EE4341/EE6341 Advanced Analog Circuits - Active Filter Design

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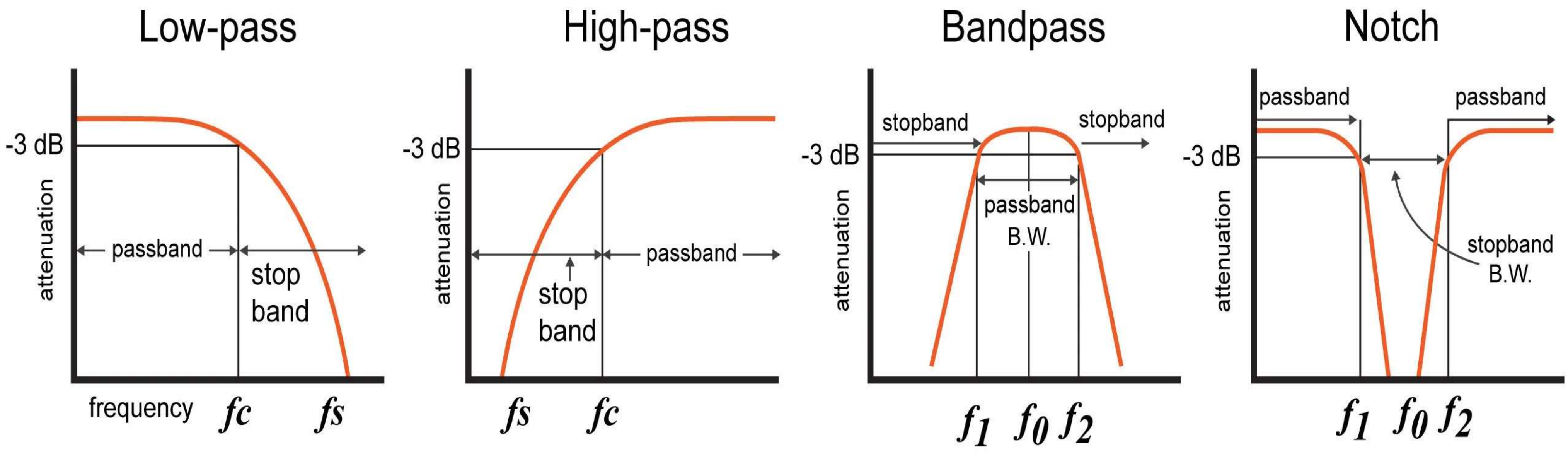
Introduction



- A filter processes signals on a frequency-dependent basis.
- It is described by its frequency response represented by its transfer function $T(j\omega)$.
- $T(j\omega)$ has both magnitude response $|T(j\omega)|$ and phase response $\angle T(j\omega)$.

Poles and Zeros of Transfer Function

A transfer function that relates output and input can be expressed by:

zeros at $z_1, z_2, \dots. \; z_n$

$$T(s) = \frac{P(s)}{Q(s)} = b_o \frac{(s - z_1)(s - z_2)(s - z_3)\cdots(s - z_n)}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)\cdots(s - \lambda_n)}$$

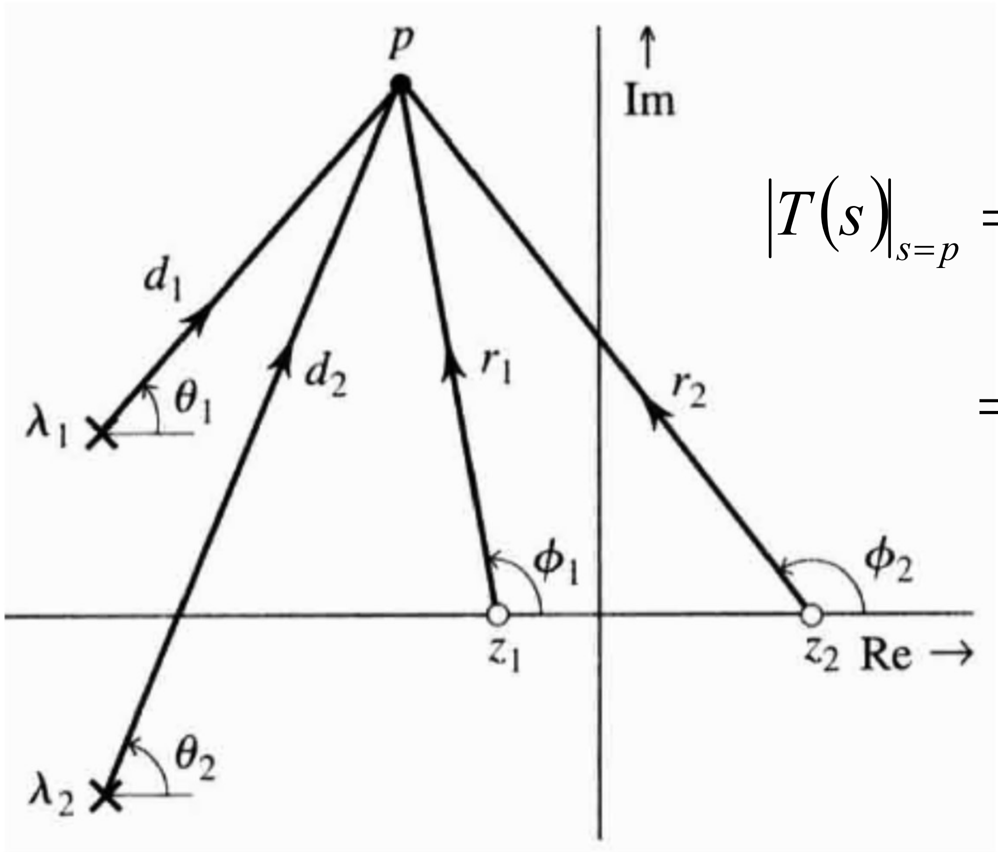
poles at $\lambda_1, \lambda_2, \dots. \; \lambda_n$

The value of the transfer function at a complex frequency $s = p$ is:

$$T(s)\big|_{s=p} = b_o \frac{(p - z_1)(p - z_2)(p - z_3)\cdots(p - z_n)}{(p - \lambda_1)(p - \lambda_2)(p - \lambda_3)\cdots(p - \lambda_n)}$$

$$= b_o \frac{\left(r_1 e^{j\phi_1}\right)\left(r_2 e^{j\phi_2}\right)\left(r_3 e^{j\phi_3}\right)\cdots\left(r_n e^{j\phi_n}\right)}{\left(d_1 e^{j\theta_1}\right)\left(d_2 e^{j\theta_2}\right)\left(d_3 e^{j\theta_3}\right)\cdots\left(d_n e^{j\theta_n}\right)}$$

For example, a transfer function has 2 zeros and 2 poles, the magnitude and phase at $s = p$ is given by:



$$\left|T(s)\right|_{s=p} = b_o \frac{r_1 r_2}{d_1 d_2}$$

$$= b_o \frac{\text{Product of distances of zeros to } p}{\text{Product of distances of poles to } p}$$

$$\angle T(s)\big|_{s=p} = (\phi_1 + \phi_2) - (\theta_1 + \theta_2)$$

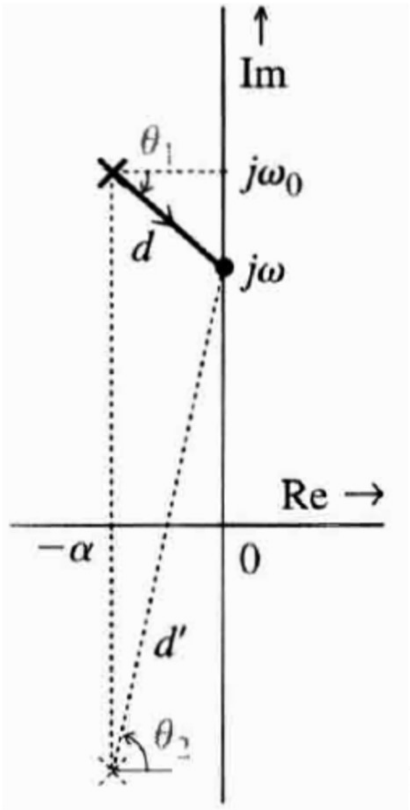
$$= \text{sum of zero angles to } p - \text{sum of pole angles to } p$$

Effects of Poles on Frequency Response

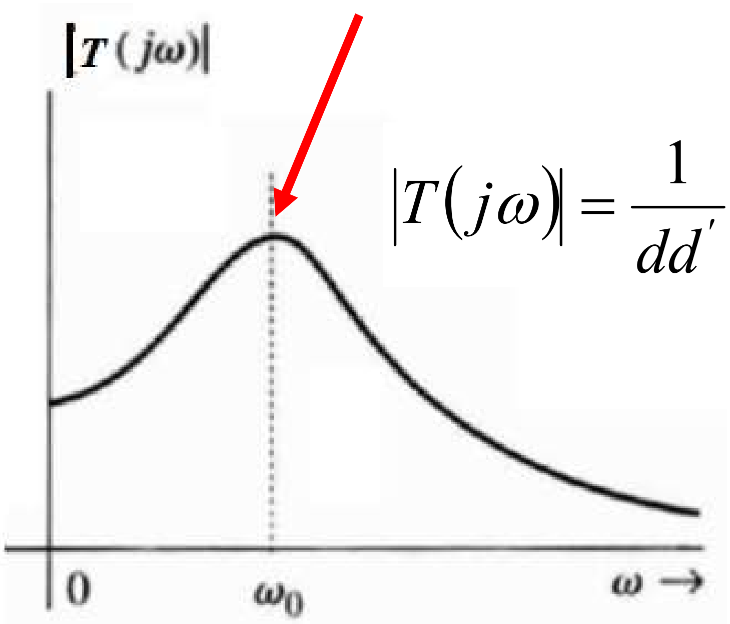
Frequency response of a system is obtained by evaluating $T(s)$ along the y-axis (i.e. $s = j\omega$)

For example, two complex poles on the frequency response:

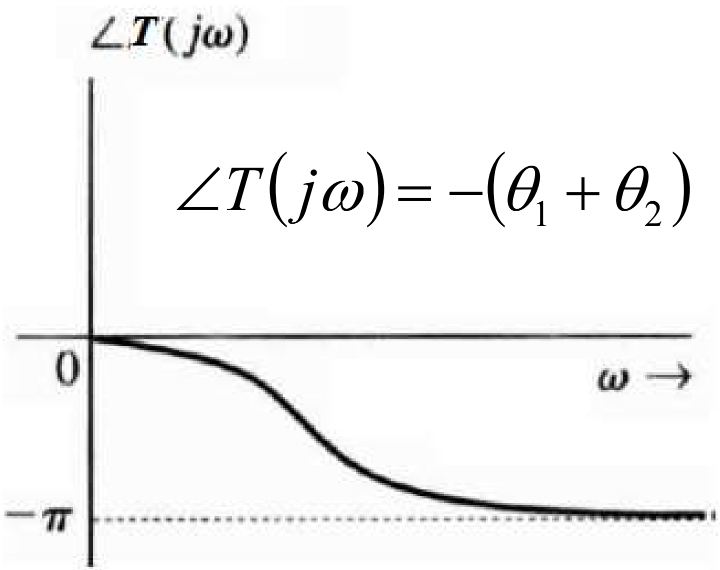
$$T(s) = \frac{1}{[s-(-\alpha + j\omega_o)][s-(-\alpha - j\omega_o)]}$$



The amplitude enhances near a pole



$$|T(j\omega)| = \frac{1}{dd'}$$

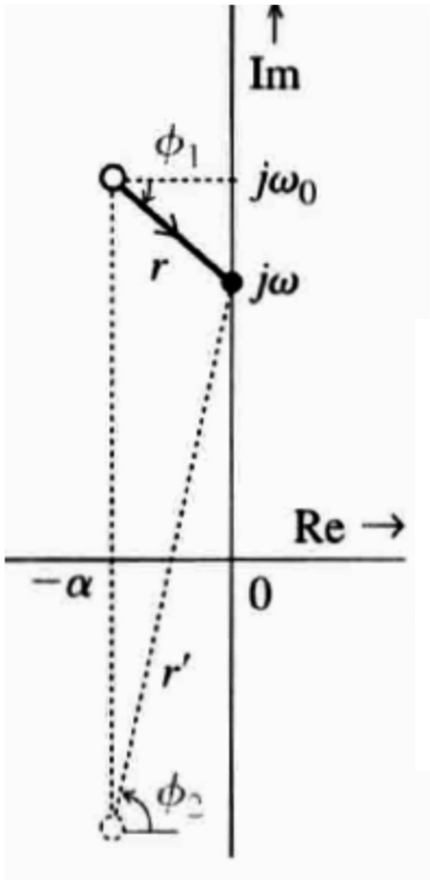


$$\angle T(j\omega) = -(\theta_1 + \theta_2)$$

Effects of Zeros on Frequency Response

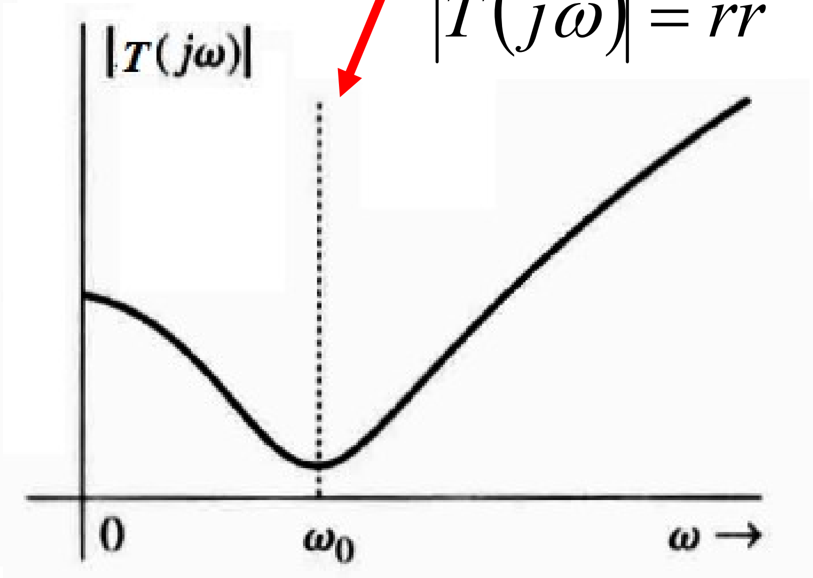
For example, two complex zeros on the frequency response:

$$T(s) = [s - (-\alpha + j\omega_o)][s - (-\alpha - j\omega_o)]$$

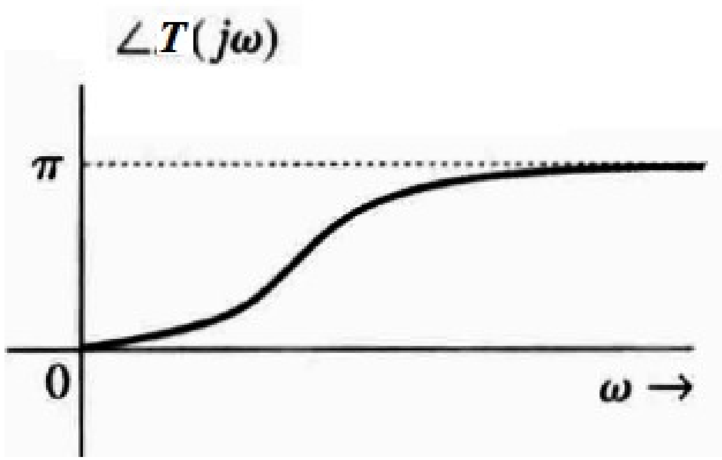


The amplitude suppresses near a zero

$$|T(j\omega)| = rr'$$



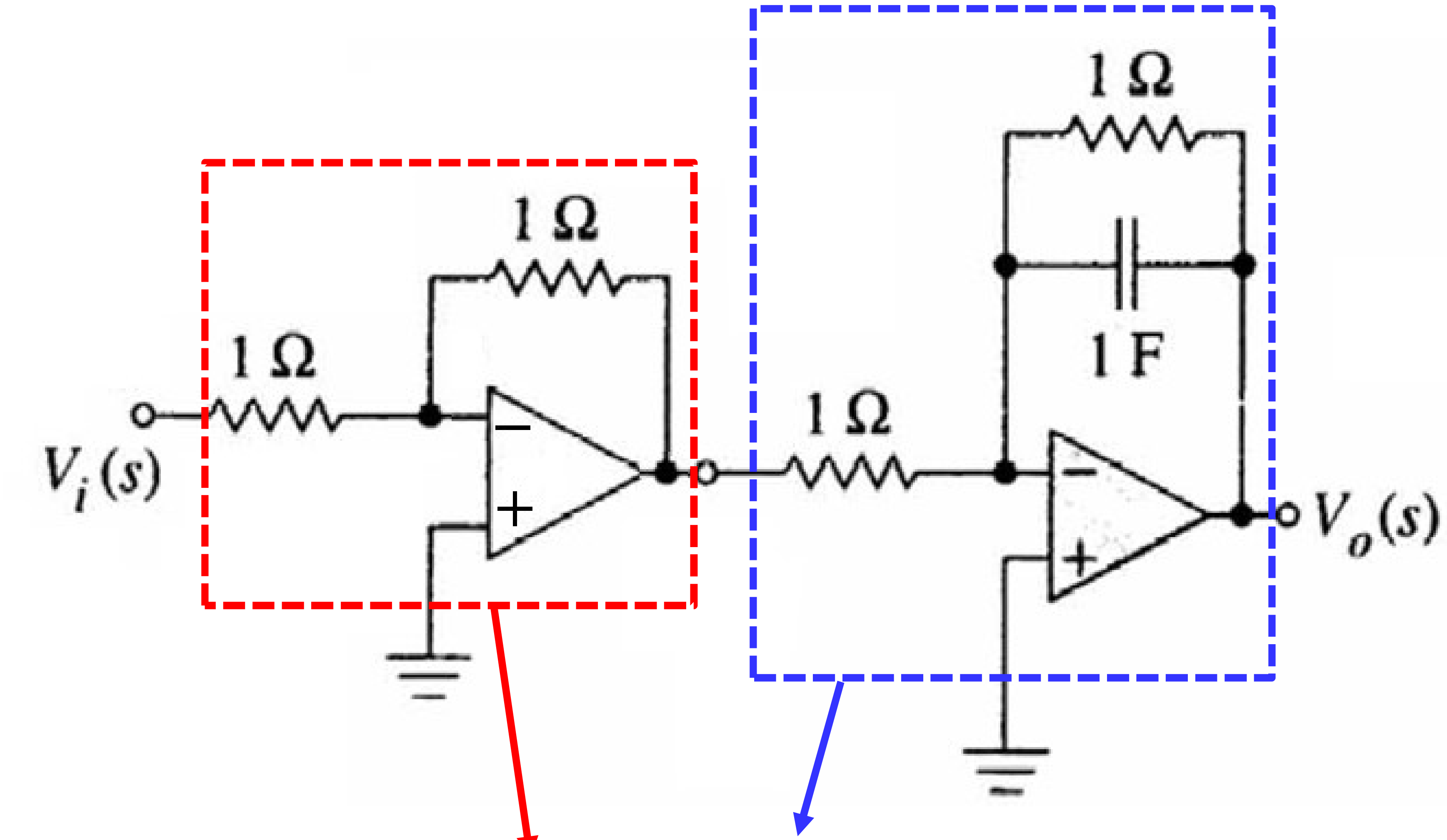
$$\angle T(j\omega) = (\phi_1 + \phi_2)$$



Why Active Filters?

- A passive filter consists of passive components, such as inductors (L), resistors (R), and capacitors (C).
- Value of L becomes large and its size bulky as frequency reduces (e.g. < 1 MHz), making compact filter design challenging.
- Active filters use operational amplifiers in combination with R and C without L, and the size of the filters become very compact.
- The modular approach used in the synthesis of higher-order active filters simplifies the design without worrying the loading effects.

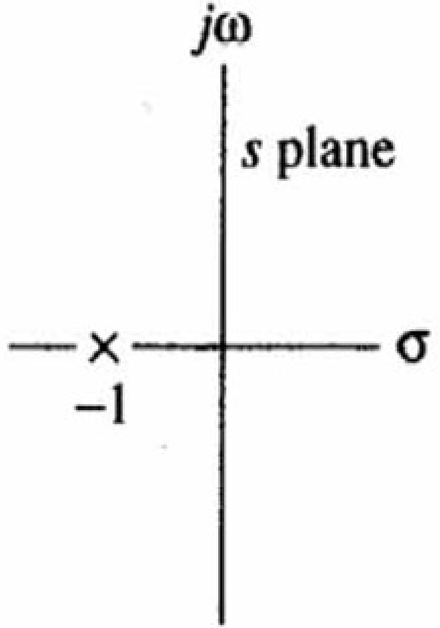
1st Order Active Low-pass Filter



$$T(s)=\frac{V_o(s)}{V_i(s)}=(-1)\left[\frac{-(1)(1/s)}{1+(1/s)}\right]=\frac{1}{s+1}$$

$s+1=0$

One pole at $s=-1$



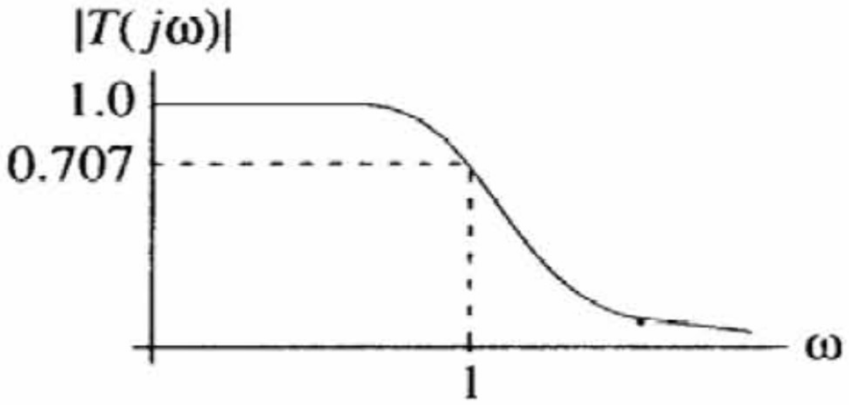
$$T(s) = \frac{1}{s+1}$$

Put $s = j\omega$ to obtain the frequency response

$$T(j\omega) = \frac{1}{1+j\omega}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

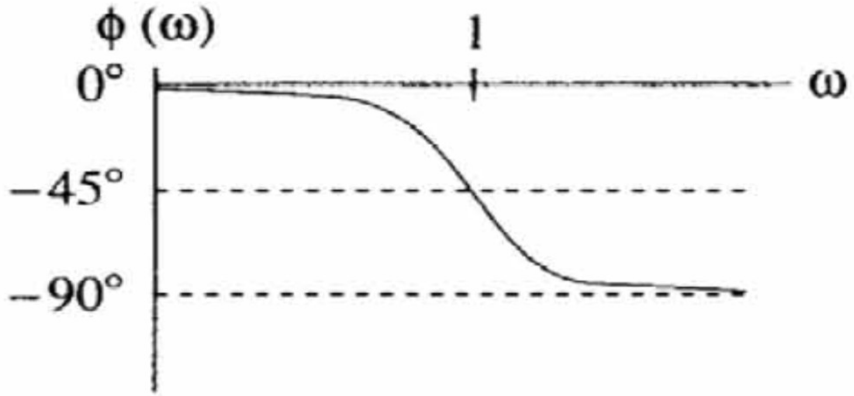
$$\angle T(j\omega) = -tan^{-1}\,\omega$$



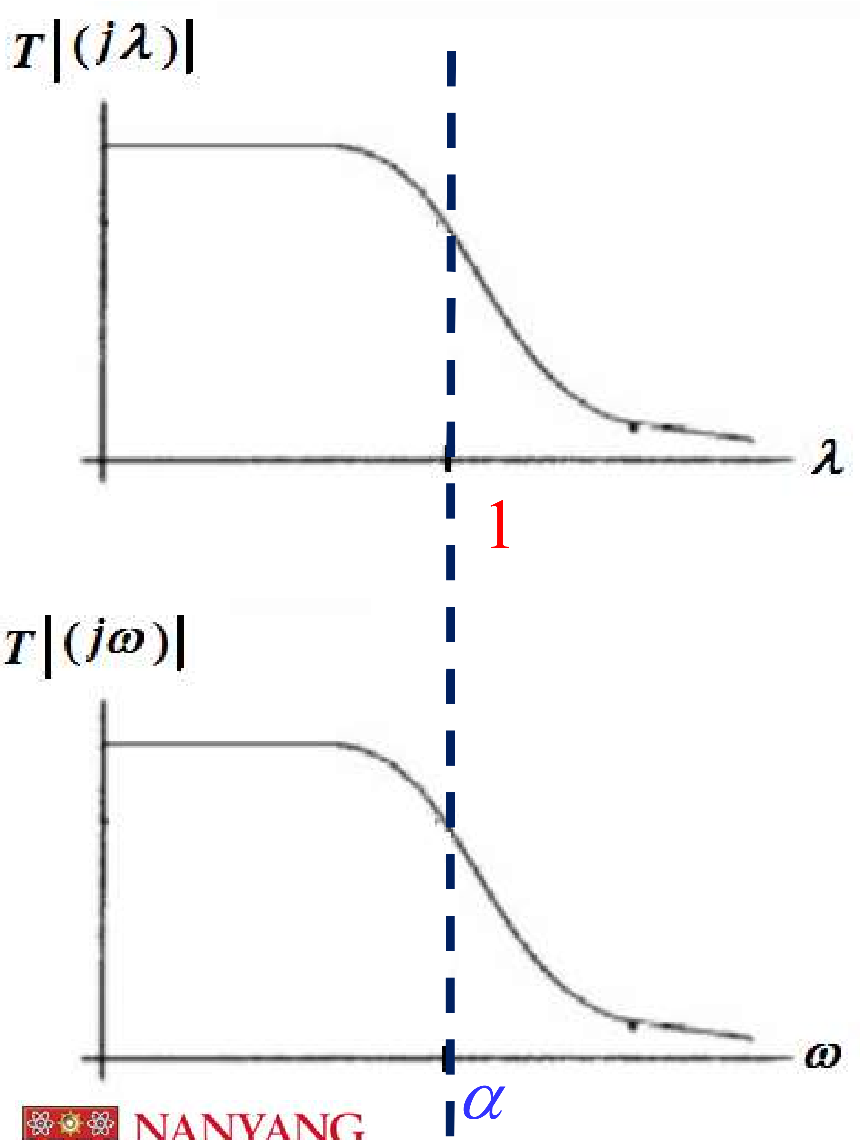
When $\omega = 1$ rad/s,

$$|T(j\omega)| = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$$

$$\angle T(j\omega) = -tan^{-1}\,1 = -45^{\circ}$$

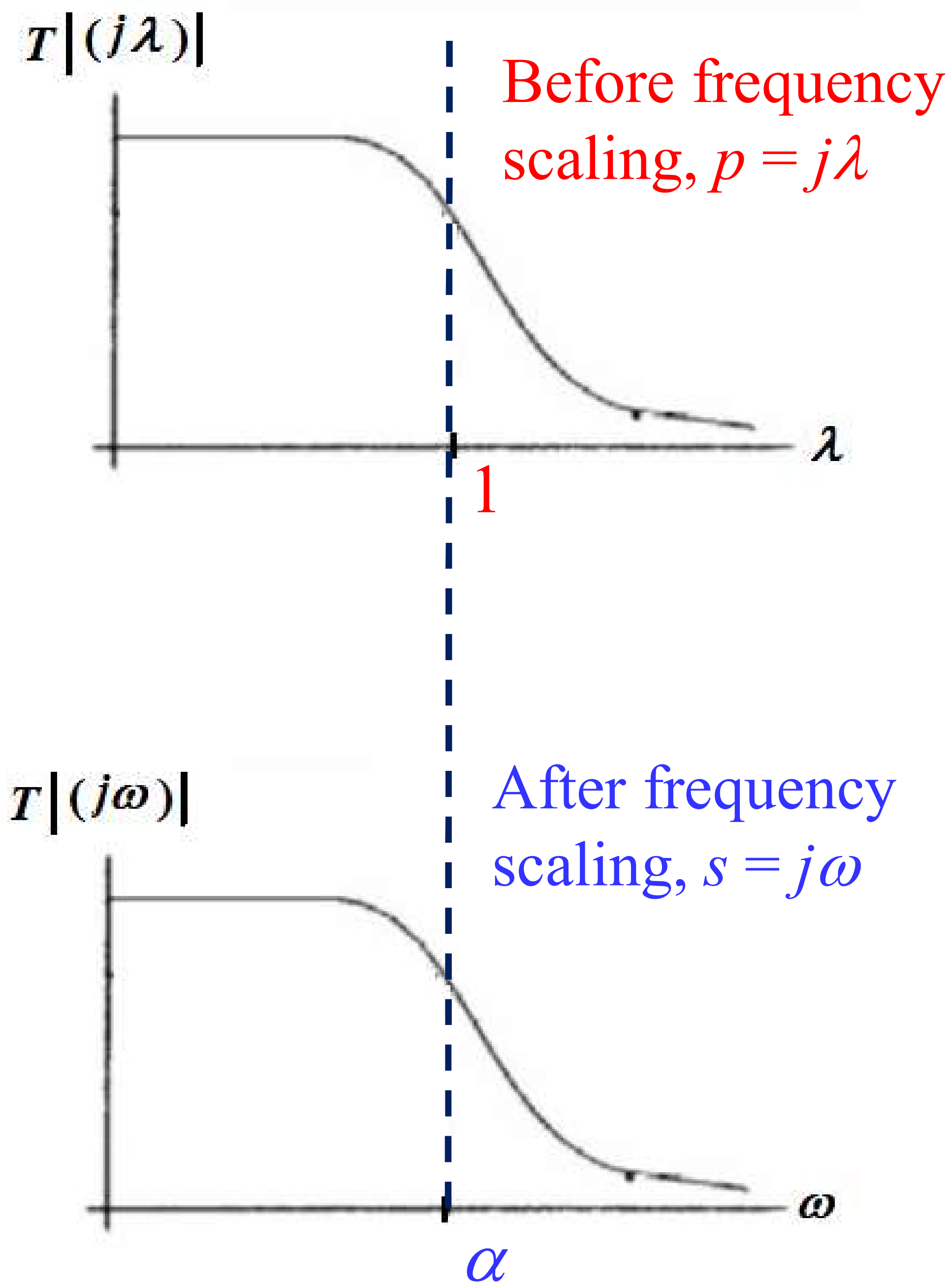


Frequency Scaling



- For ease of design, any linear filter can be designed with a reference frequency of $\omega = 1$ rad/s.

- Its magnitude and phase frequency responses can be shifted to the desired frequency $\omega = \alpha$ rad/s by applying a frequency scaling procedure.



- The frequency-scaled transfer function $T(j\omega)$ is given by:

$$T(j\omega) = T(j\lambda)\Big|_{\lambda=\frac{\omega}{\alpha}} = T\left(\frac{j\omega}{\alpha}\right)$$

where α is the frequency scaling factor and $\omega = \alpha\,\lambda\ (\alpha > 0)$.

- Frequency scaled transfer function will be identical to the transfer function before frequency scaling.

Scaled Values of C and L

$$X_C = \frac{1}{j\lambda C} = \frac{1}{\left(\dfrac{j\omega}{\alpha}\right)C} = \frac{1}{j\omega\left(\dfrac{C}{\alpha}\right)} = \frac{1}{j\omega C'}$$

Note: $\omega = \alpha\,\lambda$

Scaling to higher frequency by a factor α ($\alpha > 1$) reduces the value of C by a factor of α ($C' = C/\alpha$).

$$X_L = j\lambda L = \left(\frac{j\omega}{\alpha}\right)L = j\omega\left(\frac{L}{\alpha}\right) = j\omega L'$$

Scaling to higher frequency by a factor α ($\alpha > 1$) reduces the value of L by a factor of α ($L' = L/\alpha$).

Scaling to lower frequency by a factor α ($\alpha < 1$) increase the values of L and C by $1/\alpha$.

Standard Resistor Values

Resistance Picker

1.0Ω	1.2Ω	1.5Ω	1.8Ω	2.2Ω	2.7Ω	3.3Ω	3.9Ω
4.7Ω	5.6Ω	6.8Ω	8.2Ω	10Ω	12Ω	15Ω	18Ω
22Ω	27Ω	33Ω	39Ω	47Ω	56Ω	68Ω	82Ω
100Ω	120Ω	150Ω	180Ω	220Ω	270Ω	330Ω	390Ω
1.0kΩ	1.2kΩ	1.5kΩ	1.8kΩ	2.2kΩ	2.7kΩ	3.3kΩ	3.9kΩ
4.7kΩ	5.6kΩ	6.8Ωk	8.2kΩ	10kΩ	12kΩ	15kΩ	18kΩ
22kΩ	27kΩ	33kΩ	39kΩ	47kΩ	56kΩ	68kΩ	82kΩ
100kΩ	120kΩ	150kΩ	180kΩ	270kΩ	330kΩ	390kΩ	
470kΩ	560kΩ	680kΩ	820kΩ	1MΩ	1.2MΩ	1.5MΩ	1.8MΩ
2.2MΩ	2.7MΩ	3.3MΩ	3.9MΩ	4.7MΩ	5.6MΩ	6.8MΩ	8.2MΩ
10MΩ	12MΩ	15MΩ	18MΩ	22MΩ			

Standard Capacitor Values

STANDARD CAPACITOR VALUES

These are the EIA **standard capacitor values.** These are the values available from most vendors. Non-polarized run from 1pF to 1uf, while electrolytics are available from 0.1uF and higher (not all electrolytic values listed here).

1.0pF	10pF	100pF	.001uF	.01uF	.1uF	1.0uF	10uF
1.2pF	12pF	120pF	.0012uF	.012uF	.12uF	1.2uF	12uF
1.5pF	15pF	150pF	.0015uF	.015uF	.15uF	1.5uF	15uF
1.8pF	18pF	180pF	.0018uF	.018uF	.18uF	1.8uF	18uF
2.2pF	22pF	220pF	.0022uF	.022uF	.22uF	2.2uF	22uF
2.7pF	27pF	270pF	.0027uF	.027uF	.27uF	2.7uF	27uF
3.3pF	33pF	330pF	.0033uF	.033uF	.33uF	3.3uF	33uF
3.9pF	39pF	390pF	.0039uF	.039uF	.39uF	3.9uF	39uF
4.7pF	47pF	470pF	.0047uF	.047uF	.47uF	4.7uF	47uF
5.6pF	56pF	560pF	.0056uF	.056uF	.56uF	5.6uF	56uF
6.8pF	68pF	680pF	.0068uF	.068uF	.68uF	6.8uF	68uF

Impedance Scaling

To design a filter with available standard component values, the impedance of original component values must be scaled by a factor β ($\beta > 0$) as follows:

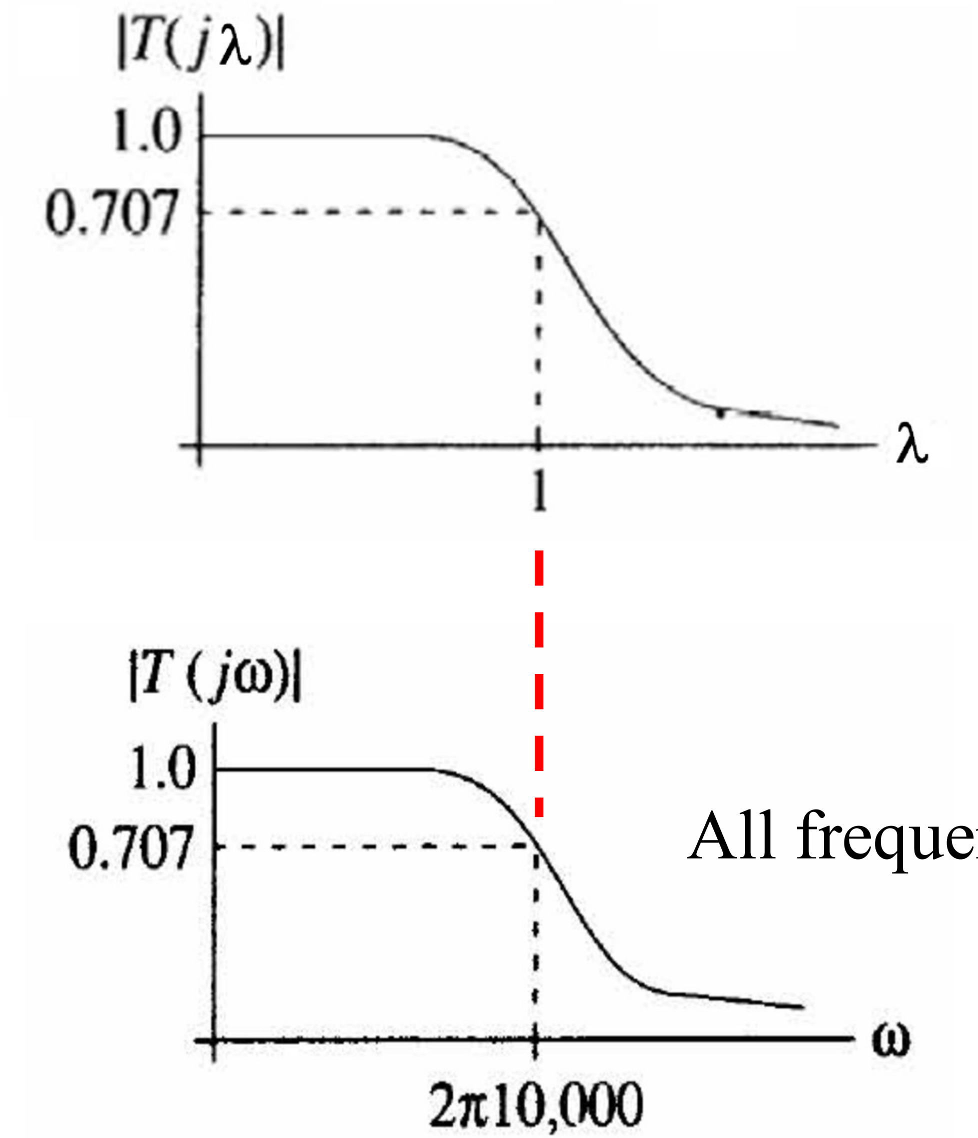
$$R' = \beta R$$

$$X_L' = \beta X_L = \beta j\omega L = j\omega\beta L = j\omega L' \qquad L' = \beta L$$

$$X_C' = \beta X_C = \frac{\beta}{j\omega C} = \frac{1}{j\omega(C/\beta)} = \frac{1}{j\omega C'} \qquad C' = \frac{C}{\beta}$$

If all the components' impedances are scaled by the same factor, the frequency response of the filter is unaffected by impedance scaling.

Exercise #1: Design a 1st order *RC* active low-pass filter with –3dB frequency of 10 kHz. Standard 10 kΩ resistor value is preferred.

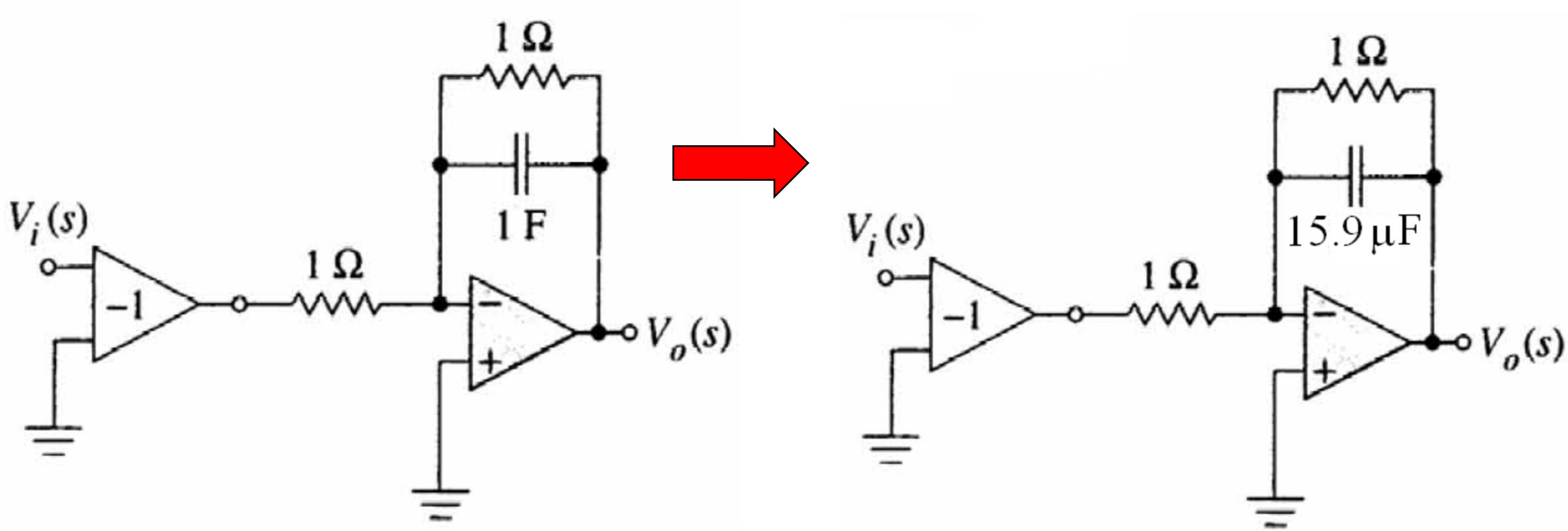


Frequency scaling factor:

$$\alpha = \frac{2\pi \times 10^4}{1} = 2\pi \times 10^4$$

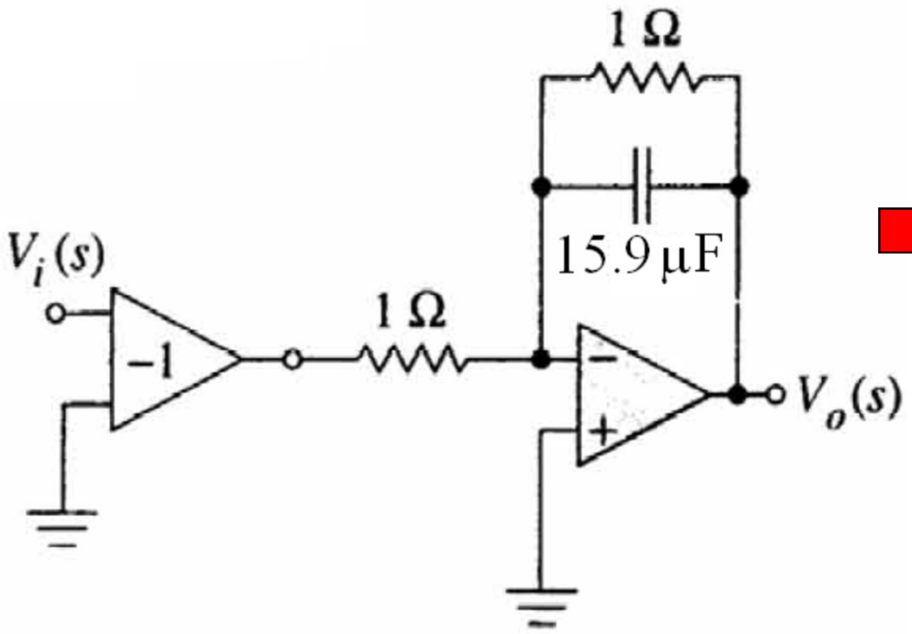
All frequencies shifts by a factor of α

Impedance scaling by $\alpha = 2\pi \times 10^4$

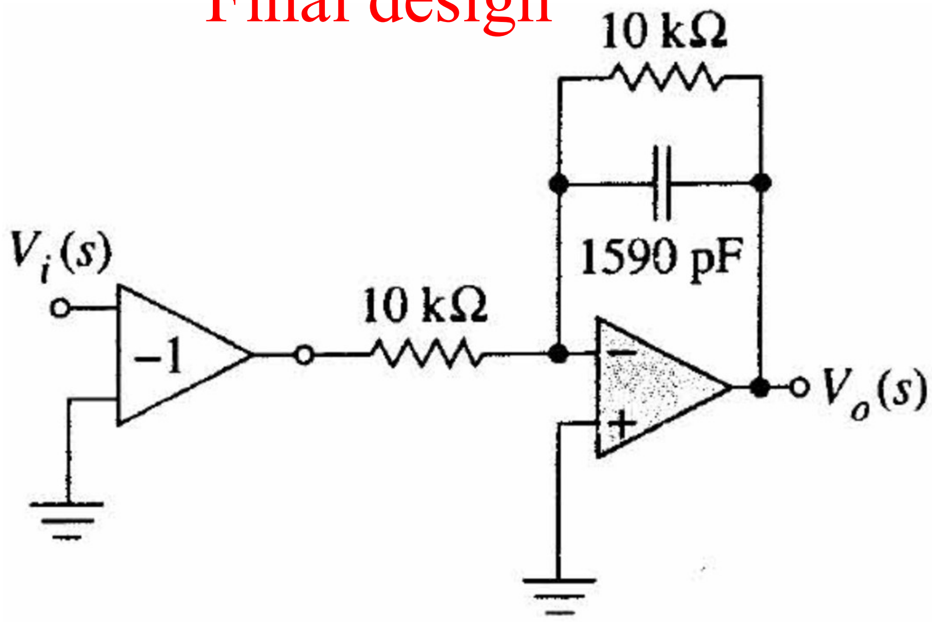


$$C' = \frac{C}{\alpha} = \frac{1\,\mathrm{F}}{2\pi \times 10^4} = 15.9\ \mu\mathrm{F}$$

Impedance scaling by $\beta = 10\text{k}\Omega / 1\Omega = 10^4$

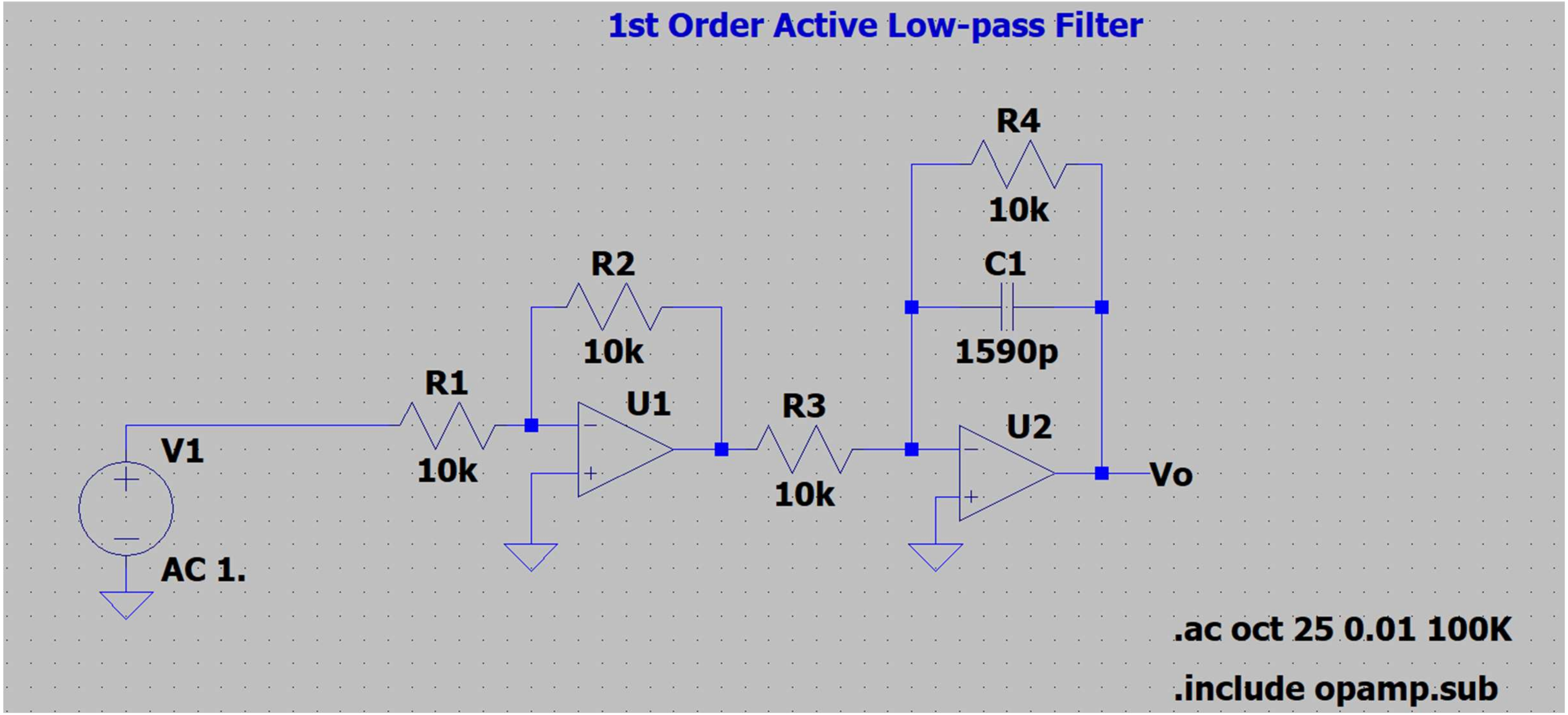


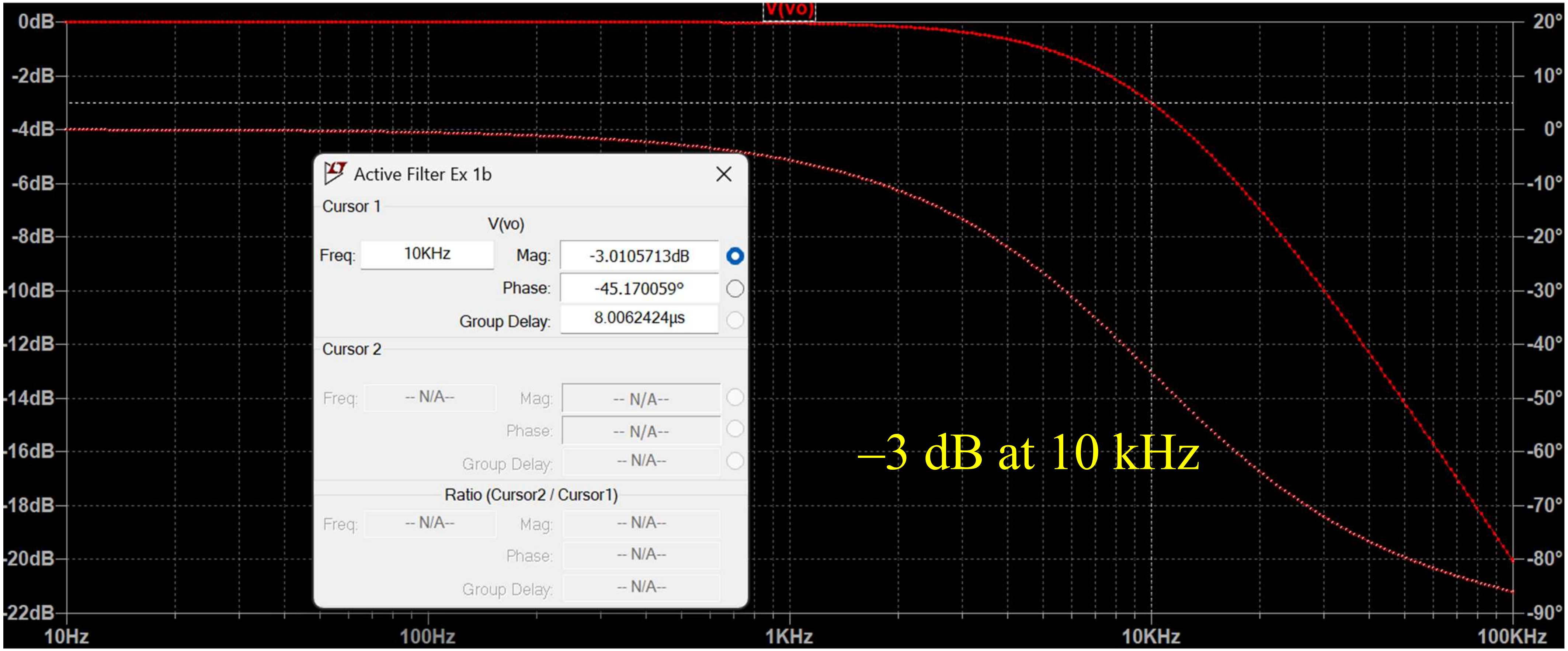
Final design



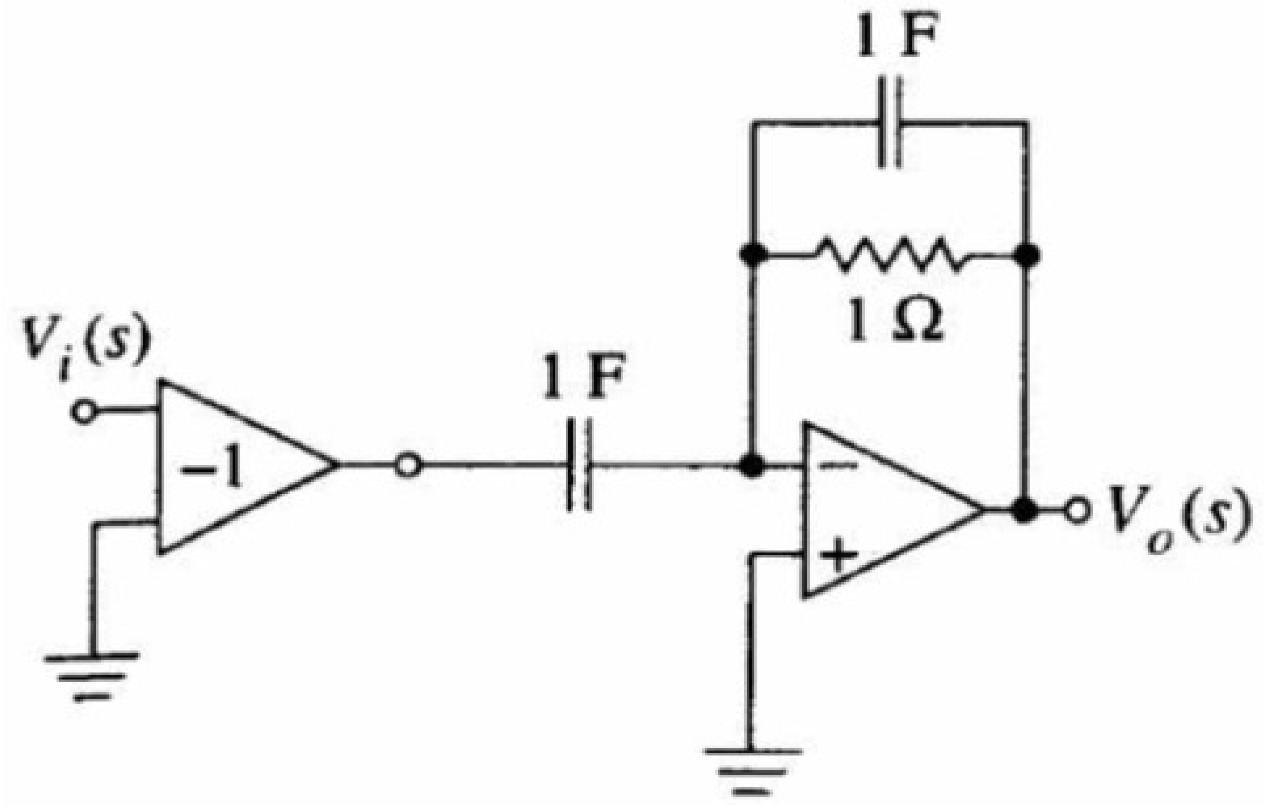
$$C'' = \frac{C'}{\beta} = \frac{15.9\ \mu\text{F}}{10^4} = 1590\ \text{pF}$$

Verification with LTSPICE simulation:



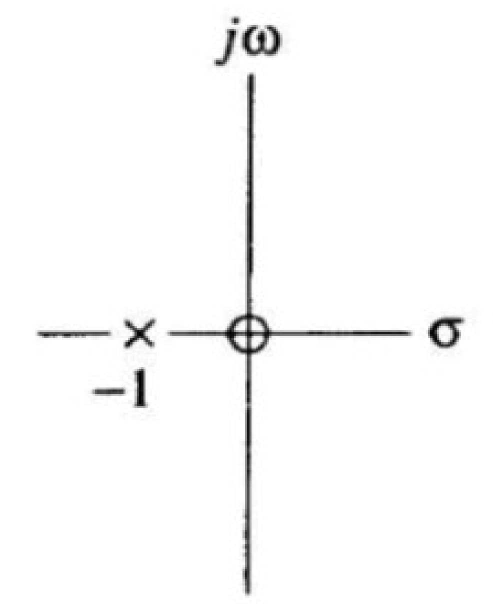

V(vo)
0dB
-2dB
-4dB
-6dB
-8dB
-10dB
-12dB
-14dB
-16dB
-18dB
-20dB
-22dB
20°
10°
0°
-10°
-20°
-30°
-40°
-50°
-60°
-70°
-80°
-90°
10Hz
100Hz
1KHz
10KHz
100KHz
Active Filter Ex 1b
Cursor 1
V(vo)
Freq: 10KHz
Mag: -3.0105713dB
Phase: -45.170059°
Group Delay: 8.0062424µs
Cursor 2
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
Ratio (Cursor2 / Cursor1)
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
–3 dB at 10 kHz

1st Order Active High-pass Filter



$$T(s)=\frac{V_o(s)}{V_i(s)}=(-1)\left[\frac{\dfrac{-(1)(1/s)}{1+(1/s)}}{(1/s)}\right]=\frac{s}{s+1}$$

One zero at $s = 0$ and one pole at $s = -1$



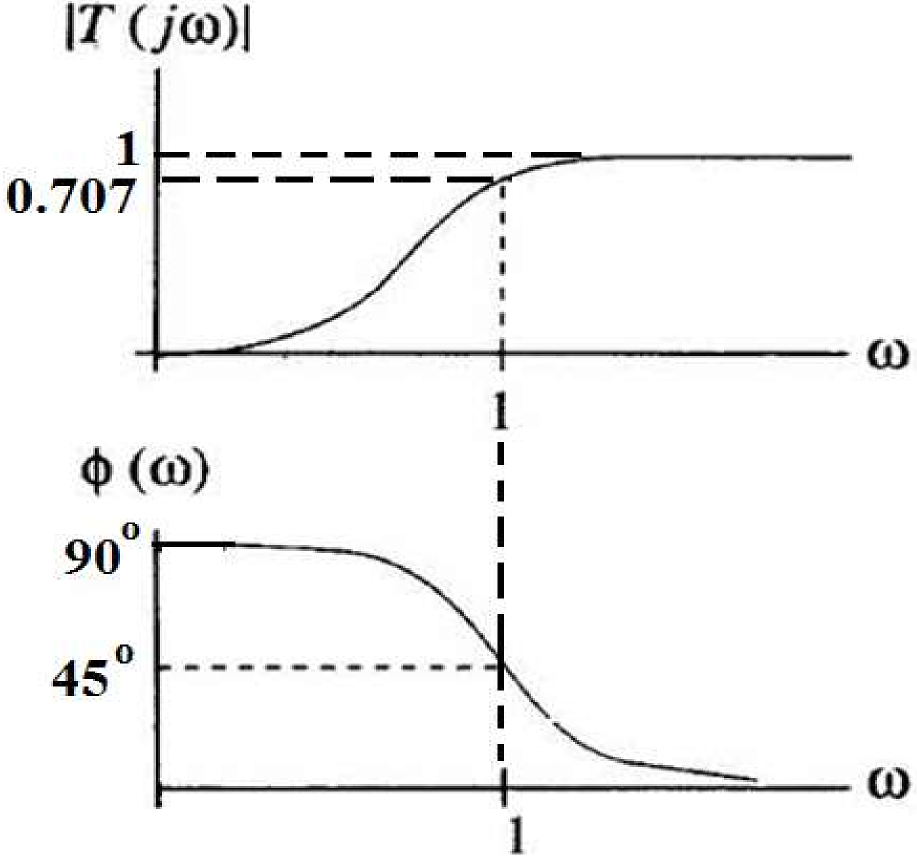
$$T(s) = \frac{s}{s+1}$$

Put $s = j\omega$ to obtain the frequency response

$$T(j\omega) = \frac{j\omega}{1 + j\omega}$$

$$|T(j\omega)| = \frac{\omega}{\sqrt{1+\omega^2}}$$

$$\angle T(j\omega) = \frac{\pi}{2} - tan^{-1}\,\omega$$



When $\omega = 1$ rad/s,

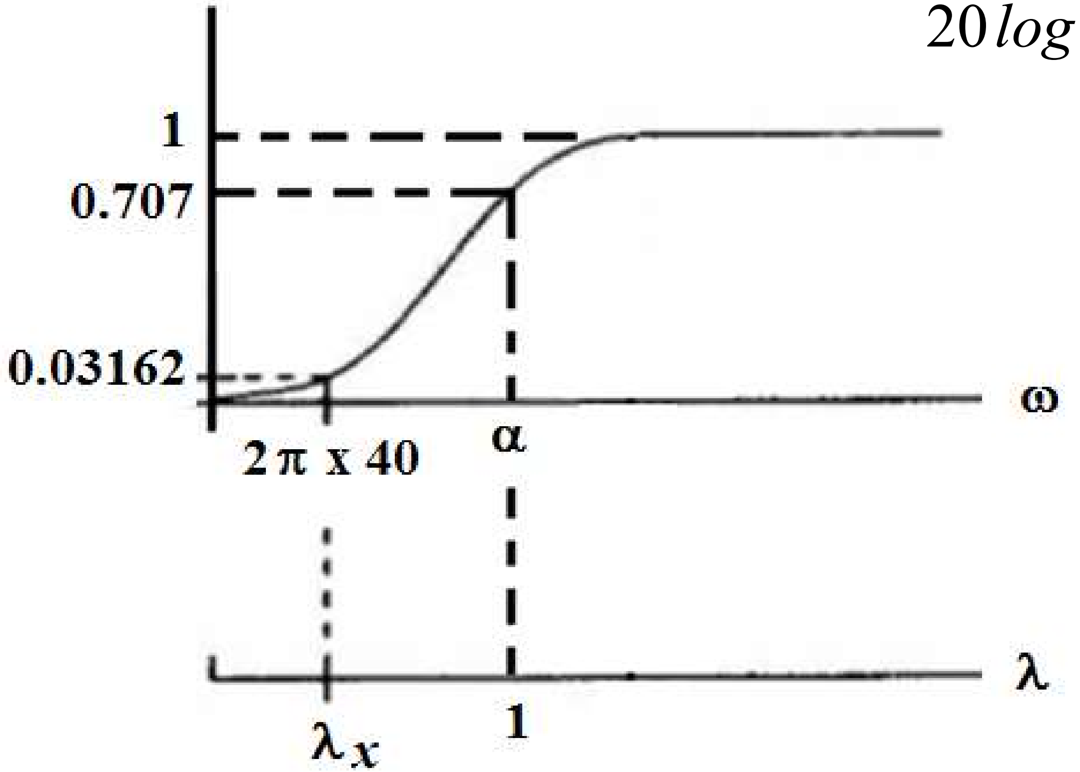
$$|T(j\omega)| = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$$

$$\angle T(j\omega) = 90^\circ - tan^{-1}\,1 = 90^\circ - 45^\circ = 45^\circ$$

Exercise #2: Design a 1st order *RC* active high-pass filter with 0 dB gain at high frequencies and attenuation of 30 dB at 40 Hz. Standard 4700 pF capacitors are preferred for the design.

$$T(j\lambda) = \frac{j\lambda}{1 + j\lambda}$$

$$|T(j\lambda)| = \frac{\lambda}{\sqrt{1^2 + \lambda^2}}$$



$$20\,log|T(j\lambda_x)| = 20\,log\frac{\lambda_x}{\sqrt{1+\lambda_x^{\,2}}} = -30\text{ dB}$$

$$\frac{\lambda_x}{\sqrt{\lambda_x^{\,2}+1}} = 10^{-30/20} = 0.03162$$

$$\lambda_x^{\,2} = 0.001\left(\lambda_x^{\,2}+1\right)$$

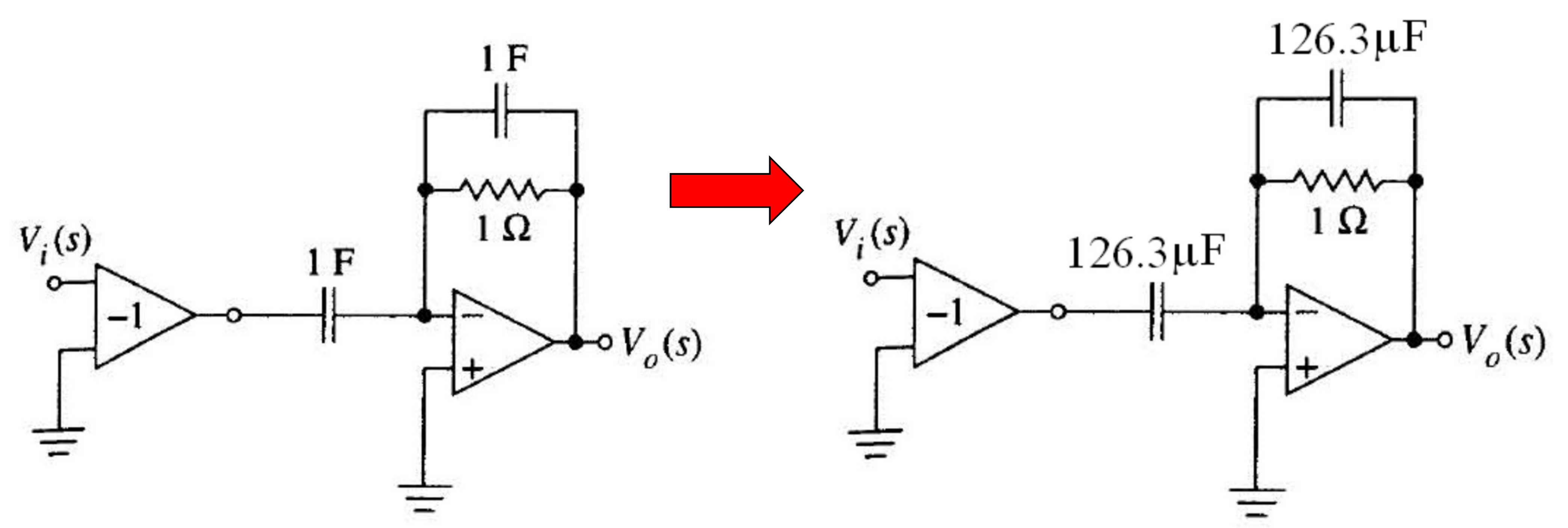
$$\therefore \lambda_x = 0.03164 \text{ rad/s}$$

$$\alpha = \frac{2\pi\times 40}{\lambda_x} = \frac{2\pi\times 40}{0.03164}$$

$$= 7.943\times 10^3$$

Frequency scaling factor $\alpha = 7.943 \times 10^3$

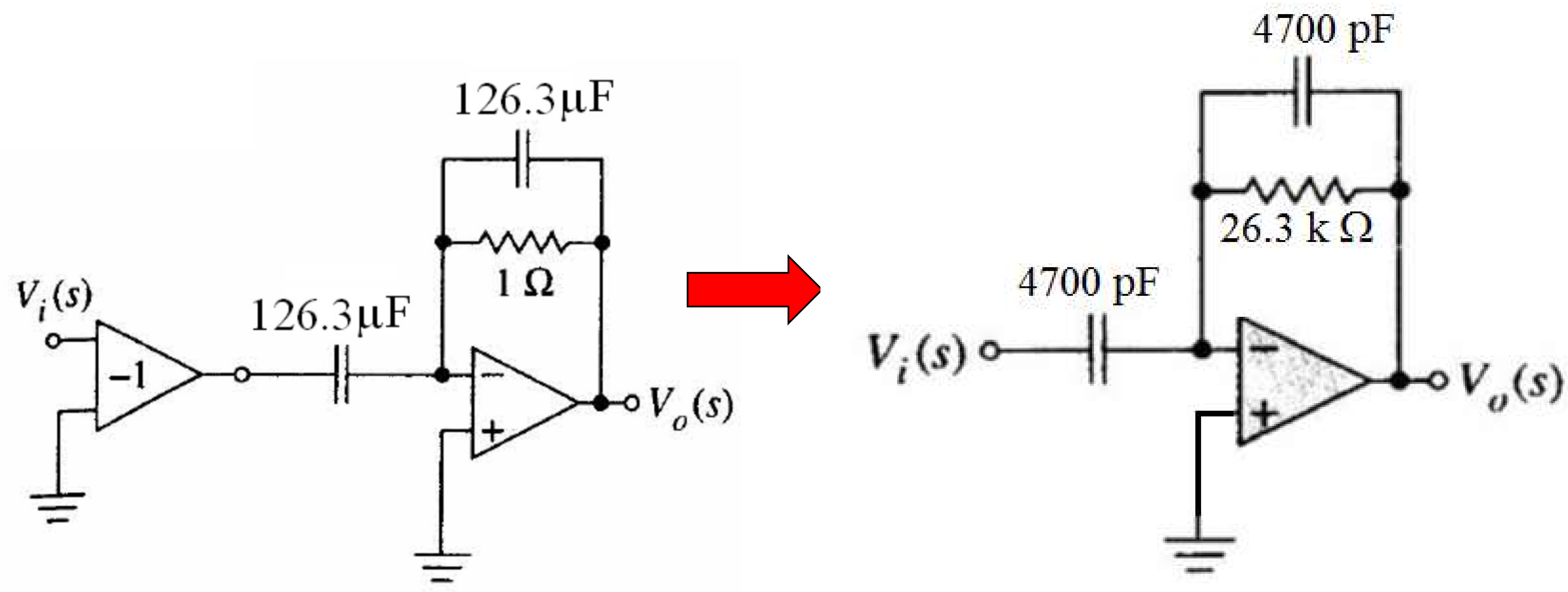
$$C' = \frac{C}{\alpha} = \frac{1\ \text{F}}{7.943 \times 10^3} = 126.3\ \mu\text{F}$$



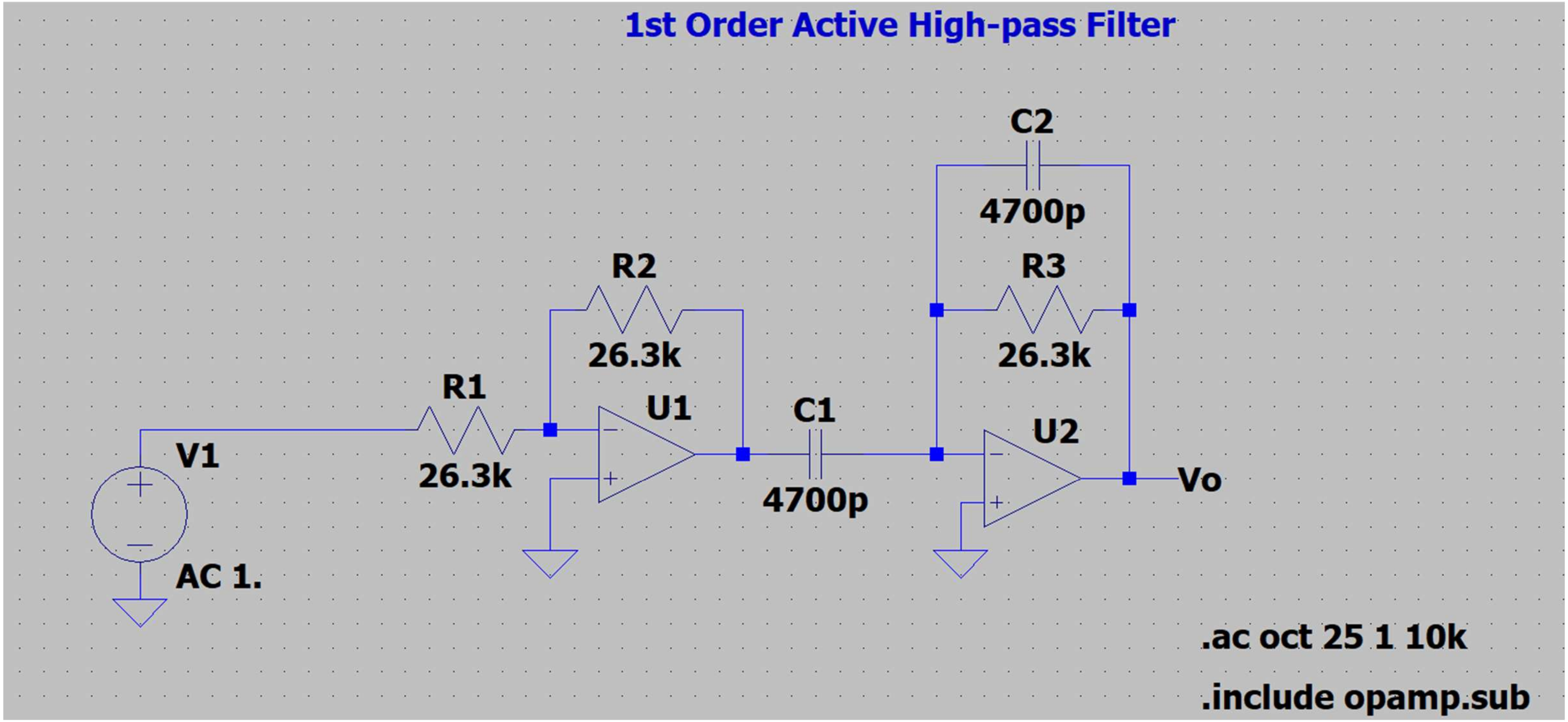
To use 4700pF standard capacitors, impedance scaling factor:

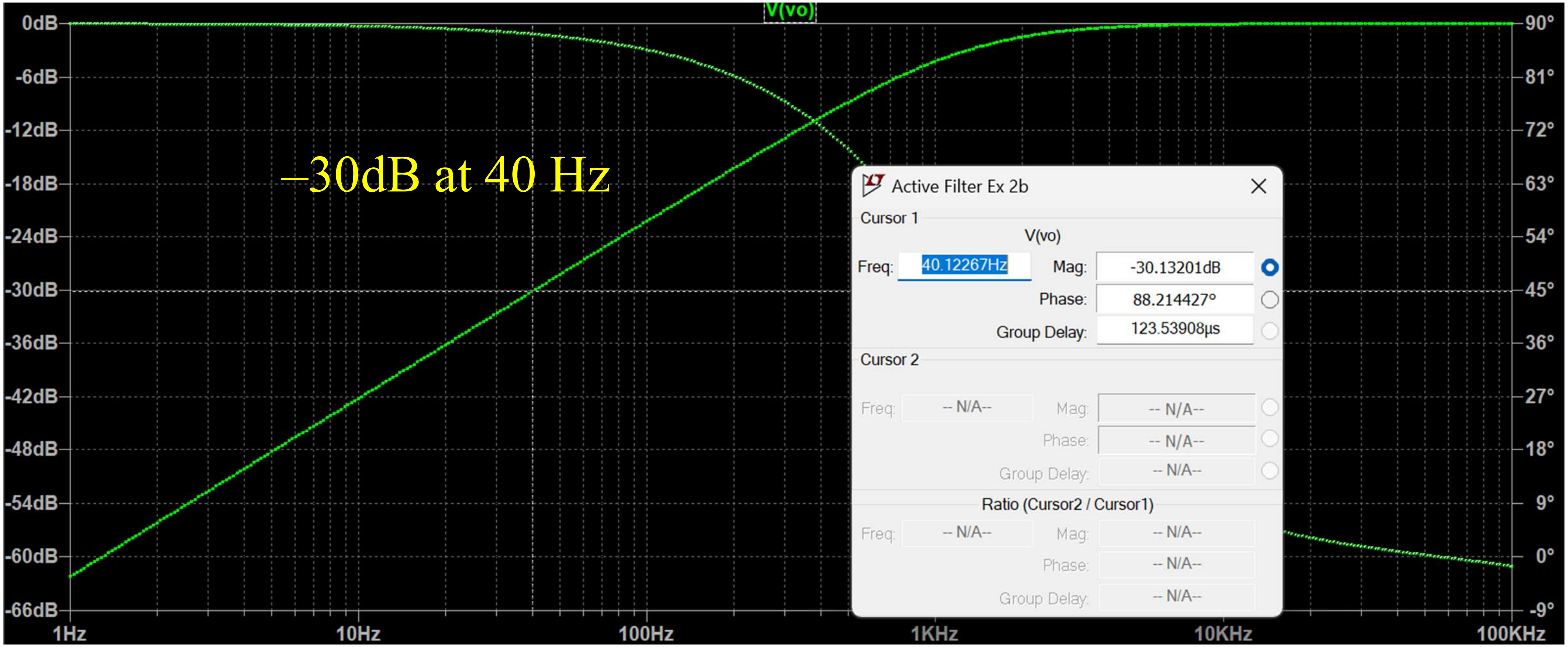
$$\beta = \frac{C'}{C''} = \frac{126.3\mu\text{F}}{4700\text{pF}} = 26.3 \times 10^3$$

$$\therefore R' = \beta R = 26.3 \times 10^3 \times 1\,\Omega = 26.3\,\text{k}\Omega$$



Verification with LTSPICE simulation:




V(vo)
0dB
-6dB
-12dB
-18dB
-24dB
-30dB
-36dB
-42dB
-48dB
-54dB
-60dB
-66dB
90°
81°
72°
63°
54°
45°
36°
27°
18°
9°
0°
-9°
1Hz
10Hz
100Hz
1KHz
10KHz
100KHz
–30dB at 40 Hz
Active Filter Ex 2b
Cursor 1
V(vo)
Freq: 40.12267Hz
Mag: -30.13201dB
Phase: 88.214427°
Group Delay: 123.53908µs
Cursor 2
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
Ratio (Cursor2 / Cursor1)
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--

2nd Order Active Low-pass Filter

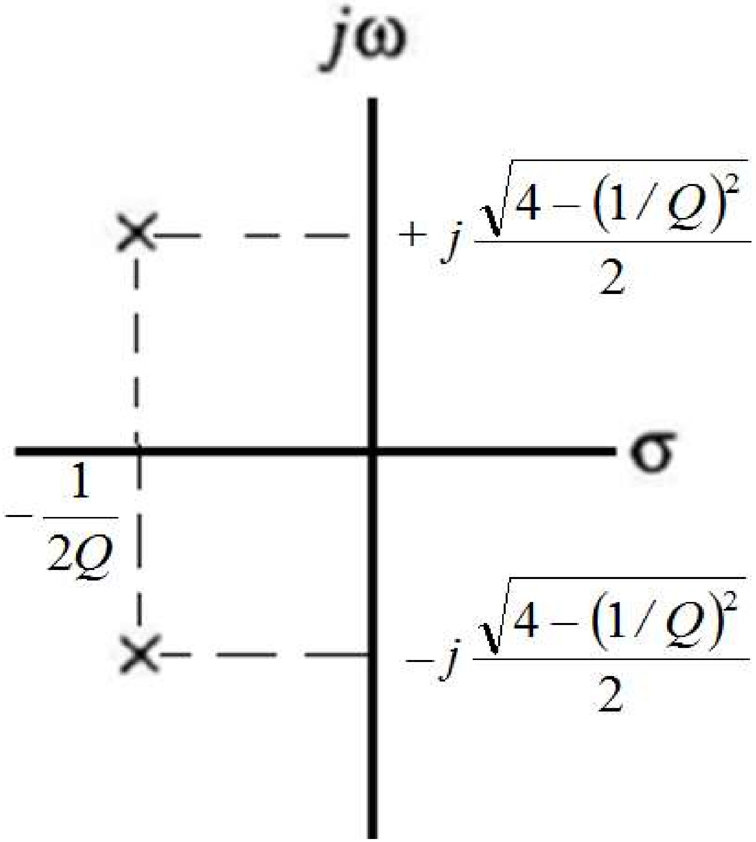
$$\frac{V_o(s)}{V_i(s)} = T(s) = \frac{1}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

where Q is a design parameter which controls the peaking in the frequency response.

$$T(j\omega) = \frac{1}{-\omega^2 + \left(\frac{1}{Q}\right)(j\omega) + 1}$$

$$|T(j\omega)| = \frac{1}{\sqrt{\left(1-\omega^2\right)^2 + \left(\frac{\omega}{Q}\right)^2}}$$

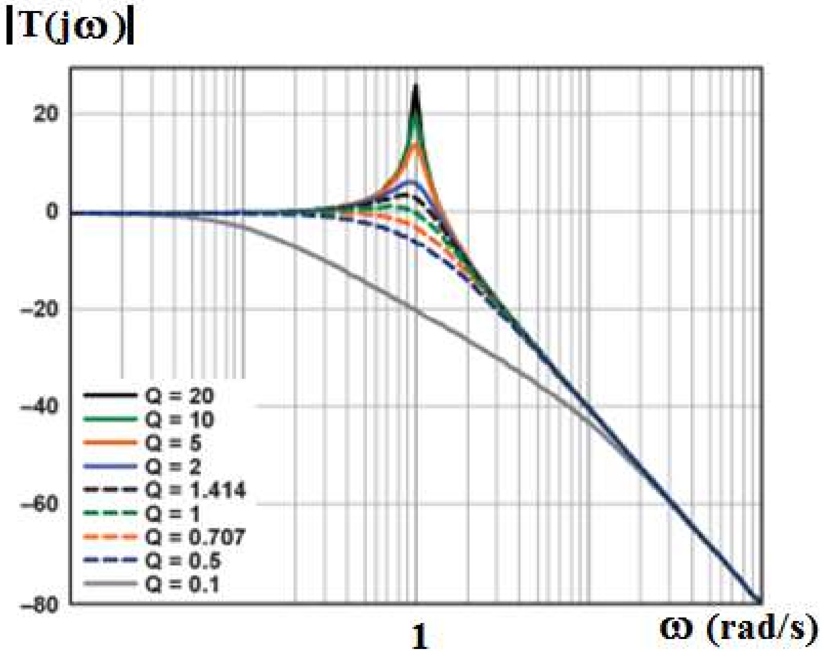
$$\phi(\omega) = -tan^{-1}\left[\frac{\omega}{Q\left(1-\omega^2\right)}\right]$$

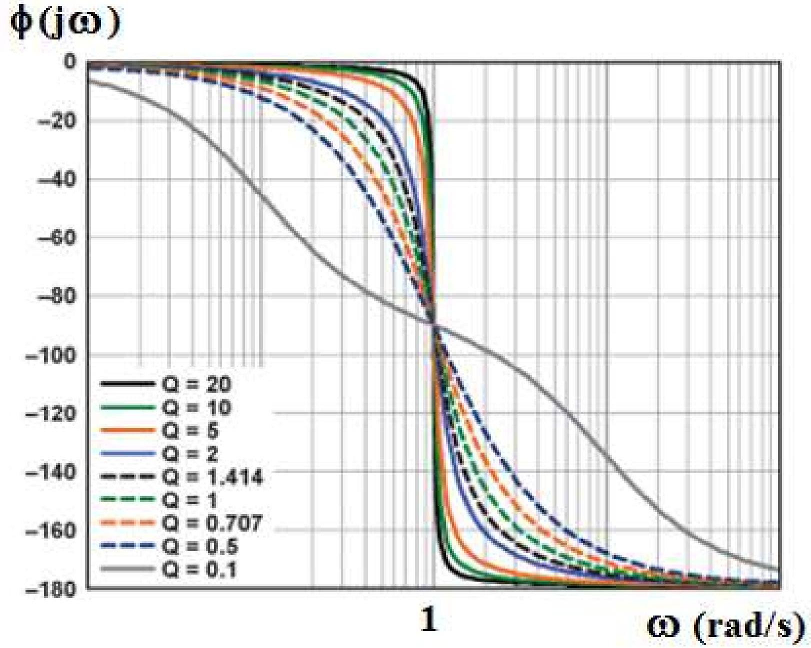


Two complex conjugate poles:

$$s = \frac{-1/Q \pm \sqrt{(1/Q)^2 - 4}}{2}$$

$$s = -\frac{1}{2Q} \pm j\frac{\sqrt{4-(1/Q)^2}}{2}$$





$$\left|T(j\omega)\right| = \frac{1}{\sqrt{\left(1-\omega^2\right)^2 + \left(\frac{\omega}{Q}\right)^2}}$$

$$\phi(\omega) = -tan^{-1}\left[\frac{\omega}{Q\left(1-\omega^2\right)}\right]$$

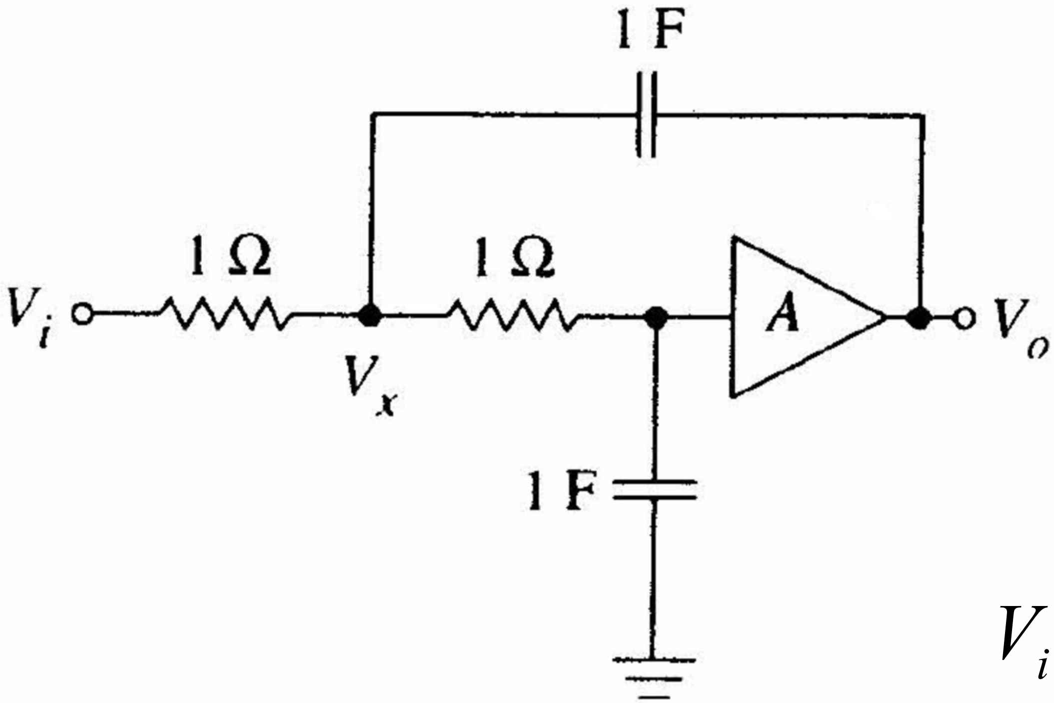
When $\omega = 1\ \text{rad/s}$:

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{\left(1-1\right)^2 + \left(\frac{1}{Q}\right)^2}} = Q$$

$$\phi(\omega) = -tan^{-1}\infty = -90^{\text{o}}$$

As Q increases, the peaking at the reference frequency becomes higher.

Sallen-Key 2nd Order Low-pass Filter

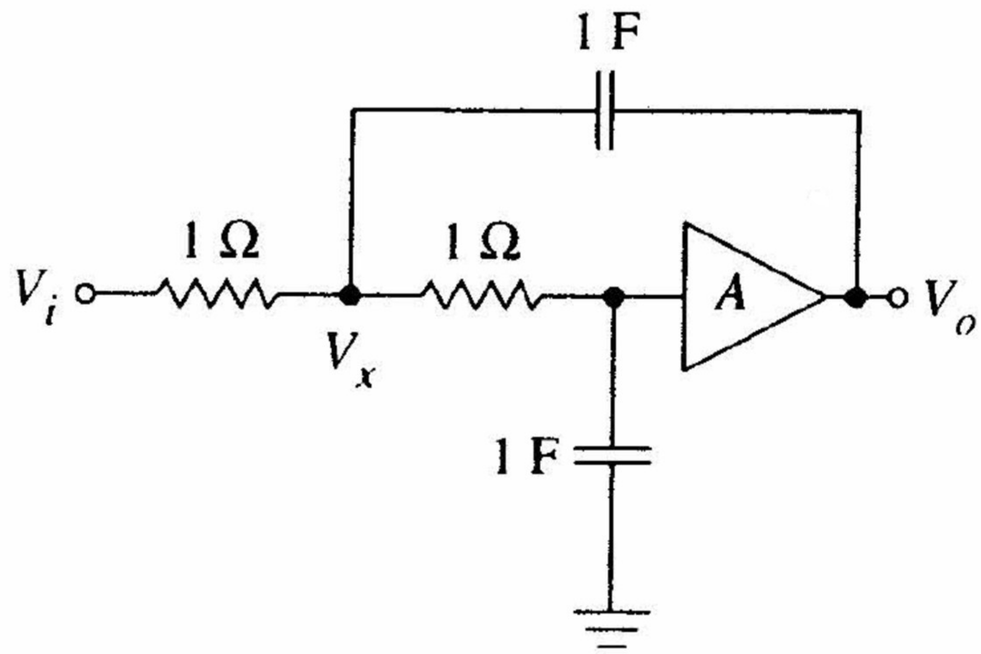


Apply KCL at the node V_x,

$$\frac{V_i - V_x}{1} + \frac{(V_o - V_x)}{1/s} = \frac{V_x}{1 + 1/s}$$

$$V_i - V_x + sV_o - sV_x = \frac{sV_x}{s+1}$$

$$V_i + sV_o = V_x\left(\frac{s^2 + 3s + 1}{s+1}\right)$$



$$V_i + sV_o = V_x\left(\frac{s^2+3s+1}{s+1}\right)$$

$$\because V_o = V_x\left(\frac{1/s}{1+1/s}\right)A = \frac{AV_x}{s+1}$$

$$\therefore V_i + sV_o = \frac{V_o(s+1)}{A}\left(\frac{s^2+3s+1}{s+1}\right)$$

$$\frac{V_o}{V_i} = T(s) = \frac{A}{s^2+(3-A)s+1}$$

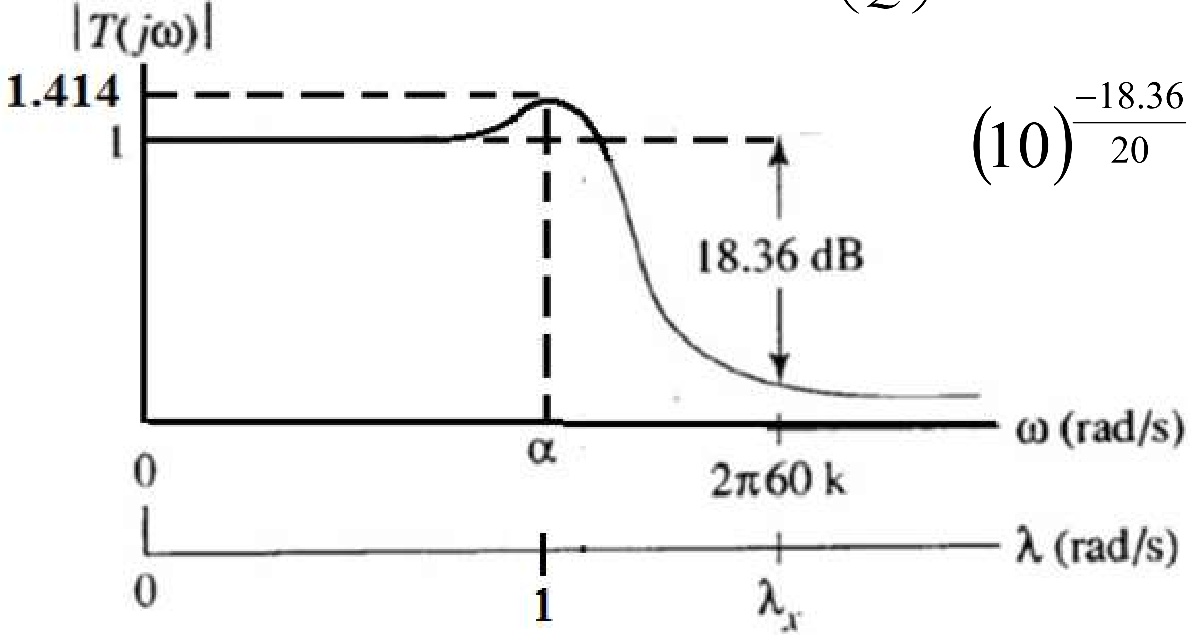
$$T(s) = \frac{1}{s^2+\left(\frac{1}{Q}\right)s+1}$$

Second order low-pass filter can be realized by a Sallen-Key filter circuit with

$$\frac{1}{Q} = 3 - A \text{ or } A = 3 - \frac{1}{Q}$$

Exercise #3: Design a 2nd order Sallen-Key low-pass filter with Q =1.414. It requires to provide an attenuation of 18.36 dB at 60 kHz. 2200 pF standard capacitors are preferred in the design.

$$T(j\lambda) = \frac{1}{(j\lambda)^2 + j\lambda\left(\frac{1}{Q}\right) + 1}, \quad \therefore |T(j\lambda)| = \frac{1}{\sqrt{(1-\lambda^2)^2 + (\lambda/Q)^2}}$$



$$(10)^{\frac{-18.36}{20}} = 0.1208 = \frac{1}{\sqrt{(1-\lambda_x{}^2)^2 + \left(\frac{\lambda_x}{\sqrt{2}}\right)^2}}$$

$$\lambda_x{}^4 - 1.5\lambda_x{}^2 - 67.53 = 0$$

$$\lambda_x{}^2 = \frac{1.5 \pm \sqrt{1.5^2 + 4(67.53)}}{2}$$

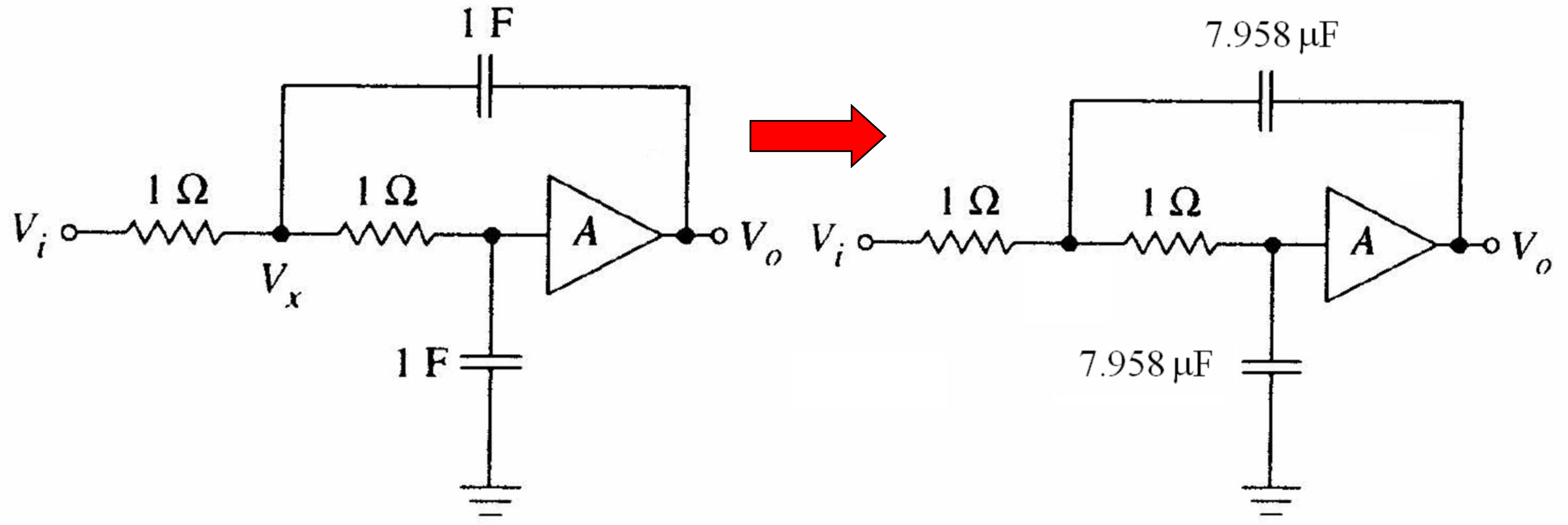
$$\therefore \lambda_x{}^2 = 9 \Rightarrow \lambda_x = 3 \text{ rad/s}$$

$$\alpha\lambda_x = 2\pi \times 60 \times 10^3$$

$$\therefore \alpha = \frac{2\pi \times 60 \times 10^3}{3} = 40\pi \times 10^3$$

Frequency scaling to obtain the new capacitor value:

$$C' = \frac{C}{\alpha} = \frac{1\,\mathrm{F}}{40\pi \times 10^3} = 7.96\ \mu\mathrm{F}$$



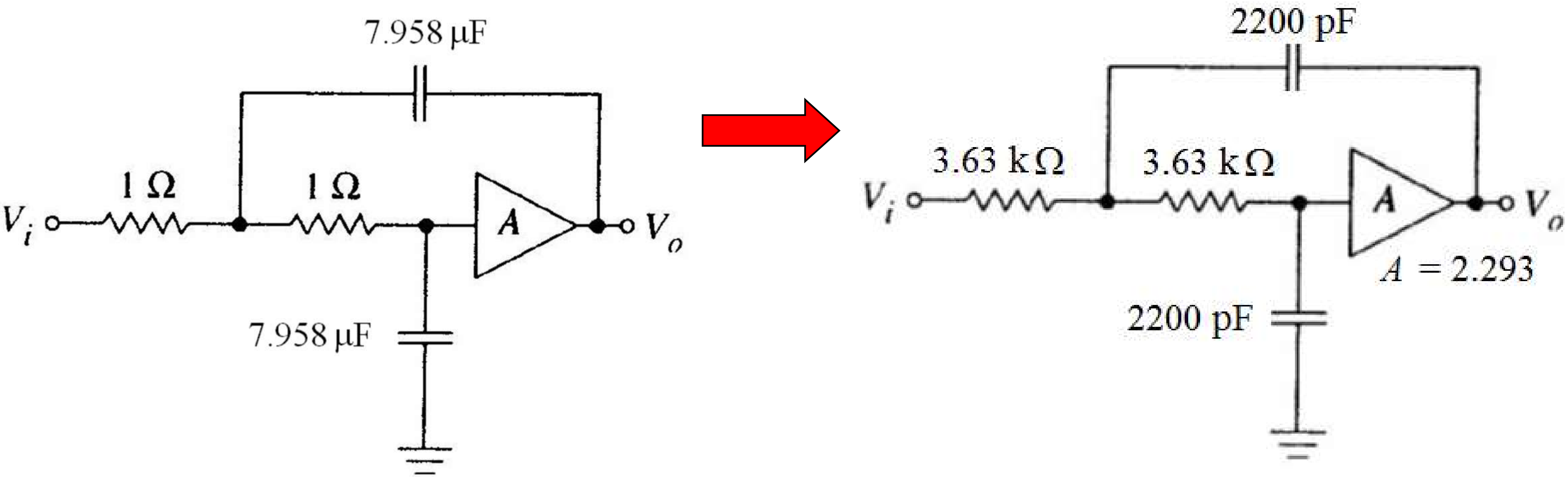
To use 2200 pF standard capacitors, the impedance-scaling factor:

$$\beta = \frac{C'}{C''} = \frac{7.985\mu\text{F}}{2200\text{pF}} \approx 3630$$

$$R' = \beta R = 3630 \times 1\Omega = 3.63\,\text{k}\Omega$$

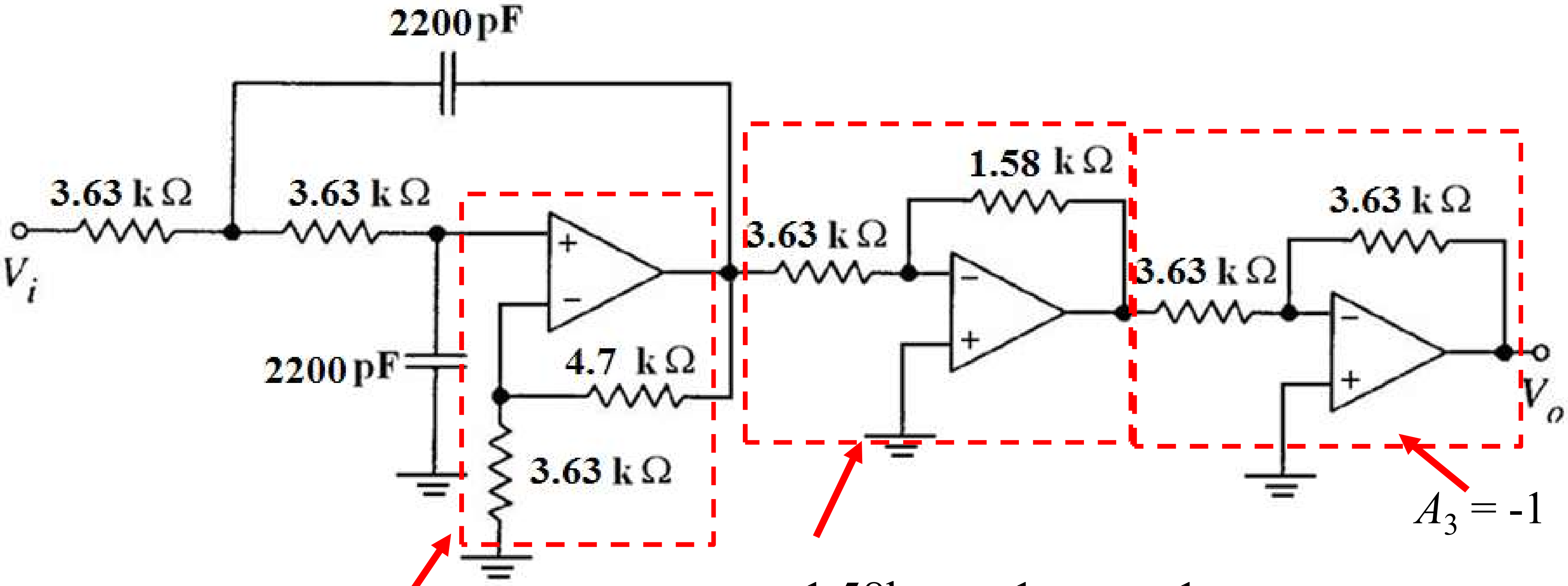
$$A = 3 - \frac{1}{Q} = 3 - \frac{1}{1.414} = 2.293$$

A non-inverting amplifier with $A = 2.293$ gives $Q = 1.414$.



Final active filter circuit:

- First stage has a non-inverting gain $A = 2.293$
- Second stage amplifier has gain of $-1/A = -0.436$ to have a passband gain of unity



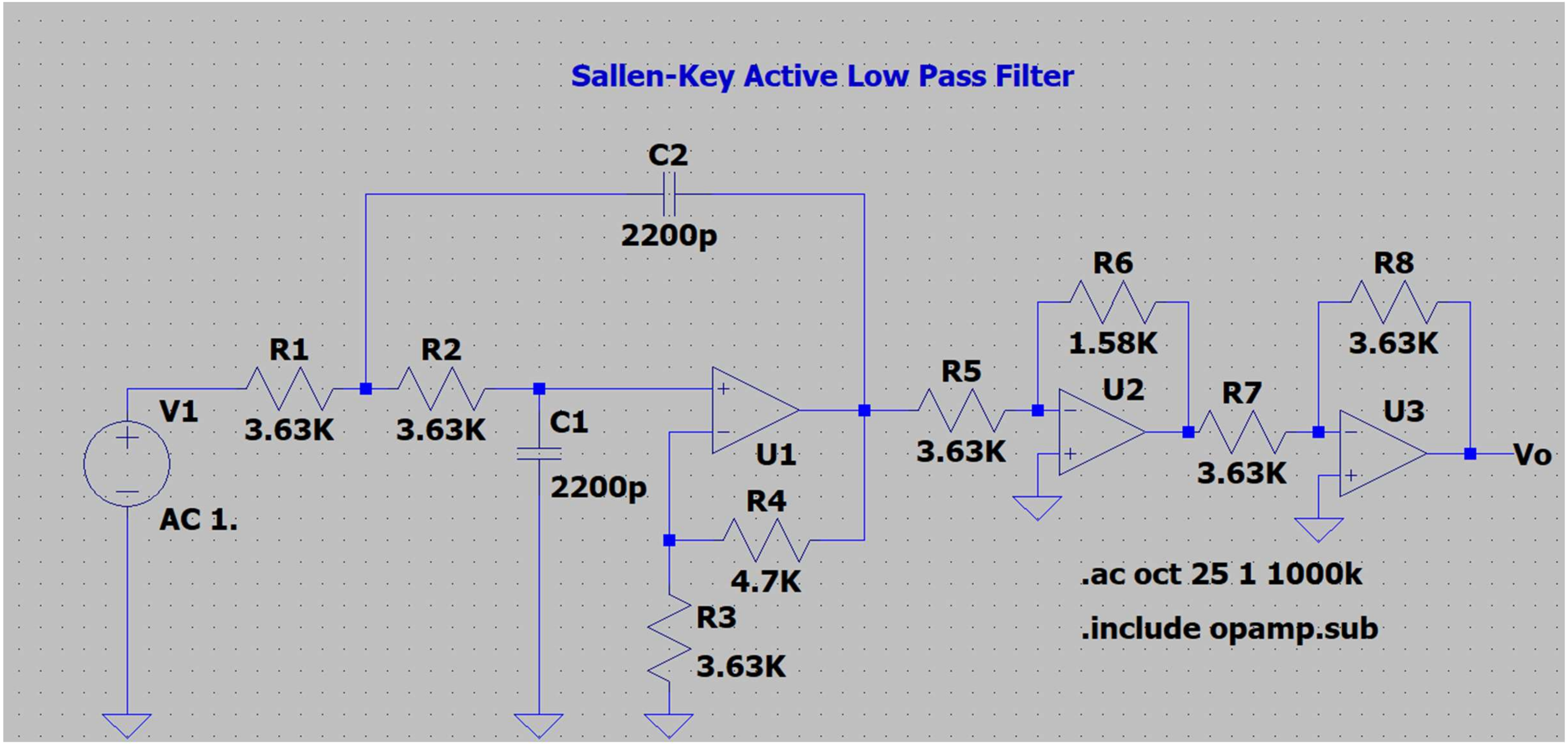
$$A_1 = 1 + \frac{4.7\text{k}}{3.63\text{k}} = 2.293$$

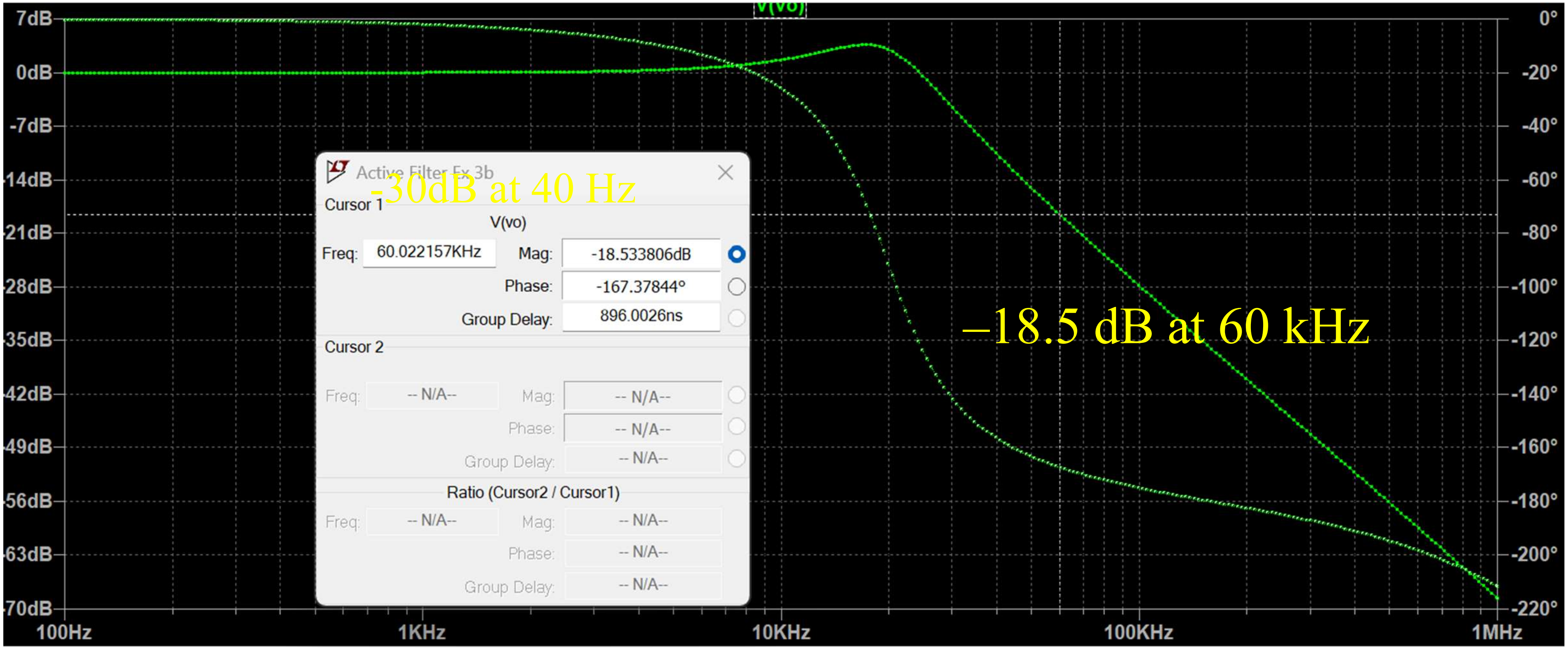
$$A_2 = -\frac{1.58\text{k}}{3.63\text{k}} = -\frac{1}{A_1} = -\frac{1}{2.293} = -0.436$$

Overall gain $A = (2.293)(-0.436)(-1) = 1$

$$\frac{V_o}{V_i} = T(s) = \frac{1}{s^2 + 0.707s + 1}$$

Verification with LTSPICE simulation:





V(vo)
7dB
0dB
-7dB
-14dB
-21dB
-28dB
-35dB
-42dB
-49dB
-56dB
-63dB
-70dB
0°
-20°
-40°
-60°
-80°
-100°
-120°
-140°
-160°
-180°
-200°
-220°
100Hz
1KHz
10KHz
100KHz
1MHz
Active Filter Ex 3b
Cursor 1
V(vo)
Freq: 60.022157KHz
Mag: -18.533806dB
Phase: -167.37844°
Group Delay: 896.0026ns
Cursor 2
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
Ratio (Cursor2 / Cursor1)
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
-30dB at 40 Hz
–18.5 dB at 60 kHz

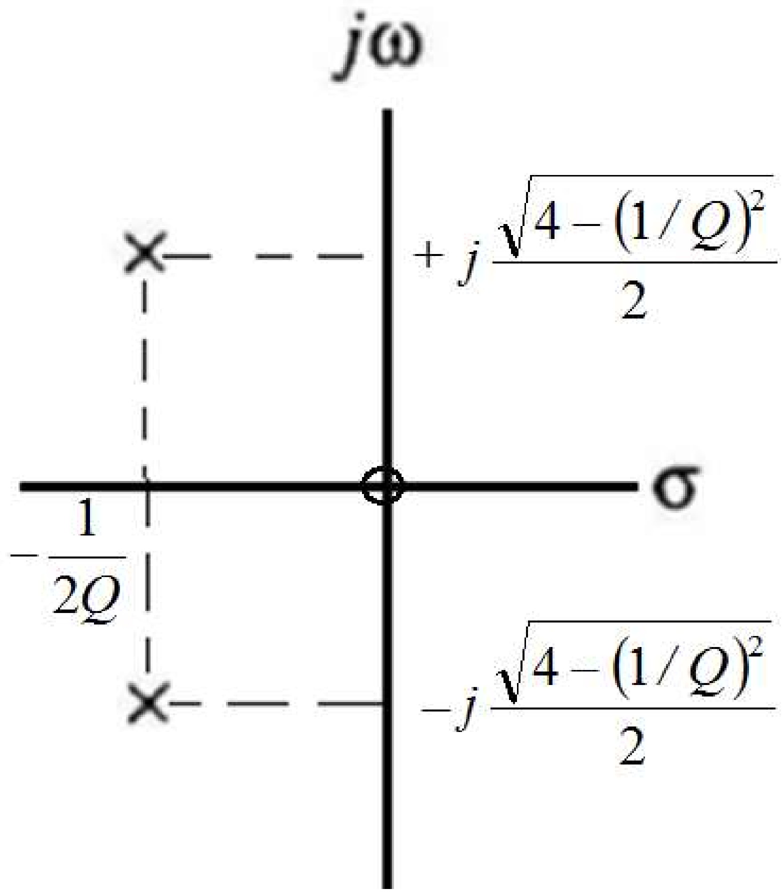
2nd Order Active High-pass Filter

$$\frac{V_o}{V_i} = T(s) = \frac{s^2}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

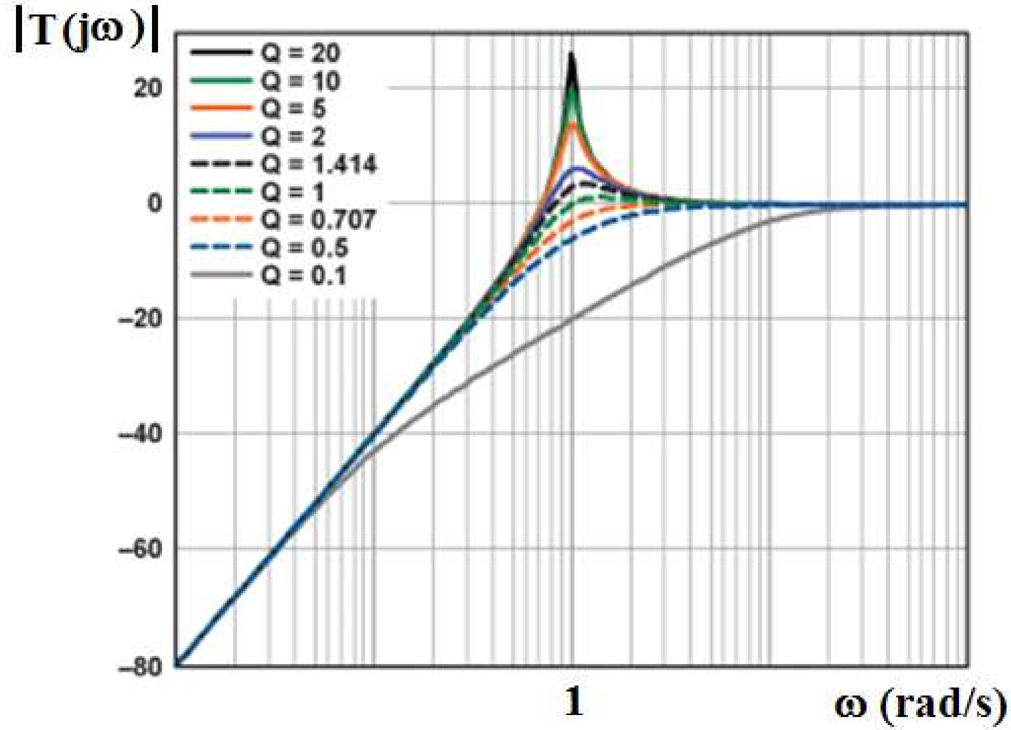
$$T(j\omega) = \frac{-\omega^2}{-\omega^2 + \left(\frac{1}{Q}\right)(j\omega) + 1}$$

$$|T(j\omega)| = \frac{\omega^2}{\sqrt{\left(1-\omega^2\right)^2 + \left(\frac{\omega}{Q}\right)^2}}$$

$$\phi(\omega) = -180^{\circ} - tan^{-1}\left[\frac{\omega}{Q\left(1-\omega^2\right)}\right]$$



Two complex conjugate poles and two zeros at $s = 0$.



$$\left|T(j\omega)\right| = \frac{\omega^2}{\sqrt{\left(1-\omega^2\right)^2 + \left(\dfrac{\omega}{Q}\right)^2}}$$

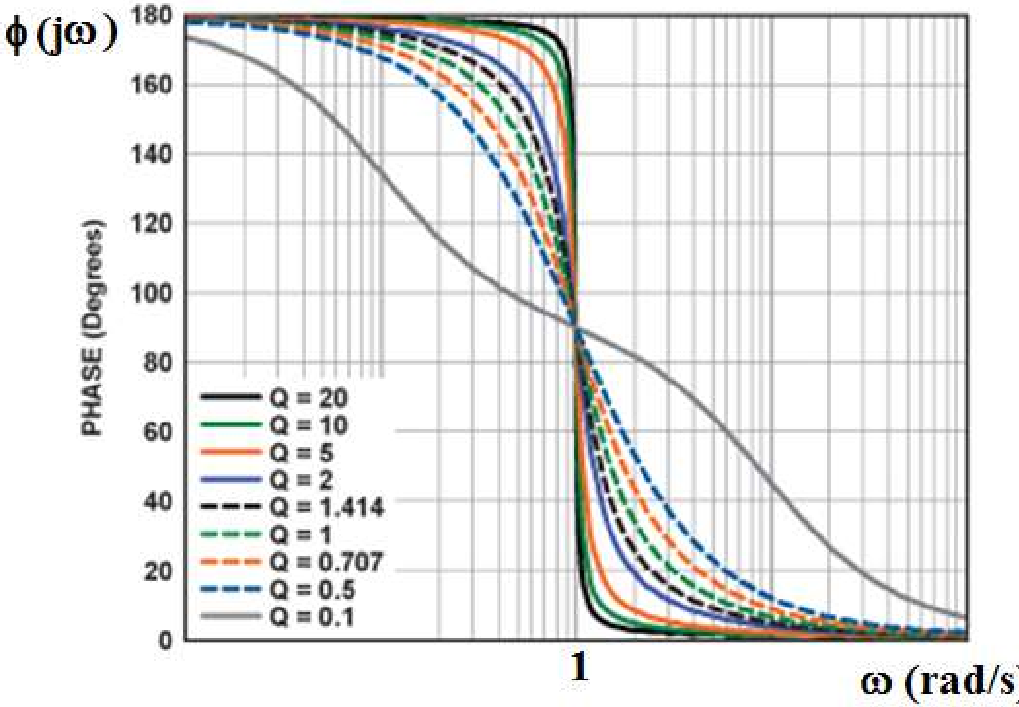
$$\angle T(j\omega) = -180^{\circ} - tan^{-1}\left[\frac{\omega^2}{Q\left(1-\omega^2\right)}\right]$$

When $\omega = 1$ rad/s,

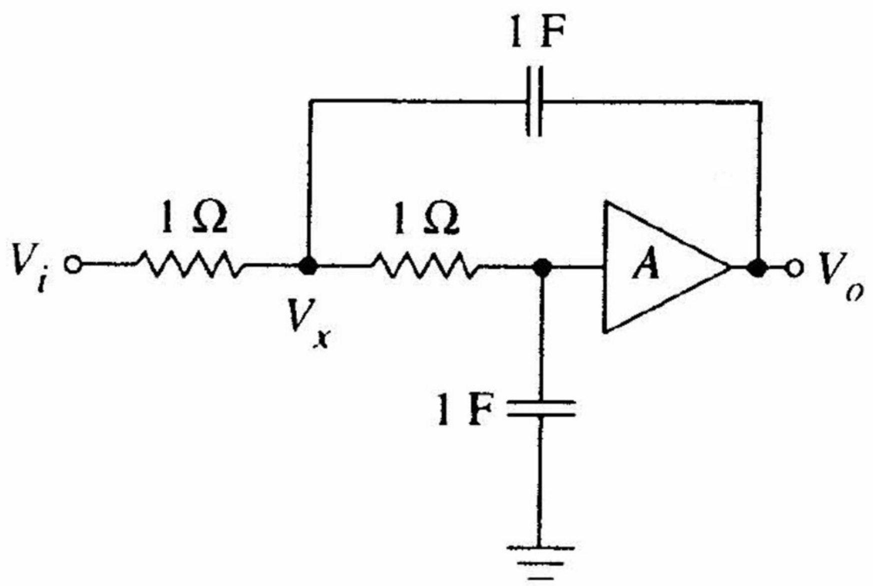
$$\left|T(j\omega)\right| = \frac{1^2}{\sqrt{\left(1-1^2\right)^2 + \left(\dfrac{1}{Q}\right)^2}} = Q$$

$$\angle T(j\omega) = -180^{\circ} - tan^{-1}\infty$$

$$= -180^{\circ} - 90^{\circ} = -270^{\circ} = +90^{\circ}$$

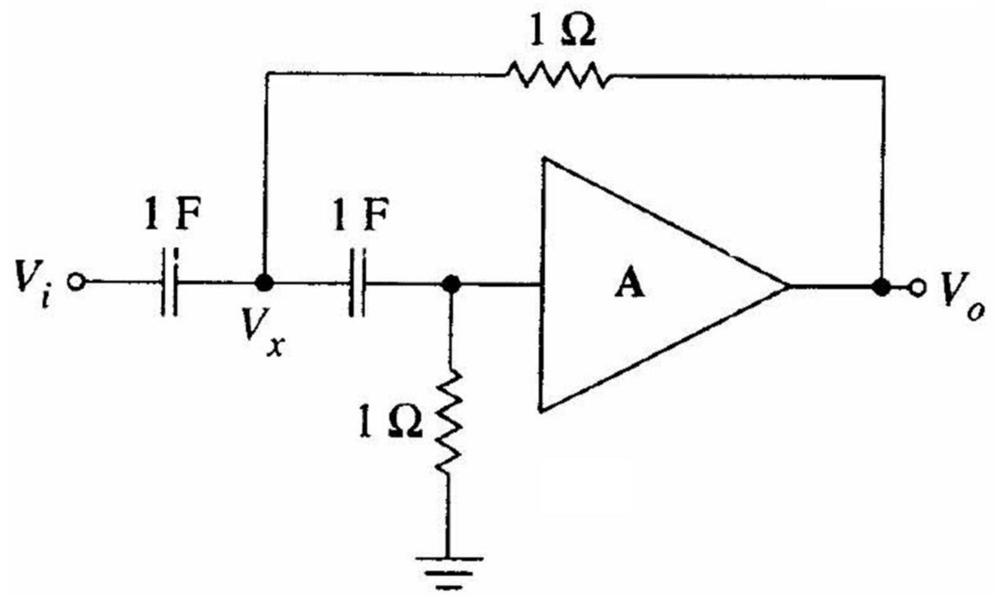


Sallen-Key 2nd Order High-pass Filter



$C \rightarrow R$

$R \rightarrow C$



$$T(s) = \frac{As^2}{s^2 + (3 - A)s + 1}$$

$$T(s) = \frac{s^2}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

$$\frac{1}{Q} = (3 - A) \text{ or } A = 3 - \frac{1}{Q}$$

2^{nd} Order Band-pass Filter

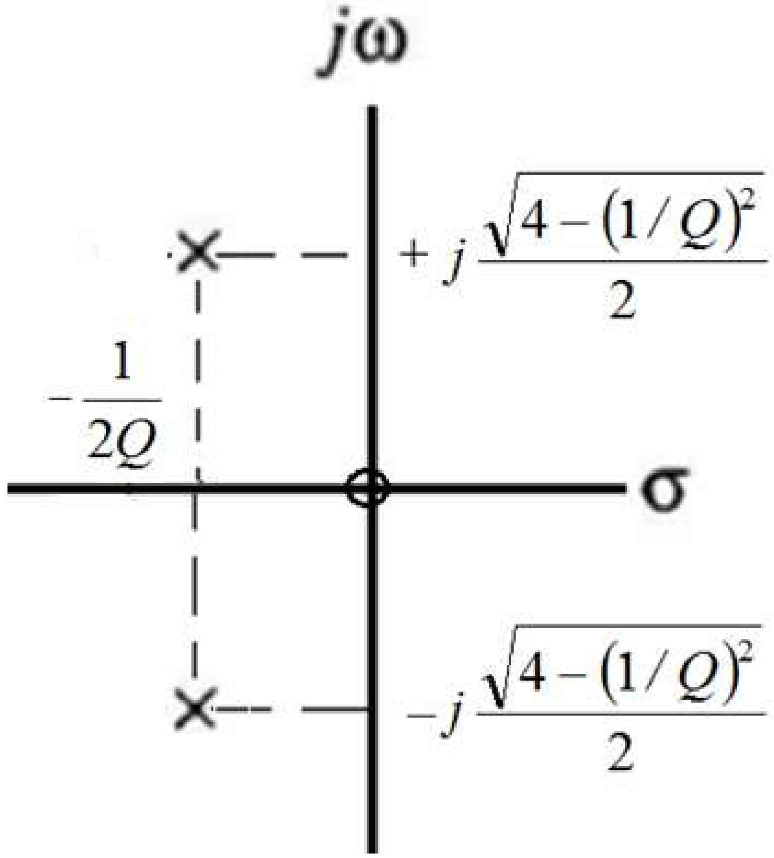
$$T(s)=\frac{s/Q}{s^2+\left(\frac{1}{Q}\right)s+1}$$

$$T(j\omega)=\frac{j\omega/Q}{1-\omega^2+j\omega/Q}=\frac{1}{1+(1-\omega^2)Q/j\omega}$$

$$=\frac{1}{1+j(\omega^2-1)Q/\omega}$$

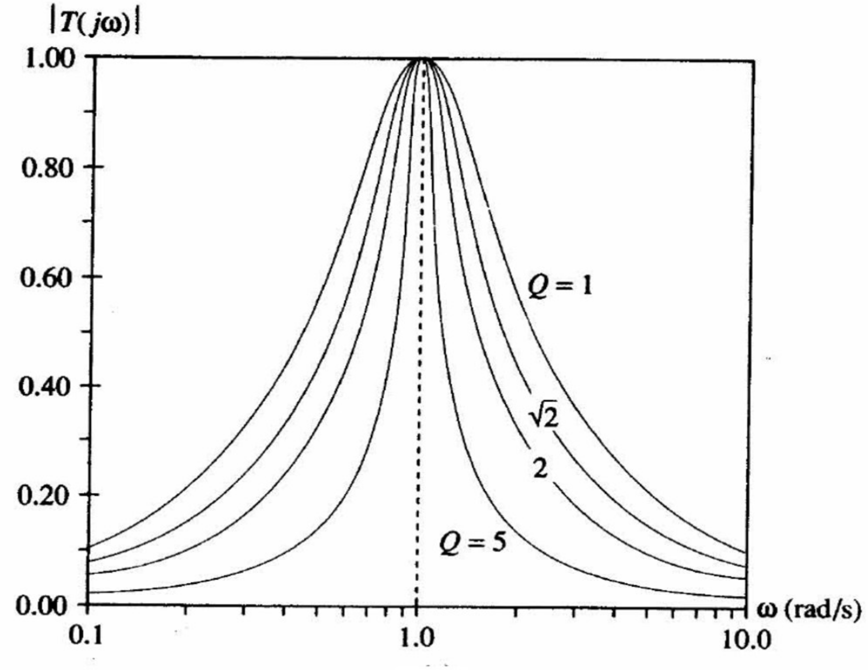
$$|T(j\omega)|=\frac{1}{\sqrt{1+\left(\frac{\omega^2-1}{\omega}\right)^2Q^2}}$$

$$\angle T(j\omega)=-tan^{-1}\left[\frac{(\omega^2-1)Q}{\omega}\right]$$



Two complex conjugate poles and one zero at $s = 0$.

2^{nd} Order Band-pass Filter

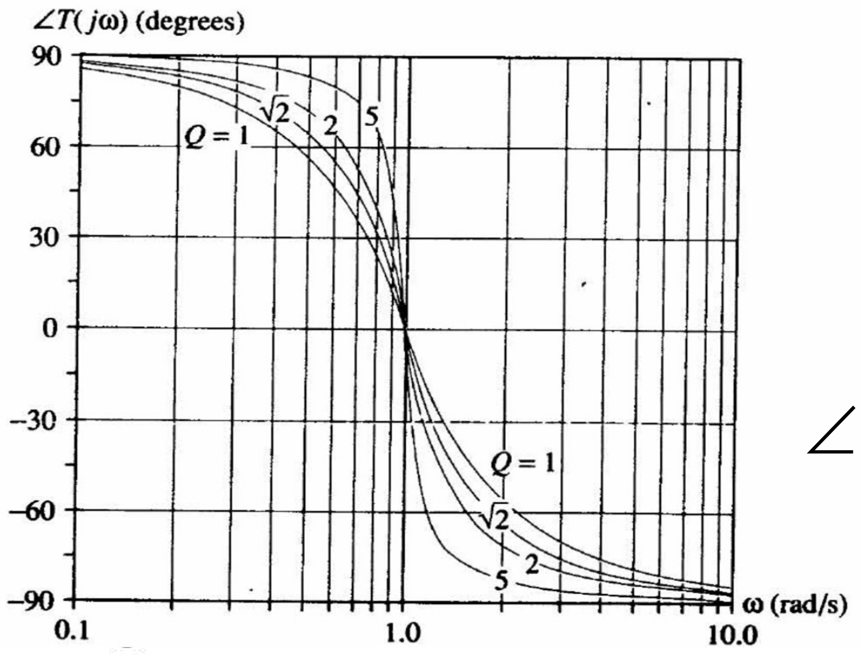


$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\left(\frac{\omega^2-1}{\omega}\right)^2 Q^2}}$$

$$\angle T(j\omega) = -tan^{-1}\left[\frac{(\omega^2-1)Q}{\omega}\right]$$

When $\omega = 1$ rad/s :

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\left(\frac{1^2-1}{1}\right)^2 Q^2}} = 1$$



$$\angle T(j\omega) = -tan^{-1}\left[\frac{(1^2-1)Q}{1}\right] = -tan^{-1}0 = 0^{\circ}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega^2-1}{\omega}\right)^2 Q^2}}$$

To determine the lower and upper -3 dB frequencies

$$|T(j\omega)| = \frac{1}{\sqrt{2}} \qquad \therefore \frac{(\omega^2-1)Q}{\omega} = \pm 1$$

$$\frac{(\omega^2-1)Q}{\omega} = \pm 1 \Rightarrow \omega^2 - 1 = \pm \frac{\omega}{Q}$$

Let the upper and lower –3 dB frequencies be u_H and u_L, then:

$$u_H{}^2 - 1 = \frac{u_H}{Q} \quad \text{------(1)}$$

$$u_L{}^2 - 1 = -\frac{u_L}{Q} \quad \text{------(2)}$$

Eq (1) – Eq(2):

$$u_H{}^2 - u_L{}^2 = \frac{(u_H + u_L)}{Q}$$

$$(u_H + u_L)(u_H - u_L) = \frac{(u_H + u_L)}{Q}$$

$$\therefore u_H - u_L = \frac{1}{Q}$$

The higher the *Q*, the narrower the -3 dB bandwidth.

From Eq (1):

$$u_H{}^2 - 1 = \frac{u_H}{Q}$$

$$u_H{}^2 - \frac{u_H}{Q} = 1$$

$$u_H\left(u_H - \frac{1}{Q}\right) = 1$$

$$\because u_H - \frac{1}{Q} = u_L$$

$$\therefore u_H u_L = 1$$

$$\sqrt{u_H u_L} = 1$$

Square-root of product of upper and lower cut-off frequencies equal to centre frequency

Biquadratic Filter

Biquadratic filter provides simultaneous outputs for 2nd order low-pass, high-pass and band-pass responses.

$$\frac{V_{HP}}{V_i} = \frac{s^2}{s^2 + \left(\frac{1}{Q}\right)s + 1} \qquad \frac{V_{BP}}{V_i} = \frac{\left(\frac{1}{Q}\right)s}{s^2 + \left(\frac{1}{Q}\right)s + 1} \qquad \frac{V_{LP}}{V_i} = \frac{1}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

By observation, the 2nd order high-pass, bandpass and low-pass filters are related as follows:

$$\frac{V_{HP}}{V_i} \times \frac{1}{s} = \frac{s}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

It is a bandpass filter response (leave the coefficient $1/Q$ out for the moment).

$$\frac{s}{s^2 + \left(\frac{1}{Q}\right)s + 1} \times \frac{1}{s} = \frac{1}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

It is a low-pass filter response

$T(s) = \frac{1}{s}$ is an integrator

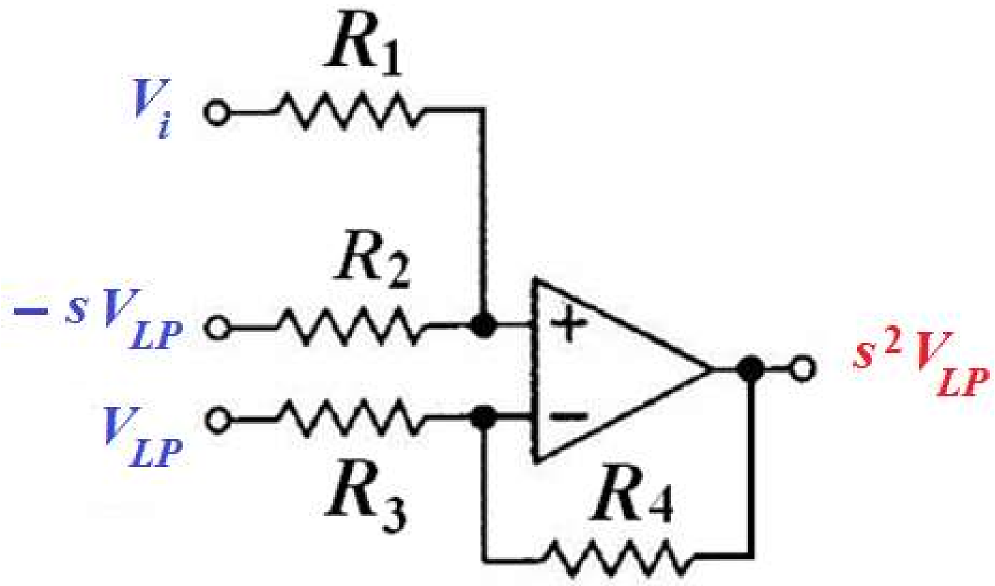
A second order low-pass filter with a passband gain of K:

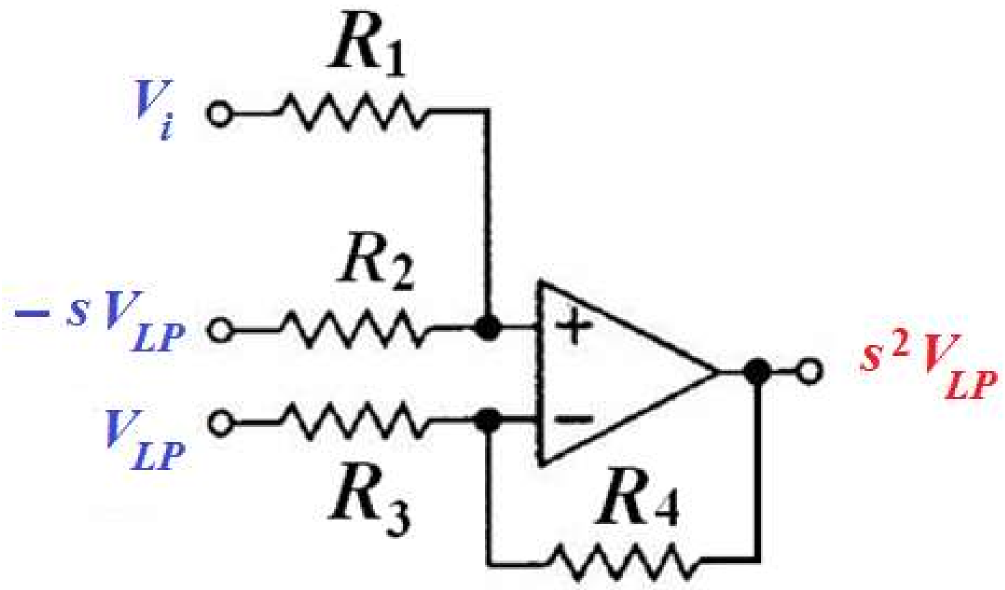
$$\frac{V_{LP}}{V_i} = \frac{K}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

$$s^2 V_{LP} + \left(\frac{1}{Q}\right) s V_{LP} + V_{LP} = K V_i$$

$$s^2 V_{LP} = K V_i - \left(\frac{1}{Q}\right) s V_{LP} - V_{LP}$$

The above function can be realized with an op-amp summing circuit with 3 inputs V_i , $-sV_{LP}$ and V_{LP} and 1 output $s^2 V_{LP}$.





Applying superposition theorem, the output due to each input can be obtained by grounding the other 2 inputs:

$$s^2V_{LP} = V_i\left(\frac{R_2}{R_1+R_2}\right)\left(1+\frac{R_4}{R_3}\right) - sV_{LP}\left(\frac{R_1}{R_1+R_2}\right)\left(1+\frac{R_4}{R_3}\right) - \left(\frac{R_4}{R_3}\right)V_{LP}$$

Let $R_1 = R_3 = R_4 = 1\,\Omega$ and keep R_2 as a variable to control the Q

$$s^2V_{LP} = V_i\left(\frac{2R_2}{1+R_2}\right) - sV_{LP}\left(\frac{2}{1+R_2}\right) - V_{LP}$$

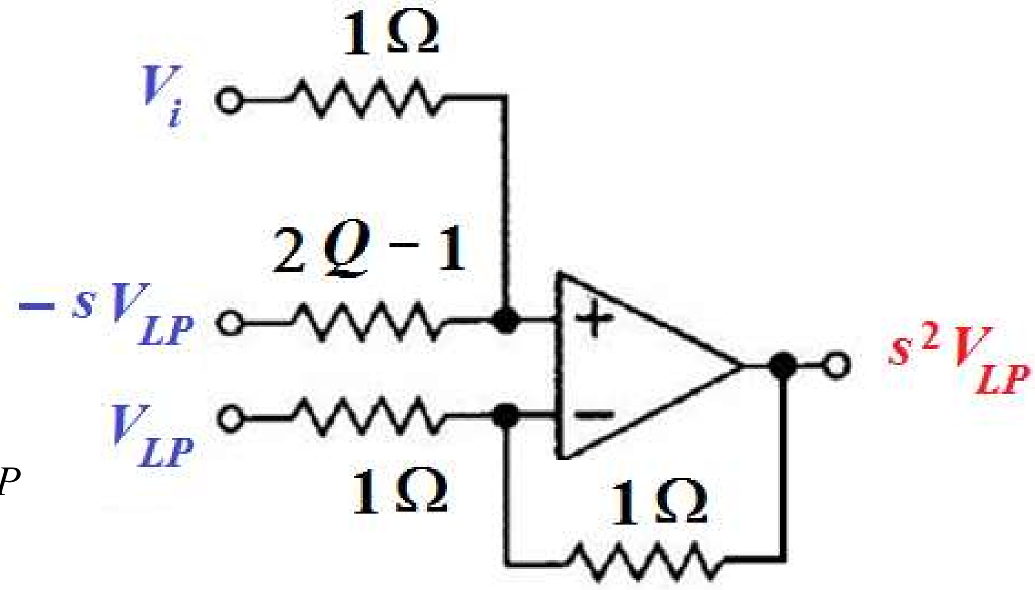
$$s^2V_{LP} = V_i\left(\frac{2R_2}{1+R_2}\right) - sV_{LP}\left(\frac{2}{1+R_2}\right) - V_{LP}$$

Comparing with the original function:

$$s^2V_{LP} = KV_i - \left(\frac{1}{Q}\right)sV_{LP} - V_{LP}$$

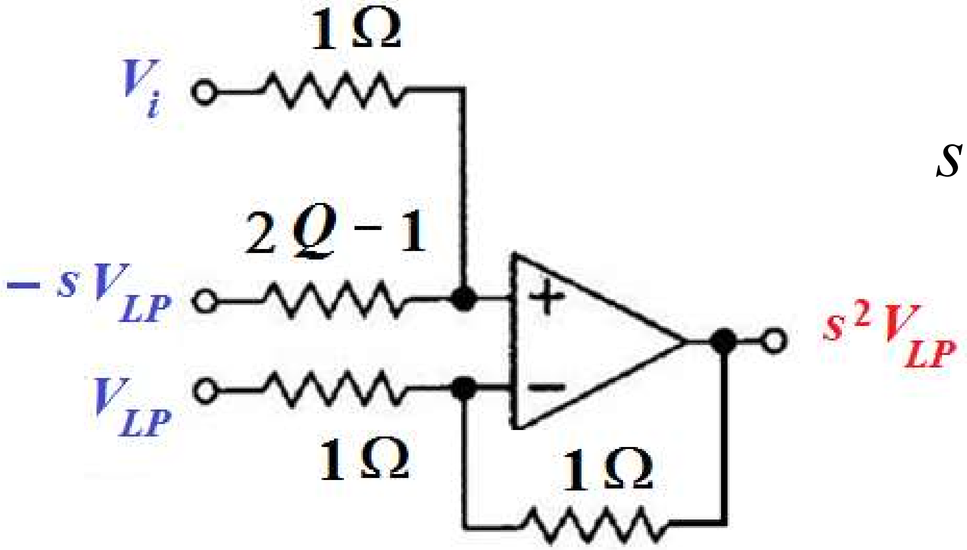
$$\frac{1}{Q} = \frac{2}{1+R_2} \Rightarrow R_2 = 2Q-1$$

The circuit becomes:



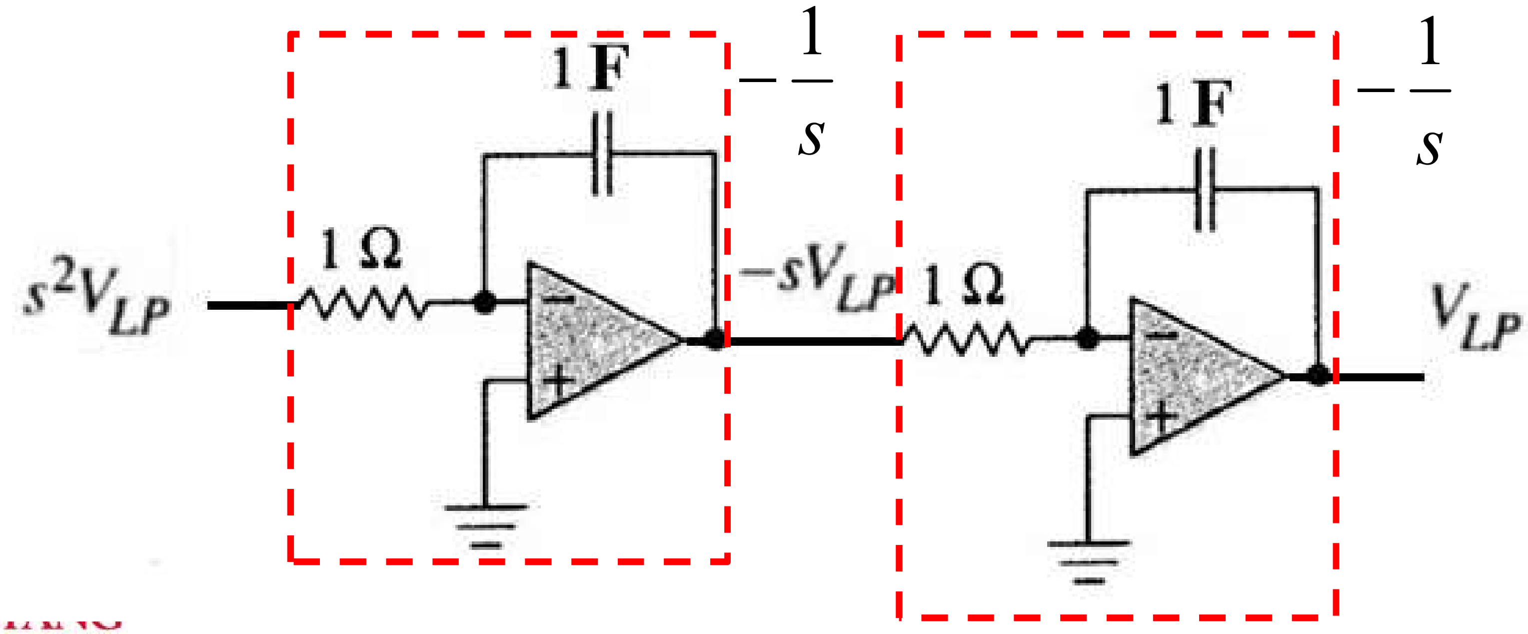
$$K = \frac{2R_2}{1+R_2} = \frac{2(2Q-1))}{1+2Q-1} = \frac{2Q-1}{Q}$$

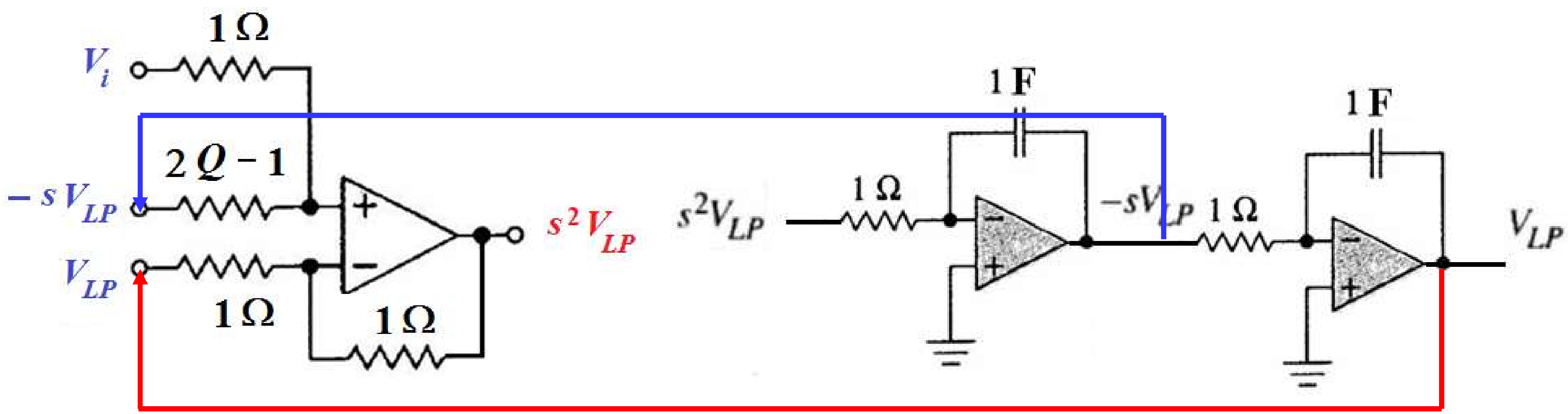
$$s^2V_{LP} = V_i\left(\frac{2Q-1}{Q}\right) - sV_{LP}\left(\frac{1}{Q}\right) - V_{LP}$$



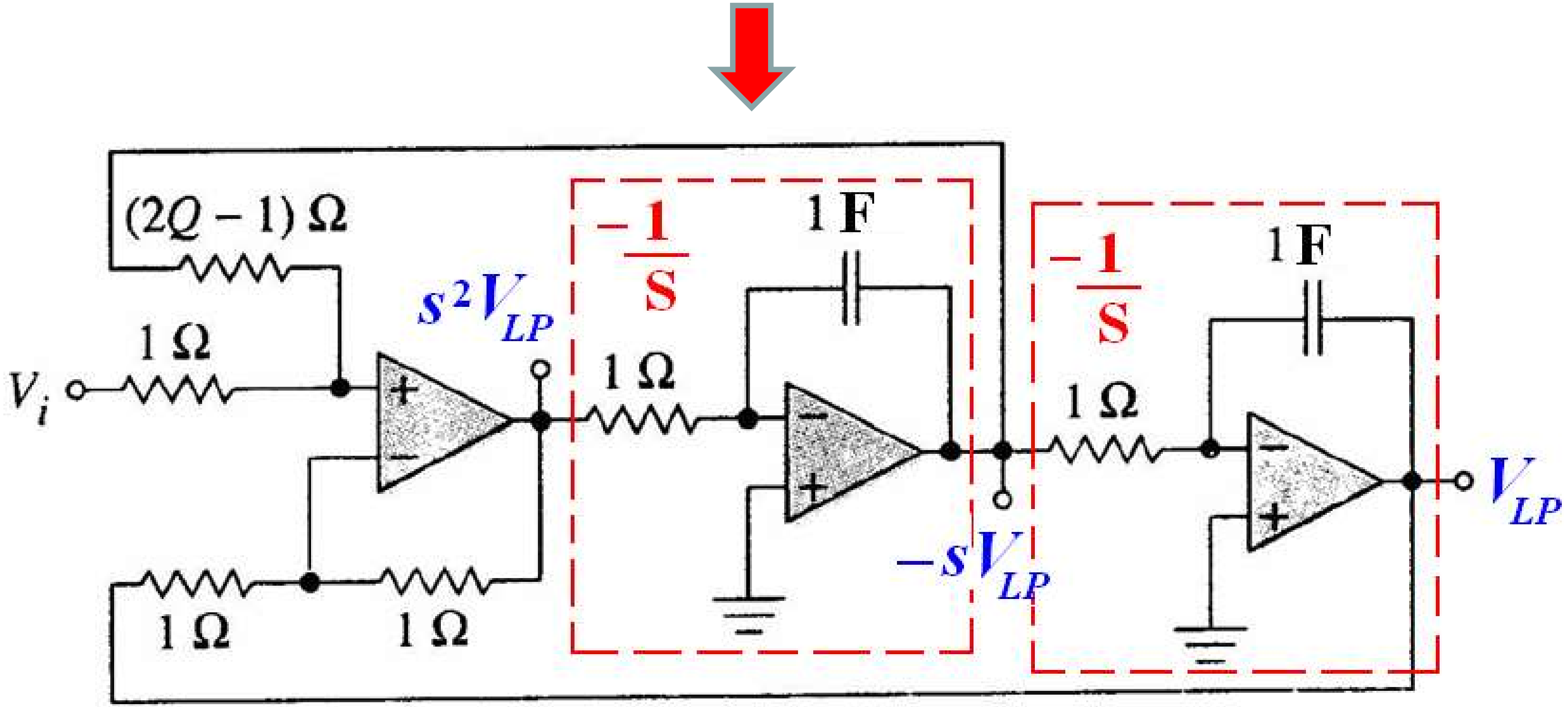
$$s^2V_{LP} = V_i\left(\frac{2Q-1}{Q}\right) - sV_{LP}\left(\frac{1}{Q}\right) - V_{LP}$$

s^2V_{LP}, sV_{LP} and V_{LP} are related as follows:

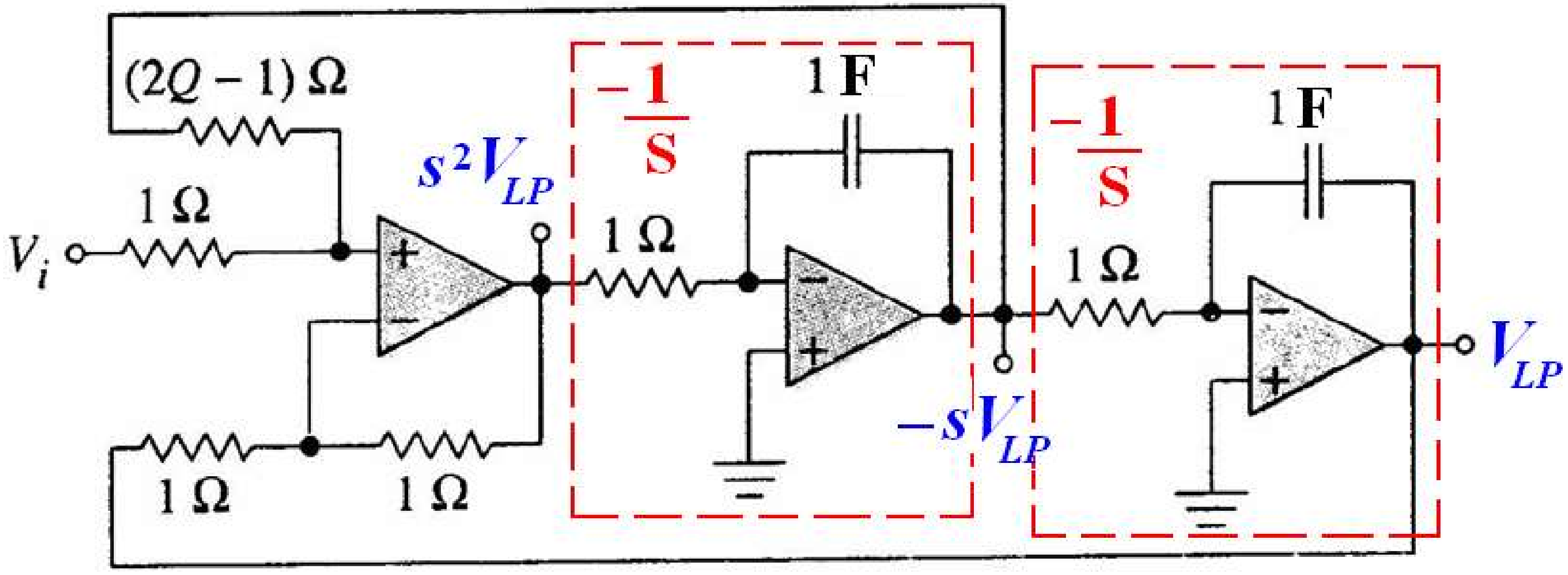




1 Ω
V_i
2 Q − 1
$-sV_{LP}$
V_{LP}
1 Ω
1 Ω
s^2V_{LP}
1 F
s^2V_{LP}
1 Ω
$-sV_{LP}$
1 Ω
1 F
V_{LP}



(2Q − 1) Ω
1 Ω
V_i
s^2V_{LP}
$-\frac{1}{S}$
1 F
1 Ω
$-sV_{LP}$
$-\frac{1}{S}$
1 F
1 Ω
V_{LP}
1 Ω
1 Ω



3 outputs with 2nd order high-pass, band-pass and low-pass responses.

$$\frac{s^2V_{LP}}{V_i} = \frac{Ks^2}{s^2+\left(\frac{1}{Q}\right)s+1}$$

High-pass filter

$$\frac{-sV_{LP}}{V_i} = \frac{-H(s/Q)}{s^2+\left(\frac{1}{Q}\right)s+1}$$

Band-pass filter

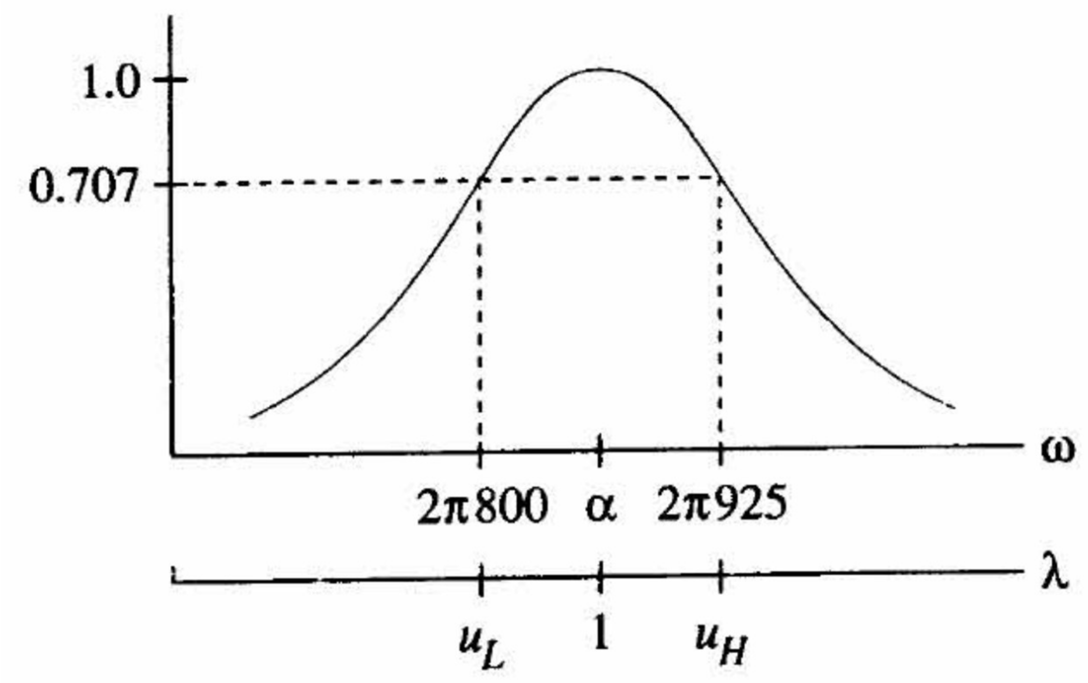
$$\frac{V_{LP}}{V_i} = \frac{K}{s^2+\left(\frac{1}{Q}\right)s+1}$$

Low-pass filter

$$K = \frac{2Q-1}{Q} \qquad H = 2Q-1$$

Exercise #4: Design a 2nd order band-pass filter with -3dB bandwidth from 800 Hz to 925 Hz. 1 kΩ standard resistors are preferred for your design.

$$\omega_H - \omega_L = 2\pi \times (925 - 800) = 785.4 \text{ rad/s}$$



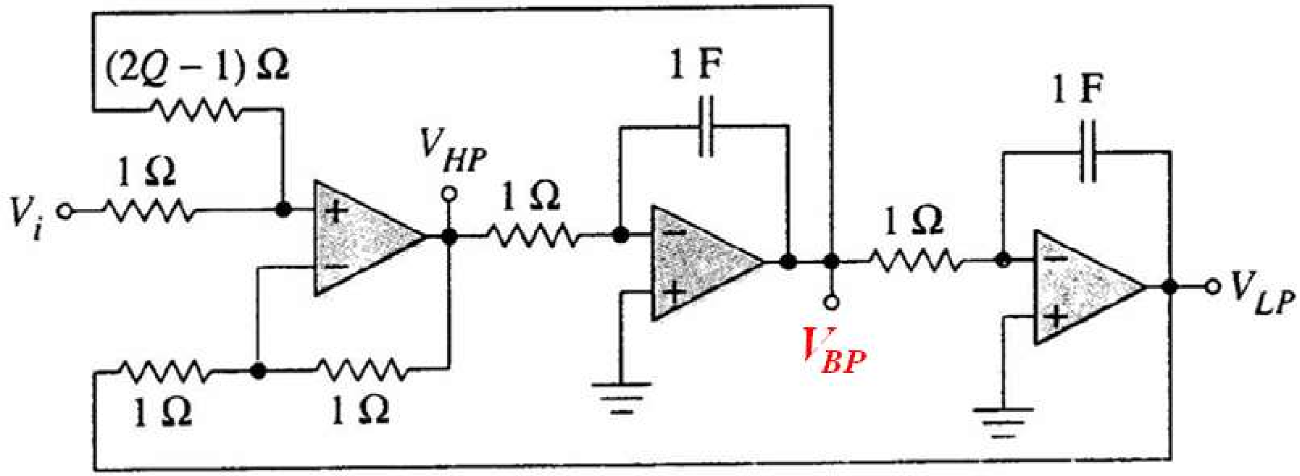
$$\omega_c = \sqrt{\omega_H \omega_L}$$
$$= 2\pi\sqrt{925 \times 800}$$
$$= 5.405 \times 10^3 \text{ rad/s}$$

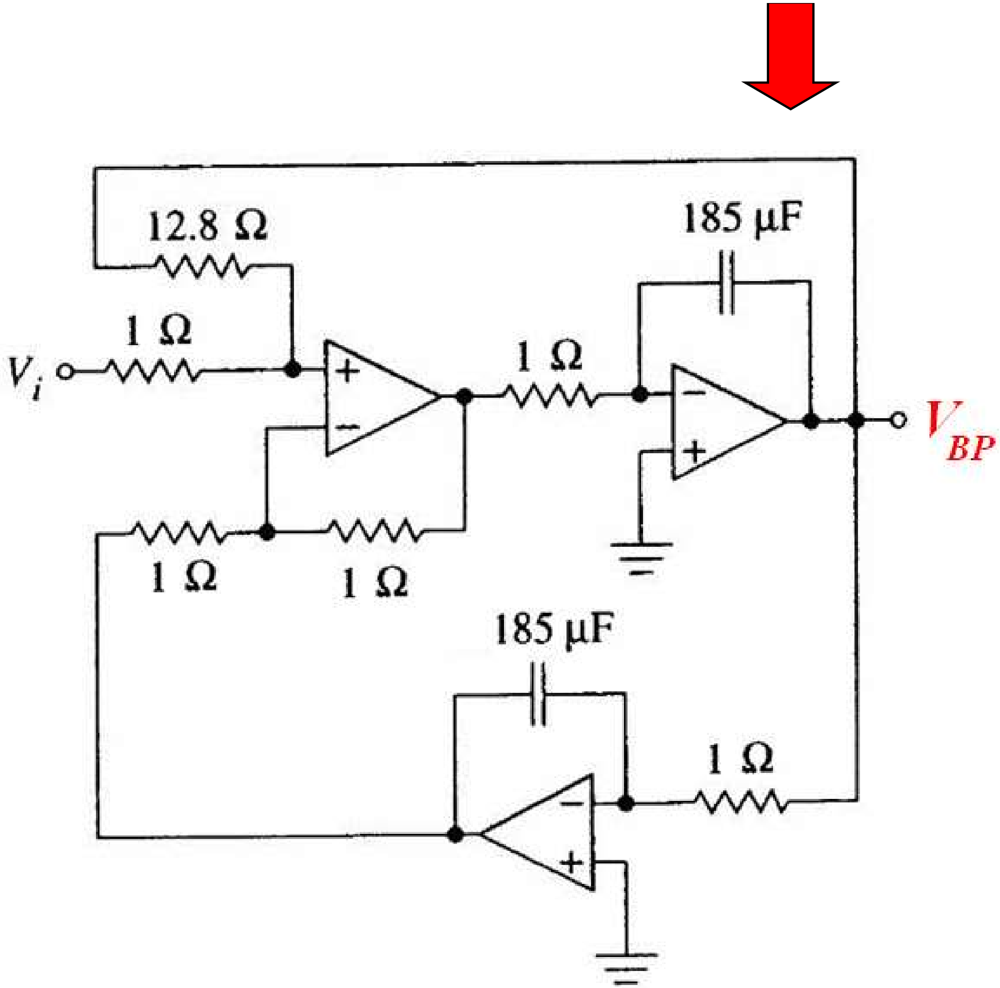
$$f_c = \frac{5.405 \times 10^3}{2\pi} = 860.23 \text{ Hz}$$

$$\alpha = 5.405 \times 10^3$$

$$u_H - u_L = \frac{\omega_H - \omega_L}{\alpha} = \frac{785.4}{5.405 \times 10^3} = 0.1453 \text{ rad/s}$$

$$Q = \frac{1}{u_H - u_L} = \frac{1}{0.1453} = 6.882$$





$$R_{(2Q-1)} = 2Q - 1$$

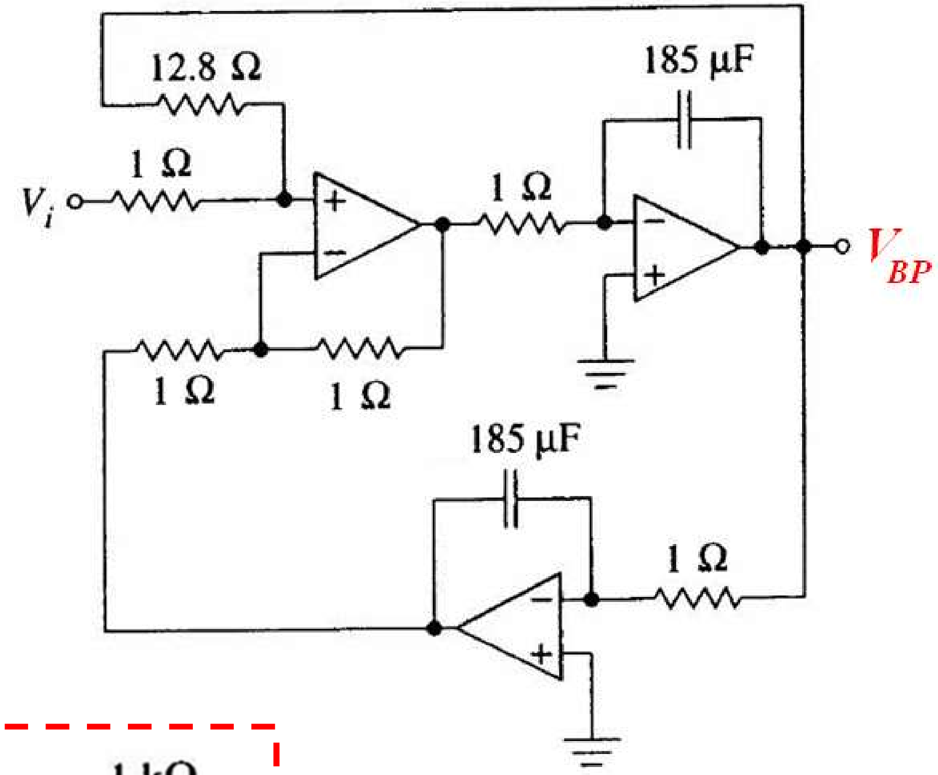
$$= 2 \times 6.882 - 1 = 12.8\,\Omega$$

Other $R = 1\,\Omega$

$$\alpha = \frac{5.405 \times 10^3}{1} = 5.405 \times 10^3$$

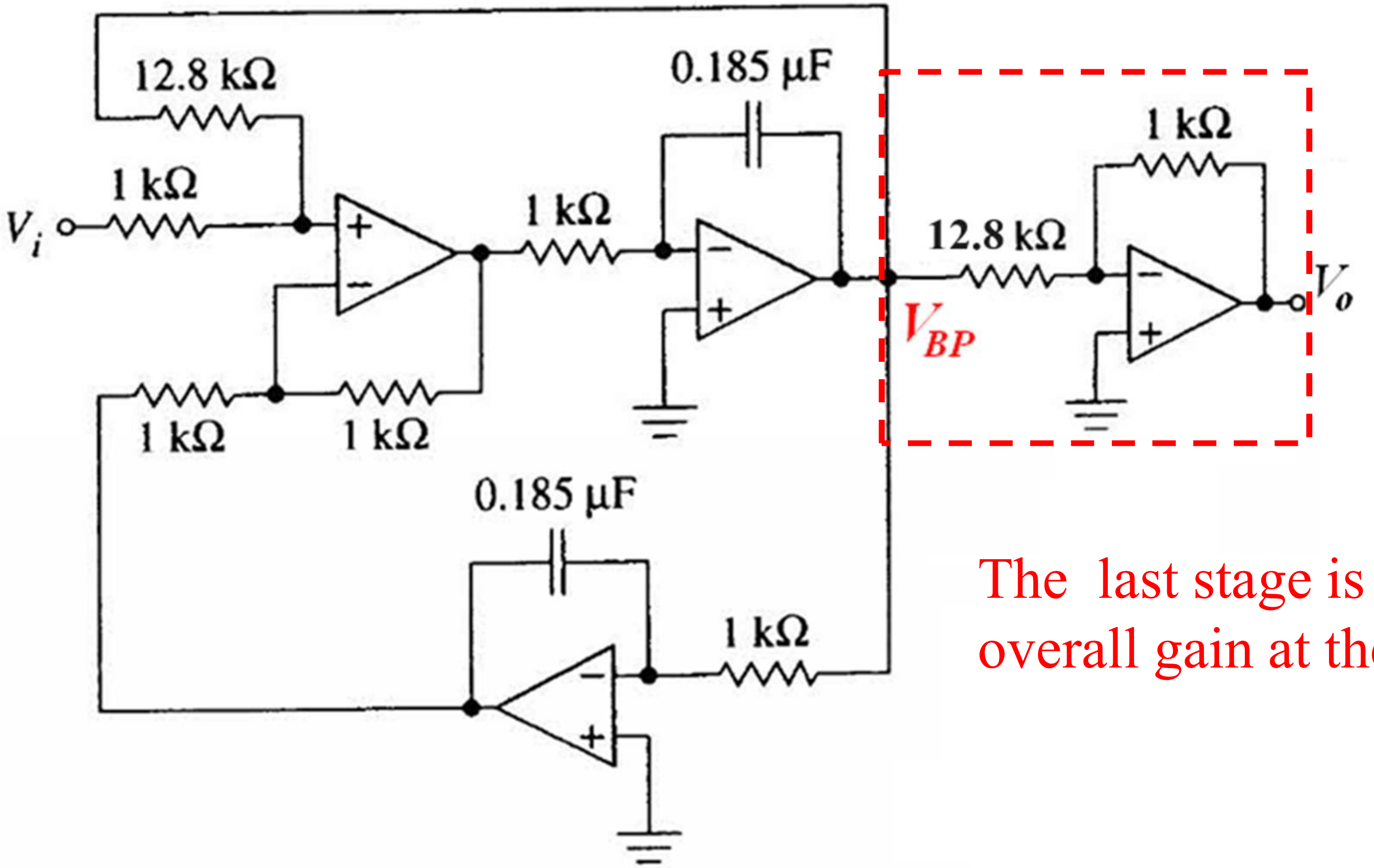
$$C' = \frac{1\ \text{F}}{5.405 \times 10^3} = 185\ \mu\text{F}$$

Impedance scaling factor: $\beta = \dfrac{1k\Omega}{1\Omega} = 10^3$



$$C' = \frac{C'}{\beta} = \frac{185\ \mu\text{F}}{10^3} = 0.185\ \mu\text{F}$$

$$R_{2Q-1}' = \beta R = 12.8 \times 10^3 = 12.8\ \text{k}\Omega$$

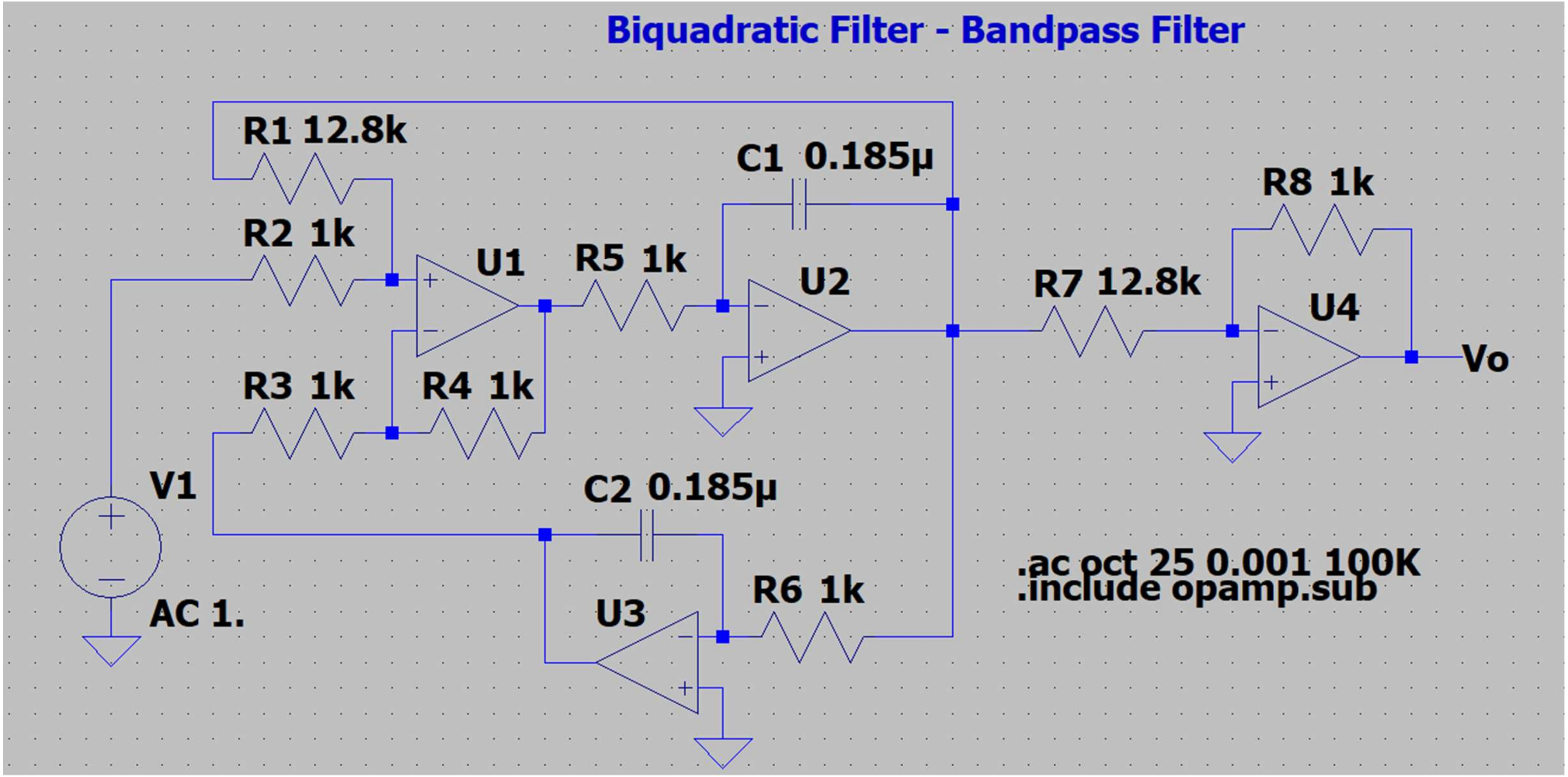


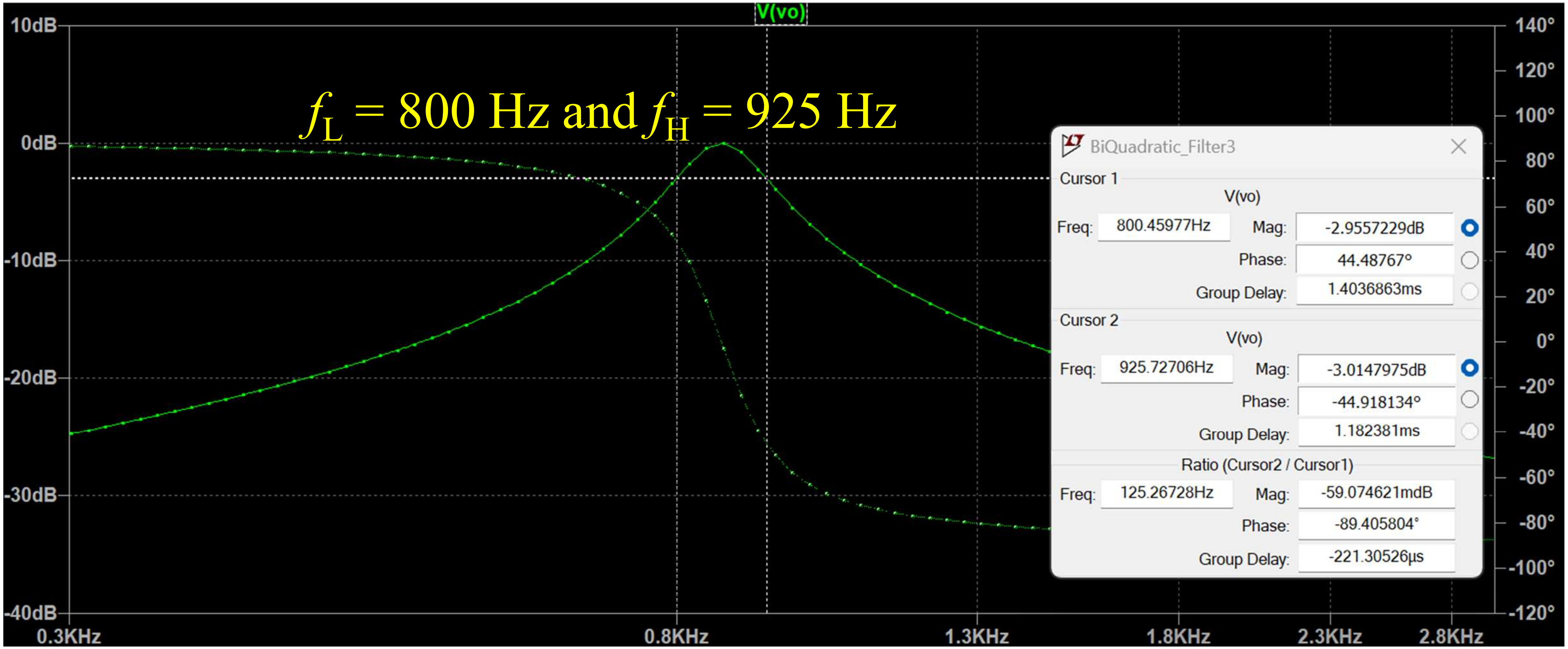
Volatge gain :

$$-\frac{1}{H} = \frac{-1}{2Q-1} = -\frac{1}{12.8}$$

The last stage is added so that the overall gain at the centre frequency = 1.

Verification with LTSPICE simulation:




V(vo)
f_L = 800 Hz and f_H = 925 Hz
10dB
0dB
-10dB
-20dB
-30dB
-40dB
0.3KHz
0.8KHz
1.3KHz
1.8KHz
2.3KHz
2.8KHz
140°
120°
100°
80°
60°
40°
20°
0°
-20°
-40°
-60°
-80°
-100°
-120°
BiQuadratic_Filter3
Cursor 1
V(vo)
Freq: 800.45977Hz
Mag: -2.9557229dB
Phase: 44.48767°
Group Delay: 1.4036863ms
Cursor 2
V(vo)
Freq: 925.72706Hz
Mag: -3.0147975dB
Phase: -44.918134°
Group Delay: 1.182381ms
Ratio (Cursor2 / Cursor1)
Freq: 125.26728Hz
Mag: -59.074621mdB
Phase: -89.405804°
Group Delay: -221.30526µs

2^{nd} Order Band-stop Filer

$$\frac{V_{BS}}{V_i} = T(s) = \frac{s^2 + 1}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

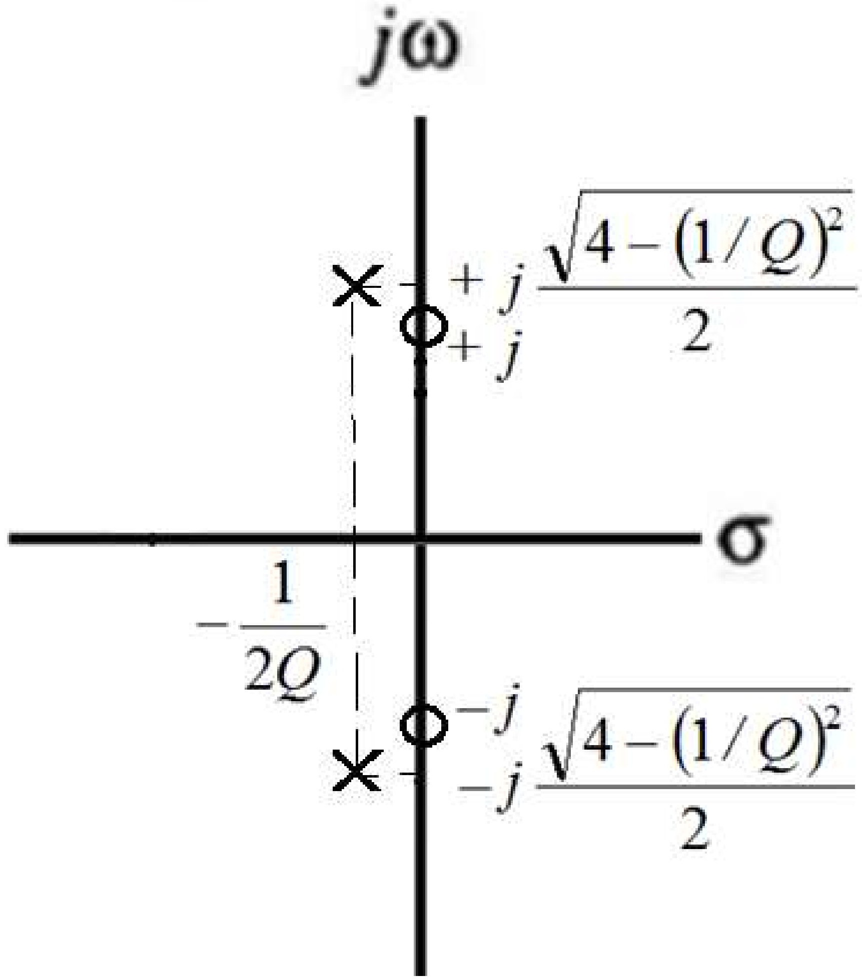
Two complex zeros:

$$s^2 + 1 = 0 \Rightarrow s = \pm j$$

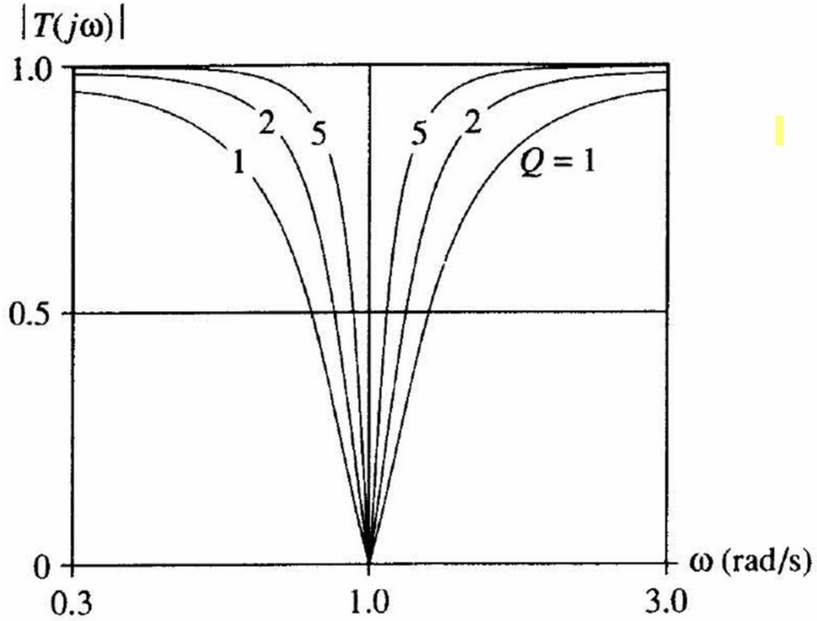
Two complex conjugate poles:

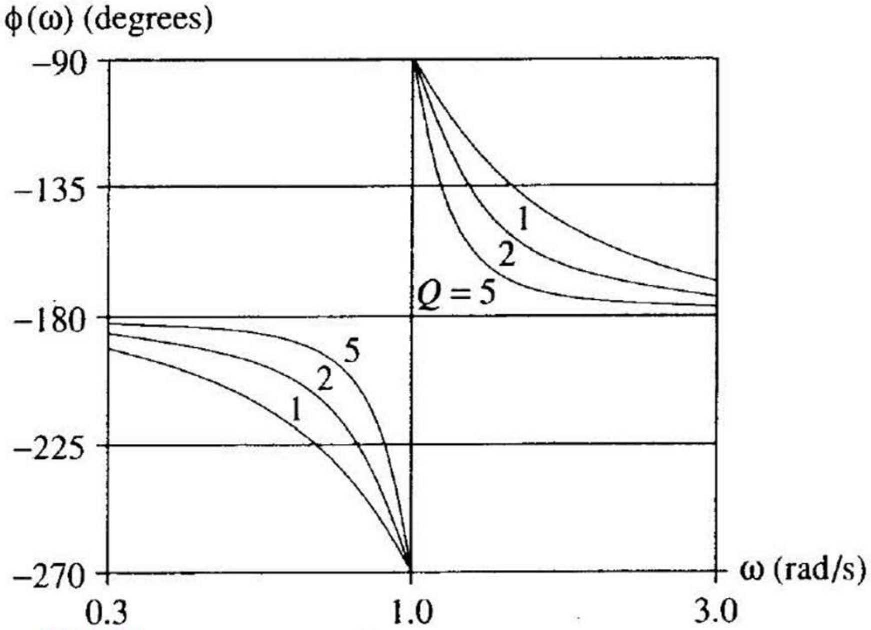
$$s^2 + \left(\frac{1}{Q}\right)s + 1 = 0$$

$$s = -\frac{1}{2Q} \pm j\frac{\sqrt{4-(1/Q)^2}}{2}$$



2^{nd} Order Band-stop Filer





The band-stop transfer function is:

$$T(s) = \frac{s^2 + 1}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

$$T(j\omega) = \frac{-\omega^2 + 1}{-\omega^2 + j\omega\left(\frac{1}{Q}\right) + 1}$$

$$|T(j\omega)| = \frac{1 - \omega^2}{\sqrt{(1-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}}$$

When $\omega = 1$ rad/s $: |T(j\omega)| = 0$

$$T_{BS}(s)=\frac{s^2+1}{s^2+\left(\frac{1}{Q}\right)s+1}=\frac{s^2}{s^2+\left(\frac{1}{Q}\right)s+1}+\frac{1}{s^2+\left(\frac{1}{Q}\right)s+1}$$

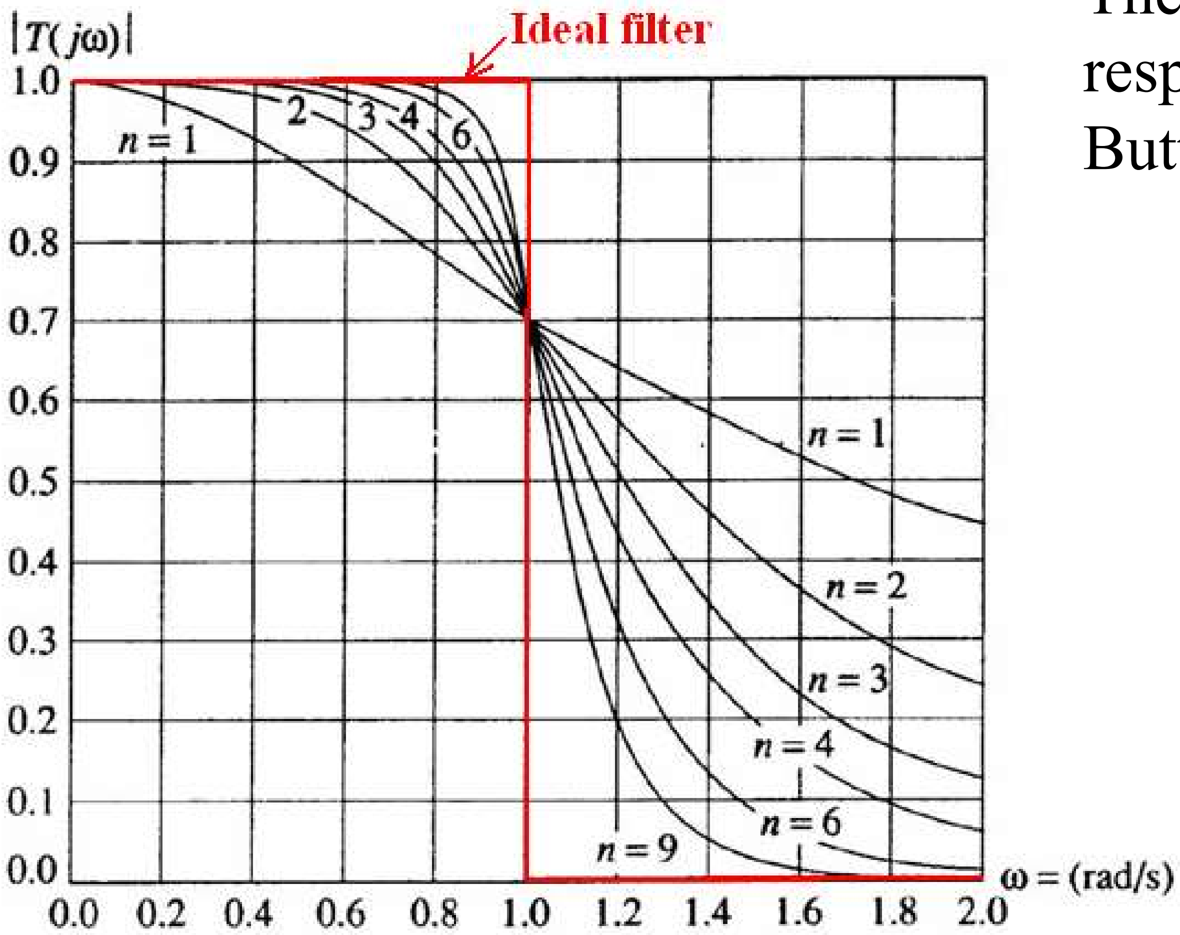
The 2nd order band-stop filter can be realized by using a summer to combine the high-pass and low-pass outputs of a bi-quad filter accordingly:

$$\frac{V_{HP}}{V_i}=\frac{Ks^2}{s^2+\left(\frac{1}{Q}\right)s+1} \qquad \frac{V_{LP}}{V_i}=\frac{K}{s^2+\left(\frac{1}{Q}\right)s+1}$$

$$T_{BS}(s)=\frac{V_{BS}}{V_i}=\frac{1}{K}\left(\frac{V_{LP}}{V_i}+\frac{V_{HP}}{V_i}\right), \quad K=\frac{2Q-1}{Q}$$

Butterworth Filter Design

Butterworth filter is designed to have a flat pass-band response.



The magnitude frequency response for the nth order Butterworth filter is:

$$|T(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

When $\omega = 1\ \mathrm{rad/s}$:

$$|T(j\omega)| = \frac{1}{\sqrt{1+1^{2n}}} = \frac{1}{\sqrt{2}}$$

$$\left|T(j\omega)\right|^2 = T(j\omega)T(-j\omega) = \frac{1}{1+\omega^{2n}}$$

$$\because s = j\omega \Rightarrow \omega = \frac{s}{j} \qquad \therefore T(s)T(-s) = \frac{1}{1+(s/j)^{2n}}$$

To obtain the poles:

$$1+(s/j)^{2n} = 0$$

$$(s/j)^{2n} = -1$$

$$s^{2n} = -1(j)^{2n}$$

$$e^{j(2k-1)\pi} = \cos(2k-1)\pi + j\sin(2k-1)\pi = -1$$

where $k = 1, 2, 3 \ldots$

$$e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2) = j$$

$$s^{2n} = -1(j)^{2n}$$

$$e^{j(2k-1)\pi} = -1 \qquad\qquad e^{j\pi/2} = j$$

$$s^{2n} = e^{j(2k-1)\pi}\left(e^{j\pi/2}\right)^{2n} = e^{j(2k-1)\pi}e^{jn\pi} = e^{j(2k-1+n)\pi}$$

$$s_k = e^{\frac{j(2k+n-1)\pi}{2n}} \quad \text{where } k = 1, 2, 3, \ldots, n$$

$$\therefore T(s) = \frac{1}{(s-s_1)(s-s_2)\ldots\ldots(s-s_n)}$$

$$s_k = \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1)$$

For example, $n = 4$ (4th order Butterworth filter):

$$s_k = \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1)$$

$$s_k = \cos\frac{\pi}{8}(2k+4-1) + j\sin\frac{\pi}{8}(2k+4-1)$$

Therefore the 4 poles are:

$$s_1 = \cos\frac{5\pi}{8} + j\sin\frac{5\pi}{8} = -0.3827 + j0.9239$$

$$s_2 = \cos\frac{7\pi}{8} + j\sin\frac{7\pi}{8} = -0.9239 + j0.3827$$

$$s_3 = \cos\frac{9\pi}{8} + j\sin\frac{9\pi}{8} = -0.9239 - j0.3827$$

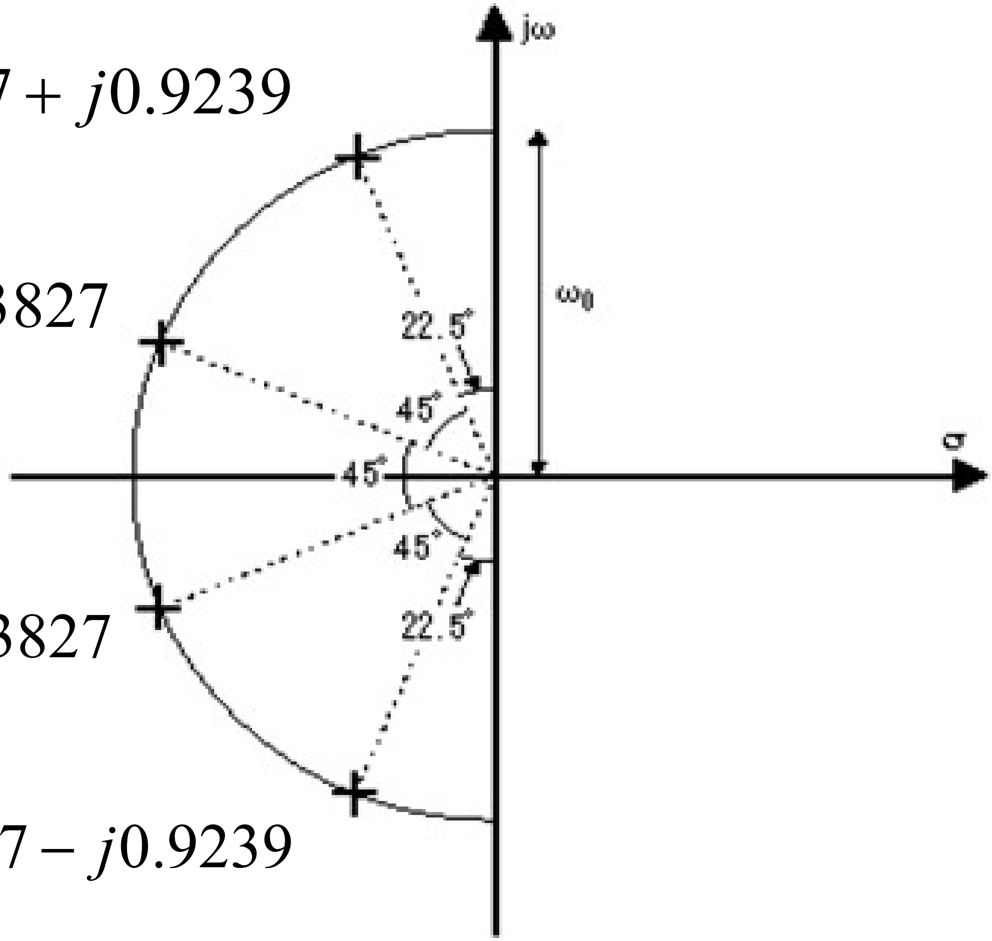
$$s_4 = \cos\frac{11\pi}{8} + j\sin\frac{11\pi}{8} = -0.3827 - j0.9239$$

$$s_1 = -0.3827 + j0.9239$$

$$s_2 = -0.9239 + j0.3827$$

$$s_3 = -0.9239 - j0.3827$$

$$s_4 = -0.3827 - j0.9239$$



$$T(s) = \frac{1}{(s + 0.3827 - j0.9239)(s + 0.3827 + j0.9239)(s + 0.9239 - j0.3827)(s + 0.9239 + j0.3827)}$$

$$T(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

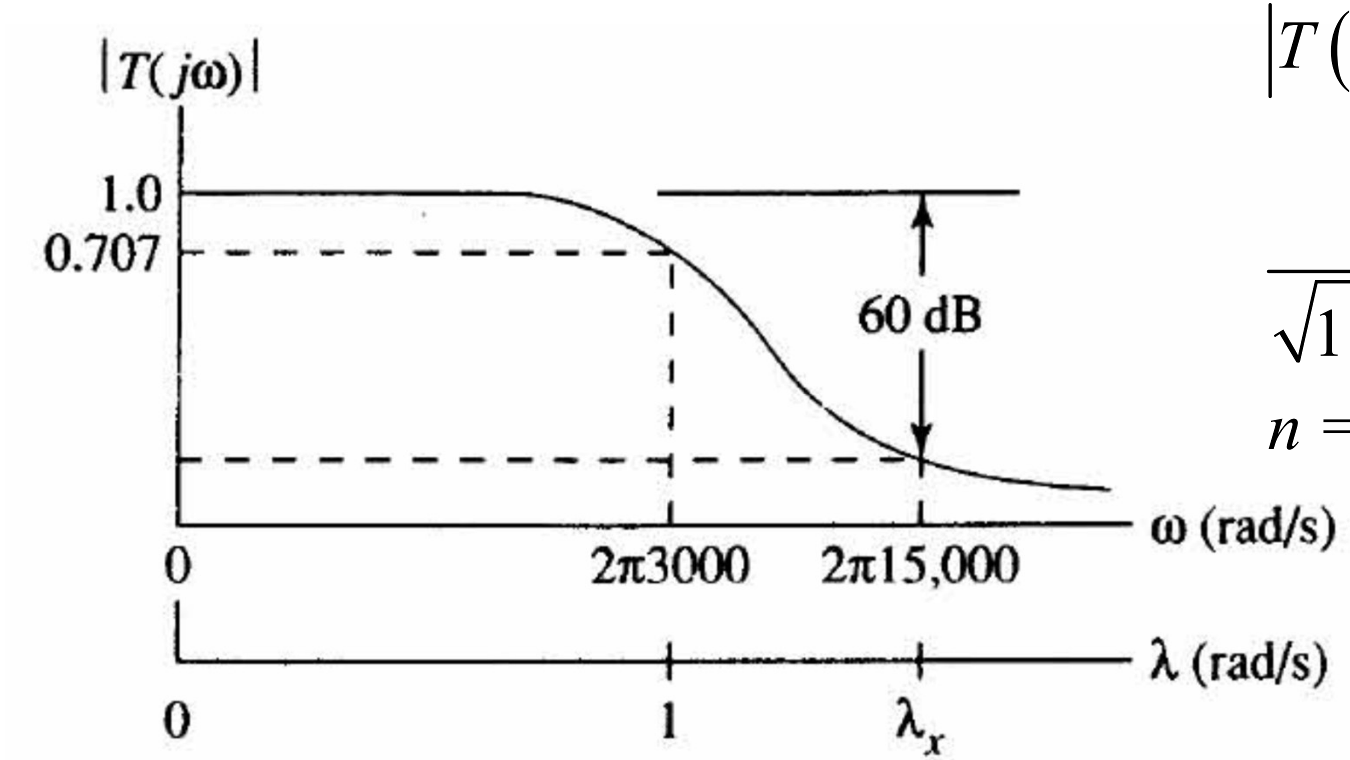
Butterworth Filter Response

The poles of nth order Butterworth filter can be determined using the method described earlier.

$$T(s) = \frac{1}{B(s)}$$

n	Factors of $B(s)$
1	$(s+1)$
2	$(s^2+1.4142s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7654s+1)(s^2+1.8478s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+1.4142s+1)(s^2+1.9318s+1)$

Exercise #5: Design a maximally flat, low-pass, active filter with -3dB bandwidth of 3 kHz and attenuation of at least 60 dB at 15 kHz. 10 kΩ resistors are preferred for your design.



$$\left|T\left(j\lambda_x\right)\right| = \frac{1}{\sqrt{1+\lambda_x^{\ 2n}}} = 10^{\frac{-60}{20}}$$

$$\frac{1}{\sqrt{1+5^{2n}}} = 0.001$$

n = 4.29 (choose n = 5)

$$\alpha = 2\pi \times 3000 = 6000\pi \qquad \lambda_{\mathrm{x}} = \frac{2\pi \times 15k}{\alpha} = 5 \text{ rad/s}$$

n	Factors of $B(s)$
1	$(s+1)$
2	$(s^2+1.4142s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7654s+1)(s^2+1.8478s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+1.4142s+1)(s^2+1.9318s+1)$

$$T(s)=\frac{1}{B(s)}$$

$$T(s)=\frac{1}{(s+1)}\frac{1}{\left(s^2+0.6180s+1\right)}\frac{1}{\left(s^2+1.6180s+1\right)}$$

$$T(s)=\frac{1}{(s+1)}\frac{1}{\left(s^2+0.6180s+1\right)}\frac{1}{\left(s^2+1.6180s+1\right)}$$

The transfer function can be realized by using one 1st order and two 2nd order Sallen-Key filters. The Q values and the corresponding gains A for Sallen-Key design are:

$$\frac{1}{Q_1}=0.6180 \Rightarrow Q_1=\frac{1}{0.6180}=1.618 \qquad A_1=3-\frac{1}{Q_1}=2.382$$

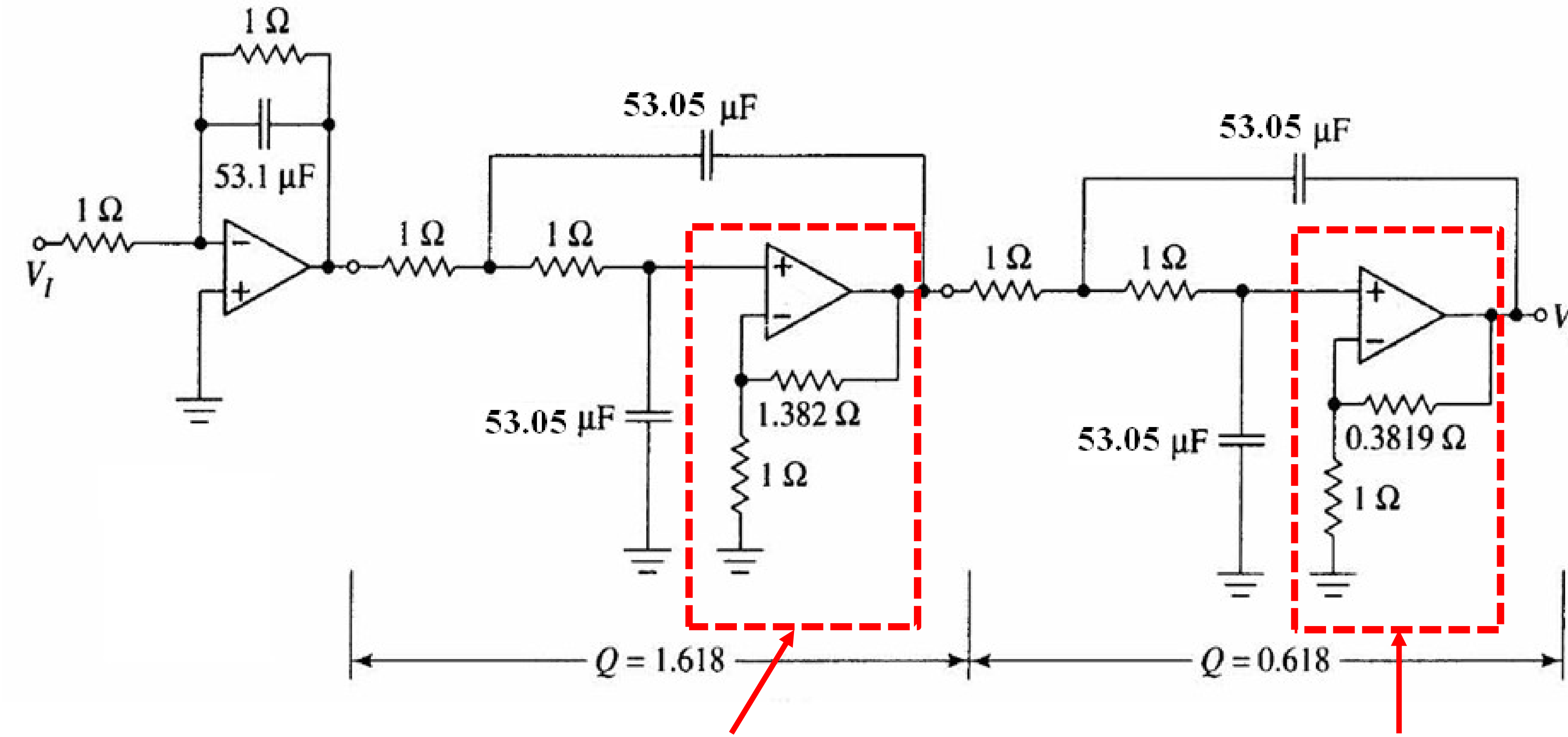
$$\frac{1}{Q_2}=1.6180 \Rightarrow Q_2=\frac{1}{1.6180}=0.6180 \qquad A_2=3-\frac{1}{Q_2}=1.3819$$

$$T(s)=\frac{1}{(s+1)}\frac{2.382}{\left(s^2+0.6180s+1\right)}\frac{1.3819}{\left(s^2+1.6180s+1\right)}$$

$$A_1A_2=2.382\times1.3819=3.3$$

After frequency-scaling by $\alpha = 2\pi \times 3000$,

$$C = \frac{1\text{F}}{2\pi \times 3000} = 53.05\ \mu\text{F}$$

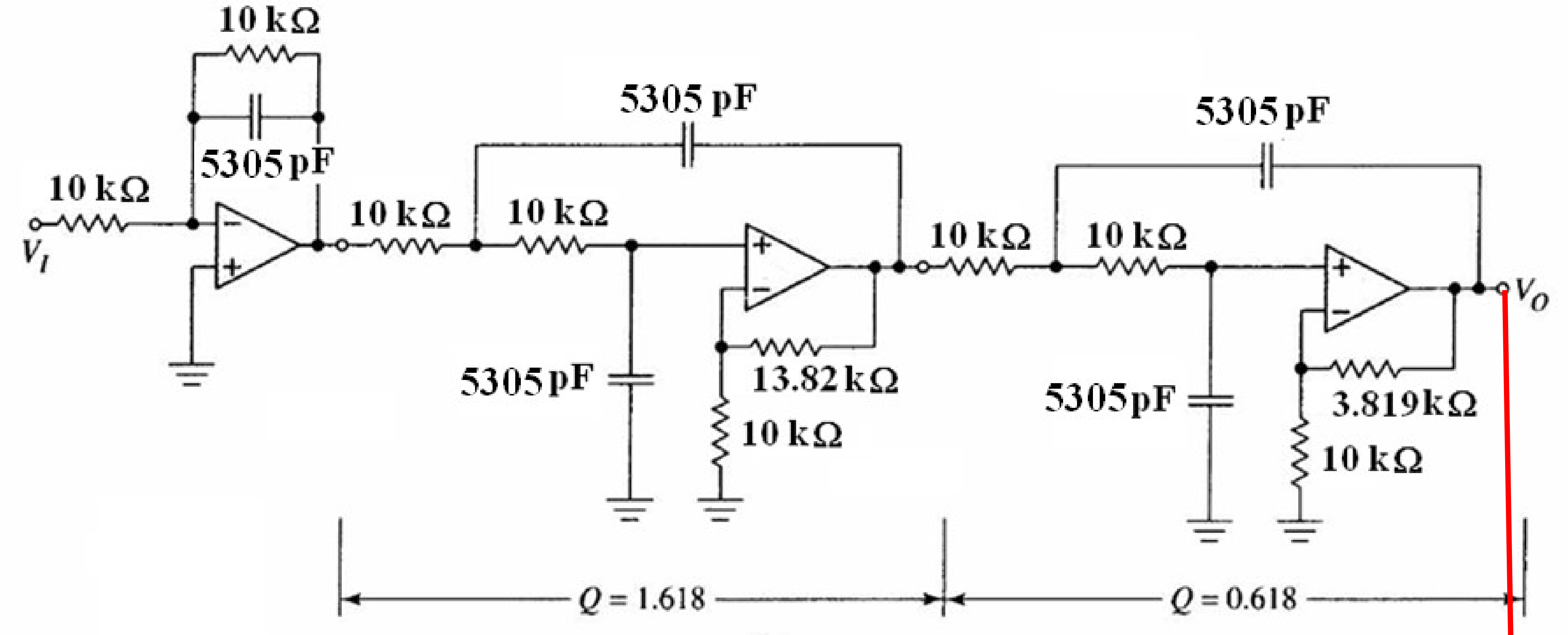


$$A_1 = 1 + \frac{1.382}{1} = 2.382$$

$$A_2 = 1 + \frac{0.3819}{1} = 1.3819$$

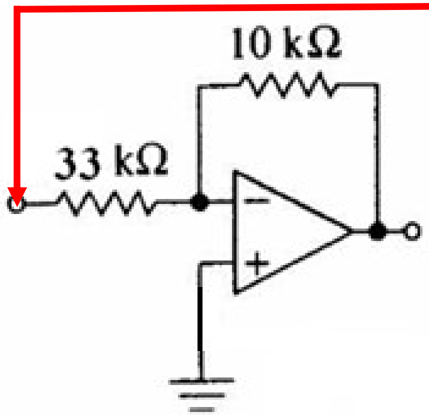
After impedance-scaling by $\beta = 10^4$

$$C = \frac{53.05\mu\text{F}}{10^4} = 5305\ \text{pF}$$

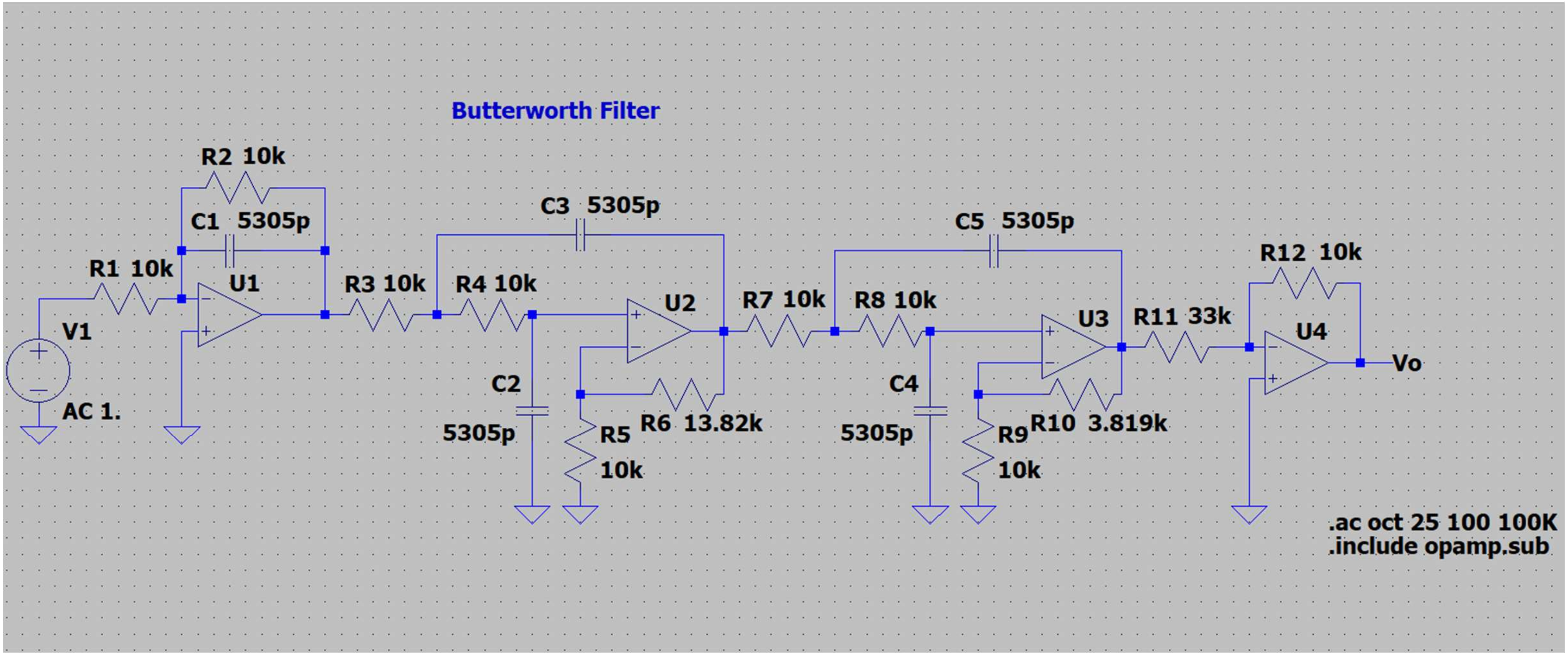


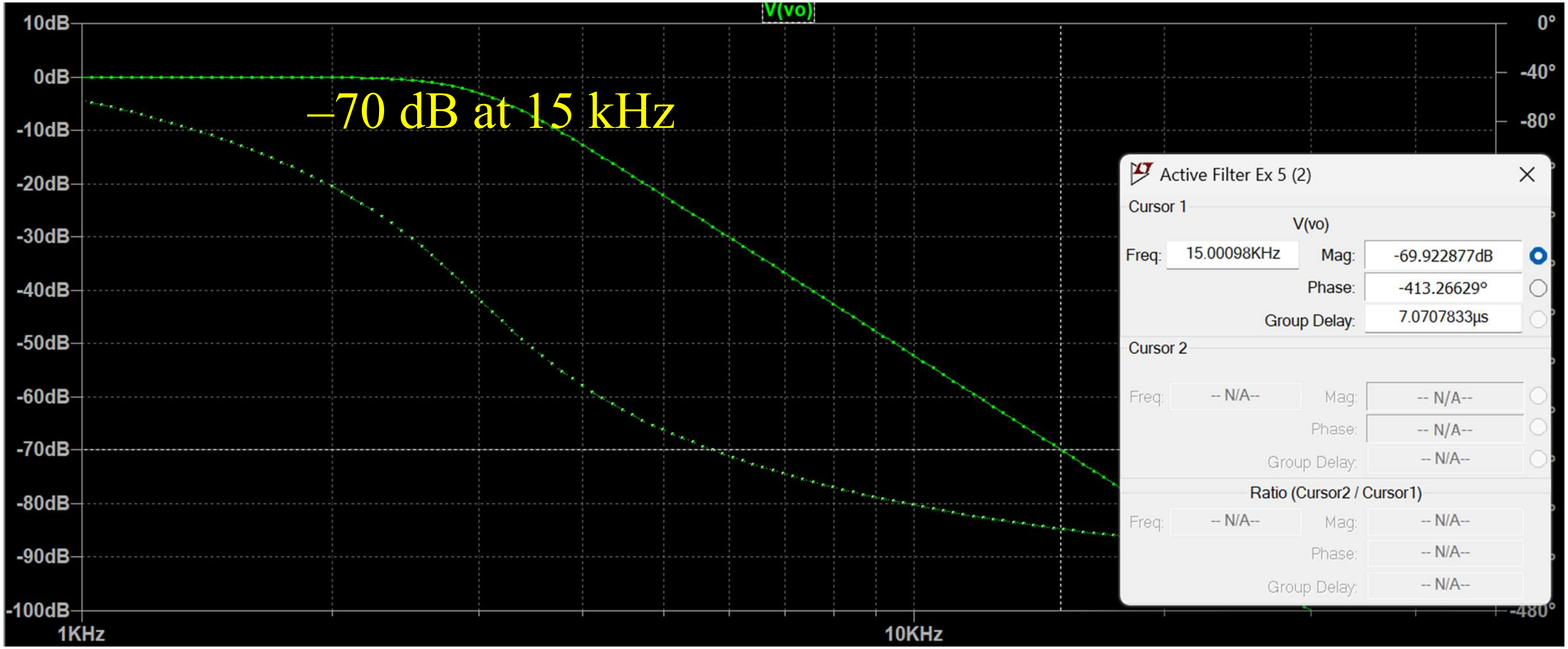
Add another inverting amplifier with the following voltage gain at the output :

$$A = \left| \frac{1}{A_1 A_2} \right| = \frac{1}{3.3}$$

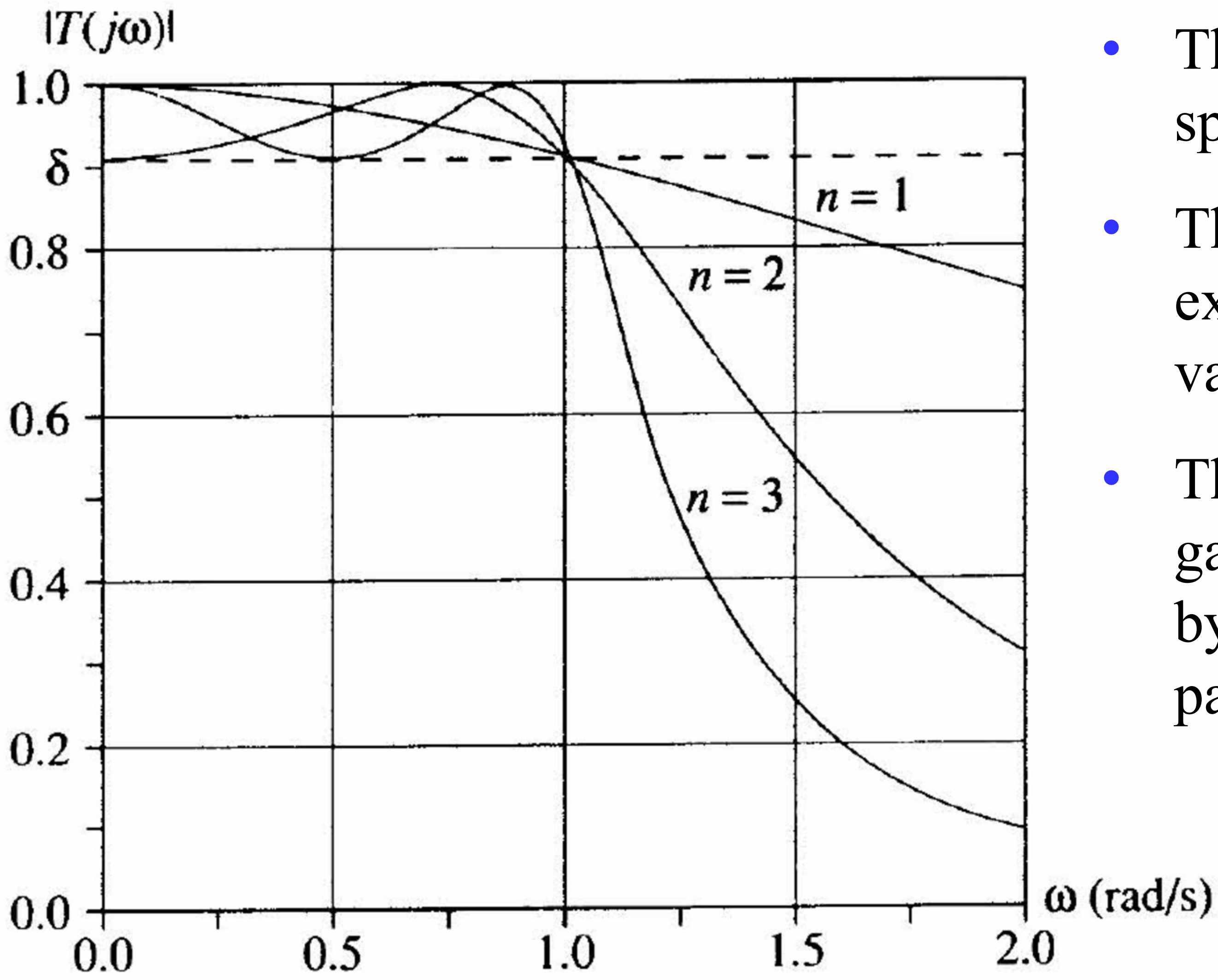


Verification with LTSPICE simulation:




V(vo)
10dB
0dB
-10dB
-20dB
-30dB
-40dB
-50dB
-60dB
-70dB
-80dB
-90dB
-100dB
0°
-40°
-80°
-480°
1KHz
10KHz
–70 dB at 15 kHz
Active Filter Ex 5 (2)
Cursor 1
V(vo)
Freq: 15.00098KHz
Mag: -69.922877dB
Phase: -413.26629°
Group Delay: 7.0707833µs
Cursor 2
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--
Ratio (Cursor2 / Cursor1)
Freq: -- N/A--
Mag: -- N/A--
Phase: -- N/A--
Group Delay: -- N/A--

Chebyshev Filter Design



- The parameter n specifies the filter order
- The transfer function exhibits a ripple or variation in gain for $n > 1$
- The minimum pass-band gain δ can be controlled by means of a Chebyshev parameter ε

$$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^{\ 2}(\omega)}}$$

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(\omega)}}$$

Chebyshev Polynomial of degree n, $C_n(\omega)$ is defined by:

$$C_n(\omega) = \cos\left(n\cos^{-1}\omega\right) \qquad C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega)$$

For examples:

$$C_0(\omega) = \cos\left(0 \times \cos^{-1}\omega\right) = \cos(0) = 1$$

$$C_1(\omega) = \cos\left(\cos^{-1}\omega\right) = \omega$$

$$C_2(\omega) = C_{1+1}(\omega) = 2\omega C_1(\omega) - C_0(\omega) = 2\omega^2 - 1$$

$$C_3(\omega) = C_{2+1}(\omega) = 2\omega C_2(\omega) - C_1(\omega) = 2\omega\left(2\omega^2 - 1\right) - \omega = 4\omega^3 - 3\omega$$

$$C_4 = 8\omega^4 - 8\omega^2 + 1$$

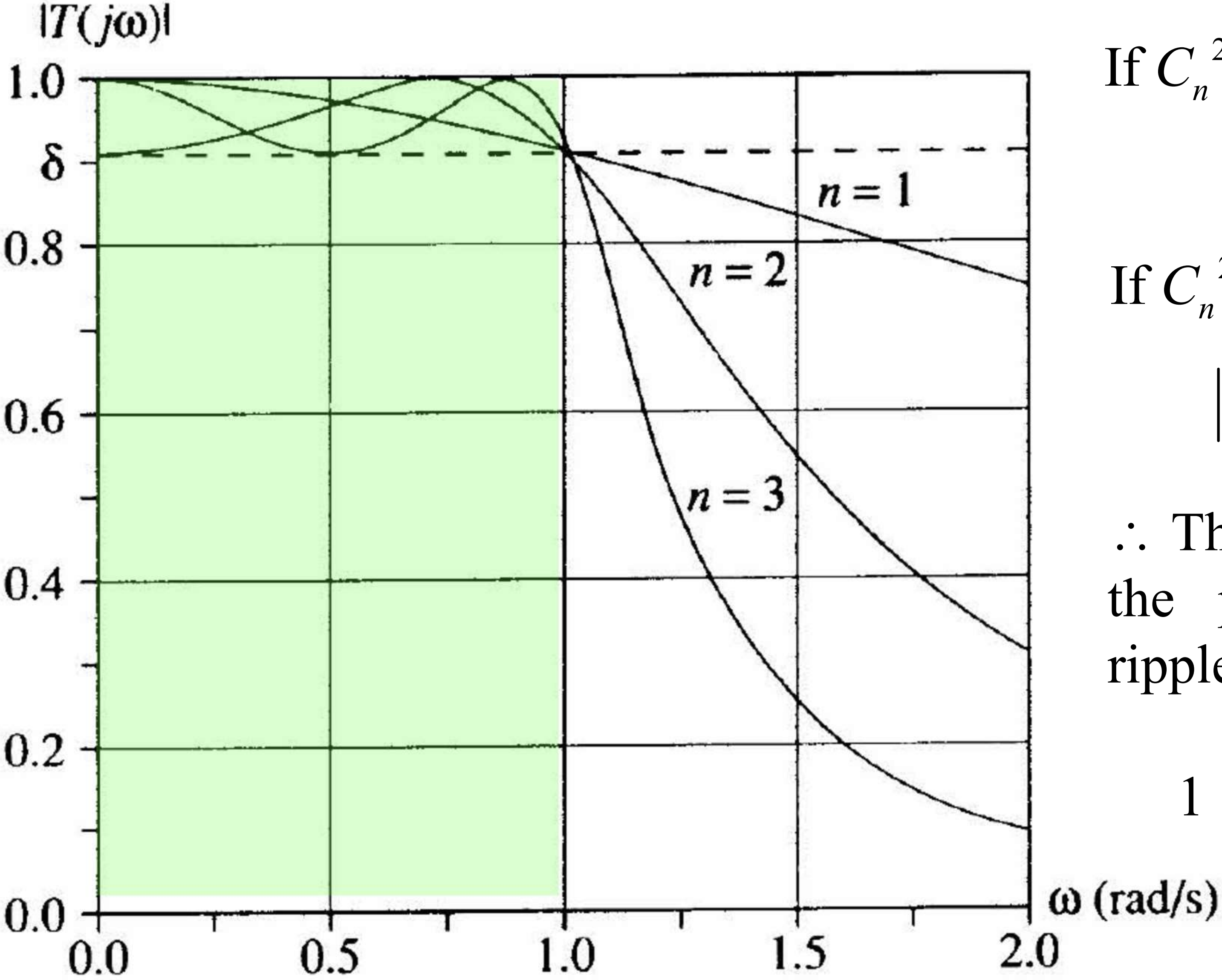
$$C_5 = 16\omega^5 - 20\omega^3 + 5\omega$$

Note: $-1 < C_n(\omega) < 1$ for $\omega \le 1$

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(\omega)}}$$

For $\omega \leq 1$: $-1 \leq C_n(\omega) \leq 1$

$\therefore\ C_n{}^2(\omega) \leq 1$ for $\omega \leq 1$



If $C_n{}^2(\omega) = 0$:

$$\left|T(j\omega)\right| = 1$$

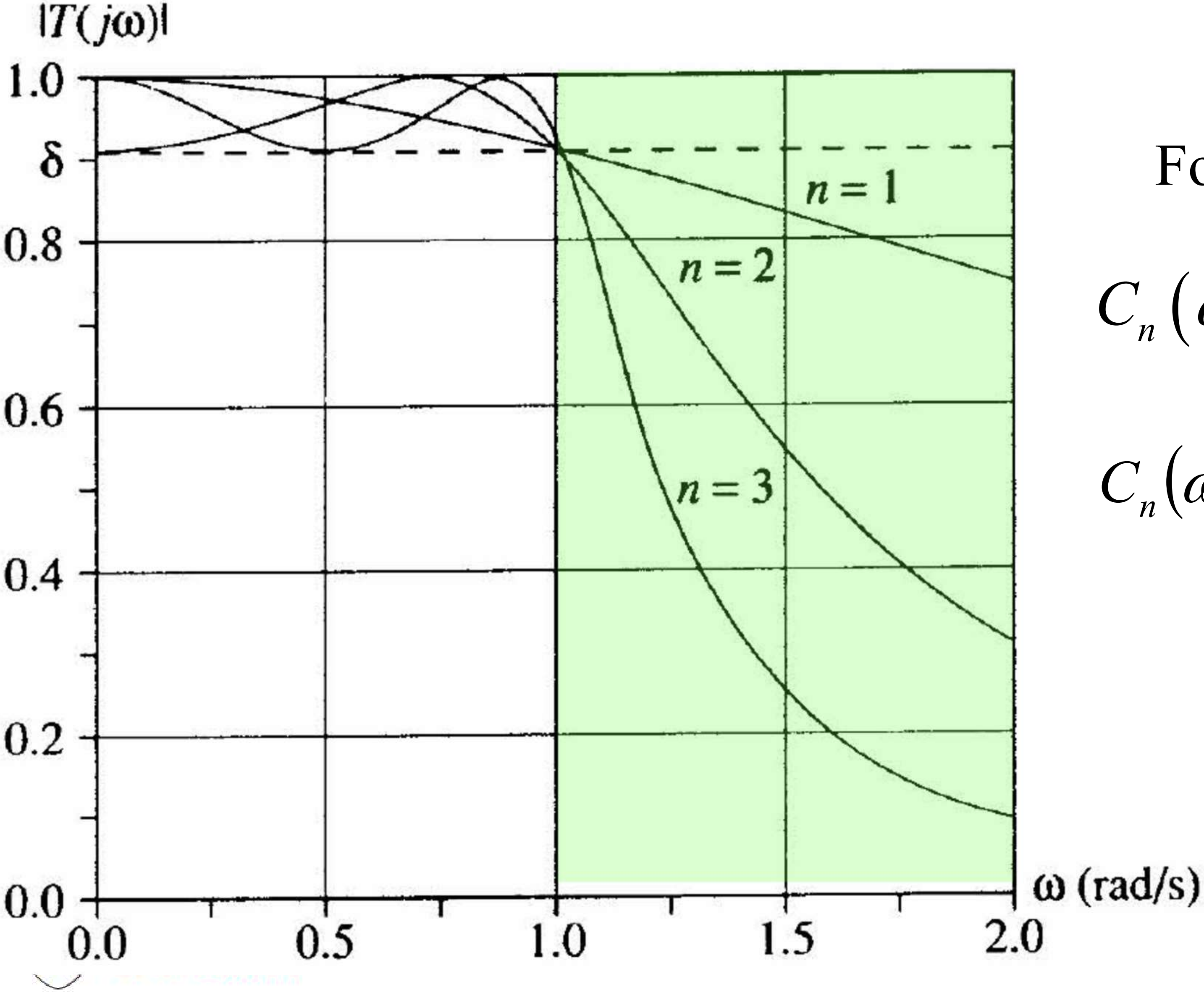
If $C_n{}^2(\omega) = 1$:

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2}}$$

$\therefore$ The magnitude within the pass-band will be rippled between:

$$1 \text{ and } \delta = \frac{1}{\sqrt{1+\varepsilon^2}}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(\omega)}}$$



For $\omega > 1$:

$$C_n(\omega) = \cos\left(n\cos^{-1}\omega\right)$$

$$C_n(\omega) = \cosh\left(n\cosh^{-1}\omega\right)$$

$$\left|T(j\omega)\right|^2 = T(j\omega)T(-j\omega) = \frac{1}{1+\varepsilon^2 C_n^{\ 2}(\omega)}$$

$$\because s = j\omega \Rightarrow \omega = \frac{s}{j} \qquad \therefore T(s)T(-s) = \frac{1}{1+\varepsilon^2 C_n^{\ 2}\left(\frac{s}{j}\right)}$$

To obtain the poles: $1+\varepsilon^2 C_n^{\ 2}\left(\frac{s}{j}\right) = 0$

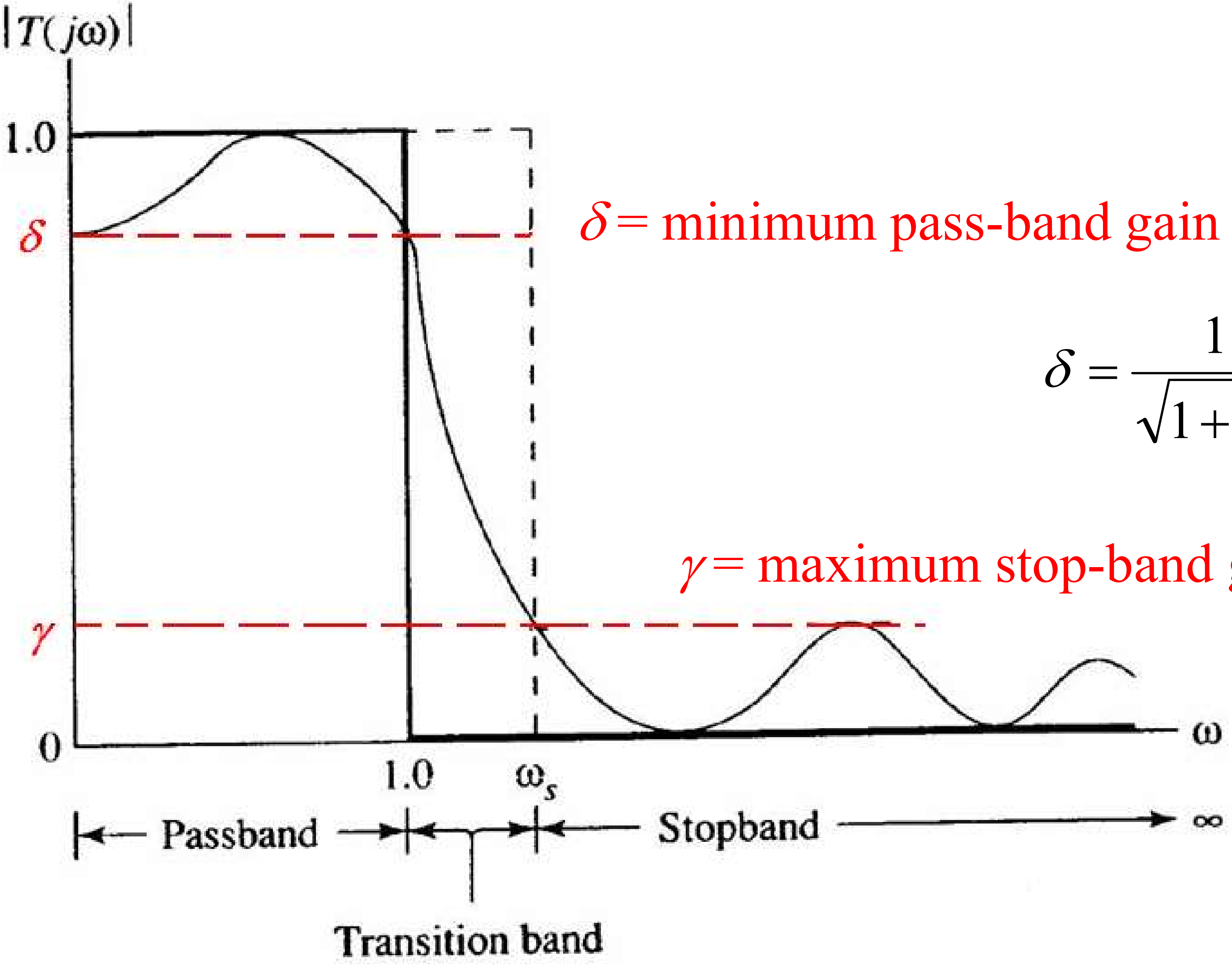
$$1+\varepsilon^2 \cos^2\left[n\cos^{-1}\left(\frac{s}{j}\right)\right] = 0$$

Skip the mathematical details, the poles will be:

$$s_k = -\sin\left(\frac{(2k-1)\pi}{2n}\right)\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\left(\frac{(2k-1)\pi}{2n}\right)\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

$$\therefore T(s) = \frac{1}{(s-s_1)(s-s_2)\ldots\ldots(s-s_n)}, \quad \text{where } k = 1,2,3\ldots,\ n$$

Chebyshev Filter Design



δ = minimum pass-band gain

$$\delta = \frac{1}{\sqrt{1+\varepsilon^2}}$$

γ = maximum stop-band gain

To design a Chebyshev filter, the specification for minimum pass-band gain δ is given.

The pass-band ripple in dB and δ is related by:

$$r_{dB} = 20\log(1) - 20\log\delta$$

$$= 0 - 20\log\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)$$

$$= -20\log\left(\frac{1}{1+\varepsilon^2}\right)^{1/2}$$

$$= -10\log\left(\frac{1}{1+\varepsilon^2}\right)$$

Compute ε when r_{dB} is specified:

$$\therefore \frac{1}{1+\varepsilon^2} = 10^{-\frac{r_{dB}}{10}}$$

$$1+\varepsilon^2 = 10^{\frac{r_{dB}}{10}}$$

$$\varepsilon = \left(10^{\frac{r_{dB}}{10}} - 1\right)^{\frac{1}{2}}$$

Another specification for maximum stop-band gain γ at stop-band frequency ω_s:

$$\left|T(j\omega_s)\right| = \gamma = \frac{1}{\sqrt{1+\varepsilon^2\left[\cosh\left(n\cosh^{-1}\omega_s\right)\right]^2}}$$

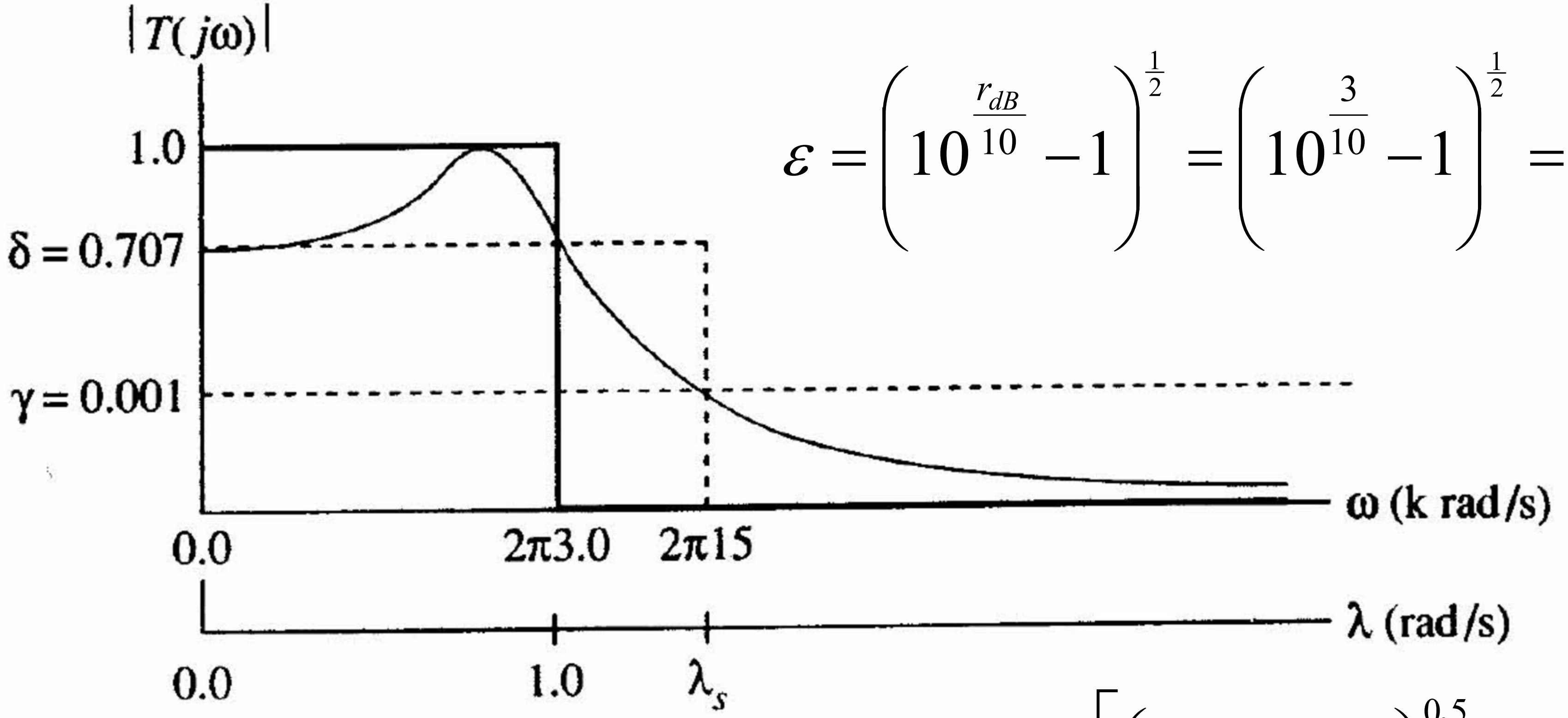
$$\gamma^2 = \frac{1}{1+\varepsilon^2\left[cosh\left(n\, cosh^{-1}\,\omega_s\right)\right]^2}$$

$$\gamma^2\left(1+\varepsilon^2\right)\left[\cosh\left(n\cosh^{-1}\omega_s\right)\right]^2 = 1$$

$$n = \frac{cosh^{-1}\left[\left(\frac{1}{\gamma^2}-1\right)^{\frac{1}{2}}/\varepsilon\right]}{cosh^{-1}\,\omega_s}$$

It allows us to solve for n when the stop-band gain γ is given.

Exercise #6: Design a Chelyshev low-pass filter with −3 dB bandwidth of 3 kHz and at least 60 dB attenuation at 15 kHz. 5 kΩ standard resistors are preferred.



$$\varepsilon = \left(10^{\frac{r_{dB}}{10}} - 1\right)^{\frac{1}{2}} = \left(10^{\frac{3}{10}} - 1\right)^{\frac{1}{2}} = 0.998$$

$$\lambda_s = \frac{2\pi \times 15k}{2\pi \times 3k} = 5 \text{ rad/s}$$

$$n = \frac{\cosh^{-1}\left[\left(\frac{1}{0.001^2} - 1\right)^{0.5} / 0.998\right]}{\cosh^{-1}(5)}$$

$$= 3.317 \qquad \therefore \text{choose n} = 4$$

$$T(s) = \frac{1}{B(s)} = \frac{K}{s^n + a_{n-1}s^{n-1} + \ldots + a_o}$$

$K = a_o$ for odd n (n=1, 3, 5...)

$K = \frac{a_o}{\sqrt{1+\varepsilon^2}}$ for even n (n = 2, 4, 6...)

Chebyshev Denominator $B(s)$

Passband Ripple (dB)	n	$B(s)$
0.5	1	$(s + 2.863)$
	2	$(s^2 + 1.426s + 1.5164)$
	3	$(s + 0.626)(s^2 + 0.626s + 1.142453)$
	4	$(s^2 + 0.350s + 1.062881)(s^2 + 0.846s + 0.35617)$
	5	$(s + 0.362)(s^2 + 0.224s + 0.012665)(s^2 + 0.586s + 0.476474)$
	6	$(s^2 + 0.156s + 1.022148)(s^2 + 0.424s + 0.589588)(s^2 + 0.58s + 0.157)$
3.0	1	$(s + 1.002)$
	2	$(s^2 + 0.2986s + 0.83950649)$
	3	$(s + 0.299)(s^2 + 0.2986s + 0.83950649)$
	4	$(s^2 + 0.17s + 0.902141)(s^2 + 0.412s + 0.1961)$
	5	$(s + 0.177)(s^2 + 0.11s + 0.936181)(s^2 + 0.288s + 0.377145)$
	6	$(s^2 + 0.076s + 0.95402)(s^2 + 0.208s + 0.522041)(s^2 + 0.286s + 0.089093)$

$$T(s) = \frac{K}{s^n + a_{n-1}s^{n-1} + \ldots + a_o}$$

If $n = 1$: $T(s) = \dfrac{a_o}{s + a_o}$

If $n = 2$: $T(s) = \dfrac{\dfrac{a_o}{\sqrt{1+\varepsilon^2}}}{s^2 + a_1 s + a_0}$

- A combination of first and second order filters can be used to realize n^th order filter. For example, if n =5, one first order and two second order filters will be used.
- Note that the denominator is not in the form of $(s+1)$ or $[s^2 + (1/Q)s + 1]$.
- Some mathematical manipulation is needed to implement Chebyshev filter using the usual first or second order filter.

$T(s)$ can be modified with the following forms:

If n =1:

$$T(s)=\frac{a_o}{s+a_o}=\frac{\frac{a_o}{\omega_\mathrm{k}}}{\frac{s}{\omega_\mathrm{k}}+\frac{a_o}{\omega_\mathrm{k}}}=\frac{1}{\frac{s}{\omega_k}+1}$$

, note: $\omega_\mathrm{k}=a_o$

If n =2:

$$T(s)=\frac{K}{s^2+a_1s+a_0}=\frac{K}{s^2+\left(\frac{\omega_k}{Q}\right)s+\omega_k{}^2}=\frac{\frac{K}{\omega_\mathrm{k}{}^2}}{\left(\frac{s}{\omega_k}\right)^2+\frac{1}{Q}\left(\frac{s}{\omega_k}\right)+1}$$

Note: $K=\frac{a_o}{\sqrt{1+\varepsilon^2}},\ \ \omega_k{}^2=a_o,\ Q=\frac{\omega_k}{a_1}=\frac{\sqrt{a_0}}{a_1}$

Now they can be realized by either first or second-order filter with reference frequency = 1 rad/s and then frequency-scale using $\alpha=\omega_k$.

Chebyshev Denominator $B(s)$

Passband Ripple (dB)	n	$B(s)$
3.0	1	$(s + 1.002)$
	2	$(s^2 + 0.2986s + 0.83950649)$
	3	$(s + 0.299)(s^2 + 0.2986s + 0.83950649)$
	4	$(s^2 + 0.17s + 0.902141)(s^2 + 0.412s + 0.1961)$
	5	$(s + 0.177)(s^2 + 0.11s + 0.936181)(s^2 + 0.288s + 0.377145)$
	6	$(s^2 + 0.076s + 0.95402)(s^2 + 0.208s + 0.522041)(s^2 + 0.286s + 0.089093)$

$$T(s) = \left(\frac{K_1}{s^2 + 0.17s + 0.902141}\right)\left(\frac{K_2}{s^2 + 0.412s + 0.1961}\right)$$

$$T(s) = \frac{K}{s^2 + a_1 s + a_0} \qquad \text{where } K = \frac{a_o}{\sqrt{1+\varepsilon^2}}$$

$$K_1 = \frac{a_o}{\sqrt{1+\varepsilon^2}} \frac{0.902141}{\sqrt{1+0.998^2}} = 0.6385 \qquad K_2 = \frac{a_o}{\sqrt{1+\varepsilon^2}} = \frac{0.1961}{\sqrt{1+0.998^2}} = 0.1388$$

$$T(s)=\frac{\dfrac{K}{\omega_{\mathrm{k}}{}^{2}}}{\left(\dfrac{s}{\omega_k}\right)^2+\dfrac{1}{Q}\left(\dfrac{s}{\omega_k}\right)+1}$$ where $\omega_k{}^2=a_o,\ Q=\dfrac{\omega_k}{a_1}=\dfrac{\sqrt{a_0}}{a_1}$

$$T(s)=\left(\frac{0.6385}{s^2+0.17s+0.902141}\right)\left(\frac{0.1388}{s^2+0.412s+0.1961}\right)$$

$$T(s)=\frac{0.6385}{s^2+0.17s+0.902141}$$

$$\omega_{k1}{}^2=a_o=0.902141$$

$$T(s)=\frac{0.1388}{s^2+0.412s+0.1961}$$

$$\omega_{k2}{}^2=a_o=0.1961$$

$$Q_1=\frac{\sqrt{0.902141}}{0.17}=\frac{0.9498}{0.17}=5.587$$

$$Q_2=\frac{\sqrt{0.1961}}{0.412}=\frac{0.44283}{0.412}=1.0748$$

$$T(s)=\frac{\dfrac{K}{\omega_k^{\;2}}}{\left(\dfrac{s}{\omega_k}\right)^2+\dfrac{1}{Q}\left(\dfrac{s}{\omega_k}\right)+1} \qquad \text{where } \omega_k^{\;2}=a_o,\; Q=\frac{\omega_k}{a_1}=\frac{\sqrt{a_0}}{a_1}$$

$$T(s)=\left(\frac{0.6385}{s^2+0.17s+0.902141}\right)\left(\frac{0.1388}{s^2+0.412s+0.1961}\right)$$

$$\frac{K_1}{\omega_{k1}^{\;2}}=\frac{0.6385}{0.902141}=0.7078 \qquad \frac{K_2}{\omega_{k2}^{\;2}}=\frac{0.1388}{0.1961}=0.7078$$

$$\left(\frac{s}{\omega_{k1}}\right)^2=\frac{s^2}{0.902141}=1.1085s^2 \qquad \left(\frac{s}{\omega_{k2}}\right)^2=\frac{s^2}{0.1961}=5.1s^2$$

$$\frac{1}{Q_1}\left(\frac{s}{\omega_{k1}}\right)=\frac{1}{5.587}\left(\frac{s}{\sqrt{0.902141}}\right)=0.1884s \qquad \frac{1}{Q_2}\left(\frac{s}{\omega_{k2}}\right)=\frac{1}{1.0748}\left(\frac{s}{\sqrt{0.1961}}\right)=2.101s$$

$$T(s)=\left(\frac{0.6385}{s^2+0.17s+0.902141}\right)\left(\frac{0.1388}{s^2+0.412s+0.1961}\right)$$

$$\Downarrow$$

$$T(s)=\left(\frac{0.7078}{1.1085s^2+0.1884s+1}\right)\left(\frac{0.7078}{5.1s^2+2.101s+1}\right)$$

Realization of the above by two Sallen-Key second order filters:

$$A_1=3-\frac{1}{Q_1}=3-\frac{1}{5.587}=2.821 \qquad A_2=3-\frac{1}{Q_2}=3-\frac{1}{1.0748}=2.07$$

$$T(s)=\left(\frac{0.7078A_1}{1.1085s^2+0.1884s+1}\right)\left(\frac{0.7078A_2}{5.1s^2+2.101s+1}\right)$$

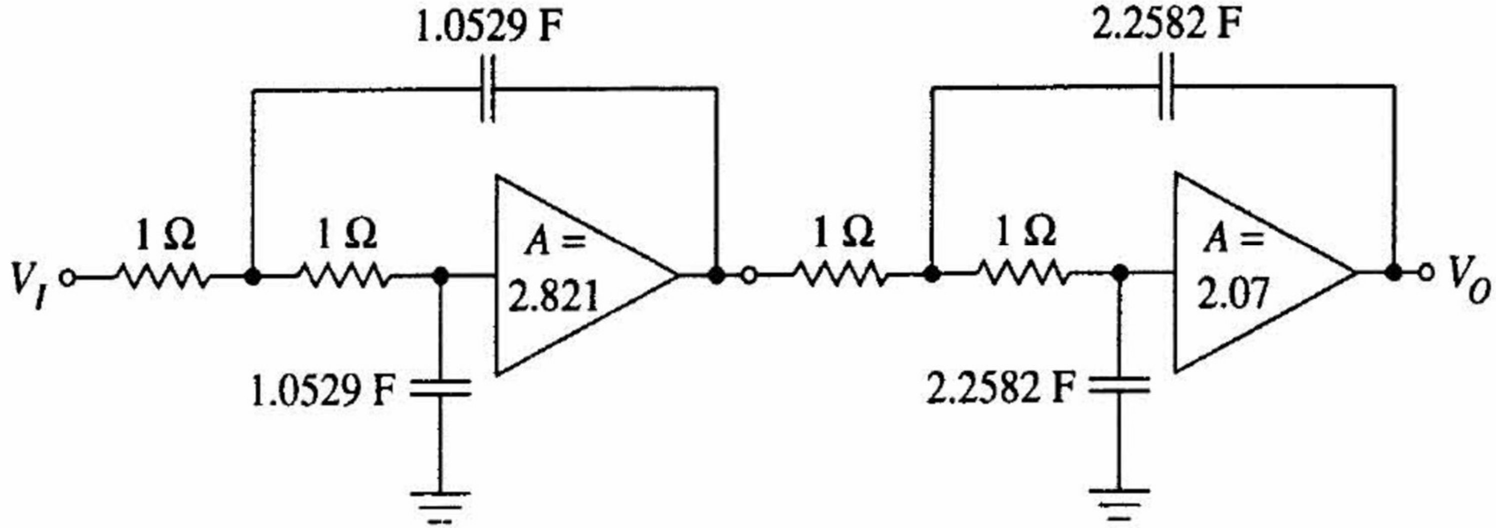
$$T(s)=\left(\frac{2.82\times 0.7078}{1.1085s^2+0.1884s+1}\right)\left(\frac{2.07\times 0.7078}{5.1s^2+2.101s+1}\right)$$

$$T(s)=5.84\left(\frac{0.7078}{\frac{s^2}{\omega_{k1}^{\ 2}}+\frac{1}{Q_1}\left(\frac{s}{\omega_{k1}}\right)+1}\right)\left(\frac{0.7078}{\frac{s^2}{\omega_{k2}}+\frac{1}{Q_2}\left(\frac{s}{\omega_{k2}}\right)+1}\right)$$

The 4th order low-pass filter can be realized by two 2nd order Sallen-Key filters with frequency scaling of ω_{k1} and ω_{k2}. Hence the capacitors must be scaled from 1 F to:

$$\frac{1}{\omega_{k1}^{\ 2}}=1.1085 \Rightarrow \omega_{k1}=\frac{1}{\sqrt{1.1085}}=0.9498 \qquad C_1=\frac{1\,\mathrm{F}}{\omega_{k1}}=\frac{1}{0.9498}=1.0529\,\mathrm{F}$$

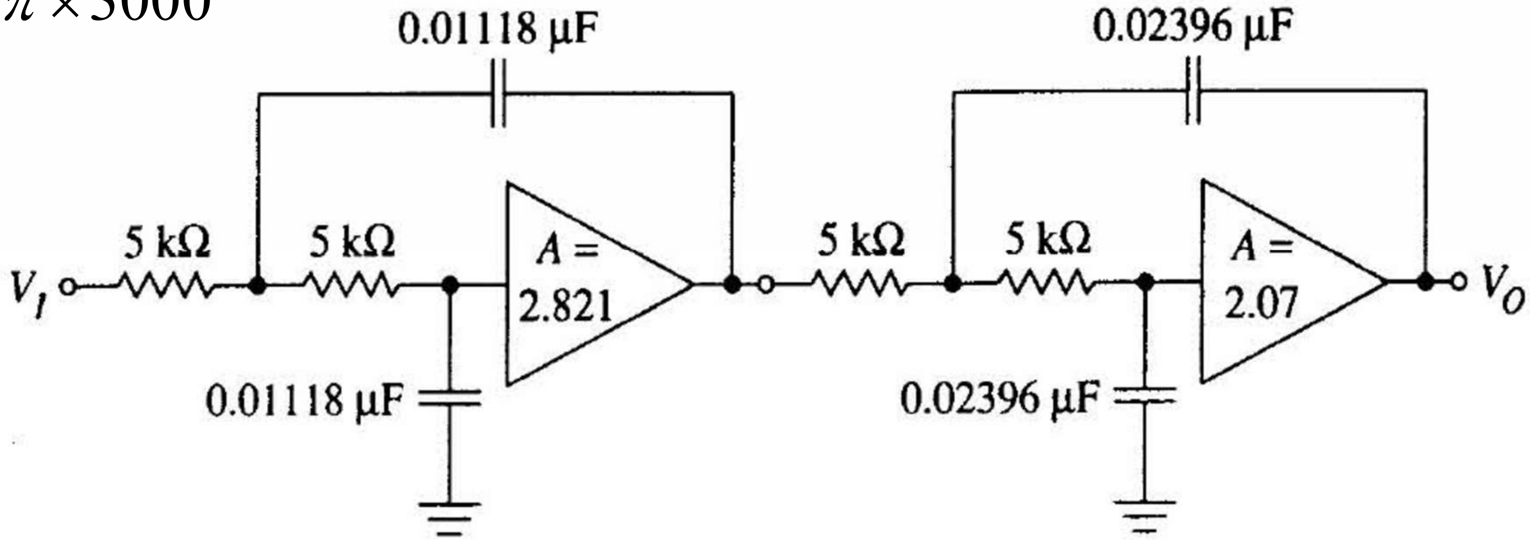
$$\frac{1}{\omega_{k2}^{\ 2}}=5.1 \Rightarrow \omega_{k2}=\frac{1}{\sqrt{5.1}}=0.4428 \qquad C_2=\frac{1\,\mathrm{F}}{\omega_{k2}}=\frac{1}{0.4428}=2.2582\,\mathrm{F}$$



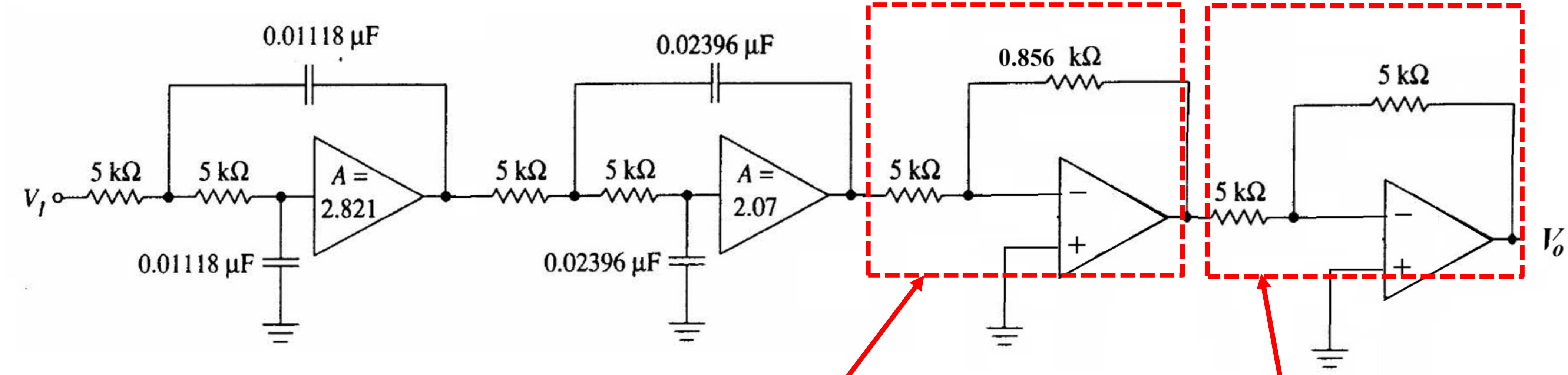
After frequency-scale by $\alpha = 2\pi \times 3000$ and impedance-scale by $\beta = 5000$:

$$C_1' = \frac{1.0529\text{F}}{2\pi \times 3000} = 55.86\mu\text{F} \qquad C_1'' = \frac{55.86\mu\text{F}}{5000} = 0.01118\mu\text{F}$$

$$C_2' = \frac{2.2582\text{F}}{2\pi \times 3000} = 119.8\mu\text{F} \qquad C_2'' = \frac{119.8\mu\text{F}}{5000} = 0.02396\mu\text{F}$$



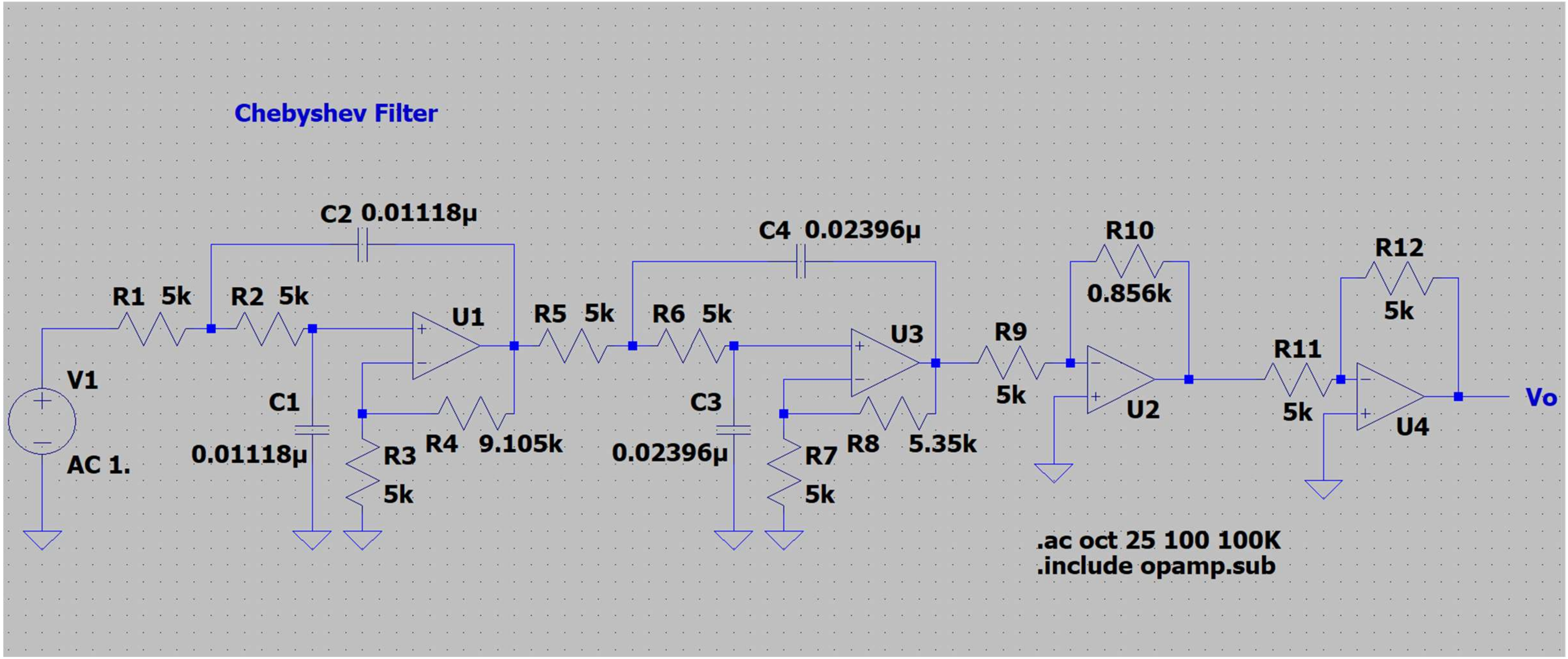
Final circuit:

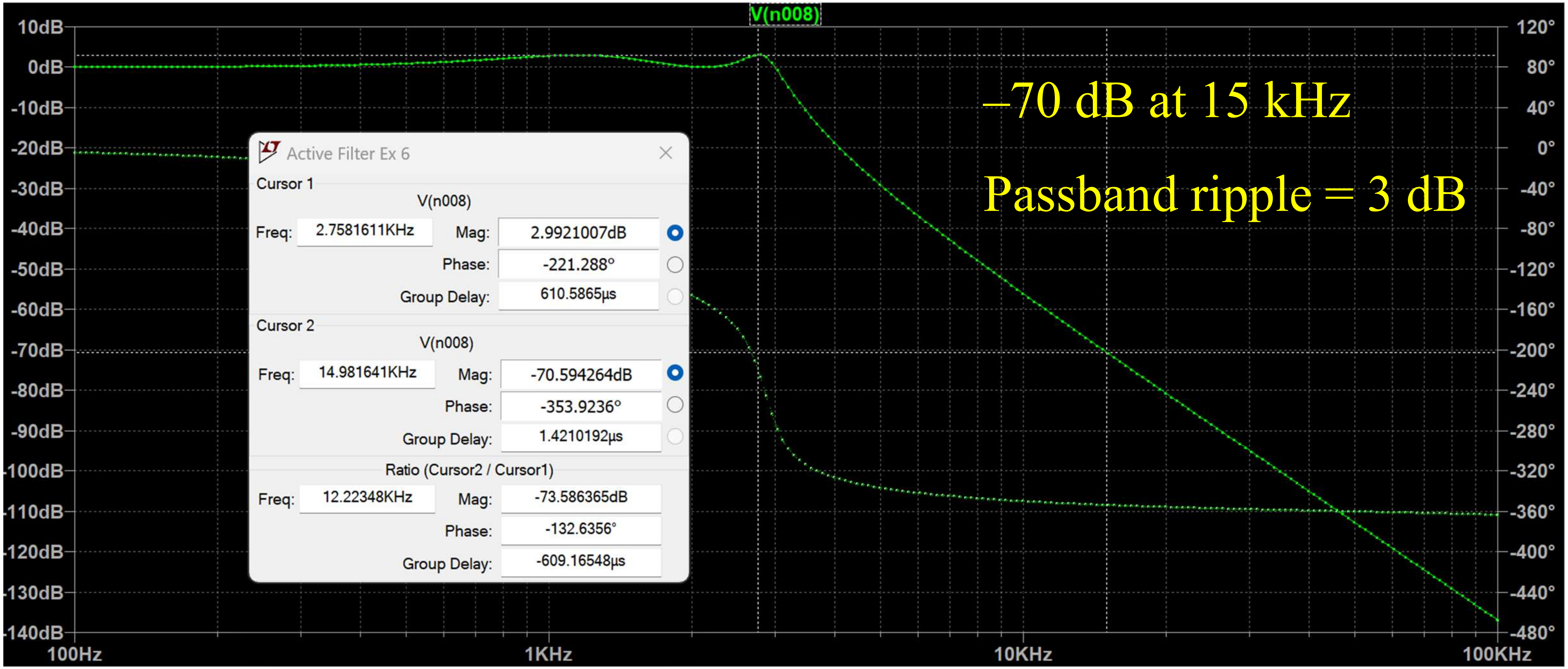


Gain = –1/5.84 = – 0.1712 to make overall passband gain = 1

Gain = –1 to make both output in phase in input

Verification with LTSpice:




V(n008)
10dB
0dB
-10dB
-20dB
-30dB
-40dB
-50dB
-60dB
-70dB
-80dB
-90dB
-100dB
-110dB
-120dB
-130dB
-140dB
100Hz
1KHz
10KHz
100KHz
120°
80°
40°
0°
-40°
-80°
-120°
-160°
-200°
-240°
-280°
-320°
-360°
-400°
-440°
-480°
Active Filter Ex 6
Cursor 1
V(n008)
Freq: 2.7581611KHz
Mag: 2.9921007dB
Phase: -221.288°
Group Delay: 610.5865µs
Cursor 2
V(n008)
Freq: 14.981641KHz
Mag: -70.594264dB
Phase: -353.9236°
Group Delay: 1.4210192µs
Ratio (Cursor2 / Cursor1)
Freq: 12.22348KHz
Mag: -73.586365dB
Phase: -132.6356°
Group Delay: -609.16548µs
–70 dB at 15 kHz
Passband ripple = 3 dB