Math 5210 Assignment 4. Monotone Convergence Theorem

1. Let $\{a_n\}$ be a sequence of non-negative real numbers. Show that the partial sums for the infinite series $\sum_{n=1}^{\infty} a_n$ form a monotone series.

2. What is the approximation property for the infimum of a set A? If $m = \inf(A)$, prove that there is a sequence $a_n \in$ with $\lim a_n = m$. Follow the proof given in class.

3. Give a detailed proof of the MCT in the case of a monotone decreasing sequence which is bounded below. Follow the proof given in class.

4. If $0 < r < 1$, show that $\lim_{n\to\infty} r^n = 0$. Note that with $a_n = r^n$, $a_{n+1} = ra_n$. What happens if $0 \le r \le 1$?

5. Let $a_1 \in (0,1)$ Prove that $a_{n+1} = \sqrt{a_n + 1} - 1$ is a increasing sequence which converges to 0. What happens if $a_0 = 0$, if $a_0 = -1$?

6. Suppose $a_1 \ge 2$ and $a_{n+1} = 2 + \sqrt{a_n - 2}$. Show that a_n converges to 2 or 3. How does the limit depend upon the value of a_1

7. (Bonus) Let $0 < b_1 < a_1$ and let

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

a) Show that $0 < b_n < a_n$

b) Show that b_n is increasing and bounded above.

c) Show that a_n is decreasing and bounded below.

d) Show that $0 < a_{n+1} - b_{n+1} < (a_1 - b_1)/2^n$

e) Conclude that the sequences a_n and b_n converge to a common limit.