Bezier Bases for Simplex and Pyramid Elements

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1 Simplices

Given the coordinates of the vertices of the reference triangle,

$$\boldsymbol{r}_V = \begin{pmatrix} -1 & -\frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 1 & -\frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{2}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & 0 & \frac{1}{2}\sqrt{6} \end{pmatrix}$$

the barycentric coordinates can be found by solving the following linear system:

$$\mathbf{A}\lambda = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} & 0 \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{6} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ r \\ s \\ t \end{pmatrix}.$$

The result is

$$\lambda = \begin{pmatrix} -\frac{1}{18}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) - \frac{1}{2}\,r + \frac{1}{2} \\ -\frac{1}{18}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) + \frac{1}{2}\,r + \frac{1}{2} \\ \frac{1}{9}\sqrt{3}\left(3\,s+\sqrt{3}\right) - \frac{1}{72}\sqrt{6}\left(6\,t+\sqrt{6}\right) \\ \frac{1}{24}\sqrt{6}\left(6\,t+\sqrt{6}\right) \end{pmatrix}.$$

Given the Duffy-type transform mapping the reference coordinates to the d-cube

$$\mathbf{a} = \begin{pmatrix} -\frac{6\,r}{2\,\sqrt{3}s+\sqrt{6}t-3} \\ -\frac{8\,\sqrt{3}s}{3\left(\sqrt{6}t-3\right)} - \frac{1}{3} \\ \frac{1}{2}\sqrt{6}t - \frac{1}{2} \end{pmatrix},$$

these equations can be solved for the representation of the rst coordinates in terms of the abc coordinates,

$$\mathbf{r} = \begin{pmatrix} \frac{1}{4}\,a(b-1)(c-1) \\ -\frac{1}{12}\,\sqrt{3}(3\,b+1)(c-1) \\ \frac{1}{6}\,\sqrt{6}(2\,c+1) \end{pmatrix}, \tag{1}$$

with the barycentric coordinates then given by

$$\lambda = \begin{pmatrix} -\frac{1}{8}\,(a-1)(b-1)(c-1) \\ \frac{1}{8}\,(a+1)(b-1)(c-1) \\ -\frac{1}{4}\,(b+1)(c-1) \\ \frac{1}{2}\,c+\frac{1}{2} \end{pmatrix}.$$

The $ijkl^{\text{th}}$ Bezier basis of order p for the simplex is given by

$$B^p_{ijkl} \coloneqq C^p_{ijkl}\lambda_0^i\lambda_1^j\lambda_2^k\lambda_3^l$$

where $i+j+k+l=p$ and

$$C^p_{ijkl} = \frac{p!}{i!j!k!l!} = \frac{(i+j)!}{i!j!}\frac{(i+j+k)!}{(i+j)!k!}\frac{(i+j+k+l)!}{(i+j+k)!l!}$$

After substitution of the barycentric coordinates in terms of the abc coordinates, (1), we obtain

$$B^p_{ijkl}(a,b,c) = C^p_{ijkl}8^{-i-j}4^{-k}2^{-l}\,(-1)^{i+k}\,(a+1)^j(a-1)^i(b+1)^k(b-1)^{i+j}(c+1)^l(c-1)^{i+j+k}.$$

Noting the definition of the ith 1D Bezier basis function of degree p,

$$B^p_i(a) = \frac{p!}{i!(p-i)!}(-\frac{1}{2}\,a+\frac{1}{2})^{p-i}(\frac{1}{2}\,a+\frac{1}{2})^i, \tag{2}$$

it is then possible to represent the simplex basis using a tensor-product of 1D Bezier basis functions:

$$B^p_{i,j,k,l}(a,b,c) = B^{i+j}_j(a)B^{i+j+k}_k(b)B^{i+j+k+l}_l(c).$$

As the gradients of the basis are taken with respect to the rst coordinates, we first note that the Jacobian of abc coordinates with respect to the rst coordinates is given by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} -\frac{6}{2\,\sqrt{3}s+\sqrt{6}t-3} & \frac{12\,\sqrt{3}r}{\left(2\,\sqrt{3}s+\sqrt{6}t-3\right)^2} & \frac{6\,\sqrt{6}r}{\left(2\,\sqrt{3}s+\sqrt{6}t-3\right)^2} \\ 0 & -\frac{8\,\sqrt{3}}{3\left(\sqrt{6}t-3\right)} & \frac{8\,\sqrt{6}\sqrt{3}s}{3\left(\sqrt{6}t-3\right)^2} \\ 0 & 0 & \frac{1}{2}\,\sqrt{6} \end{pmatrix},$$

and after substituting (1), by

$$\frac{\mathbf{da}}{\mathbf{dr}} = \begin{pmatrix} \frac{4}{(b-1)(c-1)} & \frac{4\,\sqrt{3}a}{3\,(b-1)(c-1)} & \frac{2\,\sqrt{6}a}{3\,(b-1)(c-1)} \\ 0 & -\frac{4\,\sqrt{3}}{3\,(c-1)} & -\frac{\sqrt{6}(3\,b+1)}{6\,(c-1)} \\ 0 & 0 & \frac{1}{2}\,\sqrt{6} \end{pmatrix}.$$

It is important to note that these Jacobian terms are singular at degenerate points of the mapping and that the basis gradients must be defined accordingly. Observing that the singular component will always affect the first term in (2), we note that

$$\frac{p!}{i!(p-i)!}(-\frac{1}{2}\,a+\frac{1}{2})^{p-i-1}(\frac{1}{2}\,a+\frac{1}{2})^{i}=\frac{p}{p-i}B_i^{p-1}(a). \tag{3}$$

The gradients of $B^p_{i,j,k,l}(a,b,c)$ with respect to the rst coordinates are then given by

$$\begin{aligned}\frac{\partial B^p_{i,j,k,l}(a,b,c)}{\partial r} &= (1)\left(\frac{i+j+k+l}{i+j}\right)\frac{dB_j^{i+j}(a)}{da}B_k^{i+j+k-1}(b)B_l^{i+j+k+l-1}(c)\\ &+(0)\left(\frac{i+j+k+l}{i+j+k}\right)B_j^{i+j}(a)\frac{dB_k^{i+j+k}(b)}{db}B_l^{i+j+k+l-1}(c)\\ &+(0)\,B_j^{i+j}(a)B_k^{i+j+k}(b)\frac{dB_l^{i+j+k+l}(c)}{dc}.\end{aligned}$$

$$\begin{aligned}\frac{\partial B^p_{i,j,k,l}(a,b,c)}{\partial s} &= \left(\frac{1}{3}\,\sqrt{3}a\right)\left(\frac{i+j+k+l}{i+j}\right)\frac{dB_j^{i+j}(a)}{da}B_k^{i+j+k-1}(b)B_l^{i+j+k+l-1}(c)\\ &+\left(\frac{2}{3}\,\sqrt{3}\right)\left(\frac{i+j+k+l}{i+j+k}\right)B_j^{i+j}(a)\frac{dB_k^{i+j+k}(b)}{db}B_l^{i+j+k+l-1}(c)\\ &+(0)\,B_j^{i+j}(a)B_k^{i+j+k}(b)\frac{dB_l^{i+j+k+l}(c)}{dc}.\end{aligned}$$

$$\begin{aligned}\frac{\partial B^p_{i,j,k,l}(a,b,c)}{\partial t} &= \left(\frac{1}{6}\,\sqrt{6}a\right)\left(\frac{i+j+k+l}{i+j}\right)\frac{dB_j^{i+j}(a)}{da}B_k^{i+j+k-1}(b)B_l^{i+j+k+l-1}(c)\\ &+\left(\frac{1}{12}\,\sqrt{6}(3\,b+1)\right)\left(\frac{i+j+k+l}{i+j+k}\right)B_j^{i+j}(a)\frac{dB_k^{i+j+k}(b)}{db}B_l^{i+j+k+l-1}(c)\\ &+\left(\frac{1}{2}\,\sqrt{6}\right)B_j^{i+j}(a)B_k^{i+j+k}(b)\frac{dB_l^{i+j+k+l}(c)}{dc}.\end{aligned}$$