1 FEM Solver Implementation

Full weak form:

$$-\delta W = \underbrace{\int_{\Omega_0} \boldsymbol{S} : \delta \boldsymbol{E} \mathrm{d}V}_{-\delta W_{\mathrm{int}}} \underbrace{- \int_{\Omega_0} \hat{\boldsymbol{b}}_0^T \delta \boldsymbol{u} \mathrm{d}V - \int_{\Gamma_{0;\sigma}} \hat{\boldsymbol{t}}_0^T \delta \boldsymbol{u} \mathrm{d}A}_{-\delta W_{\mathrm{ext}}} \stackrel{!}{=} 0. \tag{1.1}$$

We neglect δW_{ext} for now, and focus on internal work:

$$-\delta W_{\mathrm{int}} = \int_{\Omega_0} \boldsymbol{S} : \delta \boldsymbol{E} \mathrm{d}V \equiv \int_{\Omega_0} \delta \boldsymbol{E} : \boldsymbol{S} \mathrm{d}V = \int_{\Omega_0} \delta \boldsymbol{E}^T : \boldsymbol{S} \mathrm{d}V \stackrel{!}{=} 0, \tag{1.2}$$

due to symmetry of both tensors.

We use the linear (engineering) strain

$$\boldsymbol{E} = \mathrm{symmetrize}(\nabla \boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T), \tag{1.3}$$

as well as the linear St.Venant-Kirchoff constitutive relation

$$\boldsymbol{S} = \boldsymbol{C}_{\mathrm{VK}} : \boldsymbol{E}, \tag{1.4}$$

$$(C_{\mathrm{VK}})_{ijkl} = \lambda \delta_{ij} \delta kl + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.5}$$

with the Lamé constants λ and μ.

1.0.1 Discretization within an element

We use the discretized quantities (technically they should have an ”h” subscript, but we leave it away here because we do not handle the analytical quantities anyways)

$$u_i = \sum_{k=1}^{\mathrm{nnode}} \mathrm{N}_k \mathrm{d}_{\mathrm{nodeldof2dof}(k,i)}, \tag{1.6}$$

$$\delta u_i = \sum_{k=1}^{\mathrm{nnode}} \mathrm{N}_k \delta \mathrm{d}_{\mathrm{nodeldof2dof}(k,i)}, \tag{1.7}$$

$$x_i = \sum_{k=1}^{\mathrm{nnode}} \mathrm{N}_k \mathrm{X}_{\mathrm{nodeldof2dof}(k,i)}, \tag{1.8}$$

where

$$\mathrm{nodeldof2dof}(k, i) = k \cdot \mathrm{nnode} + i \tag{1.9}$$

is the mapping from node k and node-local (node-scoped) dof i to the corresponding element-global (element-scoped) dof. The inverse of this mapping is

$$\mathrm{dof2nodeldof}(g) = \{\mathrm{floor}(g/\mathrm{ndofn}), \mathrm{mod}(g, \mathrm{ndofn})\}. \tag{1.10}$$

We will henceforth use the shorthand notation

$$\mathrm{q}_{(k,j)} := \mathrm{q}_{\mathrm{nodeldof2dof}(k,i)}, \tag{1.11}$$

for any non-tensor vector quantity q as above.

Now, given the need for the gradient of the unknown displacement in (1.3), we need to discretize it. In index notation,

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial (\mathrm{Nd})_i}{\partial x_j} = \sum_{k=1}^{\mathrm{nnode}} \frac{\partial \mathrm{N}_k}{\partial x_j} \mathrm{d}_{(k,i)}. \tag{1.12}$$

Now, we can find the gradient of the shape functions from the parametric coordinates and (inverse) Jacobian:

$$\frac{\partial \mathrm{N}_k}{\partial x_j} = \sum_{l=1}^{\mathrm{ndofn}} \frac{\partial \mathrm{N}_k}{\partial \xi_l} \frac{\partial \xi_l}{\partial x_j}. \tag{1.13}$$

To find the inverse Jacobian, we need the Jacobian:

$$J_{lj} = \frac{\partial x_j}{\partial \xi_l} = \frac{\partial \sum_{k=1}^{\mathrm{nnode}} \mathrm{N}_k \mathrm{X}_{(k,j)}}{\partial \xi_l} = \sum_{k=1}^{\mathrm{nnode}} \frac{\partial \mathrm{N}_k}{\partial \xi_l} \mathrm{X}_{(k,j)}. \tag{1.14}$$

Then we can simply write (1.13) as

$$\frac{\partial \mathrm{N}_k}{\partial x_j} = \sum_{l=1}^{\mathrm{ndofn}} \frac{\partial \mathrm{N}_k}{\partial \xi_l} (\boldsymbol{J}^{-1})_{lj} =: (\tilde{\mathbf{N}})_{kj}. \tag{1.15}$$

Note that this computation is a completely isolated task.

But, we now look at the rest of the stuff, i.e. (1.2). We start from (1.3) basically, and insert (1.12).

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\sum_{k=1}^{\mathrm{nnode}} (\frac{\partial \mathrm{N}_k}{\partial x_j} \mathrm{d}_{(k,i)}) + \sum_{k=1}^{\mathrm{nnode}} (\frac{\partial \mathrm{N}_k}{\partial x_i} \mathrm{d}_{(k,j)})\right) \\ &= \frac{1}{2}\sum_{k=1}^{\mathrm{nnode}} \left(\frac{\partial \mathrm{N}_k}{\partial x_j} \mathrm{d}_{(k,i)} + \frac{\partial \mathrm{N}_k}{\partial x_i} \mathrm{d}_{(k,j)}\right) = \frac{1}{2}\sum_{k=1}^{\mathrm{nnode}} \left(\tilde{\mathrm{N}}_{kj} \mathrm{d}_{(k,i)} + \tilde{\mathrm{N}}_{ki} \mathrm{d}_{(k,j)}\right). \end{aligned} \tag{1.16}$$

To get this into a form like $E_{ij} = \mathrm{B}_{ijm}\mathrm{d}_m$, where $m = 1, 2, ..., 6$, we must use the inverse mapping given in (1.10), which we will just shorthand (similar to 1.11) with the vector/array $[m]$, where node is $= [m]_1$ and node-local dof is $= [m]_2$. Then,

$$E_{ij} = \frac{1}{2}\sum_{m=1}^{\mathrm{ndof}} \left(\tilde{\mathrm{N}}_{[m]_1 j} \mathrm{d}_m \delta_{[m]_2 i} + \tilde{\mathrm{N}}_{[m]_1 i} \mathrm{d}_m \delta_{[m]_2 j}\right), \tag{1.17}$$

where the diracs ensure that basically i and j in E_{ij} are basically matching how it is in (1.16). So, equivalently,

$$\mathrm{B}_{ijm} = \frac{1}{2}\left(\tilde{\mathrm{N}}_{[m]_1 j} \delta_{[m]_2 i} + \tilde{\mathrm{N}}_{[m]_1 i} \delta_{[m]_2 j}\right). \tag{1.18}$$

Finally, we can look at the whole internal work equation (1.2) and formulate its discretized version as a quadratic form, where we again leave out any explicit "h" subscript, even though it is a discretized quantity;

$$\delta W_{\text{int}}^{(e)} = \int_{\Omega_0^{(e)}} \delta(\boldsymbol{E}^T)_{ij} S_{ij} \mathrm{d}V = \int_{\Omega_0^{(e)}} \delta E_{ji} S_{ij} \mathrm{d}V = \int_{\Omega_0^{(e)}} \delta E_{ji} (C_{\mathrm{VK}})_{ijkl} E_{kl} \mathrm{d}V$$
$$= \int_{\Omega_0^{(e)}} \delta \mathrm{d}_{\underline{m}} \mathrm{B}_{ji\underline{m}} (C_{\mathrm{VK}})_{ijkl} \mathrm{B}_{kl\overline{m}} \mathrm{d}_{\overline{m}} \mathrm{d}V = \int_{\Omega_0^{(e)}} \delta \mathrm{d}_{\underline{m}} \mathrm{k}^{(e)}_{\underline{m}\overline{m}} \mathrm{d}_{\overline{m}} \mathrm{d}V = \delta \mathrm{d}_{\underline{m}} \int_{\Omega_0^{(e)}} \mathrm{k}^{(e)}_{\underline{m}\overline{m}} \mathrm{d}V \mathrm{d}_{\overline{m}}$$
$$= \delta \mathrm{d}_{\underline{m}} \mathrm{K}^{(e)}_{\underline{m}\overline{m}} \mathrm{d}_{\overline{m}} \tag{1.19}$$

Of course, this holds $\forall \delta \mathbf{d}$, so we end up with this on each element:

$$\mathrm{K}^{(e)}_{\underline{m}\overline{m}} \mathrm{d}_{\overline{m}}. \tag{1.20}$$

To get the integral in (1.19) w.r.t. $\boldsymbol{\xi}$, we recall

$$\mathrm{d}V = \det(\mathbf{J}) \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{1.21}$$

as well as the parametric domain, and hence,

$$\int_{\Omega_0^{(e)}} \mathbf{k}^{(e)} \mathrm{d}V = \int_{\xi_1=0}^{1} \int_{\xi_2=0}^{1-\xi_1} \mathbf{k}^{(e)}(\boldsymbol{\xi}) \det(\boldsymbol{J}) \mathrm{d}\xi_2 \mathrm{d}\xi_1 \tag{1.22}$$

We assemble that to get the actual linear system.