Sharpened CG Iteration Bound for Schwarz-preconditioned High-contrast Heterogeneous Scalar Elliptic PDEs

Going Beyond Condition Number

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EEMCS-DIAM Numerical Analysis MSc. Thesis Presentation, April, 2025

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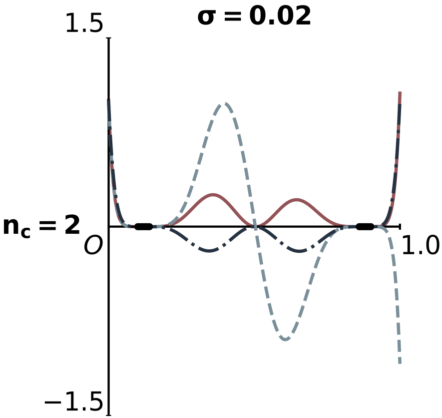

TU Delft

Opening

Darcy Problem

Opening

Conjugate Gradient Method

Opening

Condition Number

Opening

Preconditioners

Opening

Research Gap

Structure

- Research Questions
- Mathematical Background
- Related Work
- Preliminary Results
- Outlook

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Research Questions

Main Research Question

How can we refine the CG iteration bound for Schwarz-preconditioned high-contrast heterogeneous elliptic problems beyond the classical condition number-based bound?

Research Questions

Subsidiary Research Questions

Q1 What other spectral characteristics like the condition number can we consider?

Q2 How to estimate the characteristics from **Q1**?

Q3 Given a toy eigenspectrum, how can we sharpen the CG iteration bound?

Q4 How does the sharpened bound from **Q3** compare to the classical CG bound?

Q5 How does the performance described in **Q4** depend on the characteristics found in **Q1**?

Q6 Can the sharpened bound from **Q3** distinguish between preconditioners?

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Mathematical Background

Conjugate gradient method

Algorithm Conjugate Gradient Method [1]

$\mathbf{r}_0 = \mathbf{b} - A\mathbf{u}_0$, $\mathbf{p}_0 = \mathbf{r}_0$, $\beta_0 = 0$
for $j = 0, 1, 2, \ldots, m$ **do**
    $\alpha_j = (\mathbf{r}_j, \mathbf{r}_j)/(A\mathbf{p}_j, \mathbf{p}_j)$
    $\mathbf{u}_{j+1} = \mathbf{u}_j + \alpha_j \mathbf{p}_j$
    $\mathbf{r}_{j+1} = \mathbf{r}_j - \alpha_j A\mathbf{p}_j$
    $\beta_j = (\mathbf{r}_{j+1}, \mathbf{r}_{j+1})/(\mathbf{r}_j, \mathbf{r}_j)$
    $\mathbf{p}_{j+1} = \mathbf{r}_{j+1} + \beta_j \mathbf{p}_j$
end for

[1] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Second. Society for Industrial and Applied Mathematics, 2003. DOI: 10.1137/1.9780898718003. eprint: https://epubs.siam.org/doi/book/10.1137/1.9780898718003

Mathematical Background

Conjugate gradient method

- Iterative, projection method onto a Krylov subspace $\mathcal{K}_m(A_0, \mathbf{r}_0)$ given by

$$\text{span}\{\mathbf{r}_0, A\mathbf{r}_0, A^2\mathbf{r}_0, \ldots, A^{m-1}\mathbf{r}_0\}$$

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$$\text{span}\{\mathbf{r}_0, A\mathbf{r}_0, A^2\mathbf{r}_0, \ldots, A^{m-1}\mathbf{r}_0\}$$

- Approximate solution can be expressed as

$$\mathbf{u}_m = \mathbf{u}_0 + \sum_{i=0}^{m-1} c_i A^i \mathbf{r}_0 = \mathbf{u}_0 + q_{m-1}(A)\mathbf{r}_0$$

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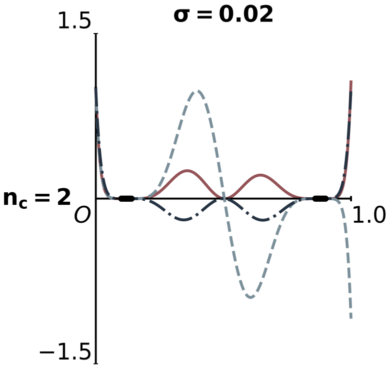

Figure: Residual polynomial $r_m(\lambda) = 1 - \lambda q_{m-1}(\lambda)$

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- Minimize residual polynomial on eigenvalues of A

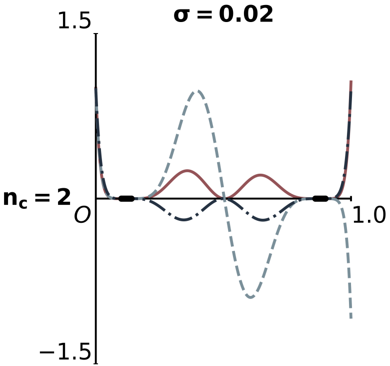

Figure: Residual polynomial $r_m(\lambda) = 1 - \lambda q_{m-1}(\lambda)$

Mathematical Background

Conjugate gradient method

- Classical (condition number) convergence bound:

Theorem

The error of the m^{th} iterate of the CG algorithm is bounded by

$$||\mathbf{e}_m|| \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m ||\mathbf{e}_0||_A,$$

where $\kappa = \lambda_{max}/\lambda_{min}$ is the condition number of (symmetric matrix) A.

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- Only sharp for uniform eigenvalue distributions!

$$||\mathbf{e}_m||_A \leq \min_{r\in\mathcal{P}_{m-1},r(0)=1}\ \max_{\lambda\in[\lambda_{\min},\lambda_{\max}]} |r(\lambda)|||\mathbf{e}_0||_A \overset{\text{uniform } \sigma(A)}{=} \frac{||\mathbf{e}_0||}{C_m\left(\frac{\kappa+1}{\kappa-1}\right)}$$

Mathematical Background

Conjugate gradient method

Setting $\lambda_{\text{min}} = 0.1$ and $\lambda_{\text{max}} = 0.9$ gives $m_{\text{classical}} = 26$. Worst case distribution:

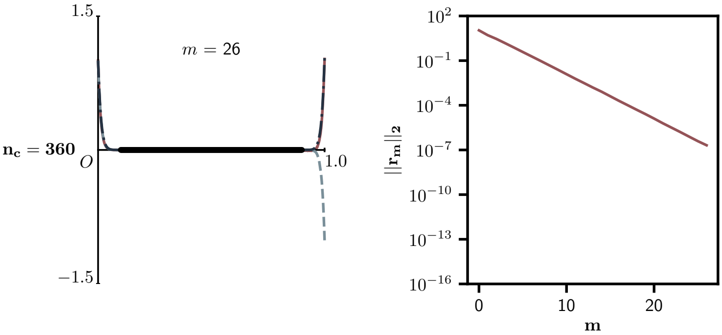

Figure: CG convergence for uniform spectrum.

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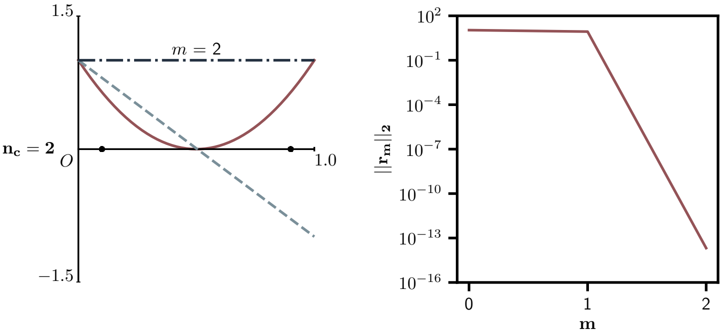

Figure: CG convergence for spectrum with two distinct eigenvalues.

Mathematical Background

Schwarz preconditioners

- Derived from the Alternating Schwarz method[2]

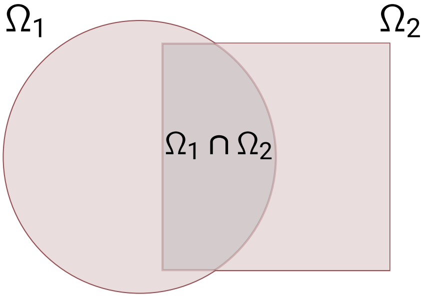

Figure: Domain decomposition with overlapping subdomains.

[2] V. Dolean, P. Jolivet, and F. Nataf. *An Introduction to Domain Decomposition Methods*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2015. DOI: 10.1137/1.9781611974065. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611974065

Mathematical Background

Schwarz preconditioners

- Derived from the Alternating Schwarz method[2]
- Convergence rate depends on the overlap δ and the wave number of eigenmodes k

2D Alternating Schwarz Example

Let $\Omega_1 = (-\infty, \delta) \times \mathbb{R}, \Omega_2 = (\delta, \infty) \times \mathbb{R}$

$$-(\eta - \Delta)u = f \text{ in } \mathbb{R}^2,$$
$$u \text{ bounded at infinity.}$$

Then the convergence rate is given by

$$\rho_{2D}(k; \eta, \delta) = e^{-\delta\sqrt{\eta+k^2}}$$

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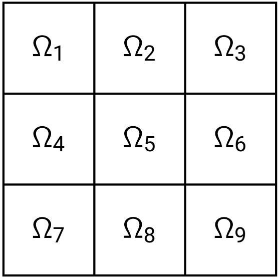

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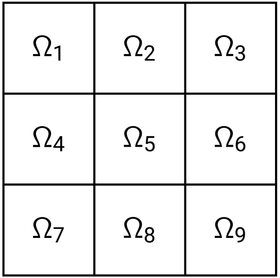

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2-level Additive Schwarz Preconditioner

$$M_{\text{ASM},2} = {R_0}^T A_0^{-1} R_0 + M_{\text{ASM}}$$

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Related Work

Tailored Coarse Spaces for High-Contrast Problems

Related Work

CG convergence in case of non-uniform spectra

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Preliminary Results

Two-cluster case

Preliminary Results

Generalization to multiple clusters

Preliminary Results

Numerical example

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Outlook

Research Question 1 answered

Outlook

Research question 3 answered

Outlook

Next steps

- Work on answering research questions 4, 5, and 6.

Outlook

Open Challenge

- How to answer research question 2?