机器人学导论

楼云江

哈尔滨工业大学（深圳）

2023春季

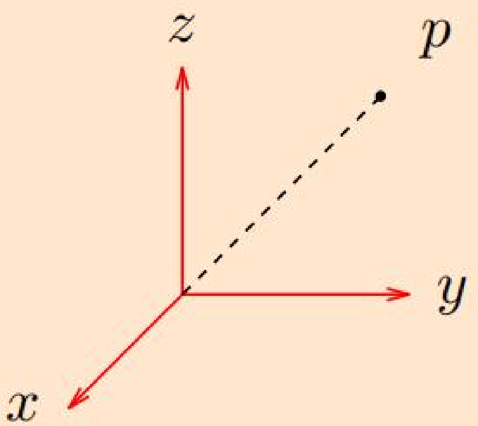
[3] 刚体运动

HIT

Rigid Body Motion

Chapter 3 Rigid Body Motion

3.1 Notations



$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \text{ or } p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

For $p \in \mathbb{R}^n, n = 2, 3 (2 \text{ for planar, } 3 \text{ for spatial})$

Point: $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$

Vector: $v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$

Matrix: $A \in \mathbb{R}^{n \times m}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

$$p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}: \text{initial position}$$

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, t \in (-\varepsilon, \varepsilon)$$

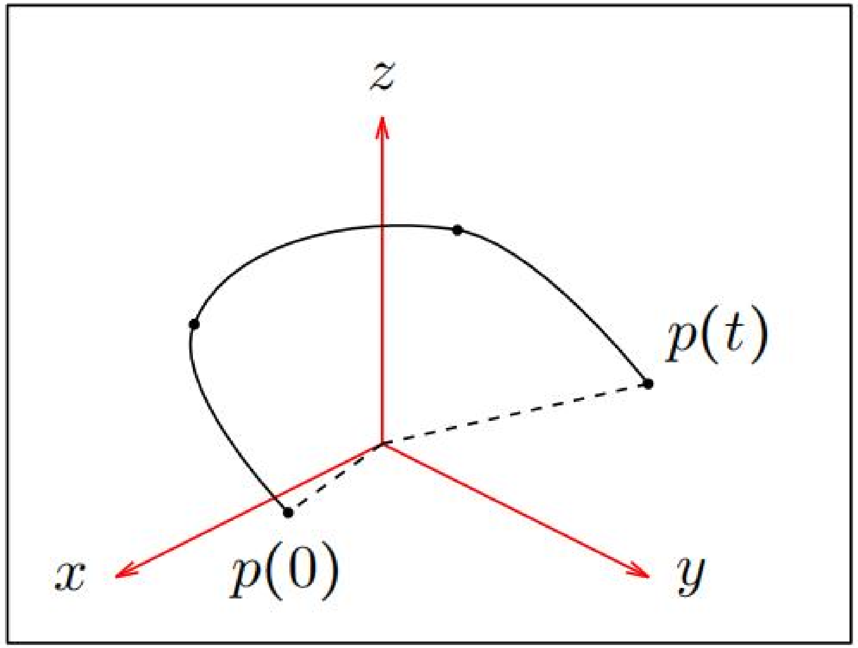


Figure 3.1

Definition: Trajectory

A **trajectory** is a curve $p : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

3.2 Rigid Body Transformations (Description of rigid body motion)

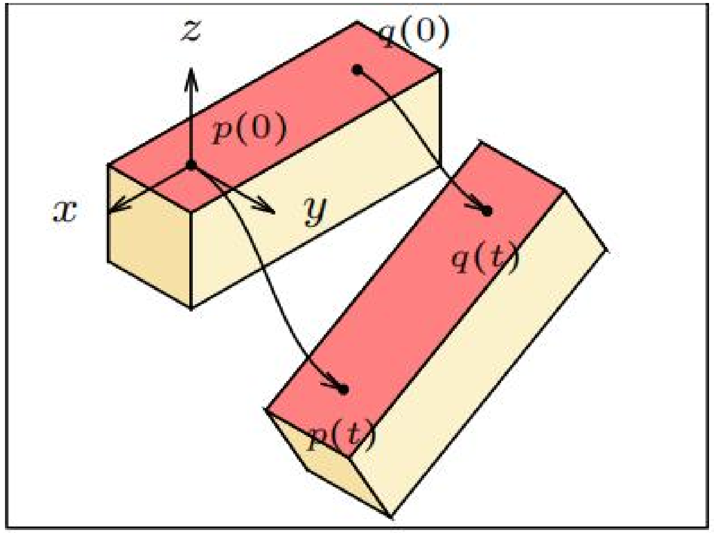


Figure 3.2

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

Definition: Rigid body transformation

$$g : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

s.t.

1. Length preserving: $\|g(p) - g(q)\| = \|p - q\|$
2. Orientation preserving: $g_*(v \times \omega) = g_*(v) \times g_*(\omega)$

A rigid body transformation ⟺ **A coordinate frame transformation**

①Choose a reference frame A (spatial frame)

②Attach a frame B to the body (body frame)

$x_{ab} \in \mathbb{R}^3$:	coordinates of x_b in frame A
$R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in \mathbb{R}^{3\times3}$:	Rotation (or orientation) matrix of B w.r.t. A

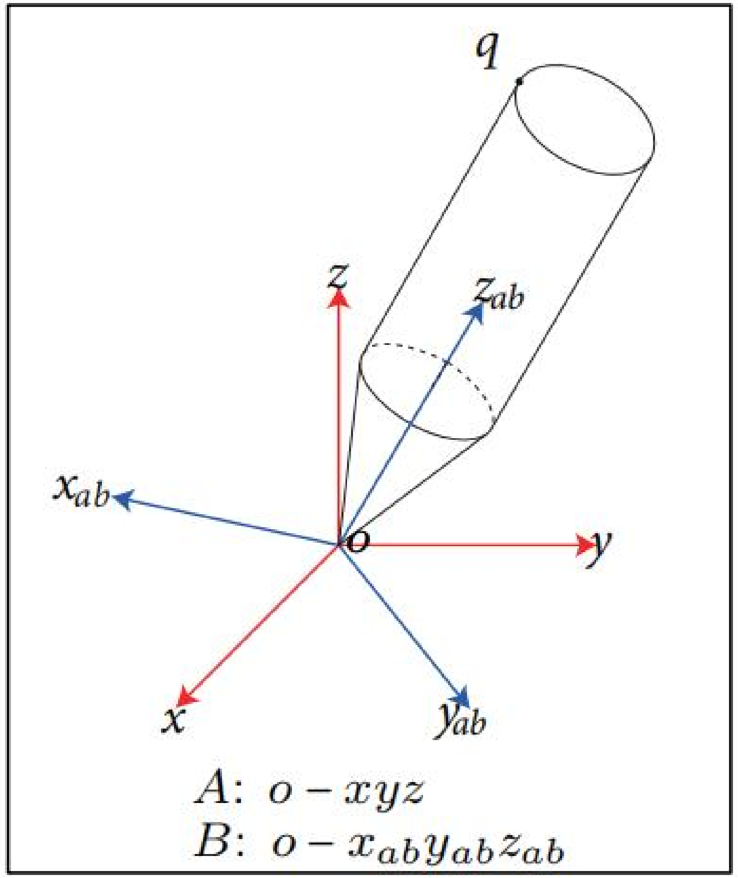


Figure 3.3

3.3 Coordinate Frame Transformations in $\mathbb{R}^3$

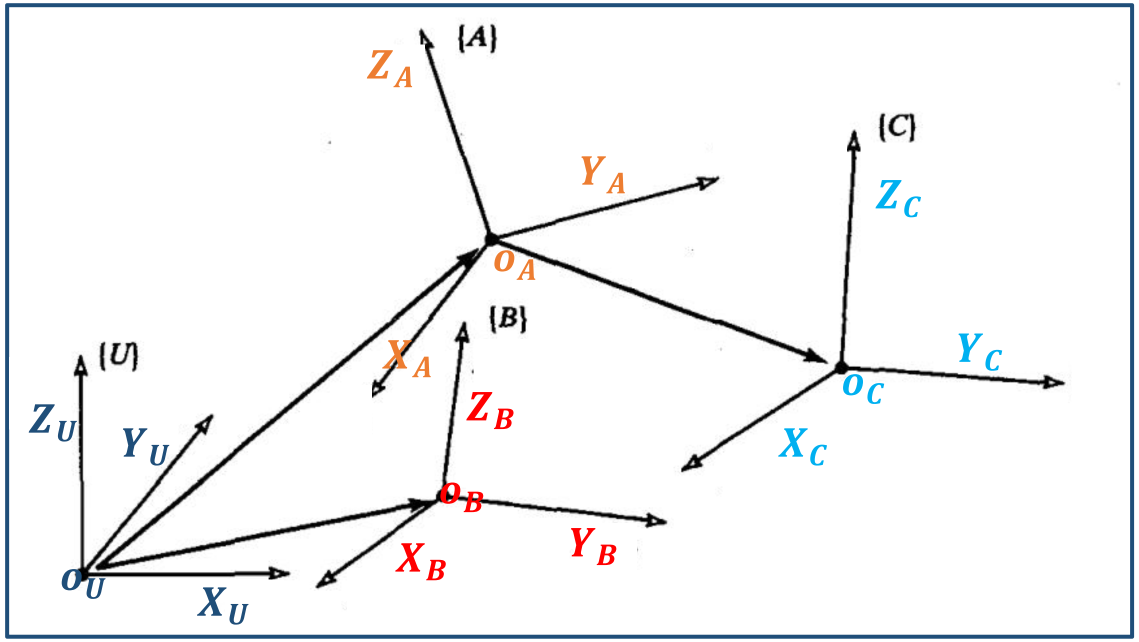


Figure 3.4

Coordinate Frame A

Transform Matrix
$T \in \mathbb{R}^{4x4}$

Coordinate Frame B

A coordinate frame $\{A\}$ consists of two parts:

- Original Point $\mathrm{o} = [o_x \quad o_y \quad o_z]^T$
- Coordinat axes $[A_x^T \quad A_y^T \quad A_z^T]$

A coordinate frame transform matrix can be divided into two parts:

- Rotation Matrix $\mathrm{R} \in \mathbb{R}^{3x3}$
- Ttranslation Vector $\mathrm{p} \in \mathbb{R}^{3x1}$

哈爾濱工業大學(深圳) HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

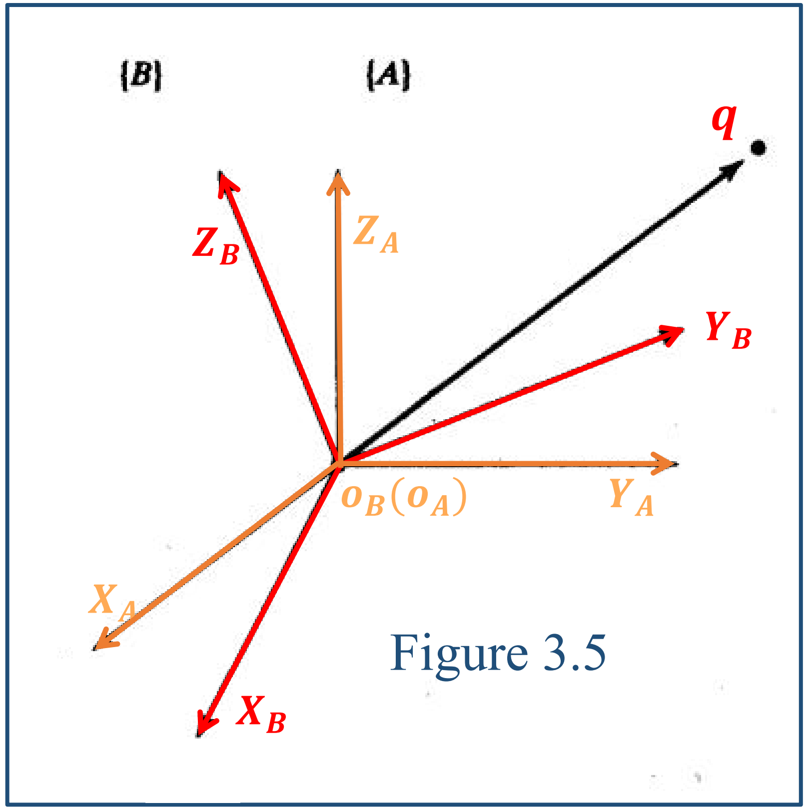


Figure 3.5

Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{B\}$

Let $q_a = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{A\}$

$$
\begin{aligned}
q_a &= x_{ab} \cdot x_b + y_{ab} \cdot y_b + z_{ab} \cdot z_b \\
&= \begin{bmatrix} x_{ab} & y_{ab} & z_{ab} \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \\
&= R_{ab} \cdot q_b
\end{aligned}
$$

Definition: $SO(3) = \{R \in \mathbb{R}^{3x3} | R^T R = I, \det R = 1\}$

$R_{ab} \in SO(3)$ **is the rotation matrix of frame$\{B\}$ w.r.t frame $\{A\}$**

3.3 Coordinate Frame Transformations in $\mathbb{R}^3$ (Property of Rotation Matrix)

Let $R = [r_1\ r_2\ r_3]$ be a rotation matrix

$$\Rightarrow r_i^T \cdot r_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

or

$$R^T \cdot R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} [r_1\ r_2\ r_3] = I$$

or $R \cdot R^T = I$

We have:

$$\det(R^T R) = \det R^T \cdot \det R = (\det R)^2 = 1, \det R = \pm 1$$

As $\det R = r_1^T (r_2 \times r_3) = 1 \Rightarrow \det R = 1$

Property 2: R_{ab} preserves distance between points and orientation.

1. $\|R_{ab} \cdot (p_b - q_b)\| = \|p_a - q_a\|$
2. $R(v \times \omega) = (Rv) \times R\omega$

Proof :

For $a \in \mathbb{R}^3$, let $\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Note that $\hat{a} \cdot b = a \times b$

1 follows from
$$\|R_{ab}(p_b - p_a)\|^2 = (R_{ab}(p_b - p_a))^T R_{ab}(p_b - p_a) = (p_b - p_a)^T R_{ab}^T R_{ab}(p_b - p_a) = \|p_b - p_a\|^2$$

2 follows from $R\hat{v}R^T = (Rv)^\wedge$ (prove it yourself) $\square$

3.3 Coordinate Frame Transformations in $\mathbb{R}^3$ (Translation Vector)

哈爾濱工業大學(深圳)
HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

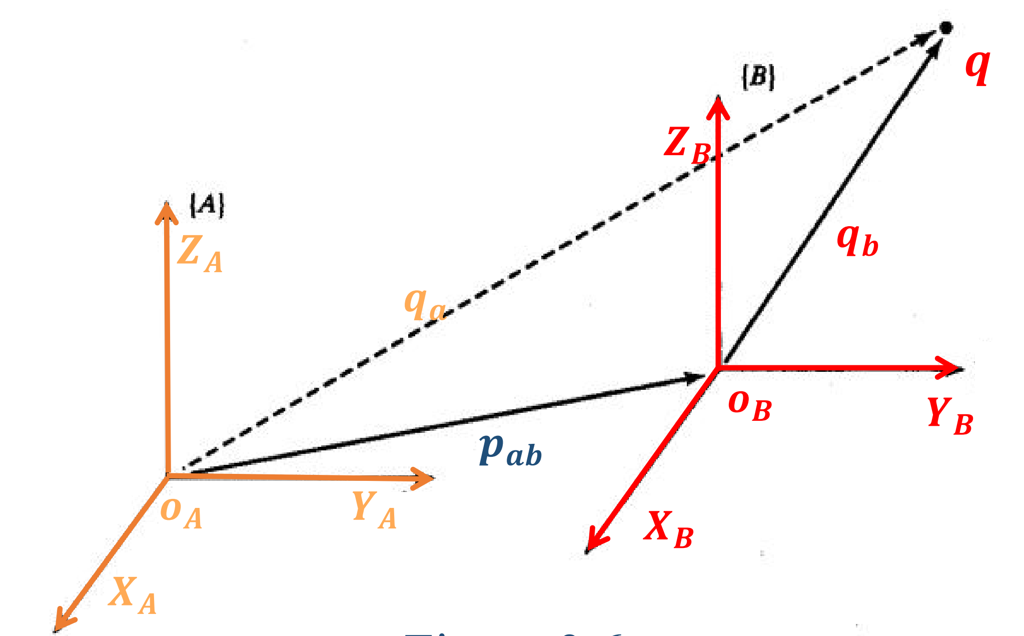


Figure 3.6

Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{B\}$

Let $q_a = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{A\}$

$$q_a = q_b + p_{ab}$$

$$p_{ab} = \begin{bmatrix} o_{Bx} - o_{Ax} \\ o_{By} - o_{Ay} \\ o_{Bz} - o_{Az} \end{bmatrix}$$

$p_{ab} \in \mathbb{R}^3$ is the translation vector of frame $\{B\}$ w.r.t frame $\{A\}$

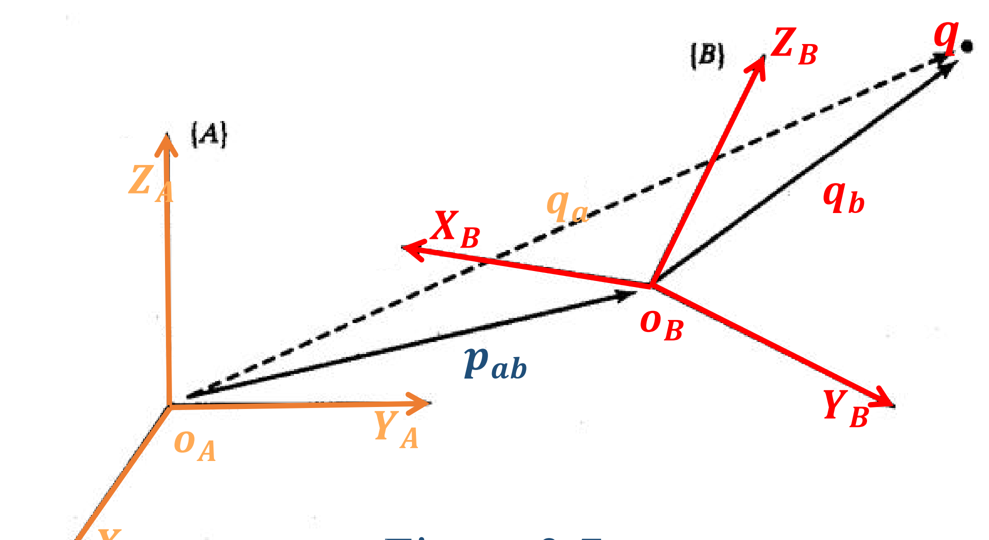


Figure 3.7

Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{B\}$

Let $q_a = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \in \mathbb{R}^3$: coordinates of point **q** in frame $\{A\}$

$$q_a = R_{ab} \cdot q_b + p_{ab}$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & & & p_{ab} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

$$\bar{q}_a = T_{ab} \cdot \bar{q}_b$$

A general coordinate frame transformation consists of two parts:

- Rotation Matrix R $\in \mathbb{R}^{3x3}$
- Ttranslation Vector p $\in \mathbb{R}^{3x1}$

$T_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0^{1x3} & 1 \end{bmatrix}$ is the transformation matrix of $\{B\}$ w.r.t $\{A\}$

3.4 Rigid Motion in $\mathbb{R}^3$

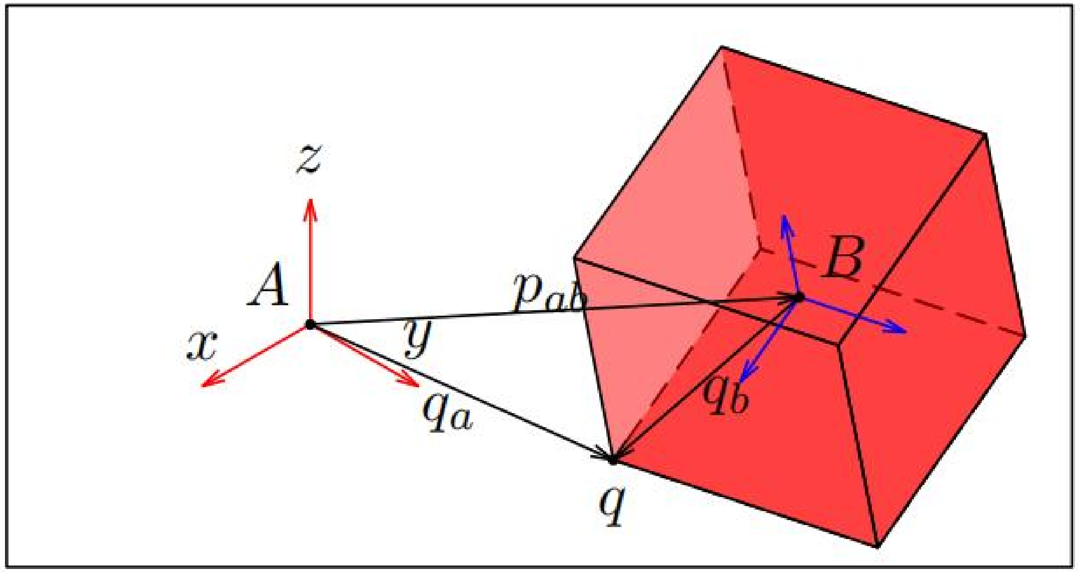


Figure 3.8

Review: $SO(3) = \{R \in \mathbb{R}^{3x3} | R^T R = I, det R = 1\}$

$p_{ab} \in \mathbb{R}^3$: Coordinates of the origin of B

$R_{ab} \in SO(3)$: Orientation of B relative to A

$SE(3) : \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| p \in \mathbb{R}^3, R \in SO(3) \right\}$: Orientation of B relative to A

Or...as a transformation:

$$g_{ab} = (p_{ab}, R_{ab}) : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

$$q_b \mapsto q_a = p_{ab} + R_{ab} \cdot q_b$$

3.4 Rigid Motion in $\mathbb{R}^3$ (Homogeneous Representation)

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \Rightarrow \quad \bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

1. Point-Point = Vector
2. Vector+Point = Point
3. Vector+Vector = Vector
4. Point+Point: Meaningless

3.4 Rigid Motion in $\mathbb{R}^3$ (Homogeneous Representation)

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$\boxed{g_{ab} = (p_{ab}, R_{ab})}$$

$$\boxed{\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}$$

- **Composition Rule:**

$$\bar{q}_b = \bar{g}_{bc} \cdot \bar{q}_c$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b = \underbrace{\bar{g}_{ab} \cdot \bar{g}_{bc}}_{\bar{g}_{ac}} \cdot \bar{q}_c$$

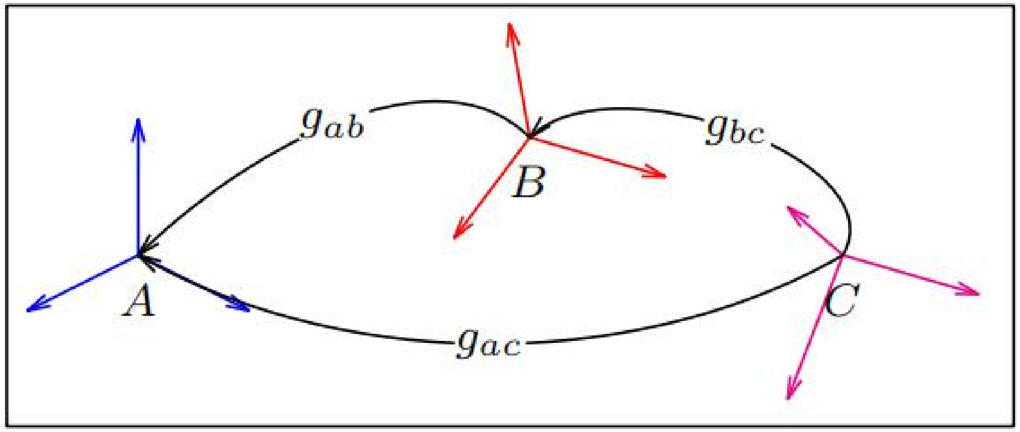


Figure 3.9

$$\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix}$$

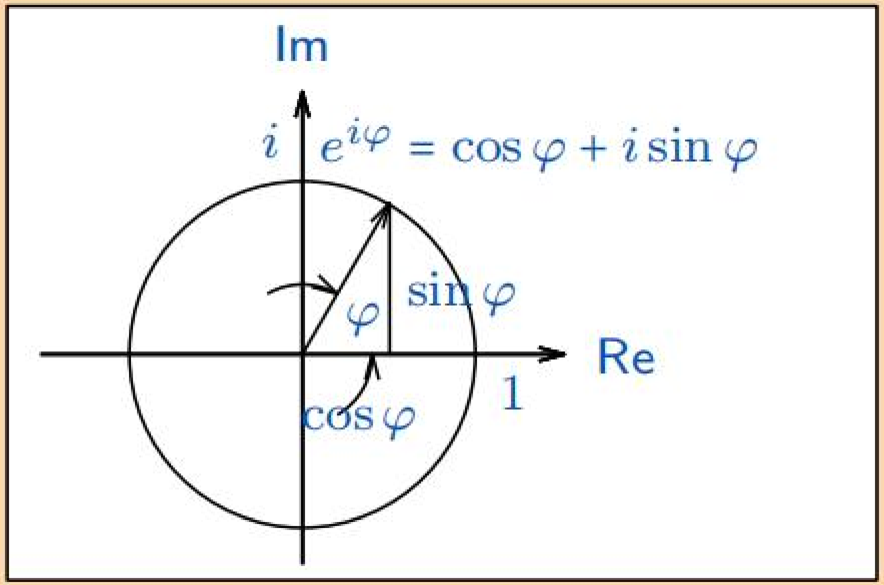


Figure 3.10

Euler's Formula

"One of the most remarkable, almost astounding, formulas in all of mathematics."

R. Feynman

◇ Review:

$$\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{at}x_0$$

3.5 Exponential Coordinates and Screw Theory (Parametrization of Rotation Matrix)

哈爾濱工業大學（深圳）
HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

$\Rightarrow$ 3 independent parameters!

Consider motion of a point q on a rotating link

$$\begin{cases} \dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

$$\Rightarrow q(t) = e^{\hat{\omega} t} q_0 \text{ where } e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$

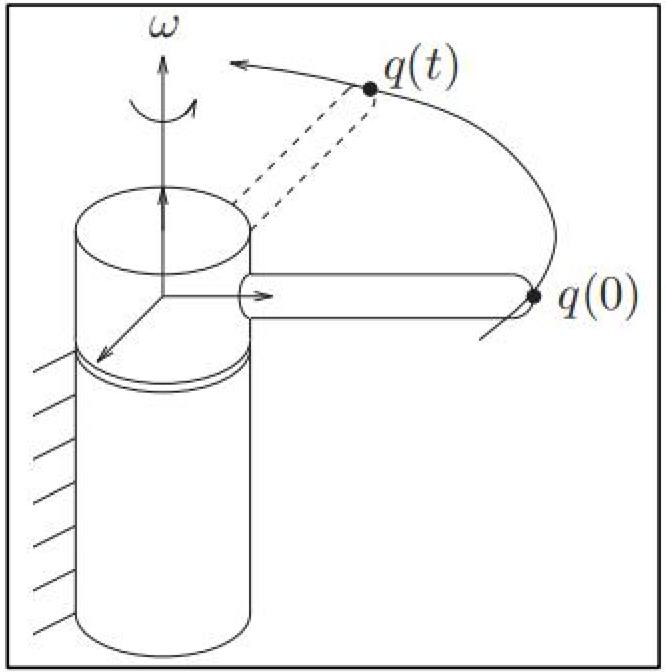


Figure 3.11

By the definition of rigid transformation, $R(\omega, \theta) = e^{\hat{\omega} \theta}$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Proof :

Let $a \in \mathbb{R}^3$, write

$$a = \omega\theta, \omega = \frac{a}{\|a\|}(\text{or } \|\omega\| = 1), \text{and } \theta = \|a\|$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \frac{(\hat{\omega}\theta)^3}{3!} + \cdots$$

As

$$\hat{a}^2 = aa^T - \|a\|^2 I, \hat{a}^3 = -\|a\|^2\hat{a}$$

we have:

$$e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^3}{5!} - \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots)\hat{\omega}^2$$

$$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

□

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

Proof for Property 3:

Let $R \triangleq e^{\hat{\omega}\theta}$, then:

$$(e^{\hat{\omega}\theta})^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = (e^{\hat{\omega}\theta})^T$$

$$\Rightarrow R^{-1} = R^T \Rightarrow R^T R = I \Rightarrow \det R = \pm 1$$

From $\det \exp(0) = 1$, and the continuity of $\det$ function w.r.t. θ, we have $\det e^{\hat{\omega}\theta} = 1, \forall \theta \in \mathbb{R}$ $\square$

- Let $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$, we have $\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{bmatrix}$ and $\widehat{\omega} = -\widehat{\omega}^T$

- Let $v\theta = 1 - cos\theta, c\theta = cos\theta, s\theta = sin\,\theta, ||\omega|| = 1$. By Rodrigues' formula,

$$e^{\widehat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v\theta + c\theta & \omega_1\omega_2 v\theta - \omega_3 s\theta & \omega_1\omega_3 v\theta + \omega_2 s\theta \\ \omega_1\omega_2 v\theta + \omega_3 s\theta & \omega_2^2 v\theta + c\theta & \omega_2\omega_3 v\theta - \omega_1 s\theta \\ \omega_1\omega_3 v\theta - \omega_2 s\theta & \omega_2\omega_3 v\theta + \omega_1 s\theta & \omega_3^2 v\theta + c\theta \end{bmatrix} = R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in SO(3)$$

- Let $tr(R) = r_{11} + r_{22} + r_{33} = \sum_{i=1}^{3} \lambda_i$,where λ_i is the eigenvalue of R, i $= 1,2,3$

Case 1: $\mathrm{tr}(R) = 3$ or $R = I$, $\theta = 0 \Rightarrow \omega\theta = 0$

Case 2: $-1 < \mathrm{tr}(R) < 3$,

$$\theta = \arccos \frac{\mathrm{tr}(R) - 1}{2} \Rightarrow \omega = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Case 3: $\mathrm{tr}(R) = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi$

3.5 Exponential Coordinates and Screw Theory (Exponential coordinates of SE(3))

For rotational motion:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\text{or } \dot{\bar{p}} = \hat{\xi} \cdot \bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi} t} \bar{p}(0)$$

$$\text{where } e^{\hat{\xi} t} = I + \hat{\xi} t + \frac{(\hat{\xi} t)^2}{2!} + \cdots$$

For translational motion:

$$\dot{p}(t) = v$$

$$\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\dot{\bar{p}}(t) = \hat{\xi} \cdot \bar{p}(t) \Rightarrow \bar{p}(t) = e^{\hat{\xi} t} \bar{p}(0)$$

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

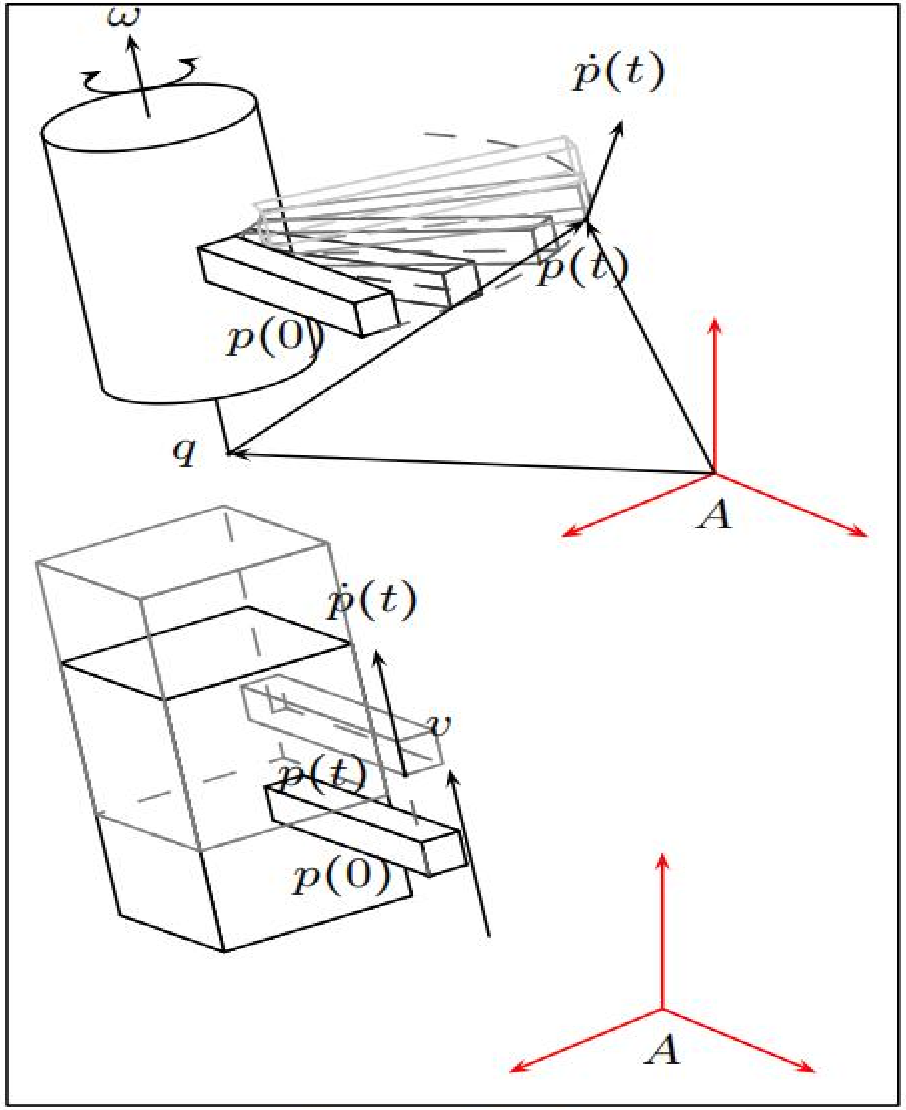


Figure 3.12

$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

- If $\omega = 0$, then $\hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0$, $e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$

- If ω is not 0, assume $\|\omega\| = 1$.

Define:

$$g_0 = \begin{bmatrix} I & \omega \times v \\ 0 & 1 \end{bmatrix}, \hat{\xi}' = g_0^{-1} \cdot \hat{\xi} \cdot g_0 = \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}$$

where $h = \omega^T \cdot v$.

$$e^{\hat{\xi}\theta} = e^{g_0 \cdot \hat{\xi}' \cdot g_0^{-1}} = g_0 \cdot e^{\hat{\xi}'\theta} \cdot g_0^{-1}$$

and as

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \hat{\xi}'^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix} \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

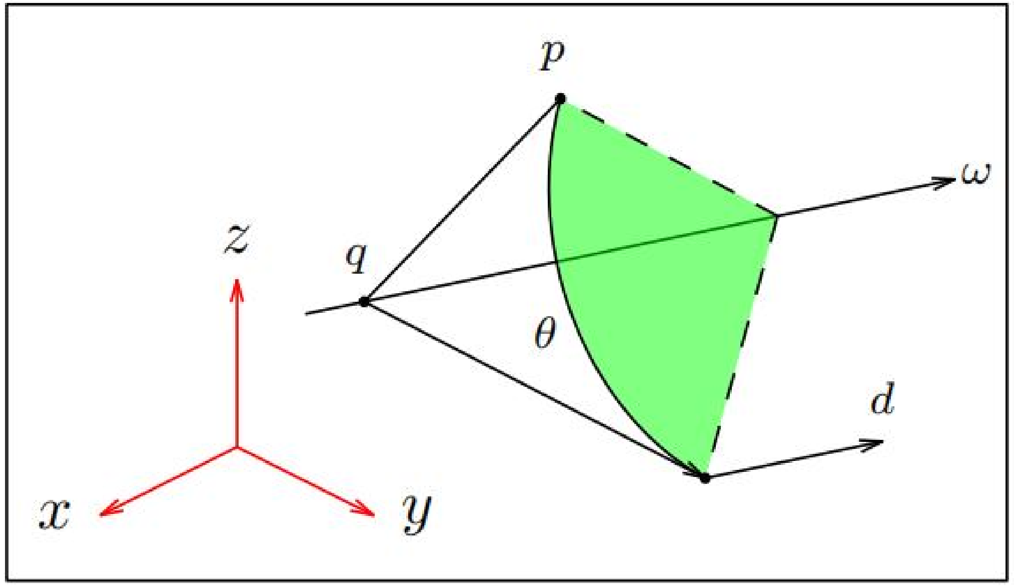


Figure 3.13

Screw attributes

Pitch: $h = \frac{d}{\theta} (\theta = 0, h = \infty), d = h \cdot \theta$

Axis: $l = \{q + \lambda\omega | \lambda \in \mathbb{R}\}$

Magnitude: $M = \theta$

Definition:

A **screw** S consists of an axis l, pitch h, and magnitude M. A **screw motion** is a rotation by $\theta = M$ about l, followed by translation by $h\theta$, parallel to l. If $h = \infty$, then, translation about v by $\theta = M$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

If we let $v = -\omega \times q + h\omega$, then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, $e^{\hat{\xi}\theta} = g$

For pure rotation ($h = 0$): $\xi = (-\omega \times q, \omega)$

For pure translation: $g = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$, $\Rightarrow \xi = (v, 0)$, and $e^{\hat{\xi}\theta} = g$

3.5 Exponential Coordinates and Screw Theory (Screw Theory)

$\xi = (v, \omega) \in \mathbb{R}^6$

1. Pitch: $h = \begin{cases} \dfrac{\omega^T v}{\|\omega\|^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0 \end{cases}$
2. Axis: $l = \begin{cases} \dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega, & \lambda \in \mathbb{R}, \text{if } \omega \neq 0 \\ 0 + \lambda v & \lambda \in \mathbb{R}, \text{if } \omega = 0 \end{cases}$
3. Magnitude: $M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|v\|, & \text{if } \omega = 0 \end{cases}$

Screw	Twist: $\hat{\xi}\theta$
Case 1: Pitch: $h = \infty$ Axis: $l = \{q + \lambda v \mid \|v\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M$, $\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$
Case 2: Pitch: $h \neq \infty$ Axis: $l = \{q + \lambda\omega \mid \|\omega\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M$, $\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q + h\omega \\ 0 & 0 \end{bmatrix}$

Special cases:

1. $h = \infty$, Pure translation (prismatic joint)
2. $h = 0$, Pure rotation (revolute joint)

Definition: Screw Motion
Rotation about an axis by $\theta = M$, followed by translation about the same axis by $h\theta$

Theorem 2 (Chasles):
Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.

1793–1880

Proof :

For $\hat{\xi} \in se(3)$:

$$\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2 = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h\omega \\ 0 & 0 \end{bmatrix}$$

$$[\hat{\xi}_1, \hat{\xi}_2] = 0 \Rightarrow e^{\hat{\xi}\theta} = e^{\hat{\xi}_1\theta} e^{\hat{\xi}_2\theta}$$

$\square$

3.6 Other Parametrizations of SO(3) (XYZ fixed angles)

哈爾濱工業大學(深圳) HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

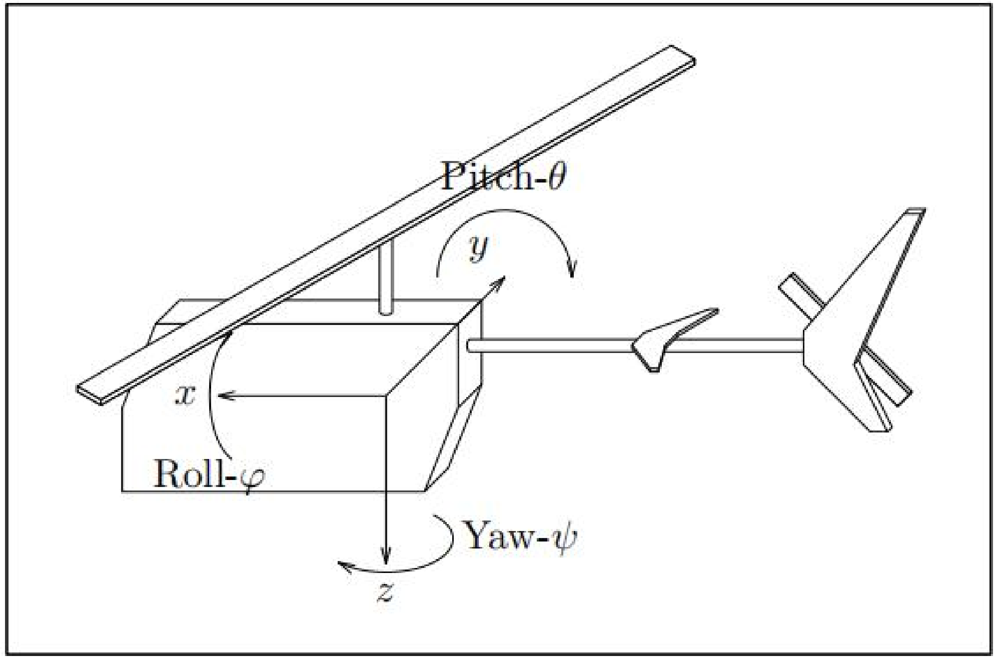


Figure 3.14

XYZ fixed angles (or Roll-Pitch-Yaw angle)

$$R_x(\varphi) := e^{\hat{x}\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$R_y(\theta) := e^{\hat{y}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) := e^{\hat{z}\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Rotate φ about x-axis
2. Rotate θ about y-axis
3. Rotate ψ about z-axis

$$R_{ab} = R_z(\psi)R_y(\theta)R_x(\varphi)$$

$$R_{ab} = \begin{bmatrix} c\psi c\theta & -s\psi c\varphi + c\psi s\theta s\varphi & s\psi s\varphi + c\psi s\theta c\varphi \\ s\psi c\theta & c\psi c\varphi + s\psi s\theta s\varphi & -c\psi s\varphi + s\psi s\theta c\varphi \\ -s\theta & c\theta s\varphi & c\theta c\varphi \end{bmatrix}$$

3.6 Other Parametrizations of SO(3) (ZYX Euler angle)

哈爾濱工業大學(深圳) HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

ZYX Euler angle

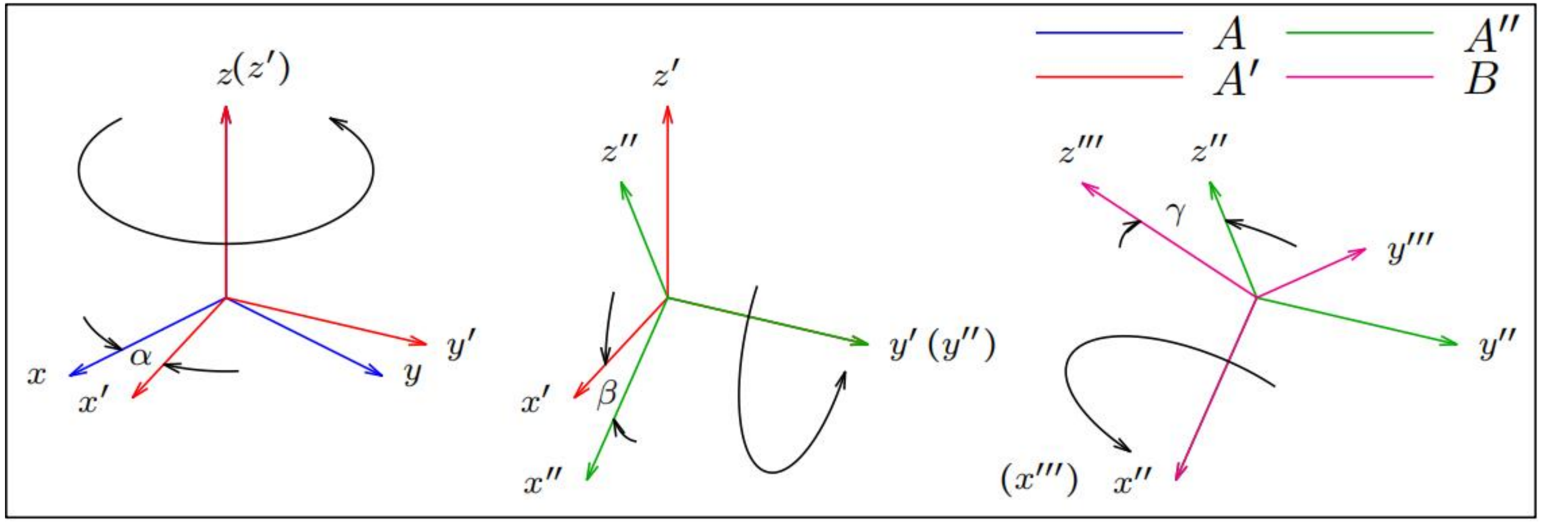


Figure 3.15

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$\alpha = \operatorname{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$$
$$\gamma = \operatorname{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$$

1. Rotate α about z-axis
2. Rotate β about y'-axis
3. Rotate γ about x''-axis

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta) \qquad R_{a''b} = R_x(\gamma)$$
$$R_{ab} = R_z(\alpha)R_y(\beta)R_x(\gamma)$$
$$R_{ab}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & -s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma \\ s_\alpha c_\beta & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & -c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

Note: When $\beta = \frac{\pi}{2}, \cos\beta = 0,\ \alpha + \gamma = \text{const} \Rightarrow$ singularity!

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$
$\|Q\|^2 = QQ^* = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$

Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0 p_0 - q \cdot p, q_0 p + p_0 q + q \times p)$

Property 3: (a) The set of unit quaternions forms a group
(b) If $R = e^{\hat{\omega}\theta}$, then $Q = (\cos\frac{\theta}{2}, \omega \sin\frac{\theta}{2})$
(c) Q acts on $x \in \mathbb{R}^3$ by QXQ^*, where $X = (0, x)$

- $$q_0 = \cos\frac{\theta}{2}, q = \omega \sin\frac{\theta}{2}$$

  and the Rodrigues' formula:

  $$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

  then

  $$R(Q) = I + 2q_0\hat{q} + 2\hat{q}^2$$
  $$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & -2q_0q_3+2q_1q_2 & 2q_0q_2+2q_1q_3 \\ 2q_0q_3+2q_1q_2 & 1-2(q_1^2+q_3^2) & -2q_0q_1+2q_2q_3 \\ -2q_0q_2+2q_1q_3 & 2q_0q_1+2q_2q_3 & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

  where $\|Q\| \triangleq q_0^2+q_1^2+q_2^2+q_3^2 = 1$

- Conversion from Roll-Pitch-Yaw angle to unit quaternions

$$Q = (\cos\frac{\varphi}{2}, x\sin\frac{\varphi}{2})(\cos\frac{\theta}{2}, y\sin\frac{\theta}{2})(\cos\frac{\psi}{2}, z\sin\frac{\psi}{2})$$

References

哈爾濱工業大學(深圳)
HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

□ Reference:

[1] Murray, R.M. and Li, Z.X. and Sastry, S.S., **A mathematical introduction to robotic manipulation.** CRC Press, 1994.

[2] Ball, R.S., **A treatise on the theory of screws.** University Press, 1900.

[3] Bottema, O. and Roth, B. , **Theoretical kinematics.** Dover Publications, 1990.

[4] Craig, J.J., **Introduction to robotics : mechanics and control, 3rd ed.** Prentice Hall, 2004.

[5] Fu, K.S. and Gonzalez, R.C. and Lee, C.S.G., **Robotics : control, sensing, vision, and intelligence.** CAD/CAM, robotics, and computer vision. McGraw-Hill, 1987.

[6] Hunt, K.H., **Kinematic geometry of mechanisms.** 1978, Oxford, New York: Clarendon Press, 1978.

[7] Paul, R.P., **Robot manipulators : mathematics, programming, and control.** The MIT Press series in artificial intelligence. MIT Press, 1981.

[8] Park, F. C., **A first course in robot mechanics.** Available online, 2006.

[9] Tsai, L.-W., **Robot analysis : the mechanics of serial and parallel manipulators.** Wiley, 1999.

[10] Spong, M.W. and Hutchinson, S. and Vidyasagar, M. , **Robot modeling and control.** John Wiley & Sons, 2006.

[11] Selig, J., **Geometric Fundamentals of Robotics**. Springer, 2008.