MATH5210 Analysis
Assignment 5
Limits of Functions
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Lemma 1.1: If α is a number s.t. $0 \leq \alpha < \epsilon$, then $\alpha = 0$

Proof: Assume that $\alpha > 0$, let $\epsilon = \frac{\alpha}{2}$.
Then

$$\alpha < \frac{\alpha}{2}$$

This conclusion is not possible, then the assumption must be incorrect. Therefore $\alpha = 0$.

Claim: Let $f : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x)$ exists, then the limit is unique.

Proof: Let $\epsilon > 0$ be given.
Then $\exists$ δ_1 s.t.

$$|f(x) - L| < \frac{\epsilon}{2}$$

for all x s.t.

$$0 < |x - x_0| < \delta_1$$

Likewise $\exists$ δ_2 s.t.

$$|f(x) - M| < \frac{\epsilon}{2}$$

for all x s.t.

$$0 < |x - x_0| < \delta_2$$

Then for $\delta = \min\{\delta_1, \delta_2\}$,

$$|L - M| = |L - f(x) + f(x) - B|$$

and then by the triangle inequality

$$|L - f(x) + f(x) - B| < |L - f(x)| + |f(x) - B|$$

which is equal to

$$= |\frac{\epsilon}{2}| + |\frac{\epsilon}{2}|$$

Since $\epsilon > 0$ was given

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So $0 \leq A - B < \epsilon$ and by lemma 1.1,

$$A - B = 0$$

$$\Rightarrow A = B$$

Therefore if $\lim_{x \to x_0} f(x)$ exists, then the limit is unique. ■

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Claim: Let $f : (a, b) \to \mathbf{R}$ and let $x_0 \in (a, b)$. If $\lim_{x \to x_0} f(x) = L$ and $L > 0$, then $\exists\ \alpha, \delta > 0$ s.t. $f(x) > \alpha\ \forall\ x \in (x_0 - \delta, x_0 + \delta)$.

Proof: Since the limit exists, this is true for $\epsilon = \frac{L}{3}$, $\exists\ \delta_1$ s.t. $0 < |x - x_0| < \delta_1$. Using the definition of convergence,

$$|f(x) - L| < \frac{L}{3}$$

$$\Rightarrow -\frac{L}{3} < f(x) - L < \frac{L}{3}$$

$$\Rightarrow L - \frac{L}{3} < f(x) < L + \frac{L}{3}$$

$$\Rightarrow \frac{2}{3}L < f(x) < \frac{4}{3}L$$

So let $\alpha = \frac{2}{3}L$, then for all x s.t. $0 < |x - x_0| < \delta_1$

$$0 < \alpha = \frac{2}{3}L < f(x)$$

Therefore, if $\lim_{x \to x_0} f(x) = L$ and $L > 0$, then $\exists\ \alpha, \delta > 0$ s.t. $f(x) > \alpha\ \forall\ x \in (x_0 - \delta, x_0 + \delta)$.

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Claim: Let $f : (a,b) \to \mathbf{R}$ and let $x_0 \in (a,b)$. If $f(x) \geq 0$ for all $x \in (a,b)$ and $\lim_{x \to x_0} f(x) = L$ exists, then $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

Proof: I will use sequential characterization of limits to prove the claim. For **all** sequences a_n s.t.

$$\lim_{n \to \infty} a_n = x_0$$

then

$$\lim_{n \to \infty} f(a_n) = L$$

Let $b_n = f(a_n)$ then

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} f(a_n) = L$$

Observe $\lim_{n \to \infty} \sqrt{b_n}$. We know $\lim_{n \to \infty} b_n$ exists, so by the square root property of sequences

$$\lim_{n \to \infty} \sqrt{b_n} = \sqrt{\lim_{n \to \infty} b_n} = \sqrt{L}$$

Therefore, if $f(x) \geq 0$ for all $x \in (a,b)$ and $\lim_{x \to x_0} f(x) = L$ exists, then $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

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Preliminary Work:
If $|x - 2| < 1$
$\Rightarrow -1 < x - 2 < 1$
$\Rightarrow 1 < x < 3$
$\Rightarrow 4 < x + 3 < 6$

Claim: $\lim_{x \to 2} x^2 + x - 5 = 1$

Proof: Let $\epsilon > 0$ be given. Choose $\delta = \min\{\frac{1}{6}\epsilon, 1\}$ then if $|x - 2| < \delta$

$$\Rightarrow |x - 2| < 1$$

Then

$$|f(x) - L| < \epsilon$$

$$\Rightarrow |x^2 + x - 5 - 1| < \epsilon$$

$$\Rightarrow |x - 2| \cdot |x + 3| < \epsilon$$

and by the preliminary work we know that for $|x-2| < 1 \Rightarrow 4 < x+3 < 6$ so

$$\delta \cdot 6 < \epsilon$$

Therefore $\lim_{x \to 2} x^2 + x - 5 = 1$

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Preliminary Work:
If $|x-1| < 1$
$\Rightarrow -1 < x-1 < 1$
$\Rightarrow 0 < x < 2$
$\Rightarrow 0 < x^2 < 4$
$\Rightarrow 4 < x^2 + 4 < 8$
$\Rightarrow 20 < 5(x^2+4) < 40$
and
If $|x-1| < 1$
$\Rightarrow -1 < x-1 < 1$
$\Rightarrow 0 < x < 2$
$\Rightarrow -4 < x-4 < -2$
$\Rightarrow |x-4| < 4$

Claim: $\lim_{x \to 1} \frac{x}{x^2+4} = \frac{1}{5}$

Proof: Let $\epsilon > 0$ be given. Choose $\delta = \min\{5\epsilon, 1\}$ then if $|x-1| < \delta$

$$\Rightarrow |x-1| < 1$$

Then

$$|f(x) - L| < \epsilon$$

$$\Rightarrow \left| \frac{x}{x^2+4} - \frac{1}{5} \right| < \epsilon$$

$$\Rightarrow \left| \frac{5x - x^2 - 4}{5x^2 + 20} \right| < \epsilon$$

$$\Rightarrow \left| \frac{-(x-1) \cdot (x-4)}{5(x^2+4)} \right| < \epsilon$$

$$\Rightarrow \frac{|x-1| \cdot |x-4|}{|5(x^2+4)|} < \epsilon$$

and by the preliminary work we know that for $|x-1| < 1 \Rightarrow 20 < 5(x^2+4) < 40$ and $\Rightarrow |x-4| < 4$ so

$$\frac{|x-1| \cdot (4)}{20} < \epsilon$$

$$\Rightarrow \frac{\delta}{5} < \epsilon$$

Therefore $\lim\limits_{x \to 1} \dfrac{x}{x^2+4} = \dfrac{1}{5}$

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