Lecture 7: Polynomial Fraction Description

Polynomial Fraction Description (PFD) is an efficacious representation for MIMO systems, and can reveal structural features. It is a transfer function matrix approach.

It deals with matrices of real-coefficient polynomials in s.

Right Polynomial Fractions

Definition 7.1: A $p \times r$ polynomial matrix $P(s)$ is a matrix with entries that are real-coefficient polynomials in s. A square ($p = r$) polynomial matrix $P(s)$ is called *nonsingular* if $\det P(s)$ is a nonzero polynomial, and *unimodular* if $\det P(s)$ is a nonzero real number.

Alternative characterizations:

- Nonsingaular iff $\det P(s_0) \neq 0$ for all but a finite number of complex numbers s_0.
- Unimodular iff $\det P(s_0) \neq 0$ for all complex numbers s_0.
- Unimodular iff both $P(s)$ and $P^{-1}(s)$ are polynomial matrices.

Example 7.1: Given the polynomial matrices

$$
\begin{aligned}
P_1(s) &= \begin{bmatrix} s+1 & s+3 \\ s^2+3s+2 & s^2+5s+4 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} s+1 & s+3 \\ s^2+3s+2 & s^2+5s+6 \end{bmatrix} \\
P_3(s) &= \begin{bmatrix} s+1 & s+2 \\ s+3 & s+4 \end{bmatrix}
\end{aligned}
$$

we have

$$\det P_1(s) = (s+1)(s^2+5s+4) - (s+3)(s^2+3s+2) = -2s-2$$

$$\det P_2(s) = (s+1)(s^2+5s+6) - (s+3)(s^2+3s+2) = 0$$

and

$$\det P_3(s) = (s+1)(s+4) - (s+2)(s+3) = -2$$

Thus, $P_1(s)$ is nonlinear, $P_2(s)$ is singular and $P_3(s)$ is unimodular. Note that $P_3^{-1}(s) = -\frac{1}{2}\begin{bmatrix} s+4 & -(s+2) \\ -(s+3) & s+1 \end{bmatrix}$ is a matrix polynomial.

Definition 7.2: A $p \times r$ polynomial matrix is said to have rank r if there exists at least one non-zero $r \times r$ minor while all minors of order $\geq (r+1) \times (r+1)$ are zeros.

Example 7.2

$$P(s) = \begin{bmatrix} s+1 & s+3 \\ s^2+3s+2 & s^2+5s+6 \end{bmatrix}$$

It is easy to see that all first order minors of $N(s)$ are not zero while $\det P(s) \equiv 0$. Thus, $rank P(s) = 1$.

Note that for a non-zero $p \times r$ polynomial matrix $P(s)$,

$$1 \leq rank P(s) \leq \min(p, r)$$

Further, if $Q(s)$ and $R(s)$ are $p \times p$ and $r \times r$ nonsingular polynomial matrices, then

$$rank P(s) = rank Q(s) P(s) = rank P(s) R(s)$$

Definition 7.3: A *right polynomial fraction* (RPF) description for the $p \times m$ strictly proper rational transfer function $G(s)$ is an expression of the form

$$G(s) = N(s)D^{-1}(s) \tag{1}$$

where $N(s)$ is a $p \times m$ polynomial matrix and $D(s)$ is an $m \times m$ nonsingular polynomial matrix.

A *left polynomial fraction* (LPF) description for $G(s)$ is

$$G(s) = D_L^{-1}(s)N_L(s) \tag{2}$$

where $N_L(s)$ is a $p \times m$ polynomial matrix and $D_L(s)$ is a $p \times p$ nonsingular polynomial matrix.

The *degree* of a RPF description is the degree of $\det D(s)$.

Similarly, the *degree* of a LPF description is the degree of $\det D_L(s)$.

Definition 7.4 The *characteristic polynomial* of a proper rational matrix $G(s)$ is defined as the least common denominator of all minors of $G(s)$. The degree of the characteristic polynomial is defined as the *McMillan degree* of $G(s)$ and is denoted by $\delta G(s)$.

These definitions are consistent with the SISO definitions.

Example 7.3

$$G_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}, \quad G_2(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Note that $G_1(s)$ has all of its first order minors being $\frac{1}{s+1}$ and $\det G_1(s) = 0$ as its minor of order 2. Thus, the characteristic polynomial of $G_1(s)$ is $s + 1$ and $\delta G_1(s) = 1$.

For $G_2(s)$, its first order minors are $\frac{2}{s+1}$ and $\frac{1}{s+1}$, and $\det G_1(s) = \frac{1}{(s+1)^2}$ as its minor of order 2. Thus, the characteristic polynomial of $G_2(s)$ is $(s + 1)^2$ and $\delta G_2(s) = 2$.

Example 7.4

$$G_3(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ -\frac{1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

Its minors of order 1 are its six entries. The three minors of order 2 are

$$\frac{s}{(s+1)^2(s+2)} + \frac{1}{(s+1)^2(s+2)} = \frac{1}{(s+1)(s+2)}$$

$$\frac{s}{s+1}\frac{1}{s} + \frac{1}{(s+1)(s+3)} = \frac{s+4}{(s+1)(s+3)}$$

$$\frac{1}{(s+1)(s+2)s} - \frac{1}{(s+1)(s+2)(s+3)} = \frac{3}{s(s+1)(s+2)(s+3)}$$

The characteristic polynomial is the least common denominator of all the minors which is $s(s+1)(s+2)(s+3)$ and $\delta G_3(s) = 4$.

Elementary Polynomial Fractions:

$$G(s) = N_d(s)[d(s)I_m]^{-1} = [d(s)I_m]^{-1}N_d(s) \tag{3}$$
$$\text{i.e.,} \qquad D(s) = D_L(s) = d(s)I_m$$

where

$$N_d(s) = d(s)G(s)$$

and $d(s)$ is the least common multiple of denominators of entries of $G(s)$, i.e., the characteristic polynomial according to Definition 7.4.

Definition 7.5: An $r \times r$ polynomial matrix $R(s)$ is called a *right divisor* of the $p \times r$ polynomial matrix $P(s)$ if $\exists$ a $p \times r$ polynomial matrix $\tilde{P}(s)$ such that

$$P(s) = \tilde{P}(s)R(s)$$

Note: $R(s)$ does not have to be nonsigular.
If $R(s)$ is nonsingular, then $P(s)R^{-1}(s)$ is a $p \times r$ polynomial matrix.
If $P(s)$ is square and nonsingular, then every right divisor of $P(s)$ is nonsingular.

Example 7.5 For the polynomial matrix

$$P(s) = \begin{bmatrix} (s+1)^2(s+2) \\ (s+1)(s+2)(s+3) \end{bmatrix} \tag{4}$$

right divisors include the 1×1 polynomial matrices

$$R_a(s) = 1, \qquad R_b(s) = s+1, \qquad R_c(s) = s+2$$

$$R_d(s) = (s+1)(s+2)$$

For the slightly less simple

$$P(s) = \begin{bmatrix} (s+1)^2(s+2) & (s+3)(s+5) \\ 0 & (s+4)(s+5) \end{bmatrix}$$

two right divisors are

$$\begin{bmatrix} s+1 & 0 \\ 0 & s+5 \end{bmatrix}, \qquad \begin{bmatrix} (s+1)^2 & 0 \\ 0 & s+5 \end{bmatrix}$$

Definition 7.6: Suppose $N(s)$ is a $p \times r$ polynomial matrix and $D(s)$ is a $r \times r$ polynomial matrix. If the $r \times r$ polynomial matrix $R(s)$ is a right divisor of both, then $R(s)$ is called a *common right divisor* (CRD) of $N(s)$ and $D(s)$.

We call $R(s)$ a *greatest common right divisor* (GCRD) of $N(s)$ and $D(s)$ if it is a right common right divisor, and if any other common right divisor of $N(s)$ and $D(s)$ is a right divisor of $R(s)$.

If all common right divisors of $N(s)$ and $D(s)$ are unimodular, then $N(s)$ and $D(s)$ are called *right coprime*.

For PFDs, one of the polynomial matrices always is nonsingular, so only non-singular common right divisors occur. Suppose

$$G(s) = N(s)D^{-1}(s)$$

and that $R(s)$ is a CRD of $N(s)$ and $D(s)$. Then

$$\tilde{N}(s) = N(s)R^{-1}(s), \qquad \tilde{D}(s) = D(s)R^{-1}(s) \tag{5}$$

are polynomial matrices, and they provide another right PFD for $G(s)$

$$\tilde{N}(s)\tilde{D}^{-1}(s) = N(s)D^{-1}(s) = G(s)$$

The degree of this new PFD is no greater than the degree of the original since

$$\deg\left[\det D(s)\right] = \deg\left[\det \tilde{D}(s)\right] + \deg\left[\det R(s)\right]$$

When $R(s)$ is a greatest common right divisor, the degree reduction is maximum. Then, no further reduction is possible, and $\tilde{N}(s)$ and $\tilde{D}(s)$ is right coprime.

Theorem 7.1: Suppose $N(s)$ is a $p \times r$ polynomial matrix and $D(s)$ is an $r \times r$ polynomial matrix. If a unimodular $(p+r) \times (p+r)$ polynomial matrix $U(s)$ and an $r \times r$ polynomial matrix $R(s)$ are such that

$$U(s)\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix} \tag{6}$$

then $R(s)$ is a greatest common right divisor of $N(s)$ and $D(s)$.

Proof: Partition $U(s)$ as

$$U(s) = \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \tag{7}$$

where $U_{11}(s)$ is $r \times r$, $U_{22}(s)$ is $p \times p$.

Partition $U^{-1}(s)$ (which is also polynomial) similarly

$$U^{-1}(s) = \begin{bmatrix} U_{11}^{-}(s) & U_{12}^{-}(s) \\ U_{21}^{-}(s) & U_{22}^{-}(s) \end{bmatrix}$$

Then (6) can be rewritten as

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} U_{11}^{-}(s) & U_{12}^{-}(s) \\ U_{21}^{-}(s) & U_{22}^{-}(s) \end{bmatrix} \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

i.e. $\qquad D(s) = U_{11}^{-}(s)R(s), \qquad N(s) = U_{21}^{-}(s)R(s)$

So $R(s)$ is a CRD of $N(s)$ and $D(s)$. But, from (6) and (7),

$$R(s) = U_{11}(s)D(s) + U_{12}(s)N(s) \tag{8}$$

so that if $R_a(s)$ is another CRD of $N(s)$ and $D(s)$, say,

$$D(s) = D_a(s)R_a(s), \qquad N(s) = N_a(s)R_a(s)$$

then, from (8),

$$R(s) = [U_{11}(s)D_a(s) + U_{12}(s)N_a(s)]\, R_a(s)$$

i.e., $R_a(s)$ is a right divisor of $R(s)$. Therefore, $R(s)$ is a GCRD of $N(s)$ and $D(s)$. $\square$

When $N(s)$ and $D(s)$ are right coprime, we have $R(s) = I$, and (8) becomes

$$U_{11}(s)D(s) + U_{12}(s)N(s) = I$$

It turns out that the existence of $U_{11}(s)$ and $U_{12}(s)$ to satisfy the above equation is a necessary and sufficient condition for $N(s)$ and $D(s)$ to be right coprime, as will be shown in Theorem 7.3.

Three Types of Elementary Row Operations on a polynomial matrix (i.e. premultiply by unimodular matrices):

- Interchange two rows, e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{r2\leftrightarrow r3}$; $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{r1\leftrightarrow r3}$,
- Multiply a row by a nonzero real number $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\alpha \neq 0$
- Add to any row a polynomial multiple of another row $\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & b(s) \\ 0 & 0 & 1 \end{array}\right]$.

Any unimodular matrix can be written as a product of the above three types of matrices.

Convention: the degree of a zero polynomial ($N(s) = 0$) is $-\infty$.

Row Hermite Form

$$\left[\begin{array}{cccccc}
c_{11}(s) & c_{12}(s) & \cdots & c_{1k}(s) & \cdots & c_{1r}(s) \\
0 & c_{22}(s) & \cdots & c_{2k}(s) & \cdots & c_{2r}(s) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_{kk}(s) & \cdots & c_{kr}(s) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & c_{rr}(s) \\
\hline
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]$$

where $c_{kk}(s)$ is monic, and deg $c_{ik}(s) <$ deg $c_{kk}(s)$ for all $i < k$ and $k = 1, \cdots, r$.

Theorem 7.2: Suppose $N(s)$ is a $p \times r$ polynomial matrix and $D(s)$ is an $r \times r$ nonsingular polynomial matrix. Then elementary row operations can be used to transform

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{9}$$

into the *row Hermite form*.

Procedure:

Step 1 In the first column of $M(s)$, use row interchange to bring the lowest degree entry to the first row (which is nonzero because $M(s)$ is nonsingular).

Step 2 For each entry $m_{i1}(s)$, use the third elmentary row operation to make its degree strictly less than deg $m_{i1}(s)$.

Step 3 Repeat Steps 1 and 2 until $m_{i1}(s) = 0$ for $i = 2, \cdots r$.

Step 4 For column 2, repeat Steps 1 to 3, while ignoring the first row. (For column j, repeat Steps 1 to 3, while ignoring the first $j-1$ rows.)

Connection between elementary row operations and unimodular matrices can be established using Theorem 7.2.

Suppose $N(s) = 0$ and $D(s)$ is unimodular. Then the row Hermite form is upper triangular and the diagonal entries are unity. Hence the row Hermite form is just the identity matrix!

Therefore, for a unimodular polynomial matrix $U(s)$, there is a sequence of elementary row operations, say E_a, E_b, $E_c(s)$, $\cdots$, E_b, such that

$$[E_a E_b E_c(s) \cdots E_b] U(s) = I \tag{11}$$

Then, obviously,

$$U(s) = E_b^{-1} \cdots E_c^{-1}(s) E_b^{-1} E_a^{-1}$$

i.e., a unimodular matrix is equiv to a sequence of elementary row operations.

Then Theorem 7.1 can be restated in terms of a sequence of elementary row operations, rather than unimodular matrix $U(s)$.

If reduction to row Hermite form is used to implement (6), then GCRD will be an upper-triangular polynomial matrix.

If $N(s)$ and $D(s)$ are coprime, then $R(s) = I$.

Example 7.6 Find a GCRD of the polynomial matrices:

$$D(s) = \begin{bmatrix} s & 3s+1 \\ -1 & s^2+s-2 \end{bmatrix}, \; N(s) = \begin{bmatrix} -1 & s^2+2s-1 \end{bmatrix}$$

$$\begin{aligned} M(s) &= \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s & 3s+1 \\ -1 & s^2+s-2 \\ -1 & s^2+2s-1 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & s^2+s-2 \\ s & 3s+1 \\ -1 & s^2+2s-1 \end{bmatrix} \\ &\Longrightarrow \begin{bmatrix} 1 & -s^2-s+2 \\ 0 & s^3+s^2+s+1 \\ 0 & s+1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -s^2-s+2 \\ 0 & s+1 \\ 0 & s^3+s^2+s+1 \end{bmatrix} \\ &\Longrightarrow \begin{bmatrix} 1 & -s^2-s+2 \\ 0 & s+1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Hence, GCRD is given by

$$R(s) = \begin{bmatrix} 1 & -s^2-s+2 \\ 0 & s+1 \end{bmatrix}$$

Example 7.7 For

$$D(s) = \begin{bmatrix} s^2+s+1 & s+1 \\ s^2-3 & 2s-2 \end{bmatrix}, \qquad N(s) = \begin{bmatrix} s+2 & 1 \end{bmatrix}$$

calculation of a GCRD via Theorem 7.2 is as follows:

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s^2+s+1 & s+1 \\ s^2-3 & 2s-2 \\ s+2 & 1 \end{bmatrix} \to \begin{bmatrix} s+2 & 1 \\ s^2-3 & 2s-2 \\ s^2+s+1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} s+2 & 1 \\ (s-2)(s+2)+1 & 2s-2 \\ (s-1)(s+2)+3 & s+1 \end{bmatrix} \to \begin{bmatrix} s+2 & 1 \\ 1 & s \\ 3 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & s \\ s+2 & 1 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & s \\ 0 & -s^2-2s+1 \\ 0 & -3s+2 \end{bmatrix} \to \begin{bmatrix} 1 & s \\ 0 & s-\frac{2}{3} \\ 0 & -s^2-2s+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & s \\ 0 & s-\frac{2}{3} \\ 0 & -(s+\frac{8}{3})(s-\frac{2}{3})-\frac{7}{9} \end{bmatrix} \to \begin{bmatrix} 1 & s \\ 0 & s-\frac{2}{3} \\ 0 & -\frac{7}{9} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & s \\ 0 & 1 \\ 0 & s-\frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, $N(s)$ and $D(s)$ are coprime.

The corresponding elementary row operations are (i.e., the unimodular matrix is)

$$
\begin{aligned}
U(s) = & \begin{bmatrix} 1 & -s & 0 \\ 0 & 1 & 0 \\ 0 & -(s-\frac{2}{3}) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{9}{7} \\ 0 & 1 & 0 \end{bmatrix} \\
& \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (s+\frac{8}{3}) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -(s+2) & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \\
& \times \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -(s-2) & 1 & 0 \\ -(s-1) & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
= & \left[\begin{array}{cc|c} -s(\frac{3}{7}s+\frac{8}{7}) & 1+\frac{6}{7}\,s & (\frac{3}{7}s^3-\frac{1}{7}s^2+\frac{6}{7}s+2) \\ (\frac{3}{7}s+\frac{8}{7}) & -\frac{6}{7} & -(\frac{3}{7}s^2-\frac{1}{7}s+\frac{13}{7}) \\ \hline -\frac{3}{7}\,(s^2+2s-1) & \frac{3}{7}\,(2s+1) & \frac{3}{7}\,(s^3-s^2+3s+1) \end{array}\right] \\
= & \left[\begin{array}{c|c} U_{11}(s) & U_{12}(s) \\ \hline U_{21}(s) & U_{22}(s) \end{array}\right]
\end{aligned}
$$

with $\det U(s) = \frac{3}{7}$.

$U(s)$ can also be obtained by the following procedure:

$$
\left[\begin{array}{c|c} D(s) & I \\ N(s) & \end{array}\right] = \left[\begin{array}{cc|ccc} s^2+s+1 & s+1 & 1 & 0 & 0 \\ s^2-3 & 2s-2 & 0 & 1 & 0 \\ s+2 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\to \left[\begin{array}{cc|ccc} s+2 & 1 & 0 & 0 & 1 \\ (s-2)(s+2)+1 & 2s-2 & 0 & 1 & 0 \\ (s-1)(s+2)+3 & s+1 & 1 & 0 & 0 \end{array}\right]
$$

$$
\to \left[\begin{array}{cc|ccc} s+2 & 1 & 0 & 0 & 1 \\ 1 & s & 0 & 1 & -(s-2) \\ 3 & 2 & 1 & 0 & -(s-1) \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & s & 0 & 1 & -(s-2) \\ s+2 & 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & -(s-1) \end{array}\right]
$$

$$
\to \left[\begin{array}{cc|ccc} 1 & s & 0 & 1 & -(s-2) \\ 0 & -s^2-2s+1 & 0 & -(s+2) & s^2-3 \\ 0 & -3s+2 & 1 & -3 & 2s-5 \end{array}\right]
$$

$$
\to \left[\begin{array}{cc|ccc} 1 & s & 0 & 1 & -(s-2) \\ 0 & s-\frac{2}{3} & -\frac{1}{3} & 1 & -\frac{3}{2}s+\frac{5}{3} \\ 0 & -(s+\frac{8}{3})(s-\frac{2}{3})-\frac{7}{9} & 0 & -(s+2) & s^2-3 \end{array}\right]
$$

$$
\to \left[\begin{array}{cc|ccc} 1 & s & 0 & 1 & -(s-2) \\ 0 & s-\frac{2}{3} & -\frac{1}{3} & 1 & -\frac{2}{3}s+\frac{5}{3} \\ 0 & -\frac{7}{9} & -(s+\frac{8}{3})\frac{1}{3} & \frac{2}{3} & \frac{4}{3}s^2-\frac{1}{9}s+\frac{13}{9} \end{array}\right]
$$

$$\to \left[\begin{array}{cc|ccc} 1 & s & 0 & 1 & -(s-2) \\ 0 & 1 & \frac{3}{7}s+\frac{8}{7} & -\frac{6}{7} & -\frac{3}{7}\,s^2+\frac{1}{7}\,s-\frac{13}{7} \\ 0 & s-\frac{2}{3} & -\frac{1}{3} & 1 & -\frac{2}{3}\,s+\frac{5}{3} \end{array}\right]$$

$$\to \left[\begin{array}{cc|ccc} 1 & 0 & -\frac{1}{7}s(3s+8) & 1+\frac{6}{7}s & \frac{1}{7}(3s^3-s^2+6s+14) \\ 0 & 1 & \frac{1}{7}(3s+8) & -\frac{6}{7} & -\frac{1}{7}(3s^2-s+13) \\ 0 & 0 & -\frac{3}{7}\,(s^2+2s-1) & \frac{3}{7}(2s+1) & \frac{3}{7}\,(s^3-s^2+3s+1) \end{array}\right]$$

Hence,

$$U = \left[\begin{array}{cc|c} -\frac{1}{7}s(3s+8) & 1+\frac{6}{7}s & \frac{1}{7}(3s^3-s^2+6s+14) \\ \frac{1}{7}(3s+8) & -\frac{6}{7} & -\frac{1}{7}(3s^2-s+13) \\ \hline -\frac{3}{7}\,(s^2+2s-1) & \frac{3}{7}(2s+1) & \frac{3}{7}\,(s^3-s^2+3s+1) \end{array}\right]$$

and it is easy to verify that

$$U_{11}(s)D(s)+U_{12}(s)N(s)=I$$

$$U_{21}(s)D(s)+U_{22}(s)N(s)=0$$

Note that, as will be shown in Theorem 7.3, $U_{22}(s)$ and $-U_{21}(s)$ actually provides a left coprime PFD for $N(s)D^{-1}(s)$:

$$N(s)D^{-1}(s)=-U_{22}^{-1}(s)U_{21}(s)$$

Two different characterizations of right coprimeness:

Theorem 7.3: For a $p \times r$ polynomial matrix $N(s)$ and a nonsingular $r \times r$ polynomial matrix $D(s)$, the following statements are equivalent.
(i) The polynomial matrices $N(s)$ and $D(s)$ are right coprime.
(ii) There exist an $r \times p$ polynomial matrix $X(s)$ and an $r \times r$ polynomial matrix $Y(s)$ satisfying the *Bezout identity*

$$X(s)N(s) + Y(s)D(s) = I_r \tag{12}$$

(iii) For every complex number s_0,

$$\text{rank} \begin{bmatrix} D(s_0) \\ N(s_0) \end{bmatrix} = r \tag{13}$$

Proof: (i) $\rightarrow$ (ii): If $N(s)$ and $D(s)$ are right coprime, then reduction to row Hermiteform as in (6) yields unimodular $U_{11}(s)$ and $U_{12}(s)$ such that

$$U_{11}(s)D(s) + U_{12}(s)N(s) = I_r$$

and this has the form of (12).

(ii) $\rightarrow$ (iii): Write condition (12) as

$$\left[\begin{array}{cc} X(s) & Y(s) \end{array}\right]\left[\begin{array}{c} D(s) \\ N(s) \end{array}\right] = I_r$$

If s_0 is a complex number for which

$$\text{rank}\left[\begin{array}{c} D(s_0) \\ N(s_0) \end{array}\right] < r$$

then we have a rank contradiction.
(iii) $\rightarrow$ (i): Suppose that (13) holds and $R(s)$ is a CRD of $N(s)$ and $D(s)$. Then for some $p \times r$ polynomial matrix $\tilde{N}(s)$ and some $r \times r$ polynomial matrix $\tilde{D}(s)$,

$$\left[\begin{array}{c} D(s) \\ N(s) \end{array}\right] = \left[\begin{array}{c} \tilde{D}(s) \\ \tilde{N}(s) \end{array}\right] R(s) \tag{14}$$

If det $R(s)$ is a polynomial of degree at least 1 and s_0 is a root of this polynomial, then $R(s_0)$ is a complex matrix of less that full rank. Thus the contradiction

$$\text{rank}\left[\begin{array}{c} D(s_0) \\ N(s_0) \end{array}\right] \leq \text{rank}\ R(s_0) < r$$

Therefore, det $R(s)$ is a nonzero constant, that is, $R(s)$ is unimodular, and hence $N(s)$ and $D(s)$ are coprime. This completes the proof. □

Example 7.8 Determine if the matrix polynomials

$$D(s) = \begin{bmatrix} s+1 & 0 \\ s^2+s-2 & s-1 \end{bmatrix}, \quad N(s) = [s+2 \;\; s+1]$$

are right coprime.
Form

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s+1 & 0 \\ s^2+s-2 & s-1 \\ s+2 & s+1 \end{bmatrix}$$

Note that

$$\det \begin{bmatrix} s+1 & 0 \\ s^2+s-2 & s-1 \end{bmatrix} = (s+1)(s-1), \quad s_1 = -1, s_2 = 1$$

$$\det \begin{bmatrix} s^2+s-2 & s-1 \\ s+2 & s+1 \end{bmatrix} = s(s-1)(s+2), \quad s_1 = 0, s_2 = 1, s_3 = -2$$

$$\det \begin{bmatrix} s+1 & 0 \\ s+2 & s+1 \end{bmatrix} = (s+1)^2, \quad s_1 = -1, s_2 = -1$$

All the 3 minors of order 2 do not have a common root. So, the $rank M(s) = 2$.
By Theorem 7.3(iii), $D(s)$ and $N(s)$ are right coprime.

Coprime Right Polynomial Fraction Description of Transfer Function Matrix

If $N(s)$ and $D(s)$ are coprime, then $N(s)D^{-1}(s)$ is a *coprime right polynomial fraction description* (*CRPFD*).

Theorem 7.4: For any two CRPFDs of a strictly-proper transfer function,

$$G(s) = N(s)D^{-1}(s) = N_a(s)D_a^{-1}(s) \qquad (*)$$

there exists a unimodular polynomial matrix $U(s)$ such that

$$N(s) = N_a(s)U(s), \qquad D(s) = D_a(s)U(s)$$

i.e., all CRPFDs of a given transfer function are equivalent up to right multiplication by unimodular matrices.

Proof: By Theorem 7.3, $\exists$ polynomial matrices $X(s)$, $Y(s)$, $A(s)$ and $B(s)$ such that

$$X(s)N_a(s) + Y(s)D_a(s) = I_m \qquad (15)$$
$$A(s)N(s) + B(s)D(s) = I_m \qquad (16)$$

By $(*)$, we have $N_a(s) = N(s)D^{-1}(s)D_a(s)$.

Subtituting this into (15) gives

$$\begin{aligned} & X(s)N(s)D^{-1}(s)D_a(s) + Y(s)D_a(s) = I_m \\ \Rightarrow \quad & X(s)N(s) + Y(s)D(s) = D_a^{-1}(s)D(s) \end{aligned}$$

A similar calculation using $N(s) = N_a(s)D_a^{-1}(s)D(s)$ in (16) gives

$$A(s)N_a(s) + B(s)D_a(s) = D^{-1}(s)D_a(s)$$

Therefore, both $D_a^{-1}(s)D(s)$ and $D^{-1}(s)D_a(s)$ are polynomial matrices, and since they are inverse of each other, both of them must be unimodular. Let

$$U(s) = D_a^{-1}(s)D(s)$$

Then

$$N(s) = N_a(s)U(s), \qquad D(s) = D_a(s)D_a^{-1}(s)D(s) = D_a(s)U(s)$$

and the proof is complete. $\square$

Example 7.9 Find a CRPFD of the transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+1}{s^2} & 0 \\ \frac{s+1}{s+2} & 0 & \frac{s}{s+2} \end{bmatrix}$$

First, we find one RPFD of $G(s)$.

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{s+2}{s(s+2)} & \frac{s+2}{s^2} & 0 \\ \frac{s(s+1)}{s(s+2)} & 0 & \frac{s}{s+2} \end{bmatrix} = \begin{bmatrix} s+2 & s+1 & 0 \\ s(s+1) & 0 & s \end{bmatrix} \begin{bmatrix} s(s+2) & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} \\ &= N(s)D^{-1}(s) \end{aligned}$$

Note that $rank \begin{bmatrix} D(0) \\ N(0) \end{bmatrix} = 2 < 3$. So, $D(s)$ and $N(s)$ are not right coprime.

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s(s+2) & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ s+2 & s+1 & 0 \\ s(s+1) & 0 & s \end{bmatrix} \Longrightarrow \begin{bmatrix} s+2 & s+1 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ s(s+2) & 0 & 0 \\ s(s+1) & 0 & s \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} s+2 & s+1 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ 0 & -s(s+1) & 0 \\ -s & -s(s+1) & s \end{bmatrix} \Longrightarrow \begin{bmatrix} s+2 & s+1 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ 0 & -s(s+1) & 0 \\ 2 & -s^2+1 & s \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & -s^2+1 & s \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ 0 & -s(s+1) & 0 \\ s+2 & s+1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & -s^2+1 & s \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ 0 & -s(s+1) & 0 \\ 0 & s(\frac{1}{2}s^2+s+\frac{1}{2}) & -\frac{1}{2}s(s+2) \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 1 & s \\ 0 & s^2 & 0 \\ 0 & 0 & s+2 \\ 0 & -s & 0 \\ 0 & \frac{1}{2}s & -\frac{1}{2}s(s+2) \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 & s \\ 0 & -s & 0 \\ 0 & 0 & s+2 \\ 0 & s^2 & 0 \\ 0 & \frac{1}{2}s & -\frac{1}{2}s(s+2) \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & -s & 0 \\ 0 & 0 & s+2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$R(s) = \begin{bmatrix} 2 & 1 & -2 \\ 0 & -s & 0 \\ 0 & 0 & s+2 \end{bmatrix}, \quad R^{-1}(s) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2s} & \frac{1}{s+2} \\ 0 & -\frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\begin{aligned} \bar{D}(s) &= D(s)R^{-1}(s) = \begin{bmatrix} \frac{1}{2}s(s+2) & \frac{1}{2}(s+2) & s \\ 0 & -s & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \bar{N}(s) &= N(s)R^{-1}(s) = \begin{bmatrix} \frac{1}{2}(s+2) & -\frac{1}{2} & 1 \\ \frac{1}{2}s(s+1) & \frac{1}{2}(s+1) & s \end{bmatrix} \end{aligned}$$

Left Polynomial Fractions

Similar development exists for *left polynomial fraction description* (LPFD).

Definition 7.7: A $q \times q$ polynomial matrix $L(s)$ is called a *left divisor* of the $q \times p$ polynomial matrix $P(s)$ if $\exists$ a $q \times p$ polynomial matrix $\tilde{P}(s)$ such that

$$P(s) = L(s)\tilde{P}(s) \tag{17}$$

Definition 7.8: If $N(s)$ is a $q \times p$ polynomial matrix and $D(s)$is a $q \times q$ polynomial matrix, then a $q \times q$ polynomial matrix $L(s)$ is call a *common left divisor* (CLD) of $N(s)$ and $D(s)$ if $L(s)$ is a left divisor of both $N(s)$ and $D(s)$.

We call $L(s)$ a *greatest common left divisor* (GCLD) of $N(s)$ and $D(s)$ if it is a CLD, and if any other CLD of $N(s)$ and $D(s)$ is a left divisor of $L(s)$.

If all CLDs of $N(s)$ and $D(s)$ are unimodular, then $N(s)$ and $D(s)$ are called *left coprime*.

Example 7.10: For the polynomial matrix

$$P(s) = \begin{bmatrix} (s+1)^2(s+2) \\ (s+1)(s+2)(s+3) \end{bmatrix} \tag{18}$$

one left divisor is

$$L(s) = \begin{bmatrix} (s+1)^2(s+2) & 0 \\ 0 & (s+1)(s+2)(s+3) \end{bmatrix}$$

where the corresponding $\tilde{P}(s) = [1\ 1]^T$. There are many other left divisors.

Theorem 7.5: Suppose $N(s)$ is a $q \times p$ polynomial matrix and $D(s)$ is a $q \times q$ polynomial matrix. If a $(q+p) \times (q+p)$ unimodular polynomial matrix $U(s)$ and a $q \times q$ polynomial matrix $L(s)$ are such that

$$\begin{bmatrix} D(s) & N(s) \end{bmatrix} U(s) = \begin{bmatrix} L(s) & 0 \end{bmatrix} \tag{19}$$

then $L(s)$ is a GCLD of $N(s)$ and $D(s)$.

The unimodular polynomial matrix $U(s)$ consists of a sequence of three types of **elementary column operations** (i.e. post-multiply by unimodular matrices):

- Interchange two columns $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

- Multiply a column by a nonzero real number $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\alpha \neq 0$
- Add a polynomial multiple of one column to another $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b(s) \\ 0 & 0 & 1 \end{bmatrix}$,

Any unimodular matrix can be written as a product of the above three types of matrices.
Convention: the degree of a zero polynomial ($P(s) = 0$) is $-\infty$.

Column Hermite Form

$$\left[\begin{array}{cccccc|ccc} c_{11}(s) & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ c_{21}(s) & c_{22}(s) & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{k1}(s) & c_{k2}(s) & \cdots & c_{kk}(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{r1}(s) & c_{r2}(s) & \cdots & c_{rk}(s) & \cdots & c_{rr}(s) & 0 & \cdots & 0 \end{array}\right]$$

where $c_{kk}(s)$ is monic, and deg $c_{ki}(s) <$ deg $c_{kk}(s)$ for all $i < k$ and $k = 1, \cdots, r$.

Theorem 7.6: Suppose $N(s)$ is a $q \times p$ polynomial matrix and $D(s)$ is a $q \times q$ nonsingular polynomial matrix. The elementary column operations can be used to transform

$$M(s) = \begin{bmatrix} D(s) & N(s) \end{bmatrix}$$

into the *column Hermite form*.

Theorem 7.7: For a $q \times p$ polynomial matrix $N(s)$ and a $q \times q$ polynomial matrix $D(s)$, the following statements are equivalent.
(i) The polynomial matrices $N(s)$ and $D(s)$ are left coprime.
(ii) There exist an $p \times q$ polynomial matrix $X(s)$ and an $q \times q$ polynomial matrix $Y(s)$ satisfying the *Bezout identity*

$$N(s)X(s) + D(s)Y(s) = I_q \tag{20}$$

(iii) For every complex number s_0,

$$\text{rank} \begin{bmatrix} D(s_0) & N(s_0) \end{bmatrix} = q \tag{21}$$

If $D(s)$ and $N(s)$ are left coprime, then $G(s) = D^{-1}(s)N(s)$ is called a *coprime left polynomial fraction description* (CLPFD).

Theorem 7.8: For any two CLPFDs of a strictly proper rational transfer function,

$$G(s) = D^{-1}(s)N(s) = D_a^{-1}(s)N_a(s)$$

there exists a unimodular polynomial matrix $U(s)$ such that

$$N(s) = U(s)N_a(s), \qquad D(s) = U(s)D_a(s)$$

Next we establish that coprime right and left polynomial fraction descriptions for the same strictly proper transfer function have the same degree. To establish this result, the following matrix inversion formula is needed.

Lemma 7.1: Suppose that $V_{11}(s)$ is an $m \times m$ nonsingular polynomial matrix, and

$$V(s) = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} \tag{22}$$

is an $(m+p) \times (m+p)$ nonsingular polynomial matrix. Then defining the matrix of rational functions

$$V_a(s) = V_{22} - V_{21}(s)V_{11}^{-1}(s)V_{12}(s)$$

the following results hold:
(i) $\det V(s) = \det [V_{11}(s)] \cdot \det [V_a(s)]$,
(ii) $\det V_a(s)$ is a nonzero rational function,
(iii) the inverse of $V(s)$ is

$$V^{-1}(s) = \left[\begin{array}{c|c} \begin{array}{c} V_{11}^{-1}(s) + V_{11}^{-1}(s)V_{12}(s) \\ V_a^{-1}(s)V_{21}(s)V_{11}^{-1}(s) \end{array} & -V_{11}^{-1}(s)V_{12}(s)V_a^{-1}(s) \\ \hline -V_a^{-1}(s)V_{21}(s)V_{11}^{-1}(s) & V_a^{-1}(s) \end{array}\right]$$

Proof: A partitioned calculation verifies

$$\begin{bmatrix} I_m & 0_{m\times p} \\ -V_{21}(s)V_{11}^{-1}(s) & I_p \end{bmatrix} V(s) = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ 0 & V_a(s) \end{bmatrix} \tag{23}$$

Then,

$$\det V(s) = \det [V_{11}(s)] \cdot \det [V_a(s)]$$

Since $V(s)$ and $V_{11}(s)$ are nonsingular polynomial matrices, $\det V_a(s)$ is a nonzero rational function.

To establish (iii), multiply (23) on the left by

$$\begin{bmatrix} V_{11}^{-1}(s) & 0 \\ 0 & V_a^{-1}(s) \end{bmatrix} \begin{bmatrix} I_m & -V_{12}(s)V_a^{-1}(s) \\ 0 & I_p \end{bmatrix}$$

to obtain

$$\left[\begin{array}{c|c} \begin{array}{c} V_{11}^{-1}(s) + V_{11}^{-1}(s)V_{12}(s) \\ V_a^{-1}(s)V_{21}(s)V_{11}^{-1}(s) \end{array} & -V_{11}^{-1}(s)V_{12}(s)V_a^{-1}(s) \\ \hline -V_a^{-1}(s)V_{21}(s)V_{11}^{-1}(s) & V_a^{-1}(s) \end{array}\right] V(s)$$
$$= \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}$$

This completes the proof. $\square$

Theorem 7.9: Suppose that a strictly-proper rational transfer function is represented by a coprime right and a coprime left polynomial fraction

$$G(s) = N(s)D^{-1}(s) = D_L^{-1}(s)N_L(s) \tag{24}$$

Then there exists a constant $\alpha \neq 0$ such that $\det D(s) = \alpha \det D_L(s)$.

Proof: Since $N(s)$ and $D(s)$ are right coprime, $\exists$ $(m+p)\times(m+p)$ unimodular polynomial matrix

$$U(s) = \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}$$

such that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{25}$$

For notational convenience, let

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}^{-1} = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix}$$

where each $V_{ij}(s)$ is a polynomial matrix.

In particular, (25) gives

$$V_{11}(s) = D(s), \qquad V_{21}(s) = N(s)$$

Therefore, $V_{11}(s)$ is nonsingular. Then, by Lemma 8.18, we have

$$U_{22}(s) = \left[V_{22}(s) - V_{21}(s)V_{11}^{-1}(s)V_{12}(s)\right]^{-1}$$

which is a polynomial matrix and is nonsingaular. Furthermore,

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}$$

of which, the (2,2)-block gives

$$U_{21}(s)V_{12}(s) + U_{22}(s)V_{22}(s) = I_p$$

By Theorem 8.16, $U_{21}(s)$ and $U_{22}(s)$ are left coprime. Also, from the (2,1)-block,

$$U_{21}(s)V_{11}(s) + U_{22}(s)V_{21}(s) = U_{21}(s)D(s) + U_{22}(s)N(s) = 0 \qquad (26)$$

Thus,

$$G(s) = N(s)D^{-1}(s) = -U_{22}^{-1}(s)U_{21}(s) \tag{27}$$

This is a coprime left polynomial fraction description for $G(s)$. Again, using Lemma 8.18, and unimodularity of $V(s)$, we have

$$\begin{aligned} \det \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} &= \det[V_{11}(s)] \cdot \det[V_{22}(s) - V_{21}(s)V_{11}^{-1}(s)V_{12}(s)] \\ &= \det[D(s)] \cdot \det[U_{22}^{-1}(s)] = \frac{\det[D(s)]}{\det[U_{22}(s)]} = \alpha \end{aligned}$$

for some nonzero constant α. Then, for the CLPFD in (27), we have

$$\det[U_{22}(s)] = \alpha \det[D(s)]$$

Finally, by Theorem 8.17, the above deteminant formula must hold for any CLPFD of $G(s)$, with possibly a different nonzero constant α. This completes the proof. $\square$

Column and Row Degrees

Recall: a constant has polynomial degree 0, and a zero polynomial has degree $-\infty$.

Definition 7.10: For a $p \times r$ polynomial matrix $P(s)$, the degree of the highest-degree polynomial in the j^{th} column of $P(s)$, written as $c_j[P]$, is called the j^{th} column degree of $P(s)$.

The *column degree coefficient matrix* for $P(s)$, written as P_{hc}, is the real $p \times r$ matrix with (i, j)-entry given by the coefficient of $s^{c_j[P]}$ in the (i, j)-entry of $P(s)$.

If $P(s)$ is square and nonsingular, then it is called *column reduced* if

$$\text{deg}\left[\text{det} P(s)\right] = c_1[P] + c_2[P] + \cdots + c_p[P] \tag{28}$$

Column degrees play an important role in system realization, similar to the SISO case. Note that, in general,

$$\text{deg}\left[\text{det} P(s)\right] \leq c_1[P] + c_2[P] + \cdots + c_p[P]$$

and the column degrees of $P(s)$ can be artificially high.

Example 7.11: Two CRPFDs for $G(s)$, say, $N(s)D^{-1}(s)$ and $N_a(s)D_a^{-1}(s)$, can have different column degrees, even though
deg[det$D(s)$] = deg[det$D_a(s)$].

Consider

$$G(s) = \left[\frac{2s-3}{s^2-1}, \frac{1}{s-1}\right] \tag{29}$$

one CRPFD is

$$N(s) = [1, 2], \qquad D(s) = \begin{bmatrix} 0 & s+1 \\ s-1 & 1 \end{bmatrix}$$

Note that $c_1[D] = 1$, $c_2[D] = 1$, and

$$\deg[\det D(s)] = 2 = c_1[D] + c_2[D],$$

hence $D(s)$ is column reduced.

Choosing the unimodular matrix

$$U(s) = \begin{bmatrix} 1 & 0 \\ s^2 - s + 1 & 1 \end{bmatrix}$$

we have another CRPFD

$$N_a(s) = N(s)U(s) = \begin{bmatrix} 2s^2 - 2s + 3, & 2 \end{bmatrix}$$

$$D_a(s) = D(s)U(s) = \begin{bmatrix} s^3 + 1, & s + 1 \\ s^2 & 1 \end{bmatrix}$$

So $c_1[D] = 3$, $c_2[D] = 1$, even though
deg$[\det D_a(s)] =$ deg$[\det D(s)] = 2$.

Characterize column-reducedness by column degree coefficient matrix:
Given a $p \times p$ polynomial matrix $P(s)$, write

$$P(s) = P_{hc} \begin{bmatrix} s^{c_1[P]} & 0 & \cdots & 0 \\ 0 & s^{c_2[P]} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{c_p[P]} \end{bmatrix} + P_l(s) \qquad (30)$$

where $P_l(s)$ is a $p \times p$ polynomial matrix in which each entry has a degree strictly less than the corresponding column degree.

Hence, for Example 7.11, we have

$$D(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$$

$$D_a(s) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ s^2 & 1 \end{bmatrix}, \qquad \det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

Theorem 7.10: If $P(s)$ is a $p \times p$ nonsingular polynomial matrix, then $P(s)$ is column reduced iff P_{hc} is invertible.

Proof: From (30), we have

$$
\begin{aligned}
& s^{-c_1[P]-c_2[P]-\cdots-c_p[P]} \det P(s) \\
&= \det P(s) \ \det \left[\textbf{diag} \ \left\{ s^{-c_1[P]}, \cdots, s^{-c_p[P]} \right\} \right] \\
&= \det \left(P_{hc} + P_l(s) \begin{bmatrix} s^{-c_1[P]} & 0 & \cdots & 0 \\ 0 & s^{-c_2[P]} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{-c_p[P]} \end{bmatrix} \right)
\end{aligned}
$$

Note that

$$
\lim_{s \to \infty} \left(s^{-c_1[P]-c_2[P]-\cdots-c_p[P]} \det P(s) \right) = \det P_{hc} \tag{31}
$$

and the result follows. $\square$

Use unimodular matrices to reduce column degrees:

Given $G(s) = N(s)D^{-1}(s)$, a CRPFD, which is not column reduced, we can find a unimodular matrix $U(s)$ such that

$$\tilde{N}(s) = N(s)U(s), \qquad \tilde{D}(s) = D(s)U(s) \tag{32}$$

with $\tilde{D}(s)$ column reduced.

As right-multiplication by unimodular matrixs is equivalent to elementary column operations, we can adopt the following procedure to make $D(s)$ column reduced:

- Perform elementary column operations on $\begin{bmatrix} D(s) \\ I \end{bmatrix}$ to make the $(1,1)$-block column reduced.
- Then the $(2,1)$-block gives the corresponding elementary column operations $U(s)$.
- Right-multiply $N(s)$ by the above $U(s)$ to get the new $N(s)$.

For Example 7.11, we have

$$\left[\frac{D_a(s)}{I}\right] = \left[\begin{array}{cc} s^3+1 & s+1 \\ s^2 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} (s+1)(s^2-s+1) & s+1 \\ s^2 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc} 0 & s+1 \\ s-1 & 1 \\ \hline 1 & 0 \\ -(s^2-s+1) & 1 \end{array}\right] \triangleq \left[\frac{D(s)}{U(s)}\right]$$

Hence,

$$D(s) = D_a(s)U(s) = \begin{bmatrix} 0 & s+1 \\ s-1 & 1 \end{bmatrix}$$

is column reduced, and the corresponding $N(s)$ is given by

$$\begin{aligned} N(s) &= N_a(s)U(s) = \begin{bmatrix} 2s^2-2s+3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(s^2-s+1) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1, & 2 \end{bmatrix} \end{aligned}$$

Example 7.12: We need to have column reduced CRPFD. Otherwise, it causes confusion. For example,

$$N(s) = \left[\, s^2 \;\; 1 \,\right], \qquad D(s) = \begin{bmatrix} s^3+1 & s+1 \\ s^2 & 1 \end{bmatrix}$$

Then

$$c_1[N] = 2, \quad c_2[N] = 0, \quad c_1[D] = 3, \quad c_2[D] = 1,$$

All column degrees of $N(s)$ are less than that of $D(s)$. However,

$$N(s)D^{-1}(s) = [0\ 1], \qquad \text{not strictly proper!}$$

This is due to $D(s)$ not being column reduced:

$$D_{hc} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{not invertible!}$$

The following Theorem gives the complete answer to the link between the column degrees of $D(s)$ and $N(s)$, and the strictly properness of $N(s)D^{-1}(s)$.

Theorem 7.11: If the polynomial fraction description $N(s)D^{-1}(s)$ is a strictly-proper rational function, then $c_j[N] < c_j[D]$, $j = 1, \cdots, m$.
If $D(s)$ is column reduced and $c_j[N] < c_j[D]$, $j = 1, \cdots, m$, then $N(s)D^{-1}(s)$ is a strictly-proper rational function.

Proof: Suppose $G(s) = N(s)D^{-1}(s)$ is strictly proper. Then $N(s) = G(s)D(s)$, and in particular,

$$N_{ij}(s) = \sum_{k=1}^{m} G_{ik}(s)D_{kj}(s), \qquad \begin{array}{l} i = 1, \cdots, p \\ j = 1, \cdots, m \end{array} \tag{34}$$

Then for fixed j,

$$N_{ij}(s)s^{-c_j[D]} = \sum_{k=1}^{m} G_{ik}(s)D_{kj}(s)s^{-c_j[D]}, \qquad i = 1, \cdots, p$$

Note that

$$\lim_{s\to\infty} D_{kj}(s)s^{-c_j[D]} = \text{finite}, \qquad \forall k$$
$$\lim_{s\to\infty} G_{ik}(s) = 0, \qquad \forall k$$

as $G_{ik}(s)$ is strictly proper.

Hence

$$\lim_{s\to\infty} N_{ij}(s)s^{-c_j[D]} = 0 \qquad\qquad i = 1, \cdots, p$$

Therefore, $\deg N_{ij}(s) < c_j[D]$ for all $i = 1, \cdots, p$, i.e.

$$c_j[N] < c_j[D], \qquad\qquad j = 1, \cdots, m$$

Now suppose that $D(s)$ is column reduced, and $c_j[N] < c_j[D]$, $j = 1, \cdots, m$. Then

$$\begin{aligned} N(s)D^{-1}(s) =& \left[N(s) \cdot \text{diagonal}\left\{s^{-c_1[D]}, \cdots, s^{-c_m[D]}\right\}\right] \\ & \times \left[D(s) \cdot \text{diagonal}\left\{s^{-c_1[D]}, \cdots, s^{-c_m[D]}\right\}\right]^{-1} \end{aligned} \tag{35}$$

Since $c_j[N] < c_j[D]$, $j = 1, \cdots, m$, and using (35), we have

$$\begin{aligned} \lim_{s\to\infty} \left[D(s) \cdot \text{diagonal}\left\{s^{-c_1[D]}, \cdots, s^{-c_m[D]}\right\}\right]^{-1} =& D_{hc}^{-1} \\ \lim_{s\to\infty} \left[N(s) \cdot \text{diagonal}\left\{s^{-c_1[D]}, \cdots, s^{-c_m[D]}\right\}\right]^{-1} =& 0 \end{aligned}$$

Hence,

$$\lim_{s\to\infty} N(s)D^{-1}(s) = 0 \cdot D_{hc}^{-1} = 0$$

i.e., $G(s) = N(s)D^{-1}(s)$ is stictly proper. The proof is complete. □

Similar development exists for left polynomial fraction descriptions. We state the results without proof.

Definition 7.11: For a $q \times p$ polynomial matrix $P(s)$, the degree of the highest-degree polynomial in the i^{th} row of $P(s)$, written as $r_i[P]$, is called the i^{th}-*row degree of* $P(s)$.
The *row degree coefficient matrix* of $P(s)$, written as P_{hr}, is the real $q \times p$ matrix with (i, j)-entry given by the coefficient of $s^{r_i[P]}$ in $P_{ij}(s)$.
If $P(s)$ is square and nonsingular, then it is called *row reduced* if

$$\deg \left[\det P(s) \right] = r_1[P] + \cdots + r_q[P] \tag{36}$$

Theorem 7.12: If $P(s)$ is a $p \times p$ nonsingular polynomial matrix, then $P(s)$ is row reduced iff P_{hr} is invertible.

Theorem 7.13: If the $p \times m$ polynomial fraction description $D^{-1}(s)N(s)$ is a strictly proper rational function, then $r_i[N] < r_i[D]$, $i = 1, \cdots, p$.
If $D(s)$ is row reduced, and $r_i[N] < r_i[D]$, $i = 1, \cdots, p$, then $D^{-1}(s)N(s)$ is a strictly proper rational function.

Finally, if $G(s) = D^{-1}(s)N(s)$ is a polynomial fraction description and $D(s)$ is not row reduced, then a unimodular matrix $U(s)$ can be computed such that $D_b(s) = U(s)D(s)$ is row reduced.

So a polynomial fraction description for a strictly proper rational transfer function $G(s)$ can be assumed as either
a coprime right polynomial fraction description
with column-reduced $D(s)$
or
a coprime left polynomial fraction description
with row-reduced $D_L(s)$
In either case, the PFDs have the same degree

$$n = c_1[D] + \cdots + c_m[D]$$

or

$$n = r_1[D_L] + \cdots + r_m[D_L]$$