Model Predictive Control — Lecture 7
Moving Horizon Estimation MHE

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The Need for State Estimation / State Observer

- If we cannot measure the full state vector, or some of the states in the state vector cannot be measured directly, then an observer/state estimator can be used to estimate them.
- In this course, instead of elaborating on state estimator theory, we will focus on using the Moving Horizon Estimation concept to implement the State Estimator/Observer
- MHE can be considered as a dual of MPC, and it is natural to use MHE to build the estimate the state of the process.

Review: Observability[a]

[a] A related concept is detectability. A system is detectable iff all of its unobservable modes are stable. Observability implies detectability.

- Consider the system with zero input

$$x_{k+1} = Ax_k; \quad y_k = Cx_k$$

- A system is said to be observable if there exist a finite N such that for every x_0, the measurements $y_0, \ y_1, \ \ldots, \ y_{N-1}$ uniquely distinguish the initial state x_0.
- Hence, from linear algebra, the necessary and sufficient condition for observability of system (A, C), where $A \in \mathcal{R}^{n \times n}$ is

$$\mathrm{rank}(\mathcal{O}) = n$$

where

$$\mathcal{O} = [C^T \ (CA)^T \ \ldots \ (CA^{N-1})^T]^T$$

Review: Observer Structure

- An observer is a copy of the plant, with feedback from measured plant output to correct the state estimate.
- Given a plant

$$x_{k+1} = Ax_k + Bu_k; \quad y_k = Cx_k$$

The equations of the observer are:

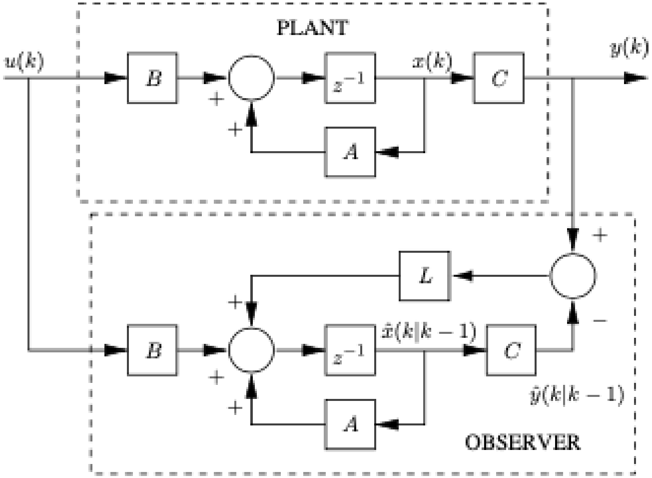
$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + \tilde{L}(y_k - \hat{y}_{k|k-1}) \\ \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\ \hat{y}_{k|k-1} &= C\hat{x}_{k|k-1} \end{aligned}$$

- $\hat{x}_{k|k}$ in the observer equations can be eliminated to give a more compact form

$$\begin{aligned} \hat{x}_{k+1|k} &= A(I - \tilde{L}C)\hat{x}_{k|k-1} + Bu_k + A\tilde{L}y_k \\ &= (A - LC)\hat{x}_{k|k-1} + Bu_k + Ly_k \end{aligned}$$

where $L = A\tilde{L}$.

Review: Observer Structure



Review: Observer Structure

- The observer is stable if the eigenvalues of $A - LC$ lie inside the unit disk. Furthermore, it can be shown that the state estimation error converges to zero:

$$e_{k+1} = (A - LC)e_k$$

where $e_k = x_k - \hat{x}_{k|k-1}$.

Moving Horizon Estimation

Receding horizon recursive state estimation KV Ling, KW Lim - IEEE Transactions on Automatic Control, 1999

- Various observer design methods: place the poles of the observer, Kalman filter (weighting matrices, noise model), etc
- MHE: Use N, the number of past measurements (window length/horizon), as a tuning parameter
- Straightforward extension to multi-rate systems[1]
- Similar to MPC, can include constraints and solve an online optimisation problem

[1] K.V. Ling, X.J. Wu, K.W. Lim, State Observers for Single Rate and Multirate Controllers, IFAC Proceedings Volumes, Volume 29, Issue 1, 1996, Pages 1321-1326, ISSN 1474-6670, https://doi.org/10.1016/S1474-6670(17)57849-0.

Moving Horizon Estimation

The same can be applied to incremental model $\xi_{k+1} = A\xi_k + B\Delta u_k; \quad y_k = C\xi_k$

Given the plant model

$$x_{k+1} = A_p x_k + B_p u_k; \quad y_k = C_p x_k$$

We can write

$$\underbrace{\begin{bmatrix} y_{k-N} \\ y_{k-N+1} \\ \vdots \\ y_k \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C_p \\ C_p A_p \\ \vdots \\ C_p A_p^N \end{bmatrix}}_{\Phi} x_{k-N} + \underbrace{\begin{bmatrix} 0 & & \\ C_p B_p & & \\ C_p A_p B_p & C_p B_p & \\ \vdots & & \\ C_p A_p^{N-1} B_p & \dots & C_p B_p \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} u_{k-N} \\ u_{k-N+1} \\ \vdots \\ u_{k-1} \end{bmatrix}}_{U}$$

$$\Rightarrow \hat{x}_{k-N} = (\Phi^T \Phi)^{-1} \Phi^T (Y - \Gamma U)$$

$$\Rightarrow \hat{x}_k = A_p^N \hat{x}_{k-N} + \underbrace{\begin{bmatrix} A_p^{N-1} B_p & A_p^{N-2} B_p & \dots & B_p \end{bmatrix}}_{F} U$$

$$= \underbrace{A_p^N (\Phi^T \Phi)^{-1} \Phi^T}_{M} Y + \underbrace{(-A_p^N (\Phi^T \Phi)^{-1} \Phi^T \Gamma) + F)}_{N} U$$

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MHE, Multi-Rate[a]

[a]K.V. Ling, X.J. Wu, K.W. Lim, State Observers for Single Rate and Multirate Controllers, IFAC Proceedings Volumes, Volume 29, Issue 1, 1996, Pages 1321-1326, ISSN 1474-6670, https://doi.org/10.1016/S1474-6670(17)57849-0.

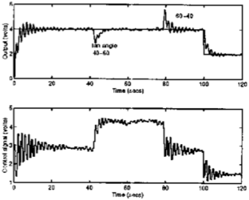


Fig. 2. Multirate output with $m = 2$, $N = 8$ (batch implementation).

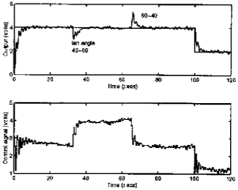
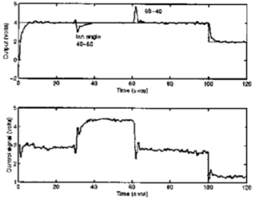


Fig. 5. Multirate output with $m = 2$, $N = 6$ (recursive implementaion).



ate output with $m = 4$, $N = 16$ (batch tation).

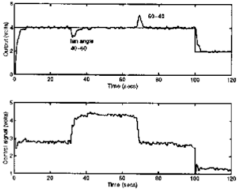


Fig. 6. Multirate output with $m = 4$, $N = 12$ (recursive implementation).

MHE and Kalman Filter

For simplicity, consider the system

$$x_{k+1} = Ax_k; \quad y_k = Cx_k$$

and the state observer

$$\hat{x}_{k+1} = (A - L_N CA)\hat{x}_k + L_N y_{k+1} - \tilde{L_N}(y_{k-N} - cA^{-N}\hat{x}_k)$$

where

$$L_N = A^N P_N (CA^N)^T, \quad \tilde{L}_N = \alpha A^N P_N (CA^{-1})^T$$

$$P_N = \left(\begin{bmatrix} CA^N \\ CA^{N-1} \\ \vdots \\ C \end{bmatrix}^T \begin{bmatrix} CA^N \\ CA^{N-1} \\ \vdots \\ C \end{bmatrix} \right)^{-1}$$

Set $N \geq n - 1$. KF ($\alpha = 0$); MHE ($\alpha = 1$).