First, we start by doing a lorentzian fit of the form:

$$d = \frac{a}{1 + \frac{(t-t_0)^2}{\omega^2}}$$

whose derivatives with respect to the parameters are:

$$\frac{\partial d}{\partial a} = \frac{1}{1 + \frac{(t-t_0)^2}{\omega^2}},$$

$$\frac{\partial d}{\partial t_0} = \frac{\frac{2a(t-t_0)}{\omega^2}}{\left(1 + \frac{(t-t_0)^2}{\omega^2}\right)^2}$$

and

$$\frac{\partial d}{\partial \omega} = \frac{\frac{2a(t-t_0)^2}{\omega^3}}{\left(1 + \frac{(t-t_0)^2}{\omega^2}\right)^2}$$

Looking at the data, we can make a rough guess for the parameters, which I estimated to be about (a, t_0, ω)=(1.5, 2.0×10^{-4}, 0.25×10^{-4}), which is not a terrible guest (Figure 1). Applying Newton's method with 10 iterations, we end up with best fit parameters of (a, t_0, ω)=(1.4228, 1.92359×10^{-4}, 0.17924×10^{-4}).

We can estimate the noise by taking $N = \langle r \rangle$ where r is our residuals, which gives us $N \approx 0.00064$.

This then allows us to calculate the error on our fit parameters using the covariance matrix:

$$C = (A_m^T N^{-1} A_m)^{-1}$$

where A_m is our gradient with respect to our parameters. The errors (square root of the diagonal elements) is thus

$$\Delta m = (4 \times 10^{-4}, 5 \times 10^{-9}, 8 \times 10^{-9})$$

We can also use numerical derivatives instead of analytical ones when calculating our fit. Using the same function as in Problem Set 1 (#2), and carrying out the fit this way, we find: $a = 1.4234 \pm 0.0005$, $t_0 = (1.92252 \pm 0.00006) \times 10^{-4}$ and $\omega = (0.17908 \pm 0.00008) \times 10^{-4}$ which are within 2σ of the ones found above (except t_0). This new fit gives $\chi^2 = 100417$ which is higher than the first fit.

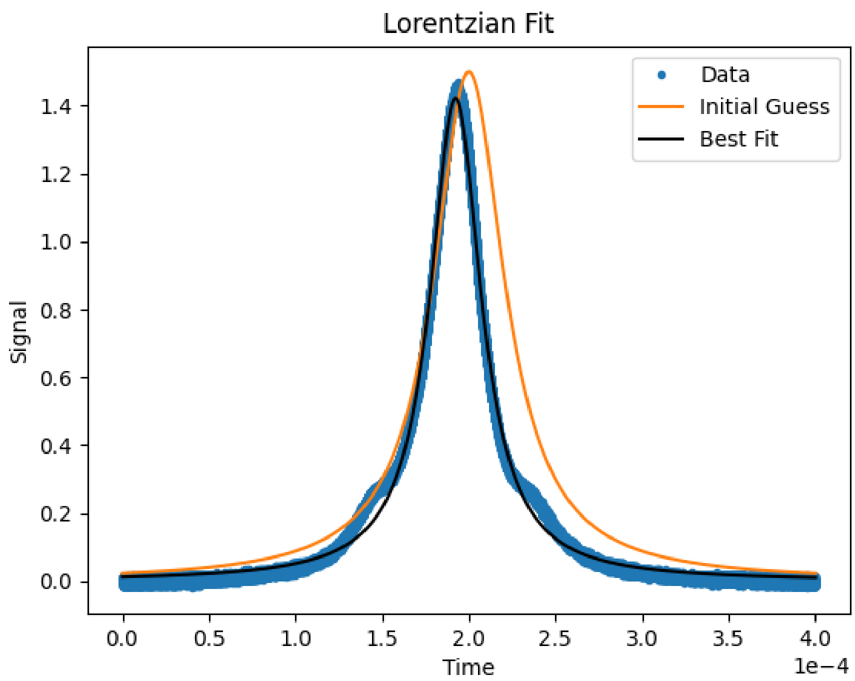

Figure 1: Lorentzian fit using Newton's method (and analytical derivatives). The best fit gives $\chi^2 = 100000$

By looking at the data, we see that there are two small bumps on the sides of the peak which cannot be modelled by a simple Lorentzian. To account for this, we can add on two Lorentzians that are displaced by dt from t_0:

$$d = \frac{a}{1 + \frac{(t-t_0)^2}{\omega^2}} + \frac{b}{1 + \frac{(t-t_0+dt)^2}{\omega^2}} + \frac{c}{1 + \frac{(t-t_0-dt)^2}{\omega^2}}$$

Again, looking at the data, we can estimate that these new parameters should be about (b, c, dt) =(0.1, 0.1, 0.50$\times 10^{-4}$). Doing the fit on with this new model (using numerical derivatives), we find: $a = 1.4413 \pm 0.0005$, $t_0 = (1.92478 \pm 0.00007) \times 10^{-4}$, $\omega = (0.1609 \pm 0.0001) \times 10^{-4}$, $b = 0.1019 \pm 0.0005$, $c = 0.0657 \pm 0.0004$ and $dt = (0.4406 \pm 0.0008) \times 10^{-4}$.

As seen from Figure 2, this gives a much netter fit than a simple Lorentzian.

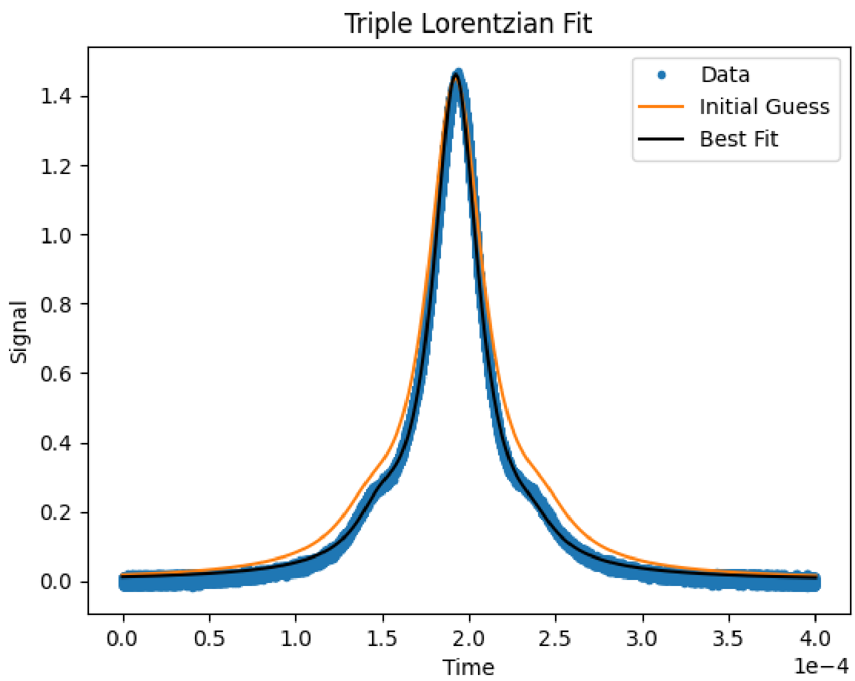

Figure 2: Triple Lorentzian fit using Newton's Method. The best fit gives $\chi^2 = 33823$

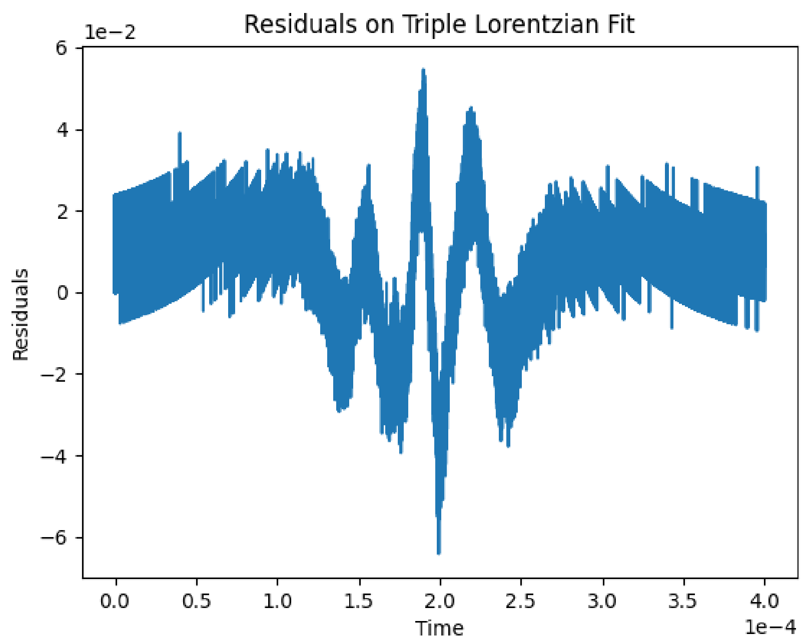

Figure 3: Residuals of the Triple Lorentzian fit

As we can see from Figure 3, the residuals don't agree with the assumption that the error bars are uniform variance, hence our model doesn't provide a complete description of the data.

Next, let's create a realisation for the errors on our best fit parameters. To do so, we perturb the ones found above by sampling from a distribution that follows the correlation of our errors and add those to the best fit solution. This is done by taking the Cholesky decomposition L of our covariance matrix $C = (A_m^T N^{-1} A_m)^{-1}$ and multiplying on the right by a 6-dimensional vector g whose components are taken from a gaussian distribution with zero-mean and unity-variance. This thus gives us:

$$m_{new} = m + Lg$$

These new parameters now give a pretty similar fit, as seen from Figure 4, whose χ^2 varies by less than ~ 100.

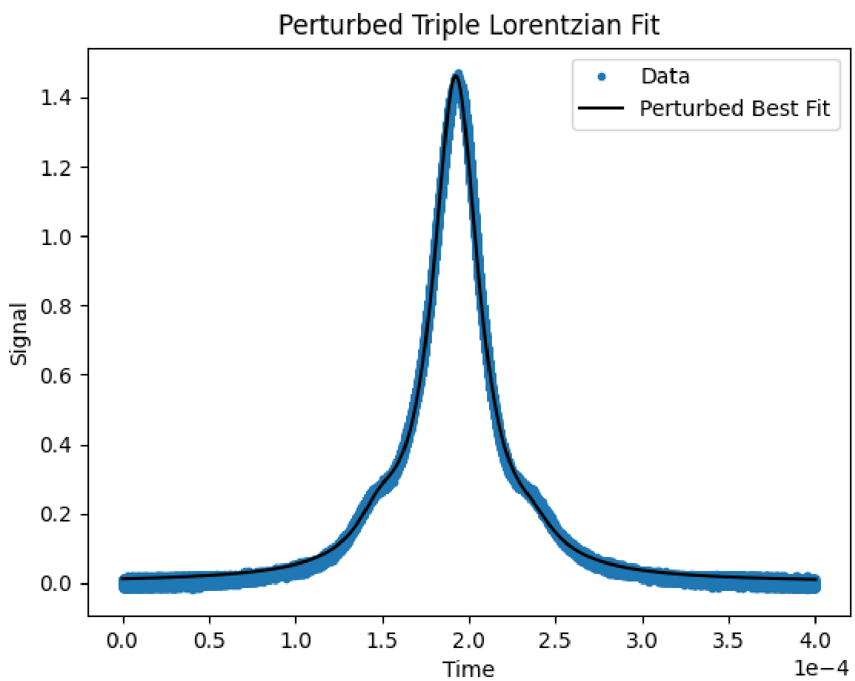

Figure 4: Triple Lorentzian fit obtained by using perturbed best fit parameters. This particular run, $\chi^2 = 33774$

While Newton's method yields acceptable results, one can use an MCMC to often get a better result and get a better idea on the error on the parameters. We will do one by following pretty much the same code as used in lecture: take a guess, take a trial step, if χ^2 decreases, take that step and if not, take it with some probability, repeat many many times and look where the parameters have converged to. However, since our parameters are correlated, it would be wise that the trial steps follow this covariance (given by `cov`, which is the C from earlier). To do this, we can use `np.random.multivariate_normal(0,cov)` which generates this desired random displacement, and we can then add it to the current position in parameter space. One could scale this displacement by a order unity factor, but as we will see shortly, we are content with a scaling of 1. For this MCMC, we will take 20000 steps. As we can see from Figure 5, the parameters converge after about ~ 2500 steps and so we can take the mean starting from this point on for each parameter and take the standard deviation to get the error. Doing so gives us the final parameters (Figure 6):

$$a = 1.4430 \pm 0.0005$$

$$t_0 = (1.92579 \pm 0.00005) \times 10^{-4}$$
$$\omega = (0.16065 \pm 0.00009) \times 10^{-4}$$
$$b = 0.1039 \pm 0.0004$$
$$c = 0.0648 \pm 0.0004$$
$$dt = (0.4457 \pm 0.0006) \times 10^{-4}$$

While these error bars are just slightly lower if not equal to those found using Newton's method, the χ^2 has decrease by about 400 which is not bad.

Also, if we keep track of how many trial steps were accepted, we get an acceptance rate of about 25% which would increase/decrease when scaling the step, hence I chose to stick with a factor of 1.

Finally, we can use these fit values to calculate the cavity resonance width. If know that the laser sidebands are separated from the main peak by 9 GHz and we assume a linear relation between time and frequency, we find that the width is 2×9 GHz and that the error is thus $\Delta d\nu = 2 \times 9\text{GHz} \times \dfrac{\Delta(dt)}{dt}$. Hence:

$$d\nu = (18.00 \pm 0.02)\text{GHz}$$

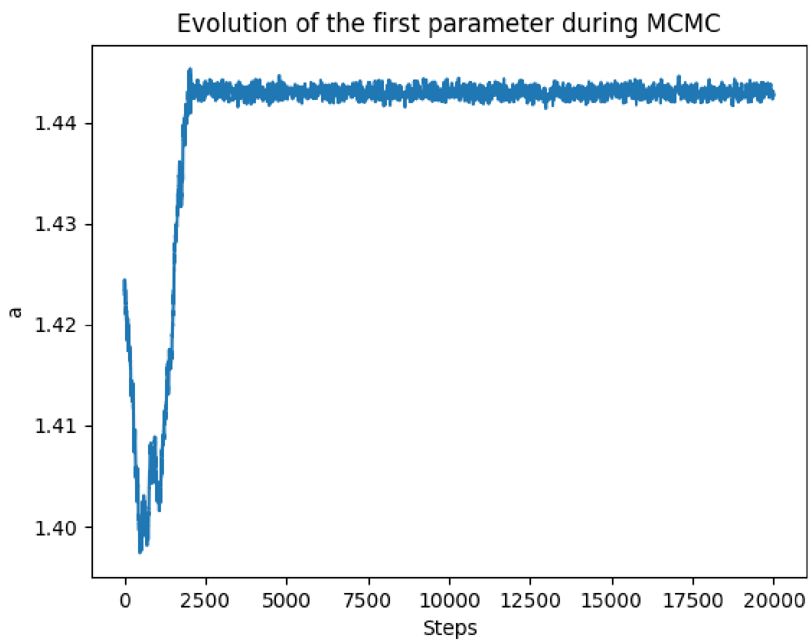

Figure 5: Convergence of the a parameter during the MCMC

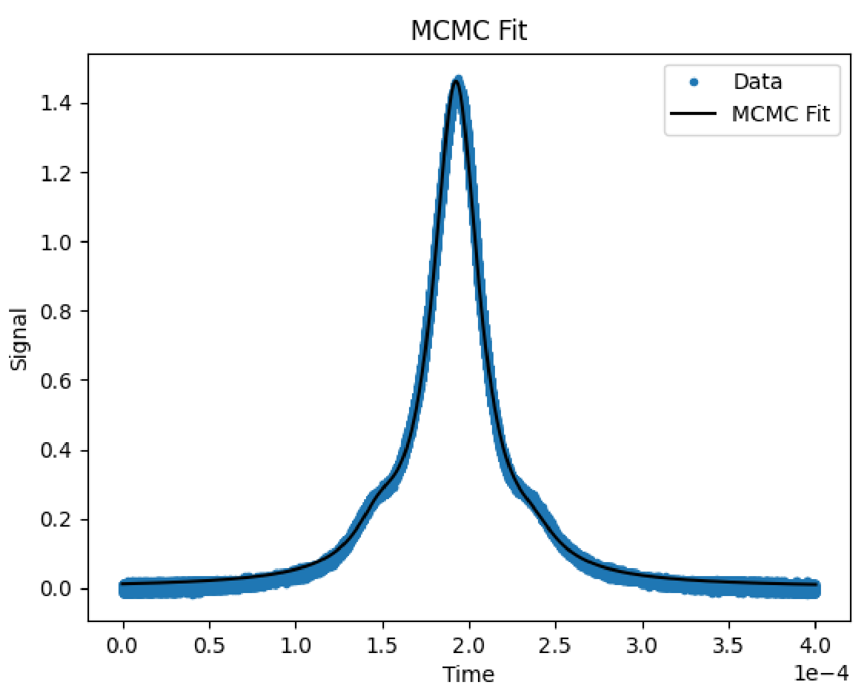

Figure 6: MCMC fit on the Triple Lorentzian. This gives $\chi^2 = 33369$