On the Selection of Finite Element Test Norms based on their Optimal Stabilization and Convergence Properties

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The success of standard Galerkin finite element methods for the solution of elliptic partial differential equations (PDEs) arises from the relation between the variational formulations and the minimization of an energy functional. In the case of convection-dominated or purely hyperbolic PDEs, additional stabilization is required in regions of high local Péclet or Reynolds number. While the need for this stabilization is understood, the specific form which it should take in order to guarantee both stable and optimally convergent discretizations is not generally provided. The goal of this work is to devise schemes for which the stabilization is optimal in the sense that the computed solution is given by the L^2 projection of the exact solution.

Consider the linear abstract variational problem

$$\text{Find } u \in U \text{ such that } b(v,u) = l(v) \ \forall v \in V.$$

We have the standard requirements:

$$|b(v,u)| \leq M||v||_V||u||_U,$$

$$\exists \gamma > 0 : \inf_{u \in U} \sup_{v \in V} \frac{b(v,u)}{||v||_V||u||_U} \geq \gamma,$$

and

$$l(v) = 0 \ \forall v \in V_0, \text{ where } V_0 \coloneqq \{v \in V : b(v,u) = 0 \ \forall u \in U\}.$$

Then by the Banach-Nečas-Babuška theorem,

$$||u||_U \leq \frac{M}{\gamma}||l||_{V'}.$$

Choosing finite dimensional subsets we have the discrete variational problem,

$$\text{Find } u_h \in U_h \text{ such that } b(v_h,u_h) = l(v_h) \ \forall v_h \in V_h.$$

If a discrete version of the inf-sup condition holds,

$$||u-u_h||_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} ||u-w_h||_U.$$

Several points are now of note. First, it can be seen that the computed solution has an error proportional to the infimum of that of all possible discrete solutions *in the trial space norm* set according to the continuity and inf-sup conditions. If the trial norm is given by the L^2 norm and the continuity and inf-sup constants are independent of the mesh size and solution regime, it is thus observed that the computed solution will converge optimally in the L2 norm. Second, a case of particular interest occurs when the continuity and inf-sup constants are equal

$$M = \gamma_h,$$

where the error incurred by the discrete approximation is smallest. It has been proven that

$$b(v,u) \leq ||v||_V||u||_U \text{ and } \exists v_u \in V \setminus \{0\} : b(v_u,u) = ||v_u||_V||u||_U \ \forall u \in U \implies \gamma_h = 1 \ (= M)$$

where v_u is termed an optimal test function for the trial function u. Further, the recentrly formulated optimal trial Petrov-Galerkin (OPG) method has been devised to satisfy the requirements of this theorem [1]. With the above abstract framework in place, we now move on to a specific model problem providing a concrete realization of the concepts presented above. Consider the linear advection equation,

$$\begin{aligned}
&\boldsymbol{b} \cdot \boldsymbol{\nabla} u = s \text{ in } \Omega, \\
&u = u_{\Gamma^-} \qquad \text{on } \Gamma^- \coloneqq \{\boldsymbol{x} \in \partial\Omega : \hat{\boldsymbol{n}} \cdot \boldsymbol{b} < 0\}.
\end{aligned}$$

In the standard manner, we obtain the weak formulation where the bilinear and linear forms are defined by

$$b(v, \boldsymbol{u}) = \sum_{\mathcal{V}} \int_{\mathcal{V}} -\boldsymbol{\nabla} v \cdot \boldsymbol{b} u \; d\omega + \sum_{\mathcal{F}} \int_{\mathcal{F}} [\![v]\!] f^* \; d\gamma, \; l(v) = \sum_{\mathcal{V}} \int_{\mathcal{V}} vs \; d\omega.$$

The OPG method corresponds to making the choice of trial space norm as the L^2 norm and achieves unity continuity and inf-sup conditions. Applying Cauchy-Schwarz to the bilinear form

$$\begin{aligned}
b(v, \boldsymbol{u}) &\leq \sum_{\mathcal{V}} || - \boldsymbol{\nabla} v \cdot \boldsymbol{b}||_{L^2(\mathcal{V})} ||u||_{L^2(\mathcal{V})} + ||[\![v]\!]||_{L^2(\mathcal{F})} ||f^*||_{L^2(\mathcal{F})} \\
&\leq \underbrace{\left(\sum_{\mathcal{V}} || - \boldsymbol{\nabla} v \cdot \boldsymbol{b}||^2_{L^2(\mathcal{V})} + ||[\![v]\!]||^2_{L^2(\mathcal{F})} \right)^{\frac{1}{2}}}_{||v||_V} \times \underbrace{\left(\sum_{\mathcal{V}} ||u||^2_{L^2(\mathcal{V})} + ||f^*||^2_{L^2(\mathcal{F})} \right)^{\frac{1}{2}}}_{||u||_U},
\end{aligned}$$

we naturally find that equality is obtained for the choice of test functions

$$\begin{aligned}
u &= -\boldsymbol{\nabla} v_u \cdot \boldsymbol{b} \quad && \text{in } \mathcal{V}, \\
f^* &= [\![v_u]\!] && \text{on } \mathcal{F}.
\end{aligned}$$

Imposing sufficient conditions on the test space, notably regarding the selection of the discrete basis and the imposition of adjoint boundary conditions (on the outflow faces of the domain), we have

$$\exists \text{ a unique } v_u \in V \text{ such that } u = B^* v_u, \; \forall u \in U.$$

An equivalent problem to the original is then be obtained where now it is the test functions *having global support* which are solved for and from which the solution is then computed through the application of the adjoint operator,

$$\begin{aligned}
&\text{Find } w_h(u_h) \in V_h \text{ such that} \\
&b(v_h, u_h) \coloneqq b(v_h, B^* w_h) \coloneqq (B^* v_h, B^* w_h) = l(v_h) \; \forall v_h \in V_h.
\end{aligned}$$

That the problem remains practical despite the global support of the test functions is in stark contrast to the necessary limited support for the test functions in the discontinuous Petrov-Galerkin (DPG) method [2] which served as inspiration for this formulation. While the OPG method as presented above has the appearance of achieving the optimal stability which is the focus of this study, we note several open problems which we have encountered during our initial implementation:

- The degree of the test space is required to be one higher than that of the trial space, leading to a global system matrix corresponding to a problem of one higher degree;
- Simply imposing the physically motivated zero Dirichlet boundary conditions along all outflow boundaries results in the values of the solution being forced to be incorrect at certain points on the boundary when a polynomial discrete test space is used. Thus even for the simple problem of advection of a constant solution along a diagonal advection field, all of the standard polynomial spaces typically used in finite element contexts are overly restrictive.

We note that the additional stability provided by the method may provide a motivation for accepting the increased cost resulting from the higher degree of the test space representation. Further, we note that it may be possible to remedy the forcing of incorrect solution values resulting from the test space boundary conditions by extending the solution in a smooth manner outside of the domain, but that this will likely lead to complications in more complex settings. Finally, we expect to extend the OPG method to nonlinear equations (beginning with Burgers' equation) by solving a series of linearized problems around the most recently computed state.

References

[1] J. Brunken, K. Smetana, and K. Urban, "(Parametrized) First Order Transport Equations: Realization of Optimally Stable Petrov-Galerkin Methods," *ArXiv e-prints*, Mar. 2018.

[2] L. Demkowicz and J. Gopalakrishnan, "A class of discontinuous petrov-Galerkin methods. part i: The transport equation," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 23âĂŞ24, pp. 1558 – 1572, 2010.