On the Selection of Finite Element Test Norms for Hyperbolic PDEs based on their Stabilization and Convergence Properties

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Motivation and Goals

The success of Bubnov-Galerkin finite element methods for the solution of elliptic partial differential equations (PDEs) arises from the relation between the variational formulations and the minimization of an energy functional.

In the case of convection-dominated or purely hyperbolic PDEs, additional stabilization is required in regions of high local Péclet number and is most commonly introduced either through the introduction of a suitably chosen numerical flux or through the modification of the test space (resulting in a Petrov-Galerkin method).

Taking the linear advection equation as a model problem, our goal is to present our initial investigation into determining the *optimal* form of stabilization.

Outline

- Presentation of methods and related theory:
  - Discontinuous Galerkin method with upwind numerical flux;
  - Discontinuous Petrov-Galerkin method with various test norms [**?**];
  - Optimal Trial Petrov-Galerkin method [**?**].
- Convergence analysis in relation to discrete inf-sup stability.
- Numerical results in one and two dimensions.
- Discussion of limitations and future directions.

Abstract Functional Setting - Continuous

Consider the *linear* abstract variational problem

$$\text{Find } u \in U \text{ such that } b(v,u) = l(v) \ \forall v \in V.$$

We have the standard requirements:

$$|b(v,u)| \leq M||v||_V||u||_U,$$

$$\exists \gamma > 0 : \inf_{u \in U} \sup_{v \in V} \frac{b(v,u)}{||v||_V||u||_U} \geq \gamma,$$

and

$$l(v) = 0 \ \forall v \in V_0, \text{ where } V_0 := \{v \in V : b(v,u) = 0 \ \forall u \in U\}.$$

Then by the Banach-Nečas-Babuška theorem,

$$||u||_U \leq \frac{M}{\gamma}||l||_{V'}.$$

Abstract Functional Setting - Discrete

Choosing finite dimensional subsets we have the discrete variational problem,

$$\text{Find } u_h \in U_h \text{ such that } b(v_h, u_h) = l(v_h) \ \forall v_h \in V_h.$$

If a discrete version of the inf-sup condition holds,

$$||u - u_h||_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} ||u - w_h||_U.$$

A case of particular interest:

$$M = \gamma_h.$$

Abstract Functional Setting - Selection of Norms

Natural strategy to choose the norm which is naturally induced by the problem. Bui-Thanh et al. [?, Theorem 2.6]:

$$M = \gamma_h = 1 \iff \exists v_u \in V \setminus \{0\} : b(v_u, u) = ||v_u||_V ||u||_U \ \forall u \in U,$$

where v_u is termed an optimal test function for the trial function u. Defining the map from trial to test space, $T : U \ni u \to Tu := v_{Tu} \in V$, by

$$(v_{Tu}, Tu)_V = b(v, u),$$

then

$$V_{\text{opt}} = \{v_{Tu} \in V : u \in U\}.$$

Discontinuous Galerkin Method

Follow the presentation of Di Pietro.

To do

- Compute inf-sup constants for the OPG method and ensure that it is 1; compare with inf-sup for DPG if possible.