

Lecture 9: State Feedback and State Estimation

9.1 State Feedback

Consider a SISO system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad \mathbf{x} \in \mathbf{R}^n \\ y(t) &= \mathbf{c}\mathbf{x}(t), \quad u, y \in \mathbf{R}\end{aligned}\tag{1}$$

State feedback law

$$u = r - \mathbf{k}\mathbf{x} = r - [k_1, k_2, \dots, k_n]\mathbf{x} = r - \sum_{i=1}^n k_i x_i\tag{2}$$

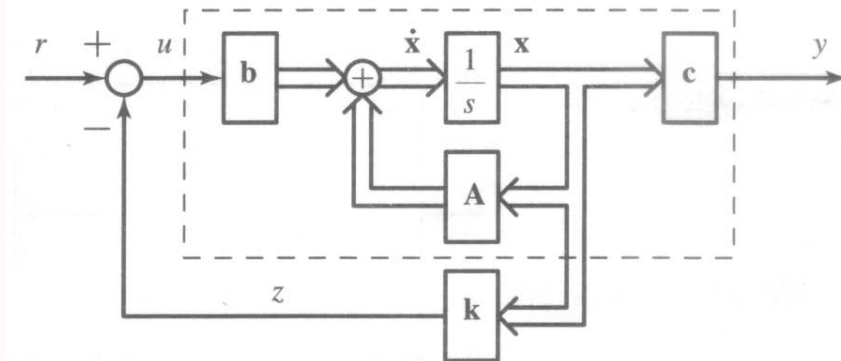


Figure 8.2 State feedback.

Closed-loop system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}(t) + \mathbf{b}r(t) \\ y(t) &= \mathbf{c}\mathbf{x}(t)\end{aligned}\tag{3}$$

Theorem 9.1 The pair $(\mathbf{A} - \mathbf{b}\mathbf{k}, \mathbf{b})$, for any $1 \times n$ real constant vector \mathbf{k} , is controllable iff (\mathbf{A}, \mathbf{b}) is controllable.

The proof follows by realizing that

$$\begin{aligned}\mathcal{C}_f &= [\mathbf{b}, (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{b}, (\mathbf{A} - \mathbf{b}\mathbf{k})^2\mathbf{b}, \dots, (\mathbf{A} - \mathbf{b}\mathbf{k})^{n-1}\mathbf{b}] \\ &= [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \dots, \mathbf{A}^n\mathbf{b}] \\ &\quad \times \begin{bmatrix} 1 & -\mathbf{k}\mathbf{b} & -\mathbf{k}(\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{b} & \dots & -\mathbf{k}(\mathbf{A} - \mathbf{b}\mathbf{k})^{n-2}\mathbf{b} \\ 0 & 1 & -\mathbf{k}\mathbf{b} & \dots & -\mathbf{k}(\mathbf{A} - \mathbf{b}\mathbf{k})^{n-2}\mathbf{b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\mathbf{k}\mathbf{b} \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}\end{aligned}\tag{4}$$

The theorem can also be proven by the definition of controllability:

If (1) is controllable, \exists input $u_1(t)$ to transfer from any initial state \mathbf{x}_0 to any final state \mathbf{x}_1 . Now, by choosing $r_1 = u_1 + \mathbf{k}\mathbf{x}$, the input r_1 transfers \mathbf{x}_0 to \mathbf{x}_1 .

Note that the external input $r(t)$ controls the state through u !

Although controllability is invariant under state feedback, observability is not!

Example 9.1 Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(t)\end{aligned}$$

With the control law

$$u = r - [3 \ 1]\mathbf{x}$$

the closed-loop system becomes

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(t)\end{aligned}$$

The controllability matrix is

$$\mathcal{C}_f = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \text{nonsingular.} \quad \text{Hence, controllable!}$$

while the observability matrix is

$$\mathcal{O}_f = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \text{singular.} \quad \text{Hence, unobservable!}$$

What can be achieved by state feedback, given controllability?

—— Arbitrary eigenvalue assignment!

Example 9.2 Consider a plant described by

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Note that the A -matrix has characteristic polynomial (CP)

$$\Delta(s) = (s - 1)^2 - 9 = (s - 4)(s + 2)$$

which has roots at 4 and -2, and hence is unstable.

Introduce the state feedback $u = r - [k_1 \ k_2]x$. Then, the closed-loop system is

$$\dot{x} = \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

which has the CP

$$\Delta_f(s) = (s - 1 + k_1)(s - 1) - 3(3 - k_2) = s^2 + (k_1 - 2)s + (3k_2 - k_1 - 8)$$

By designing k_1 and k_2 properly, we can place the closed-loop poles at any locations. For example, if we want to place the closed-loop poles at $-1 \pm j2$. Then, the desired CP is $(s + 1 + j2)(s + 1 - j2) = s^2 + 2s + 5$. By setting

$$k_1 - 2 = 2, \quad 3k_2 - k_1 - 8 = 5 \implies k_1 = 4, \quad k_2 = 17/3$$

We now present more systematic approaches for state feedback design.

Theorem 9.2 Consider the state equation (1) with $n = 4$ and its CP

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 \quad (5)$$

If (1) is controllable, \exists a state transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} \triangleq \mathbf{P}^{-1} = [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \mathbf{A}^3\mathbf{b}] \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

that transforms (1) into the controllable canonical form

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{b}}\mathbf{u}(t) = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u} \\ y(t) &= \bar{\mathbf{c}}\bar{\mathbf{x}}(t) = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4] \end{aligned} \quad (7)$$

Furthermore, we have

$$\bar{\mathbf{A}} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \mathbf{P} \mathbf{A} \mathbf{P}^{-1}, \quad \bar{\mathbf{b}} = \mathbf{Q}^{-1} \mathbf{b} = \mathbf{P} \mathbf{b} \quad \bar{\mathbf{c}} = \mathbf{c} \mathbf{Q} = \mathbf{c} \mathbf{P}^{-1}$$

and the transfer function for both (1) and (7) is

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_n}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \quad (8)$$

Proof: Let \mathcal{C} and $\bar{\mathcal{C}}$ be the controllability matrix of (1) and (7), resp. For SISO, both \mathcal{C} and $\bar{\mathcal{C}}$ are square. If (1) is controllable, then \mathcal{C} and $\bar{\mathcal{C}}$ are both nonsingular, and are related by

$$\bar{\mathcal{C}} = \mathbf{Q}^{-1} \mathcal{C} \text{ or } \bar{\mathcal{C}} = \mathbf{P} \mathcal{C}$$

Thus, we have

$$\mathbf{P} = \bar{\mathcal{C}} \mathcal{C}^{-1} \quad \text{or} \quad \mathbf{Q} = \mathcal{C} \bar{\mathcal{C}}^{-1} \quad (8a)$$

The controllability matrix of (7) can be computed as

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -\alpha_1^3 + 2\alpha_1\alpha_2 - \alpha_3 \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whose inverse turns out to be

$$\bar{\mathcal{C}}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

Substituting (9) into (8a) yield (6). Furthermore, straightforward calculation verifies (8). \square

Theorem 9.3 If (1) is controllable, then the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{k}$ can be arbitrarily assigned by using the state feedback (2), provided that complex conjugate eigenvalues are assigned in pairs.

Proof: Use $n = 4$ to illustrate. If (1) is controllable, then it can be transformed into the controllable canonical form (7).

Then, from (8a), we have

$$u = r - \mathbf{k}\mathbf{x} = r - \mathbf{k}\mathbf{P}^{-1}\bar{\mathbf{x}} \triangleq r - \bar{\mathbf{k}}\bar{\mathbf{x}}$$

with $\bar{\mathbf{k}} = \mathbf{k}\mathbf{P}^{-1}$.

Then

$$\mathbf{P}(\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{P}^{-1} = \bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}$$

From any desired set of eigenvalues, we can get

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4 \quad (10)$$

and choose

$$\bar{\mathbf{k}} = \begin{bmatrix} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{bmatrix} \quad (11)$$

i.e.,

$$\mathbf{k} = \begin{bmatrix} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{bmatrix} \bar{\mathcal{C}}\mathcal{C}^{-1} \quad (11a)$$

Then the closed-loop system becomes

$$\begin{aligned}\dot{\bar{\mathbf{x}}}(\mathbf{t}) &= (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}}(\mathbf{t}) + \bar{\mathbf{b}}r(\mathbf{t}) \\ &= \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r \\ y(t) = \bar{\mathbf{c}}\bar{\mathbf{x}}(t) &= [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]\end{aligned}\quad (12)$$

which, because of the companion form, has (10) as its characteristic polynomial, and so does $\mathbf{A} - \mathbf{b}\mathbf{k}$ (which is similar to $\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}$). Hence,

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \bar{\mathbf{k}}\bar{\mathbf{C}}\bar{\mathbf{C}}^{-1}\quad (13)$$

This completes the proof. □

An alternative proof:

$$\begin{aligned}\Delta_f(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{k}) = \det[(s\mathbf{I} - \mathbf{A})[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{k}]] \\ &= \det(s\mathbf{I} - \mathbf{A}) \det[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{k}]\end{aligned}$$

i.e.,

$$\Delta_f(s) = \Delta(s)[1 + \mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}]$$

Thus,

$$\Delta_f(s) - \Delta(s) = \Delta(s)\mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \Delta(s)\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} \quad (14)$$

Let $z(t)$ be the output of the feedback gain (as shown in Figure 2), and let

$$\bar{\mathbf{k}} = [\bar{k}_1, \bar{k}_2, \dots, \bar{k}_n]$$

Then, the transfer function from u to z is

$$\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{\bar{k}_1 s^n + \dots + \bar{k}_{n-1}s + \bar{k}_n}{\Delta(s)} \quad (15)$$

and the transfer function from u to y is

$$\bar{\mathbf{c}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{\bar{\beta}_1 s^n + \dots + \bar{\beta}_{n-1}s + \bar{\beta}_n}{\Delta(s)}$$

Substituting (15), (5) and (10) into (14) gives

$$(\bar{\alpha}_1 - \alpha_1)s^n + \dots + (\bar{\alpha}_{n-1} - \alpha_{n-1})s + (\bar{\alpha}_n - \alpha_n) = \bar{k}_1 s^n + \dots + \bar{k}_{n-1}s + \bar{k}_n$$

i.e., (11).

Comparing the transfer functions of the open loop ($\mathbf{A}, \mathbf{b}, \mathbf{c}$) and the closed-loop ($\mathbf{A} - \mathbf{b}\mathbf{k}, \mathbf{b}, \mathbf{c}$)

$$\hat{g}(s) = \frac{\beta_1 s^n + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n} \quad (16)$$

$$\hat{g}_f(s) = \frac{\beta_1 s^n + \cdots + \beta_{n-1}s + \beta_n}{s^n + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_{n-1}s + \bar{\alpha}_n} \quad (17)$$

Hence, State feedback affects only the poles of the system!

State feedback has no effect on the zeros of the system!

Unobservability results if pole-zero cancellation in $\hat{g}_f(s)$!

There are various formula for calculating the feedback gain for pole-placement:

Ackermann's Formula

$$\mathbf{k} = [0 \ 0 \ \cdots \ 0 \ 1] [\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{n-1}\mathbf{b}]^{-1} \phi(\mathbf{A})$$

where

$$\phi(\mathbf{A}) = \mathbf{A}^n + \bar{\alpha}_1 \mathbf{A}^{n-1} + \bar{\alpha}_2 \mathbf{A}^{n-2} + \cdots + \bar{\alpha}_n \mathbf{I}$$

Diagonal Canonical Form Formula

If the desired eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ of the closed-loop system are all distinct, then

$$\mathbf{k} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix}^{-1}$$

where $n \times 1$ vectors $\xi_1, \xi_2, \dots, \xi_n$ are given by

$$\xi_i = (\mathbf{A} - \mu_i \mathbf{I})^{-1} \mathbf{b}, \quad i = 1, 2, \dots, n$$

with μ_i 's the desired eigenvalues of the closed-loop system.

Note that ξ_i 's are just eigenvectors of the closed-loop system matrix $\mathbf{A} - \mathbf{b}\mathbf{k}$, i.e.,

$$(\mathbf{A} - \mathbf{b}\mathbf{k})\xi_i = \mu_i \xi_i$$

Note: This formula applies only when the desired eigenvalues are all distinct; and ξ_i 's and μ_i 's can be complex.

Of course, when calculating by hand, one would use a more direct approach, instead of these formulae:

Desired Characteristic Polynomial =

$$\Delta_f(s) = s^n + \bar{\alpha}_1 s^{n-1} + \bar{\alpha}_2 s^{n-2} + \cdots + \bar{\alpha}_{n-1} s + \bar{\alpha}_n$$

Actual Characteristic Polynomial =

$$\det[s\mathbf{I} - (\mathbf{A} - bk)]$$

Then, equating coefficients of like powers of s in the above, we get n equations in n unknowns.

Solve these n equations to get $\mathbf{k} = [k_1, k_2, \cdots, k_n]$.

Preferred Region for Eigenvalues:

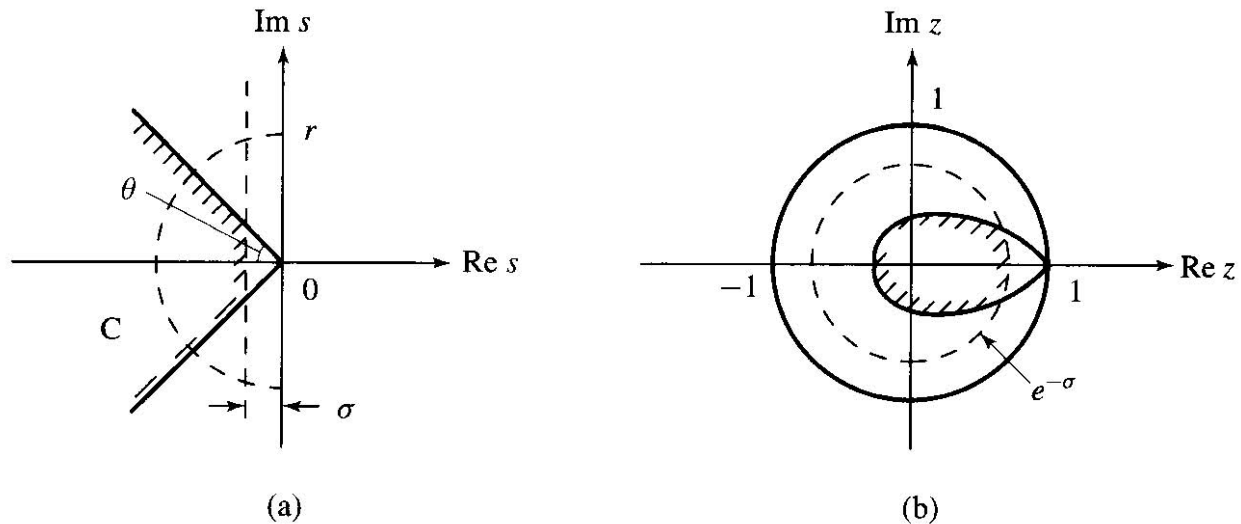


Figure 8.3 Desired eigenvalue location.

Let the desired eigenvalues be of the form

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Then

$$\sigma = \zeta\omega_n, \quad r = \omega_n$$

NOTE:

- Absolute Stability σ : the larger the $|\sigma|$, the larger the actuating signal (control u).
- Good damping range: around 0.7.
- Pole-clustering usually results in slow response and large u .
Can place eigenvalues evenly on circle of radius r inside the region.
- Radius r : the larger the radius r , the larger the actuating signal.

Solving the Lyapunov Equation

Consider controllable (\mathbf{A}, \mathbf{b}) with $\mathbf{A} \in \mathbf{R}^{n \times n}$ and $\mathbf{b} \in \mathbf{R}^{n \times 1}$. The following method of eigenvalue assignment is applicable when the selected eigenvalues DO NOT contain any eigenvalues of \mathbf{A} .

Procedure 9.1

1. Select an $n \times n$ matrix \mathbf{F} that has the set of desired eigenvalues (which does not contain any eigenvalues of \mathbf{A}).
2. Select an arbitrary $\bar{\mathbf{k}} \in \mathbf{R}^{1 \times n}$ such that $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.
3. Solve the unique \mathbf{T} in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$.
4. Computer the feedback gain $\mathbf{k} = \bar{\mathbf{k}}\mathbf{T}^{-1}$.

Justification: If \mathbf{T} is nonsingular, then $\bar{\mathbf{k}} = \mathbf{k}\mathbf{T}$, and

$$\begin{aligned} \mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}} &\implies (\mathbf{A} - \mathbf{bk})\mathbf{T} = \mathbf{TF} \\ &\text{or } (\mathbf{A} - \mathbf{bk}) = \mathbf{TFT}^{-1} \end{aligned}$$

So $(\mathbf{A} - \mathbf{bk})$ and \mathbf{F} have the same set of eigenvalues.

Why \mathbf{F} and \mathbf{A} can not have common eigenvalues?

Consider Lyapunov equation

$$\mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{B} = \mathbf{C} \quad \text{or } \mathcal{A}(\mathbf{M}) = \mathbf{C} \text{ with } \mathcal{A}(\mathbf{M}) \triangleq \mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{B}$$

where $\mathbf{A} \in \mathbf{R}^{n \times n}$, $\mathbf{B} \in \mathbf{R}^{m \times m}$, and $\mathbf{C}, \mathbf{M} \in \mathbf{R}^{n \times m}$. We can also consider $\mathcal{A} : \mathbf{R}^{n \times m} \rightarrow \mathbf{R}^{n \times m}$, or simply $\mathcal{A} : \mathbf{R}^{nm} \rightarrow \mathbf{R}^{nm}$.

Then \mathcal{A} has nm eigenvalues, and η is an eigenvalue of \mathcal{A} if

$$\mathcal{A}(\mathbf{M}) = \eta \mathbf{M}, \quad \text{for some } \mathbf{M} \neq 0, \mathbf{M} \in \mathbf{R}^{nm}$$

Suppose that

$\lambda_i, i = 1, 2, \dots, n$ are the eigenvalues of \mathbf{A} , and

$\mu_j, j = 1, 2, \dots, m$ are eigenvalues of \mathbf{B} .

Namely,

$\exists \mathbf{x}_i \in \mathbf{R}^{n \times 1}, \mathbf{x}_i \neq 0$, such that $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$, and

$\exists \mathbf{y}_j \in \mathbf{R}^{1 \times m}, \mathbf{y}_j \neq 0$, such that $\mathbf{y}_j \mathbf{B} = \mu_j \mathbf{y}_j$.

Then $\eta_k = \lambda_i + \mu_j, i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, nm$, are the eigenvalues of \mathcal{A} :

$$\mathcal{A}(\mathbf{x}_i \mathbf{y}_j) = \mathbf{A} \mathbf{x}_i \mathbf{y}_j + \mathbf{x}_i \mathbf{y}_j \mathbf{B} = \lambda_i \mathbf{x}_i \mathbf{y}_j + \mathbf{x}_i \mathbf{y}_j \mu_j = (\lambda_i + \mu_j) \mathbf{x}_i \mathbf{y}_j$$

Furthermore, $\det \mathcal{A} = \prod_{k=1}^{nm} \eta_k$. Hence, \mathcal{A} is nonsingular if $\eta_k = \lambda_i + \mu_j \neq 0$, in which case, $\mathcal{A}(\mathbf{M}) = \mathbf{C}$ has a unique solution.

Therefore, $\mathbf{A}\mathbf{T} - \mathbf{T}\mathbf{F} = \mathbf{b}\bar{\mathbf{k}}$ has a unique solution \mathbf{T} if \mathbf{A} and \mathbf{F} have no common eigenvalues.

Is the solution \mathbf{T} nonsingular? Yes, under certain conditions!

Theorem 9.4 If \mathbf{A} and \mathbf{F} have no eigenvalues in common, then the unique solution \mathbf{T} of $\mathbf{A}\mathbf{T} - \mathbf{T}\mathbf{F} = \mathbf{b}\bar{\mathbf{k}}$ is nonsingular iff (\mathbf{A}, \mathbf{b}) is controllable and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.

Proof Take $n = 4$. Let

$$\Delta(s) \triangleq \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 \quad (20)$$

$$\Delta(\mathbf{A}) = \mathbf{A}^4 + \alpha_1 \mathbf{A}^3 + \alpha_2 \mathbf{A}^2 + \alpha_3 \mathbf{A} + \alpha_4 \mathbf{I} = \mathbf{0}$$

$$\Delta(\mathbf{F}) = \mathbf{F}^4 + \alpha_1 \mathbf{F}^3 + \alpha_2 \mathbf{F}^2 + \alpha_3 \mathbf{F} + \alpha_4 \mathbf{I} \quad (21)$$

If $\bar{\lambda}_i$ is an eigenvalue of \mathbf{F} , then $\Delta(\bar{\lambda}_i)$ is an eigenvalue of $\Delta(\mathbf{F})$, and

$$\det [\Delta(\mathbf{F})] = \prod_{i=1}^n \Delta(\bar{\lambda}_i)$$

Since \mathbf{A} and \mathbf{F} have no eigenvalues in common, we have

$$\Delta(\bar{\lambda}_i) \neq 0, \quad \text{for all eigenvalues } \bar{\lambda}_i \text{ of } \mathbf{F}$$

Hence, $\Delta(\mathbf{F})$ is nonsingular. Note that

$$\begin{aligned} \mathbf{AT} - \mathbf{TF} &= \mathbf{b}\bar{\mathbf{k}}, & \mathbf{AT} &= \mathbf{TF} + \mathbf{b}\bar{\mathbf{k}} \\ \mathbf{A}^2\mathbf{T} - \mathbf{TF}^2 &= \mathbf{A}(\mathbf{TF} + \mathbf{b}\bar{\mathbf{k}}) - \mathbf{TF}^2 = \mathbf{Ab}\bar{\mathbf{k}} + (\mathbf{AT} - \mathbf{TF})\mathbf{F} \\ &= \mathbf{Ab}\bar{\mathbf{k}} - \mathbf{b}\bar{\mathbf{k}}\mathbf{F} \end{aligned}$$

Proceeding forward, we have

$$\begin{aligned} \mathbf{A}^0\mathbf{T} - \mathbf{TF}^0 &= \mathbf{0} \\ \mathbf{AT} - \mathbf{TF} &= \mathbf{b}\bar{\mathbf{k}} \\ \mathbf{A}^2\mathbf{T} - \mathbf{TF}^2 &= \mathbf{Ab}\bar{\mathbf{k}} + \mathbf{b}\bar{\mathbf{k}}\mathbf{F} \\ \mathbf{A}^3\mathbf{T} - \mathbf{TF}^3 &= \mathbf{A}^2\mathbf{b}\bar{\mathbf{k}} + \mathbf{Ab}\bar{\mathbf{k}}\mathbf{F} + \mathbf{b}\bar{\mathbf{k}}\mathbf{F}^2 \\ \mathbf{A}^4\mathbf{T} - \mathbf{TF}^4 &= \mathbf{A}^3\mathbf{b}\bar{\mathbf{k}} + \mathbf{A}^2\mathbf{b}\bar{\mathbf{k}}\mathbf{F} + \mathbf{Ab}\bar{\mathbf{k}}\mathbf{F}^2 + \mathbf{b}\bar{\mathbf{k}}\mathbf{F}^3 \end{aligned}$$

Multiplying $\alpha_4, \alpha_3, \alpha_2, \alpha_1$ and 1, resp., into the above equations and sum them up, we have

$$\begin{aligned}
\text{LHS} &= \Delta(\mathbf{A})\mathbf{T} - \mathbf{T}\Delta(\mathbf{F}) = -\mathbf{T}\Delta(\mathbf{F}) \\
\text{RHS} &= [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \mathbf{A}^3\mathbf{b}] \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{k}} \\ \bar{\mathbf{k}}\mathbf{F} \\ \bar{\mathbf{k}}\mathbf{F}^2 \\ \bar{\mathbf{k}}\mathbf{F}^3 \end{bmatrix}
\end{aligned} \tag{22}$$

As $\Delta(\mathbf{F})$ is nonsingular, we have that \mathbf{T} is nonsingular iff (\mathbf{A}, \mathbf{b}) is controllable and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable. \square

Therefore, we can select \mathbf{F} to have the desired e-values and e-vector structures (so long as \mathbf{F} and \mathbf{A} have no common e-values), and select $\bar{\mathbf{k}}$ with $(\mathbf{F}, \bar{\mathbf{k}})$ observable, then the solution \mathbf{T} of $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$ is unique and nonsingular, and the feedback gain

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{T}^{-1}$$

with $\mathbf{A} - \mathbf{b}\mathbf{k}$ similar to \mathbf{F} (hence have the same e-values).

9.2 Regulation and Tracking

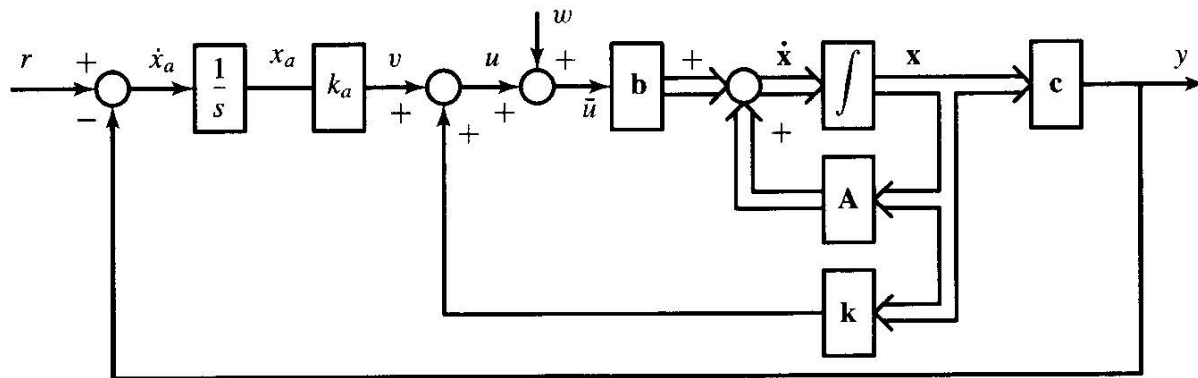
Consider the feedback system shown in Fig. 8.2. *Regulation*: When $r = 0$, the response of the system is caused by nonzero initial condition.

$$y(t) = \mathbf{c}e^{(\mathbf{A}-\mathbf{b}k)t}\mathbf{x}(0)$$

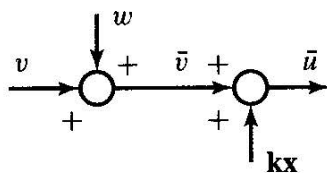
The problem is to find a state feedback so that the response will die out at a desired rate. This is called a *regulation problem*. That is to say, we need to find a state feedback to stabilize the system.

(*Asymptotic*) *Tracking*: Stabilize the system when r (the set point) is constant $\neq 0$ and further, $y(t)$ approaches r when $t \rightarrow \infty$.

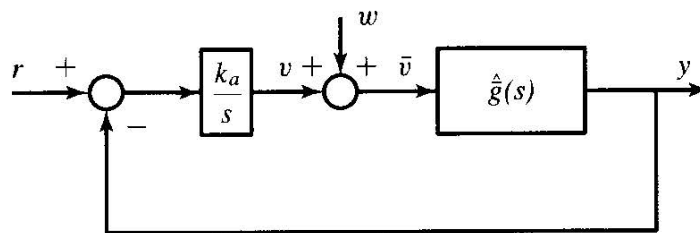
Servomechanism: Tracking time-varying reference signal $r(t)$.



(a)



(b)



(c)

Figure 8.4 (a) State feedback with internal model. (b) Interchange of two summers. (c) Transfer-function block diagram.

9.2.1 Robust Tracking and Disturbance Rejection

Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) + \mathbf{b}w(t), \quad \mathbf{x} \in \mathbf{R}^n \\ y(t) &= \mathbf{c}\mathbf{x}(t), \quad u, w, y \in \mathbf{R}\end{aligned}\tag{26}$$

where w is the disturbance.

Objective: Make $y(t)$ track asymptotically any step reference r , in the presence of disturbance w and plant parameter variations.

In order to eliminate steady-state tracking error (for constant r), integral action on tracking error $x_a = r - y$ is introduced as shown in Figure 8.4.

That is,

$$\dot{x}_a = r - y = r - \mathbf{c}\mathbf{x}\tag{27}$$

$$u = \begin{bmatrix} \mathbf{k} & k_a \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}\tag{28}$$

The complete augmented system is

$$\begin{aligned}\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_a \end{bmatrix} &= \begin{bmatrix} \mathbf{A} + \mathbf{b}\mathbf{k} & \mathbf{b}k_a \\ -\mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} w \\ y &= \begin{bmatrix} \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}\end{aligned}\tag{29}$$

Theorem 9.5 If (\mathbf{A}, \mathbf{b}) is controllable and if $\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ has no zero at $s = 0$, then all eigenvalues of $\begin{bmatrix} \mathbf{A} + \mathbf{b}\mathbf{k} & \mathbf{b}k_a \\ -\mathbf{c} & 0 \end{bmatrix}$ can be arbitrarily assigned by selecting a feedback gain $\begin{bmatrix} \mathbf{k} & k_a \end{bmatrix}$.

Proof: Need to show that $(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix})$ is controllable. Take $n = 4$ and assume \mathbf{A} , \mathbf{b} and \mathbf{c} are in controllable canonical form.

$$\begin{aligned} \bar{\mathbf{C}} &= \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} & \mathbf{A}^4\mathbf{b} \\ 0 & -\mathbf{c}\mathbf{b} & -\mathbf{c}\mathbf{A}\mathbf{b} & -\mathbf{c}\mathbf{A}^2\mathbf{b} & -\mathbf{c}\mathbf{A}^3\mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -(\alpha_1^2 - \alpha_2)\alpha_1 + \alpha_1\alpha_2 - \alpha_3 & \bar{\mathbf{C}}_{1,5} \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & \bar{\mathbf{C}}_{1,4} \\ 0 & 0 & 1 & -\alpha_1 & \bar{\mathbf{C}}_{2,4} \\ 0 & 0 & 0 & 1 & -\alpha_1 \\ 0 & -\beta_1 & \beta_1\alpha_1 - \beta_2 & -(\beta_1\alpha_1 - \beta_2)\alpha_1 + \beta_1\alpha_2 - \beta_3 & \bar{\mathbf{C}}_{5,5} \end{bmatrix} \end{aligned}$$

where

$$\bar{\mathbf{C}}_{5,5} = (\beta_1\alpha_1 - \beta_2)(\alpha_1^2 - \alpha_2) - (\beta_1\alpha_2 - \beta_3)\alpha_1 + \beta_1\alpha_3 - \beta_4$$

Adding to the last row: $\beta_1 \times \text{row 2}$, $\beta_2 \times \text{row 3}$, and $\beta_3 \times \text{row 4}$, we have

$$\begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -(\alpha_1^2 - \alpha_2)\alpha_1 + \alpha_1\alpha_2 - \alpha_3 & \bar{\mathcal{C}}_{1,5} \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & \bar{\mathcal{C}}_{1,4} \\ 0 & 0 & 1 & -\alpha_1 & \bar{\mathcal{C}}_{2,4} \\ 0 & 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 0 & -\beta_4 \end{bmatrix} \quad (31)$$

which is nonsingular iff $\beta_4 \neq 0$.

Note that $\beta_4 \neq 0$ is exactly the condition that the plant has no zero at $s = 0$. Hence, $(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix})$ is controllable, and Theorem 9.5 follows from Theorem 9.3. The proof is complete. \square

The controllability of $\left(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right)$ can also be deduced from pole-zero cancellation.

If there is a zero at $s = 0$, then there will be pole-zero cancellation with the integrator in Figure 8.4, hence, losing controllability.

If no zero at $s = 0$, then no cancellation. Hence controllable.

Let the desired characteristic polynomial for (29) be $\Delta_f(s)$. Then

$$\Delta_f(s) = \det \begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k} & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix}, \quad \text{of degree } n + 1 \quad (32)$$

Now we show output y will track asymptotically and robustly any step reference $r(t) = a$, and reject any step disturbance with unknown magnitude.

We modify the block diagram as shown in Figure 8.4(a). Then the transfer function from \bar{v} to y is

$$T_{y\bar{v}}(s) \triangleq \hat{g}(s) \triangleq \frac{\bar{N}(s)}{\bar{D}(s)} \triangleq \mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k})^{-1}\mathbf{b} \quad (33)$$

with

$$\bar{D}(s) = \det(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k})$$

Hence, we have Figure 8.4(c).

Next, we link $\Delta_f(s)$ in (32) to $\hat{g}(s)$ in (33).

Note that

$$\begin{aligned} & \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k})^{-1} & 1 \end{bmatrix} \begin{bmatrix} (s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k}) & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix} \\ &= \begin{bmatrix} (s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k}) & -\mathbf{b}k_a \\ 0 & s + \mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{k})^{-1}\mathbf{b}k_a \end{bmatrix} \end{aligned}$$

Taking the determinant and using (32) and (33), we have

$$1 \cdot \Delta_f(s) = \bar{D}(s) \left(s + \frac{\bar{N}(s)}{\bar{D}(s)} k_a \right)$$

hence

$$\Delta_f(s) = s\bar{D}(s) + k_a\bar{N}(s)$$

From Figure 8.4(c), we have

$$T_{yw}(s) = \frac{\frac{\bar{N}(s)}{\bar{D}(s)}}{1 + \frac{k_a\bar{N}(s)}{s\bar{D}(s)}} = \frac{s\bar{N}(s)}{s\bar{D}(s) + k_a\bar{N}(s)} = \frac{s\bar{N}(s)}{\Delta_f(s)}$$

For any constant disturbance $w(t) = \bar{w}$,

$$y(s) = T_{yw}(s)w(s) = \frac{s\bar{N}(s)}{\Delta_f(s)} \frac{\bar{w}}{s} = \frac{\bar{w}\bar{N}(s)}{\Delta_f(s)} \quad (34)$$

Given that $\Delta_f(s)$ is stable, the response y due to $w(t) = \bar{w}$ becomes 0 when $t \rightarrow \infty$, even when there are parameter variations in k_a and in the plant.

Finally,

$$\hat{g}_{yr}(s) = \frac{\frac{k_a}{s} \frac{\bar{N}(s)}{\Delta_f(s)}}{1 + \frac{k_a}{s} \frac{\bar{N}(s)}{\Delta_f(s)}} = \frac{k_a \bar{N}(s)}{s\bar{D}(s) + k_a \bar{N}(s)} = \frac{k_a \bar{N}(s)}{\Delta_f(s)}$$

In steady state, we have the DC gain

$$\hat{g}_{yr}(0) = \frac{k_a \bar{N}(0)}{0 \cdot \bar{D}(0) + k_a \bar{N}(0)} = 1 \quad (35)$$

so long as

$\bar{N}(0) \neq 0$ (since plant has no zero at $s = 0$), and

$k_a \neq 0$ (since the closed-loop poles (i.e. $\Delta_f(s)$) are all stable).

This completes the proof. □

Internal Model Principle: The integrator in the control loop in Figure 8.4 represents a model for constant disturbance $w(t)$. In general, a model of the disturbance must be included in the controller in order to reject asymptotically any disturbance.

Example 9.3: Consider a system described by

$$\hat{y}(s) = G(s)\hat{u}(s) + \hat{w}(s),$$

where $\hat{u}(s)$ is the control input and $\hat{w}(s)$ is the disturbance input,

$$G(s) = \frac{s - 1}{(s + 3)(s + 5)}$$

(a) Construct a minimal realization for $G(s)$.

(b) Suppose that the state variable in (a) can be measured. Design a feedback controller so that the closed-loop output $y(t)$ asymptotically tracks any step reference input $r(t)$ in the presence of unknown step disturbance $w(t)$, and closed-loop system has all its poles to the left of $s = -1$.

Note that $G(s) = \frac{s-1}{s^2+8s+15}$. A CCF form can be given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + bu \\ y &= [-1 \quad 1]x + w = Cx + w\end{aligned}$$

which obviously is a minimal realization. Then,

$$\dot{x}_a = r - y = -Cx + r - w$$

Let $u = [K \quad k_a] \begin{bmatrix} x \\ x_a \end{bmatrix}$, where $K = [k_1 \quad k_2]$. Then,

$$\begin{aligned}\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} &= \begin{bmatrix} A + bK & bk_a \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} 0 \\ -1 \end{bmatrix} w \\ y &= [C \quad 0] \begin{bmatrix} x \\ x_a \end{bmatrix} + w\end{aligned}$$

The closed-loop matrix

$$A_c = \begin{bmatrix} A + bK & bk_a \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -15 + k_1 & -8 + k_2 & k_a \\ 1 & -1 & 0 \end{bmatrix}$$

Then,

$$\begin{aligned} \Delta_f(s) &= |sI - A_c| = s[s^2 + (8 - k_2)s - k_a] + (15 - k_1)s - k_a \\ &= s^3 + (8 - k_2)s^2 + (15 - k_1 - k_a)s - k_a \end{aligned}$$

Set all the three closed-loop poles at $s = -2$. Then, $\Delta_f(s) = (s + 2)^3$. By comparing the coefficients, we get

$$k_1 = 11, \quad k_2 = 2, \quad k_a = -8$$

9.2.2 Stabilization

Controllability is sufficient for stabilization, but not necessary. One need not be able to have the ability to assign all the poles for stability, so long as the uncontrollable poles are stable, i.e. stabilizable. Consider

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c(t) \\ \dot{\bar{\mathbf{x}}}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c(t) \\ \bar{\mathbf{x}}_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c(t) \\ \mathbf{0} \end{bmatrix} u \quad (36)$$

If $\bar{\mathbf{A}}_{\bar{c}}$ is stable (and $(\bar{\mathbf{A}}_c, \bar{\mathbf{b}}_c)$ controllable), then this system is *stabilizable*, even though not controllable.

$$u = r - \mathbf{k}\mathbf{u} = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - \begin{bmatrix} \bar{\mathbf{k}}_1 & \bar{\mathbf{k}}_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c(t) \\ \bar{\mathbf{x}}_{\bar{c}}(t) \end{bmatrix}$$
$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c(t) \\ \dot{\bar{\mathbf{x}}}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c \bar{\mathbf{k}}_1 & \bar{\mathbf{A}}_{12} - \bar{\mathbf{b}}_c \bar{\mathbf{k}}_2 \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c(t) \\ \bar{\mathbf{x}}_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c(t) \\ \mathbf{0} \end{bmatrix} r \quad (37)$$

Since $(\bar{\mathbf{A}}_c, \bar{\mathbf{b}}_c)$ is controllable, there exists $\bar{\mathbf{k}}_1$ such that $\bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c \bar{\mathbf{k}}_1$ is asymptotically stable and so is the system.

9.3 State Estimator

Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad \mathbf{x} \in \mathbf{R}^n \\ y(t) &= \mathbf{c}\mathbf{x}(t), \quad u, y \in \mathbf{R}\end{aligned}\tag{38}$$

Assume only $y(t)$ is available for measurement. \mathbf{x} is not measurable.

How to implement state feedback? Use state estimator!

Open-loop State Estimator:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t)\tag{39}$$

which leads to error $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e}$, where the state estimator error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, and requires \mathbf{A} to be stable.

Closed-loop State Estimator:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t) + \mathbf{l}(y(t) - \mathbf{c}\hat{\mathbf{x}}(t))$$

i.e.

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}}(t) + \mathbf{b}u(t) + \mathbf{l}y(t)\tag{41}$$

Then, Eqns (38) and (41) lead to

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{e}\tag{42}$$

Theorem 9.6 Consider the pair (\mathbf{A}, \mathbf{c}) . All eigenvalues of $\mathbf{A} - \mathbf{l}\mathbf{c}$ can be assigned arbitrarily by selecting a real constant vector (observer gain) \mathbf{l} iff (\mathbf{A}, \mathbf{c}) is observable.

Duality: $\mathbf{A} - \mathbf{l}\mathbf{c}$ has the same eigenvalues as $\mathbf{A}^T - \mathbf{c}^T\mathbf{l}^T$. So observer design problem $\mathbf{A} - \mathbf{l}\mathbf{c}$ is the dual of the controller design problem $\mathbf{A}^T - \mathbf{c}^T\mathbf{l}^T$.

Similarly, can solve the Lyapunov equation for observer design:

Procedure 9.2

1. Select an $n \times n$ matrix \mathbf{F} that has the set of desired eigenvalues (which does not contain any eigenvalues of \mathbf{A}).
2. Select an arbitrary $\mathbf{l} \in \mathbf{R}^{n \times 1}$ such that (\mathbf{F}, \mathbf{l}) is controllable.
3. Solve the unique (nonsingular) \mathbf{T} in the Lyapunov equation $\mathbf{T}\mathbf{A} - \mathbf{F}\mathbf{T} = \mathbf{l}\mathbf{c}$.
4. The state estimator (observer) is given by

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{b}u + \mathbf{l}y \quad (43)$$

$$\hat{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{z} \quad (44)$$

9.4 Feedback from Estimated States

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad \mathbf{x} \in \mathbf{R}^n \\ y(t) &= \mathbf{c}\mathbf{x}(t), \quad u, y \in \mathbf{R}\end{aligned}\tag{51}$$

Consider the state observer:

$$\dot{\hat{\mathbf{x}}}(\mathbf{t}) = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}}(\mathbf{t}) + \mathbf{b}u(\mathbf{t}) + \mathbf{l}y\tag{52}$$

$$u = r - \mathbf{k}\hat{\mathbf{x}}\tag{53}$$

Note that if (\mathbf{A}, \mathbf{b}) is controllable, the state feedback $u = r - \mathbf{k}\mathbf{x}$ can place the eigenvalues of $(\mathbf{A} - \mathbf{b}\mathbf{k})$ in any desired locations. On the other hand, if (\mathbf{A}, \mathbf{c}) is observable, the state observer can place the eigenvalues of $(\mathbf{A} - \mathbf{l}\mathbf{c})$ at any desired locations.

The augmented system is described by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}\mathbf{k} \\ \mathbf{l}\mathbf{c} & \mathbf{A} - \mathbf{l}\mathbf{c} - \mathbf{b}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r \quad (54)$$

$$\mathbf{y} = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Using state transformation

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k} & \mathbf{b}\mathbf{k} \\ \mathbf{0} & \mathbf{A} - \mathbf{l}\mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r \quad (55)$$

$$\mathbf{y} = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

That is, the controller part $(\mathbf{A} - \mathbf{b}\mathbf{k})$ and the observer part $(\mathbf{A} - \mathbf{l}\mathbf{c})$ can be designed independently — *the separation property*.

Obviously, $\hat{\mathbf{x}}$ (or equiv \mathbf{e}) is uncontrollable, and the system can be simplified to (assuming $\mathbf{x}(0) = \hat{\mathbf{x}}(0) = 0$)

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r, \quad y = \mathbf{c}\mathbf{x}$$

and the transfer function is given by

$$G_f(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{k})^{-1}\mathbf{b}$$

Note that the observer dynamics is totally absent in the input-output description!

9.5 State Feedback — Multivariable Case

Consider a MIMO system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x} \in \mathbf{R}^n \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{u} \in \mathbf{R}^p, \quad \mathbf{y} \in \mathbf{R}^m \end{aligned} \quad (56)$$

State feedback law

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}, \quad \mathbf{K} \in \mathbf{R}^{p \times n} \quad (57)$$

Closed-loop System

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{u} \in \mathbf{R}^p, \quad \mathbf{y} \in \mathbf{R}^m \end{aligned} \quad (58)$$

Theorem 9.7 The pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$, for any $p \times m$ real constant matrix \mathbf{K} , is controllable iff (\mathbf{A}, \mathbf{B}) is controllable.

Theorem 9.8 All eigenvalues of $\mathbf{A} - \mathbf{BK}$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant \mathbf{K} iff (\mathbf{A}, \mathbf{B}) is controllable.

There are more parameters (\mathbf{K} is $p \times n$) than necessary for eigenvalue assignment.

The actual process of choosing \mathbf{K} can be by:

- (a) Lyapunov-Equation Method: similar to SISO. See CT Chen, pp259-260.
- (b) Canonical-Form Method: See WJ Rugh, Chapter 13 (notes pp63-70), or CT Chen, pp260-263.
- (c) Cyclic Design Method.

We shall present the first two methods.

9.5.1 Lyapunov-Equation Method

Consider an n -dimensional p -input pair (\mathbf{A}, \mathbf{B}) . Find a $p \times n$ real constant matrix \mathbf{K} such that $(\mathbf{A} - \mathbf{BK})$ has a desired set of eigenvalues, assuming that the set does not contain any eigenvalue of \mathbf{A} .

Procedure 9.3:

- Select an $n \times n$ matrix \mathbf{F} with the desired set of eigenvalues
- Select an arbitrary $p \times n$ matrix $\bar{\mathbf{K}}$ such that $(\mathbf{F}, \bar{\mathbf{K}})$ is observable
- Solve the unique \mathbf{T} in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{B}\bar{\mathbf{K}}$
- If \mathbf{T} is nonsingular, compute $\mathbf{K} = \bar{\mathbf{K}}\mathbf{T}^{-1}$. Otherwise, choose another $\bar{\mathbf{K}}$ and repeat the process

Unlike the SISO case, there is no guarantee that \mathbf{T} is always invertible. However, given a controllable (\mathbf{A}, \mathbf{B}) and $\bar{\mathbf{K}}$ is selected such that $(\mathbf{F}, \bar{\mathbf{K}})$ is observable, it is believed that the probability for \mathbf{T} to be nonsingular is 1.

9.5.2 Canonical-Form Method

First, transform (\mathbf{A}, \mathbf{B}) into a controllable canonical form. For example, the system has dimension 6 and the controllability indices are $\mu_1 = 4$ and $\mu_2 = 2$, respectively. The CCF can be given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & \vdots & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & \vdots & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dots & \dots \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \mathbf{x}$$

Next, from the set of desired eigenvalues, we form

$$\Delta_f(s) = (s^4 + \bar{\alpha}_{111}s^3 + \bar{\alpha}_{112}s^2 + \bar{\alpha}_{113}s + \bar{\alpha}_{114})(s^2 + \bar{\alpha}_{221}s + \bar{\alpha}_{222})$$

Then, set

$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\alpha}_{111} - \alpha_{111} & \bar{\alpha}_{112} - \alpha_{112} & \bar{\alpha}_{113} - \alpha_{113} \\ \bar{\alpha}_{211} - \alpha_{211} & \bar{\alpha}_{212} - \alpha_{212} & \bar{\alpha}_{213} - \alpha_{213} \\ \bar{\alpha}_{114} - \alpha_{114} & -\alpha_{121} & -\alpha_{122} \\ \bar{\alpha}_{214} - \alpha_{214} & \bar{\alpha}_{221} - \alpha_{221} & \bar{\alpha}_{222} - \alpha_{222} \end{bmatrix}$$

9.6 State Estimators — Multivariable Case

All discussion for state estimators in the single-input case applies to the multi-input case.

Consider the n -dimensional p -input q -output systems

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x} \in \mathbf{R}^n \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{u} \in \mathbf{R}^p, \mathbf{y} \in \mathbf{R}^q\end{aligned}\tag{73}$$

Closed-loop State Estimator is

$$\dot{\hat{\mathbf{x}}}(\mathbf{t}) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) + \mathbf{L}\mathbf{y}\tag{74}$$

The estimator Error is $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ with dynamics

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}\tag{76}$$

All eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be assigned arbitrarily by selecting a real constant vector (observer gain) \mathbf{L} iff (\mathbf{A}, \mathbf{C}) is observable.

Duality: $\mathbf{A} - \mathbf{L}\mathbf{C}$ has the same eigenvalues as $\mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$. So observer design problem $\mathbf{A} - \mathbf{L}\mathbf{C}$ is the dual of the controller design problem $\mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$.

Similarly, can solve the Lyapunov equation for observer design.

9.7 Feedback from Estimated States — MIMO

Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x} \in \mathbf{R}^n \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{u} \in \mathbf{R}^p, \mathbf{y} \in \mathbf{R}^q\end{aligned}\tag{80}$$

Consider the observer

$$\dot{\hat{\mathbf{x}}}(\mathbf{t}) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) + \mathbf{L}\mathbf{y}\tag{81}$$

$$u = r - \mathbf{K}\hat{\mathbf{x}}\tag{82}$$

The augmented system is described by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} r\tag{83}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Using state transformation

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} r \quad (84)$$
$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

That is, the controller part $(\mathbf{A} - \mathbf{BK})$ and the observer part $(\mathbf{A} - \mathbf{LC})$ can be designed independently — *the separation property*.

Obviously, $\hat{\mathbf{x}}$ (or equiv \mathbf{e}) is uncontrollable, and the system can be simplified to (assuming $\mathbf{x}(0) = \hat{\mathbf{x}}(0) = \mathbf{0}$)

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}r, \quad \mathbf{y} = \mathbf{C}\mathbf{x}$$

and the transfer function is given by

$$\hat{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$$

Note that the observer dynamics is totally absent in the input-output description!