CHOOSE AN APPROPRIATE TITLE

by

Philip Zwanenburg

Bachelor of Engineering, McGill University (2014)

A thesis submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mechanical Engineering

at

MCGILL UNIVERSITY

September 2019

Author ....

Department of Mechanical Engineering
January 1, 2019

Certified by ....

Siva Nadarajah
Associate Professor
Thesis Supervisor

Certified by ....

Mathias Legrand
Associate Professor
Thesis Supervisor

Certified by ....

Jean-Christophe Nave
Associate Professor
Thesis Supervisor

Accepted by ....

NAME
CHAIRMAN, DEPARTMENT COMMITTEE ON GRADUATE THESES

Dedication

This thesis is dedicated to those who have fuelled my interest in numerical analysis through their genious, creativity and passion. Include best graphic or logo

Acknowledgments

ToBeDone

Don’t forget NSERC+McGill Funding

Abstract

ToBeDone

Abrégé

ToBeDone

Contents

List of Figures

List of Tables

Chapter 1

Introduction

ToBeModified

Chapter 2

Methodology

In this section, the governing equations of fluid mechanics and heat transfer, as well as the associated discretizations and boundary conditions employed are outlined. As this work is concerned with the solution of these equations through variants of the finite element method, we also outline the spaces used for the discretization.

Row-vector notation is assumed throughout with the following notation employed:

Object	Description	Example
Scalar variable	italic	a
Vector variable	italic boldface lowercase	$\boldsymbol{a}$
Second-order tensor variable	italic boldface uppercase	$\boldsymbol{A}$
Vector	boldface lowercase	$\mathbf{a}$
Matrix	boldface uppercase	$\mathbf{A}$
Spaces	calligraphic uppercase	$\mathscr{A}$

Governing Equations

Following the notation of Pletcher et al. [1, Chapter 5], the continuity, Navier-Stokes and energy equations with source terms neglected are given by

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \big(\boldsymbol{F}^{\boldsymbol{i}}(\boldsymbol{w}) - \boldsymbol{F}^{\boldsymbol{v}}(\boldsymbol{w}, \boldsymbol{Q})\big) = \mathbf{0}, \tag{2.1}$$

where the vector of conservative variables and its gradients are defined as

$$\begin{aligned} \boldsymbol{w} &:= \begin{bmatrix} \rho & \rho \boldsymbol{v} & E \end{bmatrix} && \in \mathbb{R}^{d+2} \\ \boldsymbol{Q} &:= \nabla^T \boldsymbol{w} && \in \mathbb{R}^{d+2} \times \mathbb{R}^{d}, \end{aligned} \tag{2.2}$$

where d is the problem dimension, and where the inviscid and viscous fluxes are defined as

$$\boldsymbol{F^i}(\boldsymbol{w}) \coloneqq \begin{bmatrix}\rho\boldsymbol{v}^T & \rho\boldsymbol{v}^T\boldsymbol{v}+p\boldsymbol{I} & (E+p)\boldsymbol{v}^T\end{bmatrix} \in \mathbb{R}^{d+2}\times\mathbb{R}^d, \tag{2.3}$$

$$\boldsymbol{F^v}(\boldsymbol{w},\boldsymbol{Q}) \coloneqq \begin{bmatrix}\boldsymbol{0}^T & \boldsymbol{\Pi} & \boldsymbol{\Pi}\boldsymbol{v}^T-\boldsymbol{q}^T\end{bmatrix} \in \mathbb{R}^{d+2}\times\mathbb{R}^d. \tag{2.4}$$

The various symbols represent the density, ρ, the velocity, $\boldsymbol{v}$, the total energy per unit volume, E, the pressure, p, the stress tensor, $\boldsymbol{\Pi}$ and the energy flux, $\boldsymbol{q}$. The pressure is defined according to the equation of state for a calorically ideal gas,

$$p = (\gamma-1)\left(E-\frac{1}{2}\rho\boldsymbol{v}\boldsymbol{v}^T\right) \coloneqq (\gamma-1)\rho e,\ \gamma=\frac{c_p}{c_v},\ c_v=\frac{R_g}{\gamma-1},\ c_p=\frac{\gamma R_g}{\gamma-1},$$

where e represents the specific internal energy, R_g is the gas constant and the specific heats at constant volume, c_v, and at constant pressure, c_p, are constant. The stress tensor is defined as

$$\boldsymbol{\Pi} = 2\mu\left(\boldsymbol{D}-\frac{1}{3}\nabla\cdot\boldsymbol{v}\boldsymbol{I}\right),\ \boldsymbol{D}\coloneqq\frac{1}{2}\left(\nabla^T\boldsymbol{v}+\left(\nabla^T\boldsymbol{v}\right)^T\right),$$

where μ is the coefficient of shear viscosity (Add comment about how μ is determined (Sutherland, p.259 pletcher(1997))) and where the coefficient of bulk viscosity was assumed to be zero. Finally, the energy flux is defined by

$$\boldsymbol{q} = \kappa\nabla T,$$

where T represents the temperature and

$$\kappa = \frac{c_p\mu}{Pr},$$

with Pr representing the Prandtl number. In the case of the Euler equations, the contribution of the viscous flux is neglected.

Discretizations

Preliminaries

Let Ω be a bounded simply connected open subset of $\mathbb{R}^d$ with connected Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^{d-1}$. We let Ω_h denote the disjoint partion of Ω into "elements", V, and denote the element boundaries as ∂V. Elements and their boundaries are also referred to as volumes and faces respectively. We also define the following volume inner products,

$$
\begin{aligned}
(a,b)_D &= \int_D ab; \qquad a,b \in L^2(D),\\
(\mathbf{a},\mathbf{b})_D &= \int_D \mathbf{a}\cdot\mathbf{b}; \quad \mathbf{a},\mathbf{b} \in L^2(D)^m,\\
(\mathbf{A},\mathbf{B})_D &= \int_D \mathbf{A}:\mathbf{B}; \ \mathbf{A},\mathbf{B} \in L^2(D)^{m\times d},
\end{aligned}
$$

where D is a domain in $\mathbb{R}^d$, and where ':' denotes the inner product operator for two second-order tensors. Analogous notation is used for face inner products,

$$
\begin{aligned}
\langle a,b\rangle_D &= \int_D ab; \qquad a,b \in L^2(D),\\
\langle \mathbf{a},\mathbf{b}\rangle_D &= \int_D \mathbf{a}\cdot\mathbf{b}; \quad \mathbf{a},\mathbf{b} \in L^2(D)^m,\\
\langle \mathbf{A},\mathbf{B}\rangle_D &= \int_D \mathbf{A}:\mathbf{B}; \ \mathbf{A},\mathbf{B} \in L^2(D)^{m\times d},
\end{aligned}
$$

where D is a domain in $\mathbb{R}^{d-1}$. Denoting the polynomial space of order p on domain D as $\mathscr{P}^p(D)$, and letting $n = d+2$, we define the discontinuous discrete solution and gradient approximation spaces as

$$
\begin{aligned}
\mathscr{S}_h^v &= \{\boldsymbol{a} \in L^2(\Omega_h)^n : \boldsymbol{a}|_V \in \mathscr{P}^p(V)^n \ \forall V \in \Omega_h\}\\
\mathscr{G}_h^v &= \{\boldsymbol{A} \in L^2(\Omega_h)^{n\times d} : \boldsymbol{A}|_V \in \mathscr{P}^p(V)^{n\times d} \ \forall V \in \Omega_h\}.
\end{aligned}
$$

We also define discontinuous test spaces

$$\begin{aligned}
\mathscr{W}_{t\,h}^{\,v} &= \{\mathbf{a_t} \in L^2(\Omega_h)^n : \mathbf{a_t}|_V \in \mathscr{P}^{p_t}(V)^n \ \forall V \in \Omega_h\} \\
\mathscr{Q}_{t\,h}^{\,v} &= \{\mathbf{A_t} \in L^2(\Omega_h)^{n\times d} : \mathbf{A_t}|_V \in \mathscr{P}^{p_t}(V)^{n\times d} \ \forall V \in \Omega_h\},
\end{aligned}$$

where $p_t \geq p$. Will need additional spaces for DPG.

Discretized Equations

To obtain the discrete formulation, we first define a joint flux $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}) \coloneqq \boldsymbol{F^i}(\boldsymbol{w}) - \boldsymbol{F^v}(\boldsymbol{w}, \boldsymbol{Q})$ then integrate (2.2) and (2.1) with respect to test functions to obtain

$$\begin{aligned}
(\mathbf{Q_t}, \boldsymbol{Q})_V &= \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V, & \forall \mathbf{Q_t} \in \mathscr{Q}_{t\,h}^{\,v} = \mathscr{Q}_h^v \\
\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V &+ (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V = \mathbf{0}, & \forall \mathbf{w_t} \in \mathscr{W}_{t\,h}^{\,v} = \mathscr{W}_h^v.
\end{aligned}$$

Integrating by parts twice in the first equation and once in the second and choosing $p_t = p$, such that the approximation and test spaces are the same, results in the discontinuous Galerkin formulation,

$$\begin{aligned}
(\mathbf{Q_t}, \boldsymbol{Q})_V &= \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V + \langle \mathbf{Q_t}, \mathbf{n} \cdot (\boldsymbol{w}^* - \boldsymbol{w}) \rangle_{\partial V}, & \forall \mathbf{Q_t} \in \mathscr{Q}_h^v \\
\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V &- (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V + \langle \mathbf{w_t}, \mathbf{n} \cdot \boldsymbol{F}^* \rangle_{\partial V} = \mathbf{0}, & \forall \mathbf{w_t} \in \mathscr{W}_h^v.
\end{aligned}$$

where $\mathbf{n}$ denotes the outward pointing unit normal vector and where $\boldsymbol{w}^*$ and $\boldsymbol{F}^*$ represent the numerical solution and flux respectively.

Boundary Conditions

Boundary conditions are imposed weakly through the specification of a 'ghost' state for elements in which $V \cap \partial\Omega \neq \{0\}$. The following boundary conditions are supported:

Boundary Condition	Reference(s)	Comments
Riemann Invariant	[2, section 2.2]	eq. (14) should read $c_b = \frac{\gamma-1}{4}(R^+ - R^-)$
Slip-Wall	[3, eq. (10)]	
Back Pressure (Outflow)	[2, section 2.4]	
Total Temperature/Pressure (Inflow)	[2, section 2.7]	
Supersonic Inflow/Outflow		imposes the exact/extrapolated solution
No-slip Overconstrained		imposes values for all primitive variables [1]
No-slip Diabatic		imposes $\mathbf{v}$ and $(\mathbf{n} \cdot \mathbf{F}(\mathbf{W}, \mathbf{Q}))_E = \text{constant}$

[1] Add reference to Nordstrom explaining why this BC is overconstrained and add link to Taylor-Couette results where it is used.

Chapter 3

The DPG Methodology

In comparison to discontinuous Galerkin (DG) methods, the most noteworthy characteristic of the Discontinuous Petrov-Galerkin (DPG) methodology is that the optimal (to be defined below) test space is computed based on the minimization of the residual in a specified norm instead of simply being chosen to be the same as the trial space.

Linear DPG

In this section, we first outline the basic concepts of DPG methods in an abstract linear functional setting and then provide a concrete example through the application of the theory to the linear advection equation.

Abstract Functional Setting

Much of the theory outlined below is borrowed directly from Demkowicz et al. [4]. Of primary note, it is demonstrated that the DPG method can be interpreted both as a minimum residual method as well as a localization of the continuous Petrov-Galerkin (PG) method with globally optimal test functions.

DPG as a Minimum Residual Method

Potentially change title of subsection: DPG as the most accurate projection of the solution measured in the natural norm?

Consider the *linear* abstract variational problem

$$\text{Find } u \in \mathscr{U} \text{ such that} \tag{3.1}$$

$$b(v,u) = l(v) \; \forall v \in \mathscr{V}. \tag{3.2}$$

Above, $\mathscr{U}$ and $\mathscr{V}$ denote the trial and test spaces, respectively, which are assumed to be Hilbert spaces, $b(\cdot, \cdot)$ is a bilinear form on $\mathscr{V} \times \mathscr{U}$ and $l(\cdot)$ is a linear form on $\mathscr{V}$. It is assumed that the bilinear form satisfies a continuity condition with continuity constant M,

$$|b(v, u)| \leq M||u||_{\mathscr{U}}||v||_{\mathscr{V}}, \tag{3.3}$$

and an inf-sup condition with inf-sup constant γ,

$$\exists \gamma > 0 : \inf_{u \in \mathscr{U}} \sup_{v \in \mathscr{V}} \frac{b(v, u)}{||v||_{\mathscr{V}}||u||_{\mathscr{U}}} \geq \gamma. \tag{3.4}$$

Bibliography

[1] R. Pletcher, J. Tannehill, D. Anderson, Computational Fluid Mechanics and Heat Transfer, Second Edition, Series in Computational and Physical Processes in Mechanics and Thermal Sciences, Taylor & Francis, 1997.
URL `https://books.google.ca/books?id=ZJPbtHeilCgC`

[2] J.-R. Carlson, Inflow/outflow boundary conditions with application to fun3d, Tech. Rep. NASA/TM-2011-217181, NASA Langley Research Center (2011).
URL `https://ntrs.nasa.gov/search.jsp?R=20110022658`

[3] L. Krivodonova, M. Berger, High-order accurate implementation of solid wall boundary conditions in curved geometries, J. Comput. Phys. 211 (2) (2006) 492–512. `doi:10.1016/j.jcp.2005.05.029`.
URL `http://dx.doi.org/10.1016/j.jcp.2005.05.029`

[4] L. F. Demkowicz, J. Gopalakrishnan, An Overview of the Discontinuous Petrov Galerkin Method, Springer International Publishing, Cham, 2014, pp. 149–180. `doi:10.1007/978-3-319-01818-8_6`.
URL `https://doi.org/10.1007/978-3-319-01818-8_6`