

WI4680 Applications in Partial Differential Equations

Assignments

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Introduction

This report contains my treatment of the assignments. It will be updated as I progress through the course.

All my code is available [here](#).

Assignment 1

The goal of this assignment is to find the equilibria of a energy balance equation based on a very simplified model of the Earth. Additionally, we perform a one parameter analysis using continuation on a parameter that represents the greenhouse effect. This allows us to study the bifurcation diagram of the model.

Variable	Unit	Meaning	Value
θ	-	latitude	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
t	s	time	-
$x = \sin \theta$	-	latitude coordinate	$[-1, 1]$
T	K	temperature	-
R_A	$J s^{-1} m^{-2}$	effective solar radiation	-
Q	$J s^{-1} m^{-2}$	solar radiation	-
Q_0	$J s^{-1} m^{-2}$	solar radiation constant	341.3
α	-	albedo of Earth	-
α_1	-	albedo of ice	0.7
α_2	-	albedo of water	0.289
T^*	K	temperature at which ice melts	273.15
M	K^{-1}	temperature gradient (?)	-
μ	$J s^{-1} m^{-2}$	greenhouse gas & fine particle parameter	30
R_E	$J s^{-1} m^{-2}$	black body radiation	-
h_0	-	emmissivity of Earth	0.61
σ_0	$J s^{-1} m^{-2} K^{-4}$	Stefan-Boltzmann constant	$5.67 \cdot 10^{-8}$
R_D	$J s^{-1} m^{-2}$	heat dispersion	-
D	$J s^{-1} m^{-2}$	heat dispersion constant	0.3
δ	$J s^{-1} m^{-2}$	heat dispersion at poles	0
C_T	JK^{-1}	heat capacity of Earth	$5 \cdot 10^8$

Table 1: Variables and their meanings

1.1 Boundary Conditions

δ cannot be positive (resp. negative) at $x = \pm 1$ (poles), otherwise energy would be artificially entering (resp. leaving) the system. Simply said, the poles cannot be a source or sink of energy. This requires us to set $\delta = 0$ at $x = \pm 1$. Furthermore

$$\left. \frac{dT}{dx} \right|_{x=\pm 1} = 0.$$

However, we run into a problem when we combine the boundary conditions and set $\delta = 0$. The equation for the heat dispersion vanishes at the boundary and we are left with two algebraic equations in terms of $T(\pm 1)$

$$R_A(T, x) - R_E(T)|_{x=\pm 1} = 0.$$

Solving these equations for $T(\pm 1)$ gives us the equilibrium temperature at the poles

$$T_{eq}(\pm 1) \approx 220.5K,$$

for the values specified in table 1.

Alternatively, we can resort to a more basic requirement. Namely, that there must exist an equilibrium temperature T_0 at the poles,

$$\begin{aligned} F(T(x, t))|_{x=\pm 1} &= 0, \quad \forall t > 0, \\ \implies -2x \frac{dT}{dx} + R_A - R_E &\Big|_{x=\pm 1} = 0. \end{aligned}$$

Assuming we use the given expansion of T in terms of legendre polynomials, we can write the above as

$$\begin{aligned} \sum_{n=0}^{\infty} 2x \frac{\lambda_n}{D} a_n \frac{d\phi_n}{dt} - (R_A - R_E) &\Big|_{x=\pm 1} = 0, \\ \implies \begin{cases} \sum_{n=0}^{\infty} \frac{\lambda_n^2 a_n}{D} = R_A - R_E|_{x=1}, \\ \sum_{n=0}^{\infty} (-1)^n \frac{\lambda_n^2 a_n}{D} = -R_A + R_E|_{x=-1}, \end{cases} \end{aligned}$$

where we used the properties

$$\frac{d\phi_n(1)}{dx} = \frac{\lambda_n}{2} \quad \text{and} \quad \phi_n(-x) = (-1)^n \phi_n(x).$$

1.2 Discretisation

We choose to decompose the temperature field into a sum of orthogonal Legendre polynomials, i.e.

$$T(x) = \sum_{n=0}^{\infty} a_n \phi_n(x),$$

where the timedependence is dropped seeing as we are after equilibrium solutions.

Subsitution of the above in the (equilibrium) energy balance equation yields

$$-\sum_{i=0}^{\infty} \frac{\lambda_i a_i}{D} \phi_i + R_A(T, x) + R_E(T) = 0.$$

We now use Galerkin's method to project the above equation onto the Legendre basis functions. This gives

$$F_i(T(\mathbf{a})) = \int_{-1}^1 \left[-\sum_{i=0}^{\infty} \frac{\lambda_i a_i}{D} \phi_i + R_A(T, x) + R_E(T) \right] \phi_i dx = 0.$$

where F_i is the i th component of discretised equilibrium energy balance eqaution. Hence the above defines a non-linear system of equations. Note that we could have used the orthogonality of the Legendre polynomials to simplify the above, but we chose to keep the above form for clarity. Moreover, letting n_p be the number of legendre polynomials we can approximately evaluate the above (and coming) integral(s) using the Gauss-Legendre quadrature rule with $2n_p$ quadrature points.

1.3 Derivation: exact vs. numerical

We want the jacobian of the non-linear system $\mathbf{F}_T = \frac{\partial F(T(\mathbf{a}))}{\partial \mathbf{a}}$, which we may obtain directly from the previous secttion using the same Galerkin method and guassian quadrature

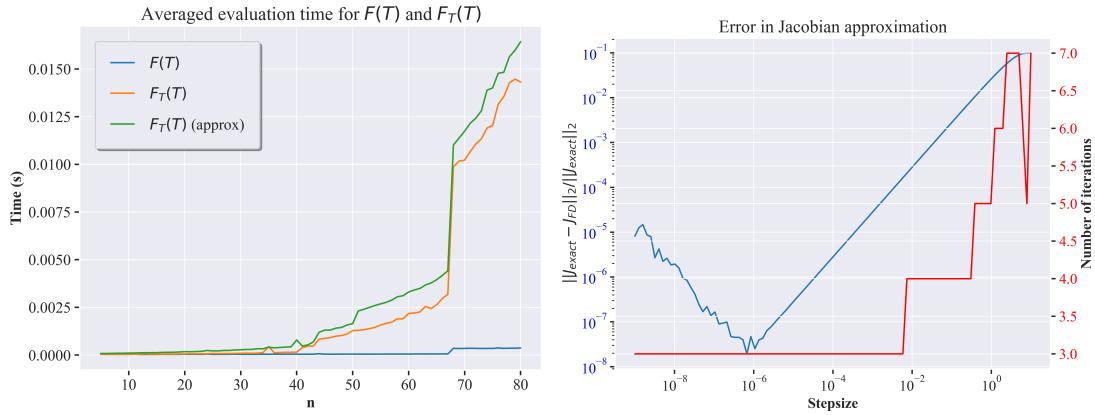
$$\frac{\partial F_i}{\partial a_j} \approx \int_{-1}^1 \left[-\frac{\lambda_i}{D} + \frac{\partial R_A(T, x)}{\partial T} + \frac{dR_E(T)}{dT} \right] \phi_j \phi_i dx = 0. \quad (1)$$

Alternatively, we can use the finite difference method to approximate the jacobian.

$$\frac{\partial F_i}{\partial a_j} = \frac{F_i(T(\mathbf{a} + h\mathbf{e}_j)) - F_i(T(\mathbf{a}))}{h} + \mathcal{O}(h). \quad (2)$$

Both methods are approximations. However, the finite difference method is fundamentally an approximation of order h , whereas Galerkin is exact up to order n_p . With regards to computational cost the finite difference method requires n_p evaluations of the non-linear system, whereas the Galerkin method requires n_p^2 numerical integrations of the derivative of the non-linear system. Hence for relatively small n_p , the Galerkin method is preferable. Since the non-linear system is not too complex, we can afford to use the Galerkin method.

With regards to computational complexity and accuracy we refer to Figures 1a and 1b. The former compares the computational time for one evaluation of the non-linear system as well as the jacobian using both the Galerkin and finite difference methods. The latter compares the correspondence between the two jacobians for different values of h .



(a) Computational time for one evaluation of the non-linear system and the jacobian using both the Galerkin and finite difference methods. (b) Comparison of the jacobian obtained using the Galerkin and finite difference methods for different values of h .

Figure 1: Comparison of the Galerkin and finite difference methods.

As explained during the lectures, there exists an optimal value of h for which the finite difference method is most accurate. From the 1b we see that the finite difference method is most accurate for $h \approx 10^{-6}$. However, we also see from 1a that the computational time for the finite difference method is comparable to the Galerkin method for problem sizes with $n_p \leq 80$. This means that as far as this assignment is concerned, the Galerkin method is preferable.

1.4 Finding equilibria: Newton and Broyden

Figure 2 shows the equilibria found using the Newton and Broyden methods.

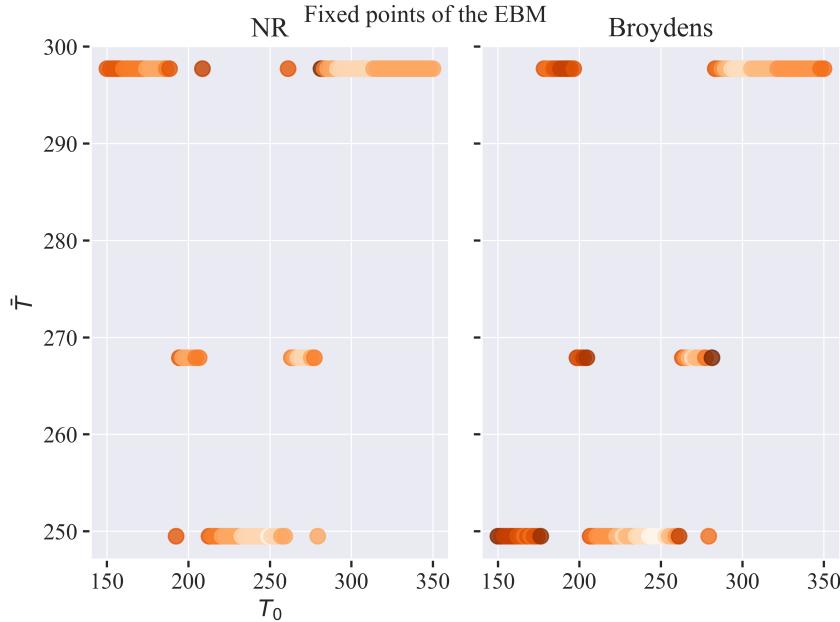


Figure 2: Equilibria found using the Newton and Broyden methods for a range of homogeneous initial temperatures T_0 . The color intensity indicates the amount of iterations required to converge. A high intensity indicates the method needed a low amount of iterations to converge.

We notice that roughly speaking the equilibria found using the Newton and Broyden methods are the same. However, the Broyden method seems to be less predictable in terms of what root it converges. It is not directly clear how to compare domains of attraction for these two methods. However, we can see that the Newton method is more predictable in terms of the number of iterations required to converge and what initial state ends up converging to what root. In other words, the Broyden method seems to be more sensitive to the initial state. This might be due to the fact that the Broyden method's linear compared to Newton method's quadratic convergence rate causes it to 'skip' roots.

In terms of the state of our model Earth, we discover three distinct average temperatures $T_{eq} \approx 250K$, $T_{eq} \approx 268K$ and $T_{eq} \approx 298K$. The first corresponds to a frozen Earth, the second to a partially frozen Water world and the third to quite a cozy Blue Marble.

1.5 Embedding

We can reduce the problem of finding the first equilibrium through the following embedding of the energy balance equation

$$f = \gamma R_D[T] + (1 - \gamma)(R_A(T, x) + R_E(T)). \quad (3)$$

Taking $\gamma = 0$ we obtain a transcendental equation for the equilibrium temperature, which we can easily solve using the Newton method. Once we have this solution we increment γ and apply some continuation scheme using the methods for evaluating the non-linear system and its jacobian. This way we can find the other equilibria. We repeat this until we reach $\gamma = 1$, at which point we find an initial equilibrium.

1.6 Bifurcation diagrams

The pseudo arclength continuation is implemented by first introducing the parametrisation $(T, \mu) \mapsto (T(s), \mu(s))$. This results in an extended system of equations, which is the original

system with an additional parametrisation equation.

$$\begin{cases} F(T(s), \mu(s)) = 0, \\ p(T, \mu, s) = 0. \end{cases},$$

where

$$p(T, \mu, s) = 2\zeta(T - \bar{T}) \frac{dT}{ds} + 2(1 - \zeta)(\mu - \bar{\mu}) \frac{d\mu}{ds} - \Delta s,$$

where $(\bar{T}, \bar{\mu})$ denote the current equilibrium, (T, μ) the continued (as of yet unknown) solution, ζ is a tune factor and Δs is the arclength or step size

The jacobian of this system is given by

$$\tilde{F}' = \begin{bmatrix} \frac{\partial F}{\partial T} & \frac{\partial F}{\partial \mu} \\ \frac{\partial p}{\partial T} & \frac{\partial p}{\partial \mu} \end{bmatrix},$$

which after discretisation becomes

$$D\tilde{F} = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{a}} & \frac{\partial \mathbf{F}}{\partial \mu} \\ 2\zeta \left(\frac{d\mathbf{T}}{ds}\right)^T & 2(1 - \zeta) \frac{d\mu}{ds} \end{bmatrix}.$$

The above system can be solved using the Newton method. The condition for convergence is such that the update

$$\begin{bmatrix} \Delta \mathbf{a} \\ \Delta \mu \end{bmatrix} = - (D\tilde{F})^{-1} \begin{bmatrix} \mathbf{F} \\ p \end{bmatrix},$$

must be smaller than some tolerance $\epsilon = 10^{-5}$.

Figure 3 shows the bifurcation diagram for the Earth model. This figure was made by rerunning the continuation scheme for different initial temperatures T_0 and plotting the equilibria found.

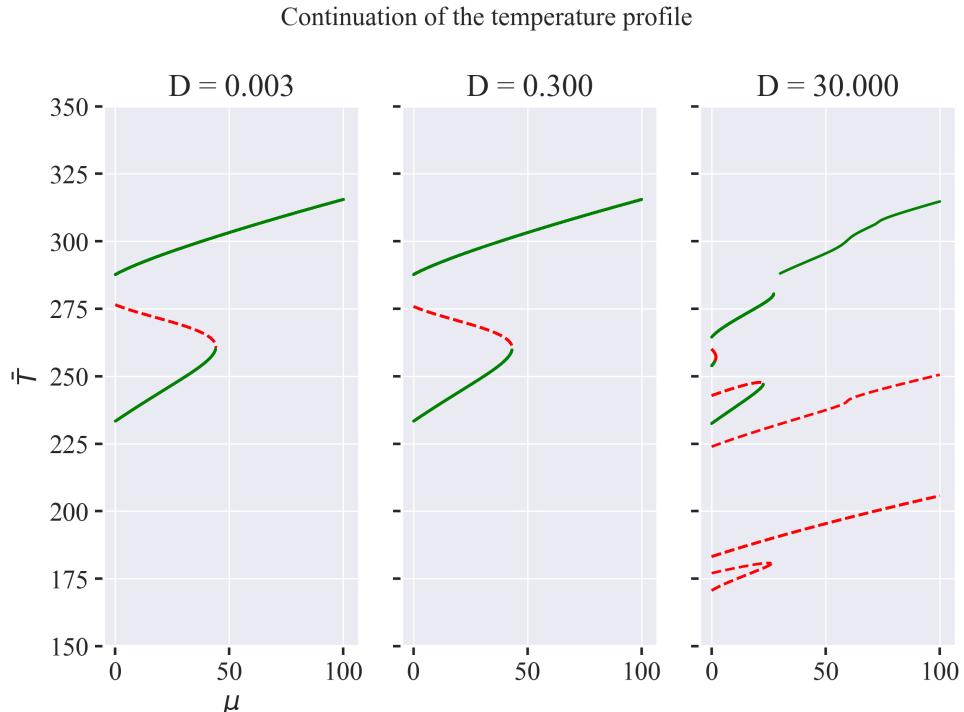


Figure 3: Bifurcation diagram for the Earth model. Green solid lines indicates a stable branch while red, dashed lines indicate an unstable branch.

A typical temperature profile is given in Figure 4. We see that the temperature profile is symmetric around the equator, which is expected.

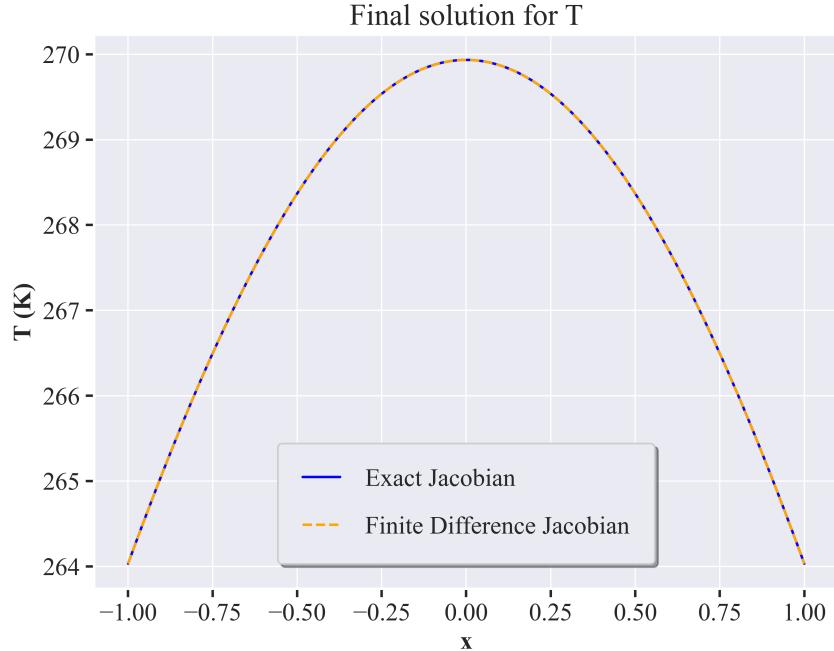


Figure 4: Typical temperature profile for the Earth model.

We see from Figure 3 that we need a small step size to accurately resolve all the branches in the bifurcation diagram ($\Delta s \leq 0.1$). Additionally, we should place emphasis on tracking small changes in the parameter ($\zeta \leq 0.001$).

In figure 3 we solely identify fold (subtle) and jump (critical) bifurcations.

Next to this we recover the three equilibria found above. For small values of the diffusion constant $D = 0.003, 0.3$ we find that for $\mu \gtrapprox 50$ there is only one stable branch. In case of the partially frozen Earth, we see that increasing μ at first has the effect of lowering the temperature. However, at some point it appears the greenhouse effect becomes strong enough to counteract the cooling effect of the ice the solution suddenly jumps to the aforementioned stable branch.

This behaviour is radically different however for $D = 30$. We suddenly see many more branches appearing. This is due to the fact that the heat dispersion is so strong that the temperature gradient is much steeper. This results in initially completely ice covered earths to become unstable.

1.7 Conclusion

We have found the equilibria of the Earth model and studied the bifurcation diagram. We have seen that for default parameter values the Earth model has three distinct equilibria, corresponding to a frozen, a partially frozen and a warm Earth. We have also seen that the bifurcation diagram is sensitive to the diffusion constant D . For small values of D we see that the bifurcation diagram is relatively simple, with only fold and jump bifurcations. However, for large values of D the bifurcation diagram becomes much more complex, with many more branches appearing. This is due to the fact that the heat dispersion is so strong that the temperature gradient is much steeper. This results in initially completely ice covered earths to become unstable.

Assignment 2

In this assignment we study the MCM system of equations, which models the interaction between three species of cells, T-, healthy- and effector cells. The corresponding system of equations is given by

$$\begin{aligned} T' &= r_1 T \left(1 - \frac{T}{k_1}\right) - a_{12} TH - a_{13} TE, \\ H' &= r_2 H \left(1 - \frac{H}{k_2}\right) - a_{21} TH, \\ E' &= \frac{r_3 TE}{T + k_3} - a_{31} TE - d_3 E, \end{aligned}$$

and its dimensionless form

$$\begin{aligned} x'_1 &= x_1 (1 - x_1) - p_1 x_1 x_2 - p_2 x_1 x_3, \\ x'_2 &= p_3 x_2 (1 - x_2) - p_4 x_1 x_2, \\ x'_3 &= \frac{p_5 x_1 x_3}{x_1 + p_6} - p_7 x_1 x_3 - p_8 x_3. \end{aligned}$$

The parameters p_i are specified in Table 2

Parameter	Definition	description	Value
p_1	$\frac{a_{12}k_2}{r_1}$	T->H competition	0.5
p_2	$\frac{a_{31}k_1}{r_1}$	E->T killing	2.5
p_3	$\frac{r_2}{r_1}$	H-T growth ratio	0.6
p_4	$\frac{a_{21}k_1}{r_1}$	T->H inactivation	1.5
p_5	$\frac{r_3k_2}{r_1}$	T->E stimulation	4.5
p_6	$\frac{k_3}{k_1}$	E-T capacity ratio	1.0
p_7	$\frac{a_{31}k_3}{r_1}$	T->E inactivation	0.2
p_8	$\frac{d_3}{r_1}$	E natural death	0.5

Table 2: Parameters of the dimensionless system

The goal of this assignment is to study the dynamics of this system as the parameter p_1 is varied. We aim to find the stationary points, bifurcations and (stable) limit cycles of the system, and study their stability.

Additionally, we aim to find the period doubling bifurcations of the limit cycles, and study the stability of the resulting period-doubled limit cycles. Eventually, we aim to determine the period doubling route to chaos.

2.1 Stationary points

Table 3 shows the stationary points of the system.

index	x_1	x_2	x_3	stability
1	-2.00	6.00	0.00	Unstable
2	0.00	0.00	0.00	Unstable
3	0.00	1.00	0.00	Unstable
4	1.33×10^{-1}	0.00	3.47×10^{-1}	Unstable
5	1.33×10^{-1}	6.69×10^{-1}	2.13×10^{-1}	Stable
6	1.00	0.00	0.00	Unstable
7	1.89×10^1	-4.62×10^1	2.09	Unstable
8	1.89×10^1	0.00	-7.15	Stable

Table 3: Stationary points of the MCM model.

2.2 Continuation of stationary points

We observe two stable stationary points (indices 5 and 8). We continue these points in the parameter $p_1 \in [0.5, 1]$ and plot the continuation in Figure 5.

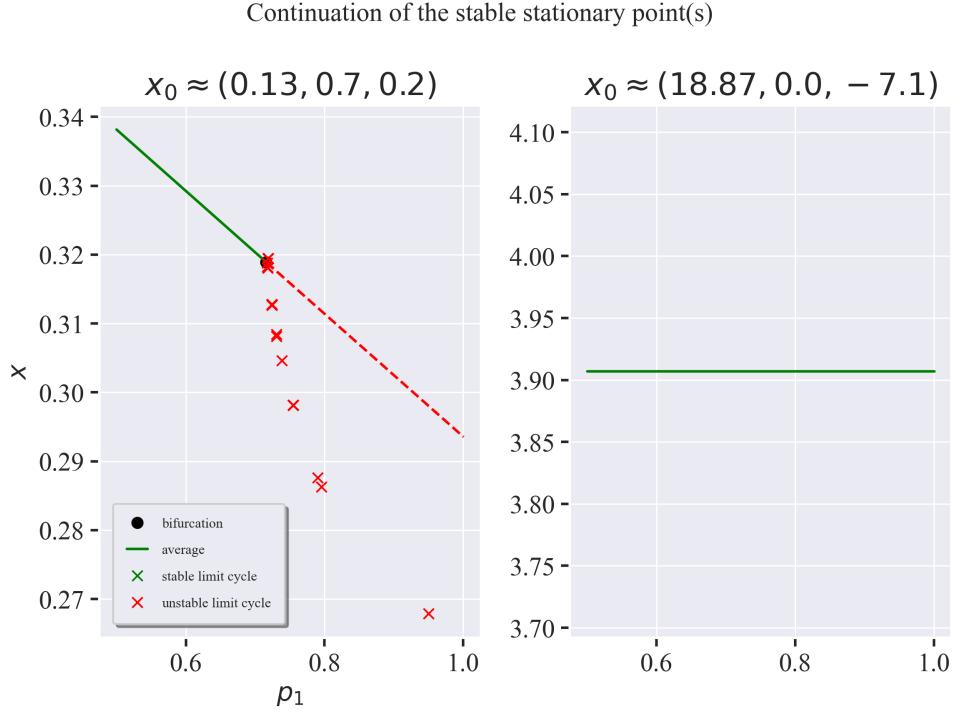


Figure 5: Continuation of the stable stationary points in the parameter p_1 , point 5 (left) and point 8 (right). The former encounters a bifurcation at $p_1 \approx 0.71$. From this Hopf-bifurcation an unstable stationary solution and a limit cycle emerges.

The bifurcation that occurs during the continuation of point 5 in Figure 5 is given by

Bifurcation type : subcritical? Hopf
Parameter : p_1
Critical value : 0.71
Eigenvalues : $0 \pm 0.36i, -0.53$
 $x_1, x_2, x_3 : 0.13, 0.67, 0.16$

This bifurcation point was found by applying an indirect method based on the interpolation

of test function values, adapted from Seydel's book (section 5.3.1). The particular test function used was the maximum of the real part of the eigenvalues of the system Jacobian.

The bifurcation is a Hopf-bifurcation, as a complex pair of eigenvalues crosses the imaginary axis with non-zero velocity. That is to say, the eigenvalues of the system jacobian at the continuation step directly after the bifurcation are

$$0.007 \pm 0.355i, -0.54,$$

which shows the eigenvalues pass through the imaginary axis with non-zero velocity.

From the hopf bifurcation we determine the first dynamical solution using the method described in section 7.6.2 of Seydel's book. In other words we solve the following BVP

$$\begin{pmatrix} \mathbf{x} \\ T \\ \lambda \\ \mathbf{h} \end{pmatrix}' = \begin{pmatrix} T\mathbf{f}(\mathbf{x}, \lambda) \\ 0 \\ 0 \\ T\mathbf{f}_x(\mathbf{x}, \lambda)\mathbf{h} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{x}(0) - \mathbf{x}(1) \\ \mathbf{h}(0) - \mathbf{h}(1) \\ \sum_i h_i \partial f_1 / \partial x_i \\ h_1(0) - 1 \end{pmatrix} = \mathbf{0},$$

using the shooting method in order to switch from the stationary solution to the limit cycle. The solution we obtain is

$$\begin{aligned} \mathbf{x}^{(0)} &= (0.13, 0.67, 0.15) \\ T^{(0)} &= 17.6 \\ p_1^{(0)} &= 0.72 \\ \mathbf{h}^{(0)} &= (1.00, -1.40, 0.00), \end{aligned}$$

from which we get an estimate for the period of the limit cycle.

2.3 Continuation of limit cycle

We use the solution of the BVP from the previous section as an initial guess for the continuation of the limit cycle. Following the method outlined in section 7.6.3 of Seydel's book, we repeatedly solve the BVP, wherein each iteration starts with the solution of the previous iteration plus a small perturbation given by \mathbf{h} (Equation 7.24, Seydel)

$$\bar{\mathbf{x}}^{(k+1)} = \mathbf{x}^{(k)} + \delta \mathbf{h}^{(k)}, \quad \bar{T}^{k+1} = T^k, \quad \bar{p}_1^{(k+1)} = p_1^{(k)},$$

for a suitably small parameter $\delta \approx 0.13$. This results a sequence of limit cycles starting from the Hopf point ($k=0$)

$$\text{cycle}^{(k)} = \left\{ \mathbf{x}^{(k)}, T^{(k)}, p_1^{(k)} \right\}, \quad k = 0, 1, 2, \dots$$

which we integrate over their respective periods and plot in Figure 6.

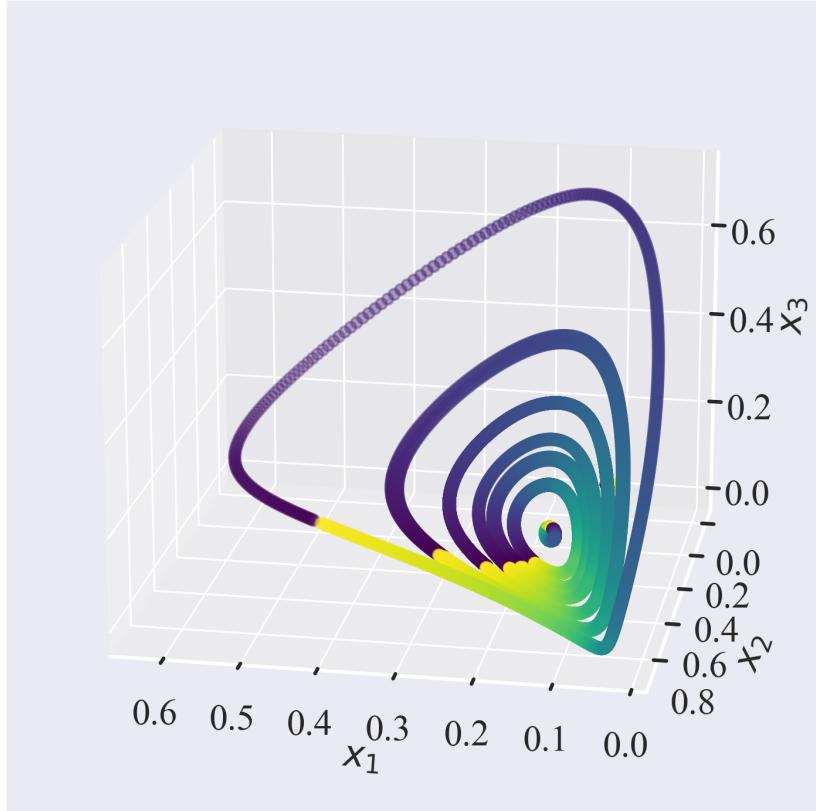


Figure 6: Continuation of the limit cycle in the parameter p_1 . The colors indicate the the flow of time from dark to light.

The limit cycles in Figure 6 are averaged both in time and in space to obtain the points in Figure 6.

For each limit cycle we calculate the monodromy matrix M using Algorithm 7.12 from section 7.5.1 of Seydel. The eigenvalues of M (Floquet multipliers) μ_j are then used to determine the stability of the limit cycle (according to Summary 7.5, p. 315, Seydel).

The stability of the limit cycles obtained from this process is accordigly shown in Figure 5, where red and green crosses indicate unstable and stable limit cycles, respectively.

At this point we observe something unexpected. The limit cycles are all unstable, which is not what we would expect from the Hopf-bifurcation. This is likely due to an incorrectly calculated monodromy matrix. This hinders our progress in finding the period-doubling bifurcations and the route to chaos, as will become apparent in the next section.

2.4 Period doubling bifurcations

Period doubling bifurcations occur at those limit cycles where there is a j for which $\mu_j = -1$. Hence, we continue the limit cycle (and calculate the monodromy matrices) until we find a multiplier close to -1 . Then, we apply the shooting method to solve the BVP in Equation 7.27 of Seydel's book. This gives a point either on original branch or on the period-doubled branch, depending on how we choose the boundary conditions and what kind of symmetry we expect (section 6.4, Seydel).

Once we have the period-doubled branch, we can continue it in the same way as the original branch. That is using the continuation method from the previous section and the solution to the BVP 7.27. We continue until we find a second period-doubling bifurcation, and so on.

Alternatively, we can predict the parameter values at which we expect period-doublings to

occur using Feigenbaum's law (Equation 7.14, Seydel)

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{p_n - p_{n-1}} = 0.21416 \dots,$$

2.5 Conclusion

In the above treatment of the MCM system we

1. Found the stationary points of the system and continued the stable ones in the parameter p_1 ;
2. Found a Hopf-bifurcation at $p_1 \approx 0.71$ and calculated the first limit cycle using the shooting method;
3. Continued the limit cycle in the parameter p_1 and calculated the stability of the limit cycles;
4. Described a method for finding period-doubling bifurcations and/or predicting them.

We have yet

1. To find the period-doubling bifurcations of the limit cycles and study the stability of the resulting period-doubled limit cycles;
2. To determine the period doubling route to chaos.

The reason for these unexplored parts is (probably) due to an incorrectly calculated monodromy matrix, which results in an incorrect stability analysis of the limit cycles. This hindered our progress in finding the period-doubling bifurcations and the route to chaos, since the proposed method for doing so assumed good approximations of the monodromy matrix.

Assignment 3

3.1 Introduction

3.2 Conclusion