WI4680 Applications in Partial Differential Equations

Assignments

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Contents

Introduction

This report contains my treatment of the assignments. It will be updated as I progress through the course.

Assignment 1

Variable	Unit	Meaning	Value
θ	-	latitude	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
t	s	time	-
$x = \sin\theta$	-	latitude coordinate	$[-1, 1]$
T	K	temperature	-
R_A	$Js^{-1}m^{-2}$	effective solar radiation	-
Q	$Js^{-1}m^{-2}$	solar radiation	-
Q_0	$Js^{-1}m^{-2}$	solar radiation constant	341.3
α	-	albedo of Earth	-
α_1	-	albedo of ice	0.7
α_2	-	albedo of water	0.289
T^*K	-	temperature at which ice melts	273.15
M	K^{-1}	temperature gradient (?)	-
μ	$Js^{-1}m^{-2}$	greenhouse gas & fine particle parameter	30
R_E	$Js^{-1}m^{-2}$	black body radiation	-
h_0	-	emmisivity of Earth	0.61
σ_0	$Js^{-1}m^{-2}K^{-4}$	Stefan-Boltzmann constant	$5.67 \cdot 10^{-8}$
R_D	$Js^{-1}m^{-2}$	heat dispersion	-
D	$Js^{-1}m^{-2}$	heat dispersion constant	0.3
δ	$Js^{-1}m^{-2}$	heat dispersion at poles	0
C_T	JK^{-1}	heat capacity of Earth	$5 \cdot 10^8$

Table 1: Variables and their meanings

1.1 Boundary Conditions

δ cannot be positive (resp. negative) at $x = \pm 1$ (poles), otherwise energy would be artificially entering (resp. leaving) the system. Simply said, the poles cannot be a source or sink of energy. This requires us to set $\delta = 0$ at $x = \pm 1$. Furthemore

$$\left.\frac{dT}{dx}\right|_{x=\pm 1} = 0.$$

However, we run into a problem when we combine the boundary conditions and set $\delta = 0$. The equation for the heat dispersion vanishes at the boundary and we are left with two algebraic equations in terms of $T(\pm 1)$

$$R_A(T, x) - R_E(T)|_{x=\pm 1} = 0.$$

Solving these equations for $T(\pm 1)$ gives us the equilibrium temperature at the poles

$$T_e q(\pm 1) \approx 220.5K,$$

for the values speciefied in table 1.

Alternatively, we can resort to a more basic requirement. Namely, that there must exist an equilibrium temperature T_0 at the poles,

$$\begin{aligned} &F(T(x,t))|_{x=\pm 1} = 0, \quad \forall t > 0, \\ \implies &-2x\frac{dT}{dx} + R_A - R_E\Big|_{x=\pm 1} = 0. \end{aligned}$$

Assuming we use the given expansion of T in terms of legendre polynomials, we can write the above as

$$\sum_{n=0}^{\infty} 2x\frac{\lambda_n}{D}a_n\frac{d\phi_n}{dt} - (R_A - R_E)\Big|_{x=\pm 1} = 0,$$
$$\implies \begin{cases} \sum_{n=0}^{\infty} \frac{\lambda_n^2 a_n}{D} = R_A - R_E|_{x=1}\,, \\ \sum_{n=0}^{\infty}(-1)^n\frac{\lambda_n^2 a_n}{D} = -R_A + R_E|_{x=-1}\,, \end{cases},$$

where we used the properties

$$\frac{d\phi_n(1)}{dx} = \frac{\lambda_n}{2} \quad \text{and} \quad \phi_n(-x) = (-1)^n\phi_n(x).$$

1.2 Discretisation

We choose to decompose the temperature field into a sum of orthogonal Legendre polynomials, i.e.

$$T(x) = \sum_{n=0}^{\infty} a_n\phi_n(x),$$

where the timedependence is dropped seeing as we are after equilibrium solutions.

Subsitution of the above in the (equilibrium) energy balance equation yields

$$-\sum_{i=0}^{\infty}\frac{\lambda_n a_n}{D}\phi_n + R_A(T,x) + R_E(T) = 0.$$

We now use Galerkin's method to project the above equation onto the Legendre basis functions. This gives

$$F_i(T(\mathbf{a})) = \int_{-1}^{1}\left[-\sum_{i=0}^{\infty}\frac{\lambda_n a_n}{D}\phi_n + R_A(T,x) + R_E(T)\right]\phi_i dx = 0.$$

where F_i is the ith component of discretised equilibrium energy balance eqaution. Hence the above defines a non-linear system of equations. Note that we could have used the orthogonality of the Legendre polynomials to simplify the above, but we chose to keep the above form for clarity. Moreover, letting n_p be the number of legendre polynomials we can approximately evaluate the above (and coming) integral(s) using the Gauss-Legendre quadrature rule with $2n_p$ quadrature points.

1.3 Derivation: exact vs. numerical

We want the jacobian of the non-linear system $\mathbf{F}_T = \frac{\partial F(T(\mathbf{a}))}{\partial \mathbf{a}}$, which we may obtain directly from the previous secttion using the same Galerkin method and guassian quadrature

$$\frac{\partial F_i}{\partial a_j} \approx \int_{-1}^{1}\left[-\frac{\lambda_n}{D} + \frac{\partial R_A(T,x)}{\partial T} + \frac{dR_E(T)}{dT}\right]\phi_j\phi_i dx = 0. \tag{1}$$

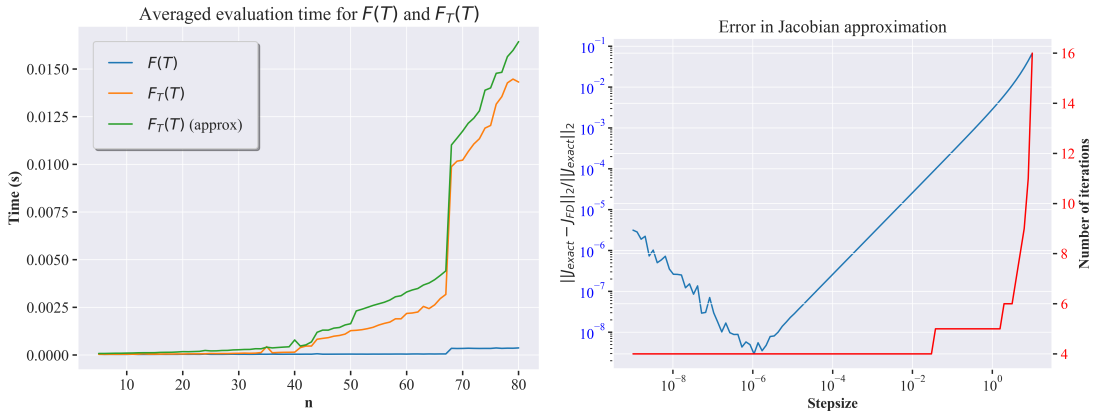
Alternatively, we can use the finite difference method to approximate the jacobian.

$$\frac{\partial F_i}{\partial a_j} = \frac{F_i(T(\mathbf{a}+h\mathbf{e_j})) - F_i(T(\mathbf{a}))}{h} + \mathcal{O}(h). \tag{2}$$

Both methods are approximations. However, the finite difference method is fundamentally an approximation of order h, whereas Galerkin is exact up to order n_p. With regards to

computational cost the finite difference method requires n_p evaluations of the non-linear system, whereas the Galerkin method requires n_p^2 numerical integrations of the derivative of the non-linear system. Hence for relatively small n_p, the Galerkin method is preferrable. Since the non-linear system is not too complex, we can afford to use the Galerkin method.

With regards to computational complexity and accuracy we refer to Figures 1a and 1b. The former compares the computational time for one evaluation of the non-linear system as well as the jacobian using both the Galerkin and finite difference methods. The latter compares the correspondence between the two jacobians for different values of h.



(a) Computational time for one evaluation of the non-linear system and the jacobian using both the Galerkin and finite difference methods.

(b) Comparison of the jacobian obtained using the Galerkin and finite difference methods for different values of h.

Figure 1: Comparison of the Galerkin and finite difference methods.

As explained during the lectures, there exists an optimal value of h for which the finite difference method is most accurate. From the 1b we see that the finite difference method is most accurate for $h \approx 10^{-6}$. However, we also see from 1a that the computational time for the finite difference method is comparable to the Galerkin method for problem sizes with $n_p \leq 80$. This means that as far as this assignment is concerned, the Galerkin method is preferrable.

1.4 Finding equilibria: Newton and Broyden

Figure 2 shows the equilibria found using the Newton and Broyden methods.

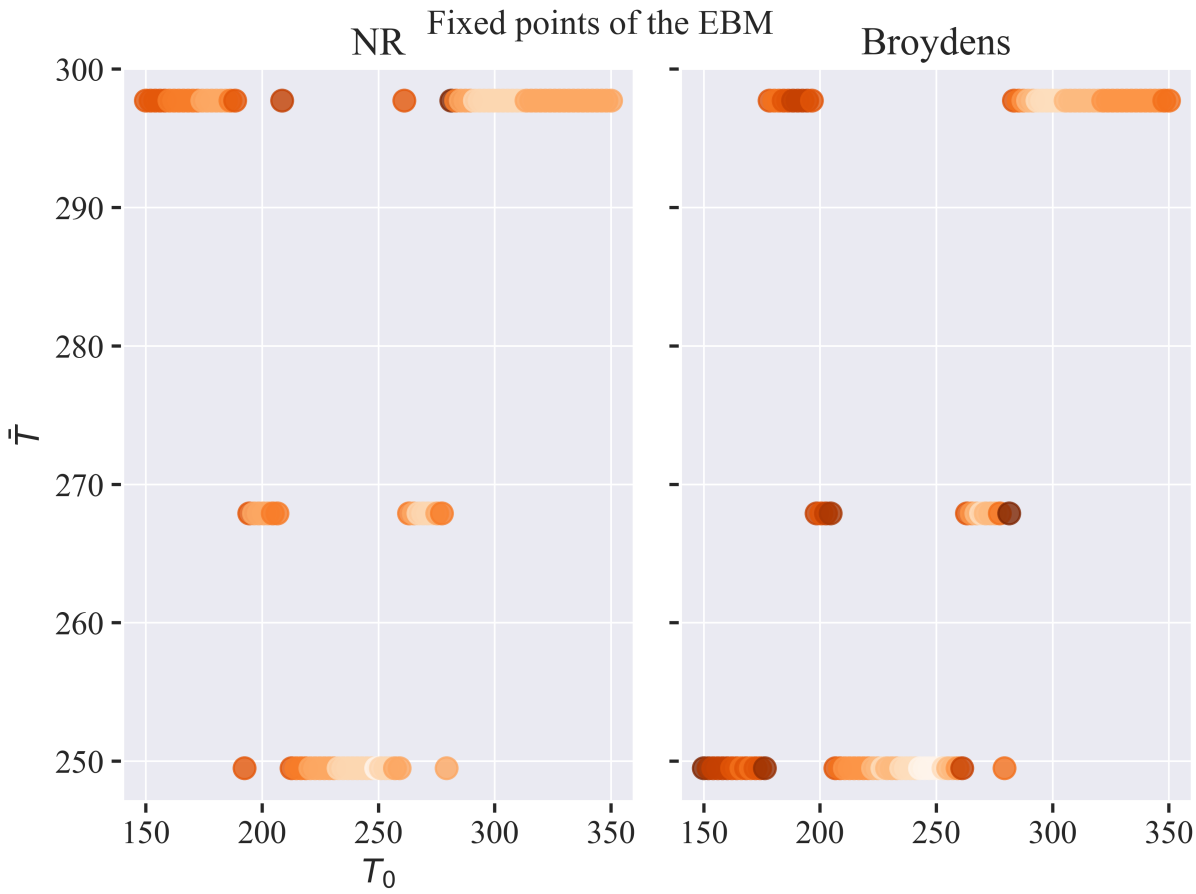


Figure 2: Equilibria found using the Newton and Broyden methods for a range of homogeneous initial temperatures T_0. The color intensity indicates the amount of iterations required to converge. a high intensity indictes the method needed a low amount of iterations to converge.

We notice that roughly speaking the equilibria found using the Newton and Broyden methods are the same. However, the Broyden method seems to be less predictable in terms of to what root it converges. It is not directly clear how to compare domains of attraction for these two methods. However, we can see that the Newton method is more predictable in terms of the number of iterations required to converge and what initial state ends up convering to what root. In other words, the Broyden method seems to be more sensitive to the initial state. This might be due to the fact that the Broyden method's linear compated to Newton method's quadratic convergence rate causes it to 'skip' roots.

In terms of the state of our model Earth, we discover three distinct average temperatures $T_{eq} \approx 250K$, $T_{eq} \approx 268K$ and $T_{eq} \approx 298K$. The first corresponds to a frozen Earth, the second to a partially frozen Water world and the third to quite a cozy Blue Marble.

1.5 Embedding

We can reduce the problem of finding the first equilibrium through the following embedding of the energy balance equation

$$f = \gamma R_D[T] + (1-\gamma)(R_A(T,x) + R_E(T)). \tag{3}$$

Taking $\gamma = 0$ we obtain a transcendental equation for the equilibrium temperature, which we can easily solve using the Newton method. Once we have this solution we increment γ and apply some continuation scheme using the methods for evaluating the non-linear system and

its jacobian. This way we can find the other equilibria. We repeat this until we reach $\gamma = 1$, at which point we find an initial equilibrium.

1.6 Bifurcation diagrams

1.7 Conclusion

Assignment 2

2.1 Introduction

2.2 Conclusion

Assignment 3

3.1 Introduction

3.2 Conclusion