Assignment 1 - Applications in PDEs (AM4680): The Earth's energy balance

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1. Instructions

Please submit your solution to this assignment in the form of a written report. The report should be professional and not longer than needed. The following guidelines apply:

- **structure:** include a brief introduction stating the subject of your work and the main goal and methods, a sensible set of sections or chapters to provide solutions to the exercises in this assignment, and a final section where you make sure you provide your solution to the main goal. Do not use the exercise numbers from the assignment, but translate the exercise numbers into sensible sections.
- **amount of text:** In each section or chapter, provide enough context so that it is clear what you are doing (i.e. what is it you want to achieve in the section, what methods do you use etc.) and briefly describe figures (i.e. what do we see, what is on the axes, and describe the main features of the figure). Please describe the meaning of any variables you use, but keep it brief (i.e. you may assume that the reader knows the context of the problem). Please do not repeat any theory unless it is explicitly asked in the exercise.
- **mathematics and code:** provide enough intermediate steps of mathematical derivations to convey which methods you are using, but you may skip smaller intermediate steps. If needed, use appendices for longer derivations that interrupt the flow of the text. Do not provide code in your report.
- use a professional language, i.e. be reasonably formal.

On the use of generative AI and other software

You are welcome to use generative AI to assist your writing, programming or visualisations. Take the following into account:

- please ensure your text remains concise. Do not elaborate just because it is easy to generate (a lot of) text. Note you can tell the AI software to provide an answer in less than a certain number of words, or summarize or shorten certain pieces of text.
- you always remain responsible for the correctness and relevance of the text.
- please indicate your use of generative AI in the report.

You are allowed to use any other software, such as computer algebra software (e.g. Mathematica, Maxima, Wolfram alpha, Maple) or programming languages.

2. Context and goal

We will consider an idealised energy balance model (EBM) for the Earth. An energy balance model tracks the amount of heat absorbed, reflected and radiated from the planet, as well as the heat redistribution over the planet. We will consider a highly simplified model. This model is particularly used to study the effect of ice on the energy balance. As (land or sea) ice has a much higher albedo, or reflectivity, than water, an ice-covered planet is considered to absorb less solar radiation and thus remain cooler. The model is based on [1], which again is inspired on various previous authors and ultimately dating back to works of Budyko [2] and Sellers [3].

The energy balance is evaluated on the timescale of a typical year, so that rotation of the Earth, seasonal variation or even yearly variations in solar radiation of cloud coverage can be ignored. It is furthermore assumed that the Earth can be represented as a sphere which is uniform in the longitudinal direction. Hence, the temperature T on earth depends only on the latitude and time. Latitude θ is rewritten to a coordinate $x = \sin(\theta)$ (dimensionless), such that $x \in [-1, 1]$ from pole to pole. Energy enters the Earth through solar radiation R_A. The R_A is expressed as

$$R_A = Q(x)\left(1 - \alpha(T(x))\right) + \mu. \tag{1}$$

Here, $Q(x)$ is the solar radiation, α is the albedo of the Earth's surface and μ is a parameter modelling the effect of greenhouse gasses and fine particles in the atmosphere on the energy budget. Solar radiation depends on latitude according to

$$Q(x) = Q_0\left(1 - 0.241\left(3x^2 - 1\right)\right) \tag{2}$$

with constant Q_0. For the albedo, it is assumed that any place on Earth is either covered by water or by ice, where ice prevails for temperatures well below 272 K (i.e. 0 deg C) and water prevails for temperatures well above 272 K. For temperatures around 272 K, the albedo is between that of water and ice, parametrizing partial or seasonal ice coverage. Correspondingly, α is defined as

$$\alpha = \alpha_1 + (\alpha_2 - \alpha_1)\,\tfrac{1}{2}\left(1 + \tanh\left(M(T - T^*)\right)\right). \tag{3}$$

Energy leaves the Earth through radiation R_E. Radiation is modelled by the Stefan-Boltzmann law for black body radiation following

$$R_E = \varepsilon_0 \sigma_0 T^4, \tag{4}$$

where ε_0 and σ_0 are respectively the emissivity and the Stefan-Boltzmann constant.

Finally, energy is redistributed assuming a dispersion process R_D with constant dispersion constant. In our coordinate system, such process is described as

$$R_D = \frac{\partial}{\partial x}\left(\left(D(1 - x^2) + \delta\right)\frac{\partial T}{\partial x}\right), \tag{5}$$

where D is a dispersion constant. The parameter δ is formally equal to zero, such that there is no change of the heat transport at the poles (verify for yourself that this should be required, see exercise 1). However, depending on the solution method used, it may be useful to assume that δ is a small positive number (see exercises below).

The energy balance is finally given by the following PDE:

$$C_T \frac{\partial T}{\partial t} = R_A - R_E + R_D, \tag{6}$$

where C_T is a heat capacity parameter. We focus on the equilibrium solutions to this model. We will thus consider the equation

$$R_A - R_E + R_D = 0, \tag{7}$$

The default values of the model parameters are given in Table 1[1]

The goal of this assignment is to evaluate the equilibrium solutions of the EBM as a function of the parameters μ and D and indicate the linear stability properties of these solutions. Please work on the exercises below to construct you evaluation of this overall goal.

[1]Note that values for α_1, α_2 were switched in [1]. This is incorrect; correct values are given in Table 1.

Parameter	Value
Q_0	341.3 Wm^{-2}
α_1	0.7
α_2	0.289
ε	0.61
σ_0	$5.67 \cdot 10^{-8}$ $\mathrm{Wm}^{-2}\mathrm{K}^{-4}$
M	0.1 K^{-1}
D	0.3 $\mathrm{Wm}^{-2}\mathrm{K}^{-1}$
μ	30 Wm^{-2}
C_T	$5 \cdot 10^{8}$ $\mathrm{Jm}^{-2}\mathrm{K}^{-1}$

Table 1. Default parameter values

3. Exercises

1) Considering the physical process that is being modelled, what boundary conditions should be prescribed at $x = \pm 1$? If $\delta = 0$, can you still prescribe these boundary conditions? If not, what do you need to prescribe instead?

2) Discretise the spatial dimension of the PDE. Suggested is to use either of the following options

- a finite difference or finite volume method. Consider if your method can work with a value $\delta = 0$ in the heat flux term or otherwise use a small value, e.g. $\delta = 10^{-3}$ $\mathrm{Wm}^{-2}\mathrm{K}^{-1}$.
- an eigenfunction projection. Please see Section 4 of this assignment.

The result of this step is a system of a finite number of non-linear algebraic equations.

3) We will work on ways of finding fixed points of the EBM.

a) Construct and implement a root-finding algorithm for the system of nonlinear equations based on the Newton-Raphson method. Please use an exact expression for the Jacobian derived by hand.

b) Compute a Jacobian by finite differences. For the default parameter values and model state of your choice (may be equilibrium or not), show a measure for the error between the numerically computed Jacobian and the hand-derived Jacobian as a function of the step size in the finite difference method. What is the best step size and very briefly comment why.

c) For the default parameter values, show the solution found using the Newton-Raphson method for a large set of initial states. The way you define the initial states is up to you. You may choose to visualise the solution as function of x or use some scalar measure for the solution, as long as you can clearly show how the solution obtained depends on the initial state chosen. Please make sure to discuss the domain of attraction of each solution. Does this correspond to the theorem(s) discussed in the course? Also qualitatively explain the number of iterations needed for convergence from each initial state.

d) Repeat c) using the Broyden method. Briefly comment on the differences in results between the methods.

e) For the equilibria found, briefly describe what you see and what this implies for ice coverage of the model planet.

4) For this particular problem, solutions can be found for a large range of initial conditions. Hence, we will not need to do a formal embedding to find a first equilibrium solution. However, please discuss how you could use the concept of embedding to find an initial equilibrium for this model instead. That is, use a parameter $\gamma \in [0,1]$ to formulate an adapted model. For $\gamma = 0$ the EBM should simplify to a system where you are sure you can easily find the solution by hand or computer. For $\gamma = 1$, you should retrieve the original model. NB. you don't need to program the actual embedding or continuation in γ.

5) We will continue by studying the sensitivity of the model to values of μ. We will consider $\mu \in [0, 100]$ Wm^{-2}.

a) Construct and program a pseudo-arclength continuation routine for continuation in the parameter μ. Briefly describe the methods you used (do not repeat equations from the theory but refer to the names of the methods instead).

b) Discuss how to measure step size, i.e. what is the norm you want to use to define step size that puts a similar weight on the step size in the direction of the solution and the direction of μ. Give a sense of a minimum and maximum values of the step size that you would want in your algorithm.

c) Perform the continuation starting from $\mu = 0$ and find a way of plotting the results in one or multiple figures. Take the following aspects into account: at least include a bifurcation diagram, indicate the stability of the branches, provide some sense of the x-structure of (some of) the solutions, indicate the extent of ice coverage as function of μ. You are free to select the number and type of figures. Aim for a limited number of figures that convey the main results.

d) what bifurcation(s) did you find? Discuss or plot the behaviour of the relevant eigenvalue(s) of the linearised system near the bifurcation.

e) Find solutions for values of $D = 30$ $\mathrm{Wm^{-2}K^{-1}}$ and $D = 0.003$ $\mathrm{Wm^{-2}K^{-1}}$. Take into account the same aspects as in question 4c and discuss the main differences for different values of D.

4. Appendix: eigenfunction expansion for the EBM

One way of discretising the EBM is to use an eigenfunction expansion. Hence, letting $T(x)$ denote the equilibrium solution, assume

$$T(x) = \sum_{n=0}^{\infty} a_n \phi_n(x). \tag{8}$$

It is suggested to use the following eigenvalue problem to define ϕ_n:

$$\frac{\partial}{\partial x}\left(\left(D(1-x^2)+\delta\right) \frac{\partial \phi_n}{\partial x} \right) = -\lambda_n \phi_n. \tag{9}$$

with $\delta = 0$ and the boundary conditions as found in exercise 1. For these boundary conditions, solutions are only possible for $\lambda_n = n(n+1)$ with $n = 0, 1, 2, \ldots$. and solutions are given by the Legendre polynomials $P_n(x)$, i.e. $\phi_n(x) = P_n(x)$. Next, substitute the eigenfunction expansion in the EBM and use Galerkin's method (review your course on partial differential equations or finite element methods if needed). Using orthogonality of the Legendre polynomials, you will find the term R_D simplifies greatly. The terms R_A and R_E are not simple though. To solve, use a finite number of Legendre polynomials and write your equations after applying the Galerkin method to a system of non-linear equations.

References

[1] R. Bastiaansen, H. A. Dijkstra, and A. S. von der Heydt. Fragmented tipping in a spatially heterogeneous world. *Environmental Research Letters*, 17(4):045006, March 2022.

[2] M. I. Budyko. The effect of solar radiation variations on the climate of the Earth. *Tellus*, 21(5):611–619, October 1969.

[3] W. D. Sellers. A global climatic model based on the energy balance of the Earth-atmosphere system. *Journal of Applied Meteorology*, 8(3):392–400, June 1969.