Target-Based Adaptation using the Discontinuous Petrov-Galerkin Method

by

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Dedication

This thesis is dedicated to those who have fuelled my interest in numerical analysis through their genious, creativity and passion. Include best graphic or logo

Acknowledgments

ToBeDone

Don't forget NSERC+McGill Funding

Abstract

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Introduction

1.1 ToBeModified

Chapter 2

Methodology

In this section, the governing equations of fluid mechanics and heat transfer, as well as the associated discretizations and boundary conditions employed are outlined. As this work is concerned with the solution of these equations through variants of the finite element method, we also outline the spaces used for the discretization.

Row-vector notation is assumed throughout with the following notation employed:

Object	Description	Example
Scalar variable	italic	a
Vector variable	italic boldface lowercase	$\boldsymbol{a}$
Second-order tensor variable	italic boldface uppercase	$\boldsymbol{A}$
Vector	boldface lowercase	$\mathbf{a}$
Matrix	boldface uppercase	$\mathbf{A}$
Spaces	calligraphic uppercase	$\mathscr{A}$

2.1 Governing Equations

Following the notation of Pletcher et al. [1, Chapter 5], the continuity, Navier-Stokes and energy equations with source terms neglected are given by

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \left(\boldsymbol{F}^i(\boldsymbol{w}) - \boldsymbol{F}^v(\boldsymbol{w}, \boldsymbol{Q}) \right) = \boldsymbol{0}, \tag{2.1}$$

where the vector of conservative variables and its gradients are defined as

$$\boldsymbol{w} := \begin{bmatrix} \rho & \rho\boldsymbol{v} & E \end{bmatrix} \in \mathbb{R}^{d+2}$$

$$\boldsymbol{Q} := \nabla^T \boldsymbol{w} \qquad \in \mathbb{R}^{d+2} \times \mathbb{R}^d, \tag{2.2}$$

where d is the problem dimension, and where the inviscid and viscous fluxes are defined as

$$\boldsymbol{F^i}(\boldsymbol{w}) := \begin{bmatrix} \rho\boldsymbol{v}^T & \rho\boldsymbol{v}^T\boldsymbol{v} + p\boldsymbol{I} & (E+p)\boldsymbol{v}^T \end{bmatrix} \in \mathbb{R}^{d+2} \times \mathbb{R}^d \tag{2.3}$$

$$\boldsymbol{F^v}(\boldsymbol{w},\boldsymbol{Q}) := \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{\Pi} & \boldsymbol{\Pi}\boldsymbol{v}^T - \boldsymbol{q}^T \end{bmatrix} \qquad \in \mathbb{R}^{d+2} \times \mathbb{R}^d. \tag{2.4}$$

The various symbols represent the density, ρ, the velocity, $\boldsymbol{v}$, the total energy per unit volume, E, the pressure, p, the stress tensor, $\boldsymbol{\Pi}$ and the energy flux, $\boldsymbol{q}$. The pressure is defined according to the equation of state for a calorically ideal gas,

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho\boldsymbol{v}\boldsymbol{v}^T\right) := (\gamma-1)\rho e, \; \gamma = \frac{c_p}{c_v}, \; c_v = \frac{R_g}{\gamma-1}, \; c_p = \frac{\gamma R_g}{\gamma-1},$$

where e represents the specific internal energy, R_g is the gas constant and the specific heats at constant volume, c_v, and at constant pressure, c_p, are constant. The stress tensor is defined as

$$\boldsymbol{\Pi} = 2\mu\left(\boldsymbol{D} - \frac{1}{3}\nabla\cdot\boldsymbol{v}\boldsymbol{I}\right), \; \boldsymbol{D} := \frac{1}{2}\left(\nabla^T\boldsymbol{v} + \left(\nabla^T\boldsymbol{v}\right)^T\right),$$

where μ is the coefficient of shear viscosity (Add comment about how μ is determined (Sutherland, p.259 pletcher(1997))) and where the coefficient of bulk viscosity was assumed to be zero. Finally, the energy flux is defined by

$$\boldsymbol{q} = \kappa\nabla T,$$

where T represents the temperature and

$$\kappa = \frac{c_p\mu}{Pr},$$

with Pr representing the Prandtl number. In the case of the Euler equations, the contribution of the viscous flux is neglected. Potentially add nondimensionalizations for external and internal flow.

2.2 Discretization

2.2.1 Preliminaries

Let Ω be a bounded simply connected open subset of $\mathbb{R}^d$ with connected Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^{d-1}$. We let Ω_h denote the disjoint partion of Ω into "elements", V, and denote the element boundaries as ∂V. We also define the following volume inner products,

$$(a, b)_D = \int_D ab; \qquad a, b \in L^2(D),$$

$$(\mathbf{a}, \mathbf{b})_D = \int_D \mathbf{a} \cdot \mathbf{b}; \quad \mathbf{a}, \mathbf{b} \in L^2(D)^m,$$

$$(\mathbf{A}, \mathbf{B})_D = \int_D \mathbf{A} : \mathbf{B}; \ \mathbf{A}, \mathbf{B} \in L^2(D)^{m\times d},$$

where D is a domain in $\mathbb{R}^d$, and where ':' denotes the inner product operator for two second-order tensors. Analogous notation is used for face inner products,

$$\langle a, b \rangle_D = \int_D ab; \qquad a, b \in L^2(D),$$

$$\langle \mathbf{a}, \mathbf{b} \rangle_D = \int_D \mathbf{a} \cdot \mathbf{b}; \quad \mathbf{a}, \mathbf{b} \in L^2(D)^m,$$

$$\langle \mathbf{A}, \mathbf{B} \rangle_D = \int_D \mathbf{A} : \mathbf{B}; \ \mathbf{A}, \mathbf{B} \in L^2(D)^{m\times d},$$

where D is a domain in $\mathbb{R}^{d-1}$. Denoting the polynomial space of order k on domain D as $\mathscr{P}^k(D)$, and letting $n = d + 2$, we define the discontinuous discrete solution and gradient approximation spaces as

$$\mathscr{W}_h^v = \{\boldsymbol{a} \in L^2(\Omega_h)^n : \boldsymbol{a}|_V \in \mathscr{P}^k(V)^n \ \forall V \in \Omega_h\}$$

$$\mathscr{Q}_h^v = \{\boldsymbol{A} \in L^2(\Omega_h)^{n\times d} : \boldsymbol{A}|_V \in \mathscr{P}^k(V)^{n\times d} \ \forall V \in \Omega_h\}.$$

We also define discontinuous test spaces

$$\mathscr{W}_{t\,h}^{\ v} = \{\mathbf{a_t} \in L^2(\Omega_h)^n : \mathbf{a_t}|_V \in \mathscr{P}^{k_t}(V)^n \ \forall V \in \Omega_h\}$$

$$Q_{t\,h}^{\ v} = \{\mathbf{A_t} \in L^2(\Omega_h)^{n \times d} : \mathbf{A_t}|_V \in \mathscr{P}^{k_t}(V)^{n \times d} \ \forall V \in \Omega_h\},$$

where $k_t \geq k$.

2.2.2 Discretized Equations

To obtain the discrete formulation, we first define a joint flux $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}) := \boldsymbol{F^i}(\boldsymbol{w}) - \boldsymbol{F^v}(\boldsymbol{w}, \boldsymbol{Q})$ then integrate (2.2) and (2.1) with respect to test functions to obtain

$$(\mathbf{Q_t}, \boldsymbol{Q})_V = \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V, \qquad\qquad \forall \mathbf{Q_t} \in Q_{t\,h}^{\ v} = Q_h^v$$

$$\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V + (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V = \mathbf{0}, \ \forall \mathbf{w_t} \in \mathscr{W}_{t\,h}^{\ v} = \mathscr{W}_h^v.$$

Integrating by parts twice in the first equation and once in the second and choosing $k_t = k$, such that the approximation and test spaces are the same, results in the discontinuous Galerkin formulation,

$$(\mathbf{Q_t}, \boldsymbol{Q})_V = \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V + \langle \mathbf{Q_t}, \mathbf{n} \cdot (\boldsymbol{w}^* - \boldsymbol{w}) \rangle_{\partial V}, \qquad \forall \mathbf{Q_t} \in Q_h^v$$

$$\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V - (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V + \langle \mathbf{w_t}, \mathbf{n} \cdot \boldsymbol{F}^* \rangle_{\partial V} = \mathbf{0}, \ \forall \mathbf{w_t} \in \mathscr{W}_h^v.$$

where $\mathbf{n}$ denotes the outward pointing unit normal vector and where $\boldsymbol{w}^*$ and $\boldsymbol{F}^*$ represent the numerical solution and flux respectively.

2.3 Boundary Conditions

Boundary conditions are imposed weakly through the specification of a 'ghost' state for elements in which $V \cap \partial\Omega \neq \{0\}$. The following boundary conditions are supported:

Boundary Condition	Reference(s)	Comments
Riemann Invariant	[2, section 2.2]	eq. (14) should read $c_b = \frac{\gamma-1}{4}(R^+ - R^-)$
Slip-Wall	[3, eq. (10)]	
Back Pressure (Outflow)	[2, section 2.4]	
Total Temperature/Pressure (Inflow)	[2, section 2.7]	
Supersonic Inflow/Outflow		imposes exact/extrapolated solution
No-slip Temperature		imposes $\mathbf{v}$ and T
No-slip Adiabatic		imposes $\mathbf{v}$ and $(\mathbf{n} \cdot \mathbf{F}(\mathbf{W}, \mathbf{Q}))_E = 0$
No-slip Flux		

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