MATH5210 Analysis
Assignment 7
Uniform Continuity
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For Problems 1 to 6, give $\epsilon - \delta$ proofs of uniform continuity.

1

Claim: $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[2, 4]$

Proof: Let $\epsilon > 0$ be given. There exists $\delta > 0$ s.t. for any $x, y \in [2, 4]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Choose $\delta = \frac{\epsilon}{10}$
Then

$$|f(x) - f(y)| = |x^2 + 2x - 3 - y^2 - 2y + 3|$$

$$= |x^2 + 2x - y^2 - 2y|$$

$$= |x^2 - y^2 + 2x - 2y|$$

$$= |(x - y)(x + y) + 2(x - y)|$$

$$= |\delta(x + y) + 2\delta|$$

$$= \delta|x + y + 2|$$

$f(x)$ is an increasing function so we need to bound x and y above and we can easily do this by examining the domain of $f(x)$ which is $[2, 4]$. So $x, y < 4$ therefore

$$< \delta|4 + 4 + 2|$$

$$= 10\delta = \epsilon$$

Therefore $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[2, 4]$.

$\blacksquare$

2

Claim: $f(x) = x^2 + 2x - 3$ is uniformly continuous on the interval $[0, 10]$

Proof:

3

Claim: $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, 5]$

Proof: Let $\epsilon > 0$ be given. There exists $\delta > 0$ s.t. for any $x, y \in [0, 5]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Choose $\delta = \epsilon$ then

$$|f(x) - f(y)| = \left| \frac{1}{x+1} - \frac{1}{y+1} \right|$$

$$= \left| \frac{y + 1 - x - 1}{(x+1)(y+1)} \right|$$

$$= \left| \frac{x - y}{(x+1)(y+1)} \right|$$

$$= \delta \left| \frac{1}{(x+1)(y+1)} \right|$$

$g(x)$ is a decreasing function so we need to bound x and y below and we can easily do this by examining the domain of $g(x)$ which is $[0, 5]$. So $x, y > 0$ therefore

$$< \delta \left| \frac{1}{(0+1)(0+1)} \right|$$

$$= \delta \left| \frac{1}{1} \right| = \epsilon$$

Therefore $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, 5]$.

$\blacksquare$

4

Claim: $g(x) = \frac{1}{x+1}$ is uniformly continuous on the interval $[0, \infty)$

Proof:

5

Claim: $h(x) = \frac{x}{x+1}$ is uniformly continuous on the interval $[0, \infty)$

Proof: Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ s.t. for all $x, y \in [0, \infty]$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Choose $\delta = \epsilon$ then

$$f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right|$$

$$= \left| \frac{x - y + xy - xy}{(x+1)(y+1)} \right|$$

$$< \delta \left| \frac{1}{(x+1)(y+1)} \right|$$

Now we will bound x and y from our knowledge of the domain. We know that $\frac{1}{(x+1)(y+1)}$ is a decreasing function so it achieves a maximum at it's smallest value, 0. So

$$< \delta \left| \frac{1}{(0+1)(0+1)} \right|$$

$$= \delta \left| \frac{1}{1} \right| = \delta = \epsilon$$

Therefore $h(x) = \frac{x}{x+1}$ is uniformly continuous on the interval $[0, \infty)$.

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6

Claim: $h(x) = \frac{x}{x^2+1}$ is uniformly continuous on the interval $(-\infty, \infty)$

Proof: Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ s.t. for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Choose $\delta = \min\{1, \epsilon\}$ then

$$|f(x) - f(y)| = \left| \frac{1}{x^2+1} - \frac{1}{y^2+1} \right|$$

$$= \left| \frac{(x-y)(x+y) + 1 - 1}{(x^2+1)(y^2+1)} \right|$$

$$= \delta \left| \frac{(x+y)}{(x^2+1)(y^2+1)} \right|$$

then we can leverage the triangle inequality

$$< \delta \left(\frac{|x|}{|x^2+1||y^2+1|} + \frac{|y|}{|x^2+1||y^2+1|} \right)$$

Aside:

Let $f(x) = \dfrac{x}{x^2+1} \Rightarrow f'(x) = \dfrac{1-x^2}{(x^2+1)^2} \Rightarrow$ critical points $x = 1, -1$

Let $g(y) = \dfrac{1}{y^2+1} \Rightarrow g'(y) = \dfrac{-2y}{(y^2+1)^2} \Rightarrow$ critical points $y = 0$

Let $i(x) = \dfrac{y}{y^2+1} \Rightarrow i'(y) = \dfrac{1-y^2}{(y^2+1)^2} \Rightarrow$ critical points $y = 1, -1$

Let $j(x) = \dfrac{1}{x^2+1} \Rightarrow j'(x) = \dfrac{-2x}{(x^2+1)^2} \Rightarrow$ critical points $x = 0$

We see from the aside that f is maximized when $x = 1, y = 0$ so

$$|x - y| = |1 - 0| = 1 = \delta$$

and g is maximized when $x = 0, y = 1$ so

$$|x - y| = |0 - 1| = 1 = \delta$$

the same is true for i and j respectively. So

$$\delta \left(\frac{|x|}{|x^2+1||y^2+1|} + \frac{|y|}{|x^2+1||y^2+1|} \right) < \delta \left(\frac{|1|}{|2||1|} + \frac{|1|}{|1||2|} \right)$$

$$= \delta \left(\frac{1}{2} + \frac{1}{2} \right) = \delta = \epsilon$$

Therefore $h(x) = \frac{x}{x^2+1}$ is uniformly continuous on the interval $(-\infty, \infty)$

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For problems 7 and 8 use the sequential characterization of uniform continuity to show that the function is not uniformly continuous.

7

Claim: $f(x) = x^2 + 2x - 3$ is not uniformly continuous on the interval $[0, \infty)$

Proof:

8

Claim: $g(x) = \frac{1}{x+1}$ is not uniformly continuous on the interval $(-1, 1)$

Proof: If $g(x)$ is uniformly continuous, then given any two sequences a_n, b_n s.t.

$$\lim_{n\to\infty} a_n - b_n = 0$$

then

$$\lim_{n\to\infty} g(a_n) - g(b_n) = 0$$

Observe the following two sequences

$$a_n = \frac{-n+1}{n} \text{ and } b_n = \frac{-n+3}{n}$$

We can see that for any constant c,

$$\lim_{n\to\infty} \frac{-n+c}{n} = \lim_{n\to\infty} -\frac{n}{n} + \frac{c}{n} = \lim_{n\to\infty} -1 + \frac{c}{n}$$

By previous proofs we know that $\lim\limits_{n\to\infty} \dfrac{c}{n} = 0$ and the limit of a constant function is the constant, so we can use the sum property of limits to say

$$\lim_{n\to\infty} -1 + \frac{c}{n} = \lim_{n\to\infty} -1 + \lim_{n\to\infty} \frac{c}{n} = -1 + 0 = -1$$

So $\lim_{n\to\infty} a_n = -1$ and $\lim_{n\to\infty} b_n = -1$ so we can use the sum property of limits to show that a_n and b_n satisfy the first condition

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = -1 - (-1) = 0$$

Perfect! Now let us consider the second condition.

$$\lim_{n\to\infty} g(a_n) - g(b_n)$$

$$= \lim_{n\to\infty} \frac{1}{a_n+1} - \frac{1}{b_n+1}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{-n+1}{n}+1} - \frac{1}{\frac{-n+3}{n}+1}$$

$$= \lim_{n\to\infty} \frac{1}{-1+\frac{1}{n}+1} - \frac{1}{-1+\frac{3}{n}+1}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}} - \frac{1}{\frac{3}{n}}$$

$$= \lim_{n\to\infty} n - \frac{n}{3}$$

$$= \lim_{n\to\infty} \frac{2}{3}n$$

Aside:
A set s is bounded if there exists M s.t. for all $x \in s$, $x \leq M$
For $n > \dfrac{3M}{2} \Rightarrow \dfrac{2}{3}n > M$ therefore $\frac{2}{3}n$ is not bounded.

From the aside we know that $\lim\limits_{n\to\infty} \dfrac{2}{3}n$ does not exist, therefore

$$\lim_{n\to\infty} g(a_n) - g(b_n) \neq 0$$

so $g(x)$ is not uniformly continuous.

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