机器人学导论

楼云江

哈尔滨工业大学（深圳）

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4 刚体速度

Rigid Body Velocity

Chapter 4 Rigid Body Velocity

4.1 Point-mass Velocity

◇ Review: Point-mass velocity

$$q(t) \in \mathbb{R}^3, t \in (-\varepsilon, \varepsilon), v = \frac{\mathrm{d}}{\mathrm{d}t} q(t) \in \mathbb{R}^3, a = \frac{\mathrm{d}^2}{\mathrm{d}t^2} q(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) \in \mathbb{R}^3$$

□ Velocity of Rotational Motion:

$$R_{ab}(t) \in SO(3), t \in (-\varepsilon, \varepsilon), \ q_a(t) = R_{ab}(t) q_b$$

$$V^a = \frac{\mathrm{d}}{\mathrm{d}t} q_a(t) = \dot{R}_{ab}(t) q_b = \dot{R}_{ab}(t) R_{ab}^T(t) R_{ab}(t) q_b = \dot{R}_{ab} R_{ab}^T q_a$$

$$R_{ab}(t) R_{ab}^T(t) = I \Rightarrow \dot{R}_{ab} R_{ab}^T + R_{ab} \dot{R}_{ab}^T = 0, \dot{R}_{ab} R_{ab}^T = -(\dot{R}_{ab} R_{ab}^T)^T$$

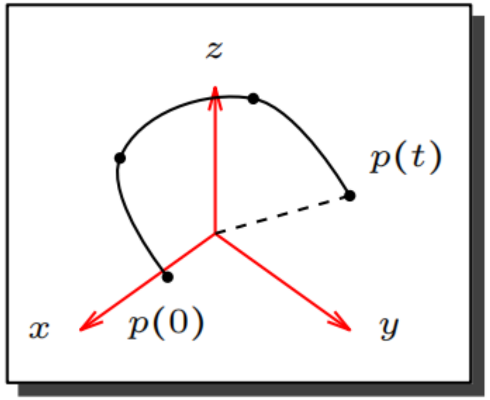

Figure 4.1

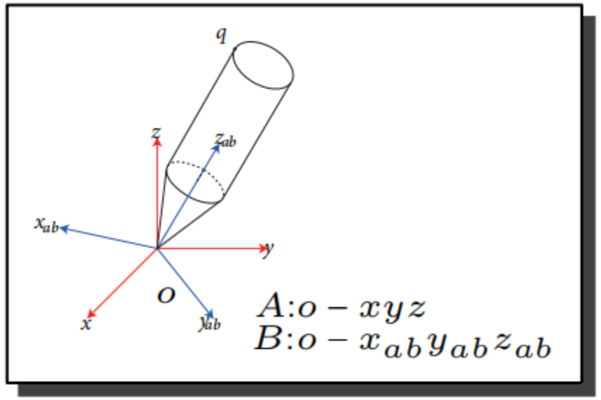

Figure 4.2

Denote spatial angular velocity by:

$$\hat{\omega}_{ab}^{s} = \dot{R}_{ab} R_{ab}^{T}, \omega_{ab} \in \mathbb{R}^3$$

Then

$$V^a = \hat{\omega}_{ab}^{s} \cdot q_a = \omega_{ab}^{s} \times q_a$$

Body angular velocity:

$$\hat{\omega}_{ab}^{b} = R_{ab}^{T} \cdot \dot{R}_{ab}, v^b \triangleq R_{ab}^{T} \cdot v^a = \omega_{ab}^{b} \times q_b$$

Relation between body and spatial angular velocity:

$$\omega_{ab}^{b} = R_{ab}^{T} \cdot \omega_{ab}^{s} \text{ or } \hat{\omega}_{ab}^{b} = R_{ab}^{T} \hat{\omega}_{ab}^{s} R_{ab}$$

□ Generalized Velocity:

$$g_{ab} = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix}, q_a(t) = g_{ab}(t) q_b$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_a(t) = \dot{g}_{ab}(t) q_b = \dot{g}_{ab} \cdot g_{ab}^{-1} \cdot g_{ab} \cdot q_b = \hat{V}_{ab}^{s} \cdot q_a$$

$$\begin{aligned} \hat{V}_{ab}^{s} = \dot{g}_{ab} \cdot g_{ab}^{-1} &= \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^{T} & -R_{ab}^{T} p_{ab} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}_{ab} R_{ab}^{T} & -\dot{R}_{ab} R_{ab}^{T} p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega}_{ab}^{s} & -\omega_{ab}^{s} \times p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \triangleq \begin{bmatrix} \hat{\omega}_{ab}^{s} & v_{ab}^{s} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

□ (Generalized) Spatial Velocity:

$$V_{ab}^{s} = \begin{bmatrix} v_{ab}^{s} \\ \omega_{ab}^{s} \end{bmatrix} = \begin{bmatrix} -\omega_{ab}^{s} \times p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^{T})^{\vee} \end{bmatrix}$$

$$v_{q_a} = \omega_{ab}^{s} \times q_a + v_{ab}^{s}$$

Note: $v_{q_b} = g_{ab}^{-1} \cdot v_{q_a} = g_{ab}^{-1} \cdot \dot{g}_{ab} \cdot q_b = \hat{V}_{ab}^{b} \cdot q_b$

□ (Generalized) Body Velocity:

$$\hat{V}_{ab}^{b} = g_{ab}^{-1} \dot{g}_{ab} = \begin{bmatrix} R_{ab}^{T} \dot{R}_{ab} & R_{ab}^{T} \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \triangleq \begin{bmatrix} \hat{\omega}_{ab}^{b} & v_{ab}^{b} \\ 0 & 0 \end{bmatrix}$$

$$V_{ab}^{b} = \begin{bmatrix} v_{ab}^{b} \\ \omega_{ab}^{b} \end{bmatrix} = \begin{bmatrix} R_{ab}^{T} \dot{p}_{ab} \\ (R_{ab}^{T} \dot{R}_{ab})^{\vee} \end{bmatrix}$$

$$\hat{V}_{ab}^s = \dot{g}_{ab} \cdot g_{ab}^{-1} = g_{ab} \cdot g_{ab}^{-1} \cdot \dot{g}_{ab} \cdot g_{ab}^{-1} = g_{ab} \cdot \hat{V}_{ab}^b \cdot g_{ab}^{-1}$$

$$= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{ab}^b & v_{ab}^b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{ab}^b R_{ab}^T & -\hat{\omega}_{ab}^b R_{ab}^T p_{ab} + v_{ab}^b \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab}\hat{\omega}_{ab}^b R_{ab}^T & -R_{ab}\hat{\omega}_{ab}^b R_{ab}^T p_{ab} + R_{ab} v_{ab}^b \\ 0 & 0 \end{bmatrix}$$

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{\mathrm{Ad}_g} V_{ab}^b$$

$$\mathrm{Ad}_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \in \mathbb{R}^{6\times 6}, \text{ for } g = (p, R)$$

4.2 Velocity of a Rigid Body (Properties of Adjoint mapping)

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$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} \mathrm{Ad}_{g^{-1}} &= \begin{bmatrix} R^T & (-R^T p)^\wedge R^T \\ 0 & R^T \end{bmatrix} \\ &= \begin{bmatrix} R^T & -R^T \hat{p} \\ 0 & R^T \end{bmatrix} = (\mathrm{Ad}_g)^{-1} \end{aligned}$$

and $\mathrm{Ad}_{g_1 \cdot g_2} = \mathrm{Ad}_{g_1} \cdot \mathrm{Ad}_{g_2}$

The map $\mathrm{Ad} : SE(3) \mapsto GL(\mathbb{R}^6), \mathrm{Ad}(g) = \mathrm{Ad}_g$ is a group homomorphism

Matrix Rep	Vector Rep
$\hat{\xi} \in se(3)$	$\xi \in \mathbb{R}^6$
$g \cdot \hat{\xi} \cdot g^{-1} \in se(3)$	$\mathrm{Ad}_g \xi \in \mathbb{R}^6$

4.3 Velocity of Screw Motion

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$$g_{ab}(\theta) = e^{\hat{\xi}\theta(t)} g_{ab}(0), \ \frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\xi}\theta(t)} = \hat{\xi}\dot{\theta}(t) e^{\hat{\xi}\theta(t)} = \dot{\theta}(t) e^{\hat{\xi}\theta(t)} \hat{\xi}$$

$$\begin{aligned} \hat{V}^s_{ab} &= \dot{g}_{ab} \cdot g_{ab}^{-1} = (\hat{\xi}\dot{\theta} e^{\hat{\xi}\theta(t)} g_{ab}(0)) \cdot (g_{ab}^{-1}(0) e^{-\hat{\xi}\theta(t)}) \\ &= \hat{\xi}\dot{\theta} \Rightarrow V^s_{ab} = \xi\dot{\theta} \end{aligned}$$

$$\begin{aligned} \hat{V}^b_{ab} &= g_{ab}^{-1} \cdot \dot{g}_{ab} = g_{ab}^{-1}(0) e^{-\hat{\xi}\theta} \cdot e^{\hat{\xi}\theta} \hat{\xi}\dot{\theta} g_{ab}(0) \\ &= g_{ab}^{-1}(0) \hat{\xi}\dot{\theta} g_{ab}(0) = (\mathrm{Ad}_{g_{ab}^{-1}(0)} \xi)^{\wedge} \dot{\theta} \Rightarrow V^b_{ab} = \mathrm{Ad}_{g_{ab}^{-1}(0)} \xi\dot{\theta} \end{aligned}$$

4.4 Metric Property of $se(3)$

Let $g_i(t) \in SE(3)$, $i = 1, 2$, be representations of the same motion, obtained using coordinate frame A and B. Then,

$$\boxed{g_2(t) = g_0 \cdot g_1(t) \cdot g_0^{-1} \Rightarrow V_2^s = \mathrm{Ad}_{g_0} \cdot V_1^s}$$

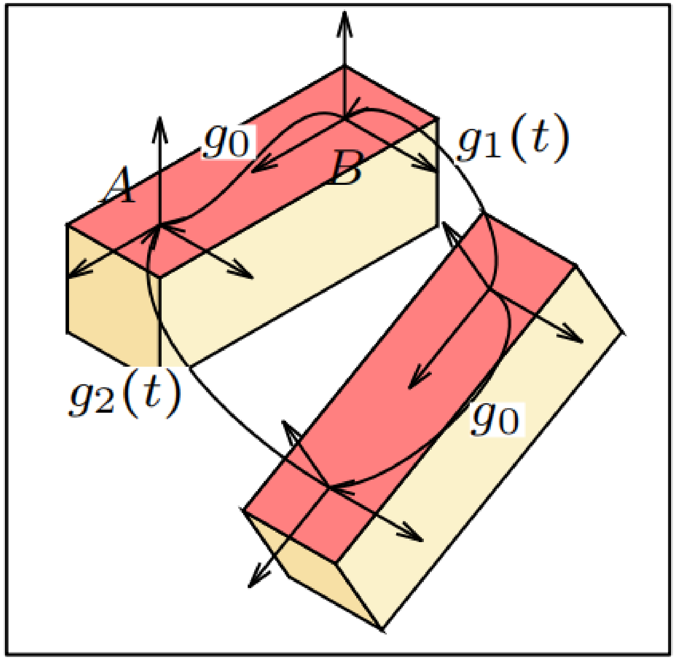

Figure 4.3

4.4 Metric Property of $se(3)$

$$\|V_2^s\|^2 = (\mathrm{Ad}_{g_0} \cdot V_1^s)^T(\mathrm{Ad}_{g_0} \cdot V_1^s) = (V_1^s)^T \mathrm{Ad}_{g_0}^T \cdot \mathrm{Ad}_{g_0} \cdot V_1^s$$

$$\begin{aligned}\mathrm{Ad}_{g_0}^T \cdot \mathrm{Ad}_{g_0} &= \begin{bmatrix} R_0^T & 0 \\ -R_0^T \hat{p}_0 & R_0^T \end{bmatrix}\begin{bmatrix} R_0 & \hat{p}_0 R_0 \\ 0 & R_0 \end{bmatrix} \\ &= \begin{bmatrix} I & R_0^T \hat{p}_0 R_0 \\ -R_0^T \hat{p}_0 R_0 & I - R_0^T \hat{p}_0^2 R_0 \end{bmatrix}\end{aligned}$$

In general, $\|V_2^s\| \neq \|V_1^s\|$, or there exists no bi-invariant metric on $se(3)$.

4.5 Coordinate Transformation

$$g_{ac}(t) = g_{ab}(t) \cdot g_{bc}(t)$$

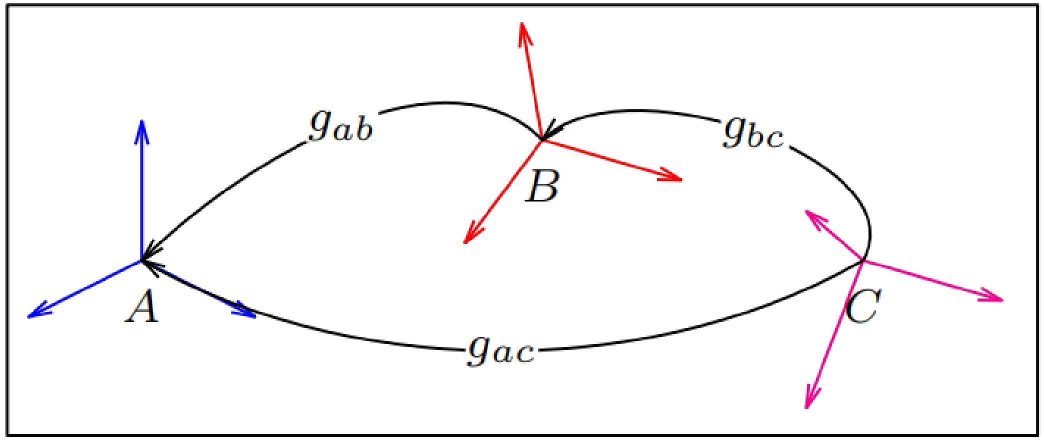

Figure 4.4

$$\begin{aligned}\hat{V}^s_{ac} &= \dot{g}_{ac} \cdot g_{ac}^{-1} \\ &= (\dot{g}_{ab} \cdot g_{bc} + g_{ab} \cdot \dot{g}_{bc})(g_{bc}^{-1} \cdot g_{ab}^{-1}) \\ &= \dot{g}_{ab} \cdot g_{ab}^{-1} + g_{ab} \cdot \dot{g}_{bc} \cdot g_{bc}^{-1} \cdot g_{ab}^{-1} = \hat{V}^s_{ab} + g_{ab}\hat{V}^s_{bc}g_{ab}^{-1}\end{aligned}$$

$$\Rightarrow V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}$$

Similarly: $V^b_{ac} = Ad_{g_{bc}^{-1}} V^b_{ab} + V^b_{bc}$

Note: $V^s_{bc} = 0 \Rightarrow V^s_{ac} = V^s_{ab}$, $V^b_{ab} = 0 \Rightarrow V^b_{ac} = V^b_{bc}$

4.6 Example

$$g_{ab}(\theta_1) = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V^s_{ab} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

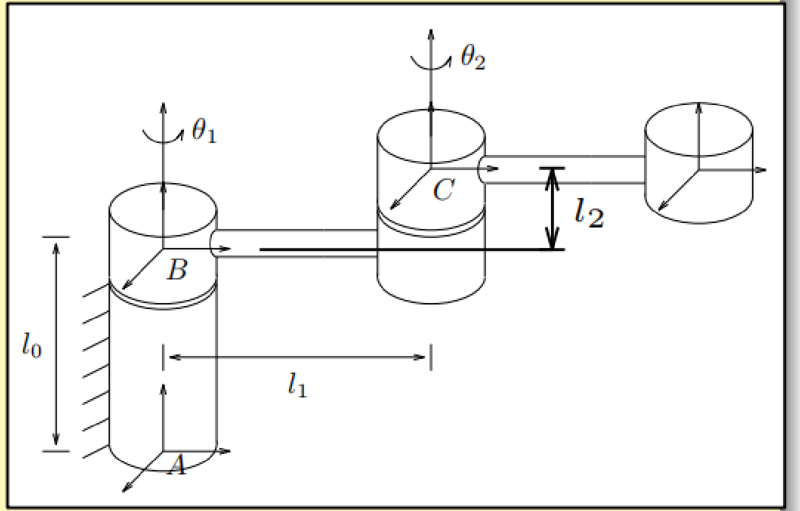

Figure 4.5

$$g_{bc}(\theta_2) = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & 0 \\ s_{\theta_2} & c_{\theta_2} & 0 & l_1 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V^s_{bc} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_2$$

$$V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} \cdot V^s_{bc} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} l_1 c_{\theta_1} \\ l_1 s_{\theta_1} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_2$$

† End of Section †