# EE4341 Advanced Analog Circuits

## Low Noise Circuit Design

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# Introduction

- Electrical noise is an important consideration in an electronic system because it provides a fundamental limitation to the minimum useful signal that can be processed.
- Noise is caused by the small voltage and current fluctuations that are generated within the devices themselves.

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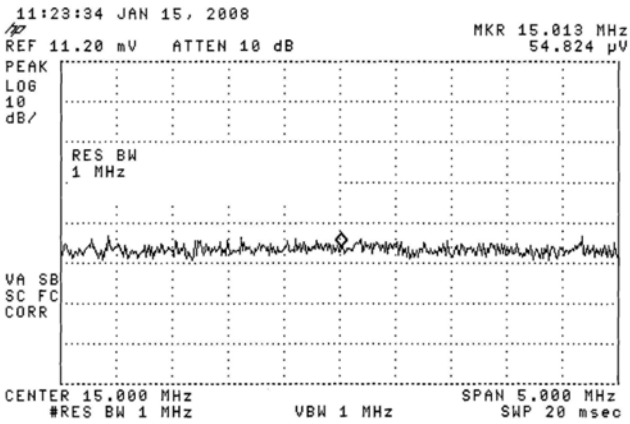
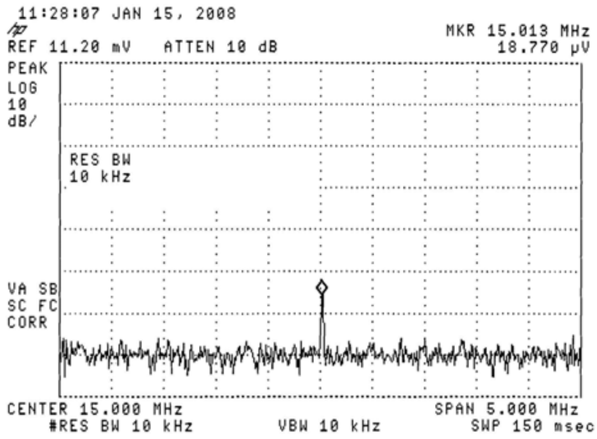



The spectrum analyzer is set at 1 MHz bandwidth and the noise floor is 54.824 μV. The noise floor is so high that it could not pick up a useful signal.

The bandwidth is reduced to 10 kHz and the noise floor is now 4.31 μV. A very clear signal at 15 MHz is detected.

**The "noise floor" determines the "sensitivity" of a system to detect a desired signal.**



# Signal Versus Noise

A deterministic signal can be described by an explicit mathematical expression with some parameters. For example, a sine wave can be described by its amplitude ($V_p$), frequency ($f$) and phase ($\phi$), with respect to a reference.

$$v(t) = V_p \sin(2\pi f t + \phi)$$

**Unfortunately, noise is a random signal and its precise instantaneous value cannot be predicted.**

# How to Describe Noise?

Noise power or intensity is usually of interest to circuit designers.

The "mean-square" value or "intensity" of a random signal $x(t)$ is the average of the squares of the instantaneous values of the random signal. It is given by:

$$\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt$$

Within a long enough period $T$, if the total number of values taken is $n$ (sample size), then the mean-square value a random signal $x(t)$ is given by:

$$\overline{X^2} = \frac{x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} + \ldots.. + x_n^{\ 2}}{n}$$



# How to Describe Noise?

The mean-square value can be separated into a time-invariant part and a time-varying part.

$$\overline{X^2} = \mu_x^{\ 2} + \sigma_x^{\ 2}$$

The time-invariant part is the square of the signal average or signal mean,

$$\mu_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt$$

The time-varying part is the signal variance, which is the mean-square value of $x(t)$ w.r.t. its mean value,

$$\sigma_x^{\ 2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T [x(t) - \mu_x]^2 dt$$

# Noise Power

The noise power dissipated by a random noise voltage on a resistor is proportional to the mean-square voltage. In most cases, the mean value (time-invariant part) of the noise voltage is zero.

$$\therefore \overline{X^2} = \sigma_x^{\ 2} \ \text{ since } \mu_x^{\ 2} = 0$$

Therefore, the noise variance is proportional to noise power.

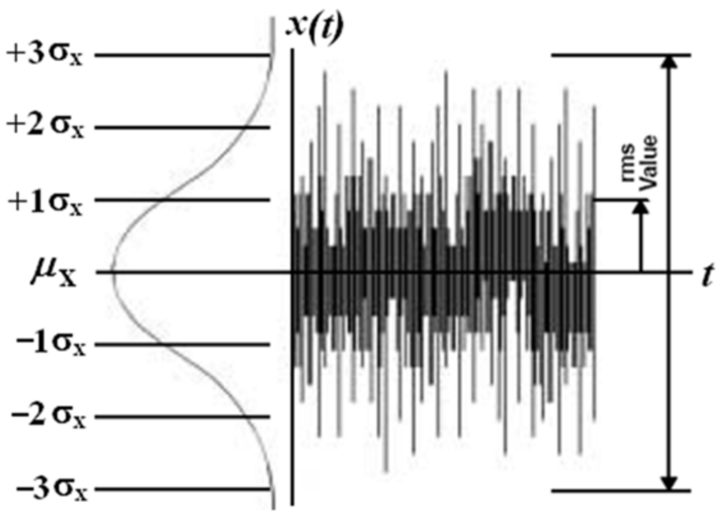
By definition, the root-mean-square value of the noise voltage is the "standard deviation" of the random signal:

$$X_{rms} = \sqrt{\overline{X^2}} = \sigma_x$$



# Probability Density Function

The amplitude distribution of a random signal is usually described by a Gaussian probability density function (PDF).




$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^{\ 2}}}$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^{\ 2}}}$$

Note that $\sigma_x = X_{rms}$

If we observe the random signal long enough, we are quite sure that 99.9% of the time the amplitude of $x$ will be within $\pm 3.3\sigma_x$

Hence, peak-to-peak noise voltage is given by: $X_{p-p} = 6.6\sigma_x = 6.6X_{rms}$

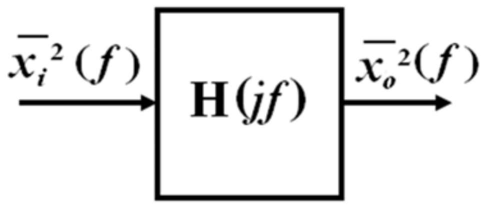
# Power Spectral Density

The PDF only characterize the random signal with respect to time.

Another important characteristic of noise is it power spectral density (PSD), it characterizes the random noise signal in frequency domain .

$$\overline{x^2}(f) = \frac{\overline{X^2}}{\Delta f} \quad \text{V}^2/\text{Hz}$$

where $\Delta f$ is the "equivalent noise bandwidth" of the RMS voltage measurement instrument.

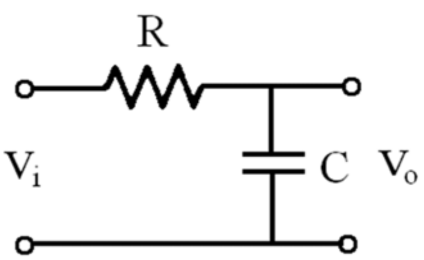



For an input noise PSD, the output noise PSD of a linear system is given by:

$$\overline{x_o^2}(f) = |H(jf)|^2 \, \overline{x_i^2}(f)$$



# Example




$$H(j\omega) = \frac{v_o}{v_i} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

$$= \frac{1}{1 + j\omega/\omega_o}$$

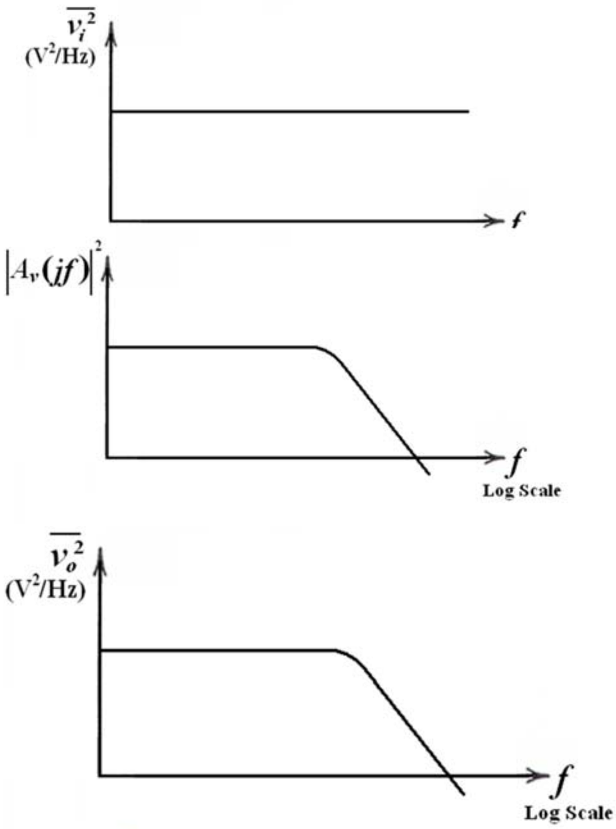
where $\omega_o = \frac{1}{RC} \Rightarrow f_o = \frac{1}{2\pi RC}$

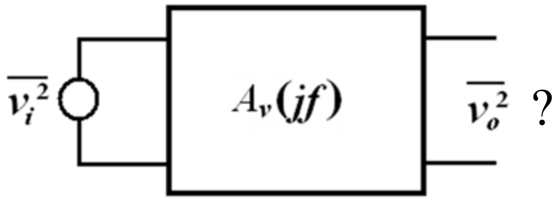
$$H(jf) = \frac{1}{1 + j(f/f_o)}, \quad \therefore |H(jf)|^2 = \frac{1}{1 + (f/f_o)^2}$$

The output noise PSD is:

$$\overline{v_{no}^2}(f) = \overline{v_{ni}^2}(f)\left|\frac{1}{1 + j(f/f_o)}\right|^2 = \overline{v_{ni}^2}(f)\left(\frac{1}{\sqrt{1^2 + (f/f_o)^2}}\right)^2 = \frac{\overline{v_{ni}^2}(f)}{1 + (f/f_o)^2} \quad \text{V}^2/\text{Hz}$$

# Calculation of RMS Noise




The output PSD in V$^2$/Hz is given by:

$$\overline{v_o^{\ 2}} = \overline{v_i^{\ 2}}\left|A_v\left(jf\right)\right|^2 \quad \text{V}^2/\text{Hz}$$

The mean-square output noise voltage:

$$\overline{V_o^{\ 2}} = \overline{v_i^{\ 2}}\int_0^{\infty}\left|A_v\left(jf\right)\right|^2 df \quad \text{V}^2$$

The RMS output noise voltage:

$$V_o = \sqrt{\overline{v_i^{\ 2}}\int_0^{\infty}\left|A_v\left(jf\right)\right|^2 df} \quad \text{V}$$



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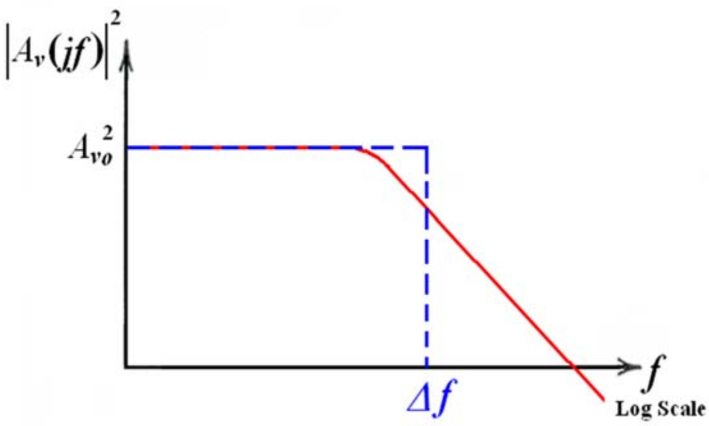
# Noise Bandwidth

$$V_o = \sqrt{\overline{v_i^{\ 2}}\int_0^{\infty}\left|A_v\left(jf\right)\right|^2 df} \quad \text{V}$$

If we could find out the ideal sharp cut-off equivalent noise bandwidth "*Δf*" of the system, the RMS output noise voltage of the system can be simplified to:

$$V_o = \sqrt{\overline{v_i^{\ 2}}A_{vo}^{\ 2}\Delta f} = A_{vo}\sqrt{\overline{v_i^{\ 2}}\Delta f} \quad \text{V}$$




$$\therefore (\Delta f)A_{vo}^2 = \int_0^{\infty}\left|A_v\left(jf\right)\right|^2 df$$

$$\Delta f = \frac{1}{A_{vo}^2}\int_0^{\infty}\left|A_v\left(jf\right)\right|^2 df$$

where $A_{vo}$ is mid-band voltage gain of the system.



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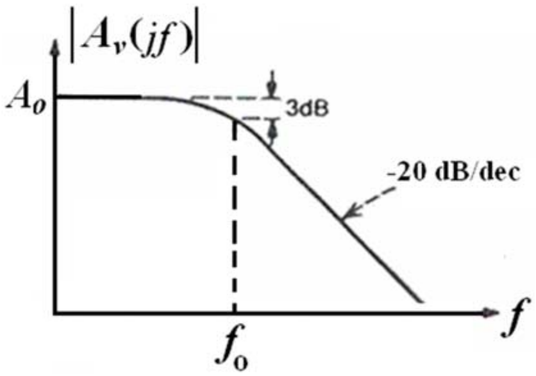
# Example

Find the equivalent noise bandwidth $\Delta f$ for an amplifier with the following transfer function:




$$A_v(jf) = \frac{A_o}{1 + jf/f_o}$$

$$|A_v(jf)| = \frac{A_o}{\sqrt{1 + (f/f_o)^2}}$$

where $f_o$ = –3dB BW

$$\Delta f = \frac{1}{A_o^{\,2}} \int_0^\infty |A_v(jf)|^2 df = \frac{1}{A_o^{\,2}} \int_0^\infty \frac{A_o^{\,2}}{1 + (f/f_o)^2} df = \int_0^\infty \frac{1}{1 + (f/f_o)^2} df$$

$$= f_o \left[ tan^{-1} \frac{\infty}{f_o} - tan^{-1} \frac{0}{f_o} \right] = \frac{\pi}{2} f_0 = 1.57 f_0$$

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# Equivalent Noise Bandwidth

| No. of Poles | Gain Transfer Function | $\Delta f$ |
|---|---|---|
| 1 | $A_v(jf) = \frac{A_o}{1 + jf/f_o}$ | $1.57f_o$ |
| 2 | $A_v(jf) = \frac{A_o}{(1 + jf/f_o)^2}$ | $1.22f_o$ |
| 3 | $A_v(jf) = \frac{A_o}{(1 + jf/f_o)^3}$ | $1.15f_o$ |
| 4 | $A_v(jf) = \frac{A_o}{(1 + jf/f_o)^4}$ | $1.13f_o$ |
| 5 | $A_v(jf) = \frac{A_o}{(1 + jf/f_o)^5}$ | $1.11f_o$ |

The sharper the roll-off, the closer the equivalent noise bandwidth $\Delta f$ towards the 3dB bandwidth

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# Thermal Noise

Thermal noise in a resistor is caused by thermal agitation of electrons. It is always present regardless of whether there is a signal current in the resistor.

The thermal noise in resistor has a constant PSD, given by

$$\overline{v^2} = 4kTR \quad \mathrm{V}^2/\mathrm{Hz}$$

where $k$ is the Boltzmann's constant = $1.38 \times 10^{-23}$ J/K
$T$ is the absolute temperature in $K$
$R$ is the resistance in $\Omega$

The mean-squared noise voltage within an equivalent noise bandwidth $\Delta f$ is:

$$\overline{V^2} = 4kTR\Delta f \quad \mathrm{V}^2$$

The RMS noise voltage in an equivalent noise bandwidth $\Delta f$ is:

$$V_{rms} = \sqrt{4kTR\Delta f} \quad \mathrm{V}$$



# Thermal Noise

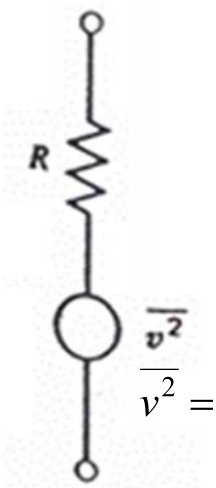
$$V_{rms} = \sqrt{4kTR\Delta f} \quad \mathrm{V}$$

**Thermal noise can be reduced by:**

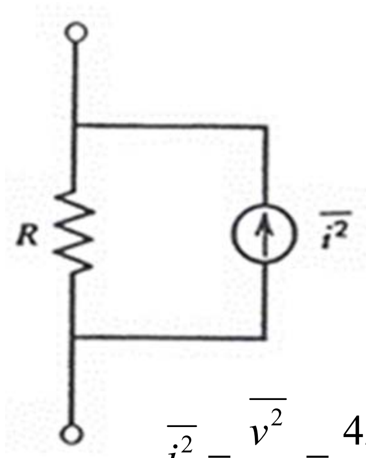
| $R \downarrow$ | $T \downarrow$ | $\Delta f \downarrow$ |
|---|---|---|
| **Avoiding large resistance** | **Lowering ambient temperature** | **Reducing operating bandwidth** |

# Thermal Noise of a Resistor

The thermal noise model of a resistor can be modeled as a noise-free resistor and a noise source. It can be represented as a noise voltage source or noise current source.




$$\overline{v^2} = 4kTR \quad \mathrm{V^2/Hz}$$




$$\overline{i^2} = \frac{\overline{v^2}}{R^2} = \frac{4kTR}{R^2} = \frac{4kT}{R} \quad \mathrm{A^2/Hz}$$



# Example

At room temperature (T = 300 K), the noise voltage PSD for a 1 kΩ resistor is:

$$\overline{v^2} = 4kTR \;\; \mathrm{V^2/Hz}$$

$$4kT = 4 \times 1.38 \times 10^{-23} \times 300 = 1.656 \times 10^{-20}$$

$$\overline{v^2} = 4kTR = 1.656 \times 10^{-20} \times 10^{3} = 1.656 \times 10^{-17} \;\; \mathrm{V^2/Hz}$$

The RMS noise in an equivalent noise bandwidth $\Delta f$ = 1 MHz is given by:

$$V_{rms} = \sqrt{\overline{v^2}\Delta f} = \sqrt{1.656 \times 10^{-17} \times 10^{6}} = 4 \;\; \mu\mathrm{V}$$

$$I_{rms} = \frac{V_{rms}}{R} = \frac{4\,\mu V}{1\,\mathrm{k}\Omega} = 4\,\mathrm{nA}$$

# Addition of Noise Sources

Different noise sources are random and uncorrelated. When two uncorrelated noise voltage sources $v_1$ and $v_2$ are connected in series, the total noise voltage $v_T$ is

$$v_T = v_1 + v_2$$

$$\overline{v_T^{\,2}} = \overline{(v_1 + v_2)^2} = \overline{v_1^{\,2}} + \overline{v_2^{\,2}} + \overline{2v_1 v_2}$$

$$\because \overline{v_1 v_2} = 0 \ \text{(uncorrelated)}$$

$$\therefore \overline{v_T^{\,2}} = \overline{v_1^{\,2}} + \overline{v_2^{\,2}}$$

Thus, independent voltage noise sources are combined in a sum of squared manner as follows:

$$\therefore \overline{v_T^{\,2}} = \overline{v_1^{\,2}} + \overline{v_2^{\,2}} + \overline{v_3^{\,2}} + \ldots\ldots + \overline{v_n^{\,2}}$$

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# Addition of Noise Sources

When two uncorrelated current noise sources $i_1$ and $i_2$ are in parallel, then

$$i_T = i_1 + i_2$$

$$\overline{i_T^{\,2}} = \overline{(i_1 + i_2)^2} = \overline{i_1^{\,2}} + \overline{i_2^{\,2}} + 2\overline{i_1 i_2}$$

$$\because \overline{i_1 i_2} = 0 \ \text{(uncorrelated)}$$

$$\therefore \overline{i_T^{\,2}} = \overline{i_1^{\,2}} + \overline{i_2^{\,2}}$$

Thus, independent current noise sources are combined in a sum of squared manner as follows:

$$\therefore \overline{i_T^{\,2}} = \overline{i_1^{\,2}} + \overline{i_2^{\,2}} + \overline{i_3^{\,2}} + \ldots\ldots + \overline{i_n^{\,2}}$$

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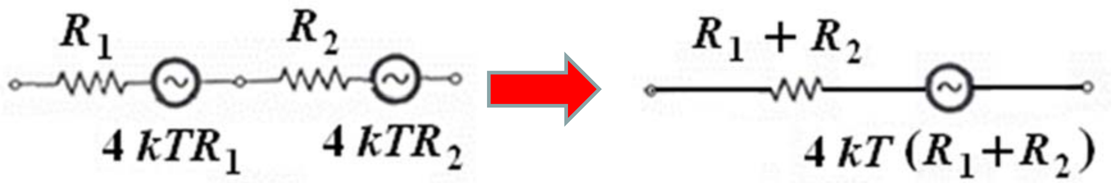
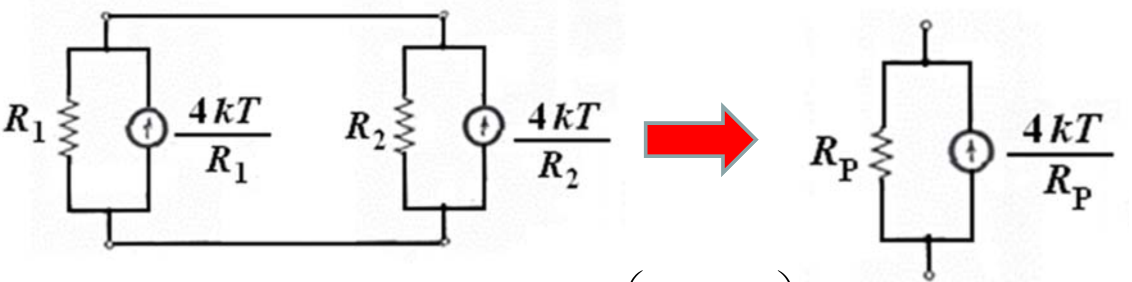
# Example:




$$\overline{v_T^{\,2}} = \overline{v_1^{\,2}} + \overline{v_2^{\,2}} = 4kTR_1 + 4kTR_2 = 4kT\left(R_1 + R_2\right)\ \mathrm{V}^2/\mathrm{Hz}$$




$$\overline{i_T^{\,2}} = \overline{i_1^{\,2}} + \overline{i_2^{\,2}} = \frac{4kT}{R_1} + \frac{4kT}{R_2} = 4kT\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{4kT}{R_P}\ \ \mathrm{A}^2/\mathrm{Hz}$$

where $R_p = R_1 // R_2$



# Shot Noise

As each electron randomly crosses a potential barrier, such as a p-n junction in a semiconductor, energy is stored and released. The aggregate effects of all the electrons shooting across the barrier is called the shot noise.

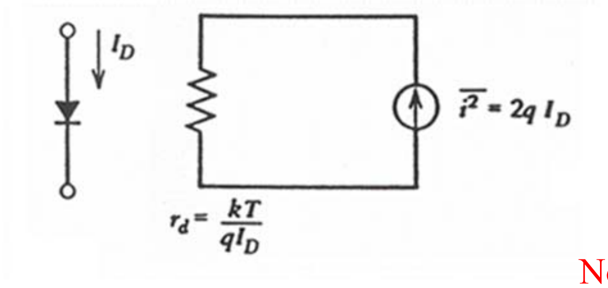
The current PSD of the shot noise is:

$$\overline{i^2} = 2qI_D \quad \mathrm{A}^2/\mathrm{Hz}$$

where $q$ is charge of one electron = $1.6 \times 10^{-19}$ C
$I_D$ is the DC biasing current in A

Shot noise is independent of ambient temperature is always associated with biasing current. Hence, there is no shot noise when the DC biasing current is zero.

# Shot Noise




$$\frac{kT}{q} = \frac{1.38\times10^{-23}\times300}{1.6\times10^{-19}} = 26\,\text{mV}$$

$$r_d = \frac{26\,\text{mV}}{I_D}$$

Note: $r_d$ is a parameter of a diode junction, it is not a "physical resistor" and therefore does not exhibit thermal noise.

$$\text{If } I_D = 0, \text{then } \overline{i^2} = 0$$

$\therefore$ the shot noise does not exist if the device has no bias current.

If $I_D$ = 50 μA, then:

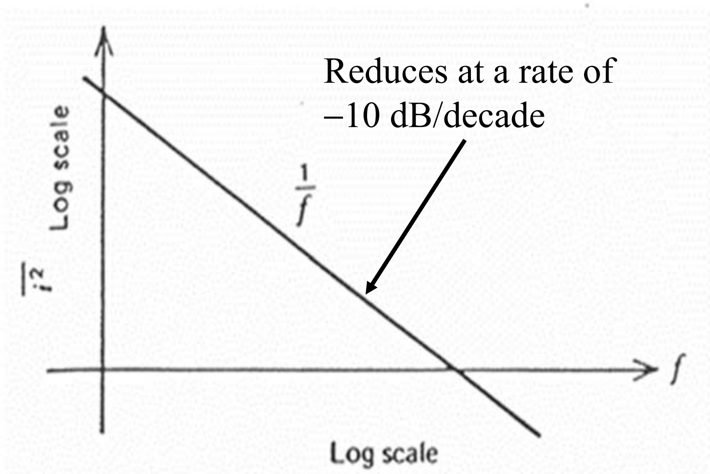
$$\overline{i^2} = 2qI_D = 3.2\times10^{-19}\times50\times10^{-6} = 1.6\times10^{-23}\,\text{A}^2/\text{Hz}$$



# Flicker Noise

It is associated with imperfections in manufacturing technology and found in all active devices as well as some discrete passive elements (e.g. excess noise in carbon resistors). It is also called 1/*f* noise.

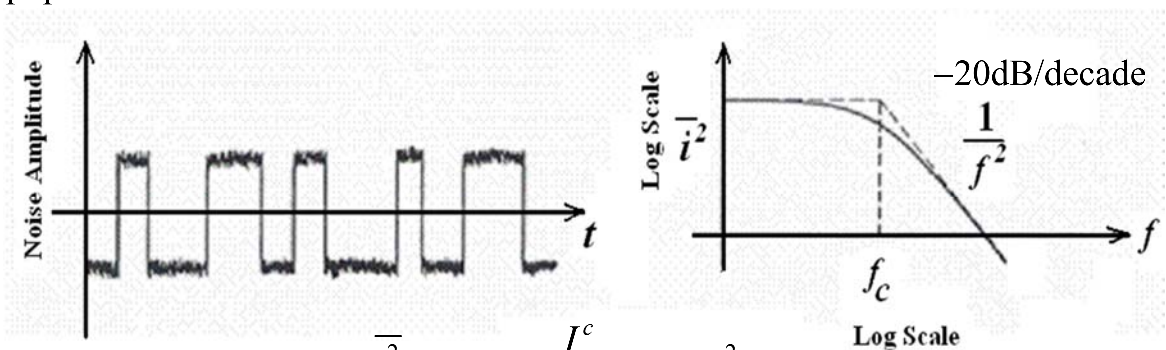
$$\overline{i^2} = k_1\left(\frac{I^a}{f}\right)\quad \text{A}^2/\text{Hz}$$




where $k_1$ = a constant for a device
$I$ = DC biasing current
$a$ = a constant coefficient between 0.5 and 2

# Burst Noise

It is found in discrete transistors and is due to the presence of heavy-metal ion contamination. It is dominant at low frequencies. Burst noise is also called the popcorn noise.




$$\overline{i^2} = k_2 \frac{I^c}{1+\left(f/f_c\right)^2} \quad \mathrm{A^2/Hz}$$

where $k_2$ = a constant for a device

$I$ = DC biasing current

$c$ = a constant between 0.5 and 2

$f_c$ = a constant value for a given process.



# Excess Noise

The flicker noise (1/*f* noise) in a resistor is also called excess noise. The total noise in a resistor consists of both thermal noise and excess noise.

The excess noise PSD in a resistor is:

$$\overline{v_{ex}^2} = \frac{m^2 V_{DC}^2}{f} \quad \mathrm{V^2/Hz}$$

Note: excess noise exists only when the resistor is biased with a dc voltage.

where $m$ = a constant coefficient

$V_{DC}$ = DC voltage across the resistor

The mean-squared excess noise between two frequencies:

$$\overline{V_{ex}^2} = \int_{f_1}^{f_2} \overline{v_{ex}^2}\, df = \int_{f_1}^{f_2} \frac{m^2 V_{DC}^2}{f}\, df = m^2 V_{DC}^2 \int_{f_1}^{f_2} \frac{1}{f}\, df = m^2 V_{DC}^2 \ln\left(\frac{f_2}{f_1}\right) \quad \mathrm{V^2}$$

The rms noise voltage between two frequencies:

$$V_{ex} = \sqrt{\overline{V_{ex}^{\,2}}} = m V_{DC} \sqrt{\ln\left(\frac{f_2}{f_1}\right)} \quad \mathrm{V}$$

# Excess Noise

Manufacturers provide Noise Index (NI) to express the amount of excess noise in a resistor. NI is defined as the rms noise in a resistor for each Volt of dc voltage across the resistor within a decade of frequency.

$$V_{ex}(\text{per decade}) = mV_{DC}\sqrt{\ln\left(\frac{f_2}{f_1}\right)} = mV_{DC}\sqrt{\ln 10} \quad \text{V} \qquad \because \frac{f_2}{f_1} = 10$$

$$NI = \frac{V_{ex}(\text{per decade})}{V_{DC}} = m\sqrt{\ln 10} \quad \text{V/V}$$

For a know dc voltage across the resistor, the excess noise voltage per decade is:

$$V_{ex}(\text{per decade}) = NI \times V_{DC} \quad \text{V}$$

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# Example

Given $R$ = 10 kΩ with $NI$ = $10^{-6}$ V/V. The DC voltage across it is 10 V. Determine the total rms noise between 10 Hz and 10 kHz. Plot the noise voltage PSD of the resistor.

$$V_{ex} = mV_{DC}\sqrt{\ln\left(\frac{f_2}{f_1}\right)} = mV_{DC}\sqrt{\ln\left(\frac{10\times 10^3}{10}\right)} = mV_{DC}\sqrt{\ln 10^3}$$

$$= \sqrt{3}V_{DC}m\sqrt{\ln 10} = \sqrt{3}V_{DC}NI = \sqrt{3}\times 10\times 10^{-6} = 17.32\ \mu\text{V}$$

$$m = \frac{NI}{\sqrt{\ln 10}} = \frac{10^{-6}}{\sqrt{\ln 10}} = 6.6\times 10^{-7}$$

$$\overline{v_{ex}^2} = \frac{m^2V_{DC}^2}{f} = \frac{(6.6\times 10^{-7})^2\times 10^2}{f} = \frac{4.356\times 10^{-11}}{f}\ \text{V}^2/\text{Hz}$$

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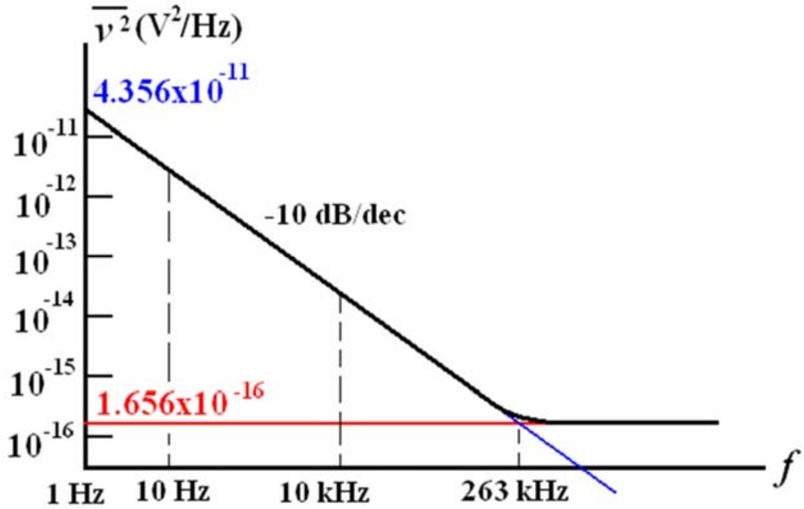
The thermal noise for R:

$$\overline{v_t^{\,2}} = 4kTR = 1.656\times10^{-20}\times10k = 1.656\times10^{-16}\ \ \mathrm{V}^2/\mathrm{Hz}$$

$$V_t = \sqrt{\overline{v_t^{\,2}}\Delta f} = \sqrt{1.656\times10^{-16}(10k-10)} = 1.286\ \mu\mathrm{V}$$

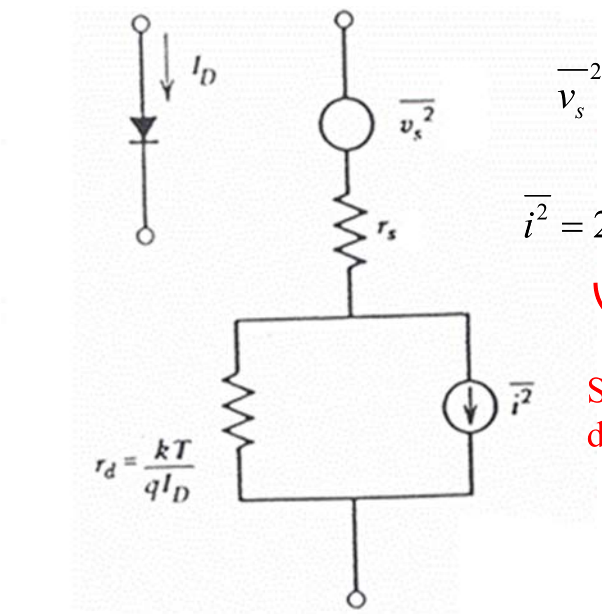
The total rms noise: $V = \sqrt{V_{ex}^{\,2} + V_t^{\,2}} = \sqrt{17.32^2 + 1.286^2} = 17.37\,\mu\mathrm{V}$






# Noise Model of Diode

The complete noise model of a diode includes a series resistance $r_{s,}$ it is a physical resistor due to the resistivity of the silicon . Hence, it exhibits thermal noise.

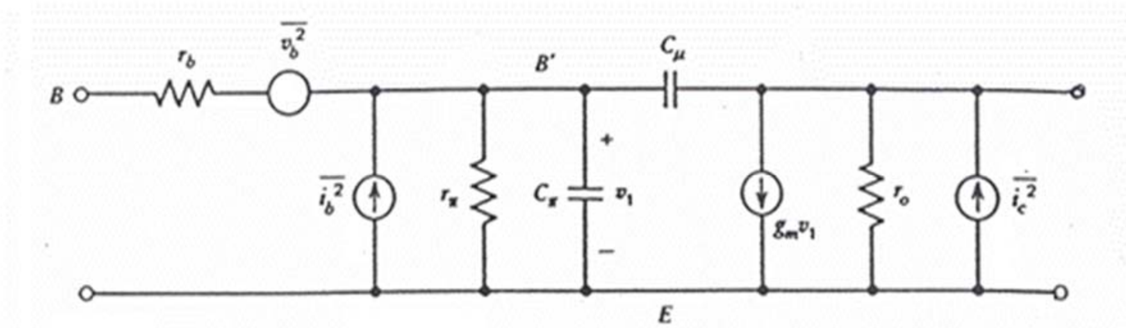



$$\overline{v_s}^{\,2} = 4kTr_s \quad \mathrm{V}^2/\mathrm{Hz}$$

$$\overline{i^2} = 2qI_D + k_1\left(\frac{I_D^{\,a}}{f}\right) + k_2\left(\frac{I_D^{\,b}}{1+(f/f_c)^2}\right)\mathrm{A}^2/\mathrm{Hz}$$

Shot noise & flicker noise are more dominant than burst noise

# Noise Model of BJT




Note: $r_\pi$ & $r_o$ are fictitious resistors that used for analysis purposes and they are not "physical resistors" and therefore do not exhibit thermal noise. However, $r_b$ is the physical resistor due to resistivity of the material.

$$\overline{v_b^2} = 4kTr_b \quad \mathrm{V^2/Hz}$$

$$\overline{i_c^2} = 2qI_c \quad \mathrm{A^2/Hz}$$

$$\overline{i_b^2} = \underbrace{2qI_B + k_1 \frac{I_B^a}{f}} + k_2 \frac{I_B^c}{1+\left(f/f_c\right)^2} \quad \mathrm{A^2/Hz}$$

Shot noise & flicker noise are more dominant than burst noise

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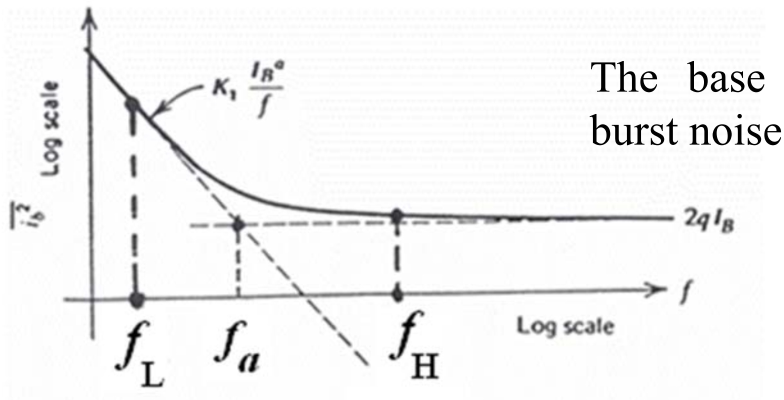
# Noise Model of BJT




The base current PSD with burst noise neglected.

Note that $f_a$ is the flicker noise corner frequency at which the $1/f$ noise is equal to the shot noise.

For low-noise design, devices with low $f_a$ are preferred. $f_a$ varies from as low as 100 Hz to as high as 10 MHz for various BJTs.

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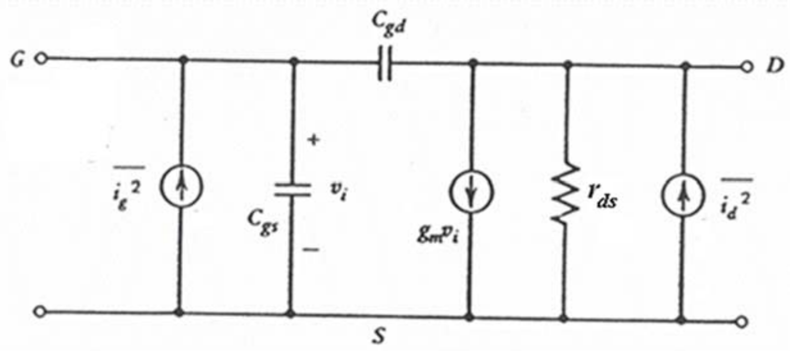
# Noise Model of FET

The resistive channel between source and drain is modulated by the gate-source voltage so as to control the drain current. Hence, the channel is highly resistive and exhibits predominantly thermal noise.




The drain - source channel resistance has been shown to be:

$$r_{ds} \approx \frac{1}{(2/3)g_m}\,\Omega^*$$

where $g_m$ is the transconductance at the biasing point

$$\overline{i_g^2} = 2qI_G \qquad \overline{i_d^2} = 4kT\left(\frac{2}{3}g_m\right) + K\frac{I_D^a}{f}$$

Shot noise due to gate current is usually insignificant as $I_G \approx 0$ (in nA)

*** Van der Ziel, "Thermal noise in field effect transistors" Proc. IEEE, Vol. 50, No. 8, Aug 1962, pp 1808-1812.**

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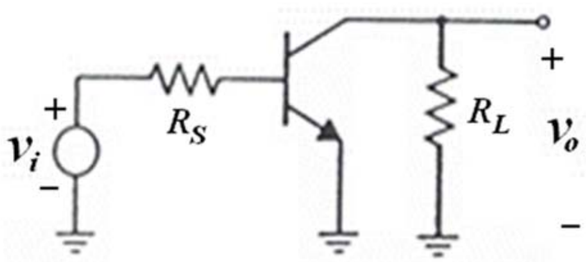
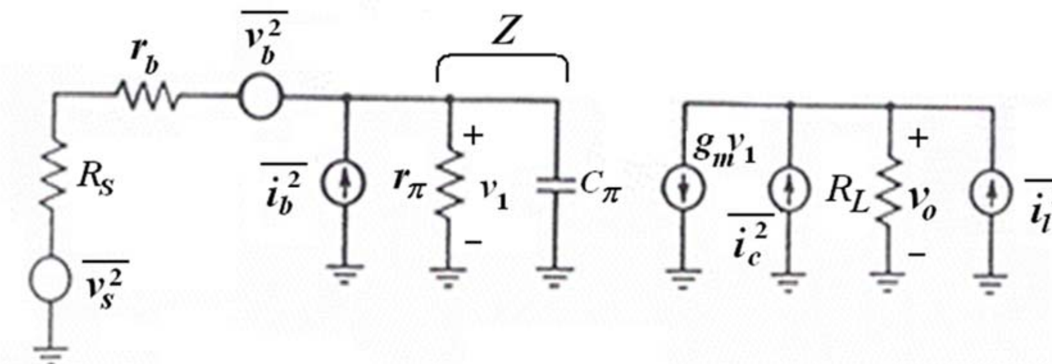
# Noise Circuit Analysis




The device noise model together with superposition principle can be used for noise analysis

The effects of uncorrelated noise sources should be combined by "sum of square" rule.

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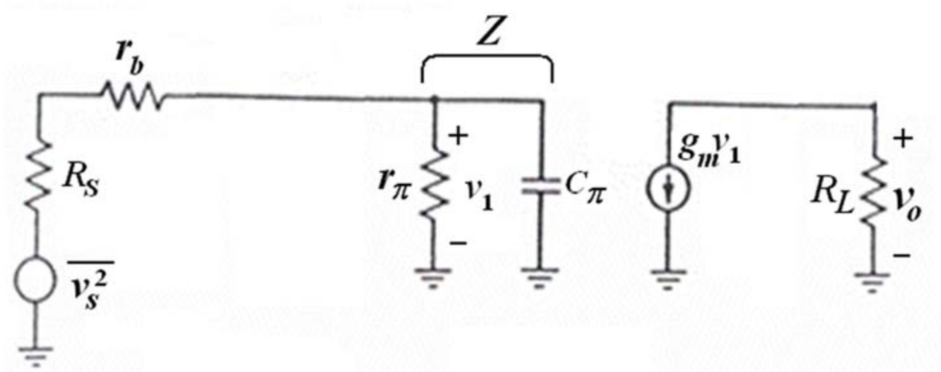


Output noise due to $\overline{v_s^2}$: $v_1 = \dfrac{Z}{Z+R_s+r_b} v_s$, where $Z = r_\pi // \left(\dfrac{1}{j\omega C_\pi}\right)$

$$v_{o1} = -g_m v_1 R_L = -g_m R_L \frac{Z}{Z+R_s+r_b} v_s$$

$$\overline{v_{o1}^2} = g_m^2 R_L^2 \left|\frac{Z}{Z+R_s+r_b}\right|^2 \overline{v_s^2}$$

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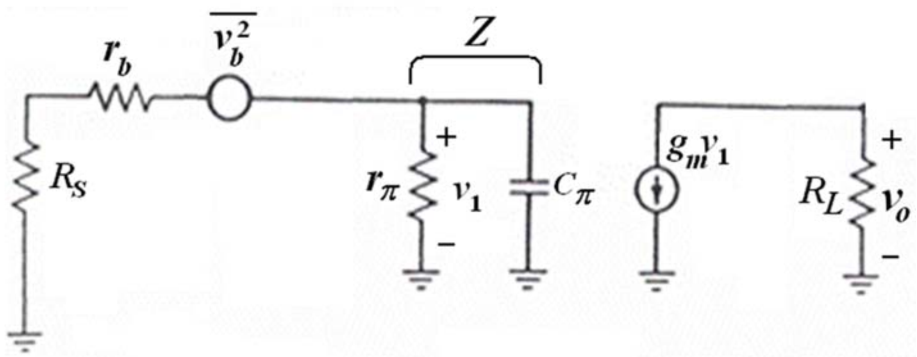


Output noise due to $\overline{v_b^2}$: $v_1 = \dfrac{Z}{Z+R_s+r_b} v_b$

$$v_{o2} = -g_m v_1 R_L = -g_m R_L \frac{Z}{Z+R_s+r_b} v_b$$

$$\overline{v_{o2}^2} = g_m^2 R_L^2 \left|\frac{Z}{Z+R_s+r_b}\right|^2 \overline{v_b^2}$$

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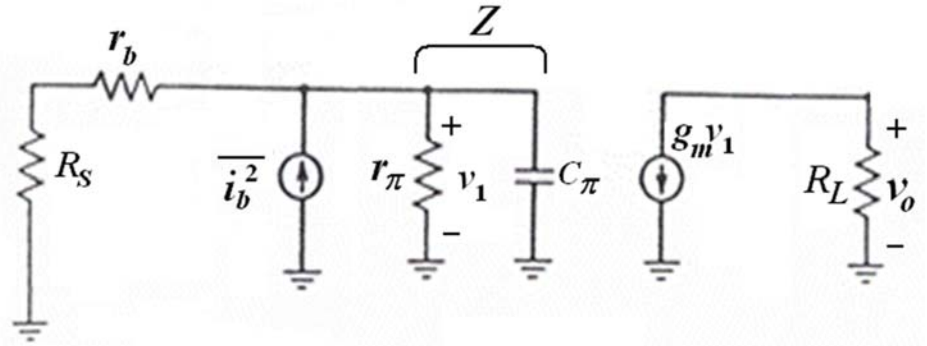


Output noise due to $\overline{i_b^{\ 2}}$: $v_1 = i_b\left[Z//\left(R_s + r_b\right)\right] = \dfrac{Z\left(R_s + r_b\right)}{Z + R_s + r_b} i_b$

$$v_{o3} = -g_m v_1 R_L = -g_m R_L \frac{Z\left(R_s + r_b\right)}{Z + R_s + r_b} i_b$$

$$\overline{v_{o3}^2} = g_m^2 R_L^2 \left(R_s + r_b\right)^2 \left|\frac{Z}{Z + R_s + r_b}\right|^2 \overline{i_b^2}$$

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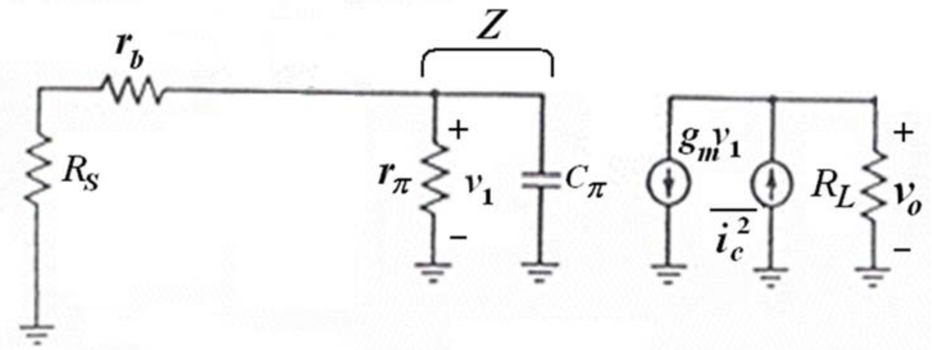


Output noise due to $\overline{i_c^{\ 2}}$: $v_o = i_c R_L$

$$v_{o4} = i_c R_L$$
$$\overline{v_{o4}^2} = \overline{i_c^2} R_L^{\ 2}$$

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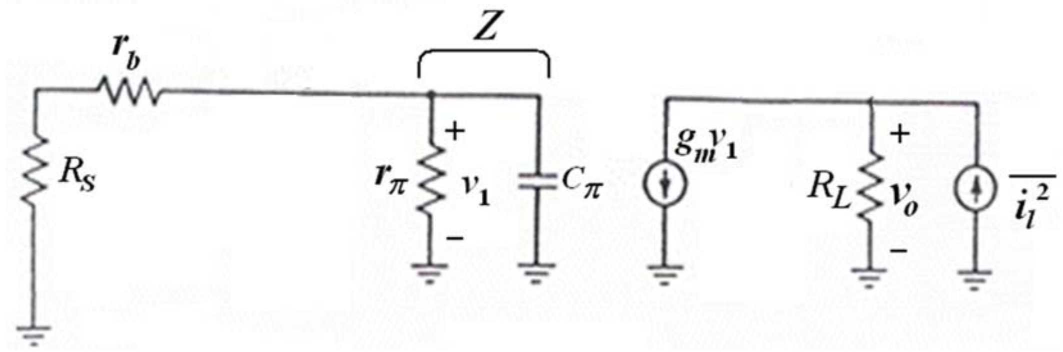



Output noise due to $\overline{i_l^{\,2}}$ :   $v_o = i_l R_L$

$$v_{o5} = i_l R_L$$

$$\overline{v_{o5}^2} = \overline{i_l^2} R_L^{\,2}$$



The total output noise is:

Note: flicker noise is neglected for simplicity.

$$\overline{v_o^2} = \overline{v_{o1}^2} + \overline{v_{o2}^2} + \overline{v_{o3}^2} + \overline{v_{o4}^2} + \overline{v_{o5}^2}$$

$$= g_m^2 R_L^2 \left| \frac{Z}{Z + R_s + r_b} \right|^2 \left( \overline{v_s^2} + \overline{v_b^2} + (R_s + r_b)^2 \overline{i_b^2} \right) + R_L^2 \left( \overline{i_l^2} + \overline{i_c^2} \right)$$

$$= g_m^2 R_L^2 \left| \frac{Z}{Z + R_s + r_b} \right|^2 \left[ 4kT(R_s + r_b) + (R_s + r_b)^2 2qI_B \right] + R_L^2 \left( \frac{4kT}{R_L} + 2qI_c \right)$$

$$Z = \frac{r_\pi (1/j\omega C_\pi)}{r_\pi + 1/j\omega C_\pi} = \frac{r_\pi}{1 + j\omega r_\pi C_\pi}$$

$$\frac{Z}{Z + r_b + R_S} = \frac{r_\pi}{r_\pi + (1 + j\omega r_\pi C_\pi)(r_b + R_S)} = \frac{r_\pi}{r_\pi + r_b + R_S + j\omega r_\pi C_\pi (r_b + R_S)}$$

$$= \left( \frac{r_\pi}{r_\pi + r_b + R_S} \right) \frac{1}{1 + \dfrac{j\omega r_\pi C_\pi (r_b + R_S)}{r_\pi + r_b + R_S}} = \left( \frac{r_\pi}{r_\pi + r_b + R_S} \right) \frac{1}{1 + \dfrac{jf}{f_1}}$$

$$f_1 = \frac{1}{2\pi [r_\pi // (R_s + r_b)] C_\pi}$$

$$\overline{v_o^{\,2}} = g_m^2 R_L^2 \left|\frac{Z}{Z+R_s+r_b}\right|^2 \left[4kT(R_s+r_b)+(R_s+r_b)^2 2qI_B\right] + R_L^2\left(\frac{4kT}{R_L}+2qI_c\right)$$

$$\because \left|\frac{Z}{Z+R_s+r_b}\right|^2 = \left(\frac{r_\pi}{r_\pi+r_b+R_S}\right)^2 \frac{1}{1+\left(\dfrac{f}{f_1}\right)^2}$$

$$\therefore \overline{v_o^2} = g_m^2 R_L^2 \left|\frac{r_\pi}{r_\pi+R_s+r_b}\right|^2 \frac{1}{1+\left(\dfrac{f}{f_1}\right)^2}\left[4kT(R_s+r_b)+(R_s+r_b)^2 2qI_B\right]$$

$$+R_L^2\left(\frac{4kT}{R_L}+2qI_c\right)$$



# Example

Assume that: $I_C$ = 100 μA, $\beta$ = 100, $r_b$ = 200 Ω, $R_s$ = 500 Ω, $C_\pi$=10 pF and $R_L$=5 kΩ. Substituting these values into the above equations and using $4kT = 1.66 \times 10^{-20}$J gives:

$$g_m = \frac{I_C}{V_T} = \frac{100\mu\text{A}}{26\text{mV}} = 3.846\,\text{mS} \qquad r_\pi = \frac{V_T}{I_B} = \frac{26\text{mV}}{1\mu\text{A}} = 26\text{k}\Omega$$

$$g_m^2 R_L^2 \left|\frac{r_\pi}{r_\pi+R_s+r_b}\right|^2 = (3.846m)^2(5k)^2\left(\frac{26k}{26k+500+200}\right)^2 = 350.85$$

$$f_1 = \frac{1}{2\pi[r_\pi // (R_s+r_b)]C_\pi} = \frac{1}{2\pi(26k // 700)(10p)} = 23.3\text{MHz}$$

$$\overline{v_o^2} = g_m^2 R_L^2 \left| \frac{r_\pi}{r_\pi + R_s + r_b} \right|^2 \frac{1}{1+\left(\dfrac{f}{f_1}\right)^2} \left[ 4kT(R_s + r_b) + (R_s + r_b)^2 2qI_B \right] + R_L^2 \left( \frac{4kT}{R_L} + 2qI_c \right)$$

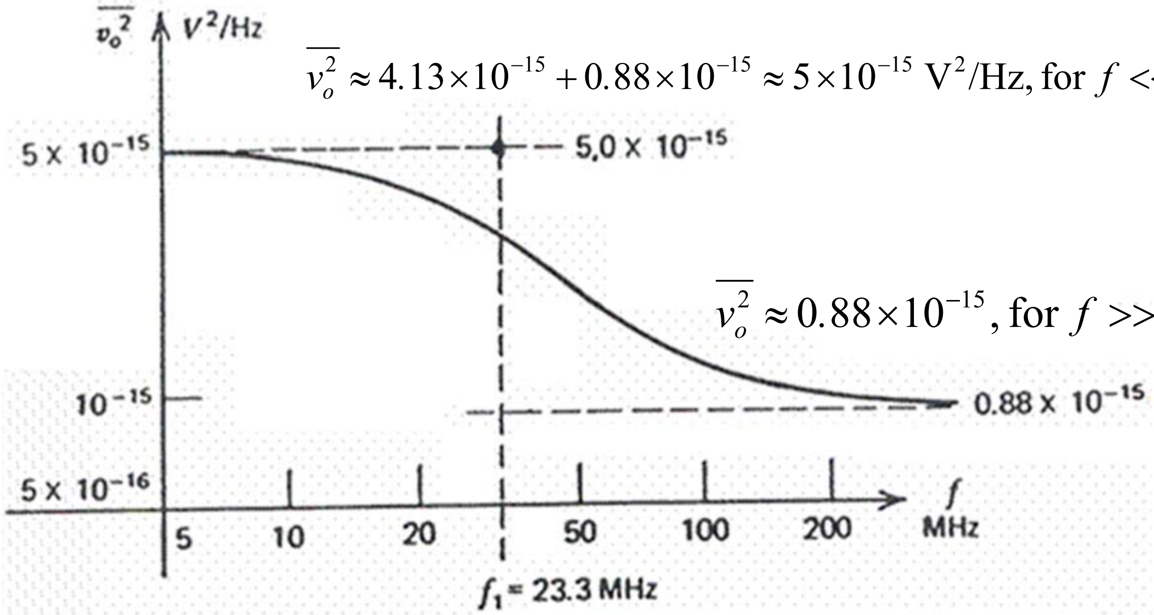
$$= (350.85) \frac{1}{1+\left(\dfrac{f}{23.3\text{MHz}}\right)^2} \left[ 1.66\times10^{-20}(700) + 700^2 \left(3.2\times10^{-25}\right) \right]$$

$$+ (5k)^2 \left( \frac{1.66\times10^{-20}}{5k} + 3.2\times10^{-23} \right)$$

$$= \frac{4.13\times10^{-15}}{1+\left(\dfrac{f}{23.3\text{MHz}}\right)^2} + 0.88\times10^{-15} \ \ \text{V}^2/\text{Hz}$$



$$\overline{v_o^2} = \frac{4.13\times10^{-15}}{1+\left(\dfrac{f}{23.3\text{MHz}}\right)^2} + 0.88\times10^{-15} \ \ \text{V}^2/\text{Hz}$$

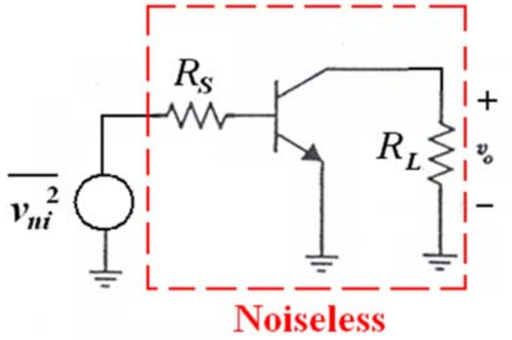



$$\overline{v_o^2} \approx 4.13\times10^{-15} + 0.88\times10^{-15} \approx 5\times10^{-15} \ \text{V}^2/\text{Hz}, \text{for } f << f_1$$

$$\overline{v_o^2} \approx 0.88\times10^{-15}, \text{for } f >> f_1$$

# Equivalent Input Noise

Let $\overline{v_{ni}^2}$ be the equivalent input noise source that produces the same output noise as all the original noise sources, the noise circuit can be simplified as




$$\overline{v_o^2} = \left|A_v(f)\right|^2 \overline{v_{ni}^2}$$

$$\overline{v_o^2} = g_m^2 R_L^2 \left|\frac{Z}{Z+R_s+r_b}\right|^2 \overline{v_{ni}^2}$$

$$\overline{v_o^2} = g_m^2 R_L^2 \left|\frac{Z}{Z+R_s+r_b}\right|^2 \left[4kT(R_s+r_b)+(R_s+r_b)^2 2qI_B\right] + R_L^2\left(\frac{4kT}{R_L}+2qI_c\right)$$

$$\therefore \overline{v_{ni}^2} = 4kT(R_s+r_b)+(R_s+r_b)^2 2qI_B + \frac{1}{g_m^2}\left|\frac{Z+R_s+r_b}{Z}\right|^2\left(4kT\frac{1}{R_L}+2qI_c\right)$$



# Example

Find the equivalent input RMS noise voltage for the previous example.

$$A_v(f) = g_m R_L\left(\frac{Z}{Z+R_s+r_b}\right) = \frac{A_{vo}}{1+\dfrac{jf}{f_1}}$$

where $A_{vo} = g_m R_L\left(\dfrac{r_\pi}{r_\pi+R_s+r_b}\right) = \left(\dfrac{26k}{26k+500+200}\right)(5k\times 3.846m) = 18.7$

$$\text{At mid-band}: \overline{v_{ni}^2} = \frac{\overline{v_o^2}}{A_{vo}^2} = \frac{5\times 10^{-15}}{18.7^2} = 1.43\times 10^{-17}\ \mathrm{V^2/Hz}$$

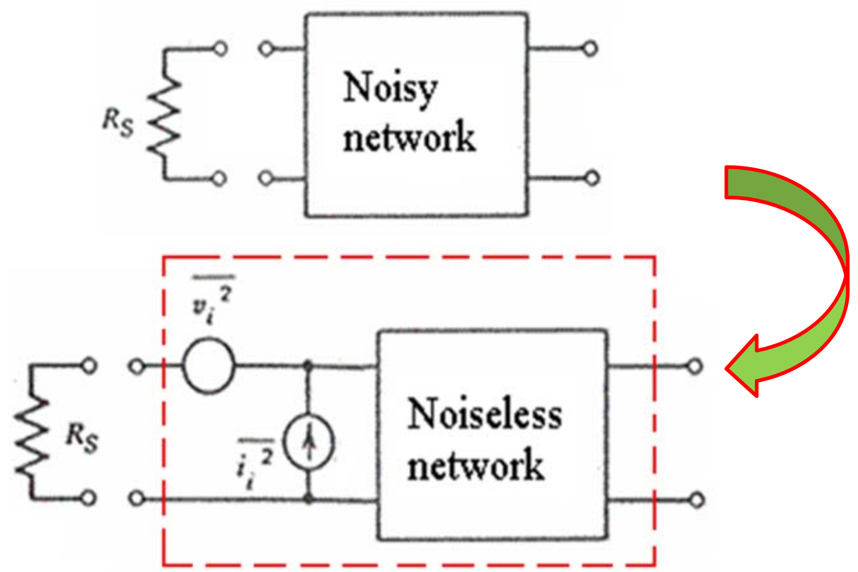
$$V_{ni}^2 = \overline{v_{ni}^2}\Delta f = \overline{v_{ni}^2}\times 1.57 f_1 = (1.43\times 10^{-17})(1.57\times 23.3M) = 5.231\times 10^{-10}\ \mathrm{V^2}$$

$$V_{in} = \sqrt{\overline{v_{ni}^2}\Delta f} = 22.87\ \mu\mathrm{V}$$

The minimum detectable input voltage signal of the amplifier is 22.87µV.

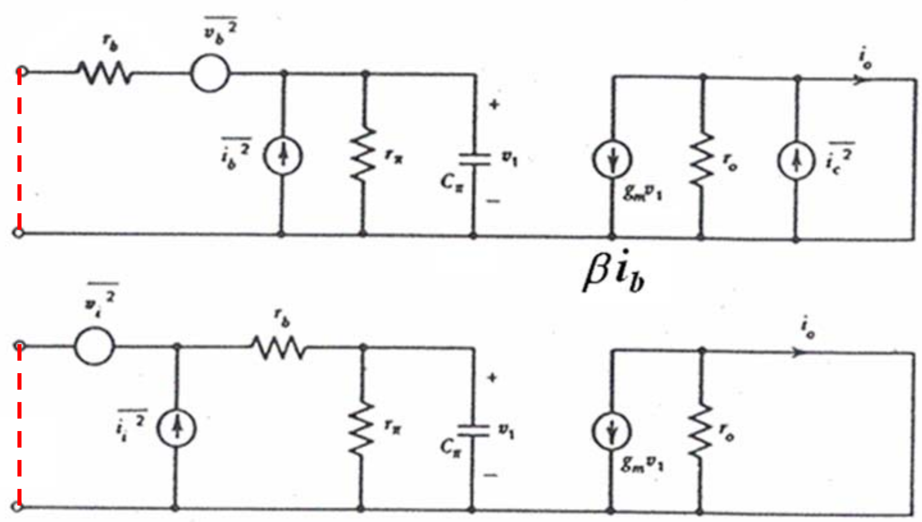
# Equivalent Input Noise Generator

Any noisy two-port network can be replaced by two equivalent input noise sources and a noise-free two-port network.






# BJT Equivalent Input Noise Source




To find $\overline{v_i^2}$ : Short circuit the inputs of both circuits and equate the output short circuit currents.

# BJT Equivalent Input Noise Source

$$g_m v_b\left(\frac{Z}{Z+r_b}\right)+i_c = g_m v_i\left(\frac{Z}{Z+r_b}\right)$$

$$\because r_b << Z,\ \therefore g_m v_b + i_c = g_m v_i \Rightarrow v_i = v_b + \frac{i_c}{g_m}$$

Since $r_b$ is small, the effect of $\overline{i_b^2}$ is negligible.

$$\overline{v_i^2} = \overline{v_b^2} + \frac{\overline{i_c^2}}{g_m^2} = 4kTr_b + \frac{2qI_c}{g_m^2} \quad \because g_m = \frac{qI_c}{kT}$$

$$\therefore \overline{v_i^2} = 4kTr_b + \frac{2qI_c}{g_m}\left(\frac{kT}{qI_c}\right)$$

$$= 4kT\left(r_b + \frac{1}{2g_m}\right) \quad \text{V}^2/\text{Hz}$$

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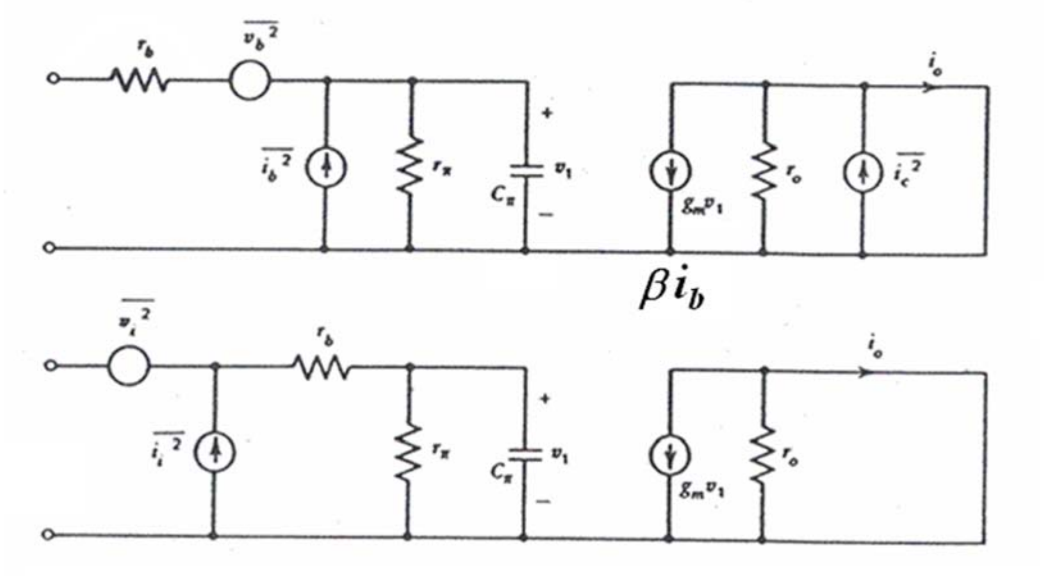
# BJT Equivalent Input Noise Source




To find $\overline{i_i^2}$ : Both inputs are open-circuited and equate the output short circuit currents.

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# BJT Equivalent Input Noise Source

$$\beta i_i = i_c + \beta i_b \Rightarrow i_i = i_b + \frac{i_c}{\beta}$$

$$\overline{i_i^2} = \overline{i_b^2} + \frac{\overline{i_c^2}}{\beta^2}$$

$$\overline{i_i^2} = 2qI_B + k_1 \frac{I_B^a}{f} + \frac{2qI_c}{\beta^2}$$

$$= 2q\left[ I_B + k_1^{'} \frac{I_B^a}{f} + \frac{I_c}{\beta^2} \right] \quad \mathrm{A}^2/\mathrm{Hz} \qquad \text{where } k_1^{'} = \frac{k_1}{2q}$$

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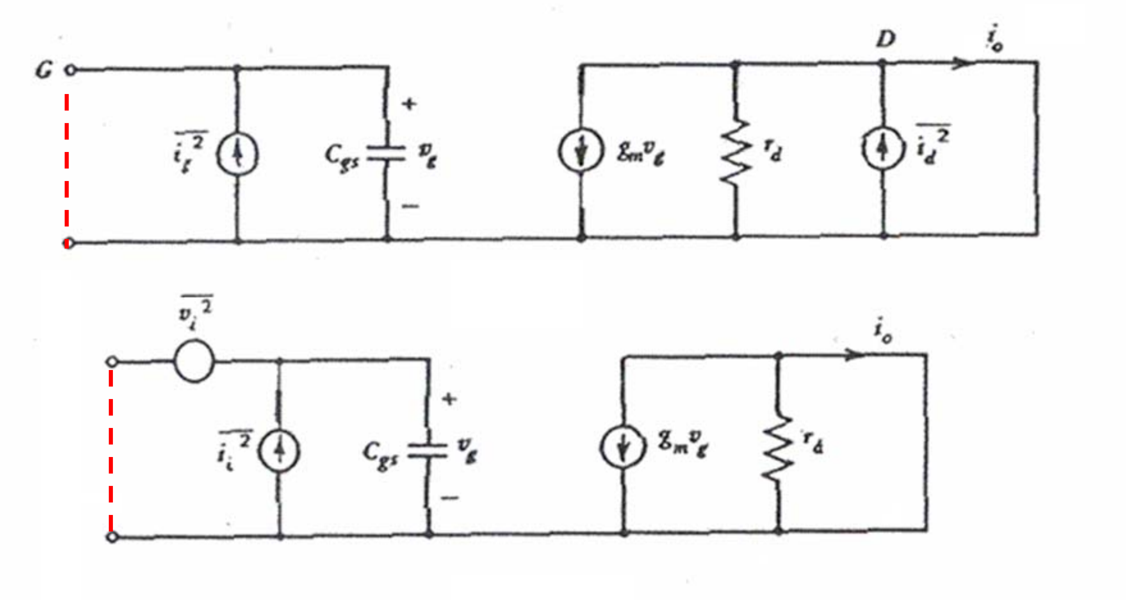
# FET Equivalent Input Noise Source




To find $\overline{v_i^2}$ : Short circuiting both inputs

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# FET Equivalent Input Noise Source

$$i_d = g_m v_g = g_m v_i$$

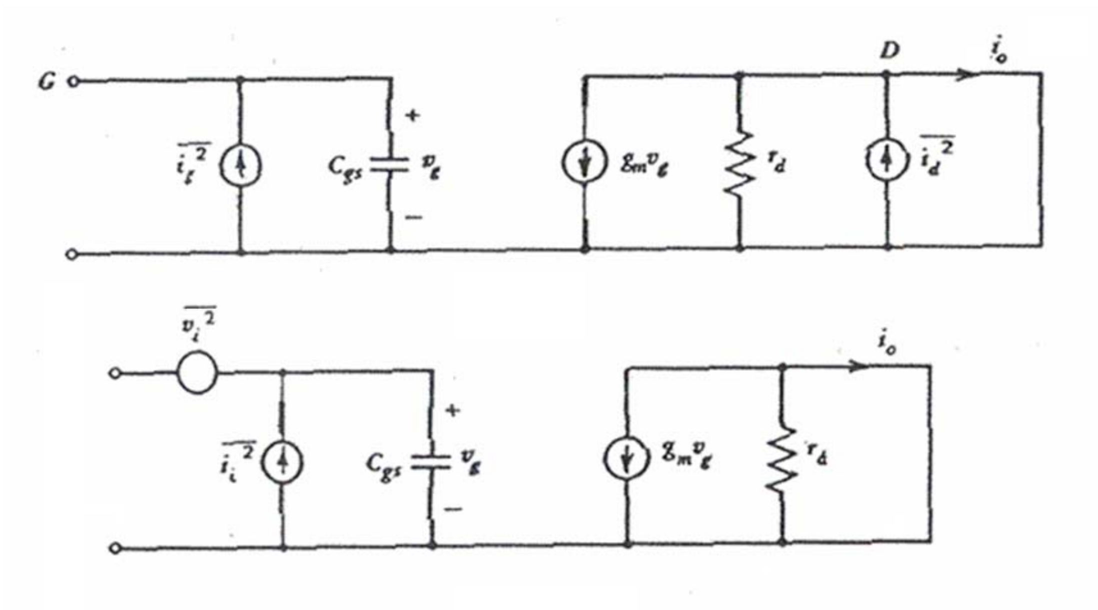
$$\therefore v_i = \frac{i_d}{g_m} \Rightarrow \overline{v_i^2} = \frac{\overline{i_d^2}}{g_m^2}$$

$$\overline{i_d^2} = \underbrace{4kT\left(\frac{2}{3}g_m\right)}_{\text{Thermal noise}} + \underbrace{k\frac{I_D^a}{f}}_{\text{Flicker noise}}$$

$$\overline{v_i^2} = 4kT\frac{2}{3}\frac{1}{g_m} + k\frac{I_D^a}{g_m^2 f}$$



# FET Equivalent Input Noise Source




To find $\overline{i_i^2}$ : By opening both inputs

# FET Equivalent Input Noise Source

$$i_i \frac{g_m}{j\omega C_{gs}} = i_g \frac{g_m}{j\omega C_{gs}} + i_d$$

$$i_i = i_g + \frac{j\omega C_{gs}}{g_m} i_d \qquad \therefore \overline{i_i^2} = \overline{i_g^2} + \frac{\omega^2 C_{gs}^2}{g_m^2} \overline{i_d^2}$$

$$\overline{i_d^2} = \underbrace{4kT\left(\frac{2}{3} g_m\right)}_{\text{Thermal noise}} + \underbrace{k\frac{I_D^a}{f}}_{\text{Flicker noise}}$$

$$\overline{i_i^2} = 2qI_G + \frac{\omega^2 C_{gs}^2}{g_m^2}\left(4KT\frac{2}{3} g_m + k\frac{I_D^a}{f}\right)$$

$$\overline{i_i^2} = 2qI_G + \frac{1}{A_i^2}\left(4KT\frac{2}{3} g_m + k\frac{I_D^a}{f}\right)$$

Note that the ac current gain of the FET is:

$$A_i = \frac{g_m}{\omega C_{gs}}$$





# Equivalent Input Noise Source

Neglecting flicker noise:

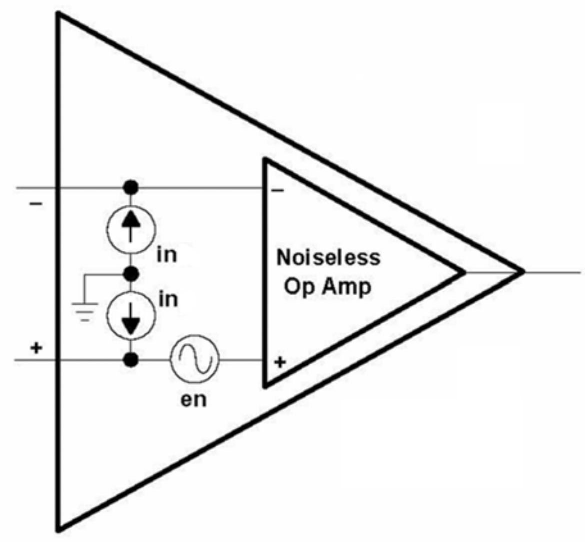
$$\overline{v_i^2} = 4kT\left(r_b + \frac{1}{2g_m}\right) \quad \text{V}^2/\text{Hz} \qquad \text{(BJT)}$$

$$\overline{v_i^2} = 4kT\left(\frac{2}{3g_m}\right) \quad \text{V}^2/\text{Hz} \qquad \text{(FET)}$$

$$\overline{i_i^2} = 2q\left[I_B + \frac{I_c}{\beta^2}\right] \quad \text{A}^2/\text{Hz} \qquad \text{(BJT)}$$

$$\overline{i_i^2} = 2qI_G + \frac{4KT}{A_i^2}\left(\frac{2}{3g_m}\right) \quad \text{A}^2/\text{Hz} \qquad \text{(FET)}$$

# Op Amp Noise Model



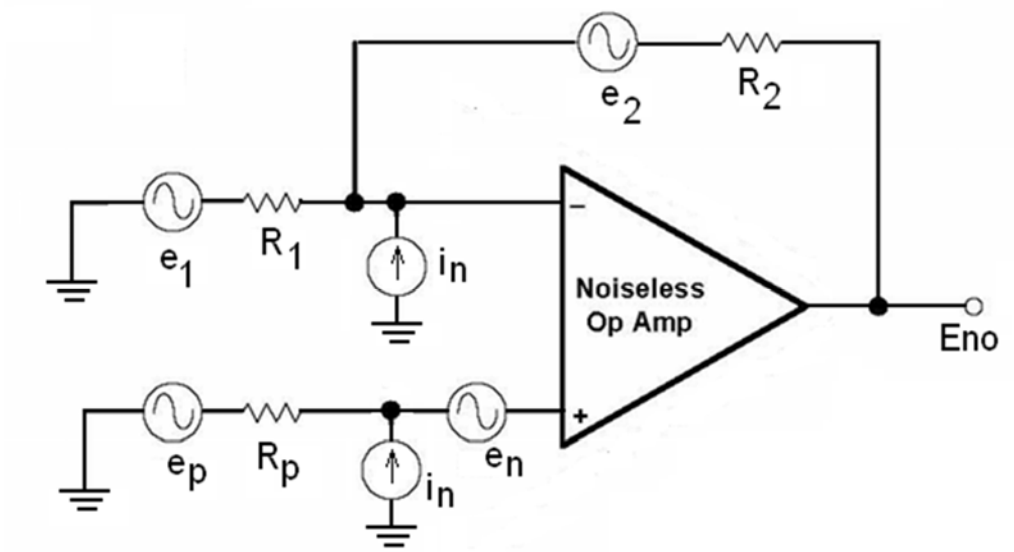

The noise specifications of an op amp refer the input noise of the op amp.

Input noise voltage is modeled by a noise voltage source in series with non-inverting input.

Input current noise is represented by currents noise sources from both input to ground.



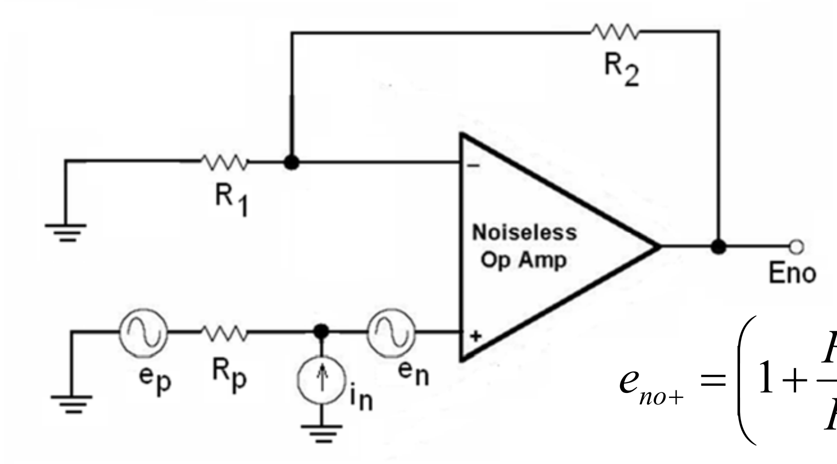
# Op-Amp Circuit Noise Analysis




$i_n$ and $e_n$ are noise sources of the op amp.

$e_1$, $e_2$ and $e_p$ are the thermal noises due to the resistors $R_1$, $R_2$ and $R_p$, respectively.

# Contribution from Non-inverting Input



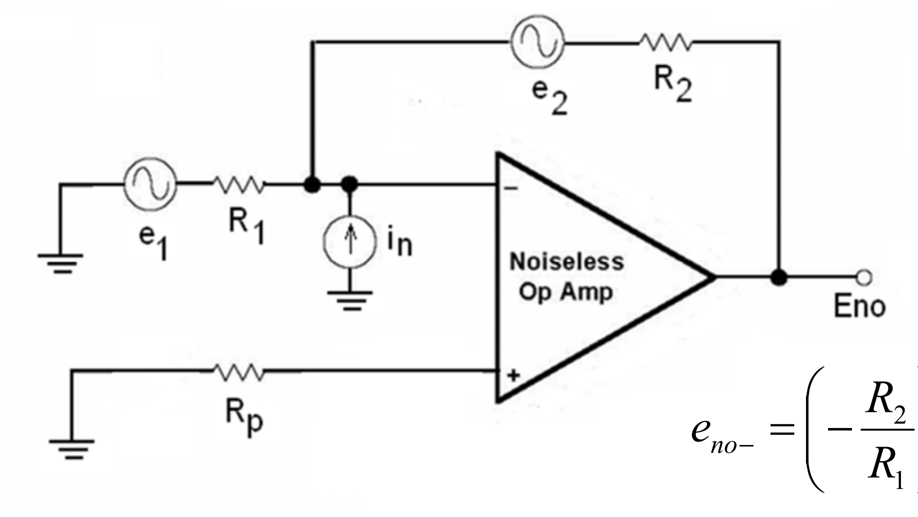

$$e_{no+} = \left(1 + \frac{R_2}{R_1}\right)\left(e_n + e_p + i_n R_P\right)$$

$$\therefore \overline{e_{no+}^2} = \left(1 + \frac{R_2}{R_1}\right)^2 \left(\overline{e_n^2} + \overline{e_p^2} + \overline{i_n^2} R_P{}^2\right)$$

where $\overline{e_p{}^2} = 4kTR_p$



# Contribution from Inverting Input

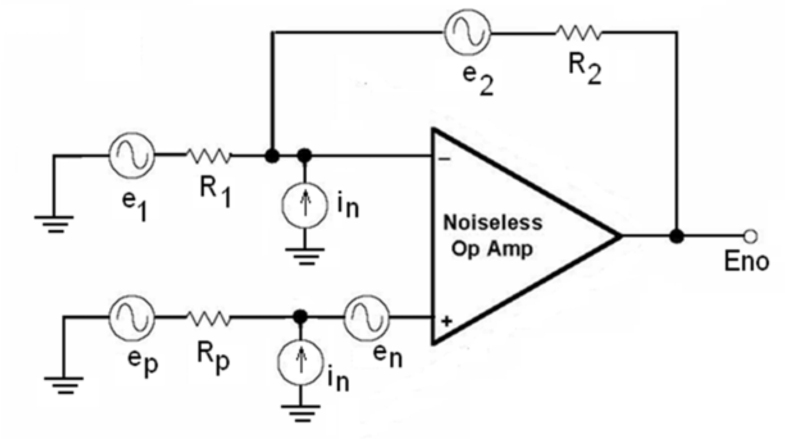



$$e_{no-} = \left(-\frac{R_2}{R_1}\right)e_1 - i_n R_2 + e_2$$

$$\therefore \overline{e_{no-}^2} = \left(\frac{R_2}{R_1}\right)^2 \overline{e_1^2} + \overline{i_n^2} R_2^2 + \overline{e_2^2}$$

where $\overline{e_1^2} = 4kTR_1$ and $\overline{e_2^2} = 4kTR_2$

# Total Noise at the Output




$$\overline{e_{no}^2} = \overline{e_{no+}^2} + \overline{e_{no-}^2}$$

$$= \left(1 + \frac{R_2}{R_1}\right)^2 \left(\overline{e_n^2} + \overline{e_p^2} + \overline{i_n^2} R_p{}^2\right) + \left(\frac{R_2}{R_1}\right)^2 \overline{e_1^2} + \overline{i_n^2} R_2{}^2 + \overline{e_2^2}$$

$$= \left(\frac{R_1 + R_2}{R_1}\right)^2 \left(\overline{e_n^2} + \overline{e_p^2} + \overline{i_n^2} R_p{}^2\right) + \left(\frac{R_2}{R_1}\right)^2 \overline{e_1^2} + \overline{i_n^2} R_2{}^2 + \overline{e_2^2}$$

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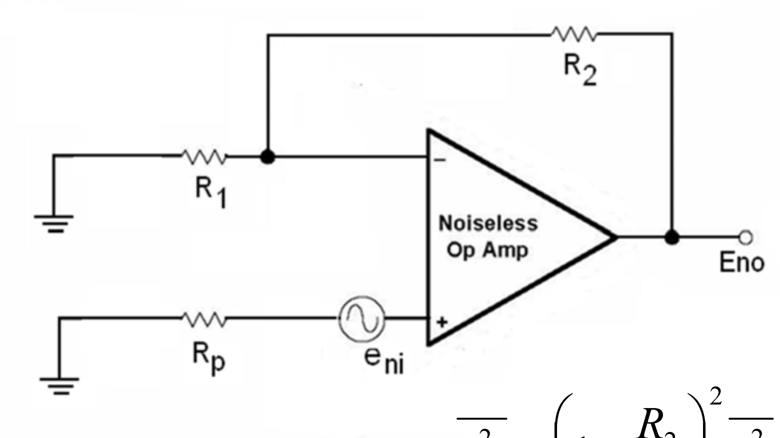
# Equivalent Input Noise




Let combine all the noises and replace it with an equivalent input voltage noise source $e_{ni}$.

$$\overline{e_{no}^2} = \left(1 + \frac{R_2}{R_1}\right)^2 \overline{e_{ni}^2} = \left(\frac{R_1 + R_2}{R_1}\right)^2 \overline{e_{ni}^2}$$

$$\therefore \overline{e_{ni}^2} = \left(\frac{R_1}{R_1 + R_2}\right)^2 \overline{e_{no}^2}$$

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# Equivalent Input Noise

$$\overline{e_{no}^2} = \left(\frac{R_1 + R_2}{R_1}\right)^2 \left(\overline{e_n^2} + \overline{e_p^2} + \overline{i_n^2} R_p{}^2\right) + \left(\frac{R_2}{R_1}\right)^2 \overline{e_1^2} + \overline{i_n^2} R_2{}^2 + \overline{e_2^2}$$

$$\overline{e_{ni}^2} = \left(\frac{R_1}{R_1 + R_2}\right)^2 \overline{e_{no}^2}$$

$$= \overline{e_n^2} + \overline{e_p^2} + \overline{i_n^2} R_p{}^2 + \left(\frac{R_1}{R_1 + R_2}\right)^2 \left(\overline{i_n^2} R_2{}^2\right) + \left(\frac{R_1}{R_1 + R_2}\right)^2 \left[\overline{e_2^2} + \left(\frac{R_2}{R_1}\right)^2 \overline{e_1^2}\right]$$

$$= \overline{e_n^2} + 4kTR_p + \overline{i_n^2} R_p{}^2 + \overline{i_n^2}\left(\frac{R_1 R_2}{R_1 + R_2}\right)^2 + \left(\frac{R_1}{R_1 + R_2}\right)^2 \left[4kTR_2 + 4kTR_1\left(\frac{R_2}{R_1}\right)^2\right]$$

$$= \overline{e_n^2} + 4kTR_p + \overline{i_n^2} R_p{}^2 + \overline{i_n^2}\left(\frac{R_1 R_2}{R_1 + R_2}\right)^2 + 4kT\left(\frac{R_1 R_2}{R_1 + R_2}\right)$$

$$= \overline{e_n^2} + \overline{i_n^2}\left(R_p{}^2 + R_n^2\right) + 4kT\left(R_p + R_n\right) \qquad \text{where } R_n = R_1 // R_2$$



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# Equivalent Input Noise

$$\overline{e_{ni}^2} = \overline{e_n^2} + \overline{i_n^2}\left(R_p{}^2 + R_n^2\right) + 4kT\left(R_p + R_n\right) \quad \mathrm{V^2/Hz}$$

$$E_{ni} = \sqrt{\left[\overline{e_n^2} + \overline{i_n^2}\left(R_p{}^2 + R_n^2\right) + 4kT\left(R_p + R_n\right)\right]\Delta f} \quad \mathrm{V_{rms}}$$

For both inverting and non-inverting amplifiers, the voltage gain for noise is:

$$A_n = \left(1 + \frac{R_2}{R_1}\right)\frac{1}{1 + jf / f_A}$$

where $f_A$ is the -3dB BW of the noise gain

The equivalent noise BW is:

$$\Delta f = 1.57 f_A$$



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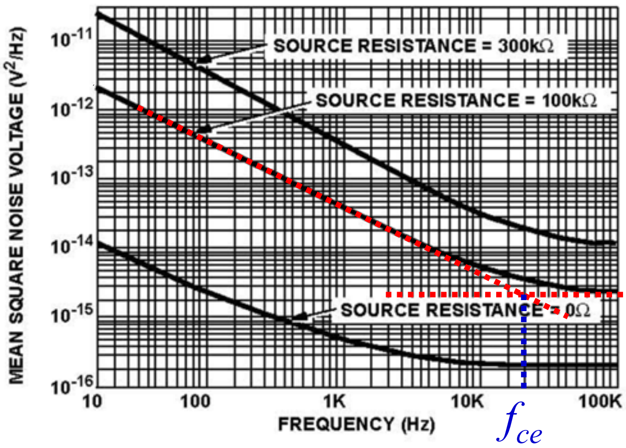
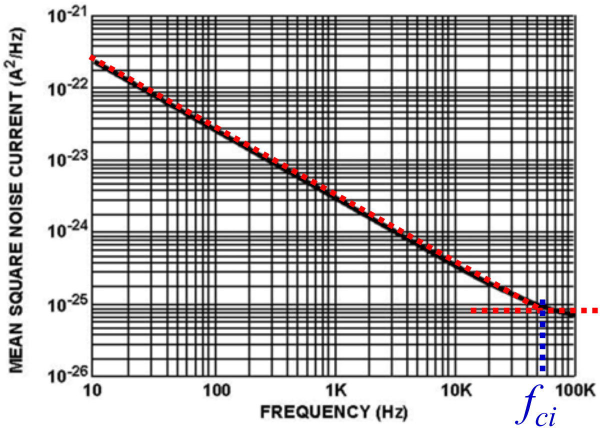
# Op Amp Noise Specifications



HA-2500/2510/2520 INPUT NOISE VOLTAGE



HA-2500/2510/2520 INPUT NOISE CURRENT

$$\overline{e_n^2} = \overline{e_{nw}^2}\left(\frac{f_{ce}}{f} + 1\right) \quad \mathrm{V^2/Hz} \qquad \overline{i_n^2} = \overline{i_{nw}^2}\left(\frac{f_{ci}}{f} + 1\right) \quad \mathrm{A^2/Hz}$$

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# Output Noise of Op Amp

$$E_{no} = |A_n| E_{ni} = \left(1 + \frac{R_2}{R_1}\right)\frac{1}{\sqrt{1 + (f/f_A)^2}} E_{ni}$$

The total noise above $f_L$ is:

$$E_{no} = \sqrt{\int_{f_L}^{\infty} \overline{e_{no}^2}\, df}$$

$$E_{no} = \left(1 + \frac{R_2}{R_1}\right)\sqrt{\int_{f_L}^{\infty} \overline{e_n^2} + \overline{i_n^2}\left(R_p^{\,2} + R_n^2\right) + 4kT\left(R_p + R_n\right)}$$

$$\therefore E_{no} = \left(1 + \frac{R_2}{R_1}\right)\left[\overline{e_{nw}^2}\left(f_{ce}\ln\frac{1.57 f_A}{f_L} + 1.57 f_A - f_L\right)\right.$$

$$\left. + \left(R_p^2 + R_n^2\right)\overline{i_{nw}^2}\left(f_{ci}\ln\frac{1.57 f_A}{f_L} + 1.57 f_A - f_L\right) + 4kT\left(\mathrm{R_p} + R_n\right)\left(1.57 f_A - f_L\right)\right]^{1/2}$$

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# Noise Factor and Noise Figure

The Noise Factor, $F$, is a quantity that compares the noise performance of a device to that of a noise-free device. It is limited to situations where $R_s$ is resistive. The definition of $F$ of a circuit or device is:

$$F = \frac{SNR_i}{SNR_o}$$

Noise Factor $F$ expressed in dB is called Noise Figure ($NF$) and is equal to:

$$NF = 10\log F = 10\log SNR_i - 10\log SNR_o$$

For example, $SNR_\text{i}$= 50 dB, circuit $NF$ = 5dB, then $SNR_o$= 45 dB



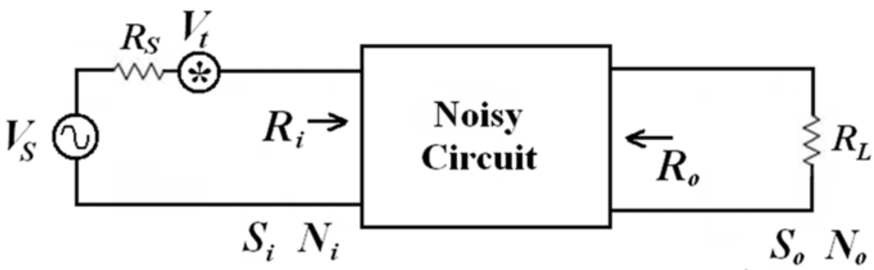


# Noise Factor

$$V_t = \sqrt{4kTR_s \Delta f}$$

$$S_i = \frac{(V_s/2)^2}{R_i} = \frac{V_s^2}{4R_i}$$

$$N_i = \frac{(V_t/2)^2}{R_i} = \frac{V_t^2}{4R_i}$$




The definition of Noise Factor of a circuit is:

$$F = \frac{S_\text{i} / N_i}{S_o / N_o}$$

Note: $N_i$ is purely contributed by the thermal noise of $R_\text{s}$

$S_i$ = maximum signal power to the input of the circuit from $V_s$ (i.e. $R_s = R_\text{i}$)
$N_i$ = maximum noise power to the input of the circuit from $V_t$ (i.e. $R_s = R_i$)

$S_o$ = maximum output power to the load (i.e. $R_L = R_o$)
$N_o$= maximum noise power to the load (i.e. $R_L = R_o$ )

Noise Figure is given by: $NF = 10\log F = 10\log\frac{S_i}{N_i} - 10\log\frac{S_o}{N_o}$ dB

# Noise Factor

$$F = \frac{S_i / N_i}{S_o / N_o} = \left(\frac{S_i}{N_i}\right)\left(\frac{N_o}{S_o}\right) = \frac{N_o}{GN_i}$$

where $G$ is the power gain of the circuit.

$N_o$ = noise power at the load contributed by the amplifier circuit + noise power at the load due to the thermal noise of the source being amplified.

$GN_i$ = noise power at the load due solely to the thermal noise of the source being amplified.

Hence, noise factor is a measure of how much "additional noise power" is contributed by the amplifier circuit during the signal amplification process.

$$F = \frac{N_o}{GN_i} = \frac{GN_i + \text{Additional Noise from Amplifier}}{GN_i}$$

For a noiseless amplifier, $F = 1$.

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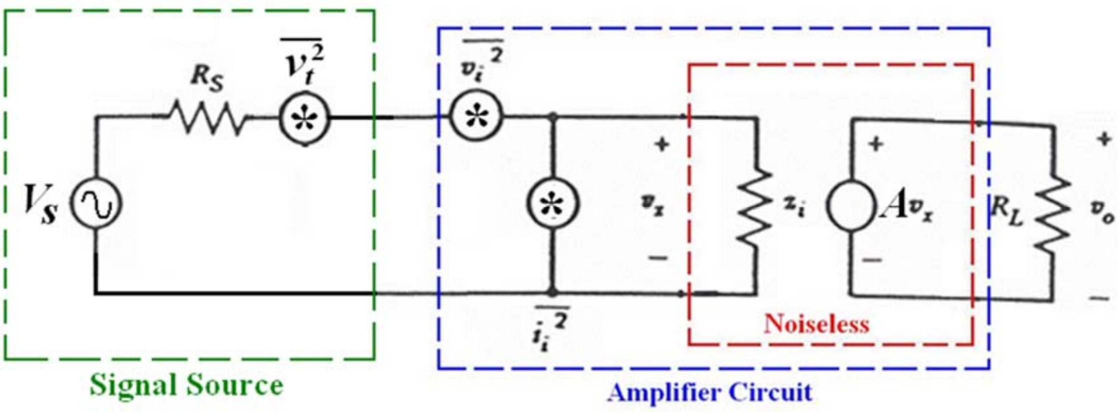
# Noise Factor




The noise voltage at the amplifier input due to $\overline{v_i^2}$ and $\overline{i_i^2}$ :

$$v_x = v_i\left(\frac{z_i}{z_i + R_s}\right) + i_i\left(\frac{R_s z_i}{R_s + z_i}\right)$$

$$\overline{v_x^2} = \left|\frac{z_i}{z_i + R_s}\right|^2 \overline{v_i^2} + \left|\frac{z_i R_s}{z_i + R_s}\right|^2 \overline{i_i^2}$$

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Noise power in $R_L$ due to the noise of the amplifier alone:

$$N_{oA} = \frac{(AV_x)^2}{R_L} = \frac{A^2\left(\overline{v_x^2}\Delta f\right)}{R_L} = \frac{A^2\Delta f}{R_L}\left(\overline{v_i^2}\left|\frac{z_i}{z_i+R_s}\right|^2 + \overline{i_i^2}\left|\frac{z_iR_s}{z_i+R_s}\right|^2\right)$$

The noise voltage at the amplifier input due to $\overline{v_t^{\,2}}$ :

$$v_x = v_t\left(\frac{z_i}{z_i+R_s}\right)$$

$$\overline{v_x^{\,2}} = \left|\frac{z_i}{z_i+R_s}\right|^2 \overline{v_t^{\,2}}$$

Noise power in $R_L$ due the thermal noise of the source resistance :

$$N_{ot} = \frac{(AV_x)^2}{R_L} = \frac{A^2\left(\overline{v_x^2}\Delta f\right)}{R_L} = \left(\frac{A^2\overline{v_t^2}\Delta f}{R_L}\right)\left|\frac{z_i}{z_i+R_s}\right|^2 = \left(\frac{A^2\Delta f}{R_L}\right)\left|\frac{z_i}{z_i+R_s}\right|^2(4kTR_s)$$

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Noise factor of the amplifier:

$$F = \frac{N_{ot}+N_{oA}}{N_{ot}} = 1 + \frac{N_{oA}}{N_{ot}}$$

$$N_{oA} = \frac{A^2\Delta f}{R_L}\left(\overline{v_i^2}\left|\frac{z_i}{z_i+R_s}\right|^2 + \overline{i_i^2}\left|\frac{z_iR_s}{z_i+R_s}\right|^2\right)$$

$$N_{ot} = \left(\frac{A^2\Delta f}{R_L}\right)\left|\frac{z_i}{z_i+R_s}\right|^2(4kTR_s)$$

$$F = 1 + \frac{N_{oA}}{N_{ot}} = 1 + \frac{\overline{v_i^2}}{4kTR_s} + \frac{\overline{i_i^2}R_S^{\,2}}{4kTR_s} = 1 + \frac{\overline{v_i^2}}{4kTR_s} + \frac{\overline{i_i^2}R_S}{4kT}$$

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# Noise Factor

$$F = 1 + \frac{\overline{v_i^2}}{4kTR_s} + \frac{\overline{i_i^2} R_S}{4kT}$$

The noise factor depends only on $R_s$ and equivalent input noise sources and is independent of circuit parameters ($G$, $Z_i$, $R_L$).

Set $(dF/dR_s) = 0$ to find the minimum $F$. The optimum $R_s$ is:

$$\frac{dF}{dR_S} = \frac{-\overline{v_i^2}}{4kT{R_s}^2} + \frac{\overline{i_i^2}}{4kT} = 0$$

$$\frac{\overline{v_i^2}}{4kT{R_s}^2} = \frac{\overline{i_i^2}}{4kT} \Rightarrow \overline{i_i^2}{R_s}^2 = \overline{v_i^2}$$

$$\therefore (R_S)_{opt} = \sqrt{\frac{\overline{v_i^2}}{\overline{i_i^2}}} = \frac{v_i}{i_i}$$

Note: $v_i$ is in $\mathrm{V}/\sqrt{\mathrm{Hz}}$

$i_i$ is in $\mathrm{A}/\sqrt{\mathrm{Hz}}$

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# Noise Factor

$$F = 1 + \frac{\overline{v_i^2}}{4kTR_s} + \frac{\overline{i_i^2} R_S}{4kT}$$

The minimum $F$ at $(R_s)_{\mathrm{opt}}$ is given by:

$$F_{min} = 1 + \frac{\overline{v_i^2}}{4kT(R_s)_{opt}} + \frac{\overline{i_i^2}(R_S)_{opt}}{4kT}$$

$$= 1 + \frac{\overline{v_i^2}}{4kT}\left(\frac{i_i}{v_i}\right) + \frac{\overline{i_i^2}}{4kT}\left(\frac{v_i}{i_i}\right)$$

$$= 1 + \frac{v_i i_i}{2kT}$$

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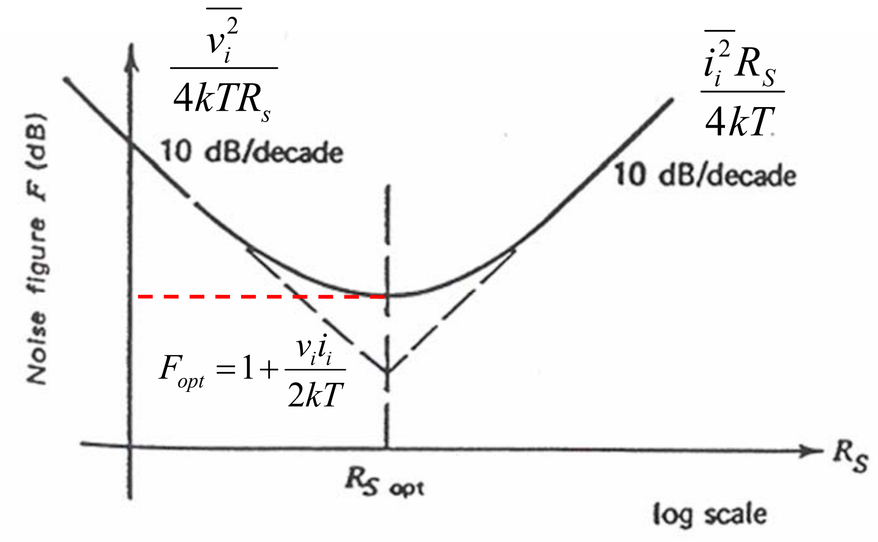
# $NF$ versus $R_s$




- Different devices have different $v_i$ and $i_i$, they have different ranges of $(R_s)_{opt}$.
- For a source with a given $R_s$, appropriate device should be selected so that its $(R_s)_{opt}$ equals to the $R_s$.



# Noise Factor for BJT

For BJTs in mid-band frequencies, neglecting flicker noise:

$$\overline{v_i^2} = 4kT\left(r_{bb'} + \frac{1}{2g_m}\right), \quad \overline{i_i^2} = 2q\left(I_B + \frac{I_C}{\beta^2}\right) = 2q\left(\frac{I_C}{\beta} + \frac{I_C}{\beta^2}\right) \approx \frac{2qI_C}{\beta}$$

$$(R_S)_{opt} = \sqrt{\frac{4kT(1+2g_m r_{bb'})}{2g_m} \times \frac{\beta}{2qI_C}} = \sqrt{\frac{kT(1+2g_m r_{bb'})}{g_m} \times \frac{\beta}{qI_C}}$$

$$= \sqrt{\frac{(1+2g_m r_{bb'})}{g_m} \times \frac{V_T\beta}{I_C}} \quad \because V_T = \frac{kT}{q}$$

$$= \sqrt{\frac{(1+2g_m r_{bb'})}{g_m} \times \frac{\beta}{g_m}}$$

$$= \frac{\sqrt{\beta}}{g_m}\sqrt{1+2g_m r_{bb'}}$$

# Noise Factor for BJT

$$F_{opt} = 1 + \frac{v_i i_i}{2kT}$$

$$= 1 + \frac{\sqrt{\frac{4kT\left(1+2g_m r_{bb'}\right)}{2g_m} \times \frac{2qI_C}{\beta}}}{2kT}$$

$$= 1 + \frac{\sqrt{4kT\left(1+2g_m r_{bb'}\right) \times \frac{qI_C}{\beta} \times \frac{V_T}{I_C}}}{2kT} \qquad \because g_m = \frac{I_C}{V_T}$$

$$= 1 + \frac{1}{\sqrt{\beta}}\sqrt{1+2g_m r_{bb'}}$$

To achieve low $F_{opt}$, choose BJT with high $\beta$ and low $r_{bb'}$. It should be biased at lower $I_C$ so that to lower $g_m$.

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# Example

Determine optimal source resistance and lowest possible noise figure for $I_c$ = 1mA, $\beta$=100 and $r_{bb'}$ = 50 Ω.

$$g_m = \frac{I_C}{V_T} = \frac{10^{-3}}{26\times10^{-3}} = 38.46 \text{ mS}$$

$$\left(R_S\right)_{opt} = \frac{\sqrt{\beta}}{g_m}\sqrt{1+2g_m r_{bb'}}$$

$$= \frac{\sqrt{100}}{38.46\times10^{-3}}\sqrt{1+2\left(38.46\times10^{-3}\right)\left(50\right)} = 572.4\Omega$$

$$F_{opt} = 1 + \frac{1}{\sqrt{\beta}}\sqrt{1+2g_m r_{bb'}}$$

$$= 1 + \frac{1}{\sqrt{100}}\sqrt{1+2\left(38.46\times10^{-3}\right)\left(50\right)} = 1.22$$

$$NF_{opt} = 10\log\left(1.22\right) = 0.86 \text{ dB}$$

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# Noise Factor for FET

For FETs in mid-band frequencies, neglecting flicker noise,

$$\overline{v_i^2} = 4kT\left(\frac{2}{3g_m}\right), \quad \overline{i_i^2} \approx 0$$

$$\left(R_S\right)_{opt} = \frac{v_i}{i_i} \approx \infty$$

$$F_{opt} = 1 + \frac{v_i i_i}{2kT} \approx 1$$

Thus the FET has excellent signal to noise ratio performance for high $R_s$. FET usually has significant lower *NF* than a BJT for $R_s$ of the order of MΩ or higher.

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# Noise Analysis

For a constant input signal $V_s$, maximum output SNR occurs when $R_s = v_i/i_i$.

In reality, usually $R_s$ has already been fixed. If the actual $R_s$ is less than $v_i/i_i$, **we should not add a physical resistor in series with $R_s$ to make it equal to $v_i/i_i$** . This will actually affect the *SNR* of the circuit as follows:

- Increase the thermal noise due to additional resistor
- Increase the noise effect of $i_i$
- Reduce the useful signal to the input of the amplifier

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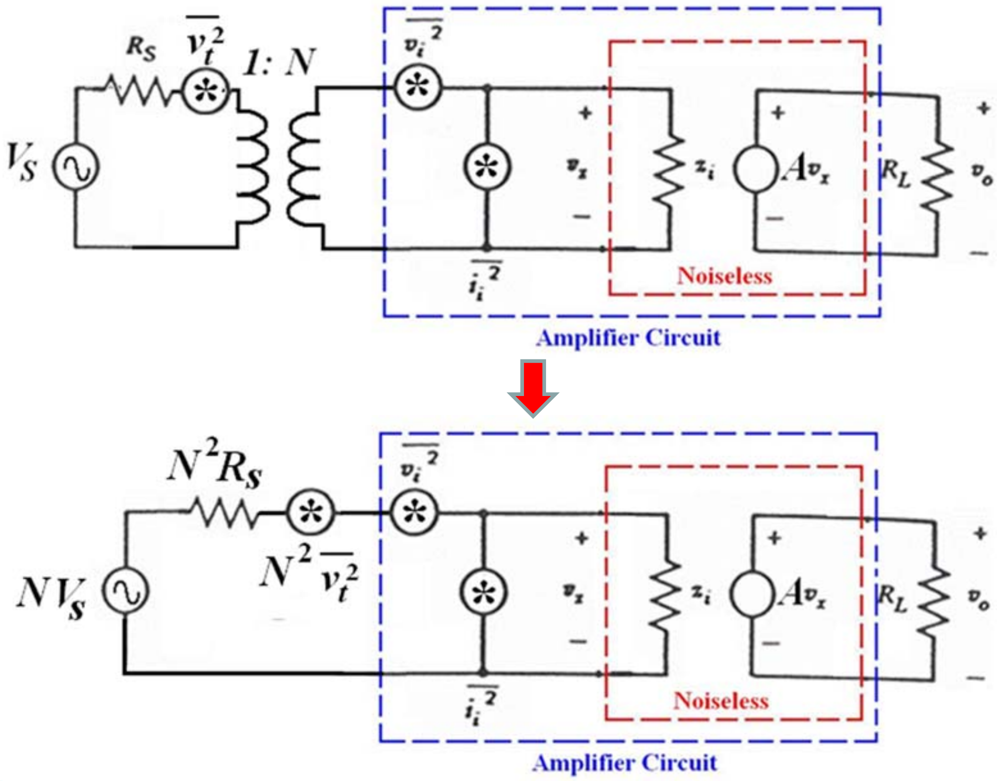
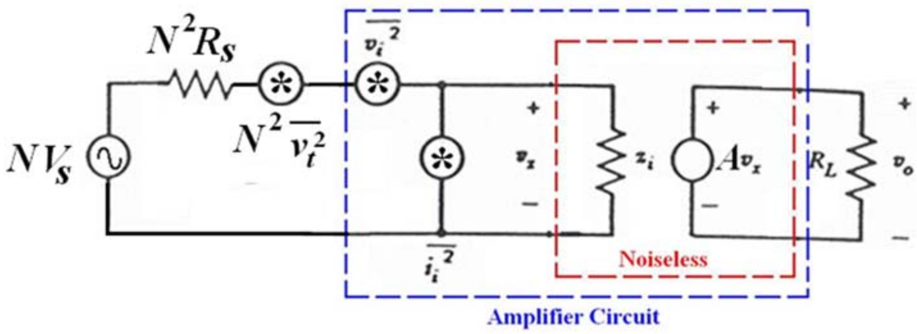
# Transformer Coupling






# Transformer Coupling




$$v_x = NV_s \times \frac{z_i}{N^2R_s + z_i}$$

$$v_x = \left(Nv_t + v_i\right)\left(\frac{z_i}{N^2R_s + z_i}\right) + i_i\left(\frac{N^2R_s z_i}{N^2R_s + z_i}\right)$$

$$S_o = \frac{A^2 v_x{}^2}{R_L} = \frac{A^2N^2V_s{}^2}{R_L}\left|\frac{z_i}{N^2R_s + z_i}\right|^2$$

$$\overline{v_x{}^2} = \left(N^2\overline{v_t{}^2} + \overline{v_i{}^2}\right)\left|\frac{z_i}{N^2R_s + z_i}\right|^2 + \overline{i_i{}^2}\left|\frac{N^2R_s z_i}{N^2R_s + z_i}\right|^2$$

$$N_o = \frac{A^2\overline{v_x{}^2}\Delta f}{R_L} = \frac{A^2\Delta f}{R_L}\left[\left(N^2\overline{v_t{}^2} + \overline{v_i{}^2}\right)\left|\frac{z_i}{N^2R_s + z_i}\right|^2 + \overline{i_i{}^2}\left|\frac{N^2R_s z_i}{N^2R_s + z_i}\right|^2\right]$$

# Transformer Coupling

$$S_o = \frac{A^2 N^2 V_s^2}{R_L}\left|\frac{z_i}{N^2 R_s + z_i}\right|^2 \qquad N_o = \frac{A^2 \Delta f}{R_L}\left[\left(N^2\overline{v_t^2} + \overline{v_i^2}\right)\left|\frac{z_i}{N^2 R_s + z_i}\right|^2 + \overline{i_i^2}\left|\frac{N^2 R_s z_i}{N^2 R_s + z_i}\right|^2\right]$$

$$\frac{S_o}{N_o} = \frac{N^2 V_s^2}{\left[\overline{v_i^2} + \overline{i_i^2}\left(N^2 R_s\right)^2 + 4kT\left(N^2 R_s\right)\right]\Delta f}$$

$$\frac{d\left(S_o/N_o\right)}{dN} = \frac{2NV_s^2 \Delta f\left[\overline{v_i^2} + 4kTN^2 R_s - \overline{i_i^2} N^4 R_s^{\,2} - 4kTN^2 R_s\right]}{\left[\overline{v_i^2} + \overline{i_i^2}\left(N^2 R_s\right)^2 + 4kT\left(N^2 R_s\right)\right]^2 (\Delta f)^2} = 0$$

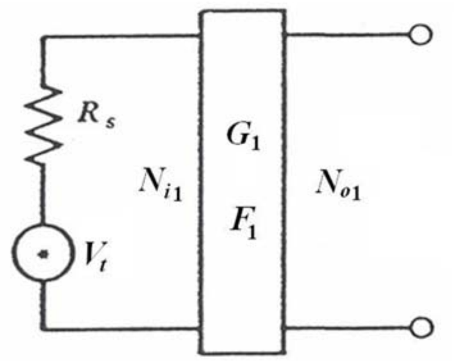
$$\text{Note}: \frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

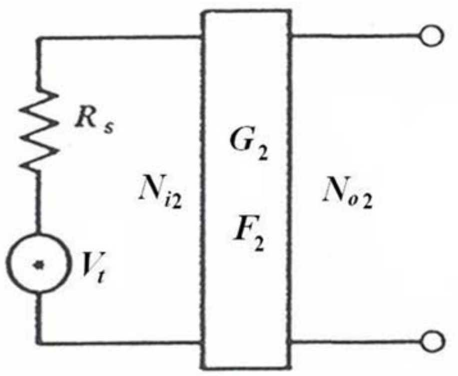
$$\overline{v_i^2} - \overline{i_i^2} N^4 R_s^{\,2} = 0 \Rightarrow N^2 R_s = \frac{v_i}{i_i}$$

$$\therefore N = \sqrt{\frac{v_i / i_i}{R_s}}$$



# Noise Analysis of Cascaded Networks




$$V_t = \sqrt{4kTR_s\Delta f}$$

Max. noise power at the input (i.e. the input resistance of the device is equal to $R_S$):

$$N_{i1} = \left(\frac{V_t}{2}\right)^2 \times \frac{1}{R_s} = \frac{V_t^2}{4R_s} = \frac{4kTR_s\Delta f}{4R_s} = kT\Delta f$$

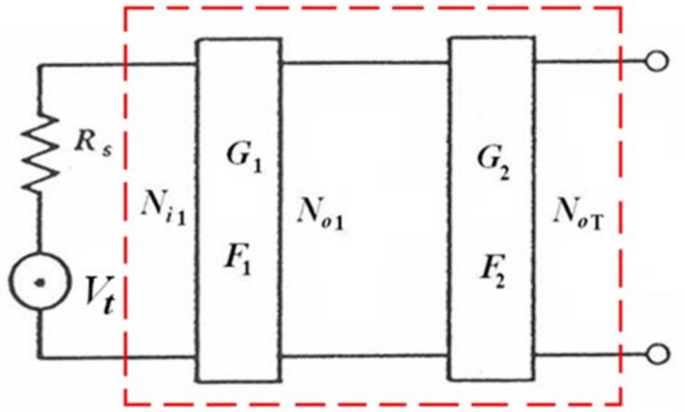
$$F_1 = \frac{N_{o1}}{N_{i1}G_1} \Rightarrow N_{o1} = F_1 G_1 N_{i1} = F_1 G_1 kT\Delta f$$

Similarly, $N_{i2} = kT\Delta f$

$$N_{o2} = F_2 G_2 N_{i2} = F_2 G_2 kT\Delta f$$

Noise power contributed by stage 2 only:

$$N_{o2} - G_2 N_{i2} = G_2\left(F_2 - 1\right)kT\Delta f$$




$$F = \frac{N_{oT}}{N_{i1}G_1G_2} = \frac{N_{oT}}{G_1G_2kT\Delta f}$$

$$N_{oT} = G_2N_{o1} + G_2(F_2 - 1)kT\Delta f$$

$$\because N_{o1} = F_1G_1kT\Delta f$$

$$\therefore N_{oT} = G_2(F_1G_1kT\Delta f) + G_2(F_2 - 1)kT\Delta f = (F_1G_1G_2 + F_2G_2 - G_2)kT\Delta f$$

$$F = \frac{N_{oT}}{G_1G_2kT\Delta f} = \frac{(F_1G_1G_2 + F_2G_2 - G_2)kT\Delta f}{G_1G_2kT\Delta f} = F_1 + \frac{F_2 - 1}{G_1}$$

The noise factor for 2-stage system is given by: $F = F_1 + \dfrac{F_2 - 1}{G_1}$



Similarly for 3-stage system:

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2}$$

For $n$ cascaded stages, the noise factor can be determined by:

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \frac{F_4 - 1}{G_1G_2G_3} \ldots\ldots + \frac{F_n - 1}{G_1G_2 \ldots G_{n-1}}$$

**The noise factor of the cascaded network is primarily influenced by the 1st stage noise.**

**To have a low overall noise factor, the power gain (or voltage gain) of the 1st stage must be made as large as possible.**

Example: A receiver consists of the following circuits in consecutive order : Two RF amplifiers (gain 10 dB each and noise figure 4 dB each), a mixer with a gain of -6 dB and noise figure of 8 dB, and two power amplifiers (gain 20 dB each and noise figure 15 dB each). Determine the system noise figure.

$$10\,log\,G_{RF} = 10\text{ dB},\ \therefore G_{RF} = 10^{10/10} = 10$$

$$NF_{RF} = 4\text{ dB},\ \therefore F_{RF} = 10^{4/10} = 2.51$$

$$10\,log\,G_{Mixer} = -6\text{ dB},\ \therefore G_{RF} = 10^{-6/10} = 0.251$$

$$NF_{Mixer} = 8\text{ dB},\ \therefore F_{Mixer} = 10^{8/10} = 6.31$$

$$10\,log\,G_{PA} = 20\text{ dB},\ \therefore G_{PA} = 10^{20/10} = 100$$

$$NF_{PA} = 15\text{ dB},\ \therefore F_{Mixer} = 10^{15/10} = 31.6$$

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$$F = F_{RF1} + \frac{F_{RF2}-1}{G_{RF1}} + \frac{F_{mixer}-1}{G_{RF1}G_{RF2}} + \frac{F_{PA1}-1}{G_{RF1}G_{RF2}G_{Mixer}} + \frac{F_{PA2}-1}{G_{RF1}G_{RF2}G_{Mixer}G_{PA1}}$$

$$\begin{aligned} F &= 2.51 + \frac{2.51-1}{10} + \frac{6.31-1}{10\times10} + \frac{31.6-1}{10\times10\times0.251} + \frac{31.6-1}{10\times10\times0.251\times100} \\ &= 2.51 + 0.151 + 0.0531 + 1.22 + 0.0122 + 0.000122 \\ &= 3.95 \end{aligned}$$

$$NF = 10\,log\,3.95 = 5.96\text{ dB}$$

The poor gain and high noise factor of the mixer deteriorate the overall system noise figure.

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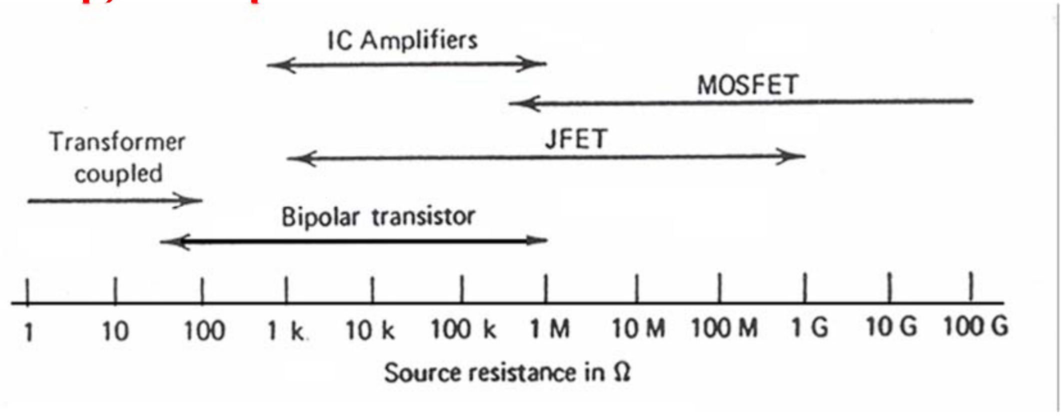
# Low Noise Amplifier Design

1. **Choose the appropriate active device (BJT, FET or Op Amp) for a specific source resistance.**




If multi-stage amplifier is designed, make the gain of the first stage as large as possible and the noise factor of first stage as low as possible.



# Low Noise Amplifier Design

2. **Choose passive components with low noise behaviors**

- Metal film resistors shall be used for input coupling circuit in order to achieve the lowest equivalent input noise. The dc voltage across the resistor shall be low to reduce the excess noise.

- Mica and ceramic capacitors are preferred in low-noise designs. When large capacitance is required, use tantalum but not aluminum electrolytic capacitor.

# Low Noise Amplifier Design

## 3. Noise matching techniques

• Adjust biasing current (e.g. $I_C$ for BJT) to make $(R_s)_{opt}$ equal to the source resistance.

• Choose the proper turn ratio if transformer coupling is used.