Bezier Bases for Simplex and Pyramid Elements

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The procedure for the derivation of the basis functions is taken directly from Chan et al. [1].

1 Simplices

Given the coordinates of the vertices of the reference simplex,

$$\boldsymbol{r}_V = \begin{pmatrix} -1 & -\frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 1 & -\frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{2}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & 0 & \frac{1}{2}\sqrt{6} \end{pmatrix},$$

the barycentric coordinates can be found by solving the following linear system

$$\mathbf{A}\lambda = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} & 0 \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{6} \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 1 \\ r \\ s \\ t \end{pmatrix}.$$

The result is

$$\lambda = \begin{pmatrix} -\frac{1}{18}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) - \frac{1}{2}\,r + \frac{1}{2} \\ -\frac{1}{18}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) + \frac{1}{2}\,r + \frac{1}{2} \\ \frac{1}{9}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) \\ \frac{1}{24}\sqrt{6}\left(6\,t+\sqrt{6}\right) \end{pmatrix}.$$

Given the Duffy-type transform mapping the reference coordinates to the d-cube,

$$\mathbf{a} = \begin{pmatrix} -\frac{6\,r}{2\sqrt{3}s+\sqrt{6}t-3} \\ -\frac{8\sqrt{3}s}{3\left(\sqrt{6}t-3\right)} - \frac{1}{3} \\ \frac{1}{2}\sqrt{6}t - \frac{1}{2} \end{pmatrix},$$

these equations can be solved for the representation of the rst coordinates in terms of the abc coordinates,

$$\mathbf{r} = \begin{pmatrix} \frac{1}{4}\,a(b-1)(c-1) \\ -\frac{1}{12}\sqrt{3}(3\,b+1)(c-1) \\ \frac{1}{6}\sqrt{6}(2\,c+1) \end{pmatrix}, \tag{1.1}$$

with the barycentric coordinates then given by

$$\lambda = \begin{pmatrix} -\frac{1}{8}\,(a-1)(b-1)(c-1) \\ \frac{1}{8}\,(a+1)(b-1)(c-1) \\ -\frac{1}{4}\,(b+1)(c-1) \\ \frac{1}{2}\,c+\frac{1}{2} \end{pmatrix}. \tag{1.2}$$

The $ijkl^{\text{th}}$ Bezier basis of order p for the simplex is given by

$$B^p_{ijkl} \coloneqq C^p_{ijkl} \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l,$$

where $i+j+k+l=p$ and

$$C^p_{ijkl} = \frac{p!}{i!j!k!l!} = \frac{(i+j)!}{i!j!} \frac{(i+j+k)!}{(i+j)!k!} \frac{(i+j+k+l)!}{(i+j+k)!l!}.$$

After substitution of (1.2), we obtain

$$B^p_{ijkl}(a,b,c) = C^p_{ijkl} 8^{-i-j} 4^{-k} 2^{-l} \left(-1\right)^{i+k} \left(a+1\right)^j \left(a-1\right)^i \left(b+1\right)^k \left(b-1\right)^{i+j} \left(c+1\right)^l \left(c-1\right)^{i+j+k}.$$

Noting the definition of the ith 1D Bezier basis function of degree p,

$$B^p_i(a) = \frac{p!}{i!(p-i)!} (-\frac{1}{2}\,a + \frac{1}{2})^{p-i} (\frac{1}{2}\,a + \frac{1}{2})^i, \tag{1.3}$$

it is then possible to represent the simplex basis using a tensor-product of 1D Bezier basis functions as

$$B^p_{ijkl}(a,b,c) = B^{i+j}_j(a) B^{i+j+k}_k(b) B^{i+j+k+l}_l(c).$$

As the gradients of the basis are taken with respect to the rst coordinates, we first note that the Jacobian of abc coordinates with respect to the rst coordinates is given by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} -\frac{6}{2\sqrt{3}s+\sqrt{6}t-3} & \frac{12\sqrt{3}r}{\left(2\sqrt{3}s+\sqrt{6}t-3\right)^2} & \frac{6\sqrt{6}r}{\left(2\sqrt{3}s+\sqrt{6}t-3\right)^2} \\ 0 & -\frac{8\sqrt{3}}{3\left(\sqrt{6}t-3\right)} & \frac{8\sqrt{6}\sqrt{3}s}{3\left(\sqrt{6}t-3\right)^2} \\ 0 & 0 & \frac{1}{2}\sqrt{6} \end{pmatrix},$$

and after substituting (1.1), by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} \frac{4}{(b-1)(c-1)} & \frac{4\sqrt{3}a}{3\,(b-1)(c-1)} & \frac{2\sqrt{6}a}{3\,(b-1)(c-1)} \\ 0 & -\frac{4\sqrt{3}}{3\,(c-1)} & -\frac{\sqrt{6}(3\,b+1)}{6\,(c-1)} \\ 0 & 0 & \frac{1}{2}\sqrt{6} \end{pmatrix}.$$

It is important to note that these Jacobian terms are singular at degenerate points of the mapping and that the basis gradients must be defined accordingly. Observing that the singular component will always affect the first term in (1.3), we note that

$$\frac{p!}{i!(p-i)!} (-\frac{1}{2}\,a + \frac{1}{2})^{p-i-1} (\frac{1}{2}\,a + \frac{1}{2})^i = \frac{p}{p-i} B^{p-1}_i(a),$$

which is used in defining the gradients of $B^p_{ijkl}(a,b,c)$ with respect to the rst coordinates as

$$\begin{aligned} \frac{\partial B^p_{ijkl}(a,b,c)}{\partial r} &= (1)\left(\frac{i+j+k+l}{i+j}\right) \frac{dB^{i+j}_j(a)}{da} B^{i+j+k-1}_k(b) B^{i+j+k+l-1}_l(c) \\ &+ (0)\left(\frac{i+j+k+l}{i+j+k}\right) B^{i+j}_j(a) \frac{dB^{i+j+k}_k(b)}{db} B^{i+j+k+l-1}_l(c) \\ &+ (0)\, B^{i+j}_j(a) B^{i+j+k}_k(b) \frac{dB^{i+j+k+l}_l(c)}{dc}, \end{aligned}$$

$$\frac{\partial B^p_{ijkl}(a,b,c)}{\partial s} = \left(\frac{1}{3}\sqrt{3}a\right)\left(\frac{i+j+k+l}{i+j}\right) \frac{dB^{i+j}_j(a)}{da} B^{i+j+k-1}_k(b) B^{i+j+k+l-1}_l(c)$$

$$+\left(\frac{2}{3}\sqrt{3}\right)\left(\frac{i+j+k+l}{i+j+k}\right)B_j^{i+j}(a)\frac{dB_k^{i+j+k}(b)}{db}B_l^{i+j+k+l-1}(c)$$
$$+(0)\,B_j^{i+j}(a)B_k^{i+j+k}(b)\frac{dB_l^{i+j+k+l}(c)}{dc},$$

$$\begin{aligned}\frac{\partial B_{ijkl}^p(a,b,c)}{\partial t} &= \left(\frac{1}{6}\sqrt{6}a\right)\left(\frac{i+j+k+l}{i+j}\right)\frac{dB_j^{i+j}(a)}{da}B_k^{i+j+k-1}(b)B_l^{i+j+k+l-1}(c)\\ &+\left(\frac{1}{12}\sqrt{6}(3\,b+1)\right)\left(\frac{i+j+k+l}{i+j+k}\right)B_j^{i+j}(a)\frac{dB_k^{i+j+k}(b)}{db}B_l^{i+j+k+l-1}(c)\\ &+\left(\frac{1}{2}\sqrt{6}\right)B_j^{i+j}(a)B_k^{i+j+k}(b)\frac{dB_l^{i+j+k+l}(c)}{dc}.\end{aligned}$$

2 Pyramids

The Bezier basis for the pyramid element is given directly by Chan et al. [1, eq. (2)]

$$B_{ijk}^p(a,b,c) = B_i^{p-k}(a)B_j^{p-k}(b)B_k^p(c),$$

where the abc coordinates are once again given by a Duffy-type transformation

$$\mathbf{a} = \begin{pmatrix} -\frac{10\,r}{5\sqrt{2}t-8} \\ -\frac{10\,s}{5\sqrt{2}t-8} \\ \sqrt{2}t-\frac{3}{5} \end{pmatrix},$$

which is determined based on making the following choice for the vertices for the reference pyramid

$$\boldsymbol{r}_V = \begin{pmatrix} -1 & -1 & -\frac{1}{5}\sqrt{2} \\ 1 & -1 & -\frac{1}{5}\sqrt{2} \\ -1 & 1 & -\frac{1}{5}\sqrt{2} \\ 1 & 1 & -\frac{1}{5}\sqrt{2} \\ 0 & 0 & \frac{4}{5}\sqrt{2} \end{pmatrix}.$$

As above, these equations can be solved for the representation of the rst coordinates in terms of the abc coordinates,

$$\mathbf{r} = \begin{pmatrix} -\frac{1}{2}\,a(c-1) \\ -\frac{1}{2}\,b(c-1) \\ \frac{1}{10}\sqrt{2}(5\,c+3) \end{pmatrix}. \tag{2.1}$$

The computation of the gradients of the basis functions with respect to the reference coordinates once again requires the Jacobian of the abc coordinates with respect to the rst coordinates, given by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} -\frac{10}{5\sqrt{2}t-8} & 0 & \frac{50\sqrt{2}r}{\left(5\sqrt{2}t-8\right)^2} \\ 0 & -\frac{10}{5\sqrt{2}t-8} & \frac{50\sqrt{2}s}{\left(5\sqrt{2}t-8\right)^2} \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

and after substituting (2.1), by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} -\frac{2}{c-1} & 0 & -\frac{\sqrt{2}a}{c-1} \\ 0 & -\frac{2}{c-1} & -\frac{\sqrt{2}b}{c-1} \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

It can once again be noted that these Jacobian terms are singular at degenerate points of the mapping and that the basis gradients must be defined accordingly. Once again using (1.3), the gradients with respect to the rst coordinates are given by

$$
\begin{aligned}
\frac{\partial B^p_{ijkl}(a,b,c)}{\partial r} &= (1)\left(\frac{p}{p-k}\right)\frac{dB^{p-k}_i(a)}{da}B^{p-k}_j(b)B^{p-1}_k(c) \\
&+ (0)\left(\frac{p}{p-k}\right)B^{p-k}_i(a)\frac{dB^{p-k}_j(b)}{db}B^{p-1}_k(c) \\
&+ (0)\,B^{p-k}_i(a)B^{p-k}_j(b)\frac{dB^{p}_k(c)}{dc},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial B^p_{ijkl}(a,b,c)}{\partial s} &= (0)\left(\frac{p}{p-k}\right)\frac{dB^{p-k}_i(a)}{da}B^{p-k}_j(b)B^{p-1}_k(c) \\
&+ (1)\left(\frac{p}{p-k}\right)B^{p-k}_i(a)\frac{dB^{p-k}_j(b)}{db}B^{p-1}_k(c) \\
&+ (0)\,B^{p-k}_i(a)B^{p-k}_j(b)\frac{dB^{p}_k(c)}{dc},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial B^p_{ijkl}(a,b,c)}{\partial t} &= \left(\frac{1}{2}\sqrt{2}a\right)\left(\frac{p}{p-k}\right)\frac{dB^{p-k}_i(a)}{da}B^{p-k}_j(b)B^{p-1}_k(c) \\
&+ \left(\frac{1}{2}\sqrt{2}b\right)\left(\frac{p}{p-k}\right)B^{p-k}_i(a)\frac{dB^{p-k}_j(b)}{db}B^{p-1}_k(c) \\
&+ \left(\sqrt{2}\right)B^{p-k}_i(a)B^{p-k}_j(b)\frac{dB^{p}_k(c)}{dc},
\end{aligned}
$$

References

[1] J. Chan, T. Warburton, A short note on a bernstein–bezier basis for the pyramid, SIAM Journal on Scientific Computing 38 (4) (2016) A2162–A2172. arXiv:https://doi.org/10.1137/15M1036397, doi:10.1137/15M1036397.
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