机器人学导论

楼云江

哈尔滨工业大学（深圳）

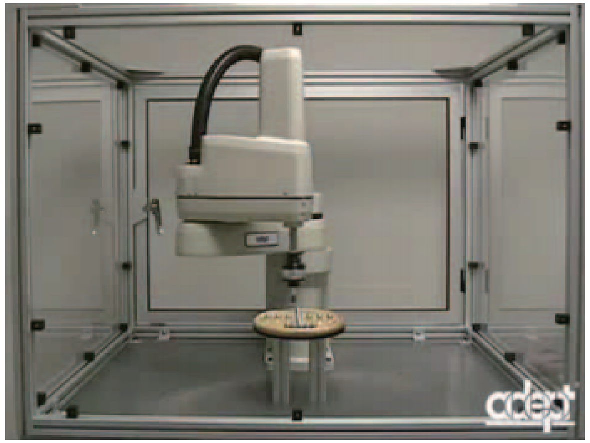
2023春季

[5] HIT 操作臂运动学 Manipulator Kinematics

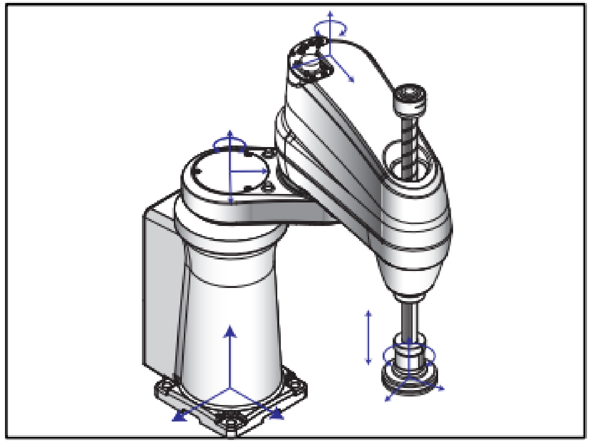
Table of Contents

Chapter 5 Manipulator Kinematics

5.1 Forward Kinematics



(a) Adept Cobra i600 (SCARA)

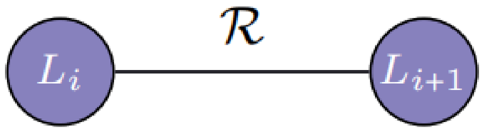


(b) Forward kinematics of SCARA

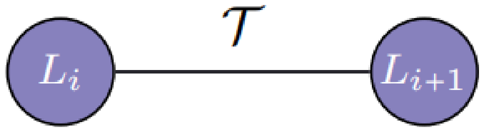
Figure 5.1

◇ Lower Pair Joints:

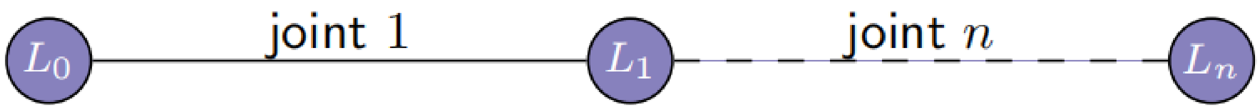
revolute joint $S^1 \mapsto SO(2)$



prismatic joint $\mathbb{R} \mapsto T(1)$



◇ Forward kinematics:



5.1 Forward Kinematics (Joint Space)

Revolute joint: $S^1, \theta_i \in S^1$ or $\theta_i \in [-\pi, \pi]$

Prismatic joint: $\mathbb{R}$

Joint space: $Q : \underbrace{S^1 \times \cdots \times S^1}_{\text{no. of } R \text{ joint}} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{no. of } P \text{ joint}}$

Adept $Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow $Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_{6}$

Reference (nominal) joint config: $\theta = (0, 0, \ldots, 0) \in Q$

Reference (nominal) end-effector config: $g_{st}(0) \in SE(3)$

Arbitrary configuration $g_{st}(\theta)$:

$$g_{st} : \theta \in Q \mapsto g_{st}(\theta) \in SE(3)$$

□ Classical Approach:

$$g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$$

Disadvantage: A coordinate frame needed for each link

□ The product of exponentials formula:

Consider **Fig 5.2.**

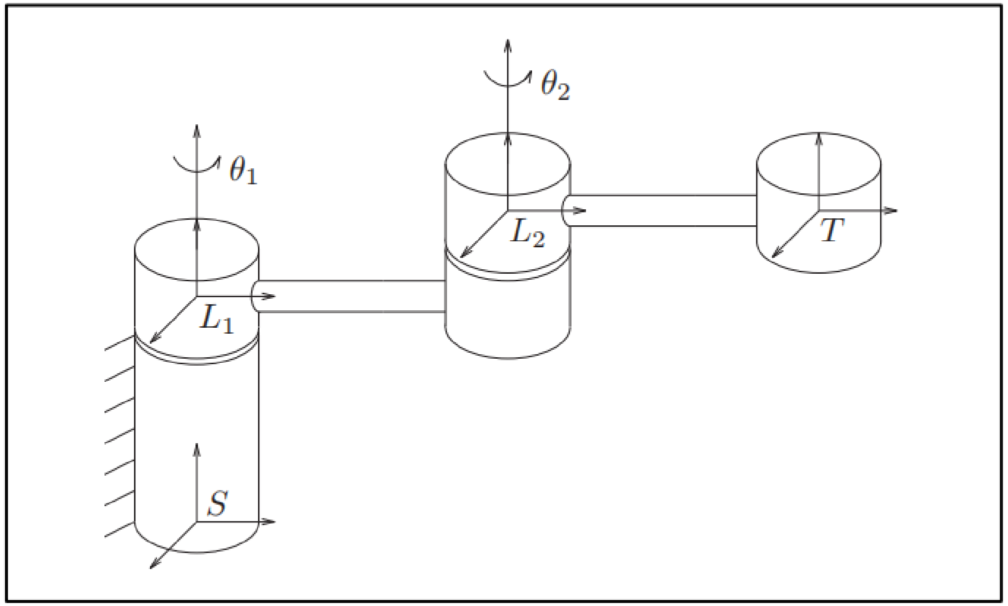


Figure 5.2: A two degree of freedom manipulator

Step 1: Rotating about ω_2 by θ_2

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

$$g_{st}(\theta_2) = e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)$$

Step 2: Rotating about ω_1 by θ_1

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} \cdot \underbrace{e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)}_{\text{offset}}$$

$$\theta : (0, 0) \mapsto (0, \theta_2) \mapsto (\theta_1, \theta_2)$$

What if another route is taken?

$$\theta : (0,0) \mapsto (\theta_1, 0) \mapsto (\theta_1, \theta_2)$$

Step 1: Rotating about ω_1 by θ_1

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0)$$

Step 2: Rotating about ω_2' by θ_2

Let $e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$

$$\omega_2' = R_1 \cdot \omega_2$$

$$q_2' = p_1 + R_1 \cdot q_2$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} -R_1\hat{\omega}_2 R_1^T(p_1 + R_1 q_2) \\ R_1\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = Ad_{e^{\hat{\xi}_1\theta_1}} \cdot \xi_2 \Rightarrow$$

$$\hat{\xi}_2' = e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2 \cdot e^{-\hat{\xi}_1\theta_1}$$

$$\begin{aligned} g_{st}(\theta_1, \theta_2) &= e^{\hat{\xi}_2'\theta_2} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2\theta_2 \cdot e^{-\hat{\xi}_1\theta_1}} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{-\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= \underbrace{e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot g_{st}(0)}_{\text{Independent of the route taken}} \end{aligned}$$

Identify a nominal configuration:

$$\Theta = (\theta_{10}, \ldots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \ldots, \theta_{n0})$$

Simplification of forward kinematics mapping:

Revolute joint: $\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}$

Choose q_i s.t. ξ_i is simple.

Prismatic joint: $\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$

Write $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_n \theta_n} \cdot g_{st}(0)$ (product of exponential mapping)

5.1 Forward Kinematics (Example: SCARA manipulator)

$$g_{st}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ l_1 + l_2 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$\omega_1 = \omega_2 = \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

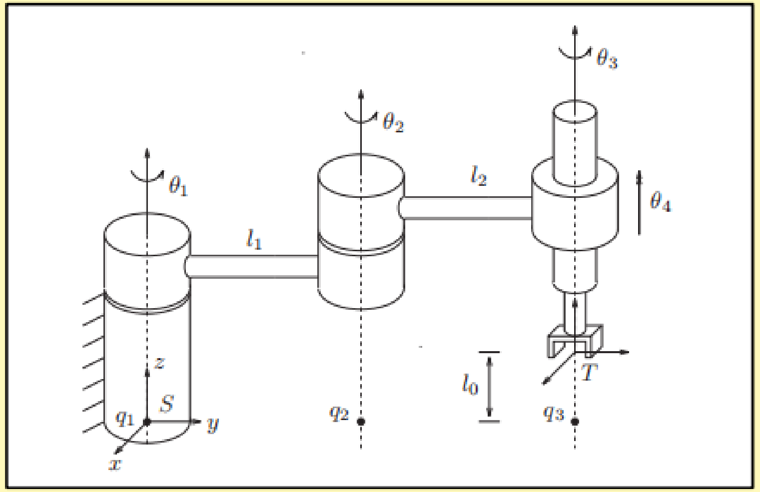


Figure 5.3

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(see next page)

5.1 Forward Kinematics (Example: SCARA manipulator)

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{\hat{\xi}_3\theta_3} \cdot e^{\hat{\xi}_4\theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 & -l_1 s_1 \\ s_2 & c_2 & 0 & l_1 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3\theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 & -l_1 s_1 - l_2 c_{12} \\ s_3 & c_3 & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_4\theta_4} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ \theta_4 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$g_{st}(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & -l_1 s_1 - l_2 s_{12} \\ s_{123} & c_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & l_0 + \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in which, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$ and $c_{12} = \cos(\theta_1 + \theta_2)$. ◇

$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

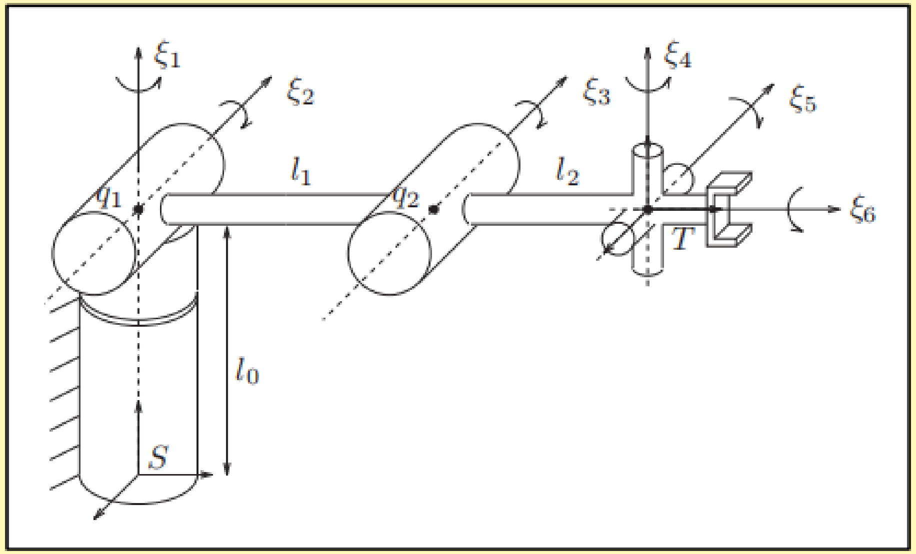


Figure 5.4

$$\xi_1 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

(Continues next slide)

5.1 Forward Kinematics (Example: Elbow manipulator)

$$\xi_2 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_4 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\xi_5 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 + l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_6 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\rightarrow g_{st}(\theta_1, \ldots \theta_6) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_6\theta_6} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$p(\theta) = \begin{bmatrix} -s_1(l_2c_2 + l_2c_{23}) \\ c_1(l_1c_2 + l_2c_{23}) \\ l_0 - l_1s_2 - l_2s_{23} \end{bmatrix}, R(\theta) = [r_{ij}]$$

(see next page)

in which,

$$r_{11} = c_6(c_1c_4 - s_1c_{23}s_4) + s_6(s_1s_{23}c_5 + s_1c_{23}c_4s_5 + c_1s_4s_5)$$
$$r_{12} = -c_5(s_1c_{23}c_4 + c_1s_4) + s_1s_{23}s_5$$
$$r_{13} = c_6(-c_5s_1s_{23} - (c_{23}c_4s_1 + c_1s_4)s_5) + (c_1c_4 - c_{23}s_1s_4)s_6$$
$$r_{21} = c_6(c_4s_1 + c_1c_{23}s_4) - (c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5)s_6$$
$$r_{22} = c_5(c_1c_{23}c_4 - s_1s_4) - c_1s_{23}s_5$$
$$r_{23} = c_6(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5) + (c_4s_1 + c_1c_{23}s_4)s_6$$
$$r_{31} = -(c_6s_{23}s_4) - (c_{23}c_5 - c_4s_{23}s_5)s_6$$
$$r_{32} = -(c_4c_5s_{23}) - c_{23}s_5$$
$$r_{33} = c_6(c_{23}c_5 - c_4s_{23}s_5) - s_{23}s_4s_6$$

Simplify forward Kinematics Map:

Choose base frame or ref. Config. s.t. $g_{st}(0) = I$ $\diamond$

5.1 Forward Kinematics (Manipulator Workspace)

$$W = \{g_{st}(\theta) | \forall \theta \in Q\} \subset SE(3)$$

- Reachable Workspace:

$$W_R = \{p(\theta) | \forall \theta \in Q\} \subset \mathbb{R}^3$$

- Dextrous Workspace:

$$W_D = \{p \in \mathbb{R}^3 | \forall R \in SO(3), \exists \theta, g_{st}(\theta) = (p, R)\}$$

(a) Workspace calculation:

$$g = (x, y, \phi)$$
$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$
$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

(b) Construction of Workspace:

(c) Reachable Workspace:

(d) Dextrous Workspace:

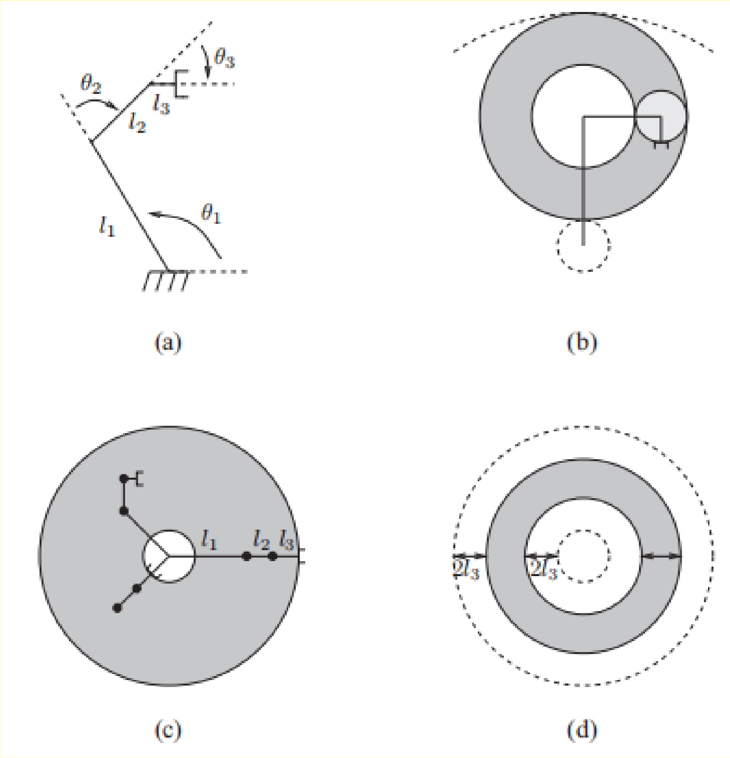


Figure 5.5

□ $6\mathcal{R}$ **manipulator with max workspace (Paden):**

Elbow manipulator and its kinematics inverse.

Definition: Inverse kinematics

Given $g \in SE(3)$, find $\theta \in Q$ s.t.

$$g_{st}(\theta) = g, \quad \text{where } g_{st} : Q \mapsto SE(3)$$

◇ Example: A planar example

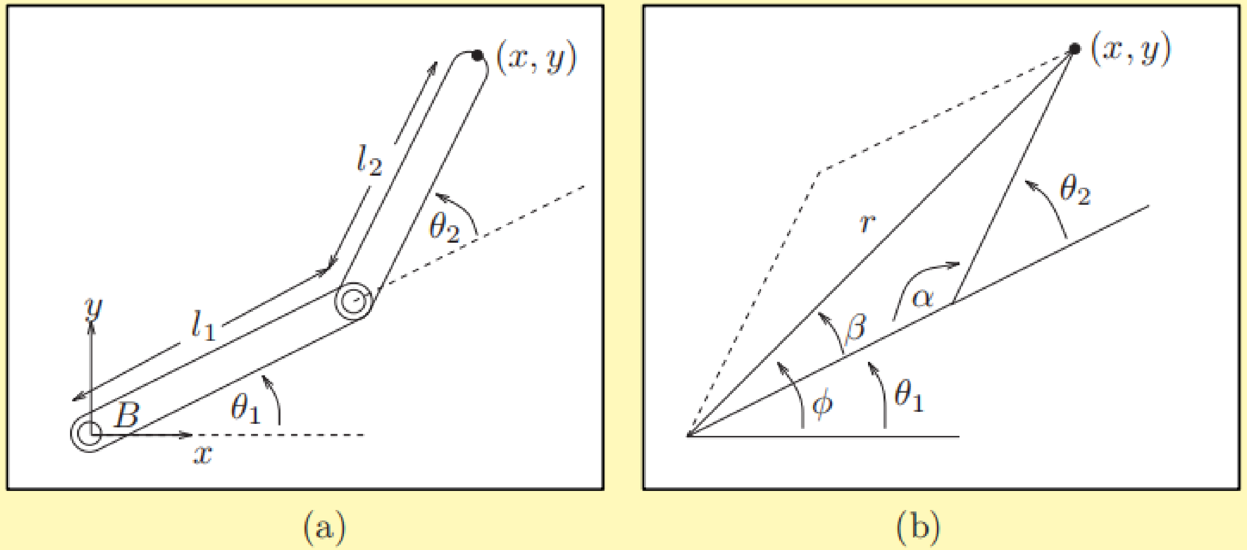


Figure 5.6

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Given (x, y), solve for (θ_1, θ_2).

5.2 Inverse Kinematics

◇ Review:

Polar Coordinates:

$$(r, \phi), r = \sqrt{x^2 + y^2}$$

Law of cosines:

$$\theta_2 = \pi \pm \alpha, \alpha = \cos^{-1} \frac{l_1^2 + l_2^2 - r^2}{2l_1 l_2}$$

Flip solution: $\pi + \alpha$

$$\theta_1 = \operatorname{atan2}(y, x) \pm \beta, \beta = \cos^{-1} \frac{r^2 + l_1^2 - l_2^2}{2l_1 r}$$

Hight Lights:

- Subproblems
- Each has zero, one or two solutions!

哈爾濱工業大學(深圳) HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist, with unit magnitude and two points $p, q \in \mathbb{R}^3$.

Find θ s.t. $e^{\hat{\xi}\theta} p = q$

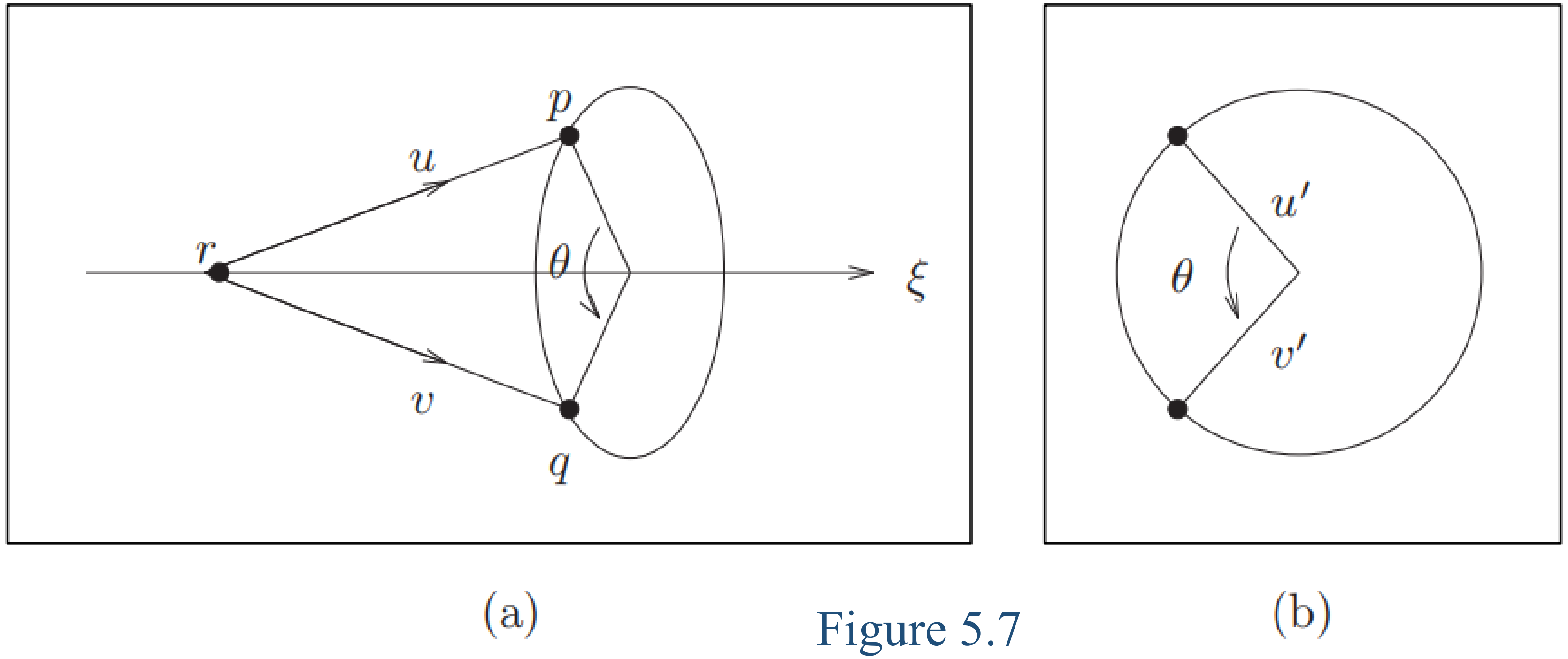


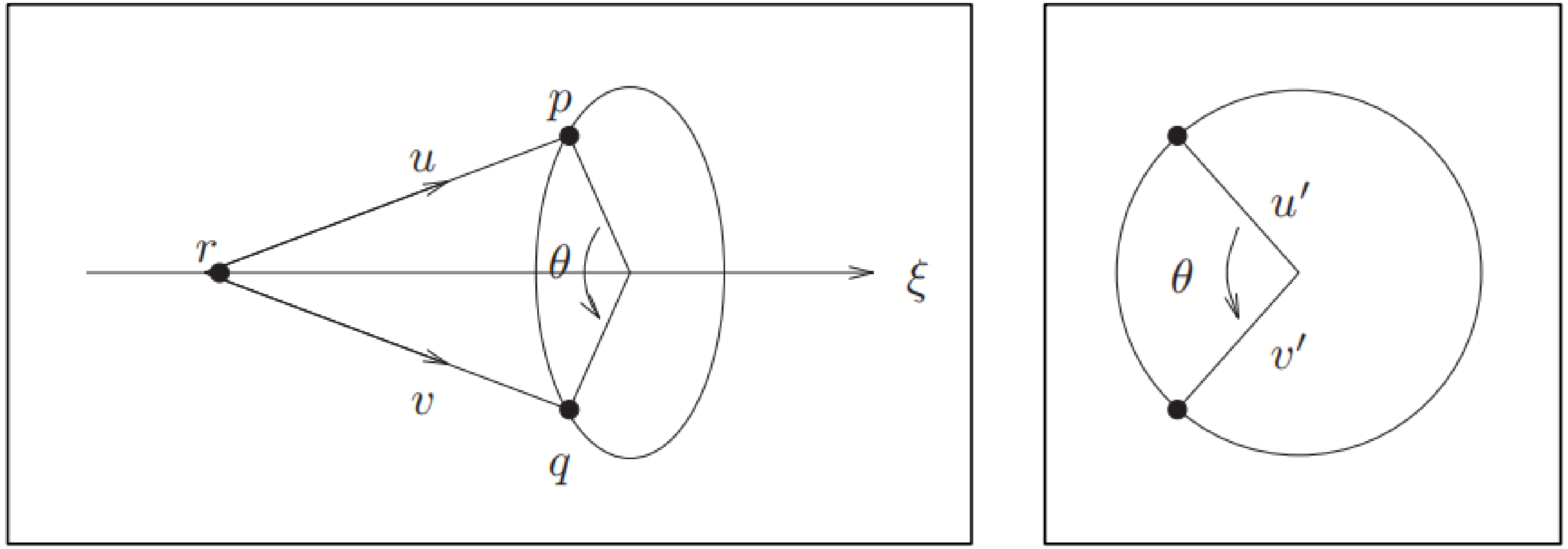
Figure 5.7

Solution: Let $r \in l_\xi$, define $u = p - r, v = q - r, e^{\hat{\xi}\theta} r = r$

Moreover,

$$\Rightarrow e^{\hat{\xi}\theta} p = q \Rightarrow e^{\hat{\xi}\theta} \underbrace{(p-r)}_{u} = \underbrace{q-r}_{v} \Rightarrow \begin{bmatrix} e^{\hat{\omega}\theta} & * \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{\hat{\omega}\theta} u = v \qquad \begin{cases} w^T u = w^T v \\ \|u\|^2 = \|v\|^2 \end{cases}$$



(a) Figure 5.7 (b)

$$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$$

The solution exists only if $\begin{cases} \|u'\|^2 = \|v'\|^2 \\ \omega^T u = \omega^T v \end{cases}$

- If $u' \neq 0$, then

$$u' \times v' = \omega \sin\theta \|u'\| \|v'\|$$
$$u' \cdot v' = \cos\theta \|u'\| \|v'\|$$

$$\Rightarrow \theta = \text{atan2}(\omega^T(u' \times v'), u'^T v')$$

- If $u' = 0$, $\Rightarrow$ Infinite number of solutions! $\diamond$

Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists, with intersecting axes, and $p, q \in R^3$. find θ_1 and θ_2 s.t. $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p = q$.

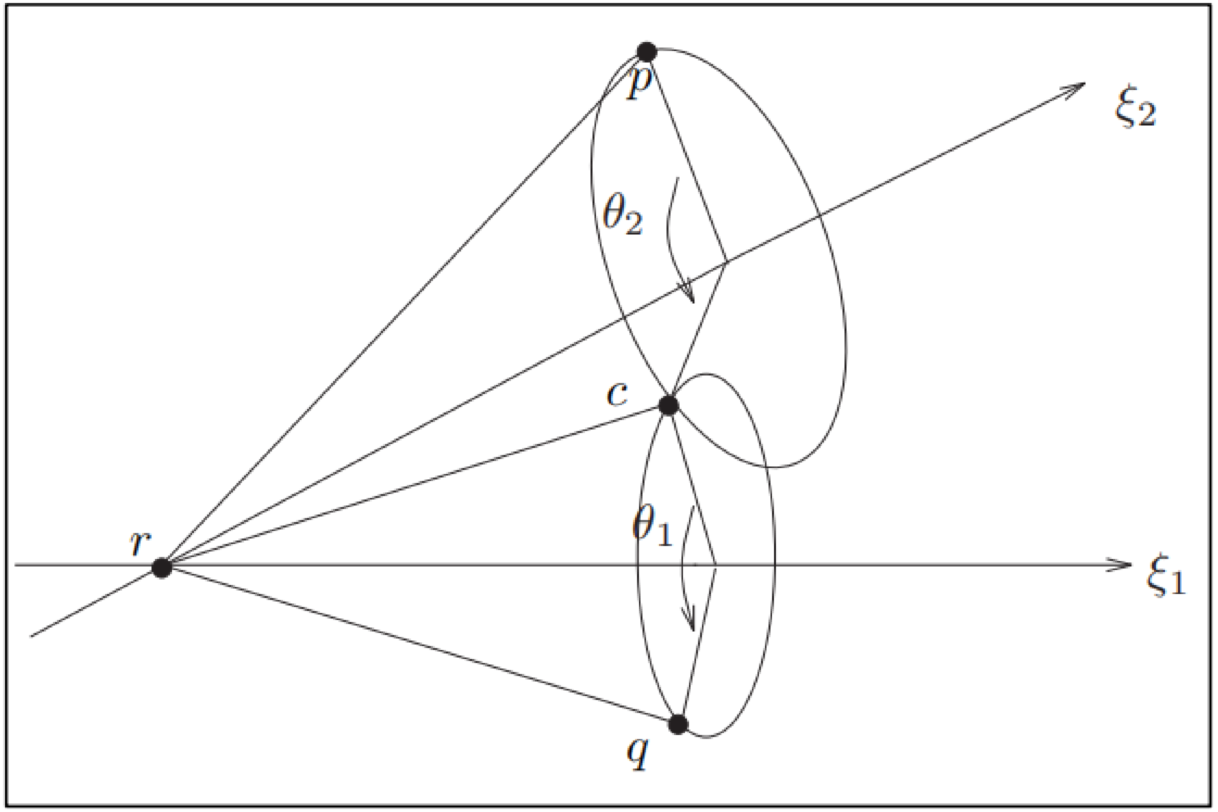


Figure 5.8

Solution: If two axes of ξ_1 and ξ_2 coincide, then we get:

Subproblem 1: $\theta_1 + \theta_2 = \theta$

If the two axes are not parallel, $\omega_1 \times \omega_2 \neq 0$, then, let c satisfy:

$$e^{\hat{\xi}_2\theta_2} p = c = e^{-\hat{\xi}_1\theta_1} q$$

Set $r \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_2\theta_2} \underbrace{p - r}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1\theta_1} \underbrace{(q - r)}_{v}, \Rightarrow e^{\hat{\omega}_2\theta_2} u = z = e^{-\hat{\omega}_1\theta_1} v$$

$$\Rightarrow \begin{cases} \omega_2^T u = \omega_2^T z \\ \omega_1^T v = \omega_1^T z \end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

As ω_1, ω_2 and $\omega_1 \times \omega_2$ are linearly independent,

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2)$$

$$\Rightarrow \|z\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\omega_1^T\omega_2 + \gamma^2\|\omega_1 \times \omega_2\|^2$$

$$\begin{array}{l} \omega_2^T u = \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v = \alpha + \beta \omega_1^T \omega_2 \end{array} \Rightarrow \begin{cases} \alpha & = \dfrac{(\omega_1^T \omega_2)\omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta & = \dfrac{(\omega_1^T \omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T\omega_2}{\|\omega_1 \times \omega_2\|^2} \quad (*)$$

$(*)$ has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases} e^{\hat{\xi}_2\theta_2} p = c \\ e^{\hat{\xi}_1\theta_1} p = c \end{cases}$$

for θ_1 and θ_2

1. Two solutions when the two circles intersect.
2. One solution when they are tangent
3. Zero solution when they do not intersect

Subproblem 3: Rotation to a given point

Given a zero-pitch twist ξ, with unit magnitude and $p, q \in \mathbb{R}^3$, find θ s.t.

$$\|q - e^{\hat{\xi}\theta} p\| = \delta$$

Define: $u = p - r, v = q - r, \|v - e^{\hat{\omega}\theta} u\|^2 = \delta^2$

$$u' = u - \omega\omega^T u$$

$$v' = v - \omega\omega^T v$$

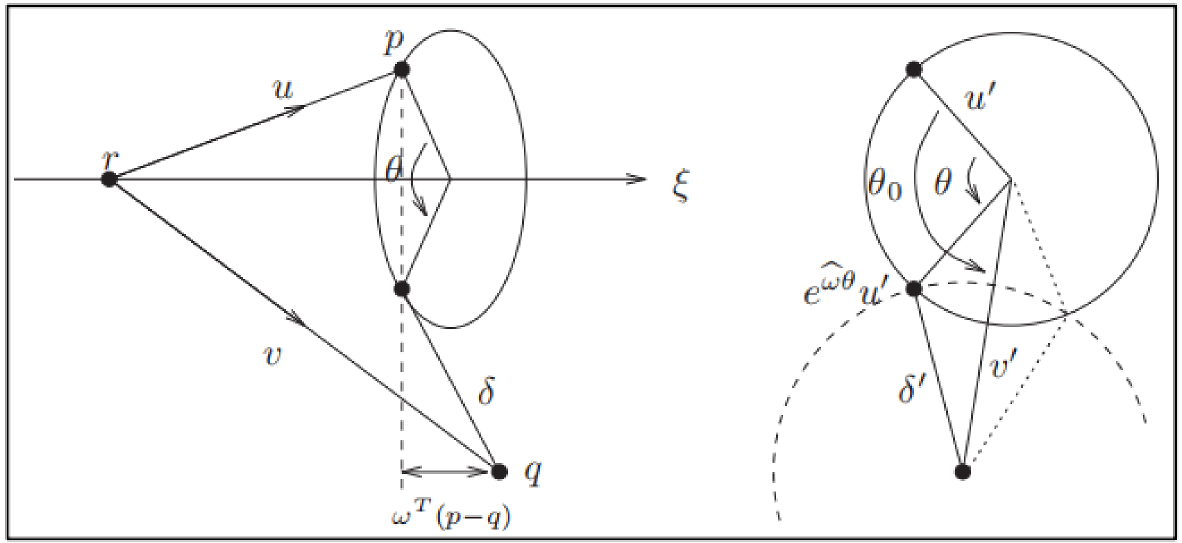


Figure 5.9

$$\Rightarrow u = u' + \omega\omega^T u, v = v' + \omega\omega^T v, \delta'^2 = \delta^2 - |\omega^T(p-q)|^2$$

$$\|(v' + \omega\omega^T v) - e^{\hat{\omega}\theta}(u' + \omega\omega^T u)\|^2 = \delta^2 \Rightarrow$$

$$\|v' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(v-u)}_{\omega\omega^T(q-p)}\|^2 = \delta^2$$

$$\|v' - e^{\hat{\omega}\theta}u'\|^2 = \delta^2 - \|\omega^T(p-q)\|^2 = \delta'^2,$$

$$\theta_0 = \operatorname{atan2}(\omega^T(u' \times v'), u'^T v'),$$

$$\phi = \theta_0 - \theta \Rightarrow \|u'\|^2 + \|v'\|^2 - 2\|u'\| \cdot \|v'\| \cos\phi = \delta'^2,$$

$$\theta = \theta_0 \pm \cos^{-1}\frac{\|u'\| + \|v'\| - \delta'^2}{2\|u'\| \cdot \|v'\|} \quad (*)$$

Zero, one or two solutions! ◊

Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta}p = p$, if $p \in l_{\xi}$. Given $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g$, select $p \in l_{\xi_3}$, $p \notin l_{\xi_1}$ or l_{ξ_2}, then:

$$gp = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p$$

Technique 2: subtract a common point

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g, q \in l_{\hat{\xi}_1} \cap l_{\hat{\xi}_2} \Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p - q = gp - q \Rightarrow$$

$$\|e^{\hat{\xi}_3\theta_3}p - q\| = \|gp - q\|$$

哈爾濱工業大學(深圳)
HARBIN INSTITUTE OF TECHNOLOGY, SHENZHEN

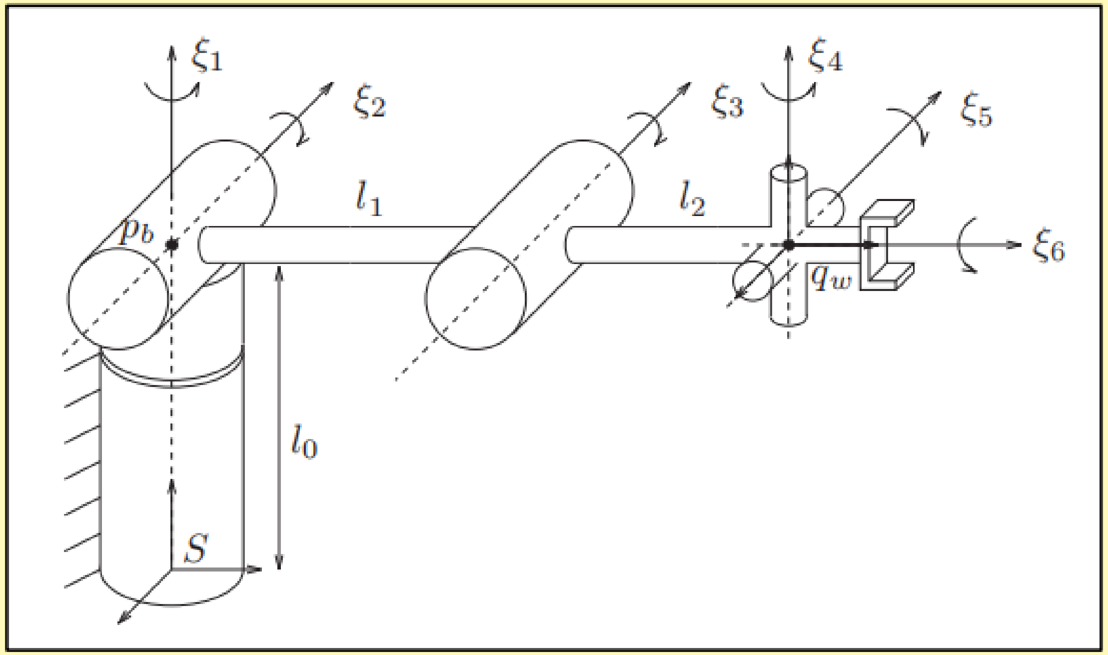


Figure 5.4

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d$$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g_d \cdot g_{st}^{-1}(0) = g_1$$

Step 1: Solve for θ_3

Let $e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_6\theta_6}q_\omega = g_1 \cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_\omega = g_1 \cdot q_\omega$$

Subtract p_b from $g_1 q_\omega$:

$$\|e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega - p_b)\| = \|g_1 q_\omega - p_b\|$$

$$\Rightarrow \|e^{\hat{\xi}_3\theta_3}q_\omega - p_b\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

Step 2: Given θ_3, solve for θ_1, θ_2

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega) = g_1 q_\omega$, Subproblem 2 $\Rightarrow \theta_1, \theta_2$

Step 3: Given $\theta_1, \theta_2, \theta_3$, solve θ_4, θ_5

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1}_{g_2}$$

let $p \in l_{\xi_6}, p \notin l_{\xi_4}$ or $l_{\xi_5}, e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = g_2 p,$
Subproblem 2 $\Rightarrow \theta_4$ and θ_5.

Step 4: Given $(\theta_1, \ldots, \theta_5)$, solve for θ_6

$$e^{\hat{\xi}_6\theta_6} = (e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_5\theta_5})^{-1} \cdot g_1 \triangleq g_3$$

Let $p \notin l_{\xi_6} \Rightarrow e^{\hat{\xi}_6\theta_6} p = g_3 \cdot p = q \Leftarrow$ Subproblem 1
Maximum of solutions: 8 $\diamond$

5.2 Inverse Kinematics
(Example: SCARA Manipulator)

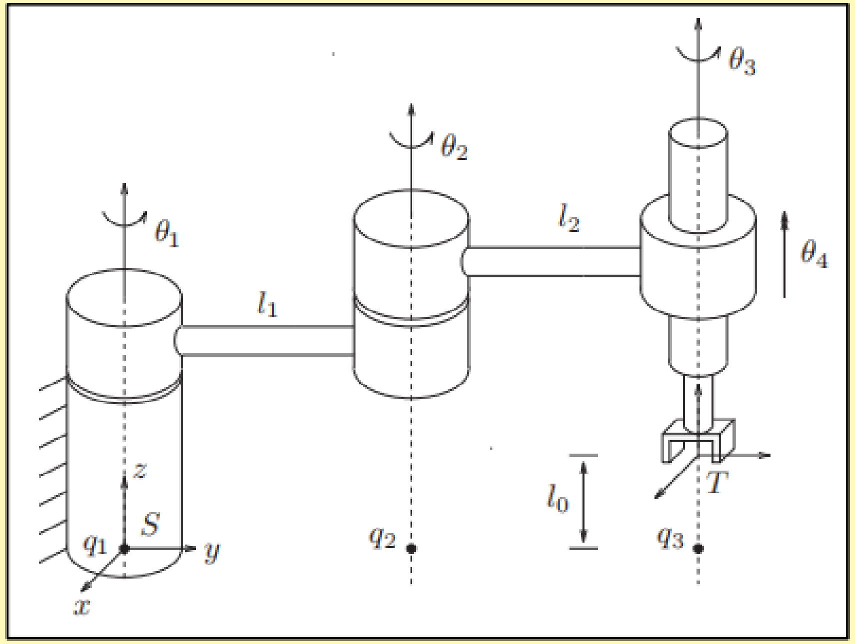


Figure 5.3

$$g_{st}(0) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_4\theta_4} g_{st}(0)$$

$$= \left[\begin{array}{ccc|c} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \triangleq g_d$$

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p(\theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ l_0 + \theta_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \theta_4 = z - l_0$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_1$$

(Continues next slide)

Let $p \in l_{\xi_3}, q \in l_{\xi_1} \Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = g_1 p,$

$$\left\| e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2} p - q) \right\| = \| g_1 p - q \|,$$

$$\left\| e^{\hat{\xi}_2\theta_2} p - q \right\| = \delta \leftarrow \text{ Subproblem 3 to get } \theta_2$$

$$\Rightarrow e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2} p) = g_1 p \Rightarrow \theta_1 \leftarrow \text{Subproblem 1 to get } \theta_1$$

$$\Rightarrow e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_2$$

$$e^{\hat{\xi}_3\theta_3} p = g_2 p, p \notin l_{\xi_3} \leftarrow \text{Subproblem 1 to get } \theta_3$$

There are a maximum of two solutions! ◇

5.2 Inverse Kinematics (Example: ABB IRB4400)

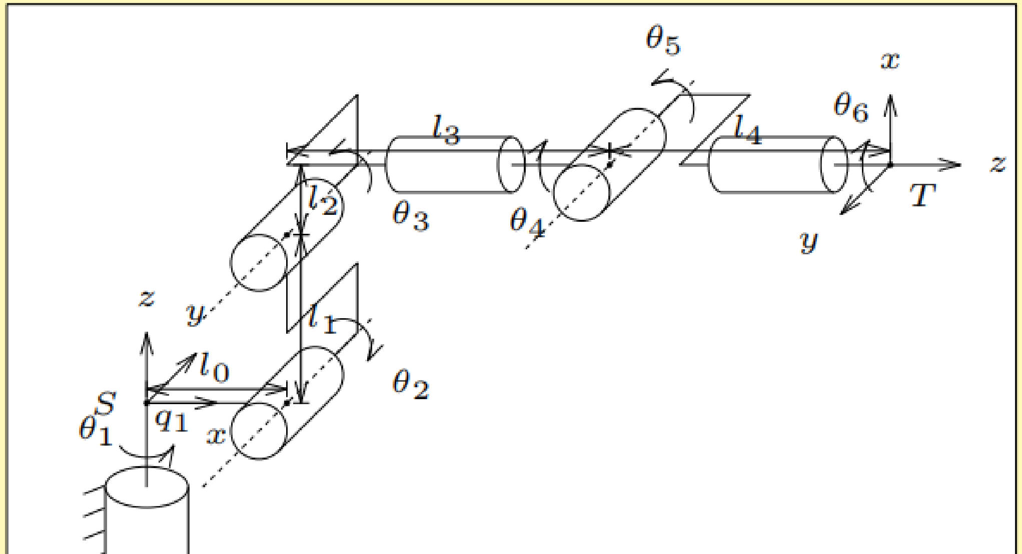


Figure 5.10

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \omega_2 = -\omega_3 = -\omega_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \omega_4 = \omega_6 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} l_0 \\ 0 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} l_0 \\ 0 \\ l_1 \end{bmatrix}, p_w := q_4 = q_5 = q_6 = \begin{bmatrix} l_0 + l_3 \\ 0 \\ l_1 + l_2 \end{bmatrix}$$

(Continues next slide)

$$g_{st}(0) = \left[\begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \xi_i = \begin{bmatrix} q_i \times \omega_i \\ \omega_i \end{bmatrix}$$

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) := g_d$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = g_d p_w =: q \Rightarrow e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = e^{-\hat{\xi}_1\theta_1} q$$

$$\Rightarrow 0 = [0\ 1\ 0] \cdot e^{-\hat{\xi}_1\theta_1} q = \cos\theta_1 q_y - \sin\theta_1 q_x, q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

$$\Rightarrow \theta_1 = \tan^{-1}(q_y/q_x)$$

$$\|e^{\hat{\xi}_3\theta_3} p_w - q_2\| = \|e^{-\hat{\xi}_1\theta_1} q - q_2\| =: \delta \leftarrow \text{Subproblem 3 to get } \theta_3$$

$$e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3} p_w) = e^{-\hat{\xi}_1\theta_1} q \leftarrow \text{Subproblem 1 to get } \theta_2$$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) =: g_2$$

Use subproblem 1,2 to solve for $\theta_4, \theta_5, \theta_6$ ◇