#1.

Plotting the random triplets using C's standard number generator in 3D and adjusting the viewing angle just a bit (Figure 1), we clearly see a set of planes: Doing the same but using

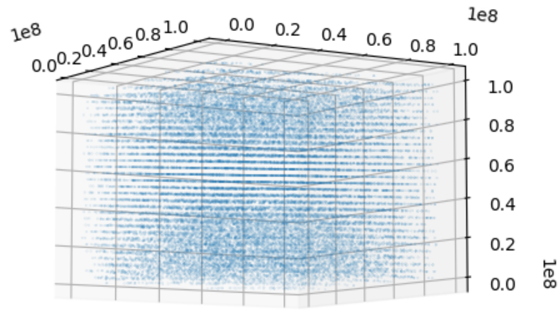

Figure 1: Apparent set of planes in C's random generated triplets. While harder to see them away from the center, looking at the edges we see that they are still planes.

Python's `np.random.rand()`, we don't see any such planes(Figure 2).

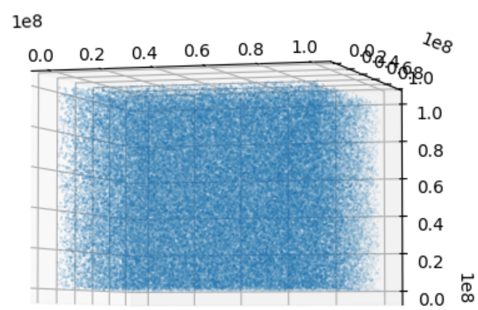

Figure 2: Random triplets generated using Python's NumPy. After looking around, we don't see any planes

#2.

If we want want to use the rejection method on an exponential, we require that our bounding distribution is less or equal to e^{-x} for all $x \geq 0$. If we try a gaussian, we find that it does not satisfy this property. To prove this, consider $g(x) = a\exp(-x^2/2\sigma^2)$. If we require $g(x) > e^{-x}$ for $x \geq 0$, we get

$$
\begin{aligned}
ae^{-\frac{x^2}{2\sigma^2}} &> e^{-x} \\
\Rightarrow e^{\frac{-x^2}{2\sigma^2}+x} &> \frac{1}{a} \\
\Rightarrow -\frac{x^2}{2\sigma^2} + x &> \ln\left(\frac{1}{a}\right) \\
\Rightarrow -\frac{x^2}{2\sigma^2} + x + \ln(a) &> 0.
\end{aligned}
$$

However, this cannot be the case for all x since no matter the choice of a and σ, the negative sign in front of the quadratic term guarantees the parabola will become negative at some point. So, a gaussian can't be use for our method.

As for a power law, $x^{-\alpha}$, since our domain is $x \geq 0$, we cannot use $\alpha < 0$ since it diverges at $x = 0$. We also cannot use $\alpha > 0$ since at $x = 0$, the power law equals 0 while $e^0 = 1 > 0$. The remaining case is $\alpha = 0$ which makes our power law a constant function. While this works, it is quite boring, so we won't look into it any further.

Now, a lorentzian would work since it satisfies the desired property. To show this, consider $f(x) = x - \ln(1+x^2)$. $f'(x) = 1 - \frac{1}{1+x^2} > 0$ for all $x \neq 0$. So f must me constantly increasing. However, since $f(0) = 0$, we conclude that $f(x) > 0$ for all $x > 0$. This then implies that for positive x values,

$$
\begin{aligned}
1 &> \ln(1+x^2) \\
\Rightarrow e^x &> 1 + x^2 \\
\Rightarrow \frac{1}{1+x^2} &> e^{-x}.
\end{aligned}
$$

We can thus use a standard lorentzian for the rejection method: We generate random numbers taken from this bounding distribution and keeping those that are less than e^{-x}. Plotting the resulting PDF, we get Figure 3, which matches the desired exponential. Computing the efficiency, we get something around 63%.

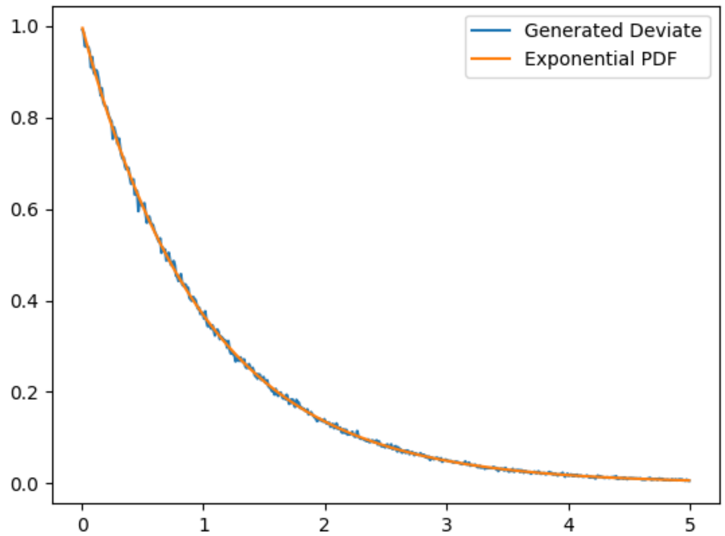

Figure 3: Random exponential deviates generated using the rejection method with a Lorentzian.

#3.

We take a region of the (u, v) plane where $0 \leq u \leq 1$ and impose a range for v such that $0 \leq u \leq \sqrt{p(v/u)} = e^{-v/2u}$. To solve for the maximum value v can take, we turn the last inequality into an equality and solve for the biggest v:

$$u = e^{-v/2u}$$

$$\Rightarrow v = -2u\ln(u)$$

$$v' = -2\ln(u_0) - 2 = 0$$

$$\Rightarrow u_0 = e^{-1}$$

$$\Rightarrow v_{max} = v(u_0) = 2e^{-1}$$

So, we will take the region $([0, 1], [0, 2e^{-1}])$. The rest is pretty straightforward: generate random pairs of u and v, if $u \leq e^{-v/2u}$, return v/u. Doing this, we get the PDF shown in Figure 4. The efficiency is easily computed and gives a value close to 68%.

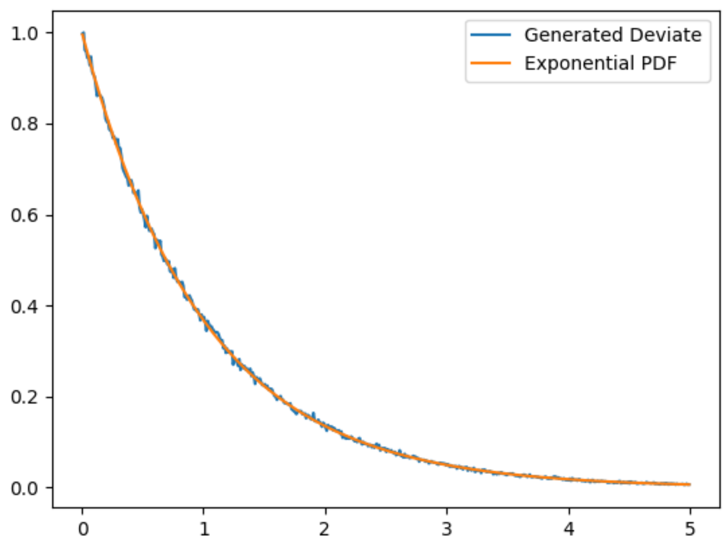

Figure 4: Random exponential deviates generated using the ratio-of-uniform method.