Optimal Test Inner Product for Linear Advection

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1 Test Functions

For the 1D case under consideration, with volumes numbered from 1 to n and face nodes numbered from 0 to n, the bilinear form associated with the steady linear advection equation (assuming a unit advection velocity) is given by

$$b(v, \boldsymbol{u}) = \sum_{i=1}^{n} - \int_{-1}^{1} v_i^{'} u_i dr - v_i(-1) f_{i-1} + v_i(1)) f_i \ \forall v \tag{1.1}$$

Above, $\boldsymbol{u} \coloneqq (u, f)$ is the group variable for the solution and trace flux components, and v is a test function and where all quantities have been transferred to the reference volume and face.

It has been noted that under specific norms, the optimal test functions are given by polynomials one degree higher than that of the corresponding solution basis [1, Section 3C]. Below, the analytical expressions for the test functions are obtained and they are then substituted into the bilinear form to determine the associated induced norms. We choose to work with the general test norm

$$(w, v) = \sum_{i=1}^{n} \int_{-1}^{1} \frac{2}{h} w_i^{'} v_i^{'} dr + a w_i(1) v_i(1) + c w_i(-1) v_i(-1) + \sum_{i=0}^{n} + b(w_i(1) - w_{i+1}(-1))(v_i(1) - v_{i+1}(-1))$$

where it is to be assumed that quantities not present in the domain are omitted in the last summation. Note that this test norm recovers that of Demkowicz et al. from the first DPG paper [1, Section 2C] when selecting the parameters in the norm above as $a = \alpha_i, b = 0, c = 0$. In this report we have taken the values of the parameters to be constant for all elements and equal to

$$[a, b, c] = [a, 0, 0].$$

Optimal test functions for a given basis function are then found by solving the following system of equations

$$(w, v) = b(w, \phi) \ \forall w \in V \tag{1.2}$$

where ϕ denotes a basis function from the trial space.

1.1 Volume Test Functions

It can be observed, when using the Legendre polynomials as volume solution basis functions, that all associated test functions except that of the constant basis are zero at both edges of the reference element [2, Section 5.1]. Further, they all satisfy (1.2) exactly when the test space is one order higher than the solution space. Consequently, only the p_0 test function needs to be computed. Noting that the p_0 test function is linear, represented as

$$v_{\phi_{i,0}} = a_0 + a_1 r,$$

it can be determined by solving the following equation for the coefficients, for the general form of the test norm

$$\int_{-1}^{1} \frac{2}{h} w_i^{'} v_i^{'} dr + (a+b)(w_i(1)v_i(1)) + (b+c)(w_i(-1)v_i(-1)) = \int_{-1}^{1} -w_i^{'}\phi dr,\ \forall w_i \in \mathcal{P}^1,$$

where $\phi_0 = \frac{1}{\sqrt{2}}$, and $\mathcal{P}^p$ is the space of all polynomials of degree less than order equal to p. Choosing $w_i = 1$ and $w_i = r$, we obtain the following equalities

$$\begin{aligned} 0 + (a+b)((1)v_i(1)) + (b+c)((1)v_i(-1)) &= 0, \\ \int_{-1}^{1} \frac{2}{h}(1)v_i^{'} dr + (a+b)((1)v_i(1)) + (b+c)((-1)v_i(-1)) &= \int_{-1}^{1} (-1)\frac{1}{\sqrt{2}} dr. \end{aligned}$$

Substituting the general expression for $v_{\phi_{i,0}}$, we obtain the coefficients by solving the following linear system

$$\mathbf{A}\hat{\mathbf{v}} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a & a \\ a & a + \frac{4}{h} \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix},\ \text{and } \hat{\mathbf{v}} = [a_0, a_1]^T.$$

The result is

$$v_{\phi_{i,0}} = -\frac{1}{4}\sqrt{2}hr + \frac{1}{4}\sqrt{2}h. \tag{1.3}$$

1.2 Face Test Functions

The linear test functions on either side of the 1D face (point) take the exact form:

$$\begin{aligned} v_{\hat{\phi}_i}^l &= a_0^l + a_1^l r, \\ v_{\hat{\phi}_i}^r &= a_0^r + a_1^r r. \end{aligned}$$

The coefficients can be computed by solving the following linear system

$$\mathbf{A}\hat{\mathbf{f}} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a & a & 0 & 0 \\ a & a + \frac{4}{h} & 0 & 0 \\ 0 & 0 & a & a \\ 0 & 0 & a & a + \frac{4}{h} \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix},\ \text{and } \hat{\mathbf{f}} = [a_0^l, a_1^l, a_0^r, a_1^r]^T.$$

The result is

$$\begin{aligned} v_{\hat{\phi}_i}^l &= \frac{1}{a}, \\ v_{\hat{\phi}_i}^r &= \frac{1}{2}hr - \frac{ah+2}{2\,a}. \end{aligned}$$

2 Implied Energy Norm

The implied energy norm is obtained by substituting the optimal test functions into the bilinear form as discussed by Demkowicz et al. [1, eq. (2.8) and Proposition 2.2]. Substituting the p_0 optimal test function, (1.3), into the bilinear form, (1.1), after multiplication by the solution coefficients

$$b(v_{u_0}, \boldsymbol{u}) = \sum_{i=1}^{n} \int_{-1}^{1} \frac{1}{2} h u_{i,0} u_i dr - (h)\, u_{i,0} f_{i-1} + (0)\, u_{i,0} f_i$$

where $u_{i,0} = \frac{1}{h} \int_{x_{i-1}}^{x_i} u dx = \frac{1}{2} \int_{-1}^{1} u dr$ denotes the average of u_i in volume i. Noting the property discussed above of all other integrated Legendre polynomials taking values of zero at $r = \pm 1$, we have

$$b(v_{u_{j>0}}, \boldsymbol{u}) = \sum_{i=1}^{n} \int_{-1}^{1} \frac{h}{2} (u_i - u_{i,0}) u_i dr.$$

Summing the two contributions

$$b(v_u, \boldsymbol{u}) = \sum_{i=1}^{n} \int_{-1}^{1} \frac{h}{2} u_i^2 dr - (h)\, u_{i,0} f_{i-1} + (0)\, u_{i,0} f_i.$$

Needs extra volume term for cases where it is not zero.

Considering the optimal test functions for the fluxes,

$$\begin{aligned} b(v_f, \boldsymbol{u}) = \sum_{i=1}^{n} \int_{-1}^{1} (0)\, f_i u_i dr - \left(\frac{1}{a}\right) f_i f_{i-1} + \left(\frac{1}{a}\right) f_i^2 \\ + \int_{-1}^{1} \left(-\frac{1}{2} h\right) f_{i-1} u_i dr - \left(-\frac{ah+1}{a}\right) f_{i-1}^2 + \left(-\frac{1}{a}\right) f_{i-1} f_i \end{aligned}$$

References

[1] L. Demkowicz, J. Gopalakrishnan, A class of discontinuous petrov-Galerkin methods. ii. optimal test functions, Numerical Methods for Partial Differential Equations 27 (1) (2011) 70–105. doi:10.1002/num.20640.
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[2] T. Bui-Thanh, L. Demkowicz, O. Ghattas, Constructively well-posed approximation methods with unity inf-sup and continuity constants for partial differential equations, Mathematics of Computation 82 (2013) 1923–1952.