Sharpened CG Iteration Bound for High-contrast Heterogeneous Scalar Elliptic PDEs

Going Beyond Condition Number

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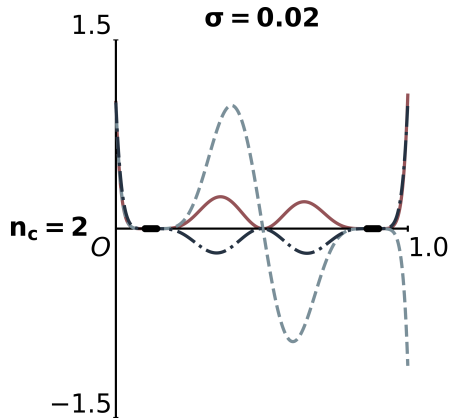

TU Delft

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- Main R.Q.: "How can we improve existing estimates on the total number of necessary CG iterations?"

Structure

- Introducing CG
- How Does CG Converge? The Role of Eigenvalues
- Preconditioning: Taming High-Contrast Problems
- High-contrast coefficients: split eigenspectrum
- Towards Sharper Iteration Bounds
- Multi-Cluster Spectra
- How Sharp Are the New Bounds?
- New Bounds in Practice: Using Ritz Values
- Conclusion: Key Takeaways & Future Directions

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- Restate R.Q.: "How do we improve/sharpen the classical bound m_1 for model problem?"

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- Restate R.Q.: "How can we construct an CG iteration bound that accounts for the influence of the eigenvalue distribution?"

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- Restate R.Q.: ”How can we construct a CG iteration bound that can distinguish between different preconditioners with similar conditioner numbers?”

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- Recap: $\epsilon_m < \min_{r \in \mathcal{P}_{m-1}, r(0)=1} \max_{\lambda \in \sigma(A)} |r(\lambda)| \leq \epsilon$

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- Note: The transition from classical bounds to multi-cluster bounds could be made more explicit—perhaps with a summary slide or visual.

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- Animation: cluster_partitioning, visualize above partitioning and subsequent calculation $m_{N_{\text{cluster}}}$.

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- Idea: simple, we split at the largest (relative) gap between subsequent eigenvalues, check if the two resulting clusters would give a sharper bound than just one uniform cluster, that is $m_2(\kappa, \kappa_l, \kappa_r) < m_1(\kappa)$. If so, we split the clusters and repeat the process for each created cluster. If not, we stop partitioning. In the process we keep track of all the extremal eigenvalues. After partitioning, we calculate $m_{N_{\text{cluster}}}$.
- Animation: cluster_partitioning, visualize above partitioning and subsequent calculation $m_{N_{\text{cluster}}}$.
- Restate R.Q.: "How much sharper is $m_{N_{\text{cluster}}}$ compared to $m_1(\kappa)$ for our model problem?"

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- Restate R.Q.: "How can we compute $m_{N_{\text{cluster}}}$ for our model problem in practice?"

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- Animation: ritz_value_migration