Model Predictive Control — Lecture 5
Incorporating Constraints in MPC

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Constraints

In practice all processes are subject to constraints. The actuators have a limited field of action and a determined slew rate, as is the case of the valves, limited by the positions of totally open or closed and by the response rate. Safety or environmental reasons also give rise to limits in the process variables such as levels in tanks, flows in piping, or maximum temperature and pressures allowable in operating the process.

$$\begin{array}{ccccc} u_{\text{MIN}} & \leq & \hat{u}(k+i|k) & \leq & u_{\text{MAX}} \\ \Delta u_{\text{MIN}} & \leq & \Delta\hat{u}(k+i|k) & \leq & \Delta u_{\text{MAX}} \\ y_{\text{MIN}} & \leq & \hat{y}(k+i|k) & \leq & y_{\text{MAX}} \end{array}$$

We could write the above constraints into standard form

$$\Omega\hat{U} \leq \beta, \quad \text{where } \hat{U} = \begin{bmatrix} \Delta\hat{u}(k|k) \\ \Delta\hat{u}(k+1|k) \\ \vdots \\ \Delta\hat{u}(k+N_u-1|k) \end{bmatrix}$$

Input increment constraints

Input increment constraints are usually represented as

$$\Delta u_{\text{MIN}} \leq \Delta \hat{u}(k+i|k) \leq \Delta u_{\text{MAX}}, \ i = 0, 1, \ldots, N_u - 1$$

Consider the set of upper bound constraints

$$\Delta \hat{u}(k+i|k) \leq \Delta u_{\text{MAX}}, \ i = 0, 1, \ldots, N_u - 1,$$

they can be written in the general form by setting

$$\Omega \leftarrow \begin{bmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}, \quad \beta \leftarrow \begin{bmatrix} \Delta u_{\text{MAX}} \\ \vdots \\ \Delta u_{\text{MAX}} \end{bmatrix}$$

Similarly, for the lower bound constraints, we set

$$\Omega \leftarrow -\begin{bmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}, \quad \beta \leftarrow -\begin{bmatrix} \Delta u_{\text{MIN}} \\ \vdots \\ \Delta u_{\text{MIN}} \end{bmatrix}$$

Input Constraints, $u_{\mathbf{MIN}} \le \hat{u}(k+i|k) \le u_{\mathbf{MAX}}$

For the set of upper bound constraints, set

$$\Omega \leftarrow \begin{bmatrix} I & & & \\ I & I & & \\ \vdots & & \ddots & \\ I & \dots & \dots & I \end{bmatrix}, \quad \beta \leftarrow \begin{bmatrix} u_{\text{MAX}} - u(k-1) \\ \vdots \\ u_{\text{MAX}} - u(k-1) \end{bmatrix}$$

For the set of lower bound constraints, set

$$\Omega \leftarrow - \begin{bmatrix} I & & & \\ I & I & & \\ \vdots & & \ddots & \\ I & \dots & \dots & I \end{bmatrix}, \quad \beta \leftarrow \begin{bmatrix} -u_{\text{MIN}} + u(k-1) \\ \vdots \\ -u_{\text{MIN}} + u(k-1) \end{bmatrix}$$

Output Constraints, $y_{\text{MIN}} \le \hat{y}(k+i|k) \le y_{\text{MAX}}$

Recall that[1]

$$\begin{bmatrix} \hat{y}(k+1|k) \\ \hat{y}(k+2|k) \\ \vdots \\ \hat{y}(k+N|k) \end{bmatrix} = \Phi\xi(k) + G\Delta u$$

Hence, for the set of upper bound constraints, set

$$E \leftarrow G, \qquad \beta \leftarrow \begin{bmatrix} y_{\text{MAX}} \\ \vdots \\ y_{\text{MAX}} \end{bmatrix} - \Phi\xi(k)$$

and for the set of lower bound constraints, set

$$\Omega \leftarrow -G, \qquad \beta \leftarrow \begin{bmatrix} -y_{\text{MIN}} \\ \vdots \\ -y_{\text{MIN}} \end{bmatrix} + \Phi\xi(k)$$

[1] ...redictions depend on the state space model used.

Ex 2.5

Suppose that we have a 1-state, 1-input model with 1 controlled variable:

$$x(k+1) = 2x(k) + u(k), \quad y(k) = 3x(k)$$

and we have the constraint

$$-1 \leq y(k+1) \leq 2$$

If $x(k) = 3$ and $u(k-1) = -1$, show that the corresponding constraints on $\Delta u(k)$ is

$$-\frac{16}{3} \leq \Delta u(k) \leq -\frac{13}{3}$$

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Example, Constraints, 1/2

Suppose we have a plant with 2 input variables and 2 controlled variables. The control horizon is $N_u = 2$ and the prediction horizons are $N_1 = 1$ and $N_2 = 2$. We have the following constraints, which are to hold at each time:

$$
\begin{aligned}
&-2 \leq \Delta u_1 \leq 2 \\
&0 \leq u_2 \leq 3 \\
&y_1 \geq 0 \\
&y_2 \geq 0 \\
&3y_1 + 5y_2 \leq 15
\end{aligned}
$$

Write the above constraints into the standard form of

$$E\hat{U} \leq \beta + Fu(k-1) + M\xi(k).$$

Example, Constraints, 2/2

In order to work out the example, we need to assume a state space model relating the variables y and u:

$$
\begin{aligned}
x(k+1) &= Ax(k) + B\Delta u(k) \\
y(k) &= Cx(k)
\end{aligned}
$$

where

$$
\Delta u(k) = \begin{bmatrix} \Delta u_1(k) \\ \Delta u_2(k) \end{bmatrix}, \qquad y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}
$$

Then the constraints on the inputs and input increments will include time instants $k+1$ to $k+N_u-1$ and constraints on outputs are from time instants $k+N_1$ to $k+N_2$.

Mini-Tutorial: Quadratic Programming, 1/2

A *Quadratic Program* is an optimisation problem of the form

$$\min_{\theta} \frac{1}{2}\theta^T H\theta + f^T\theta \qquad \text{subject to} \qquad \Omega\theta \leq \omega$$

where the Hessian $H = H^T > 0$.
As a result of QP problem being convex, the necessary and sufficient conditions for θ to be the global optimum are given by the *Karush-Kuhn-Tucker* or (KKT) conditions: there must exist vectors (Lagrange multipliers) $\lambda \geq 0$ and a vector $t \geq 0$ such that

$$H\theta + \Omega^T\lambda = -\phi, \quad -\Omega\theta - t = -\omega, \quad \text{and} \quad t^T\lambda = 0$$

Two approaches to solving the QP problem appear to offer the best performance: *Active Set* methods and the *Interior Point* methods.

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Mini-Tutorial: Quadratic Programming, 2/2

In MATLAB, the function quadprog solves the QP problem:

```
>> help quadprog
   QUADPROG Quadratic programming.
   X=QUADPROG(H,f,A,b) attempts to solve the quadratic programming
   problem:

           min 0.5*x'*H*x + f'*x   subject to:  A*x <= b
            x
```

MPC with Constraints

Recall that the MPC cost function is

$$\begin{aligned} J &= \sum_{i=N_1}^{N_2} [\hat{y}(k+i|k) - w(k+i|k)]^2 + \lambda \sum_{i=1}^{N_u} [\Delta \hat{u}(k+i-1|k)]^2 \\ &= \hat{U}^T(G^TG + \lambda I)\hat{U} + 2\hat{U}^T G^T(\hat{F} - \hat{W}) + (\hat{F} - \hat{W})^T(\hat{F} - \hat{W}) \end{aligned}$$

As shown earlier, constraints could be written in a standard form:

$$E\hat{U} \leq \beta$$

Then MPC with constraints can be formulated as a Quadratic Programming (QP) problem: $\min_\theta \frac{1}{2}\theta^T H\theta + f^T\theta,$ subject to $\Omega\theta \leq \omega$[2] by setting:

$$\begin{aligned} &H \leftarrow (G^TG + \lambda I), \quad f \leftarrow G^T(\hat{F} - \hat{W}) = G^T(\Phi\xi(k) - \hat{W}), \\ &\Omega \leftarrow E, \quad \omega \leftarrow \beta, \quad \theta \leftarrow \hat{U} \end{aligned}$$

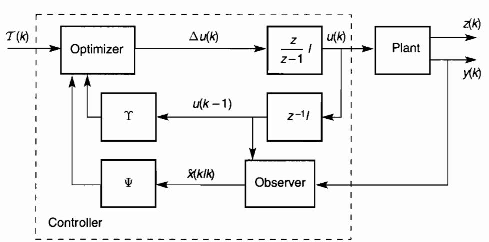
[2] see Mini-Tutorial: Quadratic Programming.

Lab Exercises 2

1. **Constraints Formulation**
   Given the state space model with incremental input write a MATLAB function to compute the matrices needed to represent the input increment, input, and output constraints. Use the following function prototype:
   `[Omega, F, M] = form_constraints(A,B,C,N1,N2,Nu,...)`
2. **Constrained receding-horizon MPC Control Law**
   Write a MATLAB function to compute the MPC control law with constraints. Use the following function prototype:
   `du = MPC2(Phi,G,Omega,F,M,lambda,...)`
   What other information is needed in order to computer $\Delta u(k)$?
3. Simulate a Constrained MPC control system.

Structure of MPC with Constraints, 1/3

The figure below shows the structure of MPC with constraints. So long as all the constraints are inactive, the solution of the predictive controller is exactly the same as in the unconstrained case. But if constraints become active, then the control becomes nonlinear because the box labelled 'Optimiser' computes a nonlinear function of its inputs.



Structure of MPC with Constraints, 2/3

We can say a little more about the structure of MPC with constraints. Suppose we knew before solving the QP problem the set of constraints that would be active at the constrained optimal, that is

$$\Omega_a \hat{U} = \omega_a$$

where Ω_a is made up of those rows of Ω which relate to the active constraints, and ω_a is made up of the corresponding elements of ω. Then we could (only in principle!) pose the optimisation problem

$$\min_{\hat{U}} J \quad \text{subject to} \quad \Omega_a \hat{U} = \omega_a$$

which could be solved by the theory of Lagrange multipliers[3], giving the optimal solution as ...

[3] see Mini-Tutorial: Lagrange Multipliers.

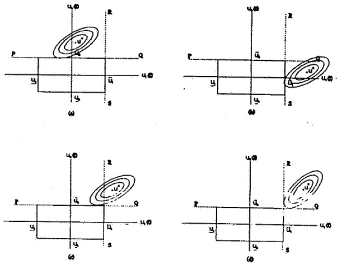
Structure of MPC with Constraints, 3/3

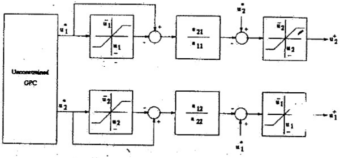
$$\begin{bmatrix} \hat{U} \\ \nu \end{bmatrix}_{\text{OPT}} = \begin{bmatrix} (G^T G + \lambda I) & \Omega_a^T \\ \Omega_a & 0 \end{bmatrix}^{-1} \begin{bmatrix} G^T(\hat{W} - \hat{F}) \\ \omega_a \end{bmatrix}$$

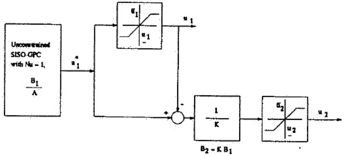
where ν is a vector of Lagrange multipliers.
Now note that the matrix to be inverted does not depend on signals at time k, so the matrix being inverted here is fixed, so long as a fixed set of constraints is active. On the other hand, $G^T(\hat{W} - \hat{F})$ and ω_a clearly does depend on the signals present at time k.
We can therefore conclude that the constrained predictive control law is a linear time-invariant control law, so long as the set of active constraints is fixed.
For the special case of a 2-input system, see: KV Ling, An Efficient Constrained Predictive Control Algorithm for 2-input Systems, 12 IFAC World Congress, Pre-print, Vol.I, pp.345-348, Syndey, Australia, 1993.

Constrained MPC for 2-input Systems[a]

[a] KV Ling, An Efficient Constrained Predictive Control Algorithm for 2-input Systems, 12 IFAC World Congress, Pre-print, Vol.I, pp.345-348, Syndey, Australia, 1993







Mini-tutorial, Lagrange Multipliers, 1/2

Consider the optimisation problem with equality constraints,

$$\min_{\theta} \frac{1}{2}\theta^T H\theta + f^T\theta \quad \text{subject to} \quad \Omega_a\theta = \omega_a$$

This problem could be solved by solving the problem $\min_{\theta,\lambda} L(\theta, \lambda)$ where $L(\theta, \lambda) = \frac{1}{2}\theta^T H\theta + f^T\theta + \lambda^T(\Omega_a\theta - \omega_a)$ and λ is known as the Lagrange multipliers. Now

$$\nabla_\theta L(\theta, \lambda) = H\theta + f + \Omega_a^T\lambda \quad \text{and} \quad \nabla_\lambda L(\theta, \lambda) = \Omega_a\theta - \omega$$

The optimal solution would be obtained when the derivatives are zero, giving

$$\begin{bmatrix} \theta \\ \lambda \end{bmatrix}_{\text{OPT}} = \begin{bmatrix} H & \Omega_a^T \\ \Omega_a & 0 \end{bmatrix}^{-1} \begin{bmatrix} -f \\ \omega_a \end{bmatrix}$$

An explicit formula, which might provide additional insight, for the constrained optimal solution is ...

Mini-tutorial, Lagrange Multipliers, 2/2

$$\begin{aligned}
\theta_{\mathrm{OPT}} &= -H^{-1}f - H^{-1}\Omega_a^T(\Omega_a H^{-1}\Omega_a^T)^{-1}(-\Omega_a H^{-1}f - \omega_a) \\
\lambda_{\mathrm{OPT}} &= (\Omega_a H^{-1}\Omega_a^T)^{-1}(-\Omega_a H^{-1}f - \omega_a)
\end{aligned}$$

Note that $-H^{-1}f$ is the unconstrained optimal.

MPC with Constraints, a case study, 1/4

The process model of an air compressor is given by:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-0.715s}}{1+4.48s+1.783s^2} & \frac{0.9222}{1+2.071s} \\ \frac{0.3378e^{-0.2999s}}{1+1.09s+0.361s} & \frac{-0.321e^{0.94s}}{1+2.4673s+0.104s^2} \end{bmatrix}$$

A sampling time of 0.05 second is chosen.
The compressor can be controlled by decoupling the process at zero frequency and using a PI controller for the first loop and a proportional controller for the second. These controllers were obtained with the help of the Inverse Nyquist Array (INA).

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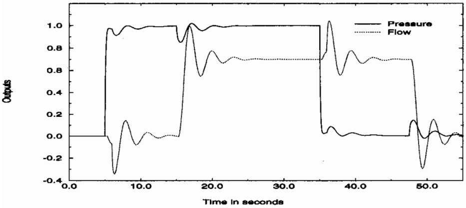
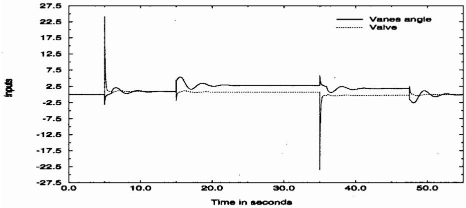
MPC with Constraints, a case study, 2/4

It can be seen that the responses are quite oscillatory. The valve position exhibits high peaks for each change in the pressure setpoint.



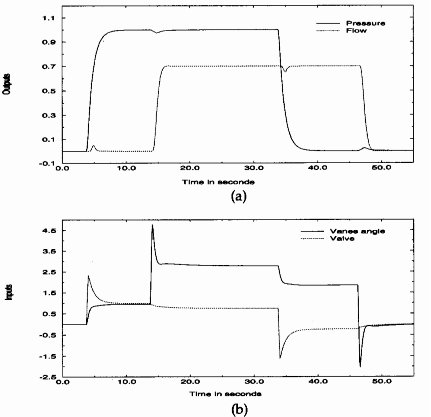
(a)



(b)

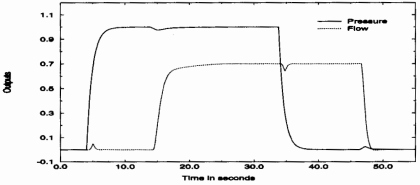
MPC with Constraints, a case study, 3/4

With unconstrained MPC, both controlled variables reach their setpoint rapidly and without oscillations. The perturbations caused in each of the controlled variables by a step change in the reference of the other variable are very small. A high peak in the valve position is observed (although much smaller than the peaks observed when using the INA controller).

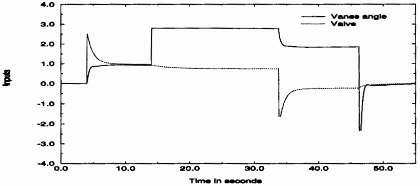


MPC with Constraints, a case study, 4/4

To reduce the manipulated variable peak, a constrained MPC can be used. The manipulated variables are restricted to being in the interval $[-2.75,\ 2.75]$. It can be seen that the response of the pressure is a bit slower than in the unconstrained case, but the manipulated variable is kept within the desired limits.



(a)



(b)

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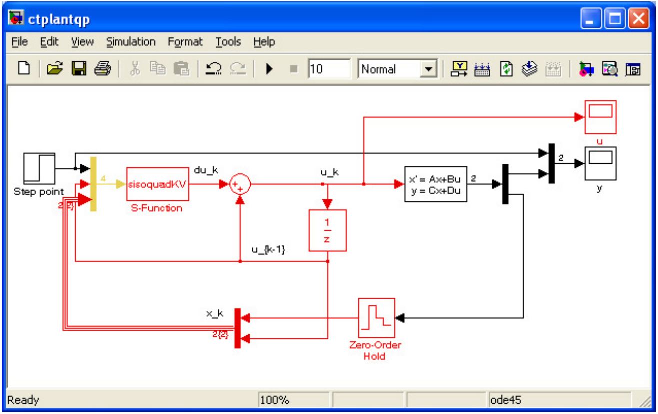
Constrained MPC simulation using S-function



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