#1.

Taking the error on our data points to be $\sigma_i = \Delta E_i/2$ where ΔE is the difference between the upper and lower uncertainties. Using this, we can estimate the χ^2 for the initial fit. Doing this with the fit parameters [69, 0.022, 0.12, 0.06, 2.1e-9, 0.95], we find

$$\chi^2_{init} = 3272.21$$

Considering that we have 2507 data points, the fit is far from being the best since we would expect a χ^2 of around 2500.

#2.

We can do a fit using Newton's method. However, I found that a plain Newton scheme would yield divergent and/or non-physical values. So, I instead did a Levenberg-Marquardt scheme where I fixed $\lambda = 0.8$. Combining this with a SVD intead of `np.linalg.inv()` to invert the necessary matrix, I got rid of this divergent behaviour. Doing a total of 15 steps, I end up with the following parameters:

$$H_0 = 67.9 \pm 0.1$$

$$\Omega_b h^2 = 0.0223 \pm 0.0001$$

$$\Omega_c h^2 = 0.1184 \pm 0.0002$$

$$\tau = 0.0576 \pm 0.0006$$

$$A_s = (2.103 \pm 0.003) \times 10^{-9}$$

$$n_s = 0.971 \pm 0.002$$

These give $\chi^2 = 2576.96$, which is much better than the initial guess.

#3.

We can also do a MCMC to estimate the fit parameters. To do this, we use the same code as the MCMC done on last problem set, but now we use the covariance matrix that was found while doing the Newton fit. After 7000 steps, I find that though χ^2 has settled around 2575 (Figure 1), not all parameters have converged in a satisfactory manner (e.g Figure 2) and some, like the optical depth, don't settle at all (see Figure 3)! Given the convergence of χ^2, I believe that the fit is not constrained enough as to give one single minimum. So, before I evaluate the dark energy density, I will first impose the appropriate constraints on τ in part 4 and use this result instead since I do not trust the result of this first MCMC fit.

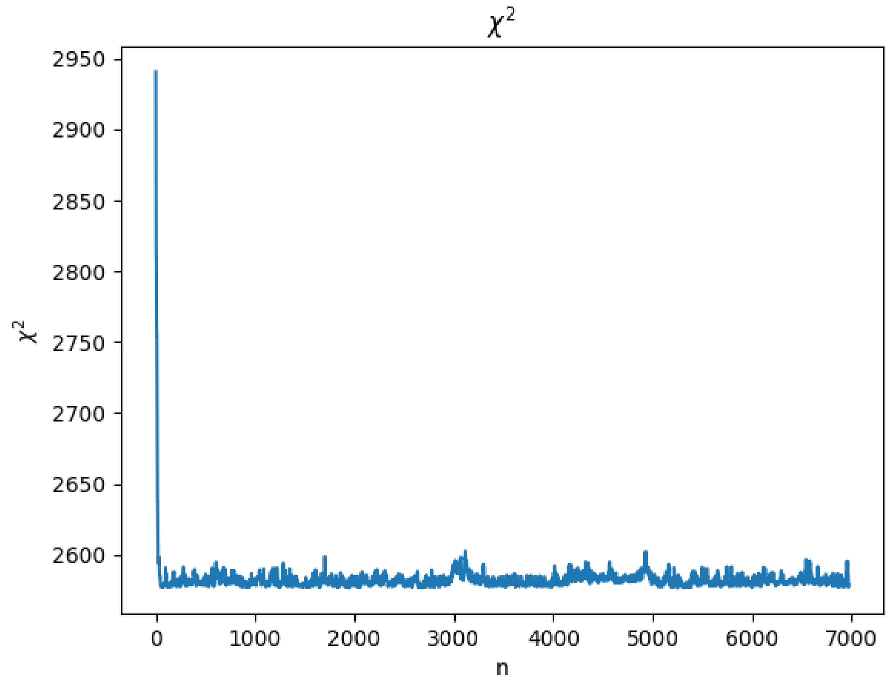

Figure 1: Convergence of χ^2 during the unconstrained MCMC

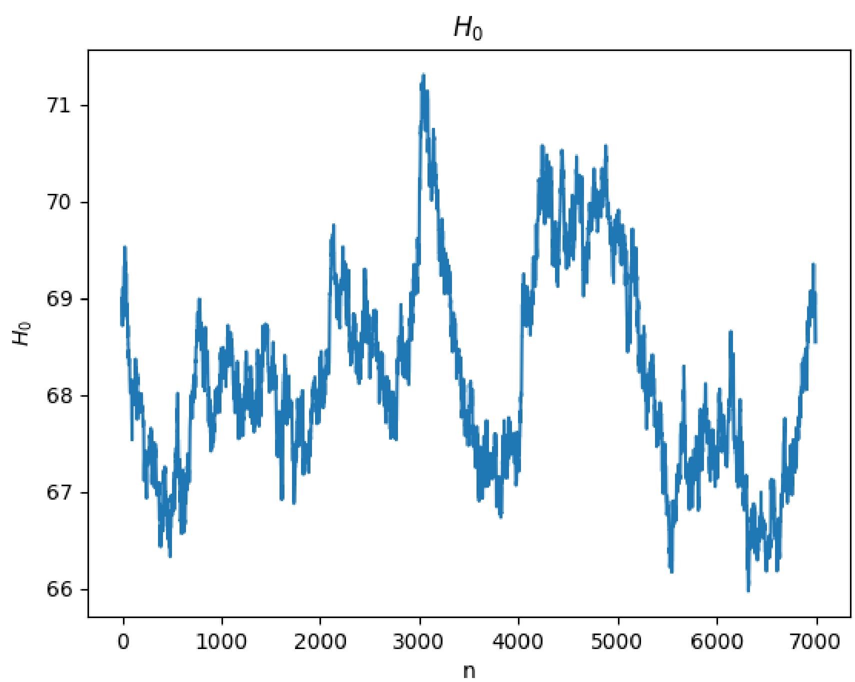

Figure 2: Evolution of H_0 during the unconstrained MCMC

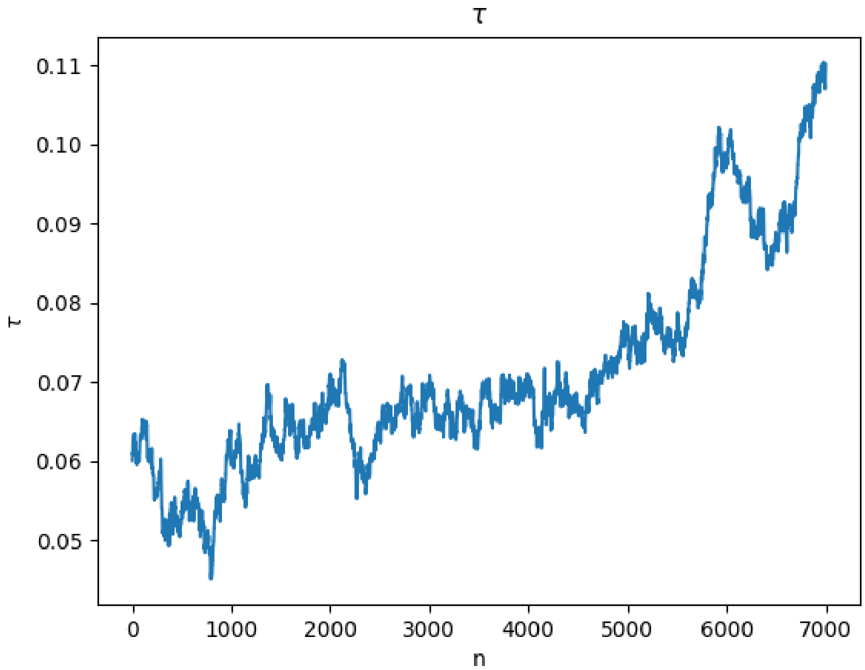

Figure 3: Evolution of τ during the unconstrained MCMC

#4.

Now, we constrain the optical depth to $\tau = 0.0540 \pm 0.00074$. To do this, we first need to redo a Newton-method fit with this new constraint, get a new covariance matrix and then use it in a new MCMC. In the first fit, when taking a step, we first look whether or not τ has exited its allowed range and if so, the step *in this direction* is 0 (the other parameters are updated regardless). This gives the following values for the fit parameters:
With the new covariance matrix, we can now redo an MCMC, but imposing the new constrain. This is simply done by refusing steps that cause τ to exit its allowed range, and the rest is the same. Doing this, we get a convergence that, though far from ideal, is is better than before (Figure 4). Using the resulting chain (omitting the first 500 steps), we can estimate the parameters to be:

$$H_0 = 67.4 \pm 0.6$$

$$\Omega_b h^2 = 0.0223 \pm 0.0003$$

$$\Omega_c h^2 = 0.1203 \pm 0.0009$$

$$\tau = 0.057 \pm 0.002$$

$$A_s = (2.11 \pm 0.01) \times 10^{-9}$$

$$n_s = 0.966 \pm 0.005$$

Which gives $\chi^2 = 2584.59$

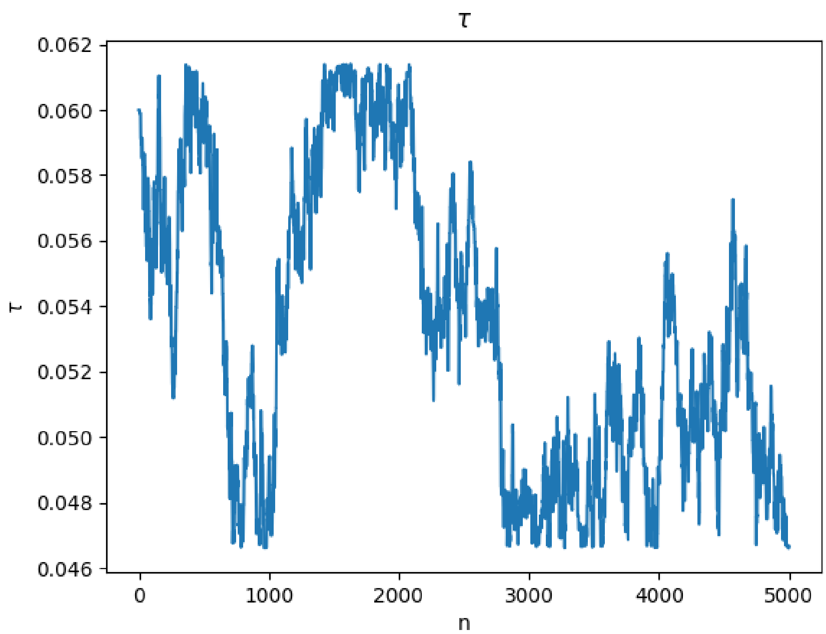

Figure 4: Evolution of τ during the *constrained* MCMC

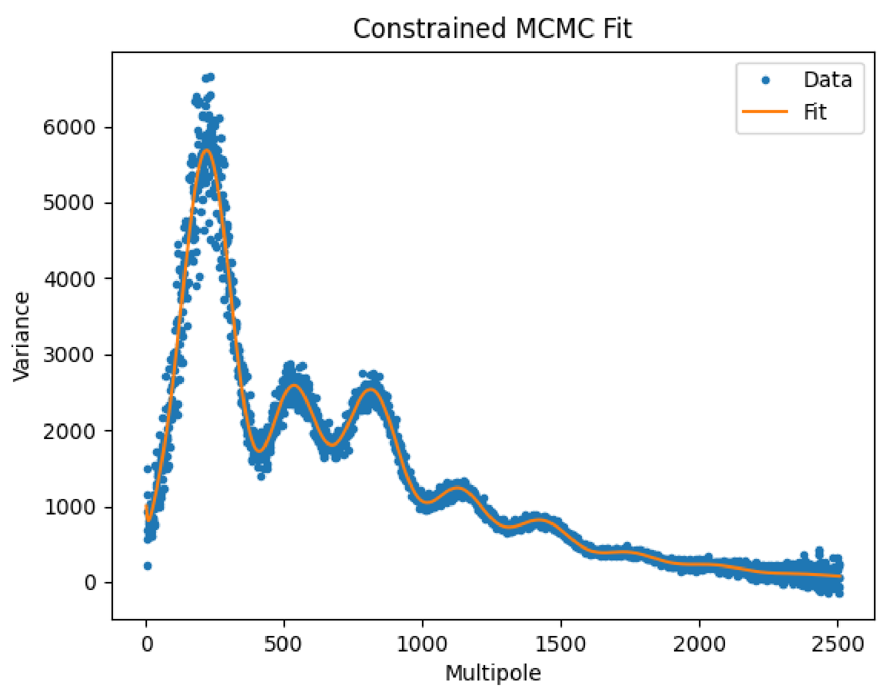

Figure 5: Fit on the data using an MCMC chain with constraint on τ imposed

These values can then be used to estimate the mean value of the dark energy Ω_Λ. Assuming a flat universe, we get:

$$\Omega_\Lambda = 1 - (\Omega_b + \Omega_c) = 1 - \frac{(\Omega_b h^2 + \Omega_c h^2)}{h^2} = 1 - 10^4 \cdot \frac{(\Omega_b h^2 + \Omega_c h^2)}{H_0^2}$$

Using our fit parameters and their uncertainties, we find:

$$\Omega_\Lambda = 0.686 \pm 0.008$$