Chapter 5 Stability

BIBO Stability of Linear Systems

- Consider a SISO LTI system described by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \tag{1}$$

where $g(t)$ is the impulse response of the system.

- The above is the zero-state response of a linear causal system.
- The system is called BIBO (bounded input bounded output) stable if for any bounded input, i.e. $|u(t)| < \infty, \forall t \geq 0$, the output $y(t)$ is bounded.

Theorem 5.1 *The system (1) is BIBO stable if and only if $g(t)$ is absolutely integrable in $[0, \infty)$, i.e.*

$$\int_0^\infty |g(t)|dt \leq M < \infty \tag{2}$$

for some constant M.

Proof Sufficiency: Suppose (2) holds. Given any $u(t)$ satisfying $|u(t)| \leq u_m$ for some $u_m > 0$. Then,

$$\begin{aligned} |y(t)| &= \left|\int_0^\infty g(\tau)u(t-\tau)d\tau\right| \leq \int_0^\infty |g(\tau)||u(t-\tau)|d\tau \\ &\leq u_m \int_0^\infty |g(\tau)|d\tau \leq u_m M. \end{aligned}$$

Necessity: By contradiction, suppose the system is BIBO stable but (2) does not hold. Then, there exists some $t_1 > 0$ such that

$$\int_0^{t_1} |g(\tau)|d\tau = \infty$$

Choose the bounded input:

$$u(t-\tau) = \begin{cases} 1 & if\ g(\tau) \geq 0 \\ -1 & if\ g(\tau) < 0 \end{cases}$$

Then,

$$y(t) = \int_0^{t_1} g(\tau)u(t_1-\tau)d\tau = \int_0^{t_1} |g(\tau)|d\tau = \infty$$

which is a contradiction.

Note that an absolutely integrable signal may not be bounded or may not approach zero.

Theorem 5.2 *If a system with impulse response $g(t)$ is BIBO stable, then, as $t \to \infty$:*

(i) The output excited by $u(t) = a,\ t \geq 0$, approaches $\hat{g}(0)a$.

(ii) The output excited by $u(t) = Asin\omega_0 t,\ t \geq 0$, approaches

$$A|\hat{g}(j\omega_0)|sin(\omega_0 t + \angle\hat{g}(j\omega_0))$$

where $\hat{g}(s)$ is the transfer function of the system, i.e.

$$\hat{g}(s) = \int_0^\infty g(\tau)e^{-s\tau}d\tau$$

Proof (i) With $u(t) = a$, as $t \to \infty$,

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau = a\int_0^t g(\tau)d\tau \to a\hat{g}(0)$$

(ii) With $u(t) = Asin\omega_0 t$,

$$
\begin{aligned}
y(t) &= A \int_0^{\tau} g(\tau) sin\omega_0(t-\tau) d\tau \\
&= A(sin\omega_0 t \int_0^t g(\tau) cos\omega_0 \tau d\tau - cos\omega_0 t \int_0^t g(\tau) sin\omega_0 \tau d\tau)
\end{aligned}
$$

As $t \to \infty$,

$$
y(t) \to A(sin\omega_0 t \int_0^t g(\tau) cos\omega_0 \tau d\tau - cos\omega_0 t \int_0^t g(\tau) sin\omega_0 \tau d\tau)
$$

On the other hand,

$$
\hat{g}(j\omega) = \int_0^{\infty} g(\tau) \left[cos\omega\tau - j sin\omega\tau d\tau \right]
$$

which implies

$$
\begin{aligned}
y(t) &\to A(sin\omega_0 t Re[\hat{g}(j\omega_0)] + cos\omega_0 t Im[\hat{g}(j\omega)]) \\
&= A|\hat{g}(j\omega_0)| sin(\omega_0 t + \angle \hat{g}(j\omega_0))
\end{aligned}
$$

Theorem 5.3 *A SISO system with proper transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part.*

Note that if $\hat{s}$ has pole p_i with multiplicity m_i, then its partial fraction expansion contains

$$\frac{1}{s-p_i}, \frac{1}{(s-p_i)^2}, \cdots, \frac{1}{(s-p_i)^{m_i}}$$

Then, its inverse Laplace transform contains

$$e^{p_i t},\ te^{p_i t},\ \cdots,\ t^{m_i-1}e^{p_i t}$$

It is clear that $g(t)$ is absolutely integrable if p_i has negative real part.

Theorem 5.4 *(i) A multivariable (MIMO) system with impulse response matrix $G(t) = [g_{ij}(t)]$ is BIBO stable if and only if every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.*
(ii) A MIMO system with proper rational transfer matrix $\hat{G}(s) = [\hat{g}_{ij}(s)]$ is BIBO stable if and only if every pole of every $g_{ij}(s)$ has a negative real part.

- Consider a state-space representation,

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{3}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{4}$$

and its transfer matrix

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

- Note that

$$\hat{G}(s) = \frac{1}{det(sI - A)}C[Adj(sI - A)]B + D$$

Every pole of $\hat{G}(s)$ is an eigenvalue of A. Thus, if all eigenvalues of A have negative real parts, the system is BIBO stable.

- But the converse is not true. For example,

$$\dot{x} = x + 0 \cdot u, \quad y = 0.5x + 0.5u$$

The transfer function is $\hat{g}(s) = 0.5$ which is BIBO stable. But the system has eigenvalue 1.

Discrete-time Case:

- Consider a discrete SISO LTI system:

$$y(k) = \sum_{m=0}^{k} g(k-m)u(m) = \sum_{m=0}^{k} g(m)u(k-m) \tag{5}$$

- The system is BIBO stable if the output due to any bounded input is bounded.
- The system is BIBO stable if and only if $g(k)$ is absolutely summable, i.e.

$$\sum_{k=0}^{\infty} |g(k)| \leq M < \infty$$

for some constant M.

Theorem 5.5 *If a discrete SISO LTI system with impulse response $g(k)$ is BIBO stable, then, as $k \to \infty$,*

- *The output excited by $u(k) = a,\ k \geq 0$, approaches $\hat{g}(1)a$.*

- The output excited by $u(k) = Asin\omega_0 k$, approaches

$$A|\hat{g}(e^{j\omega})|sin(\omega_0 k + \angle\hat{g}(e^{j\omega_0}))$$

where $\hat{g}(z)$ is the z-transform of $g(k)$, i.e.

$$\hat{g}(z) = \sum_{m=0}^{\infty} g(m)z^{-m}$$

Theorem 5.6 *A discrete time SISO system with proper rational transfer function $\hat{g}(z)$ is BIBO stable if and only if every pole of $\hat{g}(z)$ has a magnitude less than 1, or equivalently, inside the unit circle on the z-plane.*

For MIMO systems, we have

Theorem 5.7
- *A MIMO discrete-time system with impulse response sequence matrix $G(k) = [g_{ij}(k)]$ is BIBO stable if and only if every $g_{ij}(k)$ is absolutely summable.*
- *A MIMO system with proper rational transfer matrix $\hat{G}(z) = [\hat{g}_{ij}(z)]$ is BIBO stable if and only if every pole of $\hat{g}_{ij}(z)$ has a magnitude less than 1.*

Internal Stability

- Consider the general nonlinear system without input or disturbance:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{6}$$

- **Equilibrium Point.** A point $\mathbf{x}_e$ is an equilibrium point if for $\mathbf{x}(t_0) = \mathbf{x}_e$, $\mathbf{x}(t) = \mathbf{x}_e$ for any $t \geq t_0$. That is, $f(\mathbf{x}_e, t) = 0$.
  - the condition must be satisfied for all $t \geq t_0$;
  - if the systems starts at equilibrium state, it stays there.
- **Uniqueness of solution.** If f is Lipschitz w.r.t. $\mathbf{x}$, i.e.

$$\|f(t, \mathbf{x}) - f(t, \mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$

  for all $(t, \mathbf{x})$ and $(t, \mathbf{y})$ and piecewise continuous w.r.t. t in some neighborhood of $(t_0, \mathbf{x}_0)$, the system (6) has a unique solution $\mathbf{x}(t)$.
- WLOG, we can assume that $\mathbf{x}_e = 0$. Otherwise, by a change of variable, $\mathbf{x}' = \mathbf{x} - \mathbf{x}_e$.

Equilibrium State for Linear Systems

For a linear system,

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \;\; \mathbf{x}(t_0) = \mathbf{x}_0$$

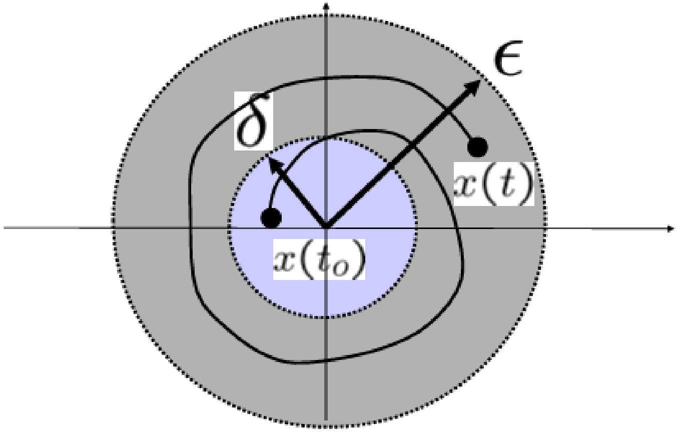
- the origin is always an equilibrium state;
- $A(t)$ is singular, multiple equilibirum states exist.

Lyapunov Stability

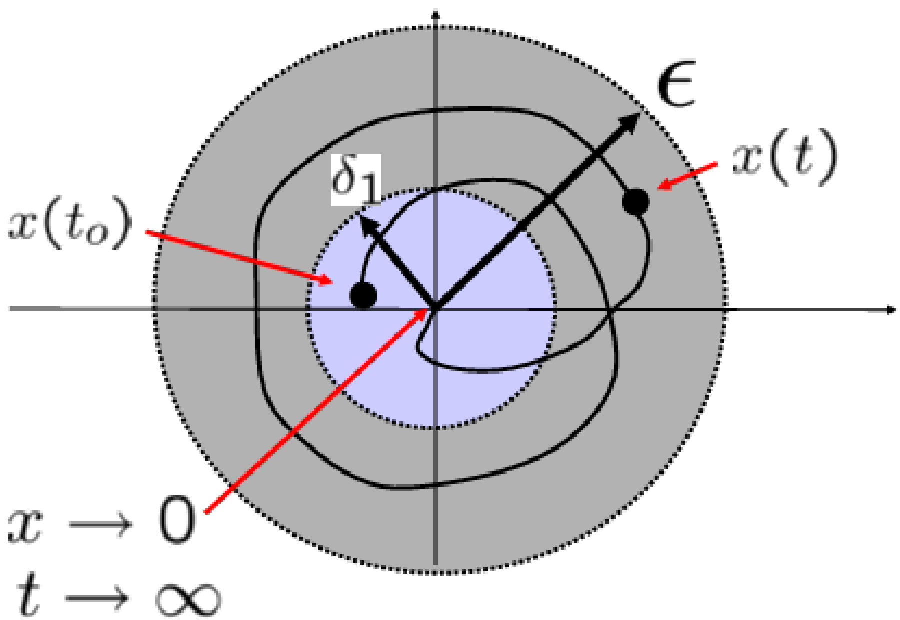
Lyapunov developed his stability theory in 1892 in Russia. The elegant theory remained unknown to the west until about 1960.

Definition 4.1 (Stability in the Sense of Lyapunov). The origin ($\mathbf{x}_e = 0$) is a *stable equilibrium* if for any given $\epsilon > 0$, there exists a number $\delta(\epsilon, t_0) > 0$ such that if $\|\mathbf{x}(t_0)\| < \delta$, then the resultant $\mathbf{x}(t)$ satisfies $\|\mathbf{x}(t)\| < \epsilon$ for all $t > t_0$.

That is, stability in the sense of Lyapunov implies that $\|x(t_0\| < \delta \implies \|x(t)\| < \epsilon, \forall t \geq t_0$.



- **Asymptotic Stability.** The origin is an *asymptotically stable equilibrium* if (a) it is stable, and in addition, (b) there exists a number $\delta(\epsilon, t_0) > 0$ such that whenever $\|\mathbf{x}(t_0)\| < \delta(\epsilon, t_0)$, $\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$.
- **Uniform stability.** If δ is not dependent on t_0, then the origin is said to be *uniformly stable* or *uniformly asymptotically stable*.



Example 5.1 *Consider the system*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

It is easy to know that $x = 0$ is the only equilibrium. Given the initial condition $x(0) = x_0$, its solution is

$$x(t; 0; x_0) = \begin{bmatrix} x_{01}cos(t) + x_{02}sin(t) \\ x_{02}cos(t) - x_{01}sin(t) \end{bmatrix}.$$

For any ϵ, take $\delta = \epsilon$. Then, if $(x_{10}^2 + x_{20}^2)^{\frac{1}{2}} < \delta$,

$$|x(t; 0; x_0) - 0| = (x_{10}^2 + x_{20}^2)^{\frac{1}{2}} < \epsilon, \ \ \forall t \geq 0.$$

Hence, the system is stable. However, it is not asymptotically stable since

$$\lim_{t \to \infty} |x(t; 0; x_0) - 0| = (x_{10}^2 + x_{20}^2)^{\frac{1}{2}} \neq 0$$

Lyapunov Stability - Direct Method

Theorem 5.8 *Consider the nonlinear time varying system (6). The equilibrium* $\mathbf{x} = 0$ *is globally asymptotically stable if there exists a scalar function* $V(x,t)$, $V(0,t) = 0$ *such that along any state trajectory of the system the following conditions hold:*

(i) there exist two continuous non-decreasing functions $\alpha(\|x\|)$ *and* $\beta(\|x\|)$ *with* $\alpha(0) = 0$, $\beta(0) = 0$ *such that* $\forall t \in [t_0, \infty)$ *and* $x \neq 0$,

$$\beta(\|x\|) \geq V(x,t) \geq \alpha(\|x\|) > 0$$

(ii) there exists a non-decreasing scalar function $\gamma(\|x\|)$ *with* $\gamma(0) = 0$ *such that*

$$\dot{V}(x,t) \leq -\gamma(\|x\|) < 0, \;\; \forall t \in [t_0, \infty), \; x \neq 0$$

(iii) When $\|x\| \to \infty$, $\alpha(\|x\|) \to \infty$.

Remark 5.1 • *$V(x,t)$ can be considered as "generalized energy function". In this sense, the Lyapunov theorem can be interpreted as "if the energy of the system is bounded and its changing rate is negative, then the system is bounded and the state will return to the equilibrium point".*

- *The choice of $V(x,t)$ is a try and error process. For simple systems one can start with a quadratic Lyapunov funtion.*
- *The theorem provides only a sufficient condition.*

For nonlinear time-invariant system,

$$\dot{x} = f(x), \;\; f(0) = 0, \;\; t \geq 0 \tag{7}$$

we have the following corollary.

Corollary 5.1 *The equilibrium $x = 0$ of the system (7) is globally asymptotically stable if there exists a scalar function $V(x)$, $V(0) = 0$ such that along any state trajectory of the system,*
(i) $V(x)$ is positive definite;
(ii) $\dot{V}(x)$ is negative definite;
(iii) When $\|x\| \to \infty$, $V(x) \to \infty$.

Example 5.2 *Consider mass-spring-damper system,*

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2\\ \dot{\mathbf{x}}_2 &= -\frac{k}{m}\mathbf{x}_1 - \frac{b}{m}\mathbf{x}_2\end{aligned}$$

- *the total energy:*

$$V(x) = potential\ energy + kinetic\ energy = \frac{1}{2}k\mathbf{x}_1^2 + \frac{1}{2}b\mathbf{x}_2^2$$

- *energy dissipation:*

$$\dot{V}(\mathbf{x}) = k\mathbf{x}_1\dot{\mathbf{x}}_1 + b\mathbf{x}_2\dot{\mathbf{x}}_2 = -b\mathbf{x}_2^2 \leq 0$$

.

- $\dot{V}(\mathbf{x}) = 0$ *only when* $\mathbf{x}_2 = 0$. *Since* $\mathbf{x} = [\mathbf{x}_1 \quad \mathbf{x}_2]^T = 0$ *is the only equilibrium state, the system will not stop at* $\mathbf{x}_2 = 0$, $\mathbf{x}_1 \neq 0$. *Thus, the energy will keep decreasing till 0, i.e.* $[\mathbf{x}_1 \quad \mathbf{x}_2]^T = 0$.
- *In fact, when* $\mathbf{x}_1 = 0$, $\dot{\mathbf{x}}_1 = 0$. *From the state equtaion, it can be easily seen that* $\mathbf{x}_2$ *must be 0.*

Example 5.3 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2)\end{aligned}$$

It is known that $x_1 = 0,\ x_2 = 0$ *is the unique equilibrium point.*

- *Choose* $V(x) = x_1^2 + x_2^2$*;*
- *Then,* $\dot{V}(x) = \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 = -2(x_1^2 + x_2^2)^2$*, which is negative definite with* $V(0) = 0$*;*
- *When* $\|x\| = \sqrt{x_1^2 + x_2^2} \to \infty$, $V(x) = \|x\|^2 = x_1^2 + x_2^2 \to \infty$.

Hence, the equilibrium $x = 0$ *is globally asymptotically stable.*

Note: The condition (ii) of Corollary (5.1) can be replaced by
(ii)' $\dot{V}(x) \leq 0$ and along any non-zero state $\dot{V}(x) \neq 0$.

Example 5.4 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (1+x_2)^2 x_2\end{aligned}$$

It is known that $x_1 = 0,\ x_2 = 0$ *is the unique equilibrium point.*

- *Choose* $V(x) = x_1^2 + x_2^2$;
- *Then,* $\dot{V}(x) = \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 = -2x_2^2(1+x_2)^2$. *Note that* $\dot{V}(x) = 0$ *implies that* $x_2 = 0$ *or* $x_2 = -1$ *but* x_1 *can be arbitrary. For* $x_2 = 0$, *from the state equation we have*

  $$\dot{x}_1 = 0,\ \ 0 = \dot{x}_2(t) = -x_1(t)$$

  Hence, x_1 *arbitrary and* $x_2 = 0$ *is not a state solution. Similarly, we can show that* x_1 *arbitrary and* $x_2 = -1$ *is not a state solution as well.*
- *When* $\|x\| = \sqrt{x_1^2 + x_2^2} \to \infty$, $V(x) = \|x\|^2 = x_1^2 + x_2^2 \to \infty$.

Hence, the equilibrium point $x = 0$ *is globally asymptotically stable.*

Lyapunov Stability of Linear Systems

- Consider the linear system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{8}$$

- The solution of (8) is given by

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$$

  where $\Phi(t, t_0)$ is the transition matrix.

- (a) The origin is stable (uniformly stable) if and only if there exists a number $\gamma(t_0)$ (γ) such that

$$\|\Phi(t, t_0)\| \leq \gamma(t_0), \quad \forall t \geq t_0 \tag{9}$$

- (b) It is asymptotically stable if and only if $\|\Phi(t, t_0)\| \to 0$ for $t \to \infty$.
- (c) If the system is asymptotically stable, it is globally asymptotically stable.

Proofs of some parts:

(a) Assume that the origin is stable. Then for $\epsilon = 1$ there exists a $\delta(1, t_0) > 0$ such that $\|\Phi(t, t_0)x(t_0)\| < 1$ for all $t > t_0$ and $\|x(t_0)\| \leq \delta(1, t_0)$. Then, for any $x(t_0) \neq 0$,

$$\|\Phi(t, t_0)x(t_0)\| = \|\Phi(t, t_0) \left(x(t_0)\frac{\delta}{\|x(t_0)\|} \right) \frac{\|x(t_0)\|}{\delta} < \frac{\|x(t_0)\|}{\delta(1, t_0)}$$

i.e.

$$\|\Phi(t, t_0)\| \leq \delta(1, t_0)^{-1} = \gamma(t_0)$$

Similarly for (c), assume that the system is asymptotically stable. Then there is an $\eta > 0$ such that when $\|x(t_0)\| \leq \eta$, $\Phi(t, t_0)x(t_0) \to 0$ as $t \to \infty$. For any $x(t_0) \neq 0$,

$$\Phi(t, t_0)x(t_0) = \Phi(t, t_0) \left(\frac{\eta x(t_0)}{\|x(t_0)\|} \right) \frac{\|x(t_0)\|}{\eta} \to 0$$

Example 5.5

$$\dot{x}(t) = [4t sin(t) - 2t]x(t), \quad x(t_0) = x_0$$

has the solution

$$x(t) = exp(4\sin t - 4t\cos t - t^2 - 4\sin t_0 + 4t_0\cos t_0 + t_0^2)x_0$$

- *It is clear that for any t_0, there exists a $\gamma(t_0)$ such that $x(t) \leq \gamma(t_0)\|x_0\|$ for any $t \geq t_0$. So, it is stable.*
- *But it is not uniformly stable. For a given x_0, we choose $t_0 = 2k\pi$, $k = 0, 1, 2, \cdots$. Then, we have*

$$x(2k\pi + \pi) = exp[(4k + 1)\pi(4 - \pi)]x_0$$

Clearly, there is no such a constant for all k.

- **Uniform exponential stability.** The linear system (8) is *uniformly exponentially stable* if there exist finite positive constants $\gamma,\ \lambda$ such that for any t_0 and $\mathbf{x}_0$, the solution satisfies

$$\|\mathbf{x}(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|\mathbf{x}_0\|, \quad t \geq t_0 \tag{10}$$

- **Theorem 5.9** *Suppose there exists a finite positive constant α such that $\|A(t)\| \leq \alpha$ for all t. Then, the system (8) is uniformly exponentially stable if and only if there exists a finite positive constant β such that*

$$\|\Phi(t,\tau)\| \leq \gamma e^{-\lambda(t-\tau)} \tag{11}$$

or equivalently

$$\int_\tau^t \|\Phi(t,\tau)\| d\sigma \leq \beta$$

for all t, τ such that $t \geq \tau$.

Proof. *Sufficiency* If (11) holds, then

$$\|\mathbf{x}(t)\| = \|\Phi(t,t_0)\mathbf{x}_0\| \le \|\Phi(t,t_0)\|\,\|\mathbf{x}_0\| \le \gamma e^{-\lambda(t-t_0)}\|\mathbf{x}_0\|, \;\; t \ge t_0$$

i.e. the system is uniformly exponentially stable.

Necessity Suppose the system is uniformly exponentially stable, i.e. (10) holds for some γ and λ. Given any t_0 and t_a, let $\mathbf{x}_a$ be such that

$$\|\mathbf{x}_a\| = 1, \;\; \|\Phi(t_a,t_0)\mathbf{x}_a\| = \|\Phi(t_a,t_0)\|$$

Such a $\mathbf{x}_a$ exists by the definition of induced norm. Now, set $\mathbf{x}(t_0) = \mathbf{x}_a$. Then,

$$\|\mathbf{x}(t_a)\| = \|\Phi(t_a,t_0)\mathbf{x}_a\| = \|\Phi(t_a,t_0)\|\,\|\mathbf{x}_a\| \le \gamma e^{-\lambda(t_a-t_0)}\|\mathbf{x}_a\|$$

Since $\|\mathbf{x}_a\| = 1$, $\|\Phi(t_a,t_0)\| \le \gamma e^{-\lambda(t_a-t_0)}$. Finally, note that t_a can be selected for any t_0 and $t_a \ge t_0$. The proof is complete.

Time-invariant Case: $A(t) = A$

- In this case, $\Phi(t, t_0) = e^{A(t-t_0)}$. The system is exponentially stable if and only if

$$\int_0^\infty \|e^{At}\| dt < \infty$$

- **Theorem 5.10** *A linear time-invariant system $\dot{x} = Ax$ is exponentially stable if and only if all eigenvalues of A have negative real parts.*

  Proof. *Sufficiency* Suppose $Re[\lambda_k] < 0$ for $k = 1, 2, \cdots, m$. Then, there exist constant matrices W_{kj} such that

$$e^{At} = \sum_{k=1}^{m} \sum_{j=1}^{\sigma_k} W_{kj} \frac{t^{j-1}}{(j-1)!} e^{\lambda_k t}$$

  Then,

$$\begin{aligned} \int_0^\infty \|e^{At}\| dt &= \int_0^\infty \| \sum_{k=1}^{m} \sum_{j=1}^{\sigma_k} W_{kj} \frac{t^{j-1}}{(j-1)!} e^{\lambda_k t} \| dt \\ &\leq \sum_{k=1}^{m} \sum_{j=1}^{\sigma_k} \|W_{kj}\| \int_0^\infty \frac{t^{j-1}}{(j-1)!} |e^{\lambda_k t}| dt \end{aligned}$$

Since $|e^{\lambda_k t}| = e^{Re[\lambda_k]t}$. If $Re[\lambda_k] < 0$, it is easy to show that the right hand side is finite.

Necessity. Suppose by contradiction that one of the eigenvalues, λ, with $Re[\lambda] \geq 0$. Let p be the associated eigenvector. Then, $Ap = \lambda p$ and

$$e^{At}p = e^{\lambda t}p$$

Set $x_0 = p$. Then, $\|x(t)\| = \|e^{At}p\| = |e^{\lambda t}|\|p\|$ does not approach zero as $t \to \infty$. Thus the system is not exponentially stable.

Remark 5.2 *The above implies that* $e^{At} \to 0$ *as* $t \to \infty$ *is a necessary and sufficient condition for uniform exponential stability. However, the result is not true for time-varying counterpart.*

For a LTI system $\dot{\mathbf{x}} = A\mathbf{x}$,

- its stability can be determined from its eigenvalues;
- it is unstable if $Re(\lambda_i) > 0$ for some λ_i or $Re(\lambda_i) \leq 0$ for all λ_i but for some repeated eigenvalue λ_m with $Re(\lambda_m) = 0$ and multiplicity m, $nullity(A - \lambda_m I) < m$ (Jordan form);
- it is stable in the sense of Lyapunov if $Re(\lambda_i) \leq 0$ for all λ_i and for all repeated eigenvalues λ_m on the imaginary axis with multiplicity m, $nullity(A - \lambda_m I) = m$ (diagonal form).
- it is asymptotically stable if $Re(\lambda_i) < 0$ for all λ_i.

Example 5.6 *Consider the LTI system:* $\dot{x} = Ax$*, where*

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda+1 & 1 & 1 \\ -1 & \lambda & -1 \\ 0 & 1 & \lambda+2 \end{vmatrix} \\ &= (\lambda+1)\begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & \lambda+2 \end{vmatrix} \\ &= (\lambda+1)(\lambda^2+2\lambda+2) = 0 \end{aligned}$$

Hence, the eigenvalues of A *are: -1,* $-1 \pm j1$*. The system is asymptotically stable.*

Example 5.7 *In a L-C circuit, choose* $x_1 = i$ *and* $x_2 = v$*, where* v *and* i *are current and voltage of capacitor and inductor, respectively. Then,*

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{L}x_2 \\ \dot{x}_2 &= \frac{1}{C}x_1 \end{aligned}$$

The total energy of the system is

$$V(x) = Li^2/2 + Cv^2/2$$

Its derivative along the state trajectory is

$$\dot{V} = Lx_1(-x_2/L) + Cx_2(x_1/C) = 0$$

The system is stable but not asymptotically stable. In fact, it can be checked that the eigenvalues of the system lie on the imaginary axis.

Lyapunov Stability for LTI Systems

- For LTI system
$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$
a good Lyapunov function candidate is the quadratic function $V(\mathbf{x}) = \mathrm{x}^T P \mathbf{x}$, where P is a positive definite matrix ($P = P^T$ and $P > 0$).

- The derivative along the state trajectory is given by
$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} \\ &= (A\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P A \mathbf{x} \\ &= \mathbf{x}^T (A^T P + PA) \mathbf{x} \end{aligned}$$

- Such a $V(\mathbf{x})$ is called a Lyapunov function for $\dot{\mathbf{x}} = A\mathbf{x}$ when $A^T P + PA \leq 0$, i.e. $\dot{V}(\mathbf{x}) \leq 0$, and the origin is said to be stable in the sense of Lyapunov.

Lyapunov Stability Theorem for LTI Systems

Theorem 5.11 *For* $\dot{\mathbf{x}} = A\mathbf{x}$ *with* $A \in \mathcal{R}^{n\times n}$*, the origin is asymptotically stable if and only if for any symmetric positive definite matrix* $Q >$*, the Lyapunov equation*

$$A^T P + PA = -Q \tag{12}$$

has a unique positive definite solution $P = P^T > 0$*. In this case, the unique positive definite solution is given by*

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{13}$$

Proof. Suppose A is asymptotically stable. That is, all the eigenvalues of A have negative real parts. Then, P of (13) is well defined and $P > 0$ since $Q > 0$. What remains is to show that P satisfies (12). To this end, substitute (13) into (12), we have

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt \\ &= \int_0^\infty \frac{d}{dt} \left[e^{A^T t} Q e^{At} \right] dt = e^{A^T t} Q e^{At} \mid_0^\infty = -Q \end{aligned}$$

To show the uniqueness of the solution, suppose P_a is also a solution. Then,

$$(P_a - P)A + A^T(P_a - P) = 0$$

which implies

$$e^{A^T t}(P_a - P)Ae^{At} + e^{A^T t}A^T(P_a - P)e^{At} = 0, \quad t \geq 0$$

that is

$$\frac{d}{dt}[e^{A^T t}(P_a - P)e^{At}] = 0, \quad t \geq 0$$

Integrating both sides leads to

$$0 = e^{A^T t}(P_a - P)e^{At} \mid_0^\infty = -(P_a - P)$$

Hence, $P_a = P$.

Conversely, if for a given $Q > 0$, there exists a $P > 0$ to the Lyapunov equation. Then, along the state trajectory

$$\dot{V}(t) = \mathbf{x}^T(t)(A^TP+PA)\mathbf{x}(t) = -\mathbf{x}^T(t)Q\mathbf{x}(t) \leq -\lambda_{\min}(Q)\|\mathbf{x}(t)\|^2 \quad (14)$$

On the other hand,

$$V(t) = \mathbf{x}^T(t)P\mathbf{x}(t) \leq \lambda_{\max}(P)\|\mathbf{x}(t)\|^2 \implies \|\mathbf{x}(t)\|^2 \geq \frac{V(t)}{\lambda_{\max}(P)}$$

Substituting the above into (14) leads to

$$\dot{V}(t) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(t)$$

Denote $\alpha = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0$. Then, we have

$$\dot{V}(t) \leq -\alpha V(t)$$

Thus,

$$V(t) \leq e^{-\alpha t}V(0) \to 0$$

or $\mathbf{x}(t) \to 0$, i.e. the system is asymptotically stable.

Example 5.8 *Consider LTI system* $\dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} x$. *Set* $Q = I$. *Solving the equation*

$$A^T P + PA = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} -2p_{11} + 4p_{22} &= -1 \\ -6p_{12} + 2p_{22} &= -1 \\ p_{11} + 2p_{12} - 4p_{22} &= 0 \end{aligned}$$

we get $p_{11} = 7/4,\ p_{12} = 3/8,\ p_{22} = 5/8$. *Hence,*

$$P = \begin{bmatrix} 7/4 & 5/8 \\ 5/8 & 3/8 \end{bmatrix} > 0$$

and the system is asymptotically stable.

- **Linear Time-varying Systems:**
  Consider

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) \tag{15}$$

Introduce quadratic function:

$$V(\mathbf{x}) = \mathbf{x}^T(t)P(t)\mathbf{x}(t) \tag{16}$$

where $P(t)$ is symmetric and continuously differentiable and satisfies $\eta I \leq P(t) \leq \rho I$ for some $\eta,\ \rho > 0$.

- **Theorem 5.12** *The system (15) is uniformly stable if there exists a Quadratic function (16) such that*

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq 0 \tag{17}$$

for all t.

Proof. From (17) we know that

$$\mathbf{x}^T(t)P(t)\mathbf{x}(t) - \mathbf{x}_0^T P(t_0)\mathbf{x}_0 = \int_{t_0}^{t} \frac{d}{d\sigma}[\mathbf{x}^T(\sigma)P(\sigma)\mathbf{x}(\sigma)]d\sigma$$

$$= \int_{t_0}^{t} x^T(\sigma)\left[\dot{P}(\sigma) + A^T(\sigma)P(\sigma) + P(\sigma)A(\sigma)\right]x(\sigma)d\sigma \leq 0,\ t \geq t_0$$

That is,

$$\mathbf{x}^T(t)P(t)\mathbf{x}(t) \leq \mathbf{x}_0^T P(t_0)\mathbf{x}_0 \leq \rho\|\mathbf{x}_0\|^2, \quad t \geq t_0$$

and

$$\eta\|\mathbf{x}(t\|^2 \leq \rho\|\mathbf{x}_0\|^2, \;\; t \geq t_0$$

or yet

$$\|\mathbf{x}(t)\| \leq \sqrt{\rho/\eta}\|\mathbf{x}_0\|, \quad t \geq t_0$$

The above holds for all $\mathbf{x}_0$ and t_0. Hence, the system is uniformly stable.

Example 5.9 *Consider the system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -a(t) \end{bmatrix} \mathbf{x}(t)$$

where $a(t)$ is a continuous function.
Choose $P(t) = I$. Then,

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = A^T(t) + A(t) = \begin{bmatrix} 0 & 0 \\ 0 & -2a(t) \end{bmatrix}$$

The system is uniformly stable if $a(t) \geq 0$.

Remark 5.3 *The above result depends on the choice of the matrix $P(t)$. A less conservative result may be achieved by a better choice of $P(t)$.*

Uniform Exponential Stability

Theorem 5.13 *The system (15) is uniformly exponentially stable if there exists a symmetric and continuously different matrix $P(t)$ such that*

$$\eta I \leq P(t) \leq \rho I \tag{18}$$

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq -\mu I \tag{19}$$

for some constants $\rho,\ \eta,\ \mu > 0$.

Proof. Note from (19) that

$$\frac{d}{dt}\left[\mathbf{x}^T(t)P(t)\mathbf{x}(t)\right] \leq -\mu\|\mathbf{x}(t)\|^2, \quad t \geq t_0$$

From (18),

$$\mathbf{x}^T(t)P(t)\mathbf{x}(t) \le \rho\|\mathbf{x}(t)\|^2$$

or

$$-\|\mathbf{x}(t)\|^2 \le -\frac{1}{\rho}\mathbf{x}^T(t)P(t)\mathbf{x}(t), \;\; t \ge t_0$$

Hence,

$$\frac{d}{dt}\left[\mathbf{x}^T(t)P(t)\mathbf{x}(t)\right] \le -\frac{\mu}{\rho}\mathbf{x}^T(t)P(t)\mathbf{x}(t), \quad t \ge t_0 \tag{20}$$

By solving the above ODE, we have

$$\mathbf{x}^T(t)P(t)\mathbf{x}(t) \le e^{-\frac{\mu}{\rho}(t-t_0)}\mathbf{x}_0^T P(t_0)\mathbf{x}_0, \;\; t \ge t_0$$

Again using (18) the above leads to

$$\|\mathbf{x}(t)\|^2 \le \frac{1}{\eta}\mathbf{x}^T(t)P(t)\mathbf{x}(t) \le \frac{1}{\eta}e^{-\frac{\mu}{\rho}(t-t_0)}\mathbf{x}_0^T P(t_0)\mathbf{x}_0, \quad t \ge t_0$$

which gives

$$\|\mathbf{x}(t)\|^2 \le \frac{\rho}{\eta}e^{-\frac{\rho}{\eta}(t-t_0)}\|\mathbf{x}_0\|^2, \;\; t \ge t_0$$

Hence, the system is uniformly exponentially stable.

Linearization

Consider a nonlinear system:

$$\dot{w} = f(w) \tag{21}$$

where $f(\cdot)$ is a differentiable function. Let $w = 0$ be the equilibrium and obtain the linearized system:

$$\dot{\mathbf{x}} = A\mathbf{x} + F(\mathbf{x}) \tag{22}$$

where $F(\cdot) = O(||\mathbf{x}||)$ and A is the Jacobian of $f(w)$, i.e. $A = \frac{\partial f}{\partial w}(0)$. The corresponding linearized system is $\dot{\mathbf{x}} = A\mathbf{x}$.

Theorem 5.14

- *If A is a Hurwitz matrix (asy. stable), the equilibrium $\mathbf{x} = 0$ of (22) (hence (21)) is exponentially stable.*
- *If A has an eigenvalue with positive real part, the equilibrium $\mathbf{x} = 0$ of (22) (hence (21)) is unstable.*
- *If A has an eigenvalue with zero real part and the rest of the eigenvalues with negative real parts, the stability of the system can only be determined by analyzing higher order terms.*

Sketch of the proof:

(a) Since A is asy. stable, given a positive definite matrix Q, there exists a positive definite solution P to

$$A^T P + PA + Q = 0$$

(b) Define Lyapunov function

$$V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$$

Then,

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T P F(\mathbf{x})$$

(c) Since $F(\mathbf{x}) = O(\mathbf{x})$ as $x \to 0$, there exists a $\delta > 0$ such that $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in B(\delta) - \{0\}$.

(d) Determine the largest $\lambda = \lambda_M$ such that $C_{\lambda_M} \subset B(\delta)$, where

$$C_\lambda = \{\mathbf{x} \in R^n | V(\mathbf{x}) < \lambda\}$$

(e) C_{λ_M} is a subset of the domain of attraction of the equilibrium $\mathbf{x} = 0$.

Example 5.11

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1(x_1^2 + x_2^2) \qquad (23) \\ \dot{x}_2 &= -x_2 + x_2(x_1^2 + x_2^2) \qquad (24) \end{aligned}$$

Clearly, $x = 0$ *is the equilibrium. Then,*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F(x) = \begin{bmatrix} x_1(x_1^2 + x_2^2) \\ x_2(x_1^2 + x_2^2) \end{bmatrix}$$

It can be easily checked that A *is asymptotically stable and thus the equilibrium is exponentially stable.*
Taking $Q = I$ *and solving* $A^T P + PA + Q = 0$*, we get* $P = (1/2)I$*. Hence,*

$$V(x) = x^T P x = \frac{1}{2}(x_1^2 + x_2^2)$$

and

$$\dot{V}(x) = -(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2$$

Clearly, $\dot{V}(x) < 0$ *when* $x \neq 0$ *and* $\|x\|_2 \leq 1$*. Hence,* $\|x\|_2 < 1$ *is a subset of the domain of attraction.*

Discrete-time Case:

Consider the discrete-time system:

$$\mathbf{x}(k+1) = A\mathbf{x}(k)$$

Theorem 5.15 *The above discrete-time system is asymptotically stable if and only if either one of the following equivalent conditions hold:*

- *All eigenvalues of A have magnitudes less than 1.*
- *For any positive definite matrix M, there exists a positive definite solution P to the discrete Lyapunov equation:*

$$A^T PA - P = -Q$$

Summary

- Concepts of BIBO stability and internal stability
- Stability of LTI system $\dot{x} = Ax$: (a) check the eigenvalues of A; (b) Lyapunov equation: $A^T P + PA = -Q$.
- LTV systems: $\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq -\mu I$
- Nonlinear system $\dot{x} = f(x)$: (a) equilibrium: $f(x_e) = 0$; Obtain linearized system $\dot{\bar{x}} = A\bar{x}$ at equilibrium and check if A is stable to determine if the nonlinear system is stable at equilibrium; (b) global stability

Exercises:

1. Is a system with impulse response $g(t) = 1/(t+1)$ BIBO stable? How about $g(t) = te^{-t}$ for $t \geq 0$.

2. Consider a discrete-time system with impulse response sequence:

$$g(k) = k(0.8)^k, \quad k \geq 0$$

Is the system BIBO stable?

3. Show that all eigenvalues of A have real parts less than $-\mu < 0$ if and only if, for any given positive definite matrix Q,

$$A^T P + PA + 2\mu P = -Q$$

has a unique positive definite solution P.

4. Show that all eigenvalues of A have magnitudes less than ρ if and only if, for any given positive definite matrix M,

$$A^T PA - \rho^2 P = \rho^2 Q$$

has a positive definite solution P.

5. Determine if the following system is: (a) asymptotically stable; (b) BIBO stable.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u, \quad y = \begin{bmatrix} -25 & 5 & 0 \end{bmatrix} x$$