

Assignment 3

Limit Problems (using limit properties)

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1.

List the properties of limits discussed in class. Give the citation for these properties in the textbook

Properties:

- #1. Uniqueness (p 26)
- #2. Bounded 2.18 (p 35)
- #3. Bounded Away 2.21 (p 36)
- #4. Preservation of Inequality 2.9 (p 28)
- #5. Bounded and Zero 2.12 (p 29)
- #6. Squeeze 2.22 (p 37)
- #7. Sum Property 2.10 (p 28)
- #8. Polynomial Property 2.17 (p 31)
- #9. Product Property 2.13 (p 30)
- #10. Quotient Property 2.15 (p 31)
- #11. Square Roots 2.17 (p 32)
- #12. Convergent Subsequence 2.33 (p 45)
- #13. Monotone Convergence 2.25 (p 38)

2.1

Claim: $\lim_{n \rightarrow \infty} \frac{5n-6}{6n+7} = \frac{5}{6}$

Proof: By the fraction property $\frac{a}{b} = \frac{ra}{rb}$ multiply by $\frac{1/n}{1/n}$

$$\lim_{n \rightarrow \infty} \frac{5n-6}{6n+7} = \lim_{n \rightarrow \infty} \frac{5 - \frac{6}{n}}{6 + \frac{7}{n}}$$

Then by the quotient property of limits

$$= \frac{\lim_{n \rightarrow \infty} 5 - \frac{6}{n}}{\lim_{n \rightarrow \infty} 6 + \frac{7}{n}}$$

Then by the summation property of limits

$$= \frac{\lim_{n \rightarrow \infty} 5 - \lim_{n \rightarrow \infty} \frac{6}{n}}{\lim_{n \rightarrow \infty} 6 + \lim_{n \rightarrow \infty} \frac{7}{n}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \frac{5 - 0}{6 + 0} = \frac{5}{6}$$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{5n-6}{6n+7} = \frac{5}{6}$

■

2.2

Claim: $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-4} = 1$

Proof: By the fraction property $\frac{a}{b} = \frac{ra}{rb}$ multiply by $\frac{1/n}{1/n}$

$$\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-4} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{1 - \frac{4}{n^2}}$$

Then by the quotient property of limits

$$= \frac{\lim_{n \rightarrow \infty} 1 + \frac{1}{n} + \frac{1}{n^2}}{\lim_{n \rightarrow \infty} 1 - \frac{4}{n^2}}$$

Then by the summation property of limits

$$= \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n^2}}{\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{4}{n^2}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \frac{1 + 0 + 0}{1 - 0} = \frac{1}{1} = 1$$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-4} = 1$

■

2.3

Claim: $\lim_{n \rightarrow \infty} \frac{\sqrt{n^3+1}}{n+2} = \infty$

Proof: By the fraction property $\frac{a}{b} = \frac{ra}{rb}$ multiply by $\frac{1/n}{1/n}$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n^3+1}}{n+2} = \lim_{n \rightarrow \infty} \frac{\sqrt{n + \frac{1}{n}}}{1 + \frac{2}{n}}$$

By the quotient property of limits

$$= \frac{\lim_{n \rightarrow \infty} \sqrt{n + \frac{1}{n}}}{\lim_{n \rightarrow \infty} 1 + \frac{2}{n}}$$

Then by the polynomial property of limits

$$= \frac{\sqrt{\lim_{n \rightarrow \infty} n + \lim_{n \rightarrow \infty} \frac{1}{n}}}{\lim_{n \rightarrow \infty} 1 + \frac{2}{n}}$$

Then by the summation property of limits

$$= \frac{\sqrt{\lim_{n \rightarrow \infty} n + \lim_{n \rightarrow \infty} \frac{1}{n}}}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{2}{n}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \frac{\sqrt{\lim_{n \rightarrow \infty} n + 0}}{1 + 0} = \frac{\sqrt{\lim_{n \rightarrow \infty} n}}{1} = \sqrt{\lim_{n \rightarrow \infty} n}$$

Then by the polynomial property

$$= \lim_{n \rightarrow \infty} \sqrt{n}$$

Then as previously proven

$$= \lim_{n \rightarrow \infty} \sqrt{n} = \infty$$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{\sqrt{n^3+1}}{n+2} = \infty$

■

2.4

Claim: $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$

Proof: We will show by the bounded and zero property of limits that $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$

Let $a_n = \sin(n)$

As we have previously discussed $|\sin(n)| \leq 1$

Thus $\sin(n)$ is bounded.

Then let $b_n = \frac{1}{n}$.

We know from previous $\epsilon - N$ proofs that $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$

■

2.5

Claim: $\lim_{n \rightarrow \infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist

Proof: We will show through the convergent sub sequence property that

$\lim_{n \rightarrow \infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist

Let $a_n = 1 + 3(-1)^n + 4(-1)^{n+1}$

Let $b_n = a_{2n} = 1 - 3 + 4$

Then the sequence $\{b_n\} = \{0, 0, 0, \dots\}$

So $\lim_{n \rightarrow \infty} b_n = 0$

Let $c_n = a_{2n+1} = 1 + 3 - 4$ Then the sequence $\{c_n\} = \{1, 1, 1, \dots\}$

So $\lim_{n \rightarrow \infty} c_n = 1$

The convergent subsequence property states that if $\lim_{n \rightarrow \infty} a_n = A$ then the limit of all subsequence of a_n also converge to A . Here we have shown two subsequence b_n and c_n which do not converge to the same value, hence $\lim_{n \rightarrow \infty} 1 + 3(-1)^n + 4(-1)^{n+1}$ does not exist.

2.6

Claim: $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$

Proof: We will show by the bounded and zero property of limits that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$

Let $a_n = (-1)^n$

Then $|a_n| = 1$

So a_n is bounded above by 1.

Then let $b_n = \frac{1}{n}$.

We know from previous $\epsilon - N$ proofs that $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$

2.7

Claim: $\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = 0$

Proof: By the squeeze property of limits we will show that $\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = 0$

Let $a_n = \frac{1}{n}$, $b_n = \frac{1}{\ln(n)}$ and $c_n = \frac{1}{\ln(\ln n)}$

As previously shown $2^n > n^2 > n$ for $n > 1$. Observe then that $e^n > 2^n > n \Rightarrow e^n > n$.

Which $\Rightarrow n > \ln n$

And $\Rightarrow \frac{1}{n} < \frac{1}{\ln n}$

Next, as we just proved $n > \ln n$

Then $\Rightarrow \ln n > \ln(\ln n)$

And $\Rightarrow \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$

Note, from previous $\epsilon - N$ proofs we know $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ and $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \frac{1}{\ln(\ln n)} = 0$

So we have two series, a_n and c_n , such that $a_n < b_n < c_n$. Since $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} c_n = 0$ then by the squeeze property of limits $\lim_{n \rightarrow \infty} b_n = 0$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = 0$

2.8

Claim: $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{3}$ does not exist

Proof: By the convergent subsequence property we will show that $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{3}$ does not exist

Let $a_n = \sin \frac{n\pi}{3}$

Let $b_n = a_{3n} = \sin(n\pi)$

Then the sequence $\{b_n\} = \{0, 0, 0, \dots\}$

So $\lim_{n \rightarrow \infty} b_n = 0$

Let $c_n = a_{6n+1} = \sin(2\pi + \frac{\pi}{3})$

Then the sequence $\{c_n\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots\}$

So $\lim_{n \rightarrow \infty} c_n = \frac{1}{2}$

The convergent subsequence property states that if $\lim_{n \rightarrow \infty} a_n = A$ then the limit of all subsequence of a_n also converge to A . Here we have shown two subsequence b_n and c_n which do not converge to the same value, hence $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{3}$ does not exist. ■

2.9

Claim: $\lim_{n \rightarrow \infty} \frac{2 \sin(n) + 3 \cos(n)}{\sqrt{n}} = 0$

Proof: We will show through the bounded and zero property of limits that

$$\lim_{n \rightarrow \infty} \frac{2 \sin(n) + 3 \cos(n)}{\sqrt{n}} = 0$$

First we will show that $|2 \sin(n) + 3 \cos(n)| \leq M$ for all n and thus is bounded.

By the triangle inequality we know $|2 \sin(n) + 3 \cos(n)| \leq |2 \sin(n)| + |3 \cos(n)|$

As previously discussed, $\sin(n)$ and $\cos(n)$ are bounded by ± 1

So $|2 \sin(n)| + |3 \cos(n)| \leq 2 \cdot 1 + 3 \cdot 1 = 5$

Thus $|2 \sin(n) + 3 \cos(n)| \leq 5$ and so it is bounded.

As previously proven, $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

Finally, by the product of limits property, we have the product of a bounded series and zero which is zero so $\lim_{n \rightarrow \infty} \frac{2 \sin(n) + 3 \cos(n)}{\sqrt{n}} = 0$ ■

2.10

Claim: $\lim_{n \rightarrow \infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = 1$

Proof: $\lim_{n \rightarrow \infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n+4}{n-12}}$

By the polynomial property of limits

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n+4}{n-12}} = \sqrt{\lim_{n \rightarrow \infty} \frac{n+4}{n-12}}$$

Then by the quotient properties of limits

$$= \sqrt{\frac{\lim_{n \rightarrow \infty} n+4}{\lim_{n \rightarrow \infty} n-12}}$$

Then by the fraction property $\frac{a}{b} = \frac{ra}{rb}$ multiply by $\frac{1/n}{1/n}$

$$\sqrt{\frac{\lim_{n \rightarrow \infty} 1 + \frac{4}{n}}{\lim_{n \rightarrow \infty} 1 - \frac{12}{n}}}$$

Then by the sum property of limits

$$= \sqrt{\frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{4}{n}}{\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{12}{n}}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \sqrt{\frac{1+0}{1-0}} = \sqrt{\frac{1}{1}} = \sqrt{1} = 1$$

Thus we have shown that $\lim_{n \rightarrow \infty} \frac{\sqrt{n+4}}{\sqrt{n-12}} = 1$

■

2.11

Claim: $\lim_{n \rightarrow \infty} \sqrt{n^2 + n + 1} - \sqrt{n^2 - n} = 1$

Proof: Using the relation $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

$$\lim_{n \rightarrow \infty} \sqrt{n^2 + n + 1} - \sqrt{n^2 - n} = \lim_{n \rightarrow \infty} \frac{n^2 + n + 1 - n^2 + n}{\sqrt{n^2 + n + 1} + \sqrt{n^2 - n}} = \lim_{n \rightarrow \infty} \frac{2n + 1}{\sqrt{n^2 + n + 1} + \sqrt{n^2 - n}}$$

Then by the fraction property $\frac{a}{b} = \frac{ra}{rb}$ multiply by $\frac{1/n}{1/n}$

$$= \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + \sqrt{1 - \frac{1}{n}}}$$

Then by the quotient property of limits

$$= \frac{\lim_{n \rightarrow \infty} 2 + \frac{1}{n}}{\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + \sqrt{1 - \frac{1}{n}}}$$

Then by the summation property of limits

$$= \frac{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + \lim_{n \rightarrow \infty} \sqrt{1 - \frac{1}{n}}}$$

Then by the polynomial property of limits

$$= \frac{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\sqrt{\lim_{n \rightarrow \infty} 1 + \frac{1}{n} + \frac{1}{n^2}} + \sqrt{\lim_{n \rightarrow \infty} 1 - \frac{1}{n}}}$$

Then by the summation property of limits

$$= \frac{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\sqrt{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n^2}} + \sqrt{\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n}}}$$

Then by common knowledge and previous $\epsilon - N$ proofs

$$= \frac{2 + 0}{\sqrt{1 + 0 + 0} + \sqrt{1 - 0}} = \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{1 + 1} = \frac{2}{2} = 1$$

Thus we have shown that $\lim_{n \rightarrow \infty} \sqrt{n^2 + n + 1} - \sqrt{n^2 - n} = 1$

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