = \lim_{n\to\infty} \sqrt{\frac{n+4}{n-12}}$

By the polynomial property of limits

$$\lim_{n\to\infty} \sqrt{\frac{n+4}{n-12}} = \sqrt{\lim_{n\to\infty} \frac{n+4}{n-12}}$$

Then by the quotient properties of limits

$$= \sqrt{\frac{\lim_{n\to\infty} n+4}{\lim_{n\to\infty} n-12}} = \sqrt{\frac{\lim_{n\to\infty} 1+\frac{4}{n}}{\lim_{n\to\infty} 1-\frac{12}{n}}}$$

Then by the sum property of limits

$$= \sqrt{\frac{\lim_{n\to\infty} 1+\lim_{n\to\infty} \frac{4}{n}}{\lim_{n\to\infty} 1-\lim_{n\to\infty} \frac{12}{n}}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \sqrt{\frac{1+0}{1-0}} = \sqrt{\frac{1}{1}} = \sqrt{1} = 1$$

Thus we have shown that $\lim_{n\to\infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = 1$

■

2.11

Claim: $\lim\limits_{n\to\infty} \sqrt{n^2+n+1} - \sqrt{n^2-n} = 1$

Proof: Using the relation $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

$$\lim_{n\to\infty} \sqrt{n^2+n+1} - \sqrt{n^2-n} = \lim_{n\to\infty} \frac{n^2+n+1-n^2+n}{\sqrt{n^2+n+1}+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{2n+1}{\sqrt{n^2+n+1}+\sqrt{n^2-n}}$$

Then multiplying by $\frac{1}{n}/\frac{1}{n}$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n}}}$$

Then by the quotient property of limits

$$= \frac{\lim\limits_{n\to\infty} 2+\frac{1}{n}}{\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n}}}$$

Then by the summation property of limits

$$= \frac{\lim\limits_{n\to\infty} 2+\lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\lim\limits_{n\to\infty} \sqrt{1-\frac{1}{n}}}$$

Then by the polynomial property of limits

$$= \frac{\lim\limits_{n\to\infty} 2+\lim\limits_{n\to\infty} \frac{1}{n}}{\sqrt{\lim\limits_{n\to\infty} 1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{\lim\limits_{n\to\infty} 1-\frac{1}{n}}}$$

Then by the summation property of limits

$$= \frac{\lim\limits_{n\to\infty} 2+\lim\limits_{n\to\infty} \frac{1}{n}}{\sqrt{\lim\limits_{n\to\infty} 1+\lim\limits_{n\to\infty} \frac{1}{n}+\lim\limits_{n\to\infty} \frac{1}{n^2}}+\sqrt{\lim\limits_{n\to\infty} 1-\lim\limits_{n\to\infty} \frac{1}{n}}}$$

Then by common knowledge and previous proofs

$$= \frac{2+0}{\sqrt{1+0+0}+\sqrt{1-0}} = \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{1+1} = \frac{2}{2} = 1$$

$\blacksquare$