EE6203
Computer Controlled Systems

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Text Book

Fadali, M. S; Visioli, A, ***Digital Control Engineering - Analysis and Design***, Elsevier, 2nd edition, 2013

Ogata, K, ***Discrete-Time Control Systems***, 2nd Edition, Prentice Hall, 1995.

References

Franklin G. F., Powell J. D. and Workman M. L., ***Digital Control of Dynamic Systems***, Addison-Wesley, 1990.

Kuo B. C., ***Digital Control Systems***, 2nd Edition, Saunders College Publishing, 1992.

Some Prerequisites:

- Engineering Math (e.g Complex analysis, Laplace Transform);
- Modelling and Classical Continuous-Time Control Systems);
- Signal and Systems

Teaching Sequence	Topic	Hours	Lecturer
1	Discrete Time System Modelling and Analysis	12	WCY
2	State-Space Design Methods and Optimal Control	15	LPH
3	Design Based on Transfer Functions	9	WCY
4	Implementation of Digital Controllers	3	WCY

EE 6203 Assessment Scheme

Continuous Assessment: 40%

- **Part 1 (WCY)**

Quiz 1 (10%) 1 Question: (Week 4)

- **Part 2 (LPH)**

Quiz 2 (15%) (to be arranged)

- **Part 3 (WCY)**

Quiz 3 (15%) (to be arranged)

Final Exam: 60%

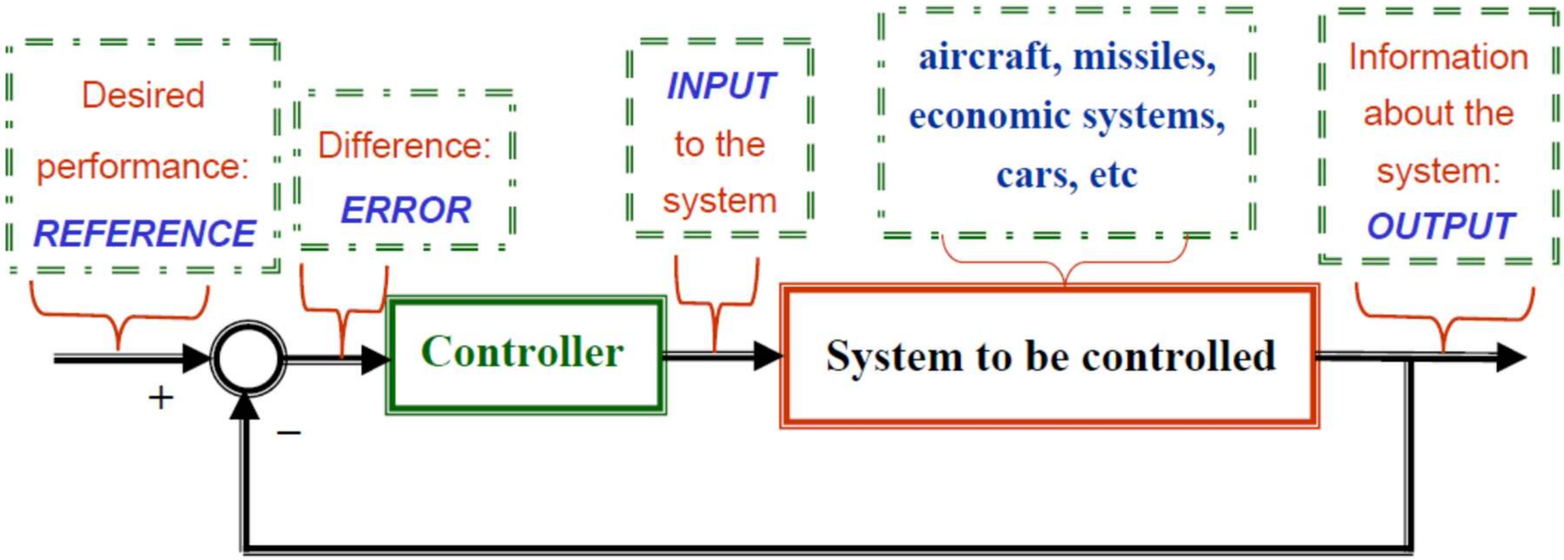
4 Questions (each 25 marks); 3 hours

Chapter 1:

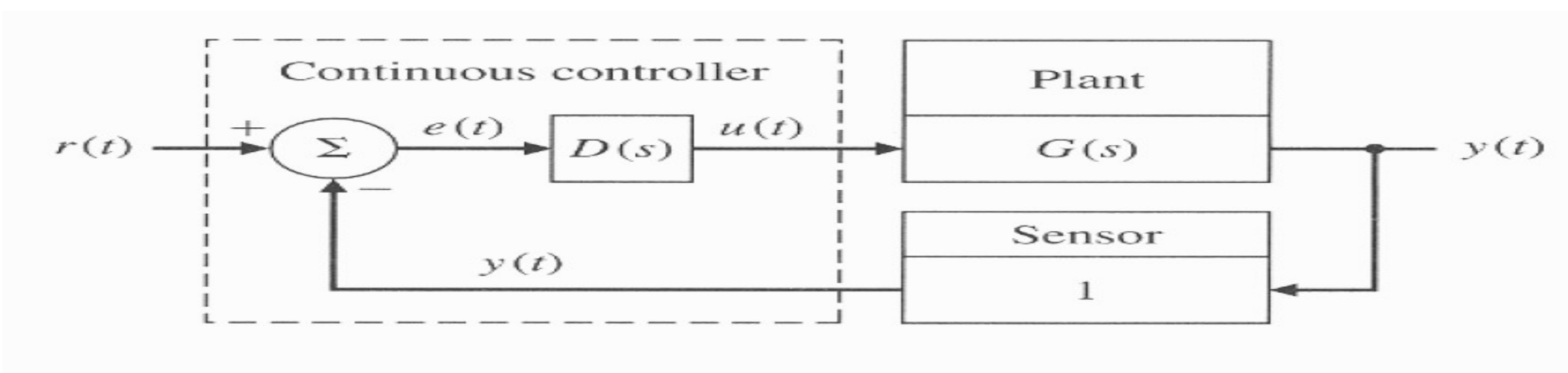
Introduction to Discrete-Time Control Systems

Chapter 1: Introduction to Discrete-Time Control Systems

Recall: a Continuous Control System is



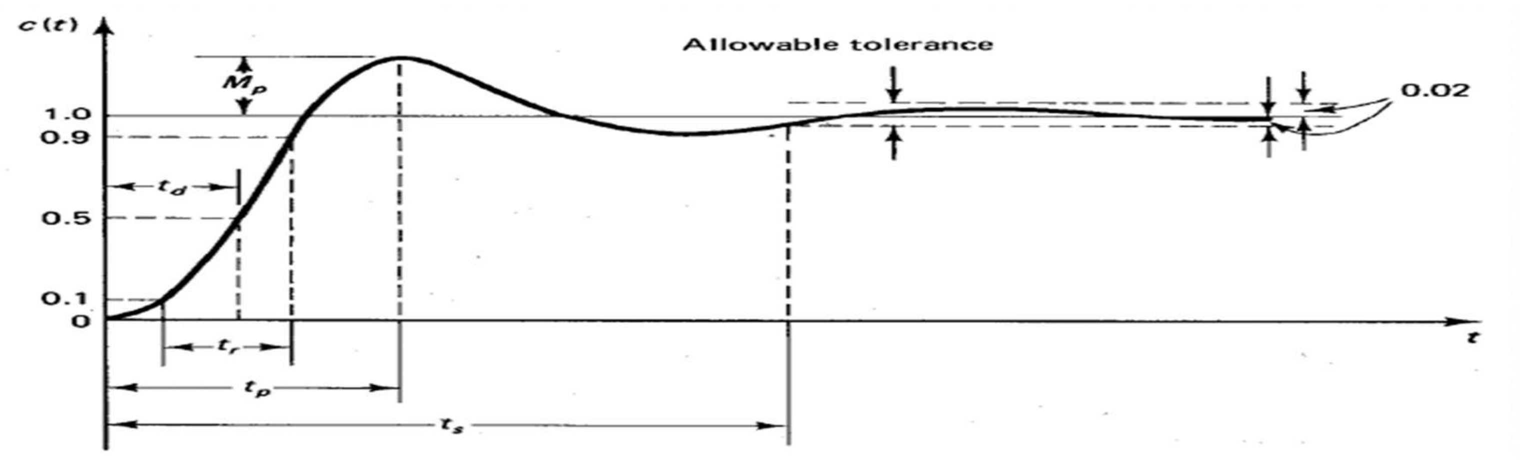
Objective: To make the system *OUTPUT* and the desired *REFERENCE* as close as possible, i.e., to make the *ERROR* as small as possible.



Requirements for Feedback Control Systems

- **Stability**
- **Transient performances**

  The specifications during transient period such as Rise time, Overshoot, Settling time ect.



- **Steady-state tracking accuracy**

  The specifications at steady state, normally when time tends to infinite.

- **Optimal Control**

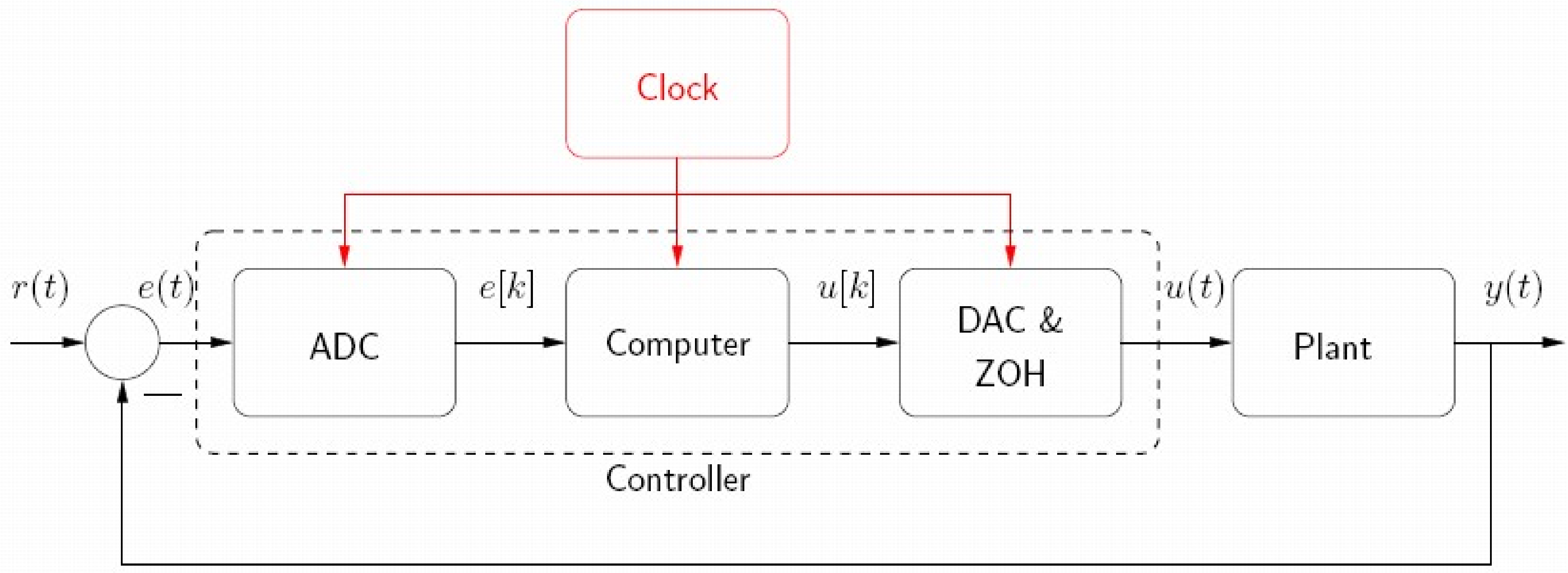
  Performance index (cost function)

Implementation of Continuous-time Controllers:

Normally using analogue devices such resistors, capacitors, inductors and operational amplifiers together with some necessary mechanical components.

These devices are neither economical nor durable.

The advances in computer technologies enables the implementation of controllers much more efficiently and economically possible.



ADC - Analog-Digital-Converter (includes sampler), DAC - Digital Analogue Converter,
ZOH - Zero Order Hold

Figure 1.2 Discrete-time Control System

This means that the controller operates in discrete time, although the controlled system (plant) usually operates in continuous time.

A number of advantages, for example,

- Often cheaper;
- Easier for implementation even for complex control algorithms;
- Re-programable;
- Usable with the same processor for different programmings,

……

1.2 Types of Signals

Signal: Any physical quantity that varies as a function of an independent variable. Independent variable is usually time but may be space, distance, …

Continuous-Time Signal: defined over a continuous range of time, with amplitude of continuous range of values or discrete values.

Analog Signal: defined over a continuous range of time with amplitude assuming a continuous range of values. That is, continuous in both time and amplitude.

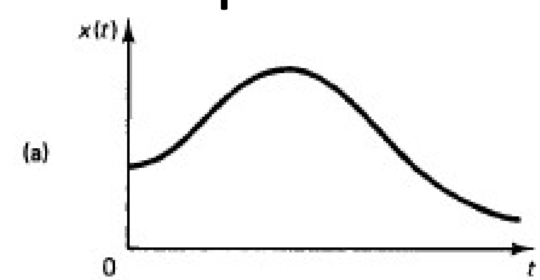


Fig 1.3(a) Continuous-Time Analog Signal

Often use "continuous-time" in lieu of "analog", even though they are not the same strictly speaking.

A signal is called **quantized** if it can only be certain values, and cannot be other values. For example, the number of students in a class is quantized as it can only be an integer.

Quantization: the process of representing a variable by a set of distinct values, to obtain quantized values of the signal. Signals to be processed with a digital computer need to be quantized.

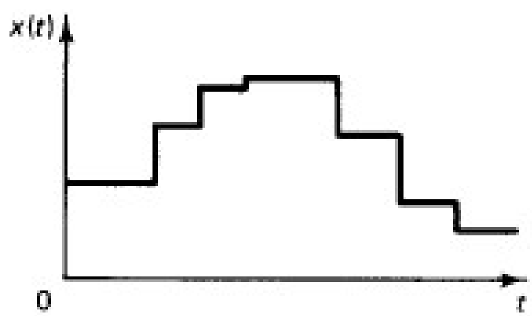


Fig 1.3(b) Continuous-Time Quantized Signal

Discrete-Time Signal: defined only at discrete instants of time (i.e., quantized time). Amplitude can be continuous and discrete (quantized).

Sampled-Data Signals: Signals with discrete-time and continuous amplitude. Obtained by sampling an analog signal at discrete instants of time.

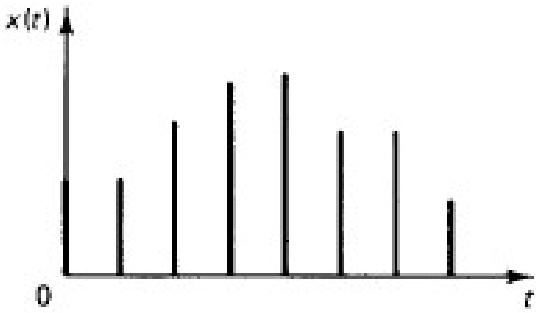


Fig 1(c) Sampled-Data Signal

Periodic sampling: sampling instants: $t_k = kT$
where k is an integer, and
T is a constant called sampling period *.*
To be studied in this course.

Digital Signals: Signals with discrete-time and discrete (quantized) amplitude. Obtained by sampling and quantizing an analog signal at discrete instants of time.

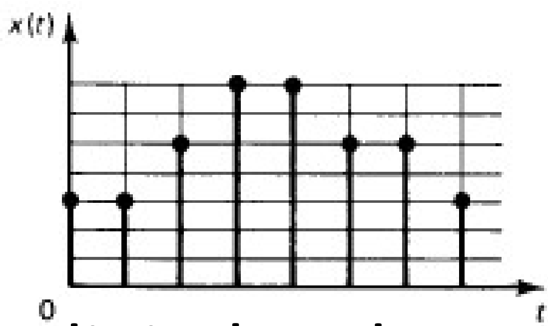


Fig 1(d) Digital Signal

Both digital and sampled-data signals are discrete-time signals.

1.3 Digital Control Systems

A typical digital control system with various signal forms is given in Figure 1.4.

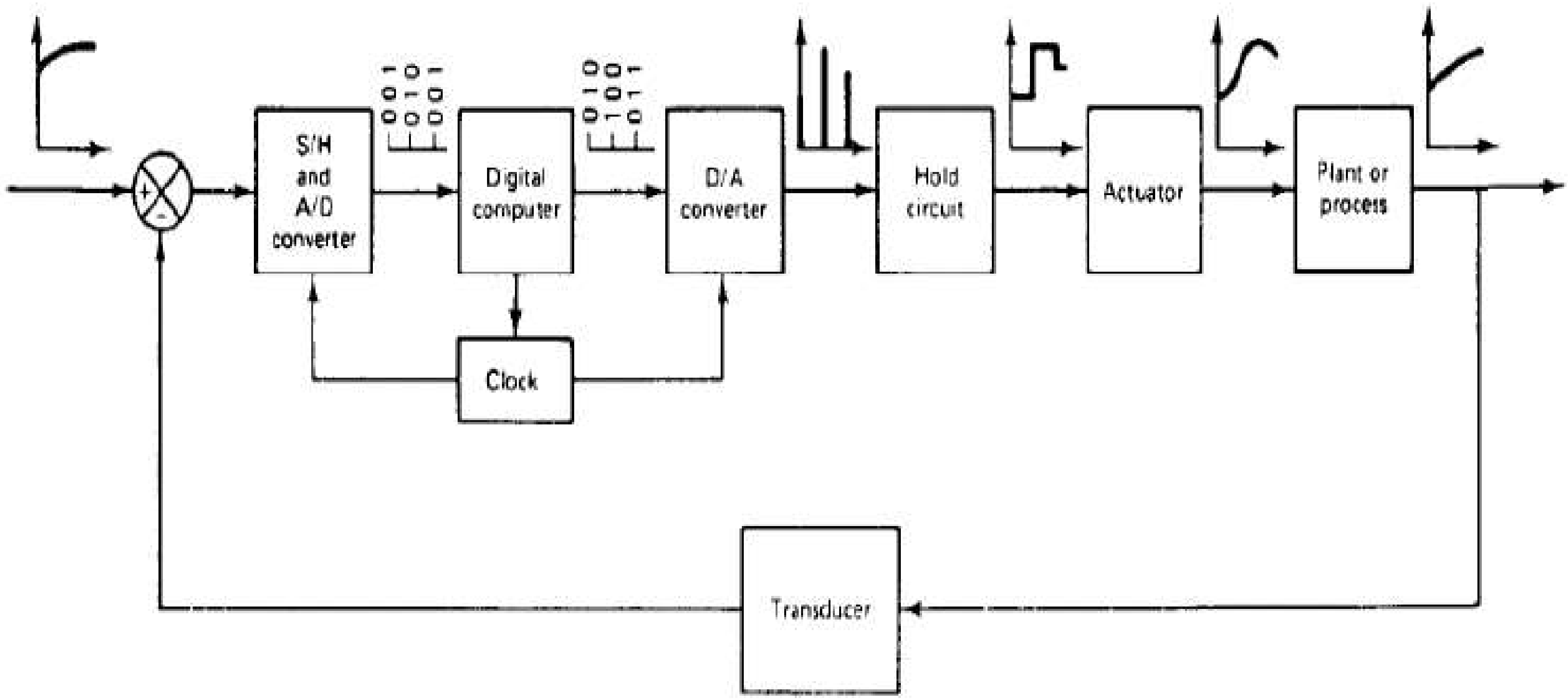


Figure 1.4, Block diagram of a digital control system showing signal in various forms

Example: D.C. Servo Motor Control

We consider the control of a d.c. servo system via a computer. A photo of a typical d.c. servo system is shown Figure 1.5.

Objective:

control the output shaft position, $y(t)$, to follow a given reference signal, $y^{*}(t)$.

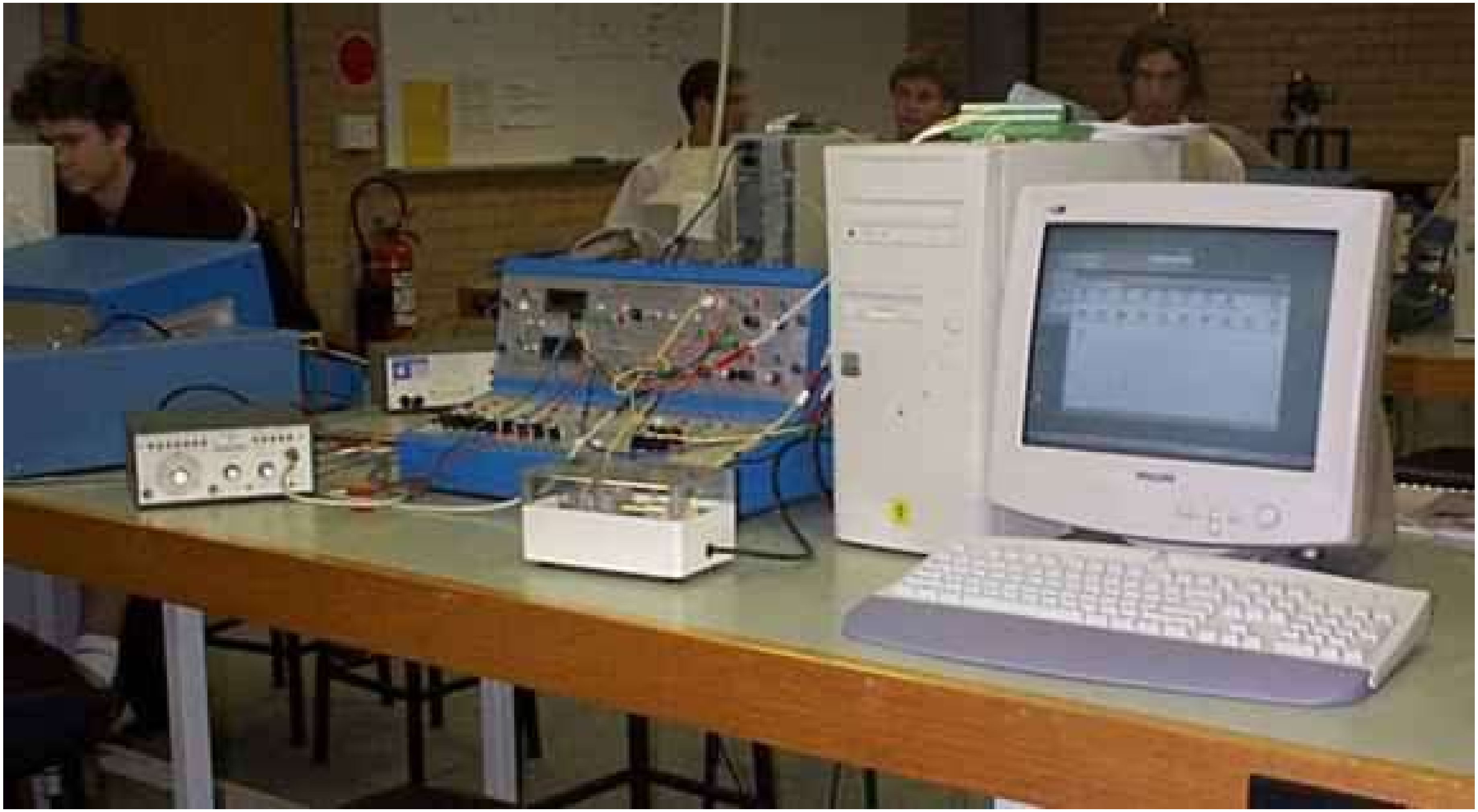


Figure 1.5 Photo of Servo Laboratory System with Digital Control via a PC

Control laws are designed for the digital controller so that the output is forced to go to the desired value.

To illustrate control performances, computer simulations are performed with *two different control laws*. Here the desired trajectory or *reference is a square wave*.

The results are shown Figures 1.7 and 1.8 respectively on the next slide.

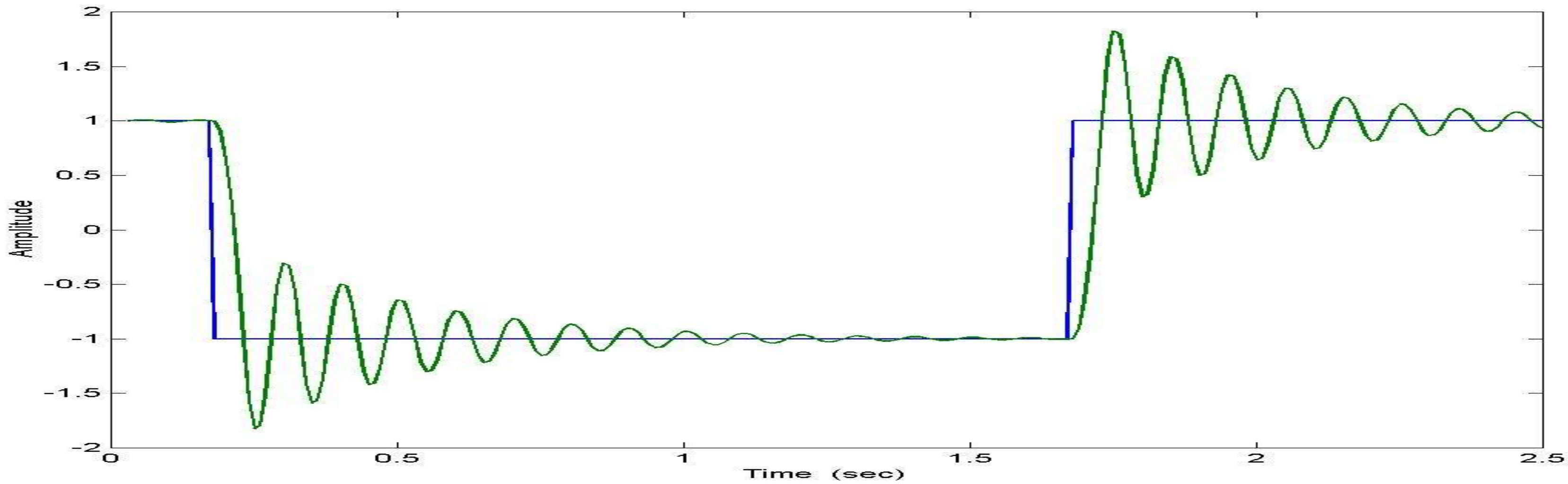


Figure 1.7 Desired trajectory and system output

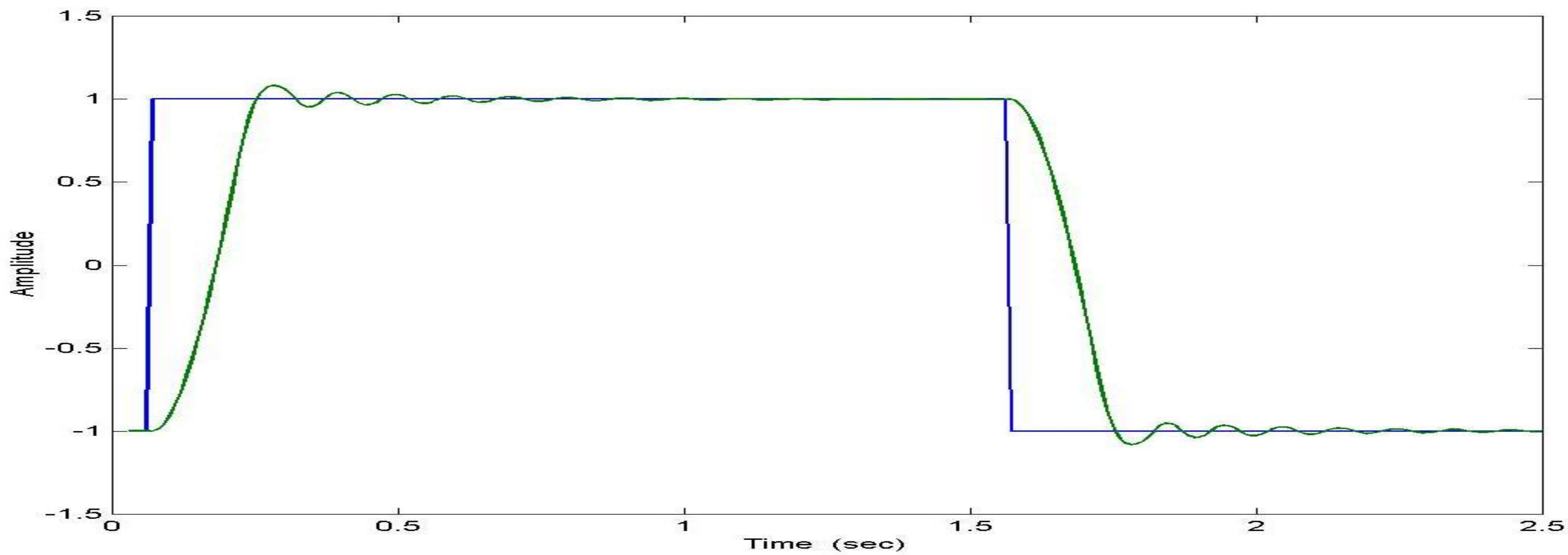


Figure 1.8 Desired trajectory and system output

Chapter 2
The $\mathcal{Z}$ Transform

2.1 Introduction

In this chapter, we study the following topics:

- Introduction
- The $\mathcal{Z}$ Transform
- $\mathcal{Z}$ Transform of Elementary Functions
- Important Properties and Theorems of the $\mathcal{Z}$ Transform
- The Inverse $\mathcal{Z}$ Transform
- $\mathcal{Z}$ Transform Method for Solving Difference Equations

- The *Z* Transform is a math tool for the analysis and synthesis of DTCS

- Similar to the Laplace Transform in Continuous-Time Control Systems, it converts difference equations to algebraic equations

Discrete-Time Signals: Given continuous-time signals $x(t)$, sampling results in discrete-time signal

$$x(0), x(T), x(2T), \cdots, x(kT), \cdots$$

where T is the sampling period. Or simply

$$x(0), x(1), x(2), \cdots, x(k), \cdots$$

2.2: The $\mathcal{Z}$ Transform

The $\mathcal{Z}$ transform for $x(t),\ t \geq 0$ (or $x(kT),\ k = 0, 1, 2, \cdots$) is

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT) z^{-k} \qquad (2\text{-}1)$$

For a sequence of numbers $x(k)$,

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad (2\text{-}2)$$

Eq (2-1) and (2-2) are called one-sided Z transform.

In general, we have two-sided $\mathcal{Z}$ transforms

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=-\infty}^{\infty} x(kT) z^{-k} \qquad (2\text{-}3)$$

or

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \qquad (2\text{-}4)$$

Expanding (2-1)

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots \quad \text{(2-5)}$$

so z^{-k} indicates the position in time at which $x(kT)$ occurs. This provides a way to do inverse $\mathcal{Z}$ transform.

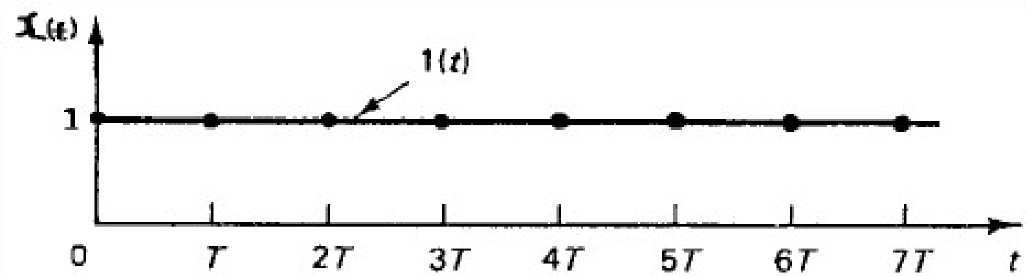
2.3 Z Transform of Elementary Functions

- ***Unit-Step Function***

$$x(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Sampling gives

$$x(kT) = 1(kT) = 1, \quad k = 0, 1, 2, 3, \cdots$$



Its $\mathcal{Z}$ transform is

$$\begin{aligned} X(z) &= \mathcal{Z}[1(t)] = \sum_{k=0}^{\infty} 1z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-k} + \cdots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \qquad \text{if } |z^{-1}| < 1 \end{aligned}$$

- ***Unit-Ramp Function***

$$x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Sampling gives

$$x(kT) = kT, \quad k = 0, 1, 2, 3, \cdots$$

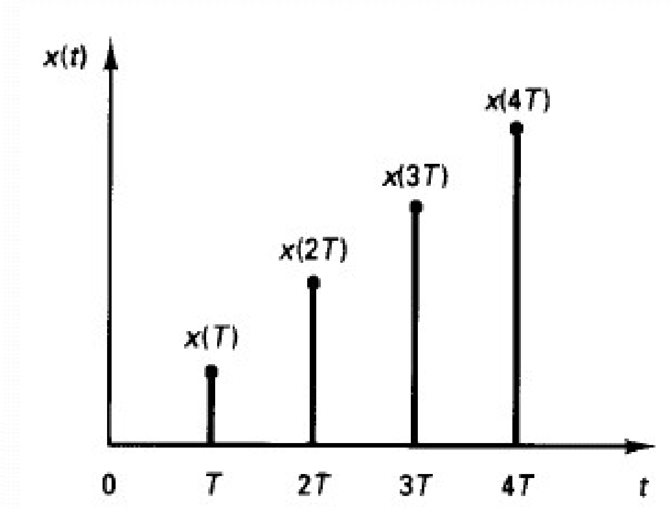


Figure 2–1 Sampled unit-ramp signal.

Its $\mathcal{Z}$ transform is

$$X(z) = \mathcal{Z}[t] = \sum_{k=0}^{\infty} x(kT)z^{-k} = \sum_{k=0}^{\infty} kTz^{-k} = T\sum_{k=0}^{\infty} kz^{-k}$$

$$= T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots + kz^{-k} + \cdots \quad)$$

$$= T\frac{z^{-1}}{(1-z^{-1})^2} = \frac{Tz}{(z-1)^2}$$

which depends on the sampling period T.

Similarly, for a number sequence

$$x(k) = k, \quad k = 0, 1, 2, 3, \cdots$$

we have

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} kz^{-k} = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

- ***Polynomial Function (a number sequence)***

$$x(k) = \begin{cases} a^k, & k = 0, 1, 2, 3, \cdots \\ 0, & k < 0 \end{cases}$$

$$\begin{aligned} X(z) &= \mathcal{Z}[a^k] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k \\ &= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots \\ &= \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \end{aligned}$$

- ***Exponential Function***

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

i.e. $x(kT) = e^{-akT}, \quad k = 0, 1, 2, 3, \cdots$

$$\begin{aligned} X(z) &= \mathcal{Z}[e^{-at}] = \sum_{k=0}^{\infty} x(kT) z^{-k} \\ &= \sum_{k=0}^{\infty} e^{-akT} z^{-k} = \sum_{k=0}^{\infty} (e^{-aT} z^{-1})^k \\ &= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}} \end{aligned}$$

Note that, by denoting $e^{-aT} = b$

we have $e^{-akT} = (e^{-aT})^k = b^k$

hence, linking the exponential function to the polynomial function.

- ***Sinusoidal Functions***

$$x(t) = \begin{cases} \sin \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$ i.e. $x(kT) = \sin \omega kT, \quad k = 0, 1, 2, 3, \cdots$

Noting that $\mathcal{Z}[e^{-at}] = \dfrac{1}{1 - e^{-aT} z^{-1}}$ $\qquad \sin \omega t = \left(e^{j\omega t} - e^{-j\omega t}\right)/j2$

we have

$$X(z) = \mathcal{Z}[\sin \omega t] = \mathcal{Z}\left[\frac{1}{j2}\left(e^{j\omega t} - e^{-j\omega t}\right)\right] = \frac{1}{j2}\left(\mathcal{Z}[e^{j\omega t}] - \mathcal{Z}[e^{-j\omega t}]\right)$$

$$= \frac{1}{j2}\left(\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega T} z^{-1}}\right) = \frac{1}{j2}\frac{(1 - e^{-j\omega T} z^{-1}) - (1 - e^{j\omega T} z^{-1})}{(1 - e^{j\omega T} z^{-1})(1 - e^{-j\omega T} z^{-1})}$$

$$= \frac{1}{j2}\frac{\left(e^{j\omega T} - e^{-j\omega T}\right) z^{-1}}{1 - \left(e^{j\omega T} + e^{-j\omega T}\right) z^{-1} + z^{-2}} = \frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$$

$$= \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

Similarly, for

$$x(t) = \begin{cases} \cos \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

we have

$$\begin{aligned} X(z) &= \mathcal{Z}[\cos \omega t] = \mathcal{Z}\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)\right] \\ &= \frac{1}{2}\left(\mathcal{Z}[e^{j\omega t}] + \mathcal{Z}[e^{-j\omega t}]\right) \\ &= \frac{1}{2}\left(\frac{1}{1 - e^{j\omega T} z^{-1}} + \frac{1}{1 - e^{-j\omega T} z^{-1}}\right) \\ &= \frac{1}{2}\, \frac{2 - \left(e^{j\omega T} + e^{-j\omega T}\right) z^{-1}}{1 - \left(e^{j\omega T} + e^{-j\omega T}\right) z^{-1} + z^{-2}} \\ &= \frac{1 - z^{-1} \cos \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}} \\ &= \frac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1} \end{aligned}$$

***Example 2-1*:** Obtain the $\mathcal{Z}$ transform of

$$X(s) = \frac{1}{s(s+1)}$$

Solution:

Firstly
$$x(t) = \mathcal{L}^{-1}[X(s)] = 1 - e^{-t}, \qquad t \geq 0$$

Hence

$$X(z) = \mathcal{Z}[1 - e^{-t}] = \mathcal{Z}[1] - \mathcal{Z}[e^{-t}] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}}$$

$$= \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}$$

$$= \frac{(1 - e^{-T})z}{(z - 1)(z - e^{-T})}$$

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\dfrac{1}{s}$	$1(t)$	$1(k)$	$\dfrac{1}{1-z^{-1}}$
4.	$\dfrac{1}{s+a}$	e^{-at}	e^{-akT}	$\dfrac{1}{1-e^{-aT}z^{-1}}$
5.	$\dfrac{1}{s^2}$	t	kT	$\dfrac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\dfrac{2}{s^3}$	t^2	$(kT)^2$	$\dfrac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\dfrac{6}{s^4}$	t^3	$(kT)^3$	$\dfrac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\dfrac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\dfrac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\dfrac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\dfrac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\dfrac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\dfrac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

TABLE 2–1 (continued)

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1,2,3,\ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k\cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t)=0$, for $t<0$.

$x(kT)=x(k)=0$, for $k<0$.

Unless otherwise noted, $k=0,1,2,3,\ldots$.

2-4 Important Properties and Theorems of the $\mathcal{Z}$ Transform

Assume $\mathcal{Z}[x(t)] = X(z)$

- ***Multiplication by a constant***

$$\mathcal{Z}[ax(t)] = a\mathcal{Z}[x(t)] = aX(z)$$

- ***Linearity of the $\mathcal{Z}$ transform***

$$\mathcal{Z}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{Z}[f(t)] + \beta\mathcal{Z}[g(t)]$$

where α and β are constants.

- ***Multiplication by*** a^k

$$\mathcal{Z}[a^k x(k)] = X(a^{-1}z) \qquad (2\text{-}6)$$

because

$$\mathcal{Z}[a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} = X(a^{-1}z)$$

- ***Shifting (or Translation) Theorem***: Suppose $x(t) = 0$ for $t < 0$ and $X(z) = \mathcal{Z}[x(k)]$. Then, for $n = 0, 1, 2, 3, \cdots$,

$$\mathcal{Z}[x(t - nT)] = z^{-n}X(z), \qquad (2\text{-}7)$$

i.e., multiplication by z^{-1} = delaying 1 step in time, and

$$\mathcal{Z}[x(t + nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right] \qquad (2\text{-}8)$$

i.e., multiplication by z = advancing 1 step in time.

Eqn (2-7) can be shown as follows:

$$\mathcal{Z}\left[x(t - nT)\right] = \sum_{k=0}^{\infty} x(kT - nT)z^{-k} = z^{-n}\sum_{k=0}^{\infty} x(kT - nT)z^{-(k-n)}$$

$$= z^{-n}\sum_{m=-n}^{\infty} x(mT)z^{-m} \qquad \text{letting } m = k - n$$

$$= z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} \qquad \text{since } x(mT) = 0 \text{ for } m < 0$$

$$= z^{-n}X(z) \qquad (2\text{-}9)$$

Eqn (2-8) can be shown as follows:

$$\begin{aligned}\mathcal{Z}\left[x(t+nT)\right] &= \sum_{k=0}^{\infty} x(kT+nT)z^{-k} \\ &= z^n \sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)} \qquad \text{letting } m=k+n \\ &= z^n \left[\sum_{m=n}^{\infty} x(mT)z^{-m} + \sum_{m=0}^{n-1} x(mT)z^{-m} - \sum_{m=0}^{n-1} x(mT)z^{-m}\right] \\ &= z^n \left[\sum_{m=0}^{\infty} x(mT)z^{-m} - \sum_{m=0}^{n-1} x(mT)z^{-m}\right] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]\end{aligned}$$

If $x(k)$ is a number sequence, then

$$\mathcal{Z}[x(t+n)] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(k)z^{-k}\right] = z^n X(z) - \sum_{k=0}^{n-1} x(k)z^{n-k} \qquad (2\text{-}10)$$

hence

$$\mathcal{Z}[x(k+1)] = zX(z) - zx(0) \qquad (2\text{-}11)$$

$$\mathcal{Z}[x(k+2)] = z^2X(z) - z^2x(0) - zx(1) \qquad (2\text{-}12)$$

$$\mathcal{Z}[x(k+n)] = z^nX(z) - z^nx(0) - z^{n-1}x(1) - \cdots - zx(n-1) \qquad (2\text{-}13)$$

Example *2-2*: Find the $\mathcal{Z}$ transform of $1(t-T)$ and $1(t-4T)$ (as shown in Figure 2-2).

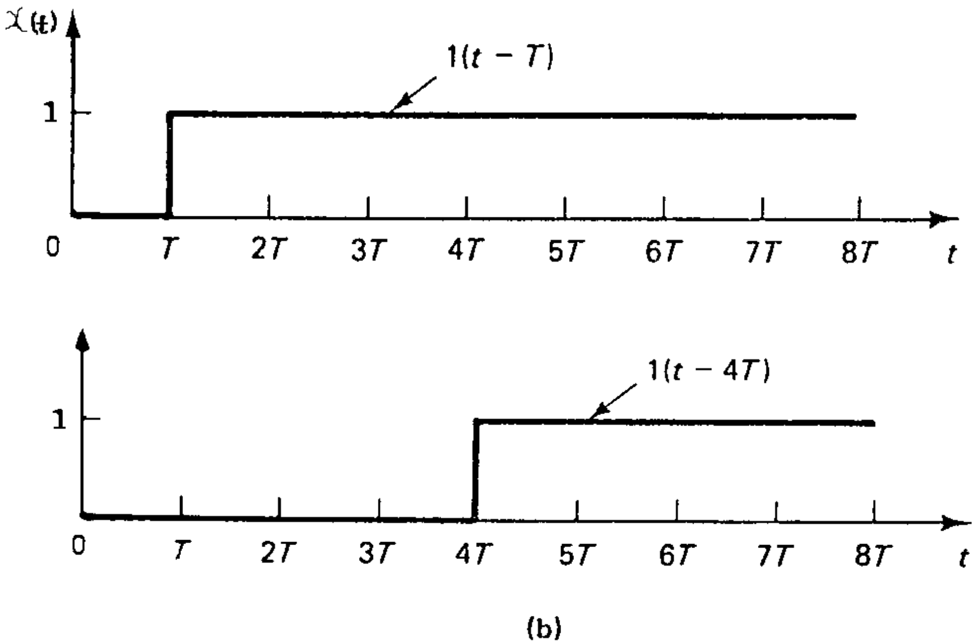


Figure 2–2 (a) Unit-step function delayed by 1 sampling period; (b) unit-step function delayed by 4 sampling periods.

Using the shifting theorem in (2-7), we have

$$\mathcal{Z}[1(t-T)] = z^{-1}\mathcal{Z}[1(t)] = z^{-1}\frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}}$$

$$\mathcal{Z}[1(t-4T)] = z^{-4}\mathcal{Z}[1(t)] = z^{-4}\frac{1}{1-z^{-1}} = \frac{z^{-4}}{1-z^{-1}}$$

Example 2-3: Obtain the $\mathcal{Z}$ transform of

$$f(k) = \begin{cases} a^{k-1}, & k = 1, 2, 3, \cdots \\ 0, & k \le 0 \end{cases}$$

Treating $f(k)$ as a^k with a one-step delay, we have

$$\mathcal{Z}[f(k)] = z^{-1}\mathcal{Z}[a^k] = z^{-1}\frac{1}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}$$

Example 2-4: Given $x(h)$, $h = 0, 1, 2, 3, \cdots$, let

$$y(k) = \sum_{h=0}^{k} x(h), \qquad k = 0, 1, 2, 3, \cdots$$

with $y(k) = 0$ for $k < 0$, find the $\mathcal{Z}$ transform $Y(z)$ of $y(k)$, assuming $\mathcal{Z}[x(k)] = X(z)$.

Solution

Note that $y(k) = \sum_{h=0}^{k} x(h) = \sum_{h=0}^{k-1} x(h) + x(k) = y(k-1) + x(k)$

i.e., $y(k) = y(k-1) + x(k)$. Taking the $\mathcal{Z}$ transform gives

$$Y(z) = z^{-1}Y(z) + X(z) \Rightarrow Y(z) = \frac{1}{1 - z^{-1}}X(z)$$

- ***Complex Translation Theorem***

$$\mathcal{Z}[e^{-at}x(t)] = X(ze^{aT})$$

because

$$\mathcal{Z}[e^{-at}x(t)] = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k} = \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k} = X(ze^{aT})$$

Alternatively, note that, after sampling

$$e^{-at}x(t) \implies e^{-akT}x(kT) = (e^{-aT})^k x(kT) = b^k x(k) \tag{2-14}$$

where $b = e^{-aT}$.

Hence, the Complex Translation Theorem can be obtained by using the multiplication-by- a^k rule.

***Example 2-5*:** Given the $\mathcal{Z}$ transform of $sin\omega t$ and $cos\omega t$, find the $\mathcal{Z}$ transform of $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$.

$$\mathcal{Z}[e^{-at}\sin\omega t] = \mathcal{Z}[\sin\omega t]\Big|_{z=ze^{aT}} = \frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}\Bigg|_{z=ze^{aT}}$$

$$= \frac{z^{-1}e^{-aT}\sin\omega T}{1-2z^{-1}e^{-aT}\cos\omega T+z^{-2}e^{-2aT}}$$

$$= \frac{ze^{aT}\sin\omega T}{z^2e^{2aT}-2ze^{aT}\cos\omega T+1}$$

$$\mathcal{Z}[e^{-at}\cos\omega t] = \mathcal{Z}[\cos\omega t]\Big|_{z=ze^{aT}} = \frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}\Bigg|_{z=ze^{aT}}$$

$$= \frac{1-z^{-1}e^{-aT}\cos\omega T}{1-2z^{-1}e^{-aT}\cos\omega T+z^{-2}e^{-2aT}}$$

$$= \frac{e^{aT}z(e^{aT}z-\cos\omega T)}{z^2e^{2aT}-2ze^{aT}\cos\omega T+1}$$

Example 2-6: Obtain the $\mathcal{Z}$ transform of te^{-at}.

Solution

$$\mathcal{Z}[te^{-at}] = \mathcal{Z}[t]\,\Big|_{z=ze^{aT}} = \frac{Tz^{-1}}{(1-z^{-1})^2}\Big|_{z=ze^{aT}} = \frac{Tz^{-1}e^{-aT}}{(1-z^{-1}e^{-aT})^2}$$

- ***Initial Value Theorem***: If $\mathcal{Z}[x(t)] = X(z)$ and $\lim_{z\to\infty} X(z)$ exists, then

$$x(0) = x(t)\,\Big|_{t=0} = \lim_{z\to\infty} X(z) \qquad (2\text{-}15)$$

This can be shown by taking the limit $z \to \infty$ of the following

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

This is very convenient for checking $X(z)$ for possible errors.

Example 2-7: Determine the initial value $x(0)$ if the $\mathcal{Z}$ transform of $x(t)$ is given by

$$X(z) = \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}$$

Solution:

By using the initial values theorem, we have

$$x(0) = \lim_{z \to \infty} \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})} = 0$$

Noting from Example 2-1 that $X(z)$ is the $\mathcal{Z}$ transform of

$$x(t) = 1 - e^{-t}$$

thus, $x(0) = 0$.

Poles and **Zeros** in the z Plane: Write $X(z)$ as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}, \quad m \le n$$

or in factored form

$$X(z) = \frac{b_0(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots\cdots(z - p_n)}$$

where $p_i \ (i = 1, 2, \cdots, n)$ are the poles of $X(z)$,

$i.e$ the solutions of $D(z)=0$,

and $z_i \ (i = 1, 2, \cdots, m)$ are the zeros,

$i.e$ the solutions of $N(z)=0$.

- ***Final Value Theorem*:** Suppose that $x(k) = 0$ for $k < 0$ and $X(z) = \mathcal{Z}[x(k)]$. Assume that all poles of $X(z)$ lie *inside the unit circle*, with the possible exception of a simple pole at $z = 1$. Then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z) \qquad (2\text{-}16)$$

To prove this, note first that assumption on poles of $X(z)$ implies $\lim_{k\to\infty} x(k)$ exists.

Then noting that

$$\mathcal{Z}[x(k)] \;=\; X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

$$\mathcal{Z}[x(k-1)] \;=\; z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)z^{-k}$$

we have

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = X(z) - z^{-1}X(z) \quad = (1 - z^{-1})X(z)$$

Taking limit yields

$$\lim_{z \to 1} \left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} \right] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

Noting that $x(k) = 0$ for $k < 0$, we have

$$\begin{aligned} \text{LHS} &= \sum_{k=0}^{\infty} [x(k) - x(k-1)] \\ &= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \cdots \\ &= -x(-1) + x(\infty) = x(\infty) \\ &= \lim_{k \to \infty} x(k) \end{aligned}$$

Remarks:

- The final value theorem allows us to calculate the limit of a sequence as *k* tends to infinity. If one is only interested in the final value of the sequence, this constitutes a significant short cut.

- The main pitfall of the theorem is that there are important cases where the limit does not exist:
    - An unbounded sequence
    - An oscillatory sequence

  Thus we should be cautioned against blindly using the final value theorem, because this can yield misleading results.

Example 2-8: Determine the value of $x(\infty)$ of

$$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}, \qquad , a > 0$$

Solution:

$$\begin{aligned}\lim_{k\to\infty} x(k) &= \lim_{z\to 1}(1-z^{-1})X(z) \\ &= \lim_{z\to 1}(1-z^{-1})\left[\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right] \\ &= \lim_{z\to 1}\left(1 - \frac{1-z^{-1}}{1-e^{-aT}z^{-1}}\right) \ = \ 1\end{aligned}$$

It is noted that $x(t) = 1 - e^{-at}$ and obviously $\lim_{t\to\infty} x(t) = 1.$

A summary of the properties and theorems of $\mathcal{Z}$ transform is given in Table 2-2.

TABLE 2–2 IMPORTANT PROPERTIES AND THEOREMS OF THE z TRANSFORM

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t + T)$ or $x(k + 1)$	$zX(z) - zx(0)$
4.	$x(t + 2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k + 2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t + kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT - T)$
7.	$x(t - kT)$	$z^{-k}X(z)$
8.	$x(n + k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k - 1)$
9.	$x(n - k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1} [(1 - z^{-1})X(z)]$ if $(1 - z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k - 1)$	$(1 - z^{-1})X(z)$
19.	$\Delta x(k) = x(k + 1) - x(k)$	$(z - 1)X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1 - z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t, a)$	$\frac{\partial}{\partial a}X(z, a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT - kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

2-5 The Inverse $\mathcal{Z}$ Transform

Question: Given a $\mathcal{Z}$ transform $X(z)$, how to find the corresponding time function $x(k)$?

$$\mathcal{Z}^{-1}[X(z)] = ?$$

Note, as $X(z)$ depends only on $x(t)$ at $t = kT$, $k = 0, 1, 2, \cdots$

- can only get $x(k)$, but not $x(t)$.
- sampling period T must also be given in order to get $x(kT)$.
- there are many $x(t)$ that can fit into $x(k)$ (see Figure 2-3)

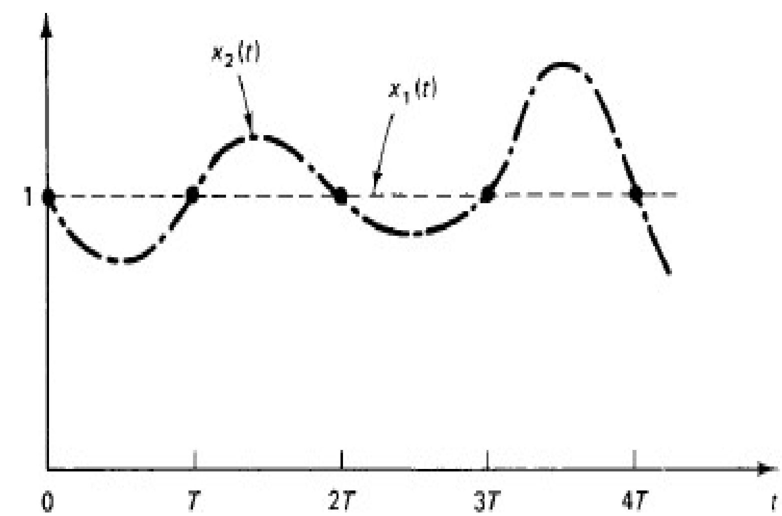


Figure 2–3 Two different continuous-time functions, $x_1(t)$ and $x_2(t)$, that have the same values at $t = 0, T, 2T, \ldots$.

We can do inverse $\mathcal{Z}$ transform by

- direct division method (long division)
- computation method
- partial fraction expansion method
  (then referring to $\mathcal{Z}$ transform table)
- inversion integral method

- ***Direct Division Method (Long Division)***

Noting that

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots$$

or

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(k)z^{-k} + \cdots$$

we can expand $X(z)$ as a series in z^{-1} and obtain $x(kT)$ or $x(k)$ from the coefficients.

For $X(z)$ given as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}, \quad m \le n$$

it can also be re-written in terms of z^{-1}

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \qquad \text{(2-18)}$$

***Example 2-10*:** Find $x(k)$ for $k = 0, 1, 2, 3, 4$ when $X(z)$ is given by

$$X(z) = \frac{10z + 5}{(z-1)(z-0.2)}$$

Solution: First, rewrite $X(z)$ in terms of z^{-1}

$$X(z) = \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

Then, dividing the numerator by the denominator, we have

$$
\begin{array}{r l}
 & 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots \\
1 - 1.2z^{-1} + 0.2z^{-2} \, \big) & 10z^{-1} + 5z^{-2} \\
 & \underline{10z^{-1} - 12z^{-2} + 2z^{-3}} \\
 & \qquad 17z^{-2} - 2z^{-3} \\
 & \qquad \underline{17z^{-2} - 20.4z^{-3} + 3.4z^{-4}} \\
 & \qquad\qquad 18.4z^{-3} - 3.4z^{-4} \\
 & \qquad\qquad \underline{18.4z^{-3} - 22.08z^{-4} + 3.68z^{-5}} \\
 & \qquad\qquad\qquad 18.68z^{-4} - 3.68^{-5} \\
 & \qquad\qquad\qquad \underline{18.68z^{-4} - 22.416z^{-5} + 3.736z^{-6}}
\end{array}
$$

Thus, $X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots$

Hence

k	0	1	2	3	4	$\cdots$
$x(k)$	0	10	17	18.4	18.68	$\cdots$

In general, no closed-form expression can be obtained for $x(k)$.

Example 2-11: Obtain the inverse $\mathcal{Z}$ transform of

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

Solution: This is a finite power series in z^{-1}, hence

k	0	1	2	3	4	5	6	$\cdots$
$x(k)$	1	2	3	4	0	0	0	$\cdots$

•***Computational Method :***Two approaches will be presented. Consider a system $G(z)$ defined by

$$G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}} \qquad (2\text{-}19)$$

In finding the inverse $\mathcal{Z}$ transform of $G(z)$, we use the Kronecker delta function $\delta(kT)$

$$\delta(kT) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

Let the input to $G(z)$ be $x(kT) = \delta(kT)$ and the output be $y(kT)$. Then

$$X(z) = 1$$

$$Y(z) = G(z)X(z) = G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}}$$

$$= \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607} \quad (2\text{-}20)$$

- ***MATLAB Approach*** (*optional*)

Use MATLAB to compute response $y(kT)$ to input $x(kT) = \delta(kT)$.

Example 2-12: The following MATLAB commands calculate the output $y(kT)$, for $k = 0, 1, 2, \cdots, 40$.

MATLAB Program 2–1

```
% --------- Finding inverse z transform ---------

% ***** Finding the inverse z transform of G(z) is the same as
% finding the response of the system Y(z)/X(z) = G(z) to the
% Kronecker delta input *****

% ***** Enter the numerator and denominator of G(z) *****

num = [0   0.4673   -0.3393];
den = [1   -1.5327   0.6607];

% ***** Enter the Kronecker delta input x and filter command
% y = filter(num,den,x) *****

x = [1   zeros(1,40)];
y = filter(num,den,x)
```

The input *x(k)* is defined through

$$x = [1 \;\; zeros(1, 40)];$$

The numerator and denominator of *G(z)* are defined

$$num = [0 \;\; 0.4673 \;\; -0.3393]; \; den = [1 \;\; -1.5327 \;\; 0.6607];$$

and the output *y(kT)* is computed through

$$y = filter(num, den, x)$$

```
y =
  Columns 1 through 7
           0    0.4673    0.3769    0.2690    0.1632    0.0725    0.0032
  Columns 8 through 14
     -0.0429   -0.0679   -0.0758   -0.0712   -0.0591   -0.0436   -0.0277
  Columns 15 through 21
     -0.0137   -0.0027    0.0050    0.0094    0.0111    0.0108    0.0092
  Columns 22 through 28
      0.0070    0.0046    0.0025    0.0007   -0.0005   -0.0013   -0.0016
  Columns 29 through 35
     -0.0016   -0.0014   -0.0011   -0.0008   -0.0004   -0.0002    0.0000
  Columns 36 through 41
      0.0002    0.0002    0.0002    0.0002    0.0002    0.0001
```

i.e.,

k	0	1	2	3	$\cdots$	40
$x(k)$	0	0.4673	0.3769	0.2690	$\cdots$	0.0001

We can also plot the values of the inverse $\mathcal{Z}$ transform of $G(z)$ as shown in Figure 2-5.

MATLAB Program 2–2

```
% ---------- Response to Kronecker delta input -----

num = [0   0.4673   -0.3393];
den = [1   -1.5327   0.6607];
x = [1   zeros(1,40)];
k = 0:40;
y = filter(num,den,x);
plot(k,y,'o')
v = [0  40  -1  1];
axis(v);
grid
title('Response to Kronecker Delta Input')
xlabel('k')
ylabel('y(k)')
```

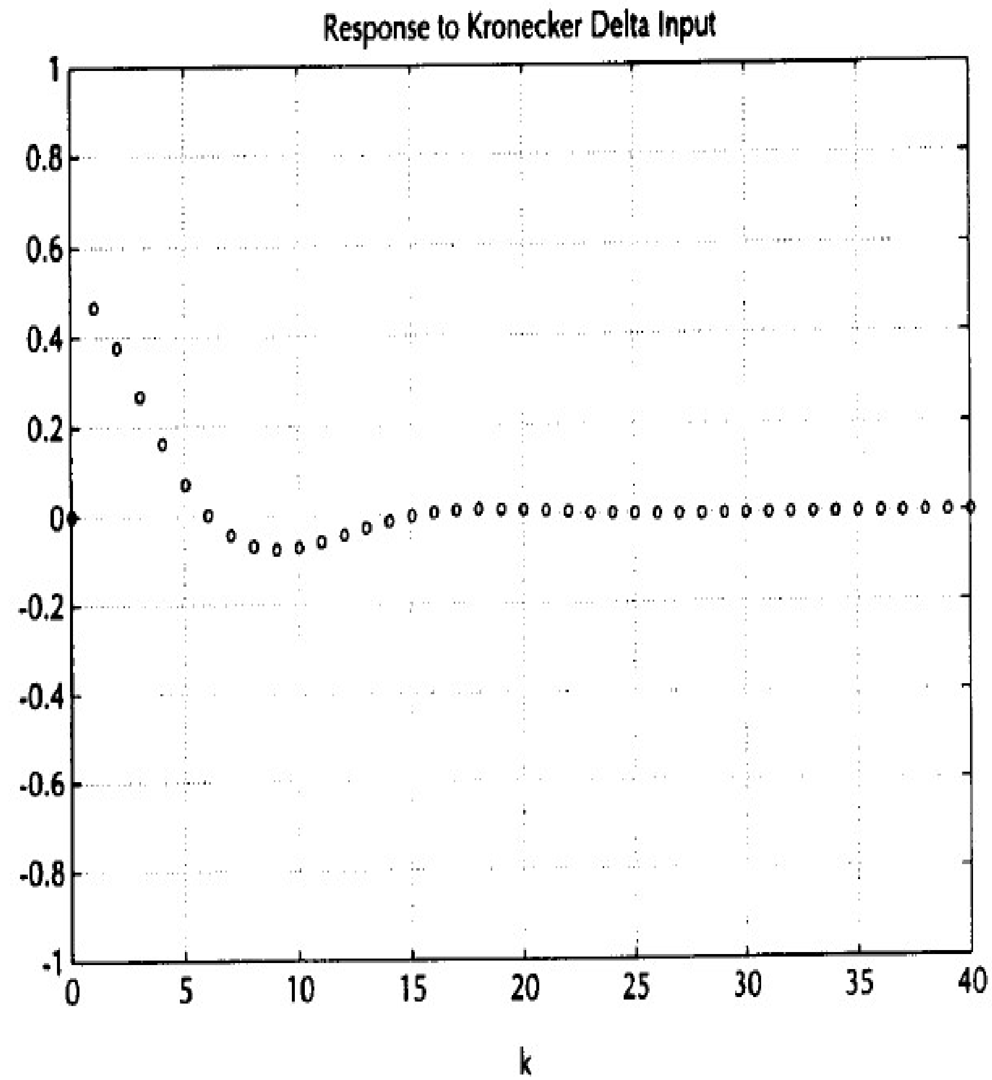


Figure 2–5 Response of the system defined by Equation (2–20) to the Kronecker delta input.

- ***Difference Equation Approach:***

Example 2-13: Rewrite eqn (2-20) as

$$(1 - 1.5327z^{-1} + 0.6607z^{-2})Y(z) = (0.4673z^{-1} - 0.3393z^{-2})X(z)$$

$$y(k) - 1.5327y(k-1) + 0.6607y(k-2) = 0.4673x(k-1) - 0.3393x(k-2)$$

$$y(k) = 1.5327y(k-1) - 0.6607y(k-2) + 0.4673x(k-1) - 0.3393x(k-2)$$

Noting that $x(0) = 1,\ x(k) = 0 \text{ for } k \neq 0\ \ y(k) = 0 \text{ for } k < 0$

we have

$$y(0) = 1.5327y(-1) - 0.6607y(-2) + 0.4673x(-1) - 0.3393x(-2) = 0$$

$$y(1) = 1.5327y(0) - 0.6607y(-1) + 0.4673x(0) - 0.3393x(-1) = 0.4673$$

$$y(2) = 1.5327y(1) - 0.6607y(0) + 0.4673x(1) - 0.3393x(0) = 0.3769$$

$$\vdots$$

The results are the same as those obtained by using the Matlab approach

• ***Partial Fraction Expansion (PFE) Method***:

Consider
$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} \qquad (m \le n)$$

Factorize it as
$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{(z-p_1)(z-p_2)\cdots\cdots(z-p_n)}$$

Then, do PFE for $X(z)/z$. If it has only simple poles, we have

$$\frac{X(z)}{z} = \frac{a_0}{z} + \frac{a_1}{z - p_1} + \ldots. + \frac{a_n}{z - p_n}$$

$$X(z) = a_0 + \frac{a_1 z}{z-p_1} + \frac{a_2 z}{z-p_2} + \cdots + \frac{a_n z}{z-p_n}$$

$$X(z) = a_0 + \frac{a_1}{1-p_1 z^{-1}} + \frac{a_2}{1-p_2 z^{-1}} + \cdots + \frac{a_n}{1-p_n z^{-1}}$$

Hence $x(k) = a_0\delta(k) + a_1 p_1^k + a_2 p_2^k + \cdots + a_n p_n^k$ (2-21)

If $X(z)/z$ involves multiple poles, for example,

$$\frac{X(z)}{z} = \frac{b_0 z + b_1}{(z-p)^2} = \frac{\overbrace{b_0}^{c_2}(z-p) + \overbrace{b_1 + b_0 p}^{c_1}}{(z-p)^2}$$

then
$$\frac{X(z)}{z} = \frac{c_1}{(z-p)^2} + \frac{c_2}{z-p}$$

$$X(z) = \frac{c_1 z^{-1}}{(1-pz^{-1})^2} + \frac{c_2}{1-pz^{-1}}$$

$$x(k) \ = \ c_1 \mathcal{Z}^{-1}\left[\frac{z^{-1}}{(1-pz^{-1})^2}\right] + c_2 \mathcal{Z}^{-1}\left[\frac{1}{1-pz^{-1}}\right]$$

From Table 2-1,

$$x(k) \ = \ c_1 k p^{k-1} + c_2 p^k \qquad k \geq 0 \qquad \text{(2-22)}$$

i.e. $x(0) \ = \ c_2, \ x(1) \ = \ c_1 + c_2 p, x(2) \ = \ 2c_1 p + c_2 p^2, \ \ldots$

Example 2-14: Given the $\mathcal{Z}$ transform

$$X(z) = \frac{(1 - e^{-aT})z}{(z-1)(z-e^{-aT})}$$

where a is a constant and T is the sampling period, determine $x(kT) = \mathcal{Z}^{-1}[X(z)]$ by using PFE method.

Solution: $$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-e^{-aT}}$$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$$

Using Table 2-1:

$$x(kT) = 1 - e^{-akT}, \qquad k = 0, 1, 2, \cdots$$

Example 2-15 *(involving complex conjugate poles):*

Use PFE to obtain the inverse $\mathcal{Z}$ transform of

$$X(z) = \frac{z^2 + z + 2}{(z-1)(z^2 - z + 1)}$$

Solution:

$$\frac{X(z)}{z} = \frac{-2}{z} + \frac{4}{z-1} + \frac{-2z-1}{z^2 - z + 1}$$

$$X(z) = -2 + \frac{4}{1-z^{-1}} - \frac{2+z^{-1}}{1-z^{-1}+z^{-2}}$$, note $\mathcal{Z}[\cos\omega kT] = \dfrac{1 - \cos\omega T\, z^{-1}}{1 - 2\cos\omega T\, z^{-1} + z^{-2}}$

$$X(z) = -2 + \frac{4}{1-z^{-1}} - \frac{2(1 - 0.5z^{-1} + 0.5z^{-1} + 0.5z^{-1})}{1 - z^{-1} + z^{-2}}$$, so $\cos\omega T = 0.5$

$$= -2 + \frac{4}{1-z^{-1}} - \frac{2(1-0.5z^{-1})}{1-z^{-1}+z^{-2}} - \frac{2z^{-1}}{1-z^{-1}+z^{-2}}$$

Also note $\mathcal{Z}[\sin\omega kT] = \dfrac{\sin\omega T\, z^{-1}}{1 - 2\cos\omega T\, z^{-1} + z^{-2}}$

and $\cos\omega T = 0.5 \implies \sin\omega T = \dfrac{\sqrt{3}}{2}$

Then $$X(z) = -2 + \frac{4}{1-z^{-1}} - \frac{2(1-0.5z^{-1})}{1-z^{-1}+z^{-2}} - \frac{4}{\sqrt{3}} \times \frac{\frac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}}$$

Hence $$x(kT) = -2\delta_0(k) + 4 - 2\cos\frac{k\pi}{3} - \frac{4}{\sqrt{3}}\sin\frac{k\pi}{3}, \qquad k \geq 0$$

We can also do the PFE in a different way to get

$$\begin{aligned} X(z) &= \frac{4}{z-1} + \frac{-3z+2}{z^2-z+1} \\ &= z^{-1}\left[\frac{4}{1-z^{-1}} - 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}} + \frac{0.5z^{-1}}{1-z^{-1}+z^{-2}}\right] \end{aligned}$$

For the terms inside the bracket, their inverse Z-transform are

$$4 * 1(k) - 3\cos\frac{k\pi}{3}\, 1(k) + \frac{1}{\sqrt{3}}\sin\frac{k\pi}{3}\, 1(k)$$

Then

$$x(kT) = 4 * 1(k-1) - 3\cos\frac{(k-1)\pi}{3}\, 1(k-1) + \frac{1}{\sqrt{3}}\sin\frac{(k-1)\pi}{3}\, 1(k-1)$$

So $x(kT)$ can take different forms, but they give us the same values!

• ***Inversion Integral Method:***

$$x(k) = x(kT) = \mathcal{Z}^{-1}[X(z)] = \frac{1}{j2\pi}\oint_C X(z)z^{k-1}dz \qquad (2\text{-}23)$$

where C is a counter-clockwise circle centered at the origin (*z = 0*) such that all poles of $X(z)z^{k-1}$ are inside it.

From the residue theory of complex functions, we have

$$x(kT)=x(k) = K_1 + K_2 + \cdots + K_m$$

$$= \sum_{i=1}^{m}\left[\text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}\right] \quad (2-24)$$

where $K_1, K_2, \cdots, K_m$ denotes the residues of $X(z)z^{k-1}$ at poles $z_1,\ z_2,\ \cdots,\ z_m,$ resp., and k ≥ 0.

If $z = z_i$ is a simple pole of $X(z)\ z^{k-1}$, then

$$K_i = \lim_{z \to z_i}\left[(z - z_i)X(z)z^{k-1}\right] \qquad (2\text{-}25)$$

If $z = z_j$ is an order q multiple pole of $X(z)\ z^{k-1}$, then

$$K_j = \frac{1}{(q-1)!}\lim_{z \to z_j}\frac{d^{q-1}}{dz^{q-1}}\left[(z - z_j)^q X(z)z^{k-1}\right] \qquad (2\text{-}26)$$

Example 2-16: Repeat Example 2-14 to compute $x(kT)$ using the inversion integral method.

$$X(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \Rightarrow X(z)z^{k-1} = \frac{(1-e^{-aT})z^k}{(z-1)(z-e^{-aT})}$$

For $k = 0, 1, 2, \cdots$, $X(z)z^{k-1}$ has only simple poles $z_1 = 1$ and $z_2 = e^{-aT}$. Hence

K_1 = [residue at simple pole $z = 1$]

$$= \lim_{z\to 1}\left[(z-1)\frac{(1-e^{-aT})z^k}{(z-1)(z-e^{-aT})}\right] = 1$$

K_2 = [residue at simple pole $z = e^{-aT}$]

$$= \lim_{z\to e^{-aT}}\left[(z-e^{-aT})\frac{(1-e^{-aT})z^k}{(z-1)(z-e^{-aT})}\right] = -e^{-akT}$$

Therefore $x(kT) = K_1 + K_2 = 1 - e^{-akT}, \qquad k = 0, 1, 2, \cdots$

2-6 $\mathcal{Z}$ Transform Method for Solving Difference Equations

Consider the following linear difference equation

$$x(k) + a_1 x(k-1) + \cdots + a_n x(k-n)$$
$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n) \qquad (2\text{-}27)$$

where $u(k)$ is the input and $x(k)$ is the output.

Let

$$\mathcal{Z}[x(k)] = X(z)$$

Taking $\mathcal{Z}$ Transform of (2-27), $x(k-1), x(k-2), x(k-3)$ ... *can be represented in terms of* $X(z)$ as shown in Table 2-3. Then, solving for $X(z)$ and computing $\mathcal{Z}^{-1}[X(z)]$, $x(kT)$ is obtained.

TABLE 2–3 z TRANSFORMS OF $x(k+m)$ AND $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

Example 2-17: Solve the difference equation using $\mathcal{Z}$ transform

$$x(k+2)+3x(k+1)+2x(k)=0, \qquad x(0)=0, \quad x(1)=1$$

Taking $\mathcal{Z}$ transform of this difference equation gives

$$\overbrace{z^2X(z)-z^2x(0)-zx(1)}^{\mathcal{Z}[x(k+2)]}+3\overbrace{[zX(z)-zx(0)]}^{\mathcal{Z}[x(k+1)]}+2X(z)=0$$

Then

$$X(z)=\frac{z}{z^2+3z+2} = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

and by looking up Table 2-1, we have

$$x(k)=\underbrace{(-1)^k}_{\#18,\ a=-1} - \underbrace{(-2)^k}_{\#18,\ a=-2}, \qquad k=0,1,2,\cdots$$

CONCLUDING COMMENTS

In this chapter the basic theory of the $\mathcal{Z}$ transform method has been presented.

- $\mathcal{Z}$ transform serves the same purpose for linear time-invariant discrete-time system as the Laplace transform for linear time-invariant continuous-time systems.
- With the $\mathcal{Z}$ transform method, linear time-invariant difference equation can be transformed into algebraic equation . This facilitates the transient response analysis of the digital control system.
- Also, the $\mathcal{Z}$ transform method allows us to use conventional analysis and design technique available to analog (continuous-time) control systems.

Chapter 3

Modeling of Digital Control Systems

3.1: Introduction

$\mathcal{Z}$ transform

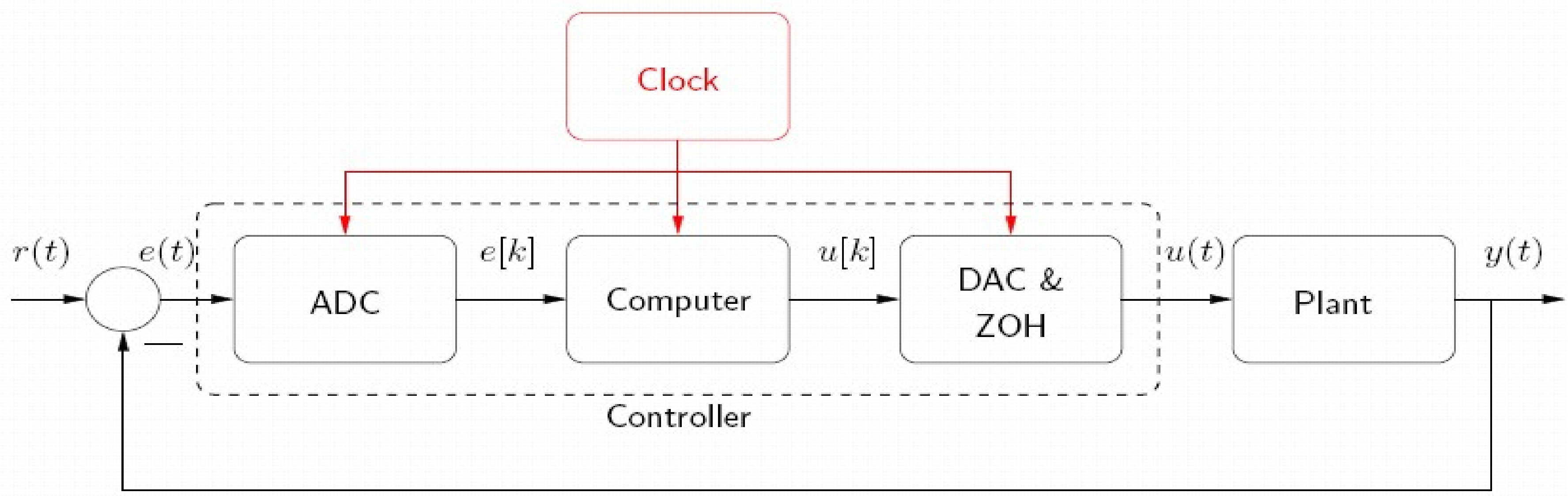
- is a math tool for the analysis and synthesis of DTCS
- is related to s plane through $z = e^{sT}$
- gives rise to methods similar to continuous-time design methods

In this chapter, we study the following topics:

- Impulse Sampling and Data Hold
- Pulse transfer function

3.2 Impulse Sampling and Data Hold

Recall a block diagram of a discrete-time control system



ADC - Analog-Digital-Converter (includes sampler), DAC - Digital Analogue Converter,
ZOH - Zero Order Hold

Discrete-time control systems are

- partly in discrete-time (digital controller)
- partly in continuous-time (plant)

Sampling and data hold play essential roles.

3.2.1 ADC Model

The ADC is usually modeled as an ideal sampler - Impulse Sampler . An impulse sampler is a fictious sampler.

Consider a continuous-time signal $x(t)$ with $x(t) = 0$ for $t < 0$. The impulse sampler is shown in Figure 3-1.

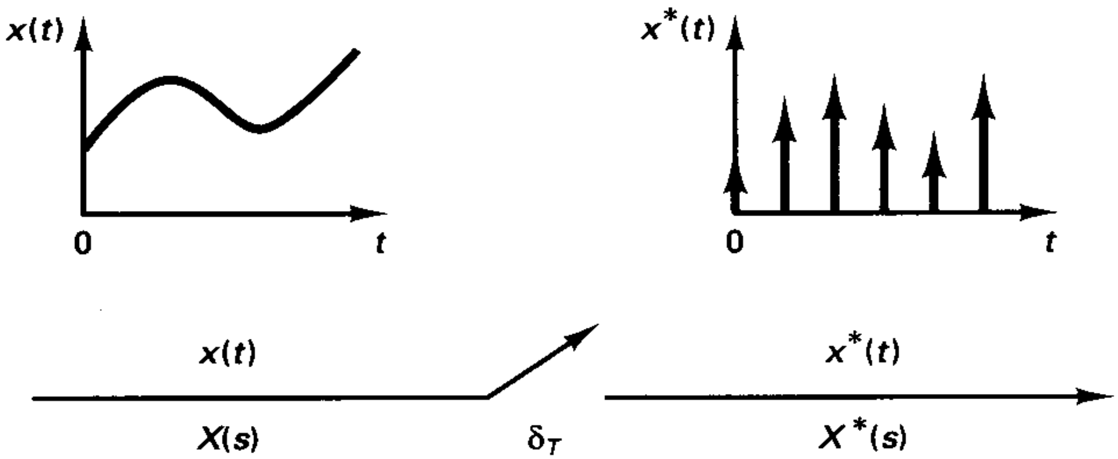


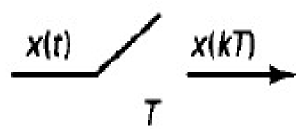
Figure 3–1 Impulse sampler.

The output of the impulse sampler is

- a train of impulses that begins with *t = 0*, 

- with sampling period *T*,
- the strength of each pulse is the sampled value of the continuous-time signal at the corresponding sampling instant.

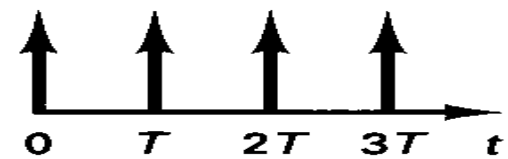
Remark:

- A real sampler 

samples the input signal $x(t)$ periodically (with period T) and produces a sequence of pulses $x(kT)$ as output.
- The width of the pulses is very small compared to the period T, but is nonetheless not zero in practice.

Recall, a unit impulse function *δ(t)* is

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$ with $$\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \qquad \text{for any } \epsilon > 0$$

Define a train of unit impulses $\delta_T(t)$ as 

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT) \;=\; \delta(t) + \delta(t - T) + \delta(t - 2T) + \cdots$$

Impulse-sampled output $x^*(t)$ is

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

or

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots \quad (3\text{-}1)$$

with the strength of each pulse equal to $x(kT)$, $k = 0, 1, 2, \cdots$.

The Laplace transform of $x^*(t)$ in (3-1) is

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t-T)] + \ldots$$

$$= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs} \quad (3\text{-}2)$$

7

By letting $e^{Ts} = z$ or $s = \frac{1}{T}\ln z$

then (3-2) becomes

$$X^*(s)\Big|_{s=1/T\ln z} = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (3\text{-}3)$$

which is exactly (2-1),

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (2\text{-}1)$$

i.e., the $\mathcal{Z}$ transform of the sequence $x(0), x(T), x(2T), \cdots$.

Hence

$$X^*(s)\,\big|_{s=1/T\ln z} = X(z)$$

and (3-3) becomes

$$X^*(s)\Big|_{s=1/T\ln z} = X^*(1/T\ln z) = X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (3\text{-}4)$$

Example 3-1:

Consider the impulse sampler in Figure 3.1 with sampling period T. If $x(t) = e^{-t}$, determine $X(s)$, $x^*(t)$, $X^*(s)$ and $X(z)$

Solution:

$$X(s) = \frac{1}{s+1}$$

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \dots$$

$$= \delta(t) + e^{-T}\delta(t-T) + e^{-2T}\delta(t-2T) + \dots$$

$$X^*(s) = 1 + e^{-T}\, e^{-sT} + e^{-2T}\, e^{-2sT} + \dots = \frac{1}{1-e^{-T}\, e^{-sT}}$$

$$X(z) = x(0) + x(T)\, z^{-1} + x(2T)\, z^{-2} + \dots$$

$$= 1 + e^{-T}\, z^{-1} + e^{-2T}\, z^{-2} + \dots = \frac{1}{1-e^{-T}\, z^{-1}}$$

Clearly $X(z) = X^*(s)\big|_{e^{sT} \to z}$

3.2.2. DAC Model

Data-Hold is a process/circuit that

• generates a continuous-time signal $h(t)$ from a discrete-time sequence $x(kT)$

The output *h(t)* of the data-hold process/circuit during $kT \leq t < (k+1)T$ can be given by a polynomial, in general, for $0 \leq \tau < T$,

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0 \qquad (3\text{-}5)$$

Note: In order to determine the $n+1$ coefficients $a_n, \cdots, a_0,$ we need $n+1$ data points:

$$x((k-n)T), \cdots, x((k-2)T), x((k-1)T), x(kT)$$

Furthermore, we must also have

$$h(kT) = x(kT)$$

Hence $a_0 = x(kT)$ and (3-5) can be written as

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT) \qquad (3\text{-}6)$$

This is called an nth-order hold.

When $n = 0$ in (3-6), we have zero-order hold:

$$h(kT + \tau) = x(kT) \qquad (3\text{-}7)$$

- ***Zero-Order Hold***

$$h(kT + \tau) = x(kT),\ 0 \leq \tau < T,\ k = 0, 1, 2, \cdots \quad (3\text{-}8)$$

That is, zero-order hold holds the value $x(kT)$ at $t = kT$ before the next sample at $t = (k + 1)T$ arrives.

Unless otherwise stated, we assume the hold circuit is of zero order.

Figure 3-3 shows a sampler and a zero-order hold, where the wave form of $h(t)$ takes a staircase form.

Zero order hold (ZOH) smoothes the sampled signal with constant (horizontal lines) in-between samples.

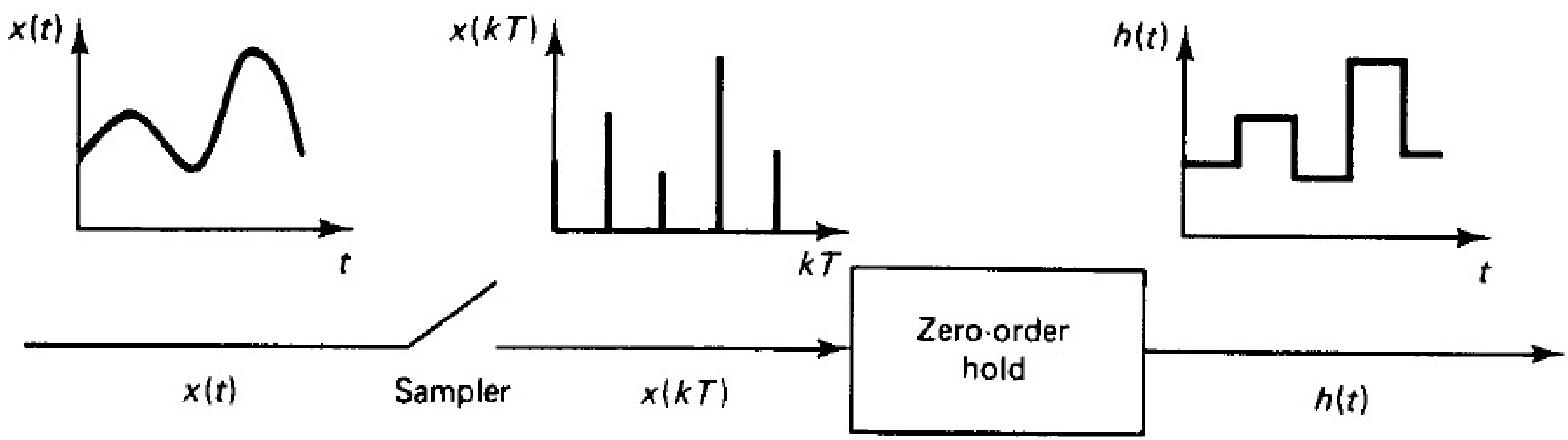


Figure 3–3 Sampler and zero-order hold.

- ***Obtain a Mathematical Model of the ZOH***

Figure 3-4(a) shows a real sampler and zero-order hold. A mathematical model is shown in Figure 3--4(b) where an impulse sampler is used to model the real sampler and $G_{h0}(s)$ is the transfer function of the ZOH.

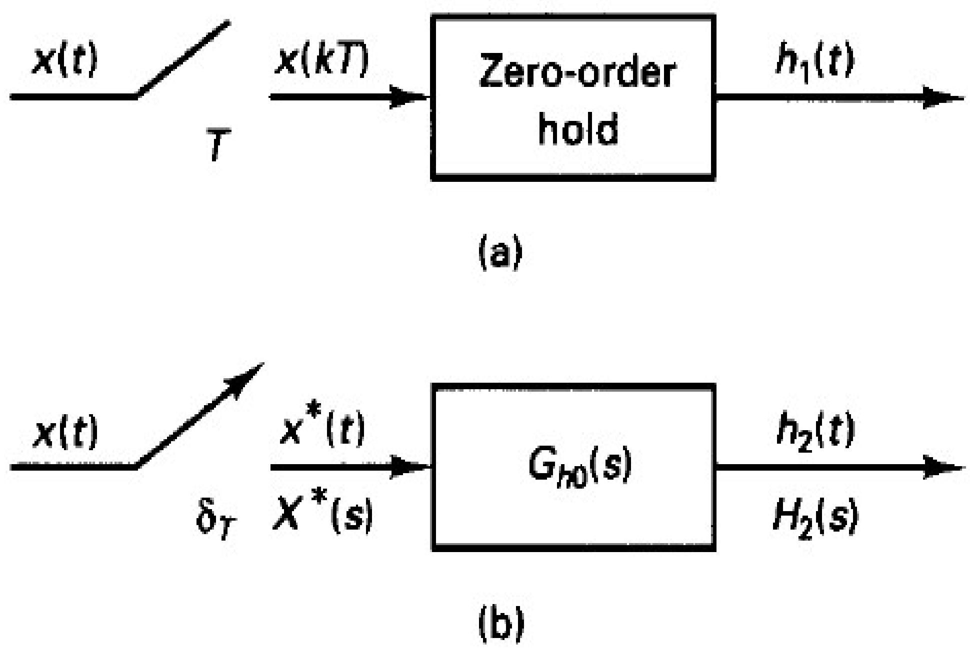


Figure 3–4 (a) A real sampler and zero-order hold; (b) mathematical model that consists of an impulse sampler and transfer function $G_{h0}(s)$.

Remark: The real sampler and the impulse sampler are distinguished by the arrow, as well as by the labeling of the signal (*x(kT)* vs *x*(t), X*(s)),* as shown in Figure 3-4

First, we consider the real sampler and ZOH in Figure 3-4(a).

If $x(t) = 0$ for $t < 0$, then the output $h_1(t)$ is given by

$$
\begin{aligned}
h_1(t) &= x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] \\
&\quad + x(2T)[1(t-2T) - 1(t-3T)] + \cdots \\
&= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)] \qquad (3\text{-}9)
\end{aligned}
$$

Since $$\mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

hence the Laplace Transform of (3-9) is

$$
\begin{aligned}
\mathcal{L}[h_1(t)] = H_1(s) &= \sum_{k=0}^{\infty} x(kT)\,\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} \\
&= \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs} \qquad (3\text{-}10)
\end{aligned}
$$

Next, we look at the mathematical model in Figure 3-4(b). The Laplace Transform of the output $h_2(t)$ is

$$\begin{aligned}\mathcal{L}[h_2(t)] &= H_2(s) = G_{h0}(s)\mathcal{L}[x^*(s)] = G_{h0}(s)X^*(s) \\ &= G_{h0}(s)\sum_{k=0}^{\infty} x(kT)e^{-kTs} \quad \text{according to (3-2)}\end{aligned}$$

As the real sampler with zero-order hold and the impulse sampler with zero-order hold should give the same output signal, we must have

$$h_1(t) = h_2(t), \qquad H_1(s) = H_2(s) \tag{3-11}$$

Hence, the transfer function of the zero-order hold is given by

$$G_{h0}(s) = \frac{1 - e^{-Ts}}{s} \tag{3-12}$$

In Summary,

- A real sampler samples the input signal $x(t)$ periodically (with period T) and produces a sequence of pulses $x(kT)$ as output.

- The width of the pulses is very small compared to the period T, but is nonetheless not zero in practice.

- These narrow pulses are abstracted as impulses $x(t)$ whose strengths are equal to the continuous-time signal $x(t)$ at $t = kT$.

So,

- We can replace the real sampler and zero-order hold with a mathematically equivalent continuous-time system that consists of an impulse sampler and a transfer function

$$\frac{1-e^{-Ts}}{s} \qquad (3\text{-}13)$$

- The real sampler and the impulse sampler are distinguished by the arrow, as well as by the labeling of the signal ($x(kT)$ vs $x^*(t)$, $X^*(s)$), as shown in Figure 3-4

- ***Transfer Function of First-Order Hold***

Although not used in practice, it is worthwhile to know that its transfer function is given by

$$G_{h1}(s) = \left(\frac{1 - e^{-Ts}}{s}\right)^2 \frac{Ts + 1}{T} \qquad (3\text{-}14)$$

This can be established as follows. From (3-5), we have

$$h(kT + \tau) = a_1\tau + x(kT), \qquad 0 \leq \tau < T, \quad k = 0, 1, 2, \cdots \quad (3\text{-}15)$$

By applying the condition that

$h((k-1)T) = x((k-1)T)$ as well as $h(kT) = x(kT)$

we have

$$h(kT + \tau)\,\big|_{\tau = -T} = -a_1 T + x(kT) \;=\; x((k-1)T)$$

(i.e., extending $h(kT + \tau)$ to $\tau = -T$ as shown in Figure 3-5.)

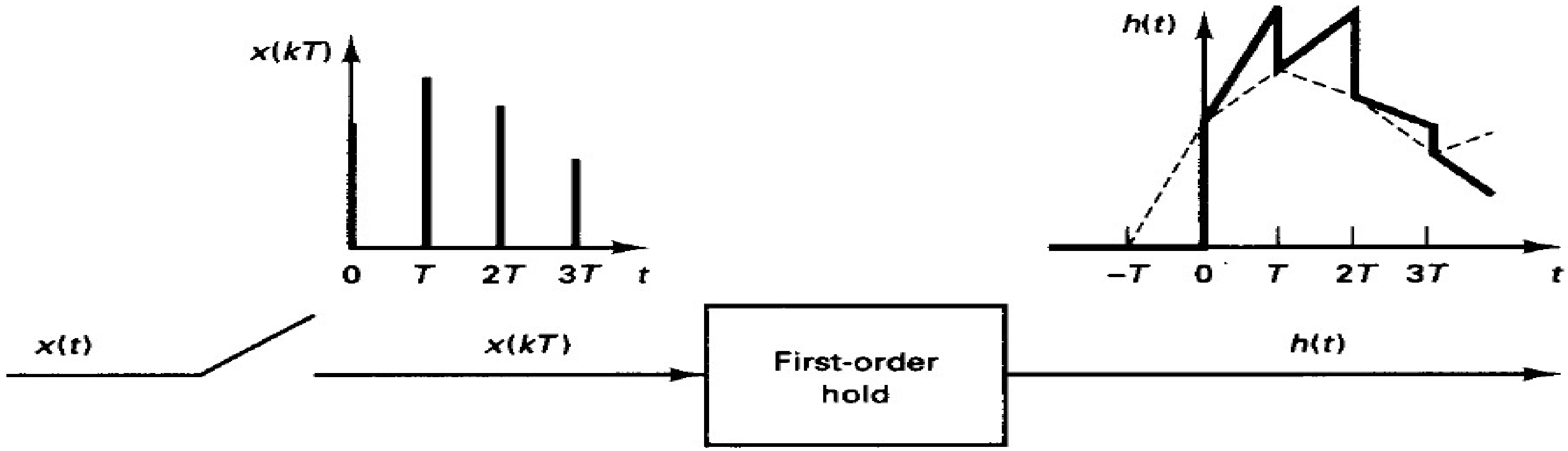


Figure 3–5 Input and output of a first-order hold.

Hence, we have $a_1 = \dfrac{x(kT) - x((k-1)T)}{T}$

and (3-15) becomes

$$h(kT + \tau) = x(kT) + \frac{x(kT) - x((k-1)T)}{T}\,\tau \qquad (3\text{-}16)$$

with $0 \le \tau < T$ and $k = 0, 1, 2, \cdots$. Putting in words, the $k+1$ value $x((k+1)T)$ is obtained by extending the straightline that joins the $(k-1)$th value $x((k-1)T)$ and the kth value $x(kT)$.

- ***Obtaining a Mathematical Model for FOH***

If we choose a unit step function as $x(t)$, then the output of the real sampler and first order hold is given in Figure 3-6(a)

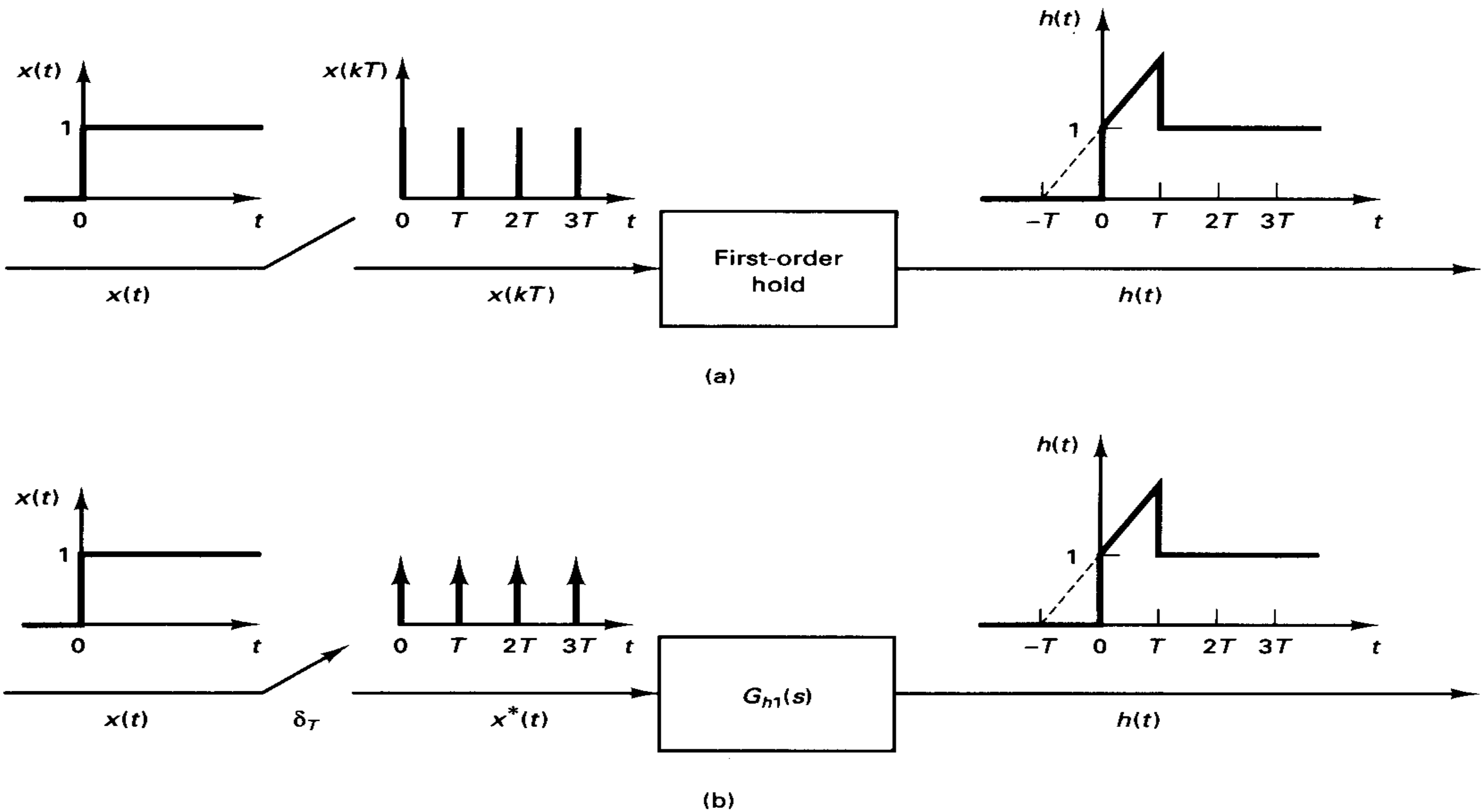


Figure 3–6 (a) Real sampler cascaded with first-order hold; (b) mathematical model consisting of impulse sampler and $G_{h1}(s)$.

i.e.,

$$\begin{aligned} h(t) &= \left(1+\frac{t}{T}\right)\,1(t)-\frac{t}{T}\,1(t-T) \\ &= \left(1+\frac{t}{T}\right)\,1(t)-\frac{t-T}{T}\,1(t-T)-1(t-T) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}[h(t)]=H(s) &= \left(\frac{1}{s}+\frac{1}{Ts^2}\right)-\frac{1}{Ts^2}e^{-Ts}-\frac{1}{s}e^{-Ts} \\ &= \frac{1-e^{-Ts}}{s}+\frac{1-e^{-Ts}}{Ts^2} \\ &= (1-e^{-Ts})\frac{Ts+1}{Ts^2} \qquad (3\text{-}17) \end{aligned}$$

From the impulse sampler and FOH, we have

$$H(s) = G_{h1}(s)X^*(s)$$

with

$$X^*(s) = \sum_{k=0}^{\infty} 1(kT)e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

Hence, comparing the above two expressions for $H(s)$, we have

$$G_{h1}(s) = \frac{H(s)}{X^*(s)} = (1-e^{-Ts})^2 \frac{Ts+1}{Ts^2}$$

i.e.,

$$G_{h1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}$$

3.3 THE PULSE TRANSFER FUNCTION

The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to the Laplace transform of the continuous-time input.

The pulse transfer function relates the $\mathcal{Z}$ transform of output at the sampling instants to the $\mathcal{Z}$ transform of the sampled inputs.

3.3.1. Convolution Summation Approach:

Consider the response of a continuous-time system driven by an impulse-sampled signal (a train of impulses) as shown in Figure 3-7.

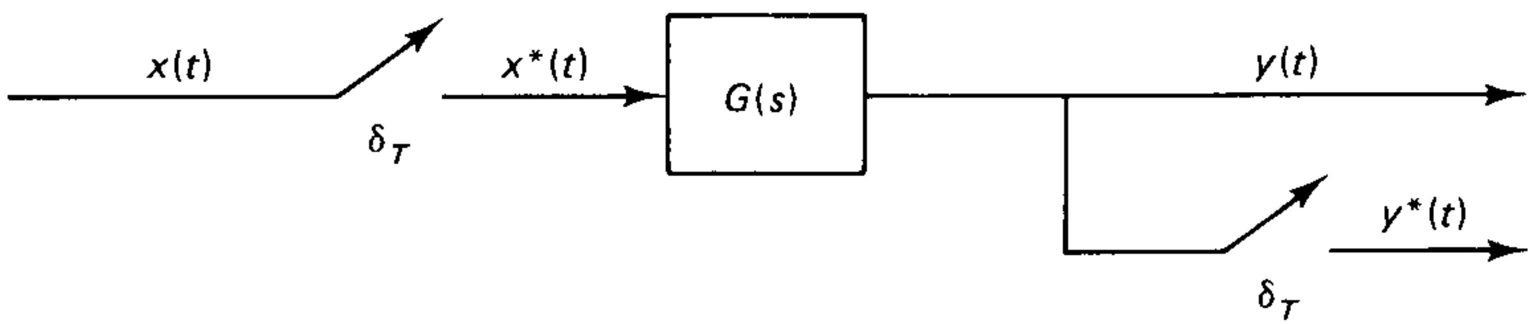


Figure 3.7

Suppose that x(t) = 0 for t < 0.

The impulse-sampled signal $x^*(t)$ is the input to the continuous-time system whose transfer function is $G(s)$.

The output of the system is a continuous-time signal $y(t)$.

If at the output there is another sampler, which is synchronized in phase with the input sampler and operates at the sample sampling period, then the output is a train of impulses.

Assume that $y(t) = 0$ for $t < 0$ (causality).

Then the $\mathcal{Z}$ transform of $y(t)$ is

$$\mathcal{Z}[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k} \qquad (3\text{-}38)$$

We observe the sequence of values taken by $y(t)$ only at instants $t = kT$.

It is also noted that the $\mathcal{Z}$ transform of the output $y^*(t)$ is also given by (3-38).

It is well known that the output $y(t)$ of a continuous-time system in response to an input $x^*(t)$ is given by

$$y(t) = \int_0^{\propto} g(t-\tau)x^*(\tau)d\tau = \int_0^{\propto} x^*(t-\tau)g(\tau)\,d\tau$$

where $g(t)$ is the impulse-response function of the system.

Note that

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

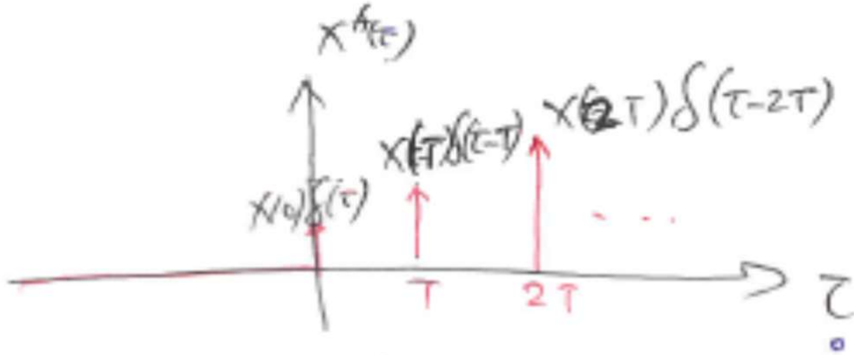
is a train of impulses. The output $y(t)$ is the sum of the individual impulse responses. Hence

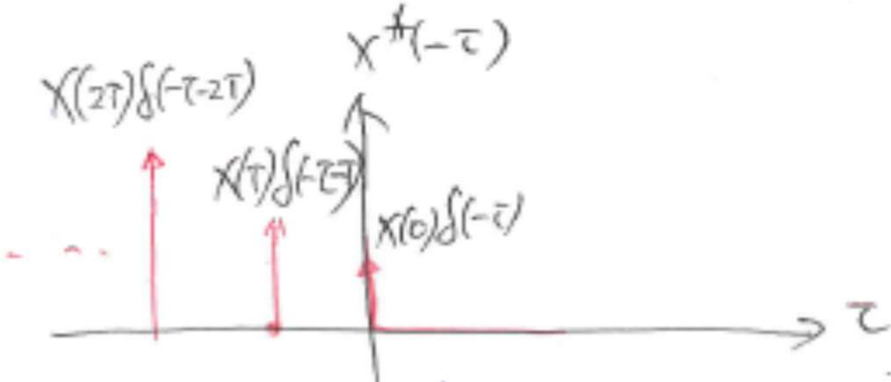
$$y(t) = g(t)x(0), \qquad 0 \le t < T$$

$$y(t) = g(t)x(0) + g(t-T)x(T), \qquad T \le t < 2T$$

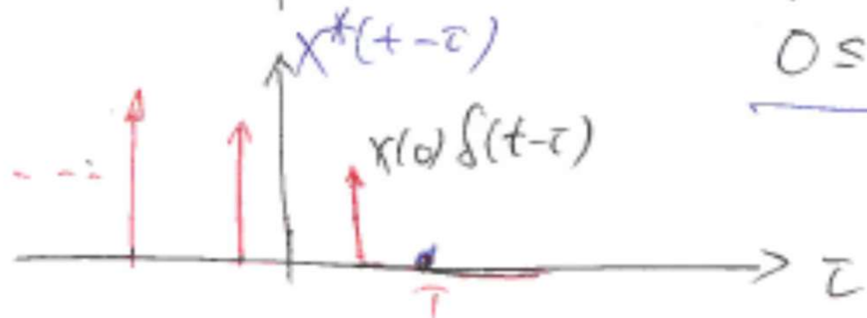
$$y(t) = g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T), \qquad 2T \le t < 3T,$$

$$y(t) = g(t)x(0) + g(t-T)x(T) + \cdots + g(t-kT)x(kT), \qquad kT \le t < (k+1)T$$



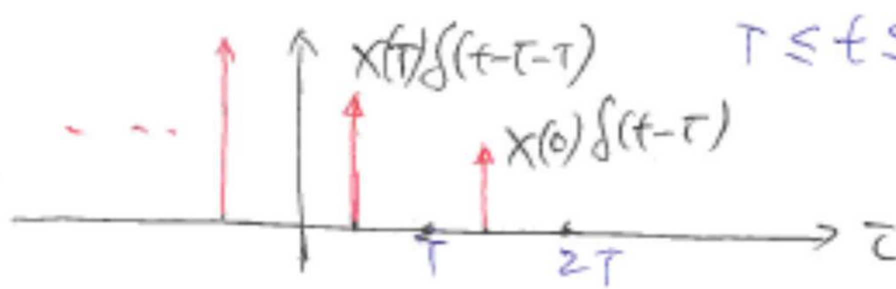


$$\delta(\tau) = \begin{cases} \infty & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$



$0 \le t \le T$

$$\int_0^\infty x(0)\,\delta(t-\tau)\,g(\tau)\,d\tau$$
$$= x(0)\int_0^\infty \delta(t-\tau)\,g(\tau)\,d\tau$$
$$= x(0)\,g(t)$$



$T \le t \le 2T$

$$\int_0^\infty x(0)\,\delta(t-\tau)\,g(\tau)\,d\tau$$
$$+ \int_0^\infty x(T)\,\delta(t-T-\tau)\,g(\tau)\,d\tau$$
$$= x(0)\,g(t) + x(T)\,g(t-T)$$

Noting that $g(t) = 0$ for $t < 0$ and hence $g(t - kT) = 0$ for $t < kT$ (causality), we have, in general

$$y(t) = \sum_{h=0}^{k} g(t - hT)x(hT), \qquad 0 \leq t < (k+1)T.$$

At sampling instants $t = kT$, we have

$$y(kT) = \sum_{h=0}^{k} g(kT - hT)x(hT) = \sum_{h=0}^{k} g(hT)x(kT - hT) \tag{3-39}$$

which are the convolution summation for discrete-time systems. In simplified notation, we write

$$y(kT) = x(kT) * g(kT) \tag{3-40}$$

Because of causality, $g(kT - hT) = 0$ for $h > k$. Then

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT) \tag{3-41}$$

$$= \sum_{h=0}^{\infty} g(hT)x(kT - hT) \tag{3-42}$$

Pulse Transfer Function. From (3-41), we have

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT), \qquad k = 0, 1, 2, \cdots$$

Hence, the $\mathcal{Z}$ transform of $y(kT)$ is

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty} y(kT)z^{-k} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} g(kT - hT)x(hT)z^{-k} \\ &= \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} g(mT)x(hT)z^{-(m+h)} \\ &= \sum_{m=0}^{\infty} g(mT)z^{-m} \sum_{h=0}^{\infty} x(hT)z^{-h} \end{aligned}$$

where $m = k - h$. Note that

$$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m}, \quad \text{and} \quad X(z) = \sum_{h=0}^{\infty} x(hT)z^{-h}$$

$$Y(z) = G(z)X(z) \qquad (3\text{-}43)$$

Hence

$$G(z) = \frac{Y(z)}{X(z)} \tag{3-44}$$

which is a rational function of z and is called the pulse transfer function.

Figure 3-8 shows a block diagram for a pulse transfer function *G(z)* with input *X(z)* and output *Y(z)*.

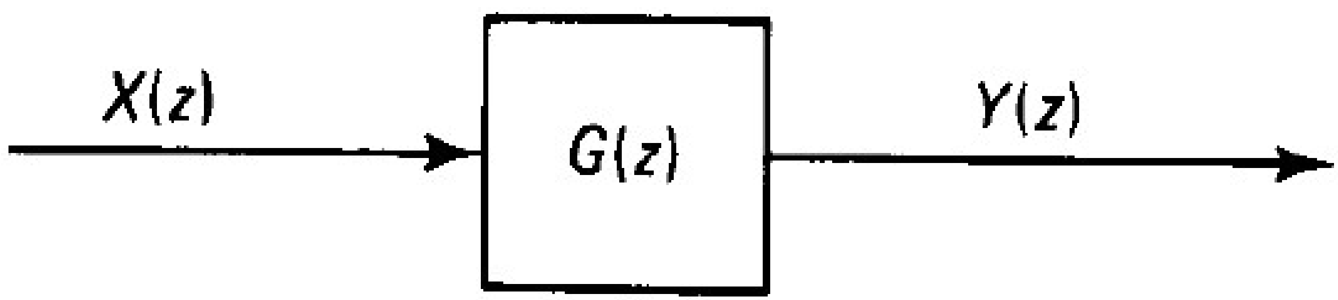


Figure 3.8 Block diagram for a pulse-transfer-function system

3.4.2 Starred Laplace Transform

In analyzing discrete-time control systems, often find mixed type of signals:

- continuous-time signals
- starred signals (i.e., impulse-sampled signals)

Consider the impulse-sampled system in Figure 3-11. Assume all initial conditions are zero.

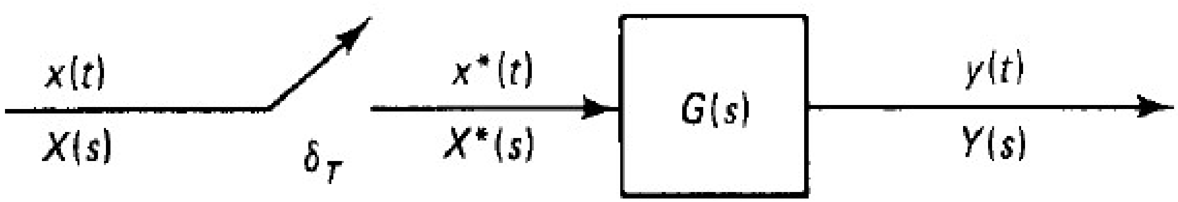


Figure 3.11

The output

$$Y(s) = G(s)X^*(s) \qquad (3\text{-}45)$$

$$Y^*(s) = [G(s)X^*(s)]^* \qquad (3.46)$$

On the other hand, note that with $e^{Ts} = z$ or $s = \frac{1}{T}\ln z$

$Y(z) = Y^*(s)|_{\frac{1}{T}lnz}$, $X(z) = X^*(s)|_{\frac{1}{T}\ln z}$, $G(z) = G^*(s)|_{\frac{1}{T}\ln z}$

Then from (3-43) we have

$$Y^*(s) = G^*(s)X^*(s) \tag{3-47}$$

Based on (3-46) and (3-47),

$$[G(s)X^*(s)]^* = G^*(s)X^*(s) \tag{3.47a}$$

Thus when taking the starred Laplace transform of *Y (s)* as in (3-46), *X*(s)* can be factored out.

(3.47a) is important to determine starred Laplace transform. Note that the relationship in (3-47) is also reversible to have (3-43). This gives an easy way to obtain pulse transfer function, as illustrated later.

Example 3-2:

In Fig 3.11, suppose that the sampling period is T and $G(s)=\frac{1}{s+1}$. If $x(t)=1(t)$, verify that $[G(s)X^*(s)]^*=G^*(s)X^*(s)$.

Solution:

Note that Y(s)= $G(s)X^*(s)$. Thus Y*(s)=$[G(s)X^*(s)]^*$

From (3-43), Y(z)=G(z)X(z)= $\frac{1}{1-e^{-T}z^{-1}}\frac{1}{1-z^{-1}}$.

Then Y*(s)= $Y(z)|_{z\to e^{sT}}=\frac{1}{1-e^{-T}e^{-sT}}\frac{1}{1-e^{-sT}}$

On the other hand, $g(t)=L^{-1}\{G(s)\}=e^{-t}$

$$g^*(t)=g(0)\delta(t)+g(T)\delta(t-T)+g(2T)\delta(t-2T)+\ldots$$
$$=\delta(t)+e^{-T}\delta(t-T)+e^{-2T}\delta(t-2T)+\ldots$$

$$G^*(s)=1+e^{-T}e^{-sT}+e^{-2T}e^{-2sT}+\ldots=\frac{1}{1-e^{-T}e^{-sT}}$$

Also

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \ldots$$
$$= \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots$$

$$X^*(s) = 1 + e^{-sT} + e^{-2sT} + \ldots = \frac{1}{1-e^{-sT}}$$

Thus $G^*(s)X^*(s) = \frac{1}{1-e^{-T}e^{-sT}} \frac{1}{1-e^{-sT}}$

Clearly $[G(s)X^*(s)]^* = G^*(s)X^*(s)$

3.3.3. General Procedure of Obtaining Pulse Transfer Functions

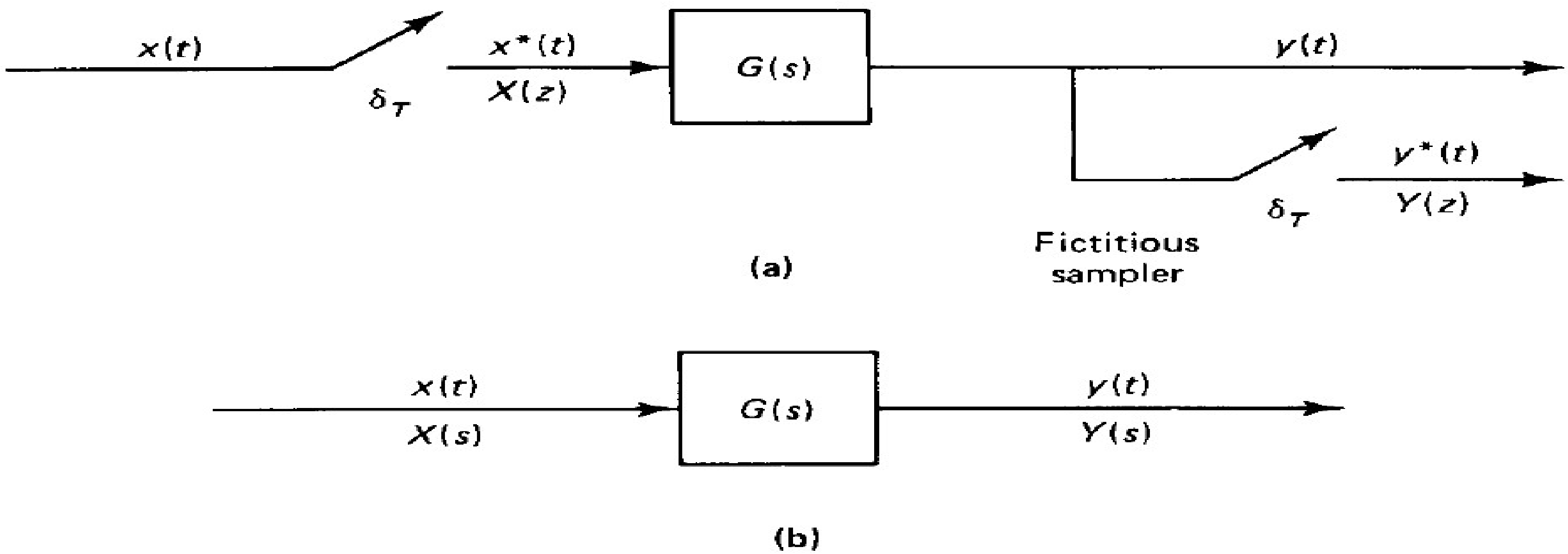


Figure 3.10 (a) Continuous-time system with an impulse sampler at the input; (b) Continuous-time system

For Figure 3-10(a), input sampler is present. Then

$$Y(s) = G(s)X^*(s)$$

So $Y^*(s) = G^*(s)X^*(s)$ and hence $Y(z) = G(z)X(z)$

For Figure 3-10(b) without input sampler,

$$Y(s) = G(s)X(s)$$

$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$$

where $G(s)$ and $X(s)$ cannot be separated! i.e.,

$$Y(z) = \mathcal{Z}[Y(s)] = \mathcal{Z}[G(s)X(s)] = \mathcal{Z}[GX(s)] = GX(z) \neq G(z)X(z)$$

Important:

The pulse transfer function for the system in Figure 3-10(b) is ***NOT*** $\mathcal{Z}[G(s)]$, because of the absence of the input sampler!

*Therefore, the presence or absence of the **input sampler** is **crucial.*** But the presence or absence of ***a sampler at the output*** of the element (or the system) ***does not affect*** the pulse transfer function.

Example 3-3**:** Obtain the pulse transfer function $G(z)$ of the system shown in Figure 3-12(a) with

$$G(s) = \frac{1}{s+a}$$

Also obtain *Y(z)* in Figure 3-12(b) if the input *x(t)=1(t).*

Noting the presence of the input sampler, we have

$$G(z) = \mathcal{Z}[G(s)]$$

By referring to Table 2-1, we have

$$\mathcal{Z}\left[\frac{1}{s+a}\right] = \frac{1}{1-e^{-aT}z^{-1}}$$

Hence

$$G(z) = \frac{1}{1-e^{-aT}z^{-1}}$$

For Figure 3-12(b),

$$Y(s) = G(s)X(s) = \frac{1}{s+a}\,\frac{1}{s}$$

Thus

$$Y(z) = \mathcal{Z}[Y(s)]$$

$$= \mathcal{Z}\left[\frac{1}{s+a}\,\frac{1}{s}\right]$$

$$= \frac{1}{a}\,\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$$

3.3.4 Pulse Transfer Function of DAC and Analog System

Recall the block diagram of a discrete-time control system shown in the figure on Page 3.

We will find the pulse transfer function of the following part of the system as in Figure 3.11

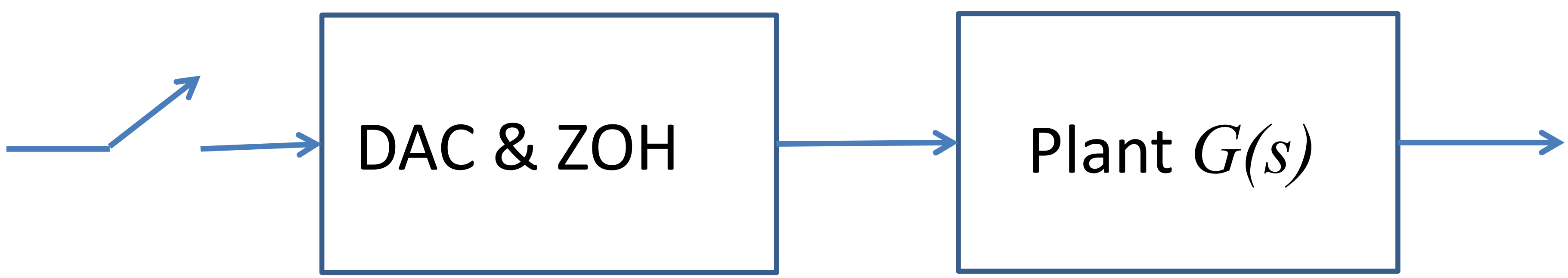


Figure 3.11

Let $X(s)$ be its Laplace transfer function, i.e.

$$X(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

with $G(s)$ a proper rational function of s. This can be written as

$$X(s) = (1 - e^{-Ts}) \frac{G(s)}{s} \triangleq (1 - e^{-Ts}) G_1(s)$$

i.e.,

$$X(s) = G_1(s) - e^{-Ts} G_1(s) \qquad (3\text{-}48)$$

Suppose

$$g_1(t) = \mathcal{L}^{-1}\left[G_1(s)\right]$$

$$x_1(t) = \mathcal{L}^{-1}\left[e^{-Ts}G_1(s)\right]$$
$$= g_1(t-T)$$

The $\mathcal{Z}$ transform of $x_1(t)$ becomes

$$\mathcal{Z}\left[x_1(t)\right] = \mathcal{Z}\left[g_1(t-T)\right] = z^{-1}\mathcal{Z}\left[g_1(t)\right]$$

Using (3-48)

$$X(z) = \mathcal{Z}\left[X(s)\right] = \mathcal{Z}\left[G_1(s) - e^{-Ts}G_1(s)\right]$$
$$= \mathcal{Z}\left[g_1(t)\right] - \mathcal{Z}\left[x_1(t)\right]$$
$$= G_1(z) - z^{-1}G_1(z) = (1 - z^{-1})G_1(z)$$

Thus in conclusion, for

$$X(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

we have

$$X(z) = \mathcal{Z}[X(s)] = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right] \qquad (3\text{-}49)$$

So, $1 - e^{-Ts}$ in $X(s)$ can simply be replaced by $1 - z^{-1}$.

Example 3-4: Obtain the pulse transfer function of the system shown in Figure 3-10(a) with

$$G(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}$$

Note that there is an input sampler in Figure 3-10(a) and a zero-order hold in *G(s)* (at the output).

Using (3-49), we have

$$G(z) = \mathcal{Z}[G(s)] = \mathcal{Z}\left[(1-e^{-Ts})\frac{1}{s^2(s+1)}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

From Table 2-1. we have

$$G(z) = (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

$$= \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})} \qquad (3\text{-}50)$$

3.3.5 Pulse Transfer Function of a Digital Controller

Suppose the input to the digital controller, for example in Figure 3-15(a), is $e(k)$ and the output is $m(k)$. In general, we have

$$m(k) + a_1 m(k-1) + a_2 m(k-2) + \cdots + a_n m(k-n)$$
$$= b_0 e(k) + b_1 e(k-1) + \cdots + b_n e(k-n) \qquad (3\text{-}51)$$

The $\mathcal{Z}$ transform of (3-51) is

$$M(z) + a_1 z^{-1} M(z) + a_2 z^{-2} M(z) + \cdots + a_n z^{-n} M(z)$$
$$= b_0 E(z) + b_1 z^{-1} E(z) + \cdots + b_n z^{-n} E(z)$$

or

$$(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}) M(z) = (b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}) E(z)$$

and the pulse transfer function is

$$G_D(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \qquad (3\text{-}52)$$

3.3.6 Pulse Transfer Function of Cascade Elements

Consider the system shown in Figure 3-12.

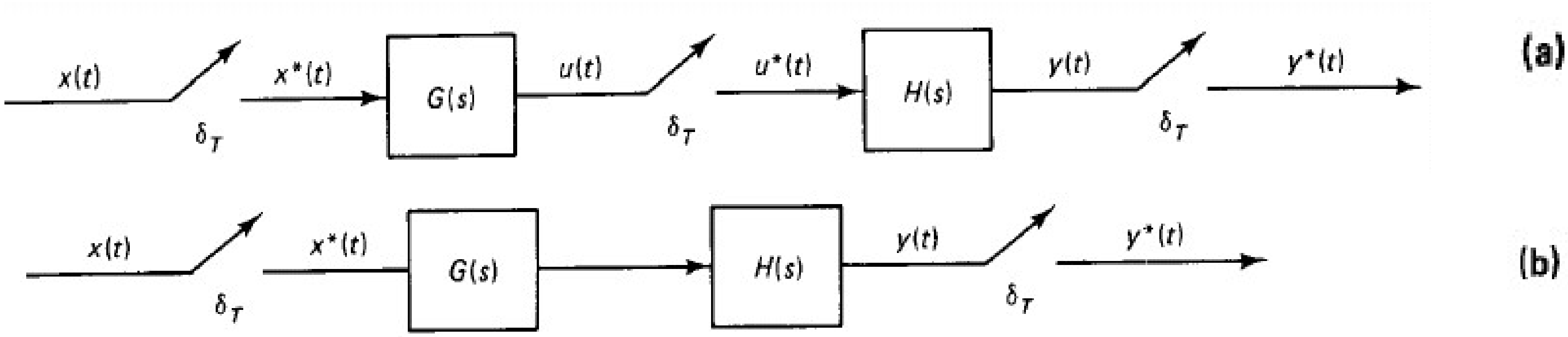


Figure 3-12

Assume that all samplers are synchronized and have the same sampling period T_s.

As we will see, the pulse transfer functions are given by

- Figure 3-12(a): $G(z)H(z)$
- Figure 3-12(b): $\mathcal{Z}[G(s)H(s)] = \mathcal{Z}[GH(s)] = GH(z) \neq G(z)H(z)$

Consider Figure 3-12(a). We have

$$U(s) = G(s)X^*(s), \; Y(s) = H(s)U^*(s)$$

By taking the starred Laplace transform, we have

$$U^*(s) = G^*(s)X^*(s), \; Y^*(s) = H^*(s)U^*(s)$$

Consequently, $Y^*(s) = H^*(s)U^*(s) = H^*(s)G^*(s)X^*(s)$

In terms of the $\mathcal{Z}$ transform, we have

$$Y(z) = G(z)H(z)X(z)$$

The pulse transfer function from $x^*(t)$ to $y^*(t)$ is

$$\frac{Y(z)}{X(z)} = G(z)H(z)$$

Next, consider Figure 3-12(b).

$$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$$

where $GH(s) = G(s)H(s)$.

Taking the starred Laplace transform, we have

$$Y^*(s) = [GH(s)]^*X^*(s)$$

In terms of $\mathcal{Z}$ transform, we have

$$Y(z) = GH(z)X(z)$$

and the pulse transfer function is

$$\frac{Y(z)}{X(z)} = GH(z) = \mathcal{Z}[GH(s)]$$

Note that

$$G(z)H(z) \neq GH(z) = \mathcal{Z}[GH(s)]$$

Example 3-5: Consider the systems in Figures 3-13(a) and 3-13(b). Obtain the pulse transfer functions.

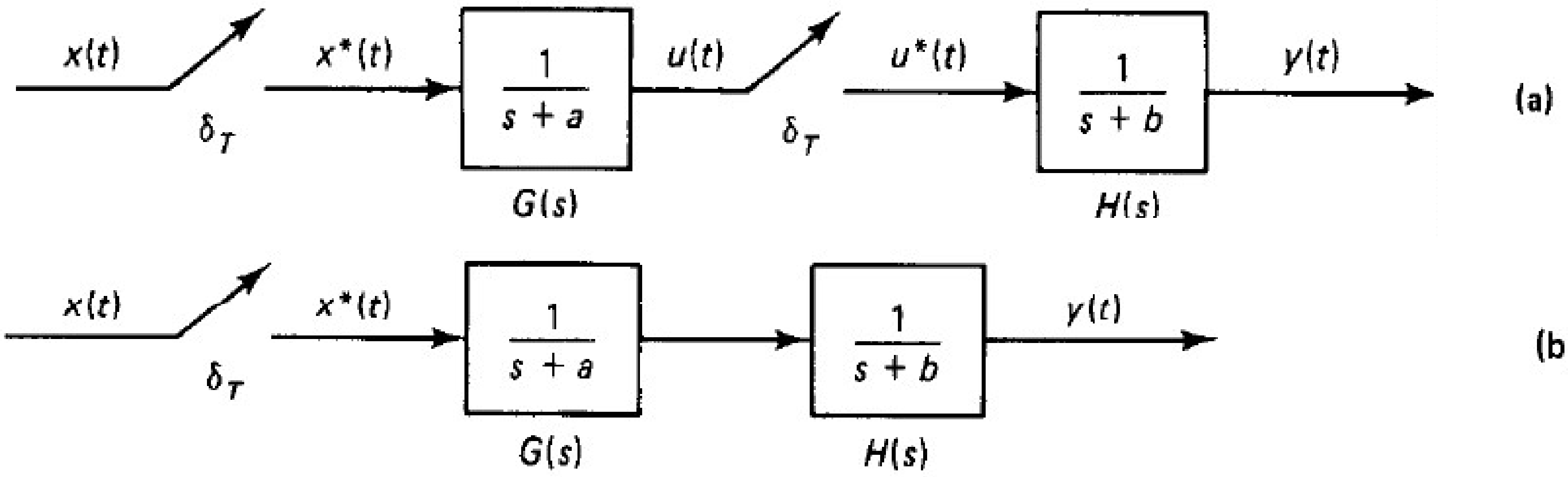


Figure 3-13

For Figure 3-13(a),

$$\frac{Y(z)}{X(z)} = G(z)H(z) = \mathcal{Z}\left[\frac{1}{s+a}\right]\mathcal{Z}\left[\frac{1}{s+b}\right]$$

$$= \frac{1}{1-e^{-aT}z^{-1}}\frac{1}{1-e^{-bT}z^{-1}}$$

For Figure 3-13(b),

$$\begin{aligned}\frac{Y(z)}{X(z)} &= \mathcal{Z}[G(s)H(s)] = \mathcal{Z}\left[\frac{1}{s+a}\frac{1}{s+b}\right] \\ &= \mathcal{Z}\left[\frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)\right] \\ &= \frac{1}{b-a}\left(\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right)\end{aligned}$$

Hence,

$$\frac{Y(z)}{X(z)} = GH(z) = \frac{1}{b-a}\left[\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}\right]$$

Clearly,

$$G(z)H(z) \neq GH(z)$$

Therefore, we must be careful to observe whether or not there is a sampler between cascaded elements.

3.3.7 Pulse Transfer Function of Closed-loop Systems

Consider the system in Figure 3-14.

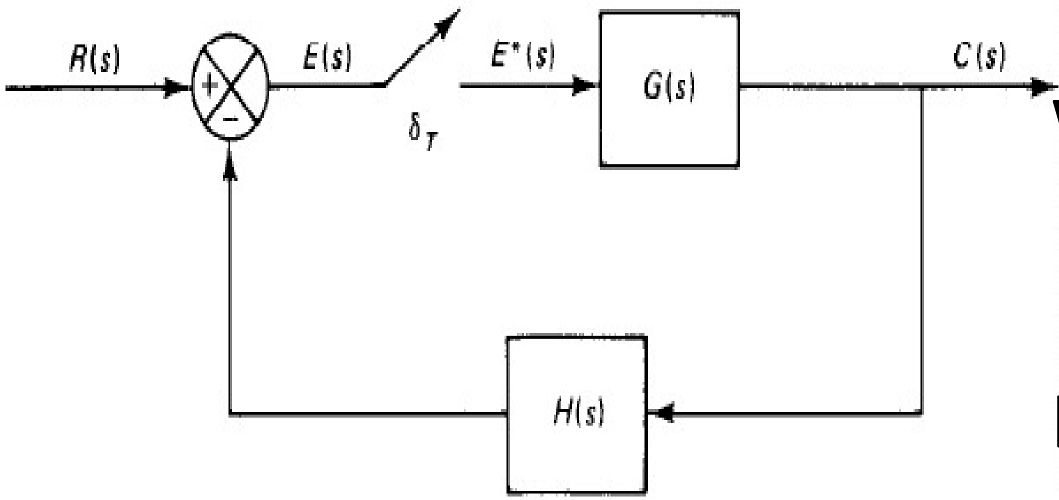


Figure 3-14

We have

$$E(s) = R(s) - H(s)C(s)$$

$$C(s) = G(s)E^*(s)$$

Hence

$$E(s) = R(s) - H(s)G(s)E^*(s)$$

Taking the starred Laplace transform, we have

$$E^*(s) = R^*(s) - GH^*(s)E^*(s) \quad \Rightarrow \quad E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

Also $C^*(s) = G^*(s)E^*(s)$ $\quad \Rightarrow \quad C^*(s) = \dfrac{G^*(s)R^*(s)}{1 + GH^*(s)}$

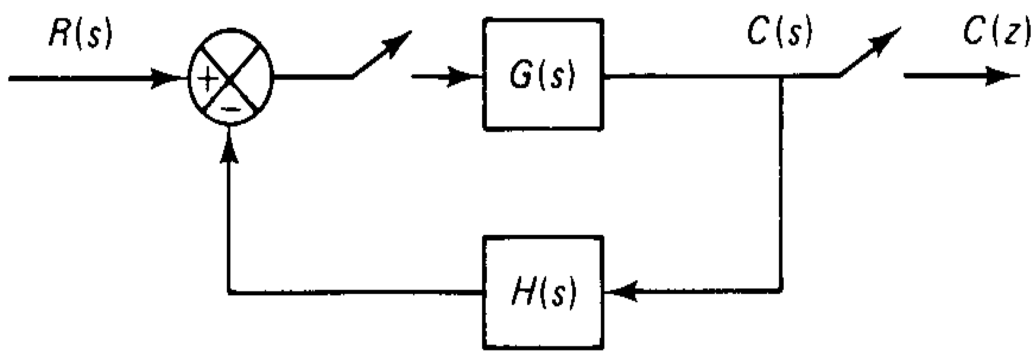
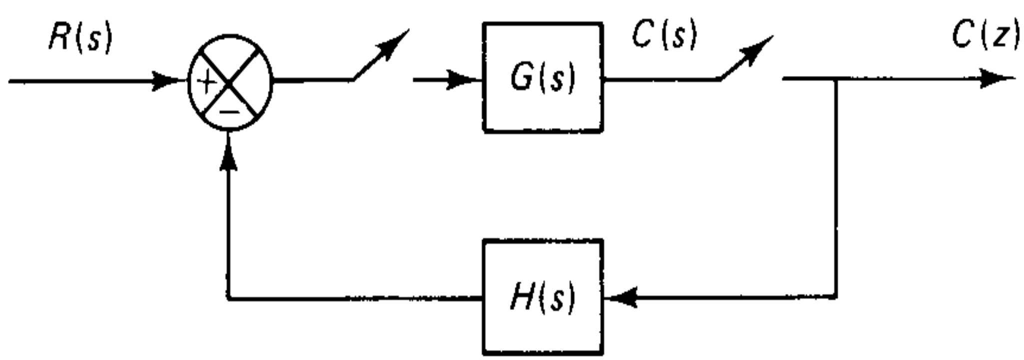
In terms of $\mathcal{Z}$ transform, we have $C(z) = \dfrac{G(z)R(z)}{1 + GH(z)}$

The pulse transfer function is given by

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

Table 3-1 shows five typical configurations for closed-loop discrete-time control systems. All samplers are assumed synchronized and have the same sampling period T_s.

TABLE 3–1 FIVE TYPICAL CONFIGURATIONS FOR CLOSED-LOOP DISCRETE-TIME CONTROL SYSTEMS

<table>
<tr>
<td>
</td>
<td>$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$</td>
</tr>
<tr>
<td>
</td>
<td>$$C(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$$</td>
</tr>
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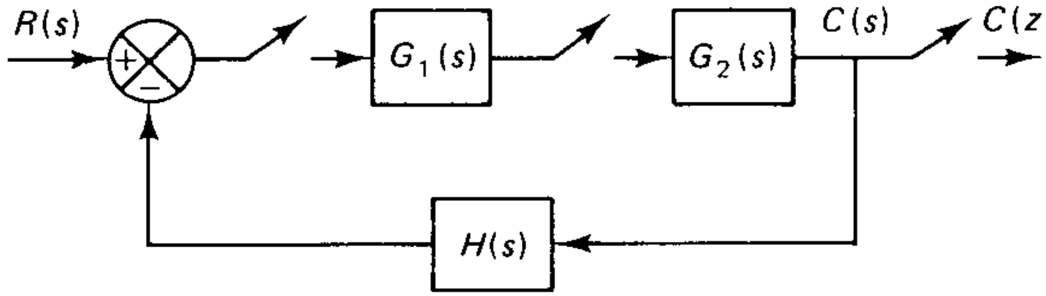
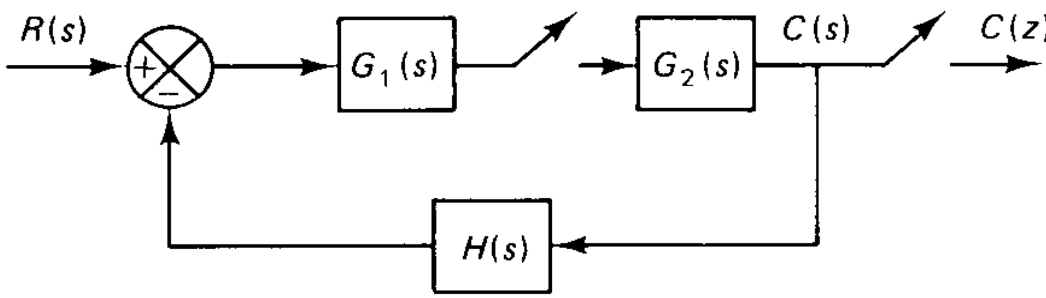
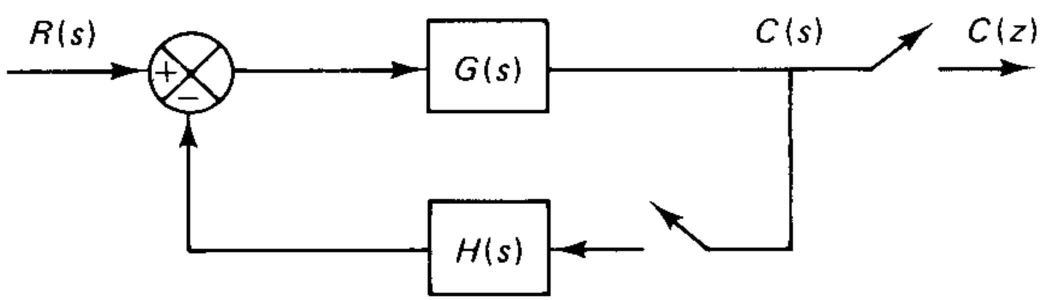


$$C(z) = \frac{G_1(z)G_2(z)R(z)}{1 + G_1(z)G_2H(z)}$$



$$C(z) = \frac{G_2(z)G_1R(z)}{1 + G_1G_2H(z)}$$



$$C(z) = \frac{GR(z)}{1 + GH(z)}$$

Note that, some discrete-time closed-loop control systems cannot be represented by $C(z)/R(z)$ because the input signal $R(s)$ cannot be separated from the system dynamics (as in Case 4 of Table 3-1).

In this case, they do not have a pulse transfer function.

Although the pulse transfer function may not exist for certain system configurations, the same technique discussed in this chapter can still be applied for analyzing them.

3.3.8 Closed-loop Pulse Transfer Function of a Digital Control System

Figure 3-15 shows a block diagram of a digital control system where the controller is $G^*_D(s)$.

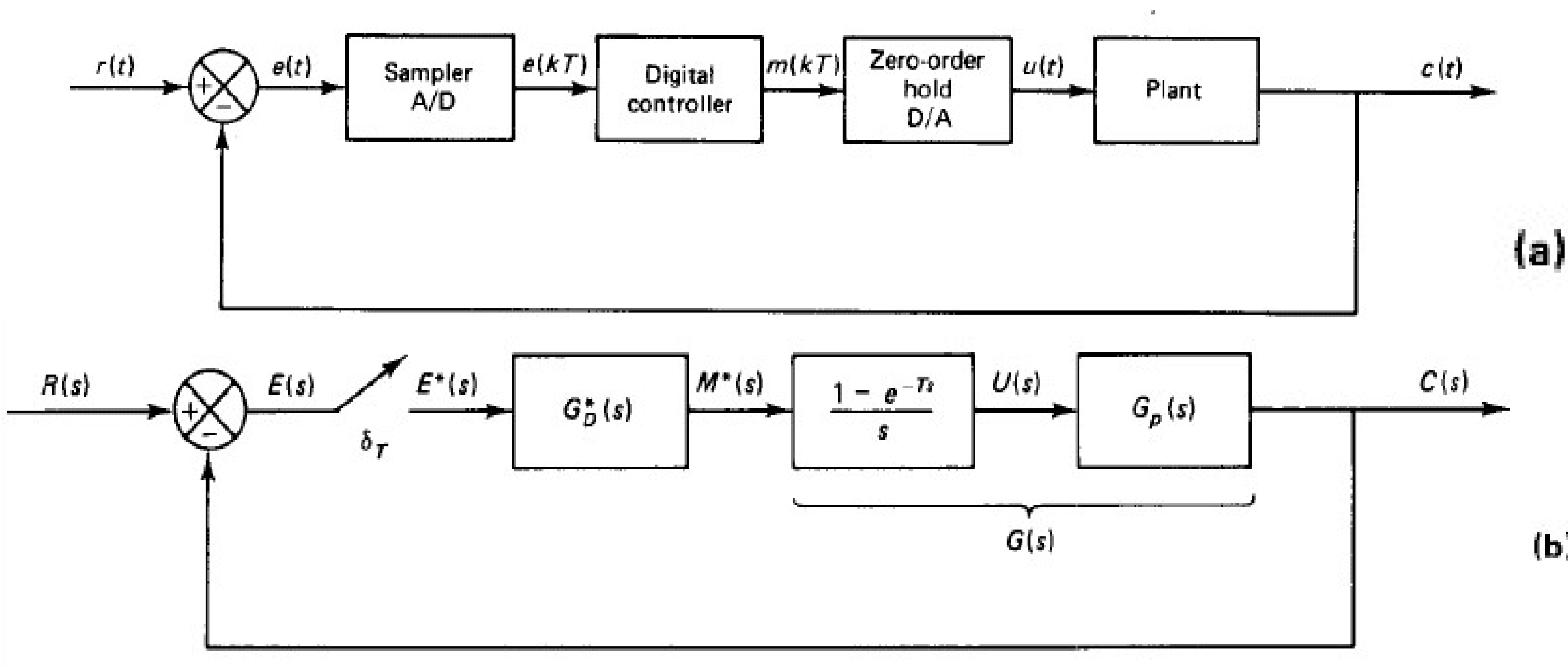


Figure 3-15 (a) Block diagram of a digital control system; (b) equivalent block diagram showing transfer functions of blocks

In the actual system, $G_D(z)$ is implemented by a digital computer that solves the corresponding difference equation.

Referring to Figure 3-15(b), let

$$\frac{1 - e^{sT}}{s} G_p(s) = G(s)$$

then

$$C(s) = G(s)G^*_D(s)E^*(s)$$

or

$$C^*(s) = G^*(s)G^*_D(s)E^*(s)$$

In terms of $\mathcal{Z}$ transform,

$$C(z) = G(z)G_D(z)E(z)$$

Since

$$E(z) = R(z) - C(z)$$

we have

$$C(z) = G_D(z)G(z)[R(z) - C(z)]$$

and therefore,

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} \qquad (3\text{-}53)$$

$G_D(z)$ can be designed so that the digital control system in Figure 3-15 has desired characteristics.

The design of $G_D(z)$ will be discussed further in Part II of this course.

Next, we study ways to discretize analog (continuous-time) PID controllers.

3.4 Pulse Transfer Function of a Digital PID Controller

The PID control action in analog controllers is given by

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right] \qquad (3\text{-}54)$$

where

e(t) is the input to the controller (the actuating error signal),

m(t) is the output of the controller,

K is the proportional gain,

T_i is the integral time (or reset time),

and T_d is the derivative time (or rate time).

By approximating the integral term with the trapezoidal summation and the derivative term with a two-point difference, we obtain

$$\begin{aligned} m(kT) &= K\left\{e(kT) + \frac{T}{T_i}\left[\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \right.\right. \\ &\qquad \left.\left.\frac{e((k-1)T)+e(kT)}{2}\right] + T_d\frac{e(kT)-e((k-1)T)}{T}\right\} \end{aligned}$$

or

$$\begin{aligned} m(kT) = K\Bigg\{e(kT) + \frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2} \\ + T_d\frac{e(kT)-e((k-1)T)}{T}\Bigg\} \qquad (3\text{-}55) \end{aligned}$$

Define

$$\frac{e((h-1)T)+e(hT)}{2} = f(hT), \qquad f(0)=0$$

Figure 3-16 shows the function $f(hT)$. Then

$$\sum_{h=1}^{k} \frac{e((h-1)T)+e(hT)}{2} = \sum_{h=1}^{k} f(hT) = \sum_{h=0}^{k} f(hT) \qquad (*)$$

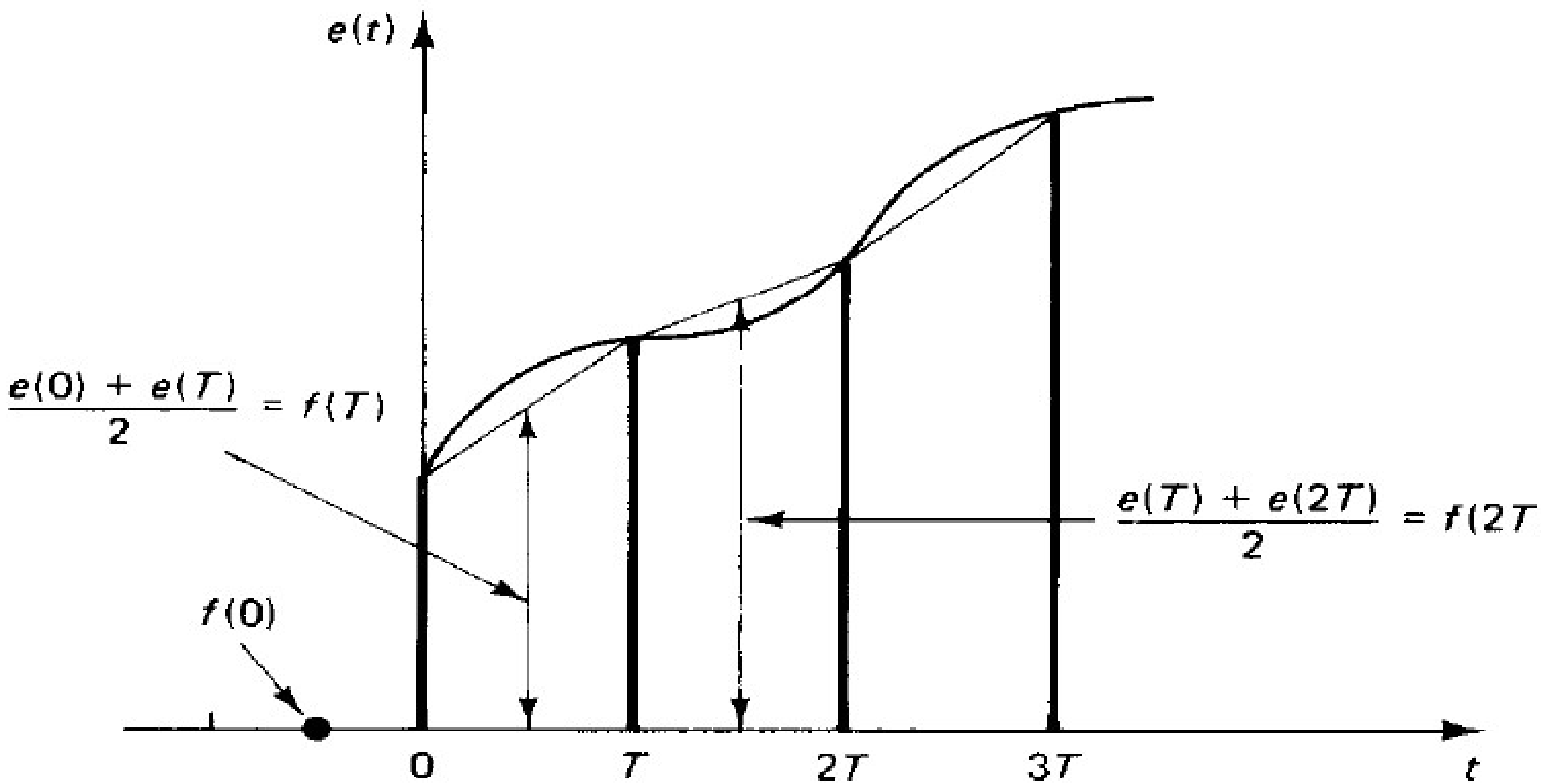


Figure 3-16: function *f(hT)*.

Let $y(k) = \sum_{h=0}^{k} f(hT)$. Then

$$y(k) - y(k-1) = f(kT)$$

Hence

$$Y(z) - z^{-1}Y(z) = F(z) \quad \Rightarrow \quad Y(z) = \mathcal{Z}\left[\sum_{h=0}^{k} f(hT)\right] = \frac{F(z)}{1 - z^{-1}}$$

with

$$F(z) = \mathcal{Z}\,[f(kT)] \quad = \mathcal{Z}\left[\frac{e((k-1)T + e(kT)}{2}\right] \quad = \frac{1 + z^{-1}}{2}E(z)$$

i.e.,

$$\mathcal{Z}\left[\sum_{h=1}^{k} \frac{e((h-1)T) + e(hT)}{2}\right] = \frac{F(z)}{1 - z^{-1}} = \frac{1 + z^{-1}}{2(1 - z^{-1})}E(z)$$

Taking the $\mathcal{Z}$ transform of (3-55), we have

$$M(z) = K\left[1 + \frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right]E(z)$$

which could be rewritten as

$$\begin{aligned} M(z) &= K\left[1 - \frac{T}{2T_i} + \frac{T}{T_i}\frac{1}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right]E(z) \\ &= \left[K_P + \frac{K_I}{1-z^{-1}} + K_D(1-z^{-1})\right]E(z) \end{aligned}$$

where

$$K_P = K - \frac{KT}{2T_i} = K - \frac{K_I}{2} = \text{proportional gain} < K$$

$$K_I = \frac{KT}{T_i} = \text{integral gain}$$

$$K_D = \frac{KT_d}{T} = \text{derivative gain}$$

The pulse transfer function is

$$G_D(z) = \frac{M(z)}{E(z)} = K_P + \frac{K_I}{1 - z^{-1}} + K_D(1 - z^{-1})$$

Example 3-6: Consider the control system with a digital PID controller shown in Figure 3-17(a).

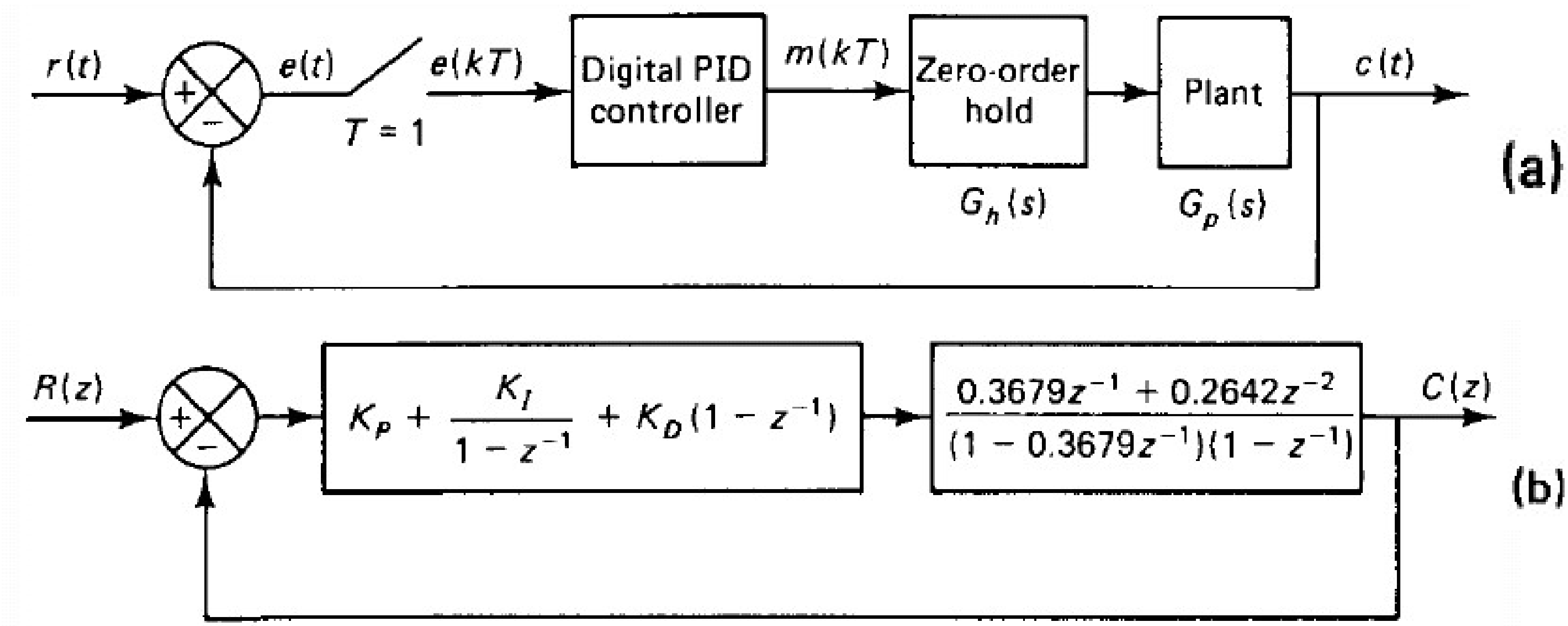


Figure 3-17 (a) Block diagram (b) equivalent block diagram

The transfer function of the plant is

$$G_p(s) = \frac{1}{s(s+1)}$$

The sampling period $T = 1$ sec and the zero-order hold is

$$G_h(s) = \frac{1-e^{-sT}}{s} = \frac{1-e^{-s}}{s}$$

Since

$$\mathcal{Z}\left[\frac{1-e^{-s}}{s}\frac{1}{s(s+1)}\right] = G(z) = \frac{0.3679z^{-1} + 0.2642z^{-2}}{(1-0.3679z^{-1})(1-z^{-1})} \quad (3\text{-}58)$$

we may redraw the block diagram as in Figure 3-17(b).

Assume that $K_P = 1$, $K_I = 0.2$, and $K_D = 0.2$. Then the pulse transfer function of the digital controller is

$$G_D(z) = \frac{1.4 - 1.4z^{-1} + 0.2z^{-2}}{1 - z^{-1}}$$

and the closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

$$= \frac{0.5151z^{-1} - 0.1452z^{-2} - 0.2963z^{-3} + 0.0528z^{-4}}{1 - 1.8528z^{-1} + 1.5906z^{-2} - 0.6642z^{-3} + 0.0528z^{-4}}$$

Assuming unit step external input $r(k) = 1$ for $k = 0, 1, 2, \cdots$, we obtain the unit step response shown in Figure 3.18.

MATLAB Program 3–1

```
% --------- Unit-step response ----------

num = [0  0.5151  -0.1452  -0.2963  0.0528];
den = [1  -1.8528  1.5906  -0.6642  0.0528];
r = ones(1,41);
k = 0:40;
c = filter(num,den,r);
plot(k,c,'o',k,c,'-')
v = [0  40  0  2];
axis(v);
grid
title('Unit-Step Response')
xlabel('k')
ylabel('c(k)')
```

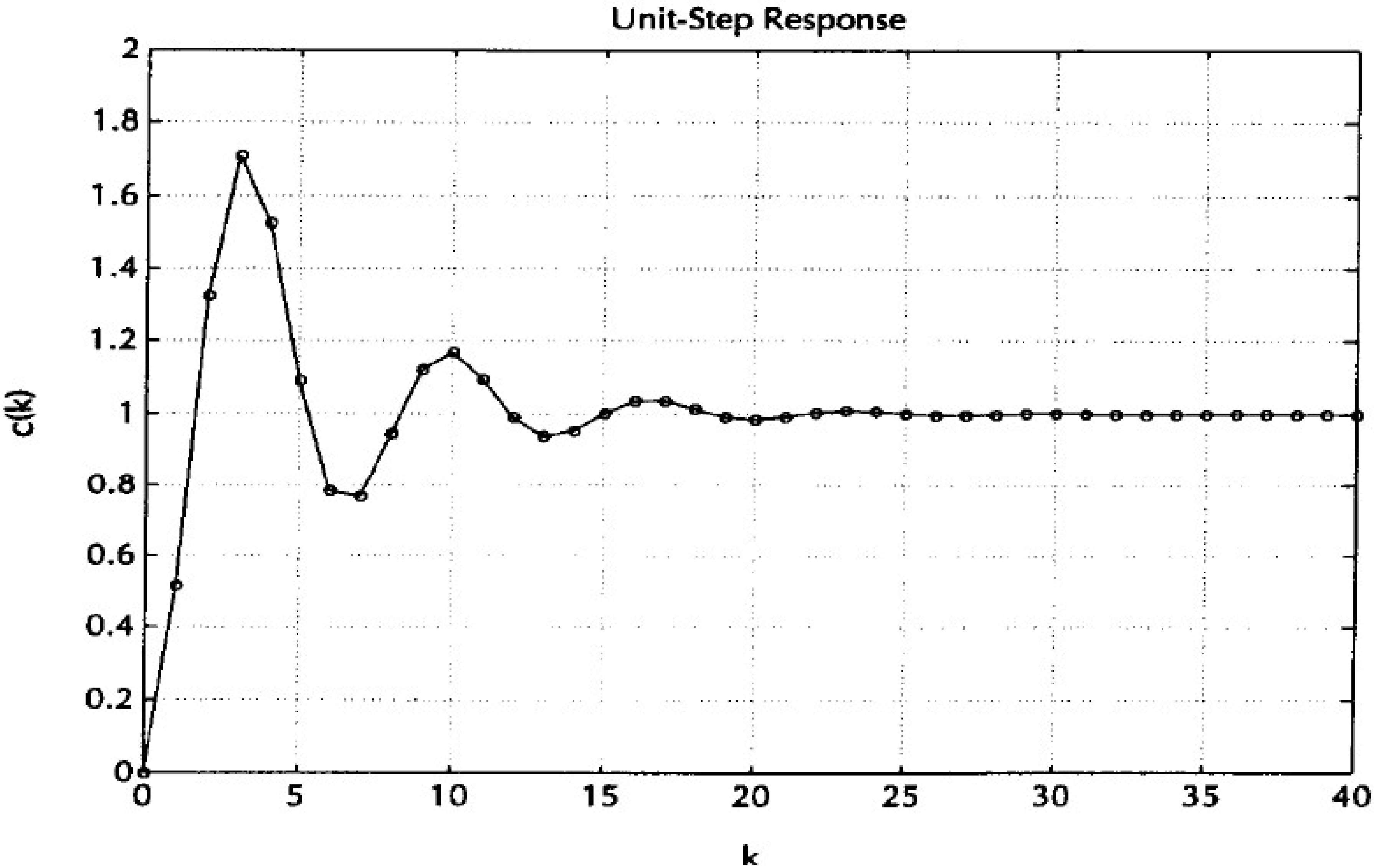


Figure 3-18 Unit-step response

The response of this system to a unit-ramp input can be obtained in a similar way, as shown in Figure 3-19.

MATLAB Program 3–2

```
% ---------- Unit-ramp response ----------

num = [0   0.5151   -0.1452   -0.2963   0.0528];
den = [1   -1.8528   1.5906   -0.6642   0.0528];
k = 0:40;
r = [k];
c = filter(num,den,r);
plot(k,c,'o',k,c,'-',k,k,'--')
v = [0   16   0   16];
axis(v);
grid
title('Unit-Ramp Response')
xlabel('k')
ylabel('c(k)')
```

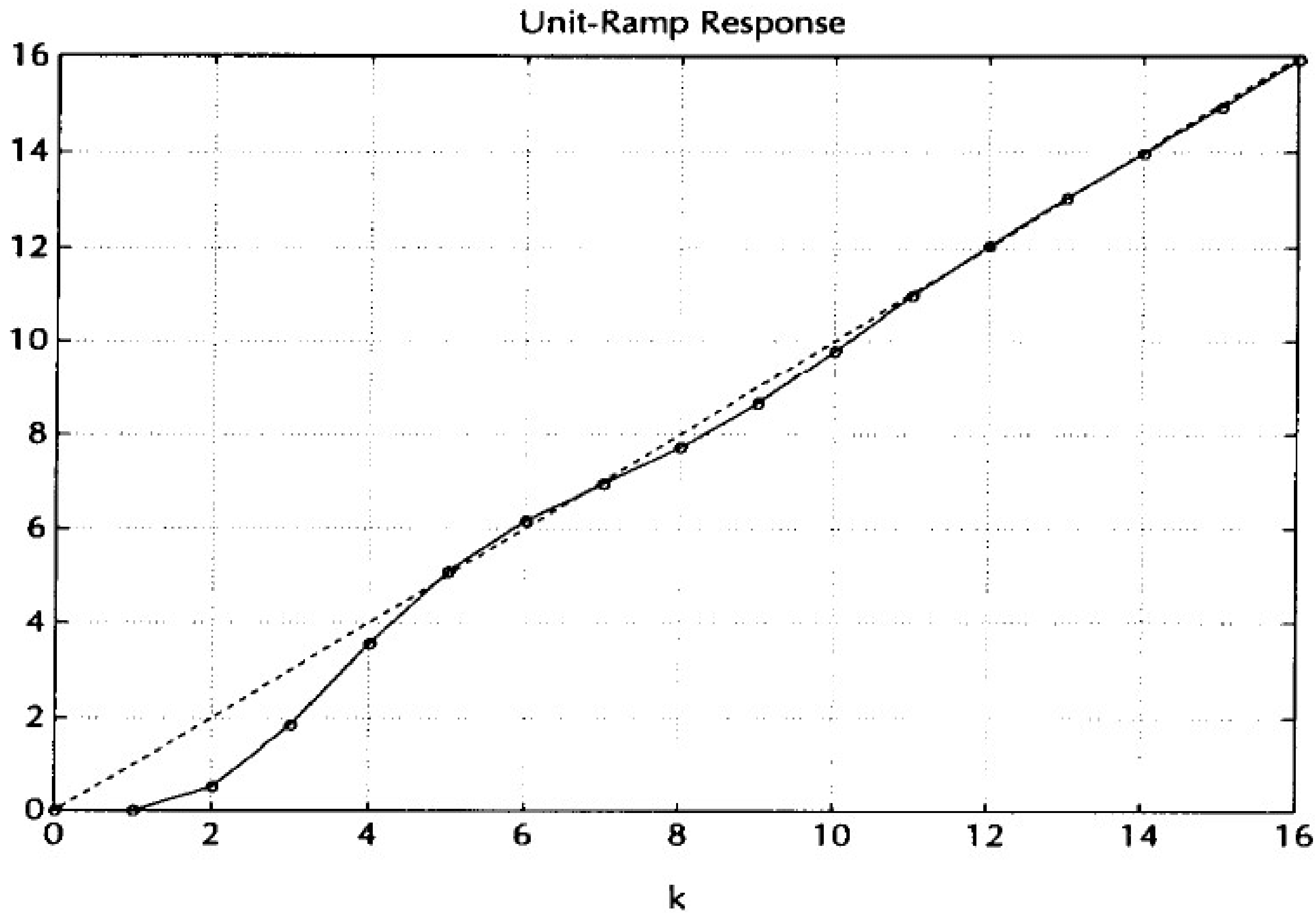


Figure 3-19 Unit-ramp response

Summary:

In this chapter, we have studied the modeling and analysis of Discrete-Time Control Systems(DTCS), including following topics:

- Impulse Sampling and Data Hold
- Pulse transfer functions

Chapter 4

Analysis of Discrete-time Control Systems by Conventional Methods

4.1 Introduction

In this chapter, we study the following topics:

- Mapping Between s and z Planes
- Stability Analysis of Closed-loop Systems in z Plane
- Transient and Steady-state Response Analysis

4.2 Mapping Between s and z Planes

For continuous-time systems, roots of the closed-loop poles in the s plane determine the absolute and relative stability of system. Similarly for discrete-time systems.

Recall from the first part of this course that the s-plane is mapped into the z-plane by

$$z = e^{sT}$$

Writing s as

$$s = \sigma + j\omega$$

we have

$$z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} \angle(T\omega) = e^{\sigma T} e^{j(\omega T + 2k\pi)}$$

for all integer values of k (i.e. $k = 0, \pm 1, \pm 2, \cdots$).

4.2.1 Mapping of Left Half s Plane into the z Plane

The left half s plane is given by

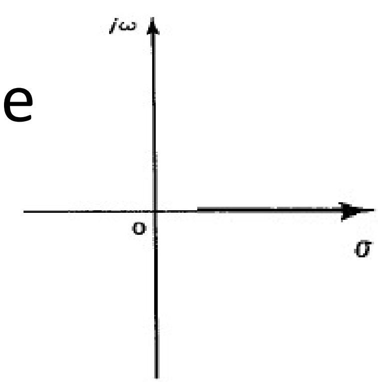
$$s = \sigma + j\omega, \qquad -\infty < \sigma < 0, \ -\infty < \omega < \infty$$

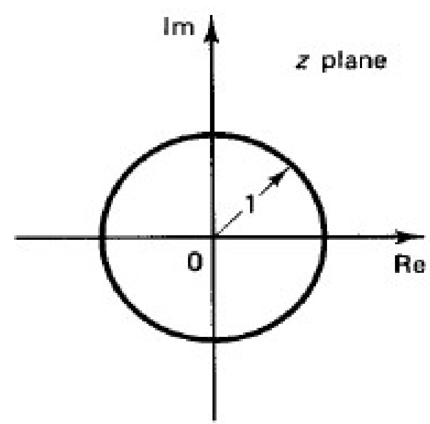
Hence the left half s plane is mapped to z plane as

$$|z| = e^{T\sigma} < 1$$

Note that $\sigma = 0$ (i.e. the $j\omega$ axis in s plane) corresponds to $|z| = 1$ (the unit circle in z plane).

s plane





z plane

4.2.2 Primary Strip and Complementary Strips

Let ω_s be the sampling frequency, i.e. $\omega_s = \dfrac{2\pi}{T}$. The mapping of s plane to z plane is shown in Figure 4-1.

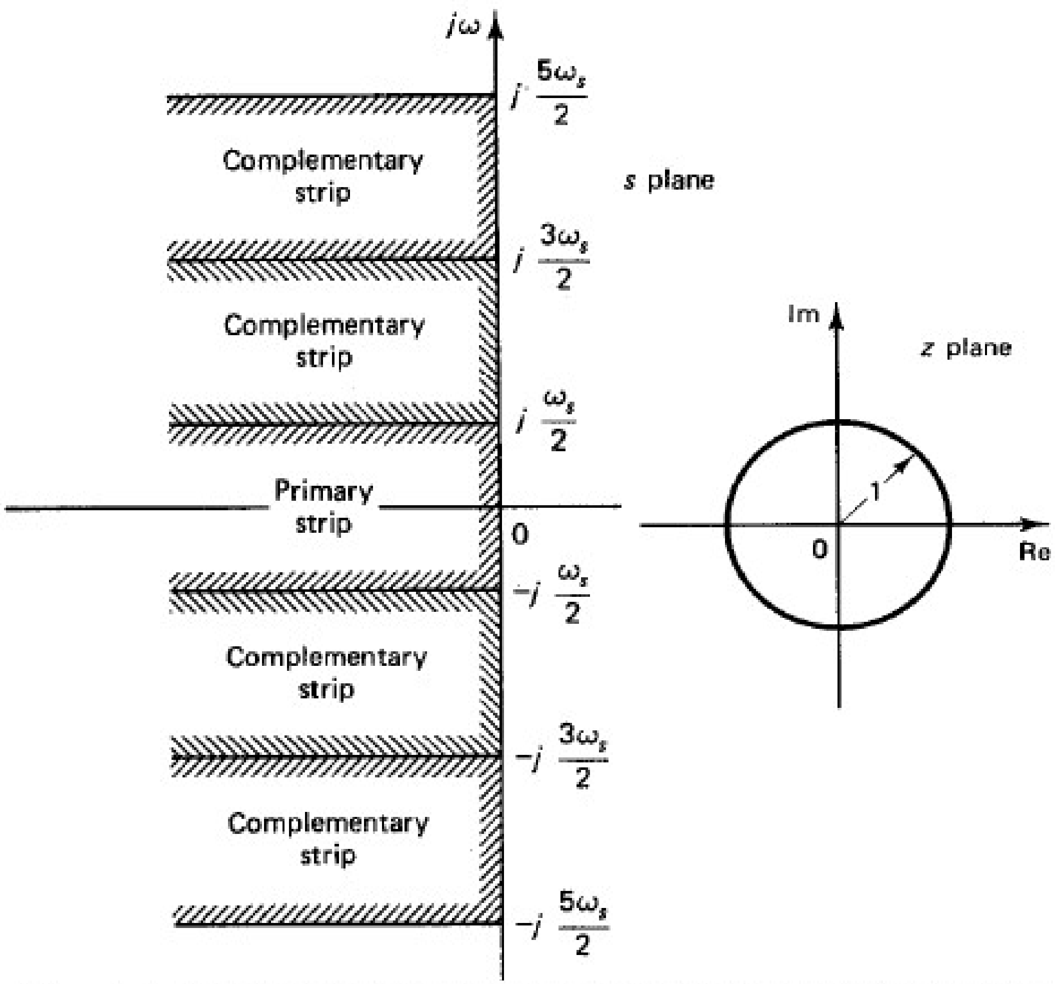


Figure 4-1 Periodic strips in the s plane and the corresponding region (unit circle centered at the origin) in the z plane.

Primary Strip: $s = j\omega,$

$$-\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}$$

is mapped into the z plane as $|z| = 1$ and $\angle z$ changes from −π to π (the complete unit circle).

Complementary Strips: $s = j\omega$**,**

$$-\frac{\omega_s}{2} + k\omega_s \le \omega \le \frac{\omega_s}{2} + k\omega_s$$

$k = \pm 1, \pm 2, \cdots$ are all mapped into the z plane as $|z| = 1$ and $\angle z$ changes from −π to π (the complete unit circle).

Hence, both the primary strip and the complementary strips are mapped into the unit disk in z plane.

Example 4.1: For the contour 1-2-3-4-5 in Figure 4-2(a) inside the primary strip, its mapping into the z plane is shown in Figure 4-2.

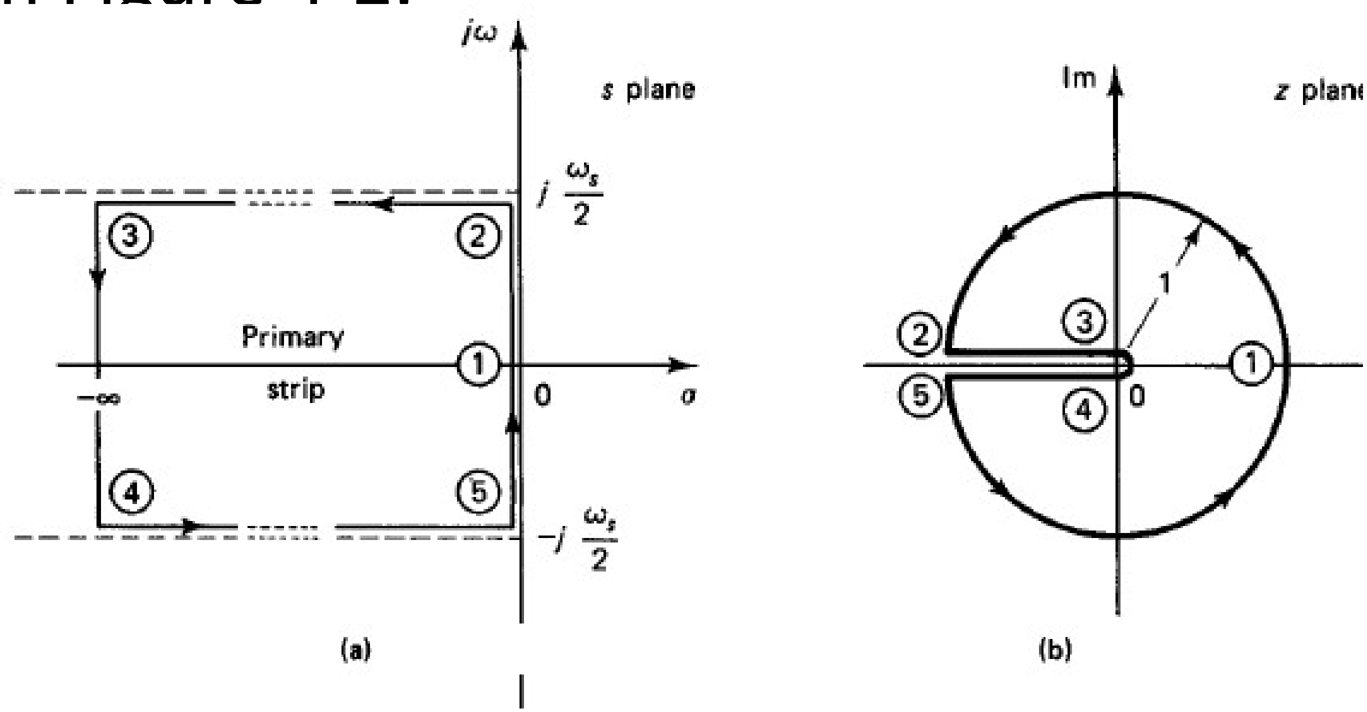


Figure 4–2 Diagrams showing the correspondence between the primary strip in the s plane and the unit circle in the z plane: (a) a path in the s plane; (b) the corresponding path in the z plane.

4.2.3 Constant-Attenuation Loci

$s = \sigma + j\omega$ with σ being constant. This is mapped into z plane as $|z| = e^{\sigma T}$, i.e., circles centered at $z = 0$. See Figure 4-3.

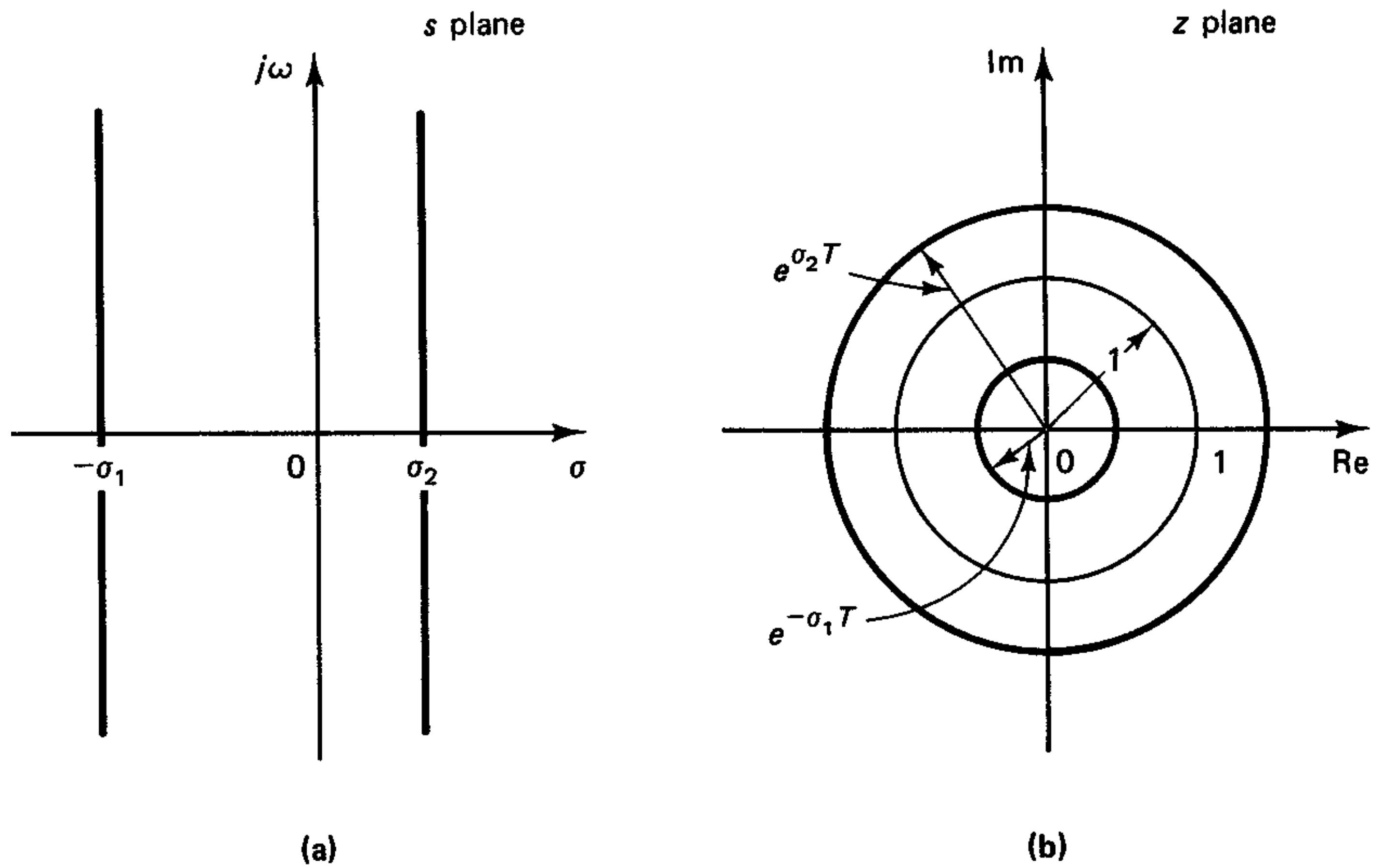


Figure 4–3 (a) Constant-attenuation lines in the *s* plane; (b) the corresponding loci in the *z* plane.

4.2.4 Settling Time T_s

Recall that, for first order systems, 4τ (98% of steady-state) is the settling time , with τ being the time-constant.

Let σ_1=1/ τ. For a given settling time T'_s, we have

$$T'_s = 4\tau = 4/\sigma_1. \longrightarrow \sigma_1 = 4/T'_s .$$

Hence, for $T_s \le T'_s = 4/\sigma_1$, we should have pole at $s = -\sigma + j\omega$, *where* $\sigma \ge \sigma_1$,

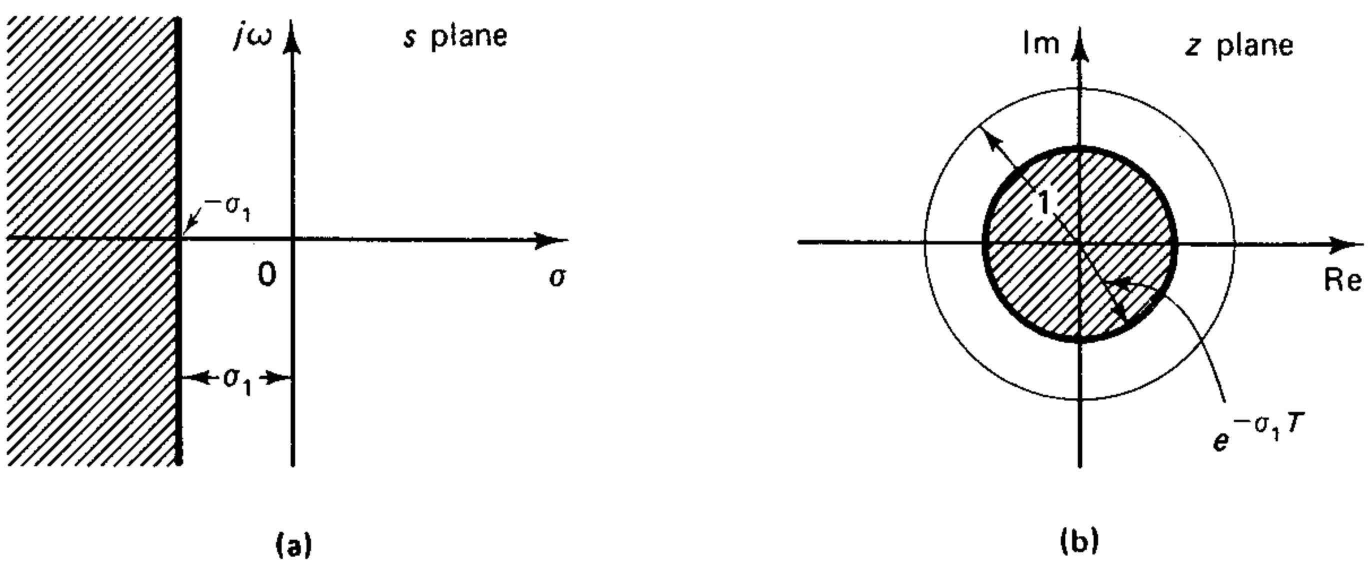


Figure 4–4 (a) Region for settling time T_s less than $4/\sigma_1$ in the s plane; (b) region for settling time T_s less than $4/\sigma_1$ in the z plane.

Example: Sketch the region of poles in the z-plane that corresponds to a second order continuous time system with settling time $T_s \leq 4$ *sec* (2% error tolerance) and with damping ratio in the range [0.707, 1]. The sampling period of the discrete-time system is 1 sec.

Assuming the continuous-time poles are

$$-\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2}$$

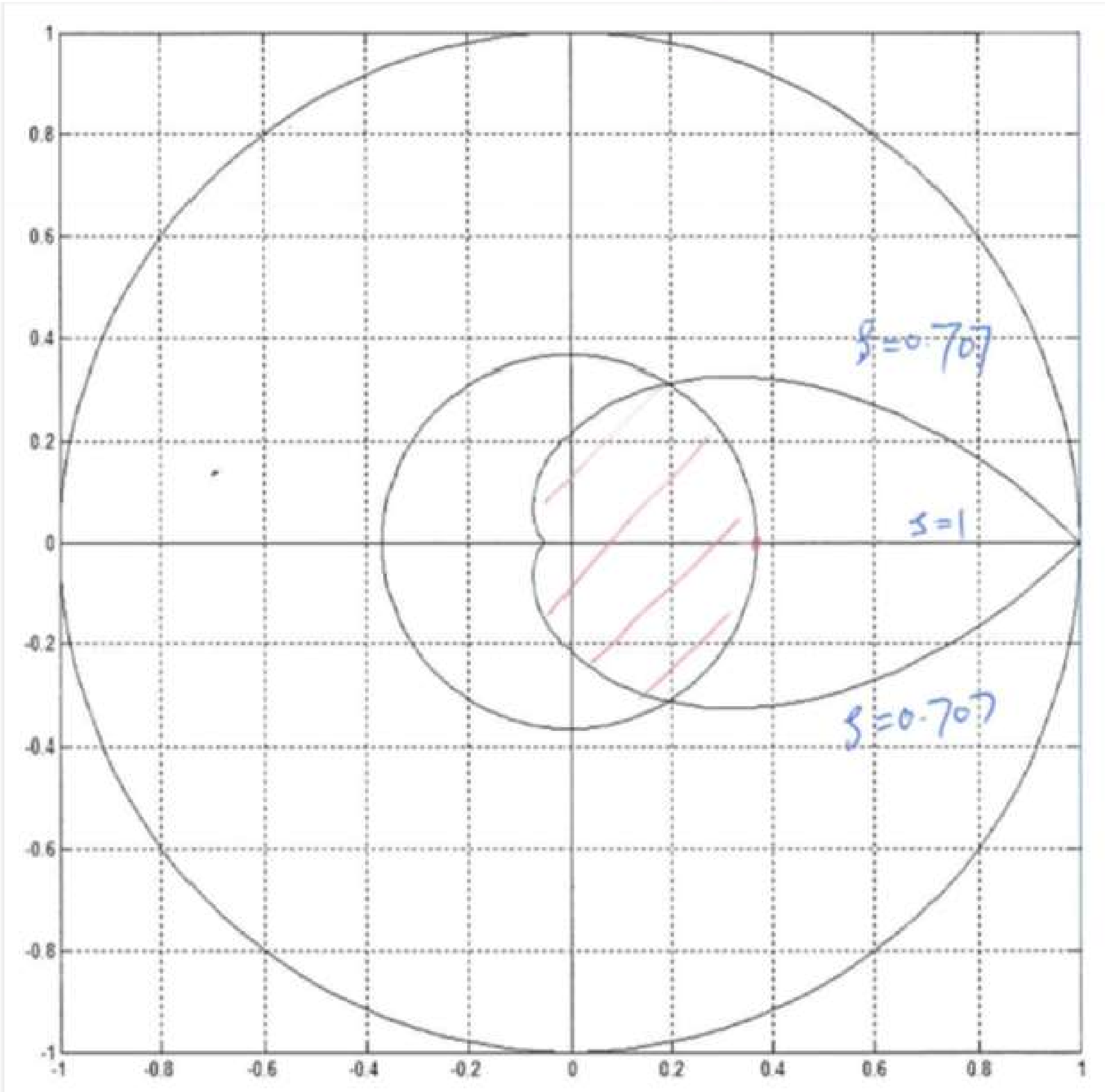
Settling time T_s ≤ 3 sec. (at 2% tolerance) implies that

$$\zeta\omega_n \geq \sigma_1 = 4/4 = 1$$

Hence, with *T* = 1 sec, we have

$$|z| = e^{-\zeta\omega_n T} \leq e^{-1} = 0.3679$$

With damping ratio $0.707 \le \zeta \le 1$,



4.2.5 Constant-Frequency Loci

s = σ+ jω with ω const. So, $z = e^{\sigma T} e^{jT\omega}$ i.e. radial line of constant angle $\angle z = T\omega$ *rad*.

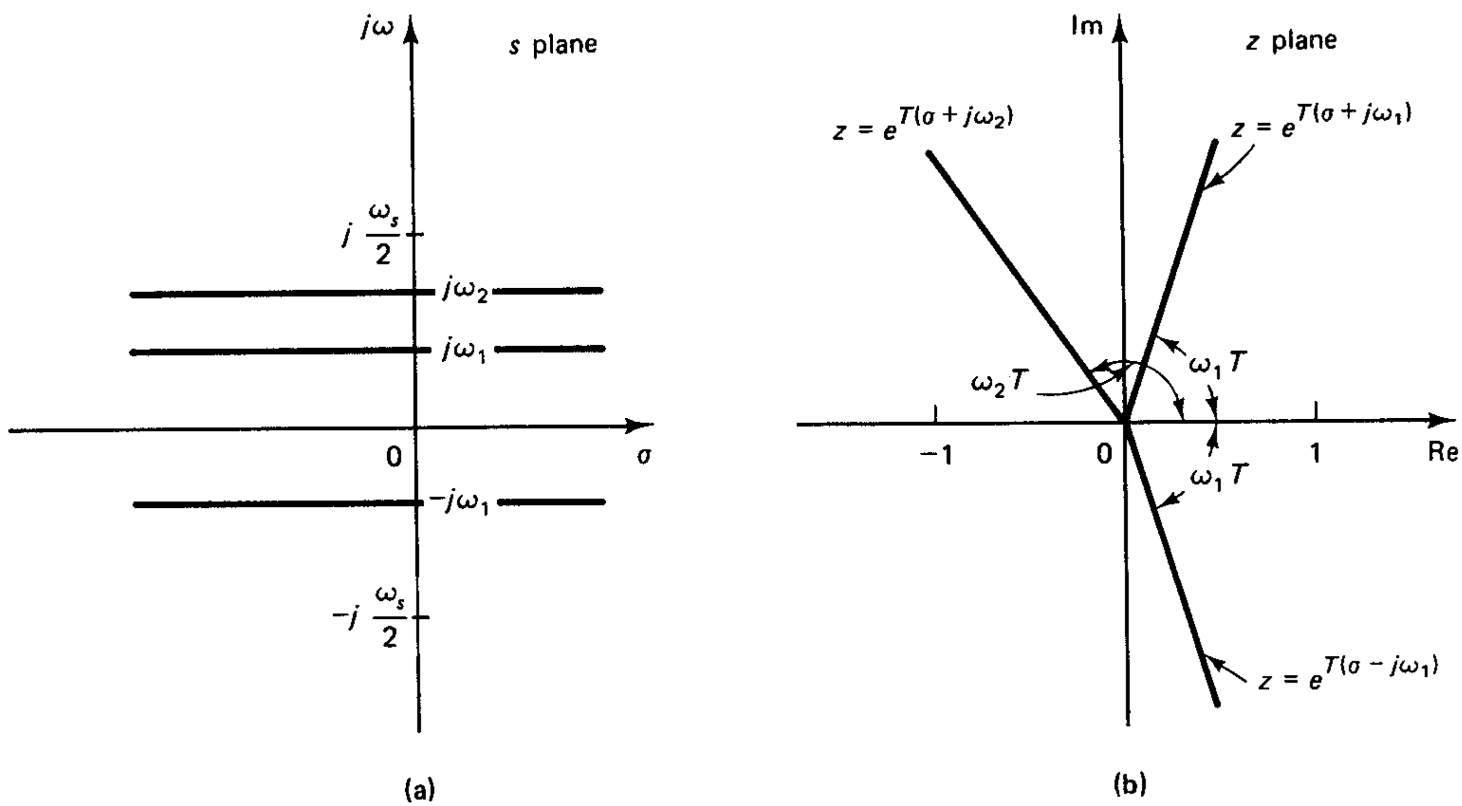


Figure 4–5 (a) Constant-frequency loci in the *s* plane; (b) the corresponding loci in the *z* plane.

Example 4.2: The region bounded by lines $\omega = \omega_1$ and $\omega = \omega_2$, and lines $\sigma=-\sigma_1$ and $\sigma=-\sigma_2$ is mapped into z plane as shown in Figure 4-6.

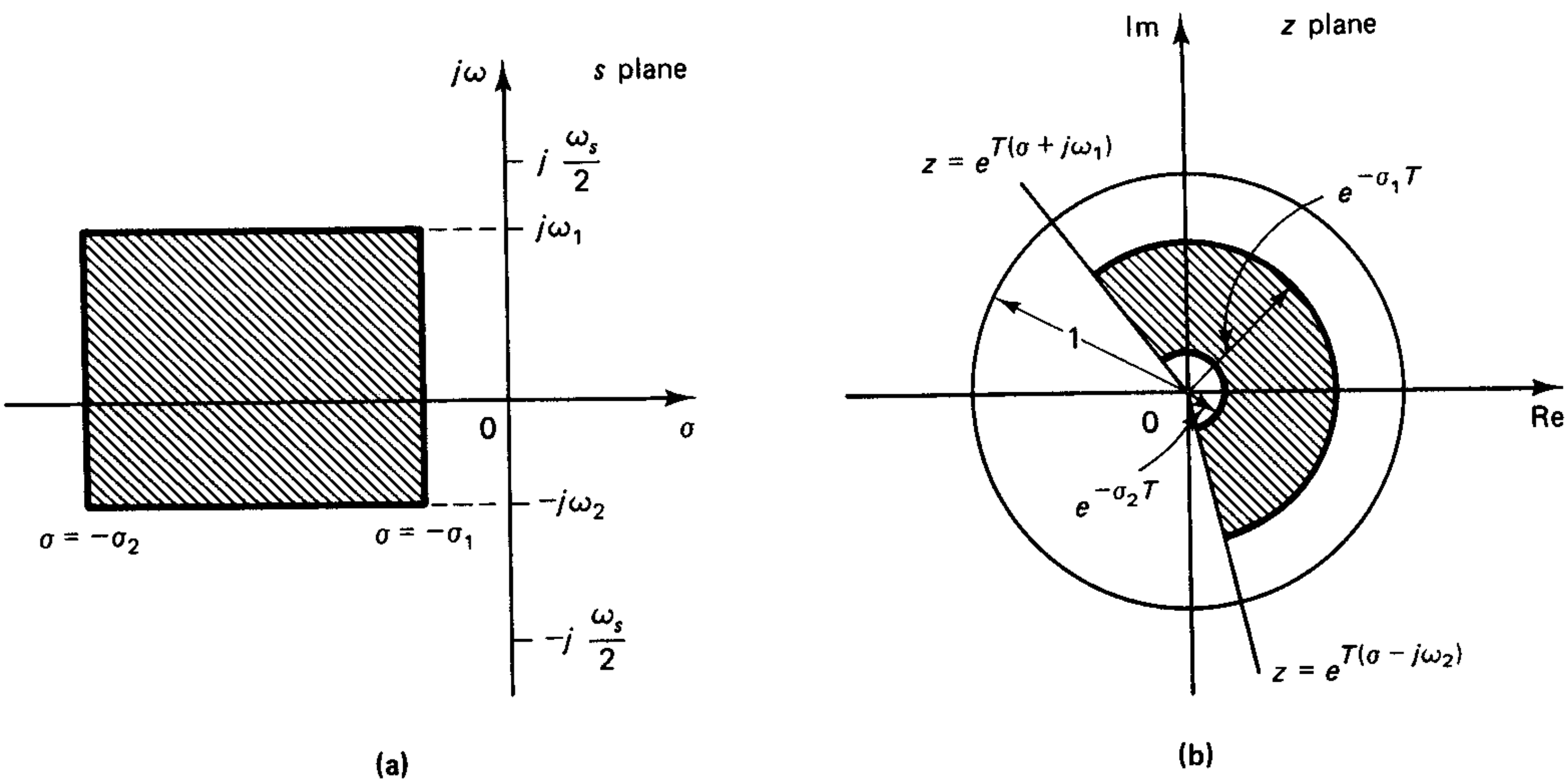


Figure 4–6 (a) Region bounded by lines $\omega = \omega_1$, $\omega = -\omega_2$, $\sigma = -\sigma_1$, and $\sigma = -\sigma_2$ in the s plane; (b) the corresponding region in the z plane.

4.2.6 Constant-Damping-Ratio Loci.

Note that for poles $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$ with ζ constant, ζ is damping ratio. Hence, in z plane,

$$z = e^{sT} = \exp\left(-\zeta\omega_n T \pm j\omega_d T\right)$$

$$= \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right) \qquad (4-1)$$

$$\angle z = 2\pi\frac{\omega_d}{\omega_s} \qquad = \omega_d T \qquad (4-2)$$

Thus if $\zeta > 0$, as $\frac{\omega_d}{\omega_s}$ increases, $|z|$ decreases exponentially while $\angle z$ increases linearly. Thus the locus in z plane becomes a logarithm spiral, as shown in Figure 4-7(b).

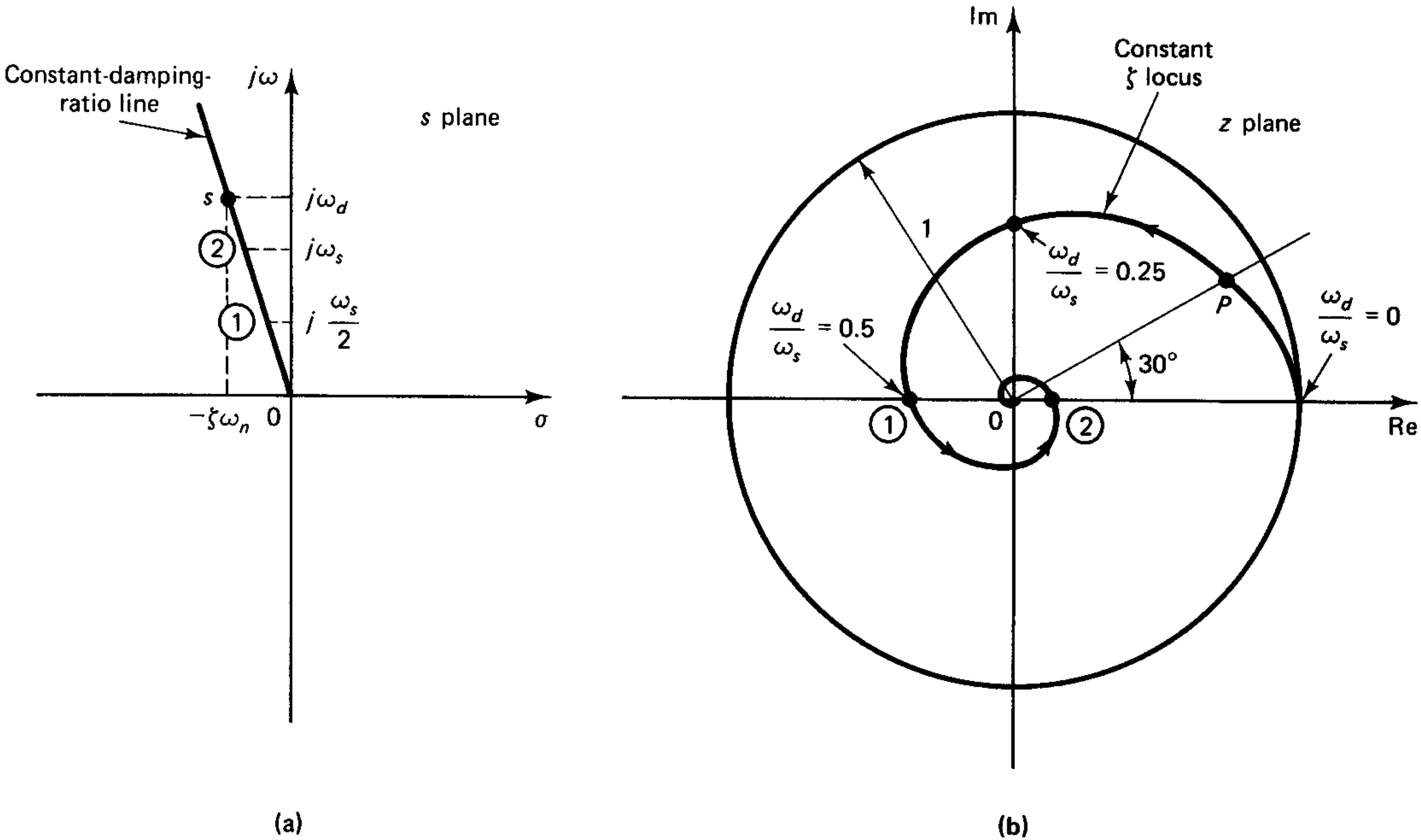


Figure 4–7 (a) Constant-damping-ratio line in the s plane; (b) the corresponding locus in the z plane.

Consider $\zeta = 0.3$. Then

$$\frac{\omega_d}{\omega_s} = 0.25 \quad \longrightarrow \quad |z| = \exp\left(-\frac{2\pi \times 0.3}{\sqrt{1-0.3^2}} \times 0.25\right) = 0.610$$

$$\angle z = 2\pi \times 0.25 = 0.5\pi = 90^\circ$$

$$\frac{\omega_d}{\omega_s} = 0.5 \quad \longrightarrow \quad |z| = \exp\left(-\frac{2\pi \times 0.3}{\sqrt{1-0.3^2}} \times 0.5\right) = 0.3725$$

$$\angle z = 2\pi \times 0.5 = \pi = 180^\circ$$

Figure 4-8 shows the constant damping ratio loci in the upper half z plane. The lower half z plane is symmetric.

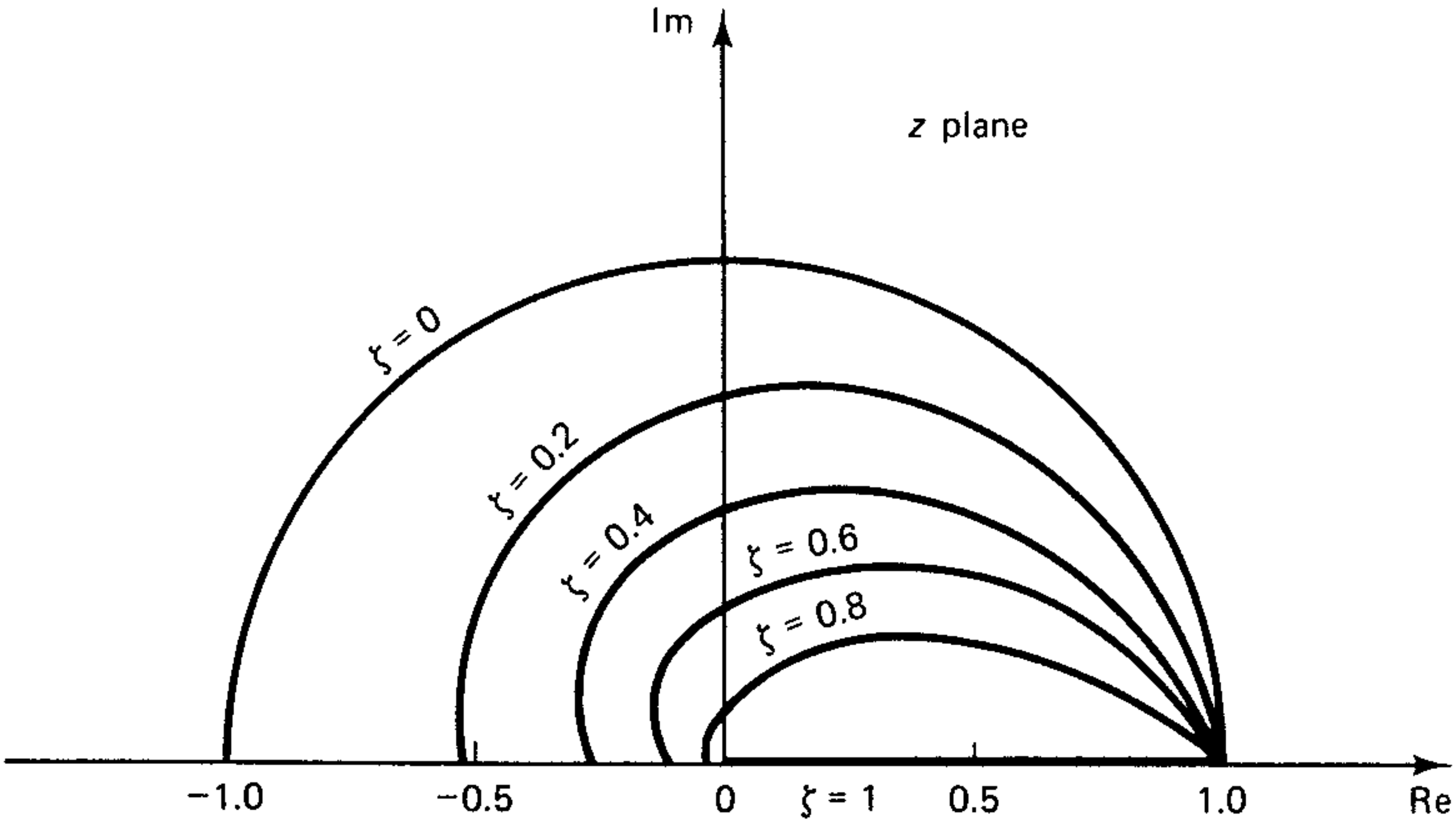


Figure 4–8 Constant-damping-ratio loci in the z plane.

4.2.7 s Plane and z Plane Regions for $\zeta > \zeta_1$

As shown in Figure 4-10, sectors in s plane are mapped into heart shapes in z plane.

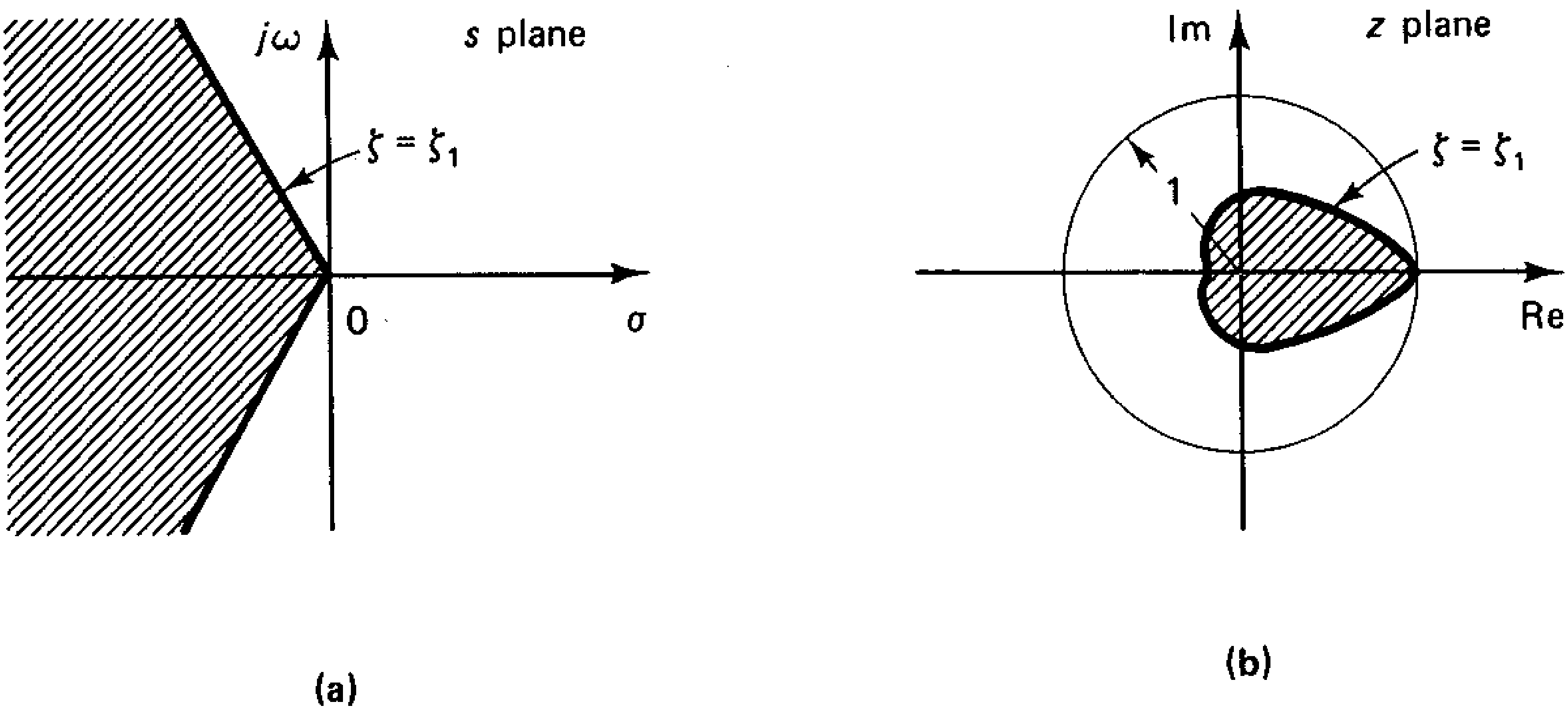


Figure 4–10 (a) Region for $\zeta > \zeta_1$ in the s plane; (b) region for $\zeta > \zeta_1$ in the z plane.

Example 4.3: Specify the region in the z plant that corresponds to a desirable region (shaded region) in the s plane bounded by lines $\omega = \pm\omega_1$, and lines $\zeta = \zeta_1$ and $\sigma = \sigma_1$, as shown in Figure 4-11(a).

Based on the previous discussions, the desirable region in z plane is shown in Figure 4-11 (b).

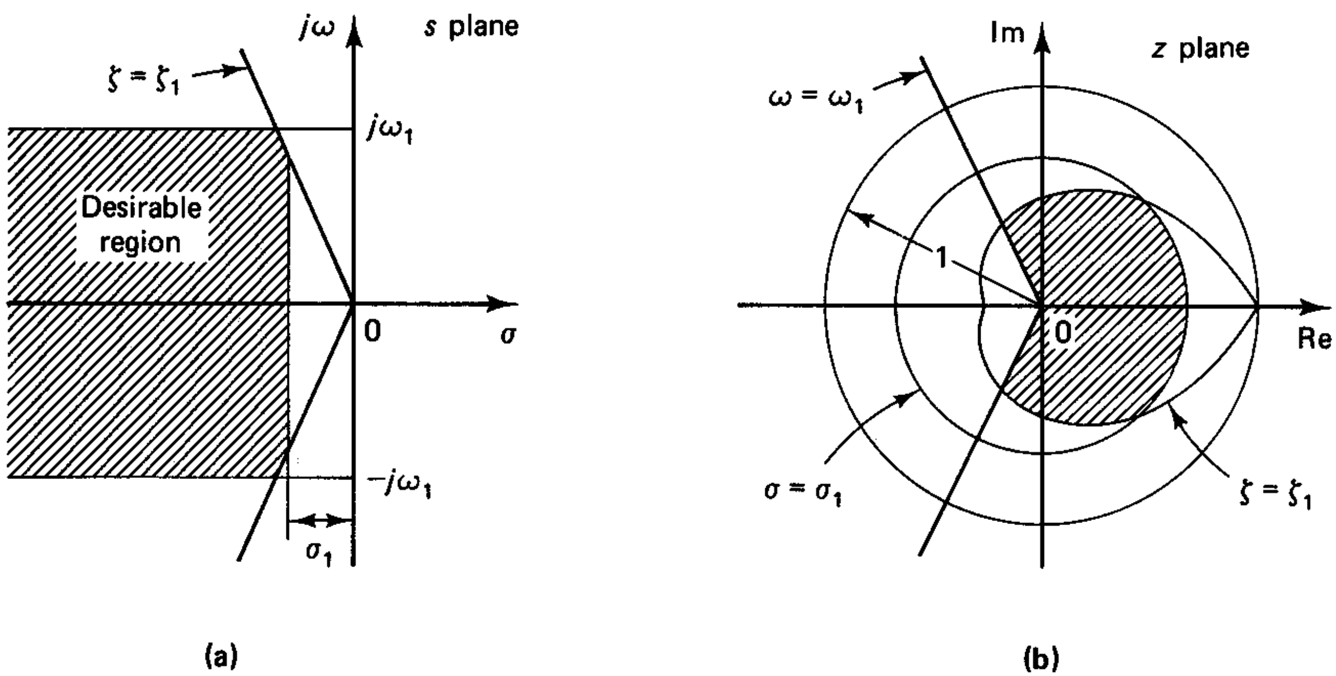


Figure 4–11 (a) A desirable region in the *s* plane for closed-loop pole locations; (b) corresponding region in the *z* plane.

4.3 Stability Analysis of Closed-loop Systems in z Plane

Consider the closed-loop system in Fig 4.13.

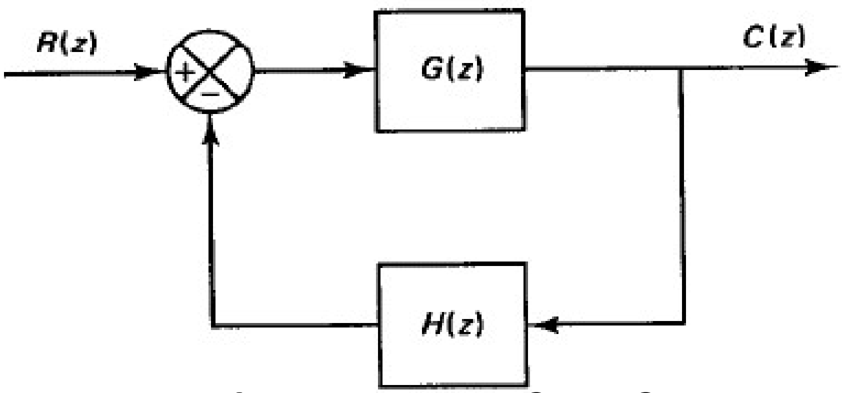


Figure 4.13: Closed loop control system

Its pulse-transfer function is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)} \qquad (4-3)$$

Stability of the system is determined by the poles of the system, i.e., the roots of the characteristic equation (CE)

$$P(z) = 1 + G(z)H(z) \quad = \quad 0$$

- To be stable, all roots of CE must lie inside the unit circle.
- Roots outside the unit circle implies instability
- A simple pole at $z = 1$ indicates critical stability. A single pair of complex conjugate poles on the unit circle also indicates critical stability. Any multiple pole on the unit circle makes system unstable.
- Closed-loop zeros have no effect on stability and can be anywhere in z plane.

We can always solve numerically for the poles. There are methods for checking stability without solving the CE. Namely, **Jury Test** and **Routh Test** *through Bilinear Transformation*. Other methods also exist.

4.3.1 Jury Stability Test.

Consider the nth order CE

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \qquad (4-5)$$

where $a_0 > 0$. The Jury table is shown in Table 4-1.

TABLE 4–1 GENERAL FORM OF THE JURY STABILITY TABLE

Row	z^0	z^1	z^2	z^3	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	b_3	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	$\cdots$	c_0		
6	c_0	c_1	c_2	c_3	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$							
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

Note that

- The 1st row consists of coefficients of *P(z)* in ascending order of powers of *z*.
- The *2*nd row is the reverse of the *1*st. In fact, all even rows are the reverse of the previous row.
- The elements in rows *3* through *(2n − 3)* are

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} = a_n a_{k+1} - a_0 a_{n-1-k}, \qquad k = 0, 1, 2, \cdots, n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} = b_{n-1} b_{k+1} - b_0 b_{n-2-k}, \qquad k = 0, 1, 2, \cdots, n-2$$

$$\vdots$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} = p_3 p_{k+1} - p_0 p_{2-k}, \qquad k = 0, 1, 2$$

Stability Criterion by the Jury Test.

A system with CE

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n, \quad a_0 > 0$$

is stable if the following conditions are all satisfied:

$$\mathbf{1.}\ |a_n| < a_0$$

$$\mathbf{2.}\ P(z)|_{z=1} > 0$$

$$\mathbf{3.}\ P(z)|_{z=-1} \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}$$

$$\mathbf{4.}\ |b_{n-1}| > |b_0|$$
$$|c_{n-2}| > |c_0|$$
$$\vdots$$
$$|q_2| > |q_0|$$

Note that, for *2*nd order ($n = 2$), only the first *3* conditions need to be checked. The last condition is null.

Example 4.4: Construct the Jury stability table for

$$P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4, \quad a_0 > 0$$

The Jury stability table and the stability conditions are as follows:

$$\mathbf{1.}\ |a_4| < a_0$$

$$\mathbf{2.}\ P(z)|_{z=1} = a_0 + a_1 + a_2 + a_3 + a_4 > 0$$

$$\mathbf{3.}\ P(z)|_{z=-1} = a_0 - a_1 + a_2 - a_3 + a_4 > 0, \qquad n = 4 = \text{even}$$

$$\mathbf{4.}\ |b_3| > |b_0|, \qquad |c_2| > |c_0|$$

Note that c_1 (or in general q_1) is not needed.

TABLE 4–2 JURY STABILITY TABLE FOR THE FOURTH-ORDER SYSTEM

Row	z^0	z^1	z^2	z^3	z^4	
	$\begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}$					$= b_3$
	$\begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}$					$= b_2$
	$\begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}$					$= b_1$
1 2	$\begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix}$					$= b_0$
	$\begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}$					$= c_2$
	$\begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}$					$= c_1$
3 4	$\begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}$					$= c_0$
5	c_2	c_1	c_0			

Example 4.5: Examine the stability of CE

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$$

$$= a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

Clearly, $|a_4| = 0.08 < 1 = a_0$ and

P(1)=1 − 1.2 + 0.07 + 0.3 − 0.08 = 0.09 > 0

P(−1)=1 + 1.2 + 0.07 − 0.3 − 0.08 = 1.89 > 0, n = 4 = even

$|b_3| = 0.994 > 0.204 = |b_0|$

$|c_2| = 0.946 > 0.315 = |c_0|$

Hence, the CE is stable.

In fact, *P(z) = (z − 0.8)(z − 0.5)(z + 0.5)(z − 0.4)*

TABLE 4–3 JURY STABILITY TABLE FOR THE SYSTEM OF EXAMPLE 4–4

Row	z^0	z^1	z^2	z^3	z^4	
	−0.08				1	$= b_3 = -0.994$
	1				−0.08	
	−0.08			−1.2		$= b_2 = 1.176$
	1			0.3		
	−0.08		0.07			$= b_1 = -0.0756$
	1		0.07			
1	−0.08	0.3				$= b_0 = -0.204$
2	1	−1.2				
	−0.994			−0.204		$= c_2 = 0.946$
	−0.204			−0.994		
	−0.994		−0.0756			$= c_1 = -1.184$
	−0.204		1.176			
3	−0.994	1.176				$= c_0 = 0.315$
4	−0.204	−0.0756				
5	0.946	−1.184	0.315			

Example 4.6:

$$P(z) = z^3 - 1.1z^2 - 0.1z + 0.2 = 0$$

So $a_0 = 1 > 0$, $a_1 = -1.1$, $a_2 = -0.1$, $a_3 = 0.2$.

Check the Jury test conditions:

1. $|a_3| = 0.2 < 1 = a_0 \quad \Longrightarrow$ OK.

2. $P(1) = 1 - 1.1 - 0.1 + 0.2 = 0 \quad \Longrightarrow$ at least one root at $z = 1$.

3. $P(-1) = -1 - 1.1 + 0.1 + 0.2 = -1.8 < 0$ $(n = 3$ odd$)$,

$$\Longrightarrow \text{OK.}$$

4.

z^0	z^1	z^2	z^3	
0.2	-0.1	-1.1	1	
1	-1.1	-0.1	0.2	
0.2			1	
1			0.2	$b_2 = -0.96$
0.2		-1.1		
1		-0.1		$b_1 = 1.08$
0.2	-0.1			
1	-1.1			$b_0 = -0.12$
-0.96	1.08	-0.12		

So $|b_2| = 0.96 > 0.12 = |b_0| \implies$ OK.

System is critically stable.

Example 4.7:

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0.$$

So $a_0 = 1 > 0$, $a_1 = -1.3$, $a_2 = -0.08$, $a_3 = 0.24$.

Check the Jury test conditions:

1. $|a_3| = 0.24 < 1 = a_0 \quad \Longrightarrow$ OK.

2. $P(1) = 1 - 1.3 - 0.08 + 0.24 = -0.14 < 0 \Longrightarrow$ unstable.

Stop.

Example 4.8: Consider the discrete-time unity-feedback control system with $T = 1$ sec and open-loop pulse transfer function

$$G(z) = \frac{K(0.3679z + 0.2642)}{(z - 0.3679)(z - 1)}$$

Determine the range of K for stability.

The closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{K(0.3679z + 0.2642)}{z^2 + (0.3679K - 1.3679)z + (0.3679 + 0.2642K)}$$

$$a_0 = 1, \quad a_1 = 0.3679K - 1.3679, \quad a_2 = 0.3679 + 0.2642K$$

This is 2nd order, so only check the first 3 conditions:

$$|a_2| < a_0, \qquad P(1) > 0, \qquad P(-1) > 0$$

That is,

$$|0.3679 + 0.2642K| < 1 \implies -5.1755 < K < 2.3925$$

$$P(1) = 0.6321K > 0 \implies K > 0$$

$$P(-1) = 2.7358 - 0.1037K > 0 \implies K < 26.382$$

Hence, for stability, we must have $0 < K < 2.3925$.

If $K = 2.3925$, the system becomes critically stable (i.e. pole on the unit circle). Substitute this value into CE, we have

$$z^2 - 0.4877z + 1 = 0 \implies z = 0.2439 \pm j0.9698$$

Note $T = 1$ sec. So from (4-2)

$$\omega_d = \frac{1}{T}\angle z = 1\angle z = \tan^{-1}\frac{0.9698}{0.2439} = 1.3244\text{rad/sec}$$

This is the frequency of the sustained oscillation.

4.3.2 Stability Analysis using Bilinear Transformation and Routh Stability Criterion

The bilinear transformation

$$z = \frac{w+1}{w-1}, \qquad \text{or} \qquad w = \frac{z+1}{z-1}$$

maps the inside of the unit circle in z plane into the left half of the w plane, as shown below.

Let w = σ± jω.

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\sigma \pm j\omega + 1}{\sigma \pm j\omega - 1}\right| < 1 \quad \Leftrightarrow \quad \frac{(\sigma+1)^2+\omega^2}{(\sigma-1)^2+\omega^2} < 1$$

$$\Leftrightarrow \quad (\sigma+1)^2+\omega^2 < (\sigma-1)^2+\omega^2 \quad \Leftrightarrow \quad \sigma < 0$$

Hence, by replacing z with $\frac{w+1}{w-1}$ in $P(z)$, we obtain a rational function of w.

Then by considering its numerator $P'(w)$, we can check the stability of the roots of $P(z)$ using the continuous-time ***Routh Stability Criterion***.

Example 4.9: Let's examine Example 4.5 again.

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$$

The bilinear transformation gives

$$P'(w) = w^4 + 14.67w^3 + 59.78w^2 + 81.33w + 21$$

Applying the Routh array method, we have

$$\begin{array}{llll} w^4 & 1 & 59.78 & 21 \\ w^3 & 14.67 & 81.33 & \\ w^2 & 54.23 & 21 & \\ w^1 & 75.65 & & \\ w^0 & 21 & & \end{array}$$

Hence all roots of *P'(w)* are in the LHP, and all roots of *P(z)* are inside the unit circle.

Some Comments:

- ***Jury*** test **does not** reveal the number of unstable roots. But ***Routh*** test **does**.
- Jury test involves less computation than bilinear transformation.
- Stability has nothing to do with the system's ability to follow a particular input.

4.4 Transient and Steady-State Response Analysis

Absolute Stability: All closed-loop poles inside the unit circle.

Relative Stability: All closed-loop poles lie in certain region(s) inside the unit circle.

Transient Response: that die out after some time.

Steady-State Response: that remains after along time.

4.4.1 Transient Response Specifications

A typical Unit step response of a system (*without integrator*) is shown in Figure 4-13.

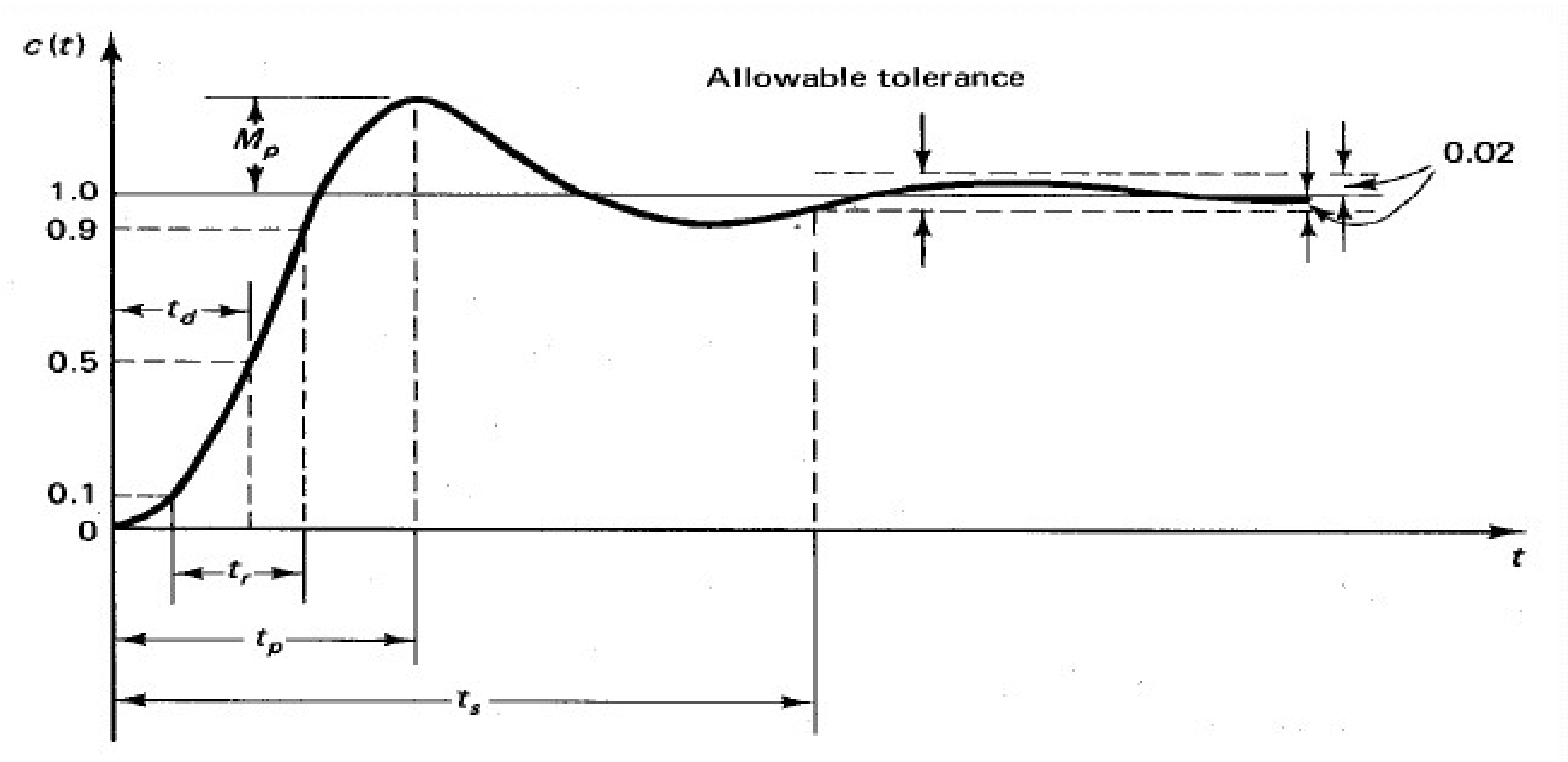


Figure 4-13. **Unit-step response curve showing transient response specifications t_d, t_r, t_p, M_p, and t_s.**

The Transient Response Specifications include:

1. **Delay time** t_d : Time required for response to reach half of the final value the very first time.
2. **Rise time** t_r: Time required for response to rise from 10% to 90% of its final value (or 5% to 95%, or 0% to 100%)
3. **Peak time** t_p: Time required for response to reach the first peak of the overshoot.
4. **Maximum overshoot** $M_p\!: = \dfrac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%.$ That is, maximum peak value of the response measure from unity.
5. **Settling time** t_s: Time required for response to reach and stay within a range about the final value of a size specified as a percentage of the final value, usually 2%.

We can use Matlab to obtain unit-step response.

4.4.2 Steady-state error analysis

Steady-state performance of a stable control system is generally judged by the steady-state error due to step, ramp and acceleration inputs.

Recall: For continuous-time system with open-loop transfer function

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{s^N(1+T_1 s)(1+T_2 s)\cdots(1+T_p s)}$$

The number N denotes the type of the system.

For discrete-time system, assume the open-loop transfer function

$$GH(z) = \frac{1}{(z-1)^N}\frac{B(z)}{A(z)}$$

where $B(z)/A(z)$ has neither poles nor zeros at $z = 1$. The number N denotes the type of the discrete-time system.

Consider

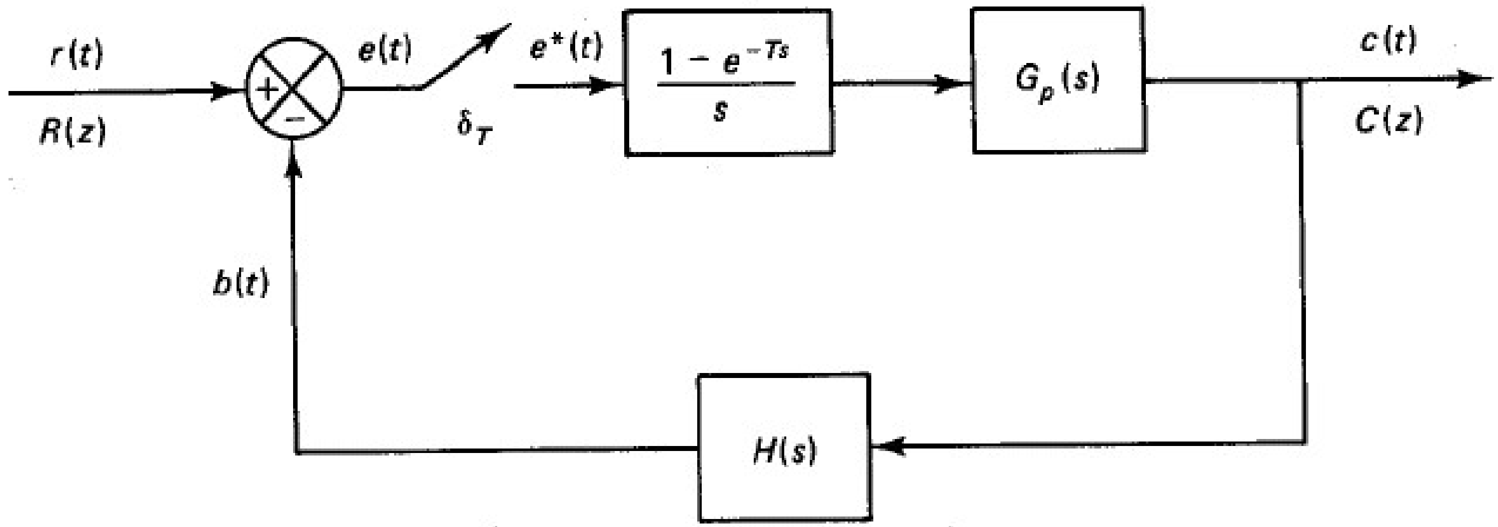


Figure 4-14: Discrete-time control system

The actuating error is

$$e(t) = r(t) - b(t),\ E(z) = R(z) - B(z)$$

and the steady-state error at the sampling instants is:

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1}[(1 - z^{-1})E(z)]$$

Note that

$$E(z) = R(z) - B(z) = R(z) - GH(z)E(z)$$

and

$$GH(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G_P(s)H(s)}{s}\right]$$

Hence

$$E(z) = \frac{1}{1 + GH(z)}R(z)$$

and

$$e_{ss} = \lim_{z \to 1}\left[(1 - z^{-1})\frac{1}{1 + GH(z)}R(z)\right]$$

4.4.3 Static Position Error Constant

For a unit step input, we have

$$R(z) = \frac{1}{1 - z^{-1}}$$

So

$$e_{ss} = \lim_{z\to 1}\left[(1 - z^{-1})\frac{1}{1 + GH(z)}R(z)\right] = \lim_{z\to 1}\frac{1}{1 + GH(z)} = \frac{1}{1+\lim\limits_{z\to 1} GH(z)}$$

Define the position error constant K_p as

$$K_p = \lim_{z\to 1} GH(z) \qquad (4.6)$$

Then

$$e_{ss} = \frac{1}{1 + K_p} \qquad (4.7)$$

4.4.4 Static Velocity Error Constant

For a unit ramp input, we have

$$R(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}$$

So

$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} R(z) \right]$$

$$= \lim_{z \to 1} \frac{Tz^{-1}}{(1 - z^{-1})GH(z) + (1 - z^{-1})}$$

$$= \frac{T}{\lim_{z \to 1} (1 - z^{-1})GH(z)} = \frac{1}{\lim_{z \to 1} \frac{(1 - z^{-1})GH(z)}{T}}$$

Define the velocity error constant K_v as

$$K_v = \lim_{z \to 1} \frac{(1 - z^{-1})GH(z)}{T} \qquad (4.8)$$

Then

$$e_{ss} = \frac{1}{K_v} \qquad (4.9)$$

4.4.5 Static Acceleration Error Constant

For a unit acceleration input, we have

$$R(z) = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3}$$

So

$$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}R(z)\right]$$

$$= \lim_{z \to 1}\frac{T^2}{(1-z^{-1})^2 GH(z)} = \frac{1}{\lim_{z \to 1}\frac{(1-z^{-1})^2 GH(z)}{T^2}}$$

Define the acceleration error constant K_a as

$$K_a = \lim_{z \to 1}\frac{(1-z^{-1})^2 GH(z)}{T^2} \quad (4.10)$$

Then

$$e_{ss} = \frac{1}{K_a} \quad (4.11)$$

The results are summarized in Table 4-4.

TABLE 4–4 SYSTEM TYPES AND THE CORRESPONDING STEADY-STATE ERRORS IN RESPONSE TO STEP, RAMP, AND ACCELERATION INPUTS FOR THE DISCRETE-TIME CONTROL SYSTEM SHOWN IN FIGURE 4–18

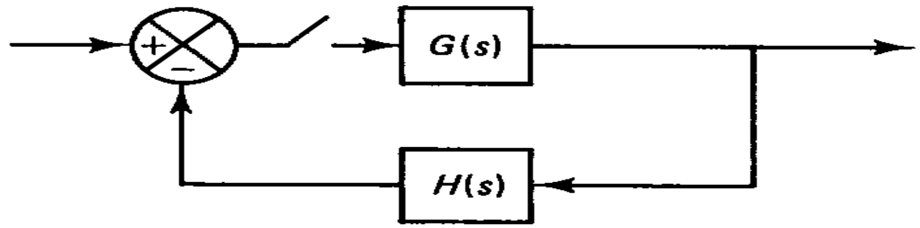
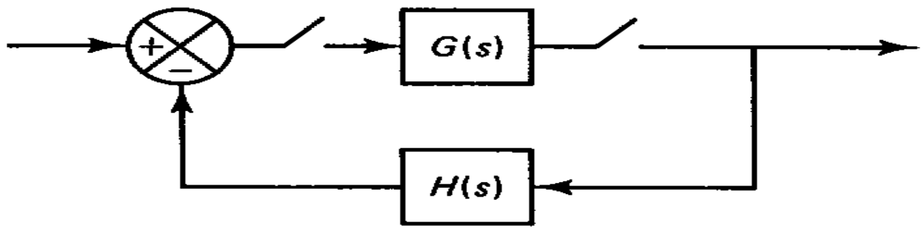
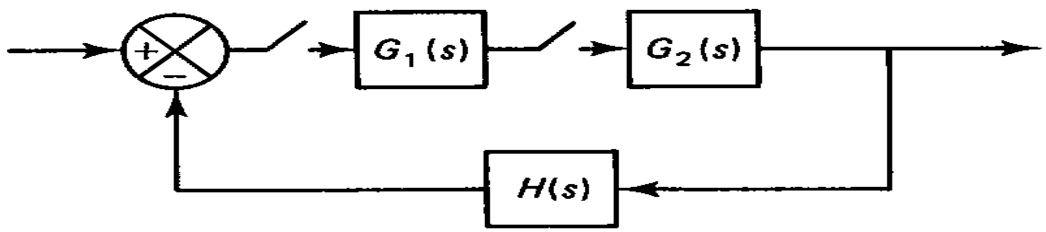
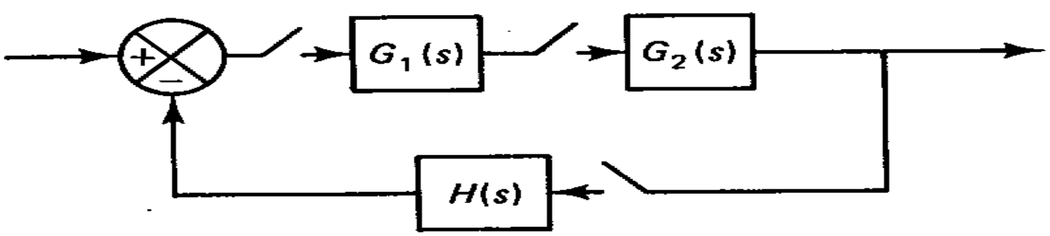
System	Steady-state errors in response to		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Acceleration input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\dfrac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\dfrac{1}{K_v}$	∞
Type 2 system	0	0	$\dfrac{1}{K_a}$

The above analysis applies to the system in Figure 4-14.

It is important to note that the above $E(z)$ is the actuating error $E(z) = R(z) - B(z)$. This is different from the tracking error $R(z) - C(z)$!

For other system configurations where the sampler(s) are placed differently, the results have to be modified. A few examples are given in Table 4-5.

TABLE 4–5 STATIC ERROR CONSTANTS FOR TYPICAL CLOSED-LOOP CONFIGURATIONS OF DISCRETE-TIME CONTROL SYSTEMS

Closed-loop configuration	Values of K_p, K_v, and K_a
	$K_p = \lim_{z\to 1} GH(z)$ $K_v = \lim_{z\to 1} \frac{(1-z^{-1})GH(z)}{T}$ $K_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 GH(z)}{T^2}$
	$K_p = \lim_{z\to 1} G(z)H(z)$ $K_v = \lim_{z\to 1} \frac{(1-z^{-1})G(z)H(z)}{T}$ $K_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 G(z)H(z)}{T^2}$
	$K_p = \lim_{z\to 1} G_1(z)HG_2(z)$ $K_v = \lim_{z\to 1} \frac{(1-z^{-1})G_1(z)HG_2(z)}{T}$ $K_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 G_1(z)HG_2(z)}{T^2}$
	$K_p = \lim_{z\to 1} G_1(z)G_2(z)H(z)$ $K_v = \lim_{z\to 1} \frac{(1-z^{-1})G_1(z)G_2(z)H(z)}{T}$ $K_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 G_1(z)G_2(z)H(z)}{T^2}$

4.4.6 Response to Disturbances

Transient response and steady-state errors are also affected by disturbance inputs. Consider the system shown in Figure 4-15.

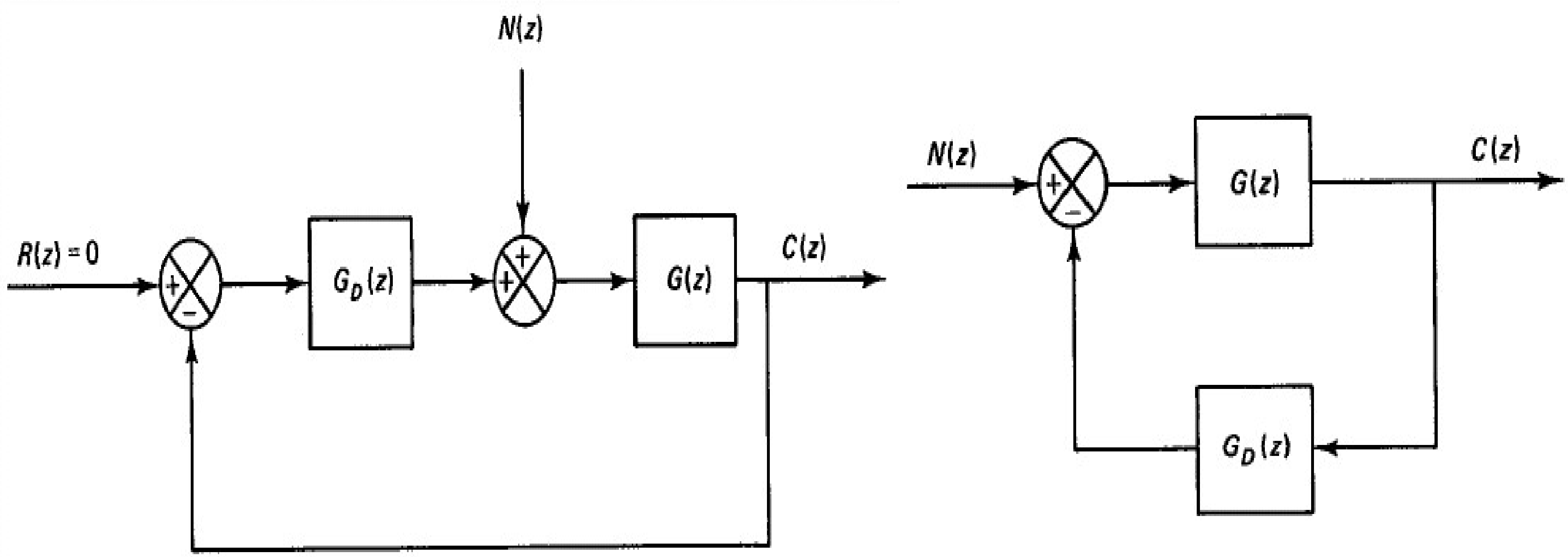


Figure 4-15 (a) Digital closed-loop control system subjected to reference input and disturbance ; (b) modified block diagram where disturbance is considered as the input to the system.

Assume $R(z) = 0$. Then

$$E(z) = R(z) - C(z)$$

$$= -C(z)$$

$$= \frac{-G(z)}{1 + G_D(z)G(z)} N(z) \qquad (4.10)$$

If $G_D(z)G(z) >> 1$, then

$$E(z) \approx -\frac{1}{G_D(z)} N(z) \qquad (4.11)$$

So large gain of $G_D(z)$ gives small error $E(z)$.

Result:

If $G_D(z)$ includes a pure integrator (i.e. it has a pole at $z = 1$) and $G(z)$ has no zero at $z=1$, then steady-state error due to constant disturbance is zero.

Proof: Note that

$$G_D(z) = \frac{\hat{G}_D(z)}{z-1} = \frac{\hat{G}_D(z)z^{-1}}{1-z^{-1}},$$

where $\hat{G}_D(z)$ has no zero at z = 1.

Now suppose that the value of constant disturbance is C. Then

$$N(z) = \frac{C}{1-z^{-1}}$$

Thus, using (4-10), the steady-state effect due to a constant disturbance is

$$e_{ss} = \lim_{z\to 1}\left[(1-z^{-1})E(z)\right]$$

$$= \lim_{z\to 1}(1-z^{-1})\frac{-G}{1+\hat{G}_D\ (z)z^{-1}/(1-z^{-1})G}N(z)$$

$$= \lim_{z\to 1}(1-z^{-1})\frac{-G}{1+\hat{G}_D\ (z)z^{-1}/(1-z^{-1})G}\frac{C}{1-z^{-1}}$$

$$= \lim_{z\to 1}\frac{-G(1-z^{-1})C}{(1-z^{-1})+\hat{G}_D\ (z)z^{-1}G}$$

$$= \frac{0}{0+\hat{G}_D\ (1)z^{-1}G(1)}$$

$$=0$$

SUMMARY of Chap 1 - 4

- Chapter 1: Introduction
- Chapter 2: The Z Transform
- Chapter 3: z-Plane Analysis
- Chapter 4: Sections 4-1 to 4-4, Mapping Between z and s Planes and Stability Analysis

Chapter 2: The $\mathcal{Z}$ Transform

In this chapter, we study the following topics:

- Introduction
- The $\mathcal{Z}$ Transform
- $\mathcal{Z}$ Transform of Elementary Functions
- Important Properties and Theorems of the $\mathcal{Z}$ Transform
- The Inverse $\mathcal{Z}$ Transform
- $\mathcal{Z}$ Transform Method for Solving Difference Equations

2.2: The $\mathcal{Z}$ Transform

The $\mathcal{Z}$ transform for $x(t)$, $t \geq 0$ (or $x(kT)$, $k = 0, 1, 2, \cdots$) is

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (2\text{-}1)$$

For a sequence of numbers $x(k)$,

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k} \qquad (2\text{-}2)$$

Eq (2-1) and (2-2) are called one-sided Z transform.

In general, we have two-sided $\mathcal{Z}$ transforms

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k} \qquad (2\text{-}3)$$

or

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \qquad (2\text{-}4)$$

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1 - z^{-1}}$
4.	$\frac{1}{s + a}$	e^{-at}	e^{-akT}	$\frac{1}{1 - e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3 z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$
8.	$\frac{a}{s(s + a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$
9.	$\frac{b - a}{(s + a)(s + b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}$
10.	$\frac{1}{(s + a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s + a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1 - (1 + aT)e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$

TABLE 2–1 (continued)

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\dfrac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\dfrac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\dfrac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\dfrac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\dfrac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\dfrac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
15.	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\dfrac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
16.	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\dfrac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
17.	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\dfrac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
18.			a^k	$\dfrac{1}{1-az^{-1}}$
19.			a^{k-1} $k = 1, 2, 3, \ldots$	$\dfrac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\dfrac{z^{-1}}{(1-az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\dfrac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\dfrac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\dfrac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\dfrac{1}{1+az^{-1}}$
25.			$\dfrac{k(k-1)}{2!}$	$\dfrac{z^{-2}}{(1-z^{-1})^3}$
26.		$\dfrac{k(k-1)\cdots(k-m+2)}{(m-1)!}$		$\dfrac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\dfrac{k(k-1)}{2!}a^{k-2}$	$\dfrac{z^{-2}}{(1-az^{-1})^3}$
28.	$\dfrac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$			$\dfrac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t) = 0$, for $t < 0$.

$x(kT) = x(k) = 0$, for $k < 0$.

Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$.

TABLE 2–2 IMPORTANT PROPERTIES AND THEOREMS OF THE z TRANSFORM

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t + T)$ or $x(k + 1)$	$zX(z) - zx(0)$
4.	$x(t + 2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k + 2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t + kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT - T)$
7.	$x(t - kT)$	$z^{-k}X(z)$
8.	$x(n + k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k - 1)$
9.	$x(n - k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\dfrac{d}{dz}X(z)$
11.	$kx(k)$	$-z\dfrac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\dfrac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\dfrac{d}{dz}X\left(\dfrac{z}{a}\right)$
16.	$x(0)$	$\lim\limits_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim\limits_{z\to 1} [(1 - z^{-1})X(z)]$ if $(1 - z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k - 1)$	$(1 - z^{-1})X(z)$
19.	$\Delta x(k) = x(k + 1) - x(k)$	$(z - 1)X(z) - zx(0)$
20.	$\sum\limits_{k=0}^{n} x(k)$	$\dfrac{1}{1 - z^{-1}}X(z)$
21.	$\dfrac{\partial}{\partial a}x(t, a)$	$\dfrac{\partial}{\partial a}X(z, a)$
22.	$k^mx(k)$	$\left(-z\dfrac{d}{dz}\right)^m X(z)$
23.	$\sum\limits_{k=0}^{n} x(kT)y(nT - kT)$	$X(z)Y(z)$
24.	$\sum\limits_{k=0}^{\infty} x(k)$	$X(1)$

2-5 The Inverse $\mathcal{Z}$ Transform

Question: Given a $\mathcal{Z}$ transform $X(z)$, how to find the corresponding time function $x(k)$?

$$\mathcal{Z}^{-1}[X(z)] = ?$$

Note, as $X(z)$ depends only on $x(t)$ at $t = kT$, $k = 0, 1, 2, \cdots$

- can only get $x(k)$, but not $x(t)$.
- sampling period T must also be given in order to get $x(kT)$.
- there are many $x(t)$ that can fit into $x(k)$ (see Figure 2-3)

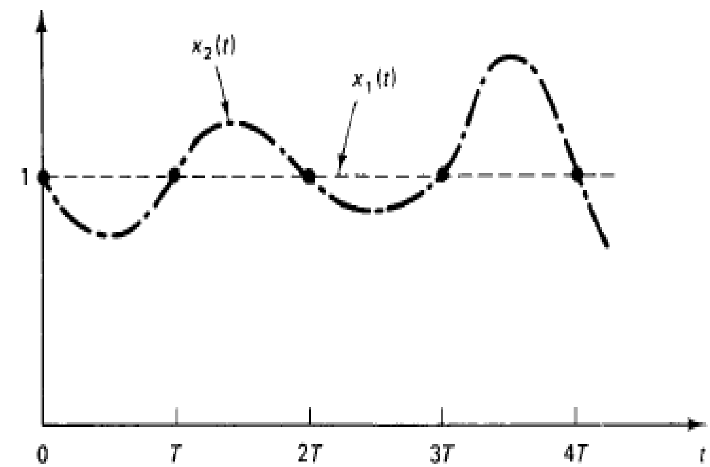


Figure 2-3 Two different continuous-time functions, $x_1(t)$ and $x_2(t)$, that have the same values at $t = 0, T, 2T, \ldots$.

We can do inverse $\mathcal{Z}$ transform by

- direct division method (long division)

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(k)z^{-k} + \cdots$$

- computation method (optional)

  MATLAB Approach

  Difference Equation Approach

- partial fraction expansion method
  (then referring to $\mathcal{Z}$ transform table)

- inversion integral method

- **Partial Fraction Expansion (PFE) Method:**

Consider $$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} \qquad (m \le n)$$

Factor it as $$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{(z-p_1)(z-p_2)\cdot\cdots\cdot(z-p_n)}$$

Then, do PFE for $X(z)/z$. If it has only simple poles, we have

$$\frac{X(z)}{z} = \frac{a_0}{z} + \frac{a_1}{z-p_1} + \dots + \frac{a_n}{z-p_n}$$

$$X(z) = a_0 + \frac{a_1 z}{z-p_1} + \frac{a_2 z}{z-p_2} + \cdots + \frac{a_n z}{z-p_n}$$

$$X(z) = a_0 + \frac{a_1}{1-p_1 z^{-1}} + \frac{a_2}{1-p_2 z^{-1}} + \cdots + \frac{a_n}{1-p_n z^{-1}}$$

Hence $$x(k) = a_0 \delta(k) + a_1 p_1^k + a_2 p_2^k + \cdots + a_n p_n^k \qquad \text{(2-21)}$$

If $X(z)/z$ involves multiple poles, say,

$$\frac{X(z)}{z} = \frac{b_0 z + b_1}{(z-p)^2} = \frac{\overbrace{b_0}^{c_2}(z-p) + \overbrace{b_1 + b_0 p}^{c_1}}{(z-p)^2}$$

then $$\frac{X(z)}{z} = \frac{c_1}{(z-p)^2} + \frac{c_2}{z-p}$$

$$X(z) = \frac{c_1 z^{-1}}{(1-pz^{-1})^2} + \frac{c_2}{1-pz^{-1}}$$

$$x(k) \;=\; c_1 \mathcal{Z}^{-1}\left[\frac{z^{-1}}{(1-pz^{-1})^2}\right] + c_2 \mathcal{Z}^{-1}\left[\frac{1}{1-pz^{-1}}\right]$$

From Table 2-1,

$$x(k) \;=\; c_1 k p^{k-1} + c_2 p^k \qquad k \geq 0 \qquad \text{(2-22)}$$

i.e. $x(0) \;=\; c_2,\ x(1) \;=\; c_1 + c_2 p, x(2) \;=\; 2c_1 p + c_2 p^2,\ \ldots$

•**Inversion Integral Method:**

$$x(k) = x(kT) = \mathcal{Z}^{-1}[X(z)] = \frac{1}{j2\pi}\oint_C X(z)z^{k-1}dz \qquad (2\text{-}23)$$

where C is a counter-clockwise circle centered at the origin (*z = 0*) such that all poles of $X(z)z^{k-1}$ are inside it.

From the residue theory of complex functions, we have

$$x(kT) = x(k) = K_1 + K_2 + \cdots + K_m$$
$$= \sum_{i=1}^{m}\left[\text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}\right] \quad (2-24)$$

where $K_1, K_2, \cdots, K_m$ denotes the residues of $X(z)z^{k-1}$ at poles $z_1,\ z_2,\ \cdots,\ z_m,$ resp., and $k \geq 0$.

If $z = z_i$ is a simple pole of $X(z)\ z^{k-1}$, then

$$K_i = \lim_{z \to z_i}\left[(z - z_i)X(z)z^{k-1}\right] \qquad (2\text{-}25)$$

If $z = z_j$ is an order q multiple pole of $X(z)\ z^{k-1}$, then

$$K_j = \frac{1}{(q-1)!}\lim_{z \to z_j}\frac{d^{q-1}}{dz^{q-1}}\left[(z - z_j)^q X(z)z^{k-1}\right] \qquad (2\text{-}26)$$

2-6 $\mathcal{Z}$ Transform Method for Solving Difference Equations

Consider the following linear difference equation

$$x(k) + a_1 x(k-1) + \cdots + a_n x(k-n)$$
$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n) \qquad \text{(2-27)}$$

where *u(k)* is the input and *x(k)* is the output.

Let

$$\mathcal{Z}[x(k)] = X(z)$$

Taking $\mathcal{Z}$ Transform of (2-27), *x(k-1), x(k-2), x(k- 3)* ... *can be represented in terms of X(z)* as shown in Table 2-3. Then, solving for *X(z)* and computing $\mathcal{Z}^{-1}[X(z)]$, *x(kT)* is obtained.

Chapter 3: Modeling of Digital Control Systems

3-1: Introduction

$\mathcal{Z}$ transform

- is a math tool for the analysis and synthesis of DTCS
- is related to s plane through $z = e^{sT}$
- gives rise to methods similar to continuous-time design methods

In this chapter, we study the following topics:

- Impulse Sampling and Data Hold
- Pulse transfer function

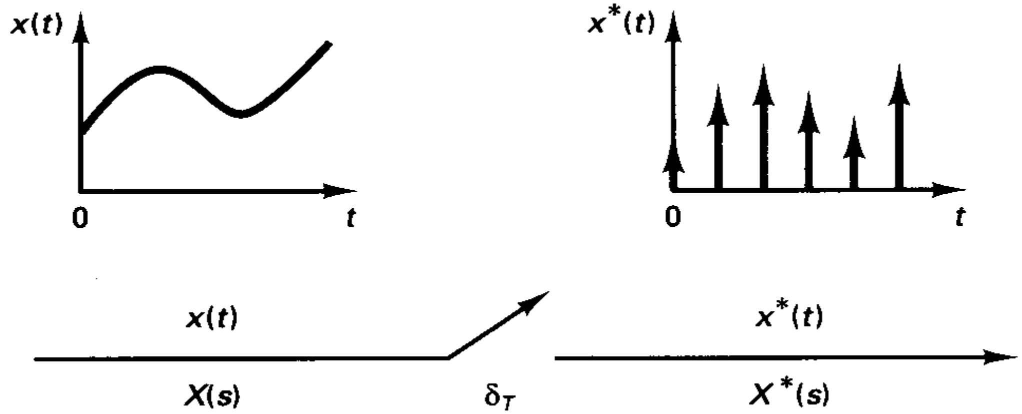


Figure 3–1 Impulse sampler.

Impulse-sampled output *x*(t)* is

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots \quad (3\text{-}1)$$

$$X^*(s) \;=\; \mathcal{L}[x^*(t)]$$

By letting $e^{Ts} = z$ or $s = \dfrac{1}{T}\ln z$

$$X^*(s)\,\Big|_{\,s=1/T\ln z} = X(z)$$

Zero order hold (ZOH):

Zero order hold (ZOH) smoothes the sampled signal with constant (horizontal lines) in-between samples.

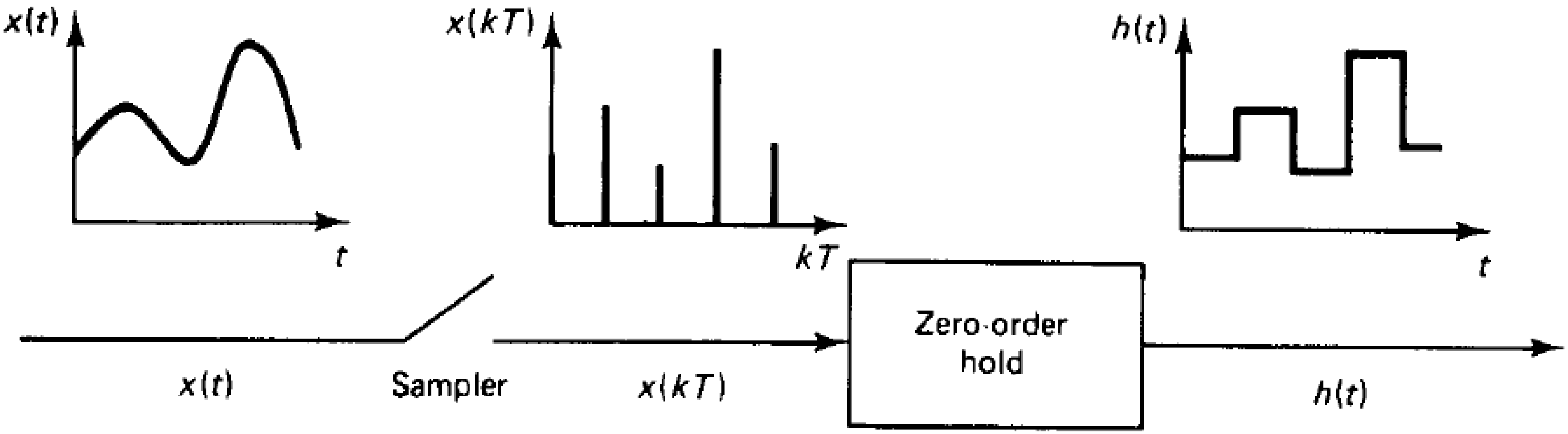


Figure 3–3 Sampler and zero-order hold.

$$G_{h0}(s) = \frac{1 - e^{-Ts}}{s}$$

3.3 THE PULSE TRANSFER FUNCTION

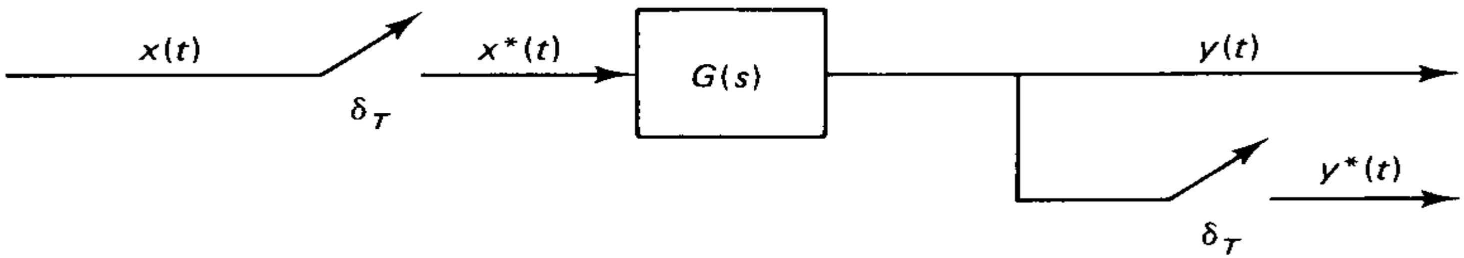
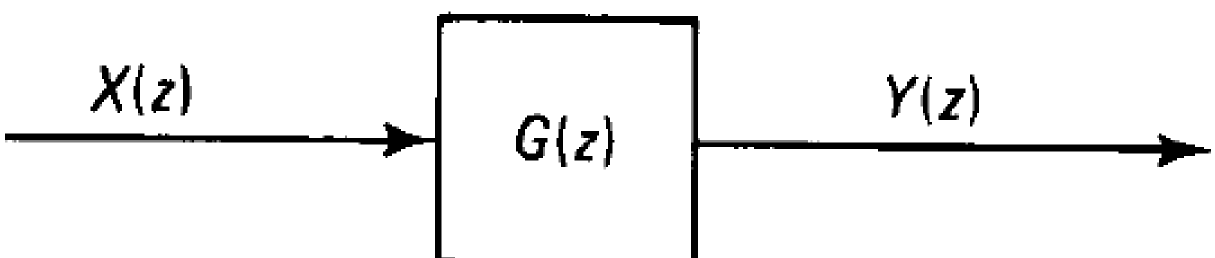


Figure 3–20 Continuous-time system $G(s)$ driven by an impulse-sampled signal.

$$Y(z) = G(z)X(z) \quad \text{or} \quad G(z) = \frac{Y(z)}{X(z)}$$

where

$$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m}, \quad \text{and} \quad X(z) = \sum_{h=0}^{\infty} x(hT)z^{-h}$$



Starred Laplace Transform of the Signal Involving Both Ordinary and Starred Laplace Transforms.

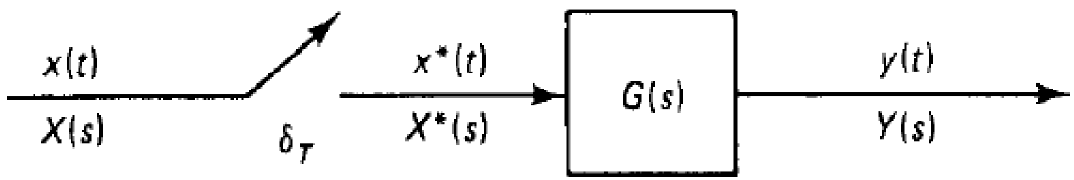


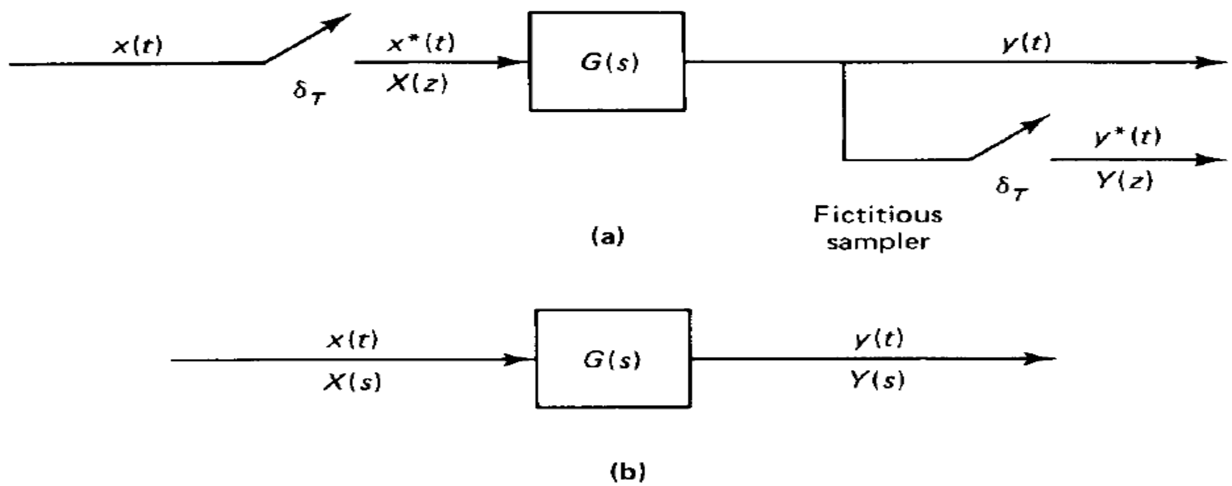
Figure 3–23 Impulse-sampled system.

$$Y(s) = G(s)X^*(s) \qquad (3\text{-}45)$$

We have

$$Y^*(s) = G^*(s)X^*(s) \qquad (3\text{-}47)$$

General Procedure of Obtaining Pulse Transfer Functions.



The presence or absence of the input sampler is crucial.

For Figure 3-24(a), input sampler is present.

$$Y(z) = G(z)X(z)$$

For Figure 3-24(b) without input sampler,

$$Y(z) = \mathcal{Z}[G(s)X(s)] = \mathcal{Z}[GX(s)] = GX(z) \neq G(z)X(z)$$

The presence or absence of a sampler at the output of the element (or the system) does not affect the pulse transfer function

Pulse Transfer Function of DAC and Analog System

If a zero-order hold is included, $X(s)$ will take the following form

$$X(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

with $G(s)$ a proper rational function of s.

$$X(z) = \mathcal{Z}\left[X(s)\right] = (1 - z^{-1})\mathcal{Z}\left|\frac{G(s)}{s}\right| \qquad (3\text{-}32)$$

So, $1 - e^{-Ts}$ **in** $X(s)$ **can simply be replaced by** $1 - z^{-1}$

Pulse Transfer Function of Cascade Elements.

Consider the system shown in Figure 3-25.

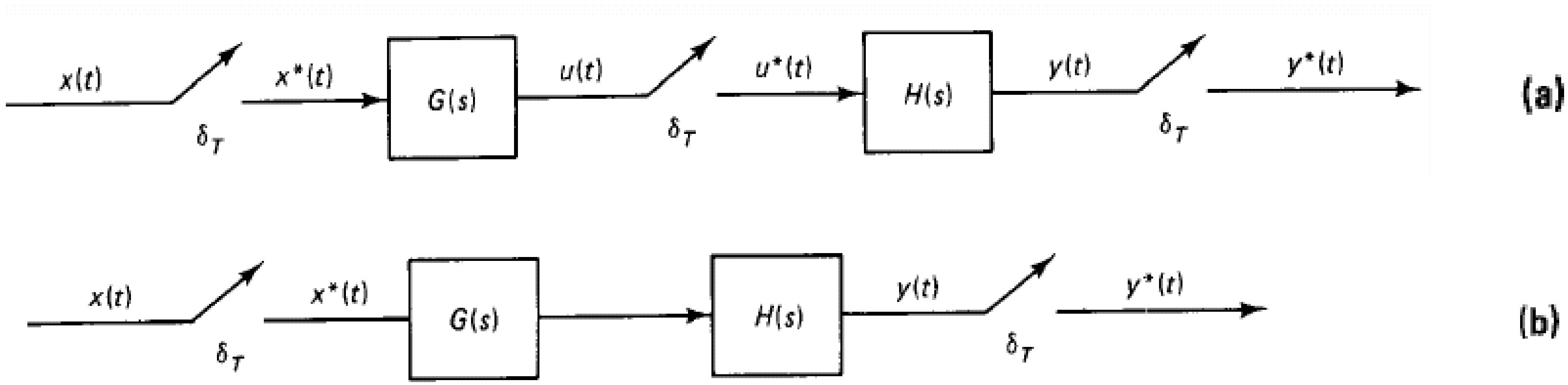


Figure 3–25 (a) Sampled system with a sampler between cascaded elements $G(s)$ and $H(s)$; (b) sampled system with no sampler between cascaded elements $G(s)$ and $H(s)$.

Assume that all samplers are synchronized and have the same sampling period T_s.

As we will see, the pulse transfer functions are given by

- Figure 3-25(a): $G(z)H(z)$
- Figure 3-25(b): $\mathcal{Z}[G(s)H(s)] = \mathcal{Z}[GH(s)] = GH(z) \neq G(z)H(z)$

Pulse Transfer Function of Closed-loop Systems.

Consider the system in Figure 3-27.

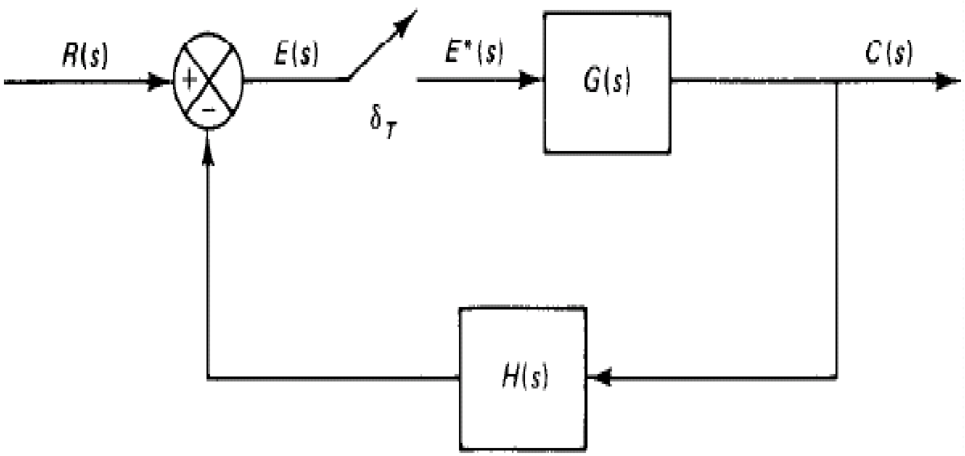


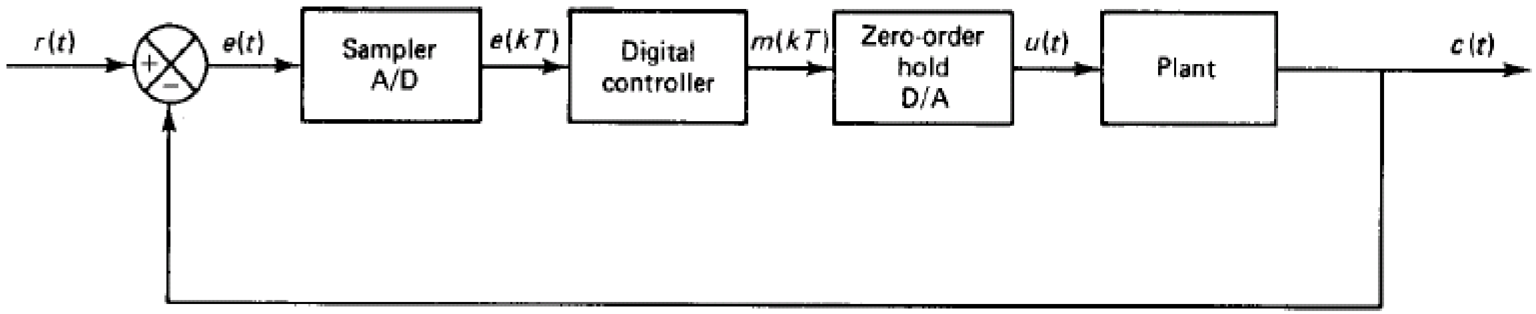
Figure 3–27 Closed-loop control system.

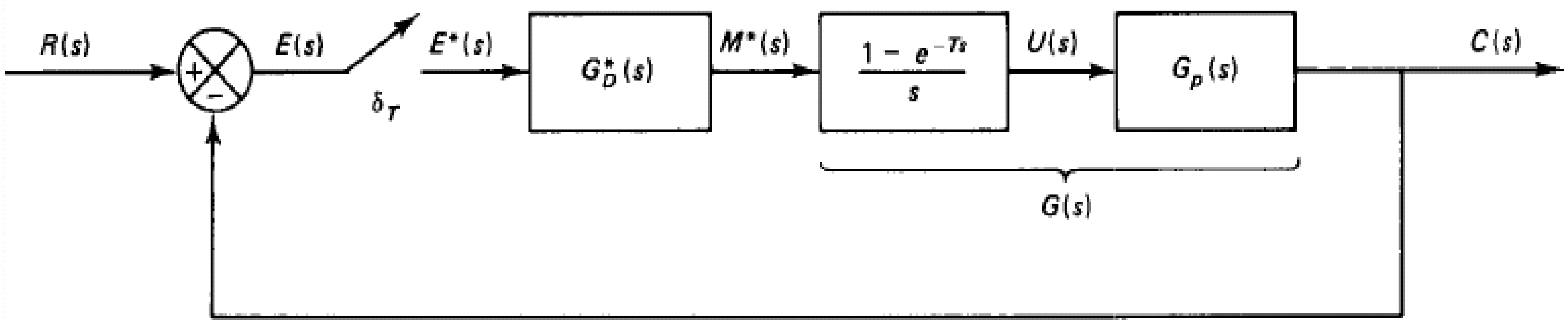
$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

Pulse Transfer Function of a Digital Controller.

$$G_D(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \qquad (3\text{-}52)$$

Closed-loop Pulse Transfer Function of a Digital Control System.





$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} \qquad (3\text{-}53)$$

Pulse Transfer Function of a Digital PID Controller.

$$G_D(z) = \frac{M(z)}{E(z)} = K_P + \frac{K_I}{1 - z^{-1}} + K_D(1 - z^{-1})$$

Chapter 4: Analysis of Discrete-time Control Systems by Conventional Methods

In this chapter, we study the following topics:

- Mapping Between s and z Planes
- Stability Analysis of Closed-loop Systems in z Plane
- Transient and Steady-state Response Analysis

4-2.1 Mapping of Left Half s Plane into the z Plane.

4-2.2 Primary Strip and Complementary Strips.

4-2.3 Constant-Attenuation Loci

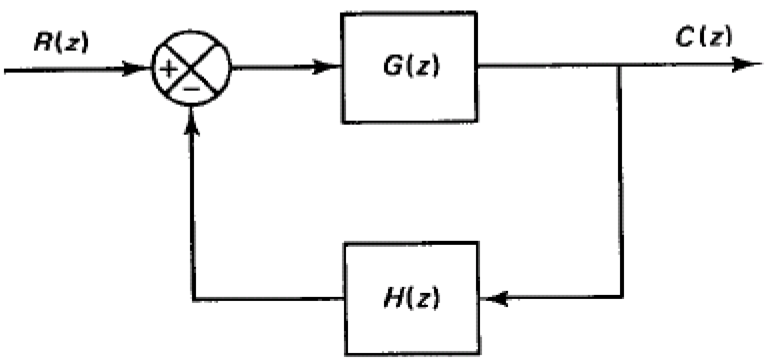
4-2.4 Settling Time t_s **or** T_s

4-2.5 Constant-Frequency Loci.

4-2.6 Constant-Damping-Ratio Loci.

4-2.7 s Plane and z Plane Regions for $\zeta > \zeta_1$

Section 4-3: Stability Analysis of Closed-loop Systems in z Plane



Stability of the system is determined by the poles of the system, i.e., the roots of the characteristic equation (CE) $P(z) = 1 + G(z)H(z) = 0$

- To be stable, all roots of CE must lie inside the unit circle.
- Roots outside the unit circle implies instability
- A simple pole at z = 1 indicates critical stability. A single pair of complex conjugate poles on the unit circle also indicates critical stability. Any multiple pole on the unit circle makes system unstable.
- Closed-loop zeros have no effect on stability and can be anywhere in z plane.

Stability Criterion by the Jury Test.

A system with CE

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n, \quad a_0 > 0$$

is stable if the following conditions are all satisfied:

1. $|a_n| < a_0$

2. $P(z)|_{z=1} > 0$

3. $P(z)|_{z=-1} \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}$

4. $|b_{n-1}| > |b_0|$

$|c_{n-2}| > |c_0|$

$\vdots$

$|q_2| > |q_0|$

Note that, for *2*nd order (*n = 2*), only the first *3* conditions need to be checked. The last condition is null.

Stability Analysis using Bilinear Transformation and Routh Stability Criterion.

4-3.2 Stability Analysis using Bilinear Transformation and Routh Stability Criterion.

The bilinear transformation

$$z = \frac{w+1}{w-1}, \qquad w = \frac{z-1}{z+1}$$

maps the inside of the unit circle in z plane into the left half of the w plane. Let w = σ± jω. Then the unit circle in z plane is |z| < 1, i.e.

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\sigma + j\omega + 1}{\sigma + j\omega - 1}\right| < 1 \qquad \Leftrightarrow \quad \frac{(\sigma+1)^2 + \omega^2}{(\sigma-1)^2 + \omega^2} < 1$$

$$\Leftrightarrow \quad (\sigma+1)^2 + \omega^2 < (\sigma-1)^2 + \omega^2 \quad \Leftrightarrow \quad \sigma < 0$$

Hence, by replacing z with $\frac{w+1}{w-1}$ in P(z), we can check the stability of the roots of P(z) using the continuous-time ***Routh Stability Criterion***.

4-4.1 Transient Response Specifications.

A typical Unit step response of a system (*without integrator*) is shown in Figure 4-13.

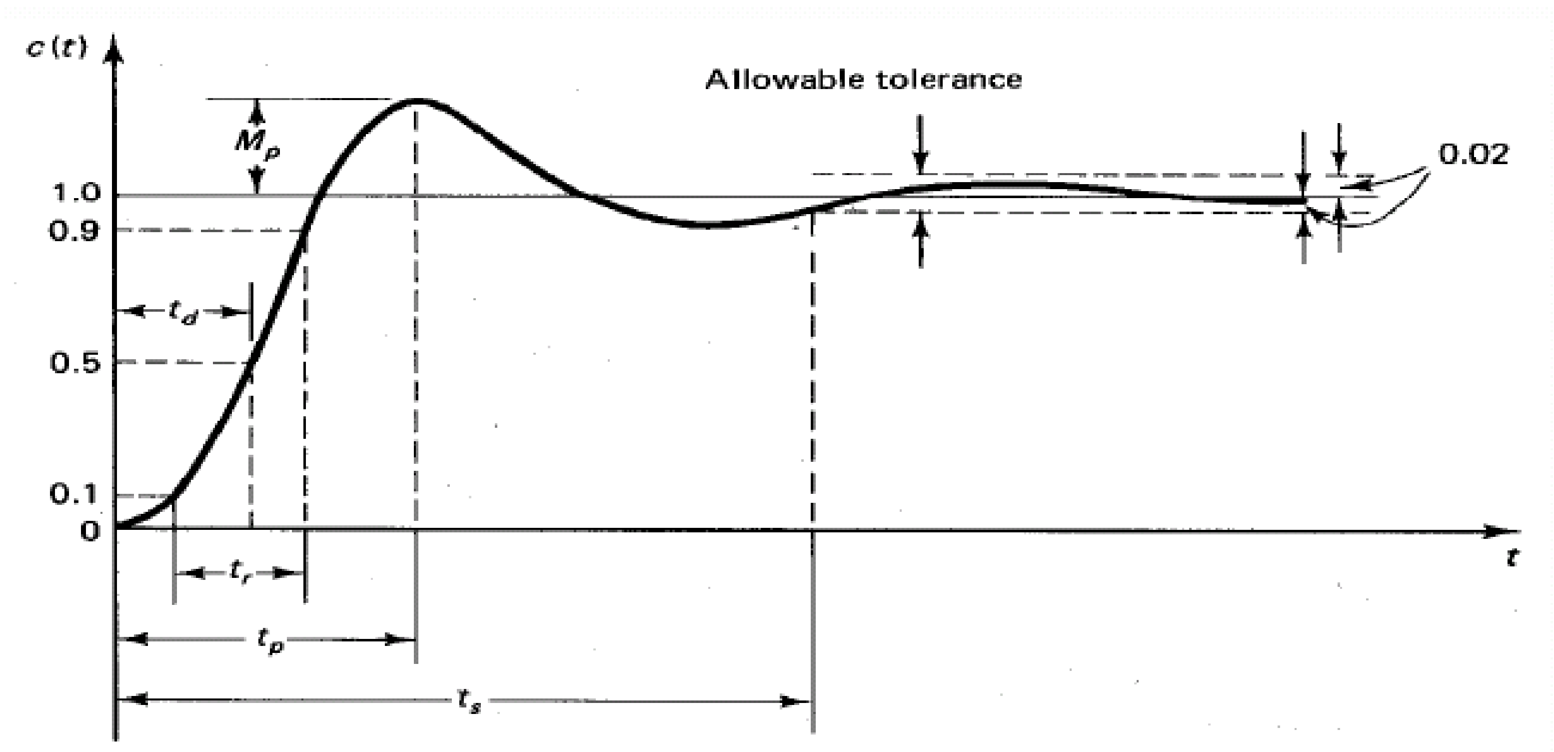


Figure 4-13. **Unit-step response curve showing transient response specifications t_d, t_r, t_p, M_p, and t_s.**

The Transient Response Specifications

Steady-state error analysis

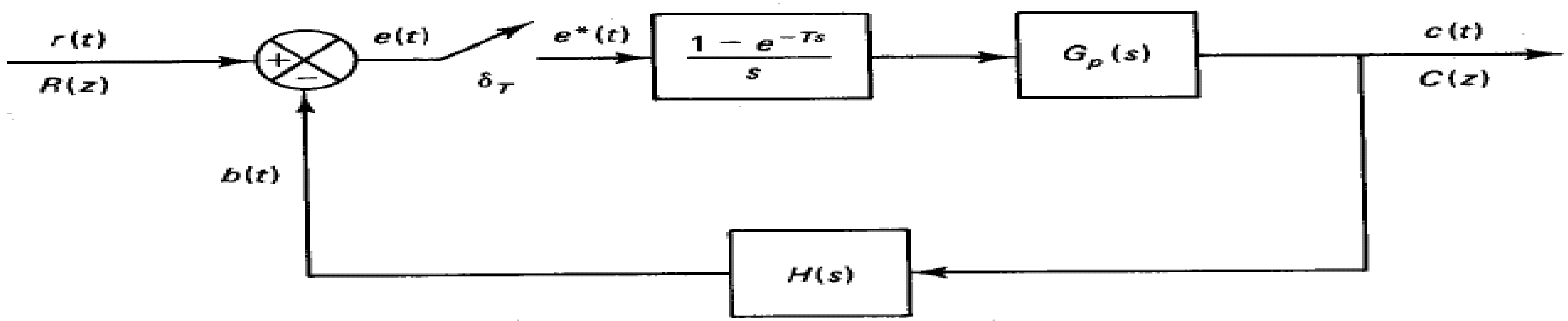
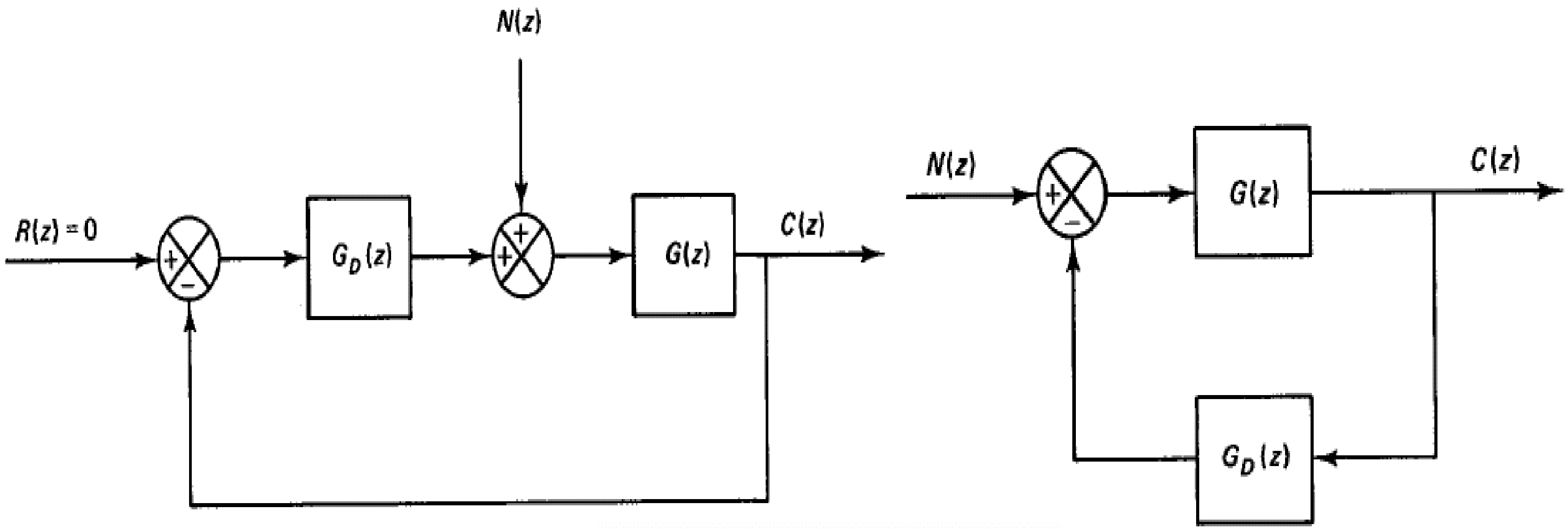


TABLE 4–4 SYSTEM TYPES AND THE CORRESPONDING STEADY-STATE ERRORS IN RESPONSE TO STEP, RAMP, AND ACCELERATION INPUTS FOR THE DISCRETE-TIME CONTROL SYSTEM SHOWN IN FIGURE 4–18

System	Steady-state errors in response to		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Acceleration input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\frac{1}{K_v}$	∞
Type 2 system	0	0	$\frac{1}{K_a}$

Response to Disturbances



If $G_D(z)G(z) >> 1$, then $E(z) \approx -\frac{1}{G_D(z)}N(z)$

So large gain of $G_D(z)$ gives small error $E(z)$.

If $G_D(z)$ includes a pure integrator (i.e. it has a pole at $z = 1$), then steady-state error due to constant disturbance is zero.