CHOOSE AN APPROPRIATE TITLE

by

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Bachelor of Engineering, McGill University (2014)

A thesis submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mechanical Engineering

at

MCGILL UNIVERSITY

January 2019

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Dedication

This thesis is dedicated to those who have fuelled my interest in numerical analysis through their genious, creativity and passion. Include best graphic or logo

Acknowledgments

ToBeDone

Don't forget NSERC+McGill Funding

Abstract

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Methodology

In this section, the governing equations of fluid mechanics and heat transfer, as well as the associated discretizations and boundary conditions employed are outlined. As this work is concerned with the solution of these equations through variants of the finite element method, we also outline the spaces used for the discretization.

Row-vector notation is assumed throughout with the following notation employed:

Object	Description	Example
Scalar variable	italic	a
Vector variable	italic boldface lowercase	$\boldsymbol{a}$
Second-order tensor variable	italic boldface uppercase	$\boldsymbol{A}$
Vector	boldface lowercase	$\mathbf{a}$
Matrix	boldface uppercase	$\mathbf{A}$
Spaces	calligraphic uppercase	$\mathscr{A}$

Governing Equations

Following the notation of Pletcher et al. [1, Chapter 5], the continuity, Navier-Stokes and energy equations with source terms neglected are given by

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \big(\boldsymbol{F}^{\boldsymbol{i}}(\boldsymbol{w}) - \boldsymbol{F}^{\boldsymbol{v}}(\boldsymbol{w}, \boldsymbol{Q})\big) = \mathbf{0}, \tag{2.1}$$

where the vector of conservative variables and its gradients are defined as

$$\begin{aligned} \boldsymbol{w} &:= \begin{bmatrix} \rho & \rho\boldsymbol{v} & E \end{bmatrix} && \in \mathbb{R}^{d+2} \\ \boldsymbol{Q} &:= \nabla^T \boldsymbol{w} && \in \mathbb{R}^{d+2} \times \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where d is the problem dimension, and where the inviscid and viscous fluxes are defined as

$$\begin{aligned}
\boldsymbol{F}^{\boldsymbol{i}}(\boldsymbol{w}) &:= \begin{bmatrix}\rho\boldsymbol{v}^T & \rho\boldsymbol{v}^T\boldsymbol{v}+p\boldsymbol{I} & (E+p)\boldsymbol{v}^T\end{bmatrix} \in \mathbb{R}^{d+2}\times\mathbb{R}^d,\\
\boldsymbol{F}^{\boldsymbol{v}}(\boldsymbol{w},\boldsymbol{Q}) &:= \begin{bmatrix}\boldsymbol{0}^T & \boldsymbol{\Pi} & \boldsymbol{\Pi}\boldsymbol{v}^T-\boldsymbol{q}^T\end{bmatrix} \qquad\qquad \in \mathbb{R}^{d+2}\times\mathbb{R}^d.
\end{aligned}$$

The various symbols represent the density, ρ, the velocity, $\boldsymbol{v}$, the total energy per unit volume, E, the pressure, p, the stress tensor, $\boldsymbol{\Pi}$ and the energy flux, $\boldsymbol{q}$. The pressure is defined according to the equation of state for a calorically ideal gas,

$$p = (\gamma-1)\left(E-\frac{1}{2}\rho\boldsymbol{v}\boldsymbol{v}^T\right) := (\gamma-1)\rho e,\ \gamma=\frac{c_p}{c_v},\ c_v=\frac{R_g}{\gamma-1},\ c_p=\frac{\gamma R_g}{\gamma-1},$$

where e represents the specific internal energy, R_g is the gas constant and the specific heats at constant volume, c_v, and at constant pressure, c_p, are constant. The stress tensor is defined as

$$\boldsymbol{\Pi} = 2\mu\left(\boldsymbol{D}-\frac{1}{3}\nabla\cdot\boldsymbol{v}\boldsymbol{I}\right),\ \boldsymbol{D} := \frac{1}{2}\left(\nabla^T\boldsymbol{v}+\left(\nabla^T\boldsymbol{v}\right)^T\right),$$

where μ is the coefficient of shear viscosity (Add comment about how μ is determined (Sutherland, p.259 pletcher(1997))) and where the coefficient of bulk viscosity was assumed to be zero. Finally, the energy flux is defined by

$$\boldsymbol{q} = \kappa\nabla T,$$

where T represents the temperature and

$$\kappa = \frac{c_p\mu}{Pr},$$

with Pr representing the Prandtl number. In the case of the Euler equations, the contribution of the viscous flux is neglected.

Discretizations

Preliminaries

Let Ω be a bounded simply connected open subset of $\mathbb{R}^d$ with connected Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^{d-1}$. We let Ω_h denote the disjoint partion of Ω into "elements", V, and denote the element boundaries as ∂V. Elements and their boundaries are also referred to as volumes and faces respectively. We also define the following volume inner products,

$$\begin{aligned}
(a, b)_D &= \int_D ab; \qquad a, b \in L^2(D), \\
(\mathbf{a}, \mathbf{b})_D &= \int_D \mathbf{a} \cdot \mathbf{b}; \quad \mathbf{a}, \mathbf{b} \in L^2(D)^m, \\
(\mathbf{A}, \mathbf{B})_D &= \int_D \mathbf{A} : \mathbf{B}; \ \mathbf{A}, \mathbf{B} \in L^2(D)^{m\times d},
\end{aligned}$$

where D is a domain in $\mathbb{R}^d$, and where ':' denotes the inner product operator for two second-order tensors. Analogous notation is used for face inner products,

$$\begin{aligned}
\langle a, b\rangle_D &= \int_D ab; \qquad a, b \in L^2(D), \\
\langle \mathbf{a}, \mathbf{b}\rangle_D &= \int_D \mathbf{a} \cdot \mathbf{b}; \quad \mathbf{a}, \mathbf{b} \in L^2(D)^m, \\
\langle \mathbf{A}, \mathbf{B}\rangle_D &= \int_D \mathbf{A} : \mathbf{B}; \ \mathbf{A}, \mathbf{B} \in L^2(D)^{m\times d},
\end{aligned}$$

where D is a domain in $\mathbb{R}^{d-1}$. Denoting the polynomial space of order p on domain D as $\mathscr{P}^p(D)$, and letting $n = d + 2$, we define the discontinuous discrete solution and gradient approximation spaces as

$$\begin{aligned}
\mathscr{S}_h^v &= \{\boldsymbol{a} \in L^2(\Omega_h)^n : \boldsymbol{a}|_V \in \mathscr{P}^p(V)^n \ \forall V \in \Omega_h\} \\
\mathscr{G}_h^v &= \{\boldsymbol{A} \in L^2(\Omega_h)^{n\times d} : \boldsymbol{A}|_V \in \mathscr{P}^p(V)^{n\times d} \ \forall V \in \Omega_h\}.
\end{aligned}$$

We also define discontinuous test spaces

$$\mathscr{W}_{t\,h}^{\,v} = \{\mathbf{a_t} \in L^2(\Omega_h)^n : \mathbf{a_t}|_V \in \mathscr{P}^{p_t}(V)^n \ \forall V \in \Omega_h\}$$
$$\mathscr{Q}_{t\,h}^{\,v} = \{\mathbf{A_t} \in L^2(\Omega_h)^{n\times d} : \mathbf{A_t}|_V \in \mathscr{P}^{p_t}(V)^{n\times d} \ \forall V \in \Omega_h\},$$

where $p_t \geq p$. Will need additional spaces for DPG.

Discretized Equations

To obtain the discrete formulation, we first define a joint flux $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}) \coloneqq \boldsymbol{F}^{\boldsymbol{i}}(\boldsymbol{w}) - \boldsymbol{F}^{\boldsymbol{v}}(\boldsymbol{w}, \boldsymbol{Q})$ then integrate (2.2) and (2.1) with respect to test functions to obtain

$$\begin{aligned}
(\mathbf{Q_t}, \boldsymbol{Q})_V &= \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V, & \forall \mathbf{Q_t} \in \mathscr{Q}_{t\,h}^{\,v} = \mathscr{Q}_h^v \\
\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V &+ (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V = \mathbf{0}, & \forall \mathbf{w_t} \in \mathscr{W}_{t\,h}^{\,v} = \mathscr{W}_h^v.
\end{aligned}$$

Integrating by parts twice in the first equation and once in the second and choosing $p_t = p$, such that the approximation and test spaces are the same, results in the discontinuous Galerkin formulation,

$$\begin{aligned}
(\mathbf{Q_t}, \boldsymbol{Q})_V &= \left(\mathbf{Q_t}, \nabla^T \boldsymbol{w}\right)_V + \langle \mathbf{Q_t}, \mathbf{n} \cdot (\boldsymbol{w}^* - \boldsymbol{w})\rangle_{\partial V}, & \forall \mathbf{Q_t} \in \mathscr{Q}_h^v \\
\left(\mathbf{w_t}, \frac{\partial \boldsymbol{w}}{\partial t}\right)_V &- (\mathbf{w_t}, \nabla \cdot \boldsymbol{F}(\boldsymbol{w}, \boldsymbol{Q}))_V + \langle \mathbf{w_t}, \mathbf{n} \cdot \boldsymbol{F}^* \rangle_{\partial V} = \mathbf{0}, & \forall \mathbf{w_t} \in \mathscr{W}_h^v.
\end{aligned}$$

where $\mathbf{n}$ denotes the outward pointing unit normal vector and where $\boldsymbol{w}^*$ and $\boldsymbol{F}^*$ represent the numerical solution and flux respectively.

Boundary Conditions

Boundary conditions are imposed weakly through the specification of a 'ghost' state for elements in which $V \cap \partial\Omega \neq \{0\}$. The following boundary conditions are supported:

Boundary Condition	Reference(s)	Comments
Riemann Invariant	[2, section 2.2]	eq. (14) should read $c_b = \frac{\gamma-1}{4}(R^+ - R^-)$
Slip-Wall	[3, eq. (10)]	
Back Pressure (Outflow)	[2, section 2.4]	
Total Temperature/Pressure (Inflow)	[2, section 2.7]	
Supersonic Inflow/Outflow		imposes the exact/extrapolated solution
No-slip Overconstrained		imposes values for all primitive variables [1]
No-slip Diabatic		imposes $\mathbf{v}$ and $(\mathbf{n} \cdot \mathbf{F}(\mathbf{W}, \mathbf{Q}))_E =$ constant

[1] Add reference to Nordstrom explaining why this BC is overconstrained and add link to Taylor-Couette results where it is used.

Chapter 3

The DPG Methodology for Linear PDEs

In comparison to discontinuous Galerkin (DG) methods, the noteworthy characteristic of the Discontinuous Petrov-Galerkin (DPG) methodology is that the optimal (to be defined below) test space is computed based on the minimization of the residual in a specified norm instead of simply being chosen to be the same as the trial space. Following Demkowicz et al. [4], DPG is generally referred to as a methodology, as opposed to a method, as different methods are obtained depending on the choice of inner product in the test space. As it is heavily relied on throughout the presentation, it is assumed that all spaces considered are Hilbert spaces.

We first outline the basic concepts of DPG methods in an abstract linear functional setting and then provide a concrete example through the application of the theory to the linear advection equation.

Abstract Functional Setting

Much of the theory outlined below is borrowed directly from the works of Demkowicz et al. [5, 4]. Of primary note, it is demonstrated that each DPG method can be interpreted as the *localization* of a method achieving optimal discrete stability through the choice of an optimal conforming test space.

A Petrov-Galerkin Variational Methodology with Optimal Stability

Consider the *linear* abstract variational problem

$$\text{Find } u \in U \text{ such that } b(v,u) = l(v) \ \forall v \in V, \tag{3.1}$$

where U and V denote the trial and test spaces, respectively, which are assumed to be Hilbert spaces, and where the bilinear form $b(\cdot,\cdot)$ acting on $V \times U$ and the linear form $l(\cdot)$ acting on V correspond to a particular variational formulation. It is assumed that the bilinear form satisfies a continuity condition with continuity constant M,

$$|b(v,u)| \leq M||v||_V||u||_U,$$

and an inf-sup condition with inf-sup constant γ,

$$\exists\gamma > 0 : \inf_{u\in U} \sup_{v\in V} \frac{b(v,u)}{||v||_V||u||_U} \geq \gamma. \tag{3.2}$$

Further, it is assumed that the linear form is continuous and satisfies the following compatibility condition

$$l(v) = 0 \ \forall v \in V_0, \text{ where } V_0 \coloneqq \{v \in V : b(v,u) = 0 \ \forall u \in U\}.$$

Then, by the Banach-Nečas-Babuška theorem (Add reference to Brener-Scott/Ciarlet), (3.1) has a unique solution, u, that depends continuously on the data,

$$||u||_U \leq \frac{M}{\gamma}||l||_{V'},$$

where V' denotes the dual space of V. Now let $U_h \subseteq U$ and $V_h \subseteq V$ be finite dimensional trial and test spaces and consider the finite dimensional variation problem

$$\text{Find } u_h \in U_h \text{ such that } b(v_h,u_h) = l(v_h) \ \forall v_h \in V_h. \tag{3.3}$$

If the form satisfies the discrete inf-sup condition with inf-sup constant γ_h,

$$\exists\gamma_h > 0 : \inf_{u_h\in U_h} \sup_{v_h\in V_h} \frac{b(v_h,u_h)}{||v_h||_{V_h}||u_h||_{U_h}} \geq \gamma_h, \tag{3.4}$$

then Babuška's theorem [6, Theorem 2.2] demonstrates that the discrete problem, (3.3), is well-posed with the Galerkin error satisfying the error estimate,

$$||u - u_h||_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} ||u - w_h||_U. \tag{3.5}$$

where the original constant in the bound, $\left(1 + \frac{M}{\gamma_h}\right)$ [6, eq. (2.14)], has been sharpened to $\frac{M}{\gamma_h}$ as demonstrated to be possible by Xu et al. [7, Theorem 2]. Generally, the well-posedness of the continuous problem does not imply the well-posedness of the discrete problem (i.e. (3.2) $\not\Longrightarrow$ (3.4)), and the fundamental motivation for the DPG methodology is then to choose the test space such that the supremum in the discrete inf-sup condition, (3.4), is obtained. (Potentially refer to where it is proven that DPG test functions are chosen in this way (following the demonstration in Demkowicz et al. [4, Section 4.1]))

A case of particular interest is then when the continuity and discrete inf-sup constants can be made equal,

$$M = \gamma_h, \tag{3.6}$$

so that the error incurred by the discrete approximation in (3.5) is smallest. As it is not immediately clear which norms should be selected for the trial and test spaces, the simplest strategy is to let the norm be chosen as that which is naturally induced by the problem such that (3.6) is satisfied. Bui-Thanh et al. [8, Theorem 2.6] have proven that $M = \gamma = 1$ if

$$\exists v_u \in V \setminus \{0\} : b(v_u, u) = ||v_u||_V ||u||_U \ \forall u \in U \setminus \{0\}, \tag{3.7}$$

where v_u is termed an optimal test function for the trial function u. Further, assuming that (3.7) holds, (3.6) is satisfied when the discrete test space is defined by

$$V \supset V_h = \text{span}\{v_{u_h} \in V : u_h \in U_h \subseteq U, \ b(v_{u_h}, u_h) = ||v_{u_h}||_V ||u_h||_U\}; \tag{3.8}$$

see Bui-Thanh et al. [8, Lemma 2.8]. Defining the map from trial to test space, $T : U \ni u \to Tu \coloneqq v_{Tu} \in V$, by

$$(v, Tu)_V = b(v, u),$$

where $(\cdot, \cdot)_V$ represents the test space inner product, then the discrete test space, (3.8), is equivalently defined as

$$V_h = \operatorname{span}\{v_{Tu_h} \in V : u_h \in U_h\}; \tag{3.9}$$

see Bui-Thanh et al. [8, Theorem 2.9]. Defining the Riesz operator for the test inner product,

$$R_V : V \ni v \to (v, \cdot) \in V',$$

which is an isometric isomorphism ([9, Theorem 4.9-4]), the test functions spanning V_h, which are henceforth referred to as *optimal* test functions, can be computed through the inversion of the Riesz operator by solving the auxiliary variational problem

$$\text{Find } v_{Tu_h} \in V \text{ such that } (w, v_{Tu_h})_V = b(w, u_h), \ \forall w \in V. \tag{3.10}$$

DPG as a Localization of the Optimal Conforming PG Methodology

Note that no assumptions regarding the conformity of the trial and test spaces were imposed in §3.1.1. Specifically, the specification of the 'D' (discontinuous) in DPG, referring to a discontinuous test space, has not yet been made and the methodology described is thus of a general Petrov-Galerkin form. Denote the trial *graph* space over the domain Ω, $H_b(\Omega)$, as that of the solution of (3.1),

$$H_b(\Omega) \coloneqq \{u \in (L^2(\Omega))^n : b(v, u) \in (L^2(\Omega))^n \ \forall v \in V\},$$

where n denotes the number of scalar variables. Integration by parts of (3.1) leads to the formal L^2-adjoint and a bilinear form representing the boundary terms

$$b(v,u) = b^*(v,u) + c(\mathrm{tr}_A^* v, \mathrm{tr}_A u)$$

where v is in the graph space for the adjoint

$$H_b^*(\Omega) \coloneqq \{v \in (L^2(\Omega))^n : b^*(v,u) \in (L^2(\Omega))^n \ \forall u \in U\}.$$

See Demkowicz et al. [5, eq. (4.18)] for a concrete example of these operators. When setting $V = H_b^*(\Omega)$, we say that the test space is H_b-conforming.

As the eventual goal of the methodology is to solve (3.3) over a tessellation, $\mathscr{T}_h$, of the discretized domain, Ω_h, consisting of elements (referred to as volumes) $\mathscr{V}$, we note that using an H_b-conforming test space results in each of the optimal test functions computed by (3.10) having global support (i.e. potentially over all of Ω_h). To make the methodology practical, broken energy spaces are introduced such that the required inversion of the Riesz operator can be done elementwise, *localizing* (3.10),

$$\text{Find } v_{Tu_h} \in V(\mathscr{V}) \text{ such that } (w, v_{Tu_h})_{V(\mathscr{V})} = b(w, u_h), \ \forall w \in V(\mathscr{V}) \tag{3.11}$$

where

$$\begin{aligned} V(\Omega_h) &\coloneqq \{v \in L^2(\Omega) : v|_{\mathscr{V}} \in H_b^*(\mathscr{V}) \ \forall \mathscr{V} \in \mathscr{T}_h\}, \\ (w,v)_{V(\Omega_h)} &\coloneqq \sum_{\mathscr{V}} (w|_{\mathscr{V}}, v|_{\mathscr{V}})_{V(\mathscr{V})} \end{aligned}$$

and $V(\mathscr{V})$ is the volume test space. Finally, it must be noted that the variational problem for the test functions, (3.11), is infinite dimensional. In practice, it must be solved approximately, for *approximate optimal test functions*, in an approximate volume test space $\tilde{V} \subseteq V$,

$$\text{Find } \tilde{v}_{Tu_h} \in \tilde{V}(\mathscr{V}) \text{ such that } (\tilde{w}, \tilde{v}_{Tu_h})_{\tilde{V}(\mathscr{V})} = b(\tilde{w}, u_h), \; \forall \tilde{w} \in \tilde{V}(\mathscr{V}), \tag{3.12}$$

where the corresponding approximate optimal test space is, analogous to (3.9), defined by

$$\tilde{V}_h = \text{span}\{\tilde{v}_{Tu_h} \in \tilde{V} : u_h \in U_h\}.$$

Potentially add comment regarding account for the approximation of optimal test functions [5, eqs. (4.30) - (4.32)].

Noting that the test graph space is a subset of $(L^2(\Omega))^n$, it has been shown that the PG methodology of §3.1.1 is, in fact, a subset of this practical DPG methodology where L^2-conforming test spaces are used as shown in the proof of Proposition 3.1.1. This however comes at the cost of introducing trace unknowns over the interior volume boundaries, in addition to the already existing trace unknowns on the domain boundary, both of which are subsequently denoted by $\hat{u}$.

Defining the group variable $\boldsymbol{u} \coloneqq (u, \hat{u})$ containing both the solution components in L^2 as well as those defined on traces (boundary and internal), we separate the bilinear form into the following components

$$b(v, \boldsymbol{u}) \coloneqq b(v, (u, \hat{u})) \coloneqq \bar{b}(v, \boldsymbol{w}) + \langle\langle v, \hat{w} \rangle\rangle \tag{3.13}$$

where $\bar{b}(v, \boldsymbol{w})$ includes all terms present in the PG methodology outlined in §3.1.1 and $\langle\langle v, \hat{w} \rangle\rangle$ accounts for newly introduced terms arising as a result of the usage of the broken tests space. Above, w includes the solution component in L^2, u, as well as the trace component on the domain boundary while $\hat{w}$ includes only the trace component on the internal volume boundaries. Following the previous notation, discrete solution variables are represented by

$\boldsymbol{w}_h \in U_h \times \hat{U}_h$ and $\hat{w}_h \in \hat{W}_h \subset \hat{U}_h$. Defining the weakly conforming optimal test space as

$$\bar{V}_h = \{v \in \tilde{V}_h : \langle\langle v, \hat{w}_h \rangle\rangle = 0 \ \forall \hat{w}_h \in \widetilde{\widehat{W}}\},$$

we have the following

Proposition 3.1.1 (PG Test Space as a Strict Subset of DPG Test Space)**.** $\bar{V}_h \subset \tilde{V}_h$.

Proof. We briefly reproduce the proof of Demkowicz et al. [5, Section 6]. As $\tilde{V}$ is a finite dimensional Hilbert space and $\tilde{V}_h \subseteq \tilde{V}$, the direct sum theorem [9, Theorem 4.5-2] allows for its decomposition as

$$\tilde{V} = \tilde{V}_h + \tilde{V}_h^{\perp}$$

where $\tilde{V}_h^{\perp}$ is the $\tilde{V}$-orthogonal complement of $\tilde{V}_h$ in $\tilde{V}$. This can be seen from

$$\begin{aligned}
\tilde{V}_h^{\perp} &\coloneqq \{\tilde{v} \in \tilde{V} : (\tilde{x}, \tilde{v})_{\tilde{V}(\mathscr{V})} = b(\tilde{x}, \boldsymbol{u}_h),\ \forall \boldsymbol{u}_h \in U \setminus U_h,\ \forall \tilde{x} \in \tilde{V}(\mathscr{V})\} && \text{(using (3.12))} \\
&= \{\tilde{v} \in \tilde{V} : (\tilde{x}, \tilde{v})_{\tilde{V}(\mathscr{V})} = 0,\ \forall \tilde{x} \in \tilde{V}(\mathscr{V})\}. && \text{(by Galerkin orthogonality)}
\end{aligned}$$

Think about the implication above that $(\tilde{x}, \tilde{v})_{\tilde{V}(\mathscr{V})} \neq 0 \ \forall \tilde{v} \in \tilde{V}_h,\ \forall \tilde{x} \in \tilde{V}(\mathscr{V})$.

Let $\bar{V}_h \ni \bar{v} = \{v \in \tilde{V} : (\tilde{x}, v)_{\tilde{V}(\mathscr{V})} = \bar{b}(\tilde{x}, \boldsymbol{w}_h) \ \forall \tilde{x} \in \tilde{V}(\mathscr{V})\}$. Since $\bar{v} \in \tilde{V}$, it can be decomposed as

$$\bar{v} = \bar{v}_h + \bar{v}_h^{\perp},\ \bar{v}_h \in \tilde{V}_h,\ \bar{v}_h^{\perp} \in \tilde{V}_h^{\perp}.$$

Since, $T\hat{w}_h \in \tilde{V}_h$, it follows that

$$\begin{aligned}
0 &= (\bar{v}_h^{\perp}, T\hat{w}_h)_{\tilde{V}(\mathscr{V})} && \text{(using } \tilde{V}\text{-orthogonality)} \\
&= b(\bar{v}_h^{\perp}, (0, \hat{w}_h)) && \text{(using (3.12))} \\
&= \langle\langle \bar{v}_h^{\perp}, \hat{w}_h \rangle\rangle && \text{(using (3.13))}
\end{aligned}$$

and consequently that $\bar{v}_h^\perp \in \bar{V}_h$. Because $T\boldsymbol{w}_h \in \tilde{V}_h$, then, as above,

$$0 = \bar{b}(\bar{v}_h^\perp, \boldsymbol{w}_h).$$

By definition of $\bar{v}$

$$\begin{aligned} 0 &= (\bar{v}_h^\perp, \bar{v})_{\tilde{V}(\mathscr{V})} && \text{(by definition)} \\ &= (\bar{v}_h^\perp, \bar{v}_h^\perp)_{\tilde{V}(\mathscr{V})} && \text{(using } \tilde{V}\text{-orthogonality).} \end{aligned}$$

and it is immediate that $\bar{v} = \bar{v}_h \in \tilde{V}_h$.

$\square$

If the optimal conforming PG methodology, §3.1.1, and the DPG methodology are both uniquely solvable, then their solutions are the same because substitution of conforming test functions into the DPG formulation immediately recovers the PG formulation.

A Concrete Example: Linear Advection

Consider the steady linear advection equation as a model problem

$$\begin{aligned} \boldsymbol{b} \cdot \nabla u &= s && \text{in } \Omega, && \text{(3.14a)} \\ u &= u_{\Gamma^\mathrm{i}} && \text{on } \Gamma^\mathrm{i} \coloneqq \{\boldsymbol{x} \in \partial\Omega : \hat{\boldsymbol{n}} \cdot \mathbf{b} < 0\}, \end{aligned}$$

where $\boldsymbol{b}$ is the advection velocity and $\hat{\boldsymbol{n}}$ is the outward pointing normal vector. Partitioning the domain into non-overlapping volumes, $\mathscr{V}$, with faces, $\mathscr{F} \coloneqq \partial\mathscr{V}$, (3.14a) is multiplied by a test function v and integrated by parts to give the bilinear and linear forms

$$\begin{aligned} b(v, \boldsymbol{u}) &= \sum_{\mathscr{V}} \int_{\mathscr{V}} -\nabla v \cdot \boldsymbol{b} u \; d\mathscr{V} + \int_{\mathscr{F} \setminus \Gamma^\mathrm{i}} v f^* \; d\mathscr{F}, && \text{(3.15a)} \\ l(v) &= \sum_{\mathscr{V}} \int_{\mathscr{V}} v s \; d\mathscr{V} + \int_{\mathscr{F} \cap \Gamma^\mathrm{i}} v f_{\Gamma^\mathrm{i}} \; d\mathscr{F}, \end{aligned}$$

where $f_{\Gamma^{\rm i}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{b} u_{\Gamma^{\rm i}}$ and where the single-valued trace normal fluxes, $f^* := \hat{\boldsymbol{n}} \cdot \boldsymbol{b} u|_{\mathscr{F}}$, have been introduced as part of the group variable $\boldsymbol{u} := (u, f^*)$. The selection of f^* instead of u^* as the trace unknown is made because of the possible degeneration of $\hat{\boldsymbol{n}} \cdot \boldsymbol{b}$. (3.15a) can be expressed more compactly as

$$b(v, \boldsymbol{u}) = \sum_{\mathscr{V}} \int_{\mathscr{V}} -\nabla v \cdot \boldsymbol{b} u \; d\mathscr{V} + \frac{1}{2} \int_{\mathscr{F}} [\![v]\!] f^* \; d\mathscr{F}, \tag{3.16}$$

after introducing the *jump* operator, $[\![v]\!] = v^- - v^+$, with "$-$" and "$+$" referring to the volumes adjacent to the face with the normal vector pointing outwards/inwards, respectively, and with the additional specification of $v^+ = \pm v^-$ on inflow/outflow boundaries, respectively. Following the motivation of pursuing norms where the continuity and inf-sup constants are equal, the Cauchy-Schwarz inequality can be applied to (3.16) to obtain

$$\begin{aligned} b(v, \boldsymbol{u}) \leq & \sum_{\mathscr{V}} || - \nabla v \cdot \boldsymbol{b} ||_{L^2(\mathscr{V})} ||u||_{L^2(\mathscr{V})} + \frac{1}{2} || [\![v]\!] ||_{L^2(\mathscr{F})} ||f^*||_{L^2(\mathscr{F})} \\ \leq & \underbrace{\left(\sum_{\mathscr{V}} || - \nabla v \cdot \boldsymbol{b} ||^2_{L^2(\mathscr{V})} + \frac{1}{\sqrt{2}} || [\![v]\!] ||^2_{L^2(\mathscr{F})} \right)^{\frac{1}{2}}}_{||v||_V} \times \underbrace{\left(\sum_{\mathscr{V}} ||u||^2_{L^2(\mathscr{V})} + \frac{1}{\sqrt{2}} ||f^*||^2_{L^2(\mathscr{F})} \right)^{\frac{1}{2}}}_{||\boldsymbol{u}||_U} . \end{aligned} \tag{3.17}$$

Above, $\boldsymbol{u} \in U = L^2(\Omega_h) \times L^2(\Gamma_h^+)$, and $v \in V = H^1_{\mathbf{b}}(\Omega_h)$ where the h subscripts denote discretization, Γ_h^+ is the union of all volumes faces including those of the boundary and the internal skeleton, and where the spaces are defined according to

$$\begin{aligned} L^2(\Omega_h) &= \{ u : u \in L^2(\mathscr{V}) \; \forall \mathscr{V} \in \Omega_h \}, \\ L^2(\Gamma_h^+) &= \{ \hat{u} : \hat{u} \in L^2(\mathscr{F}) \; \forall \mathscr{V} \in \Omega_h \}, \\ H^1_{\mathbf{b}}(\Omega_h) &= \{ v : v \in L^2(\Omega_h), \; \mathbf{b} \cdot \nabla v \in L^2(\Omega_h) \}, \end{aligned}$$

Note that the space chosen above for U implies a weak imposition of the inflow boundary condition, and it is imagined that a *ghost* volume neighbouring the inflow boundaries allows for the imposition of the boundary conditions. Equality in (3.17) is attained, as desired due to (3.7), if the test functions are chosen such that

$$\begin{aligned} u &= -\nabla v_u \cdot \mathbf{b} && \text{in } \mathscr{V}, \\ f^* &= [\![v_u]\!] && \text{on } \mathscr{F}. \end{aligned}$$

In general, for basis functions of the form $\hat{\boldsymbol{\phi}} = (0, \hat{\phi}) \in U$,

$$0 = -\nabla v_{\hat{\phi}} \cdot \mathbf{b} \qquad \text{in } \mathscr{V}, \tag{3.18}$$

$$\hat{\phi} = [\![v_{\hat{\phi}}]\!] \qquad \text{on } \mathscr{F}, \tag{3.19}$$

and for basis functions $\boldsymbol{\phi} = (\phi, 0) \in U$,

$$\begin{aligned} \phi &= -\nabla v_{\phi} \cdot \mathbf{b} && \text{in } \mathscr{V}, \\ 0 &= [\![v_{\phi}]\!] && \text{on } \mathscr{F}. \end{aligned} \tag{3.20}$$

Considering the test function for the L^2 component of the solution, it can be noted that (3.20) imposes a conformity constraint on the test space, resulting in a specific PG method from §3.1.1. However, recalling Proposition 3.1.1, the same solution is obtained using the practical DPG methodology of §3.1.2, and this can be achieved by omitting the conformity constraint and limiting the support of each of the optimal test functions,

$$\begin{aligned} \operatorname{supp}(v_{\hat{\phi}\in\mathscr{F}}) &= \{\mathscr{V}_i \in \Omega_h : \mathscr{V}_i \cap \mathscr{F} \neq \emptyset\} := \{\mathscr{V}_{\mathscr{F}}^- \cup \mathscr{V}_{\mathscr{F}}^+\}, \\ \operatorname{supp}(v_{\phi\in\mathscr{V}}) &= \{\mathscr{V}_i \in \Omega_h : \mathscr{V}_i \cap \mathscr{V} \neq \emptyset\}, \end{aligned}$$

such that they can be computed in a local manner.

Proposition 3.2.1. *Defining the "-" volume in relation to a specific face as that which* *satisfies* $\hat{\mathbf{n}} \cdot \mathbf{b} \geq 0$*, then given the localized test norm (inner product)*

$$(w,v)_{V(\mathscr{V})} = \int_{\mathscr{V}} (\nabla w \cdot \mathbf{b})(\mathbf{b} \cdot \nabla v) d\mathscr{V} + \int_{\mathscr{F}} [\![w]\!] |\hat{\mathbf{n}} \cdot \mathbf{b}| [\![v]\!] d\mathscr{F} + \int_{\mathscr{F} \cap \mathscr{V}_{\mathscr{F}}^{+}} w |\hat{\mathbf{n}} \cdot \mathbf{b}| v d\mathscr{F}$$

Note that the above is not a norm when $\hat{\mathbf{n}} \cdot \mathbf{b} = 0$*. This is not a problem for the method as the trace solution component is not required on such faces, but is a problem for the rigour. Change the norm on such faces? Add an L2 component?*

where integration is performed over all volumes and faces in the support of the local test function, then the computed test functions for the trace unknowns satisfy

$$||v_{\hat{\phi}} - v_{\hat{\phi}_h}||_{L^2(\Omega_h)} \leq \inf_{w_{\hat{\phi}_h} \in L^2(\mathscr{V}_{\mathscr{F}}^{-} \cup \mathscr{V}_{\mathscr{F}}^{+})} ||v_{\hat{\phi}} - w_{\hat{\phi}_h}||_{L^2(\Omega_h)},$$

where the exact test functions satisfy (3.18) *and* (3.19)*. Further, the computed trace unknowns obtained by solving* (3.1) *after substituting* $v_{\hat{\phi}}$ *into* (3.15) *satisfy*

$$||u^* - \hat{u}_h||_{L^2(\Gamma_h^+)} \leq \inf_{\hat{w}_h \in L^2(\Gamma_h^+)} ||u^* - \hat{w}_h||_{L^2(\Gamma_h^+)}.$$

Proof. Solving for the approximate optimal test functions using (3.12) with $\boldsymbol{u}_h = (0, \hat{\phi})$,

1

□

Remark 3.2.1. *The physical intuition behind the choice of test inner product in Proposition 3.2.1 is that the resultant test functions serve to propagate solutions taking the form* $\hat{\phi}$ *from inflow to outflow volume faces along the flow characteristics (in the advection direction) while simultaneously adding the* L^2 *projection of the source onto the trace basis function lifted into the volume along the characteristics. (Add figure).*

Where are we going with this?

- Bui-Thanh2013's continuous method results in the recovery of the solution as the propagation of the boundary along the characteristics.

- Using the localized test norm: H1-semi + trace term, the trace test functions computed are exactly those satisfying the constraints above. Inserting these into the bilinear form results in pointwise exact solution for the trace unknowns (1D).
- Equivalent with DG using lowest order test?

Ask Legrand if he is interested in this result once finished typesetting (and likely after 2D attempt is made).

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