On the Selection of Finite Element Test Norms for Hyperbolic PDEs based on their Stabilization and Convergence Properties

Philip Zwanenburg and Siva Nadarajah

Computational Aerodynamics Group
McGill University

July 11, 2018

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory
Abstract Setting
Model Problem
Methods
DPG
OPG
DG
Discrete inf-sup Testing

Results

References

Motivation and Goals

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

The success of Bubnov-Galerkin finite element methods for the solution of elliptic partial differential equations (PDEs) arises from the relation between the variational formulations and the minimization of an energy functional.

In the case of convection-dominated or purely hyperbolic PDEs, additional stabilization is required in regions of high local Péclet number and is most commonly introduced either through the introduction of a suitably chosen numerical flux or through the modification of the test space (resulting in a Petrov-Galerkin method).

Taking the linear advection equation as a model problem, our goal is to present our initial investigation into determining the *optimal* form of stabilization.

Outline

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

▶ Presentation of methods and related theory:
  ▶ Discontinuous Galerkin method with upwind numerical flux;
  ▶ Discontinuous Petrov-Galerkin method with various test norms [1];
  ▶ Optimal Trial Petrov-Galerkin method [2].
▶ Convergence analysis in relation to discrete inf-sup stability.
▶ Numerical results in one and two dimensions.
▶ Discussion of limitations and future directions.

Abstract Functional Setting - Continuous

Consider the *linear* abstract variational problem

Find $u \in U$ such that $b(v, u) = l(v) \ \forall v \in V$.

We have the standard requirements:

$$|b(v, u)| \leq M ||v||_V ||u||_U,$$

$$\exists \gamma > 0 : \inf_{u \in U} \sup_{v \in V} \frac{b(v, u)}{||v||_V ||u||_U} \geq \gamma,$$

and

$l(v) = 0 \ \forall v \in V_0$, where $V_0 := \{v \in V : b(v, u) = 0 \ \forall u \in U\}$.

Then by the Banach-Nečas-Babuška theorem,

$$||u||_U \leq \frac{M}{\gamma} ||l||_{V'}.$$

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Abstract Functional Setting - Discrete

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Choosing finite dimensional subsets we have the discrete variational problem,

Find $u_h \in U_h$ such that $b(v_h, u_h) = l(v_h) \ \forall v_h \in V_h$.

If a discrete version of the inf-sup condition holds,

$$||u - u_h||_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} ||u - w_h||_U.$$

Abstract Functional Setting - Quick Proof

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Beginning with the discrete inf-sup condition we have, for all $w_h \in U_h$,

$$\begin{aligned}
\gamma_h ||w_h - u_h||_U &\leq \sup_{v_h \in V_h} \frac{b(v_h, w_h - u_h)}{||v_h||_V} \\
&= \sup_{v_h \in V_h} \frac{b(v_h, w_h - u) + b(v_h, u - u_h)}{||v_h||_V} \\
&= \sup_{v_h \in V_h} \frac{b(v_h, w_h - u)}{||v_h||_V} \\
&\leq \sup_{v_h \in V_h} \frac{M||v_h||_V ||w_h - u||_U}{||v_h||_V} \\
&= M||w_h - u||_U.
\end{aligned}$$

Abstract Functional Setting - Quick Proof

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Using the triangle inequality and that above we thus find

$$||u_h - u||_U \leq ||u_h - w_h||_U + ||w_h - u||_U \qquad \forall w_h \in U_h$$

$$\leq \frac{M}{\gamma_h}||w_h - u||_U + ||w_h - u||_U \qquad \forall w_h \in U_h$$

$$= \left(1 + \frac{M}{\gamma_h}\right)||w_h - u||_U \qquad \forall w_h \in U_h$$

$$\rightarrow ||u_h - u||_U \leq \left(1 + \frac{M}{\gamma_h}\right) \inf_{w_h \in U_h} ||w_h - u||_U,$$

where the 1 can be removed in the Hilbert space setting [3].

Abstract Functional Setting - Selection of Norms

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

A case of particular interest:

$$M = \gamma_h.$$

▶ Error incurred by the discrete approximation is smallest.

Bui-Thanh et al. [4, Theorem 2.6] have proven that

$$M = \gamma_h = 1 \iff \exists v_u \in V \setminus \{0\} : b(v_u, u) = ||v_u||_V ||u||_U \ \forall u \in U,$$

where v_u is termed an optimal test function for the trial function u.

Abstract Functional Setting - Selection of Norms

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory
Abstract Setting
Model Problem
Methods
DPG
OPG
DG
Discrete inf-sup Testing

Results

References

One choice of strategy is to select the norm which is naturally induced by the problem. Defining the map from trial to test space, $T : U \ni u \to Tu := v_{Tu} \in V$, by

$$(v_{Tu}, Tu)_V = b(v, u),$$

then

$$V_{\text{opt}} = \{v_{Tu} \in V : u \in U\}.$$

In the discrete trial space optimal test functions are determined according to

Find $v_{Tu_h} \in V$ such that $(w, v_{Tu_h})_V = b(w, u_h)$, $\forall w \in V$.

Model Problem - Steady Linear Advection

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} u = s \qquad \text{in } \Omega,$$
$$u = u_{\Gamma^-} \qquad \text{on } \Gamma^- := \{\boldsymbol{x} \in \partial\Omega : \hat{\boldsymbol{n}} \cdot \boldsymbol{b} < 0\}.$$

Weak formulation:

$$b(v, \boldsymbol{u}) = \sum_{\mathcal{V}} \int_{\mathcal{V}} -\boldsymbol{\nabla} v \cdot \boldsymbol{b} u \, d\omega + \int_{\partial \mathcal{V} \backslash \Gamma^-} v f^* \, d\gamma,$$

$$l(v) = \sum_{\mathcal{V}} \int_{\mathcal{V}} vs \, d\omega + \int_{\mathcal{F} \cap \Gamma^-} v f_{\Gamma^-} \, d\gamma.$$

Convenient reformulation:

$$b(v, \boldsymbol{u}) = \sum_{\mathcal{V}} \int_{\mathcal{V}} -\boldsymbol{\nabla} v \cdot \boldsymbol{b} u \, d\omega + \sum_{\mathcal{F}} \int_{\mathcal{F}} [\![v]\!] f^* \, d\gamma,$$

$$l(v) = \sum_{\mathcal{V}} \int_{\mathcal{V}} vs \, d\omega.$$

The Discontinuous Petrov-Galerkin (DPG) Method

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory
Abstract Setting
Model Problem
Methods
DPG
OPG
DG
Discrete inf-sup Testing

Results

References

Demkowicz et al. (2010) [1].

Motivated by optimal solution convergence in the L^2 norm, we would like to move gradients in the bilinear form to the test space.

From the formal L^2-adjoint and a bilinear form representing the boundary terms

$$b(v, u) = b^*(v, u) + c(\text{tr}_A^* v, \text{tr}_A u)$$

we obtrain the graph space for the adjoint

$$H_b^*(\Omega) \coloneqq \{v \in (L^2(\Omega)) : B^* v \in (L^2(\Omega)) \ \forall u \in U\}.$$

When setting $V = H_b^*(\Omega)$, we say that the test space is H_b-conforming.

The DPG Method - Broken Test Space

No assumptions yet made regarding conformity of trial and test spaces.

The goal of the methodology is to solve for the solution over a tessellation, $\mathcal{T}_h$, of the discretized domain, Ω_h, consisting of elements (referred to as volumes), $\mathcal{V}$.

To make the method practical, the DPG method uses broken test spaces such that test functions can be computed elementwise,

Find $v_{Tu_h} \in V(\mathcal{V})$ such that $(w, v_{Tu_h})_{V(\mathcal{V})} = b(w, u_h), \ \forall w \in V(\mathcal{V})$

where

$$V(\Omega_h) \coloneqq \{v \in L^2(\Omega) : v|_{\mathcal{V}} \in H_b^*(\mathcal{V}) \ \forall \mathcal{V} \in \mathcal{T}_h\},$$
$$(w, v)_{V(\Omega_h)} \coloneqq \sum_{\mathcal{V}} (w|_{\mathcal{V}}, v|_{\mathcal{V}})_{V(\mathcal{V})}.$$

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

The DPG Method - Investigated Norms

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

$L^2 : (w|_{\mathcal{V}}, v|_{\mathcal{V}})_{V(\mathcal{V})} = (w|_{\mathcal{V}}, v|_{\mathcal{V}}).$

$H^1 : (w|_{\mathcal{V}}, v|_{\mathcal{V}})_{V(\mathcal{V})} = (w|_{\mathcal{V}}, v|_{\mathcal{V}}) + (\boldsymbol{\nabla} w|_{\mathcal{V}}, \boldsymbol{\nabla} v|_{\mathcal{V}}).$

$H^1_{\boldsymbol{b}} : (w|_{\mathcal{V}}, v|_{\mathcal{V}})_{V(\mathcal{V})} = (w|_{\mathcal{V}}, v|_{\mathcal{V}}) + (\boldsymbol{b} \cdot \boldsymbol{\nabla} w|_{\mathcal{V}}, \boldsymbol{b} \cdot \boldsymbol{\nabla} v|_{\mathcal{V}}).$

$H^+_{\boldsymbol{b}} : (w|_{\mathcal{V}}, v|_{\mathcal{V}})_{V(\mathcal{V})} = \langle w|_{\mathcal{V}}, |\boldsymbol{b} \cdot \hat{\boldsymbol{n}}| v_{\mathcal{V}} \rangle_{\partial \mathcal{V}^+} + (\boldsymbol{b} \cdot \boldsymbol{\nabla} w|_{\mathcal{V}}, \boldsymbol{b} \cdot \boldsymbol{\nabla} v|_{\mathcal{V}}).$

The Discontinuous Petrov-Galerkin Method

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Additional characteristics:

- General Petrov-Galerkin methodology outlined in the abstract setting is a subset of the practical DPG methodology. When both formulations are uniquely solvable, their solutions coincide.
- The required selection of a suitable numerical flux has been replaced with the required selection of a suitable test norm.
- When also using a discontinuous trial space, additional trace unknowns are introduced allowing for static condensation of volume unknowns in the global solve.

The Optimal Trial Petrov-Galerkin (OPG) Method

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory
  Abstract Setting
  Model Problem
  Methods
  DPG
  OPG
  DG
  Discrete inf-sup Testing

Results

References

Brunken et al. (2018) [1].

Similar motivation to that of the DPG method but with test norm chosen as that which is naturally induced by the problem such that the method is optimal in the sense of having unity continuity and inf-sup constants, and a trial norm corresponding to the L2 norm.

Main difference, global solve for the optimal test space followed by computation of discrete trial space through the application of the adjoint operator.

The OPG Method - Formulation

Cauchy-Schwarz on bilinear form:

$$b(v, \boldsymbol{u}) \leq \sum_{\mathcal{V}} || - \boldsymbol{\nabla} v \cdot \boldsymbol{b}||_{L^2(\mathcal{V})}||u||_{L^2(\mathcal{V})} + ||[\![v]\!]||_{L^2(\mathcal{F})}||f^*||_{L^2(\mathcal{F})}$$

$$\leq \underbrace{\left(\sum_{\mathcal{V}} || - \boldsymbol{\nabla} v \cdot \boldsymbol{b}||^2_{L^2(\mathcal{V})} + ||[\![v]\!]||^2_{L^2(\mathcal{F})} \right)^{\frac{1}{2}}}_{||v||_V} \times$$

$$\underbrace{\left(\sum_{\mathcal{V}} ||u||^2_{L^2(\mathcal{V})} + ||f^*||^2_{L^2(\mathcal{F})} \right)^{\frac{1}{2}}}_{||u||_U}.$$

Equality obtained (unity continuity and inf-sup constants) for the choice of test functions:

$$u = -\boldsymbol{\nabla} v_u \cdot \boldsymbol{b} \quad \text{in } \mathcal{V},$$
$$f^* = [\![v_u]\!] \qquad \text{on } \mathcal{F}.$$

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

The OPG Method - Formulation

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory

Abstract Setting
Model Problem
Methods
DPG
OPG
DG
Discrete inf-sup Testing

Results

References

In this case, a conforming test space is selected and the common
flux component of the group variable eliminated.

Imposing sufficient constraints on the test space,

$$\exists \text{ a unique } v_u \in V \text{ such that } u = B^* v, \ \forall u \in U.$$

The equivalent problem can then be obtained:

Find $w_h(u_h) \in V_h$ such that
$$b(v_h, u_h) := b(v_h, B^* w_h) := (B^* v_h, B^* w_h) = l(v_h) \ \forall v_h \in V_h,$$

and subsequently compute the solution.

The Discontinuous Galerkin (DG) Method

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

The DG method with upwind numerical flux corresponds to:

$$b(v, u) = \sum_{\mathcal{V}} \int_{\mathcal{V}} -\boldsymbol{\nabla} v \cdot \boldsymbol{b} u \ d\omega + \sum_{\mathcal{F}} \int_{\mathcal{F}} [\![v]\!] f^* \ d\gamma,$$

where $f^* = \hat{\boldsymbol{n}} \cdot \boldsymbol{b} u_{\text{upwind}}$. Equivalently,

$$b(v, u) = \sum_{\mathcal{V}} \int_{\mathcal{V}} -\boldsymbol{\nabla} v \cdot \boldsymbol{b} u \ d\omega$$
$$+ \sum_{\mathcal{F}} \int_{\mathcal{F}} [\![v]\!] \left((\boldsymbol{b} \cdot \hat{\boldsymbol{n}}) \{\!\{u\}\!\} + \frac{1}{2} |\boldsymbol{b} \cdot \hat{\boldsymbol{n}}| [\![u]\!] \right) d\gamma.$$

► Central flux term for discrete coercivity in
$||v||_{\text{cf}}^2 = ||v||_{L^2(\Omega)}^2 + \int_{\partial\Omega} \frac{1}{2} |\boldsymbol{b} \cdot \hat{\boldsymbol{n}}| v^2 \ d\Gamma$.

► Additional penalization term added to strengthen the stability
such that improved error estimates are obtained.

The DG Method - Coercivity Norms

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Discrete coercivity leading to a quasi-optimal error estimate can
be proven using the following norm

$$||v||^2_{uw_1} = ||v||^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2}|\boldsymbol{b}\cdot\hat{\boldsymbol{n}}|v^2 \, d\Gamma$$
$$+ \sum_{\mathcal{F}} \int_{\mathcal{F}} \frac{1}{2}|\boldsymbol{b}\cdot\hat{\boldsymbol{n}}|[\![v]\!]^2 d\gamma + \sum_{\mathcal{V}} h_{\mathcal{V}}||\boldsymbol{b}\cdot\boldsymbol{\nabla}v||^2_{L^2(\mathcal{V})}.$$

Continuity requires the addtional terms

$$||v||^2_{uw_2} = ||v||^2_{uw_1} + \sum_{\mathcal{V}} \left(h_{\mathcal{V}}^{-1}||v||^2_{L^2(\mathcal{V})} + ||v||^2_{L^2(\partial\mathcal{V})} \right).$$

Reference: Di Pietro et al. [5].

To do

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

- Compute inf-sup constants for the OPG method and ensure
  that it is 1; compare with inf-sup for DPG if possible.

References I

Test Norms for
Hyperbolic PDEs -
Stabilization and
Convergence

P. Zwanenburg and S.
Nadarajah

Overview

Theory

Theory

Abstract Setting
Model Problem
Methods
DPG
OPG
DG
Discrete inf-sup Testing

Results

References

📄 L. Demkowicz and J. Gopalakrishnan, "A class of discontinuous petrov-Galerkin methods. ii. optimal test functions," *Numerical Methods for Partial Differential Equations*, vol. 27, no. 1, pp. 70–105, 2011.

📄 J. Brunken, K. Smetana, and K. Urban, "(Parametrized) First Order Transport Equations: Realization of Optimally Stable Petrov-Galerkin Methods," *ArXiv e-prints*, Mar. 2018.

📄 J. Xu and L. Zikatanov, "Some observations on babuÅąka and brezzi theories," vol. 94, pp. 195–202, 03 2003.

📄 T. Bui-Thanh, L. Demkowicz, and O. Ghattas, "Constructively well-posed approximation methods with unity inf-sup and continuity constants for partial differential equations," *Mathematics of Computation*, vol. 82, p. 1923–1952, 2013.

📄 D. Di Pietro and A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*.
Mathématiques et Applications, Springer Berlin Heidelberg, 2011.