EE 6225 – Multivariable Control System

Part I – Advanced Process Control

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Course Outline

1.Dynamic Model, MIMO, Poles and Zeros, Trade-Offs and Limits of Performance

2.SISO - PID, DSM, IMC, IMC-Based PID, Smith Predictor

3.SISO - Cascade, Feedforward, Ratio, Split Range, Selective Control

4,5. MIMO – Multiloop Interaction Dynamics, RGA, Decoupling, Multivariable Feedback Control

6.System Identification, Linear Regression, RLS, 4SID

4. Multivariable Process (Example: Distillation Column)

- **Controlled Variables:** $x_D, x_B, P, h_D, \text{and } h_B$
- **Manipulated Variables:** R, $\mathrm{Q_D}$, $\mathrm{Q_B}$

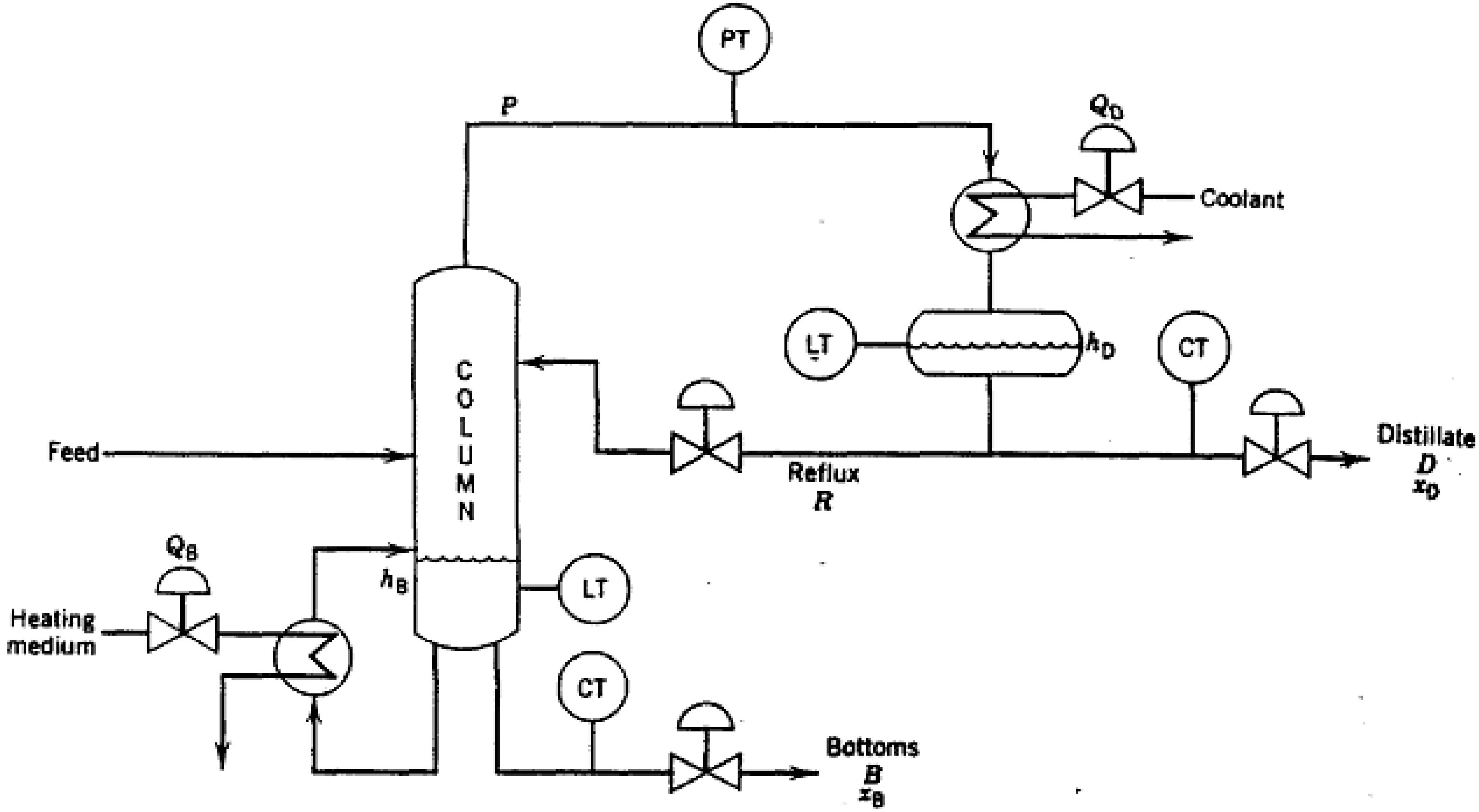


Figure 18.8. Controlled and manipulated variables for a typical distillation column.

4. Learning Objectives

- Multivariable system will exhibit different behavior from single-loop behavior due to off-diagonal interaction
- Design of multiloop (multiple single-loop) decentralized controllers for multivariable systems
  i. Characterizing process interactions and to select an appropriate multiloop control configuration
  ii. Controller tuning for each individual loop in multiloop decentralized control system design
- Decoupling strategy for multiple single-loop design (diagonal)

Outline of Lecture 4

4.1 Control of Multivariable Processes

- In practical control problems there typically are a number of process variables which must be controlled and a number which can be manipulated

- Example: product quality and throughput must usually be controlled.

- Several simple physical examples are shown in Fig. 4.1. Note "process interactions" between controlled and manipulated variables.

4.1 MIMO Processes

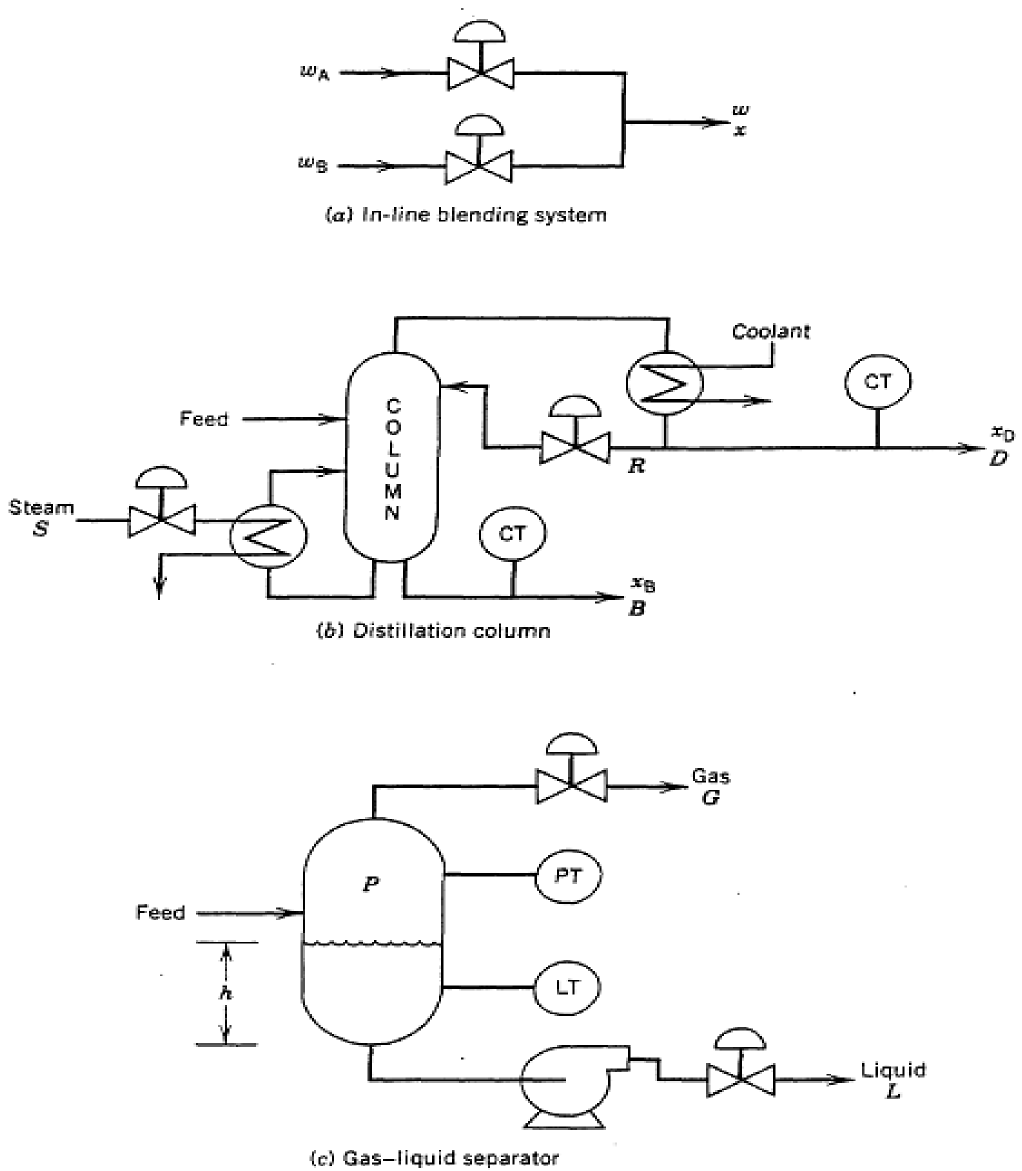
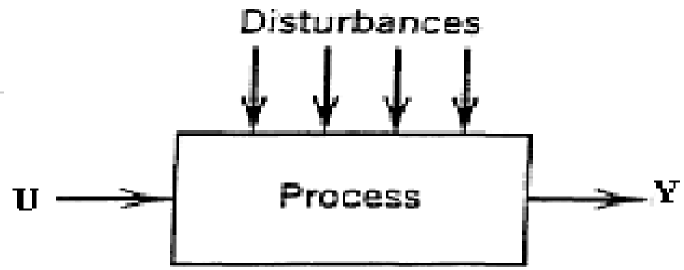


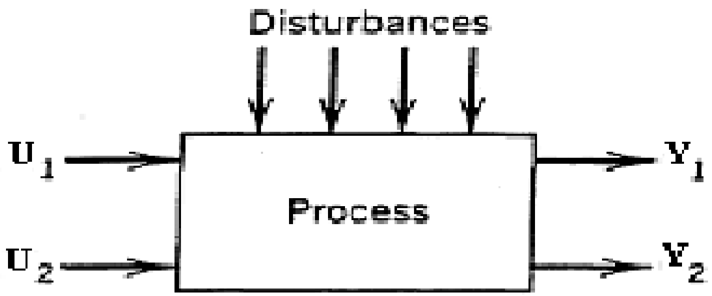
Figure 18.1. Physical examples of multivariable control problems.

FIGURE 4.1

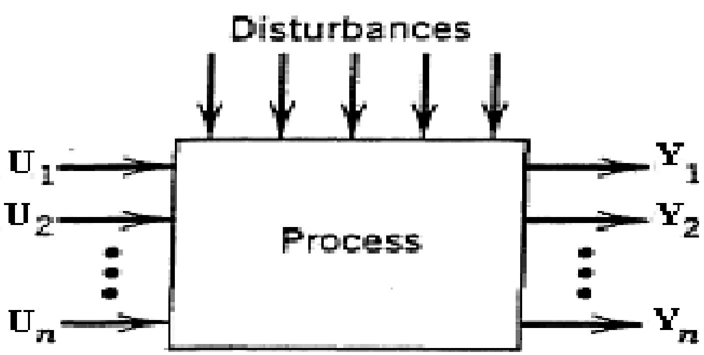
4.1 SISO vs MIMO Process



(a) Single-input, single-output process with multiple disturbances



(b) Multiple-input, multiple-output process (2 × 2)



(c) Multiple-input, multiple-output process (n × n)

4. Multiloop vs Multivariable Control Strategy

- If process interactions are significant, even the best multiloop control system (Lecture 4) may not provide satisfactory control.

- In these situations there are incentives for considering multivariable control strategies (Lecture 5)

Definitions:

•Multiloop control: Each manipulated variable depends on only a single controlled variable, i.e., a set of conventional feedback controllers.

•Multivariable Control: Each manipulated variable can depend on two or more of the controlled variables.

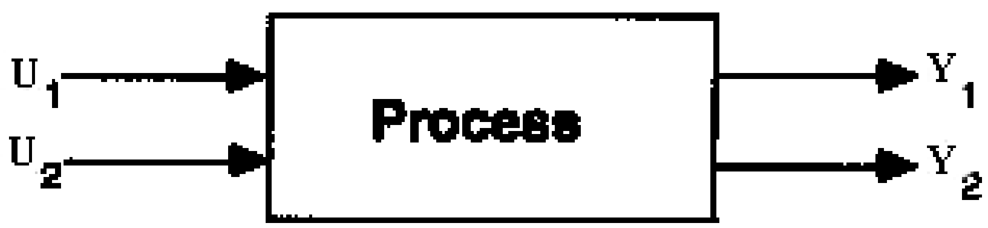
Examples: decoupling control, model predictive control

4. Multiloop Control Strategy

- Typical industrial approach
- Consists of using n standard FB controllers (e.g. PID), one for each controlled variable.

- **<u>Control system design</u>**
  1. Select controlled and manipulated variables.
  2. Select pairing of controlled and manipulated variables.
  3. Specify types of FB controllers.

Example: 2 x 2 system



Two possible controller pairings:

U_1 with Y_1, U_2 with Y_2 …<u>or</u>

U_1 with Y_2, U_2 with Y_1

Note: For n x n system, n! possible pairing configurations.

4. Transfer Function Model (2 x 2 system)

Two controlled variables and two manipulated variables (4 transfer functions required)

$$\frac{Y_1(s)}{U_1(s)} = G_{P11}(s), \quad \frac{Y_1(s)}{U_2(s)} = G_{P12}(s)$$

$$\frac{Y_2(s)}{U_1(s)} = G_{P21}(s), \quad \frac{Y_2(s)}{U_2(s)} = G_{P22}(s)$$

Thus, the input-output relations for the process can be written as:

$$Y_1(s) = G_{P11}(s)U_1(s) + G_{P12}(s)U_2(s)$$

$$Y_2(s) = G_{P21}(s)U_1(s) + G_{P22}(s)U_2(s)$$

4. Transfer Function Matrix Gp(s)

Or in vector-matrix notation as,

$$\underline{Y}(s) = \underline{\underline{G_P}}(s)\underline{U}(s)$$

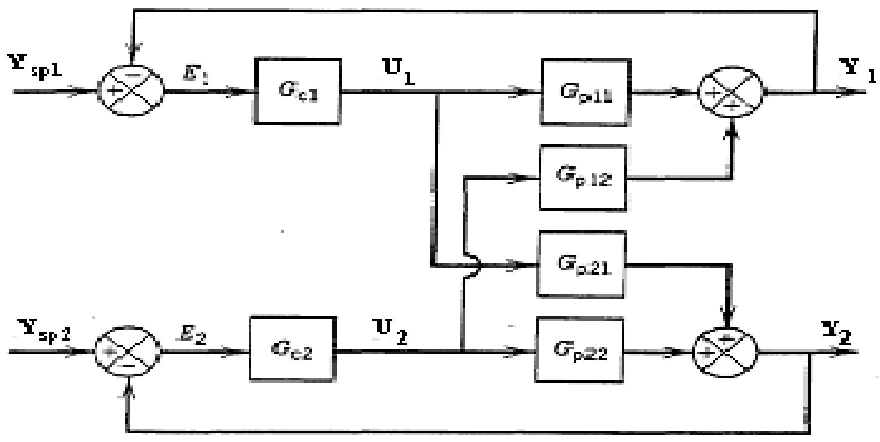
where $\underline{Y}(s)$ and $\underline{U}(s)$ are vectors,

$$\underline{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}, \quad \underline{U}(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

And $G_p(s)$ is the transfer function matrix for the process

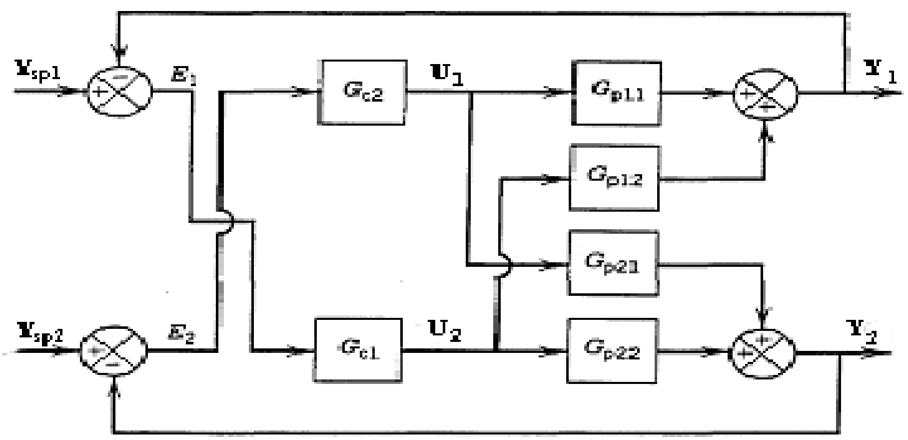
$$\underline{\underline{G_P}}(s) = \begin{bmatrix} G_{P11}(s) & G_{P12}(s) \\ G_{P21}(s) & G_{P22}(s) \end{bmatrix}$$

4.1 2 X 2 Multiloop Control Problems



(a) 1–1/2–2 controller pairing

$y1$ by $u1$ and $y2$ by $u2$ (the 1–1/2–2 pairing)



(b) 1–2/2–1 controller pairing

Figure 18.3. Block diagrams for 2 X 2 multiloop control schemes

$y1$ by $u2$ and $y2$ by $u1$ (the 1–2/2–1 pairing)

4.1 Control Loop Interactions

- Process interactions may induce undesirable interactions between two or more control loops.

- Example: 2 x 2 system: Control loop interactions are due to the presence of a third feedback loop.

- Problems arising from control loop interactions
  i) Closed -loop system may become destabilized.
  ii) Controller tuning becomes more difficult

4.1 Block Diagram Analysis

For the multiloop control configuration the transfer function between a controlled and a manipulated variable depends on whether the other feedback control loops are open or closed.

Example: 2 x 2 system, 1-1/2 -2 pairing

From block diagram algebra we can show

$$\frac{Y_1(s)}{U_1(s)} = G_{P11}(s), \qquad \text{(second loop open)}$$

$$\frac{Y_1(s)}{U_1(s)} = G_{P11} - \frac{G_{P12}G_{P21}G_{C2}}{1+G_{C2}G_{P22}} \qquad \text{(second loop closed)}$$

Note that the last expression contains G_{C2} .

$$\text{Note:} Y_1(s) = G_{P11}(s)U_1(s) + G_{P12}(s)U_2(s)$$

$$Y_2(s) = G_{P21}(s)U_1(s) + G_{P22}(s)\left[-G_{C2}(s)Y_2(s)\right]$$

$$\Rightarrow Y_2(s) = \frac{G_{P21}(s)U_1(s)}{1+G_{P22}(s)G_{C2}(s)}, U_2(s) = \frac{-G_{C2}(s)G_{P21}(s)U_1(s)}{1+G_{P22}(s)G_{C2}(s)}$$

$$\therefore, Y_1(s) = \left(G_{P11}(s) - \frac{-G_{P12}(s)G_{C2}(s)G_{P21}(s)}{1+G_{P22}(s)G_{C2}(s)}\right)U_1(s)$$

4.1 1-1/2-2 Controller Pairing

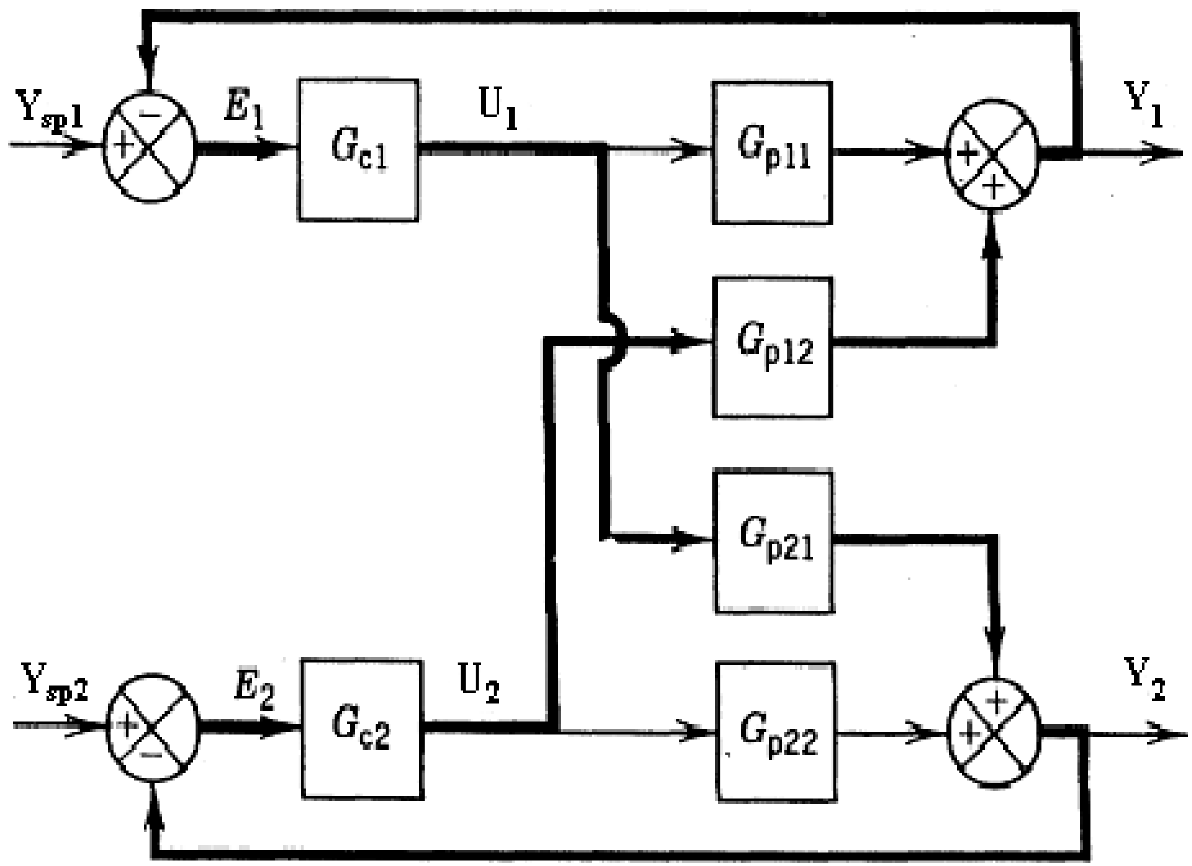


Figure 18.4. The hidden feedback control loop (in dark lines) for a 1-1/2-2 controller pairing.

4.1 Example: Interaction Effects

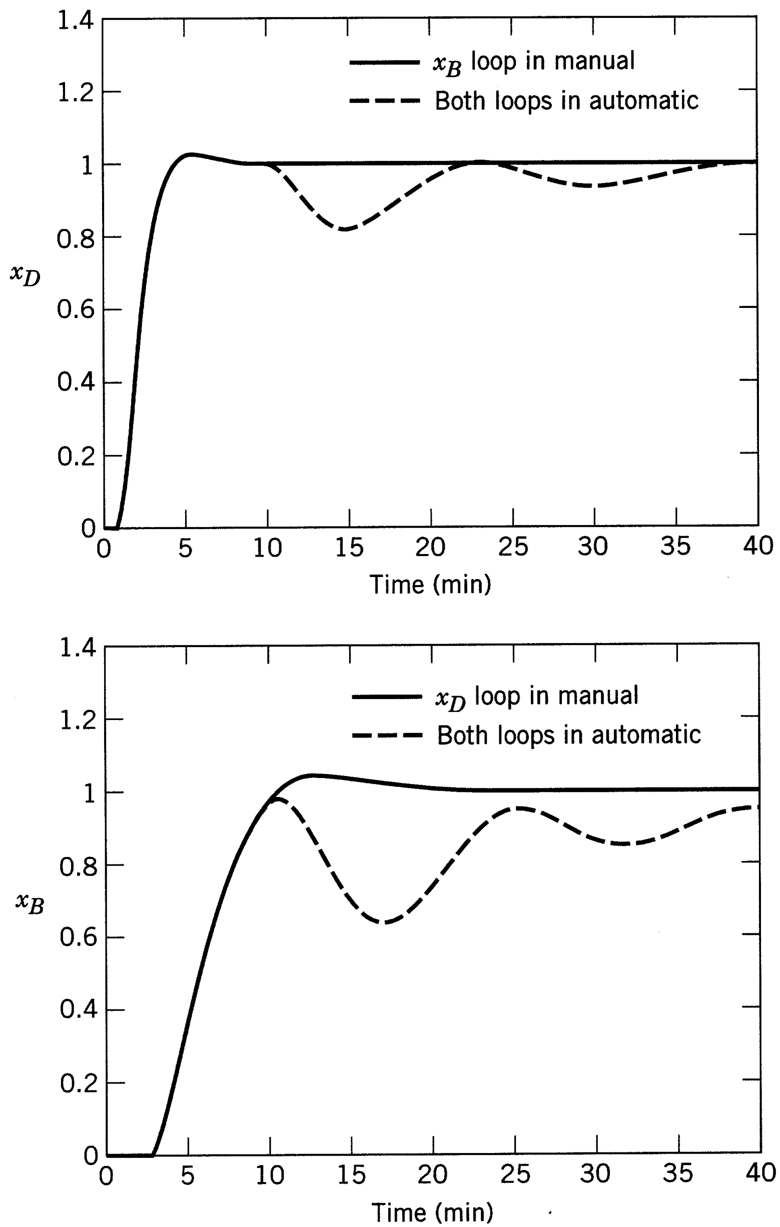


Figure 18.5 Set-point responses for Example 18.1 using ITAE tuning.

4.1 Example: Interaction Effects

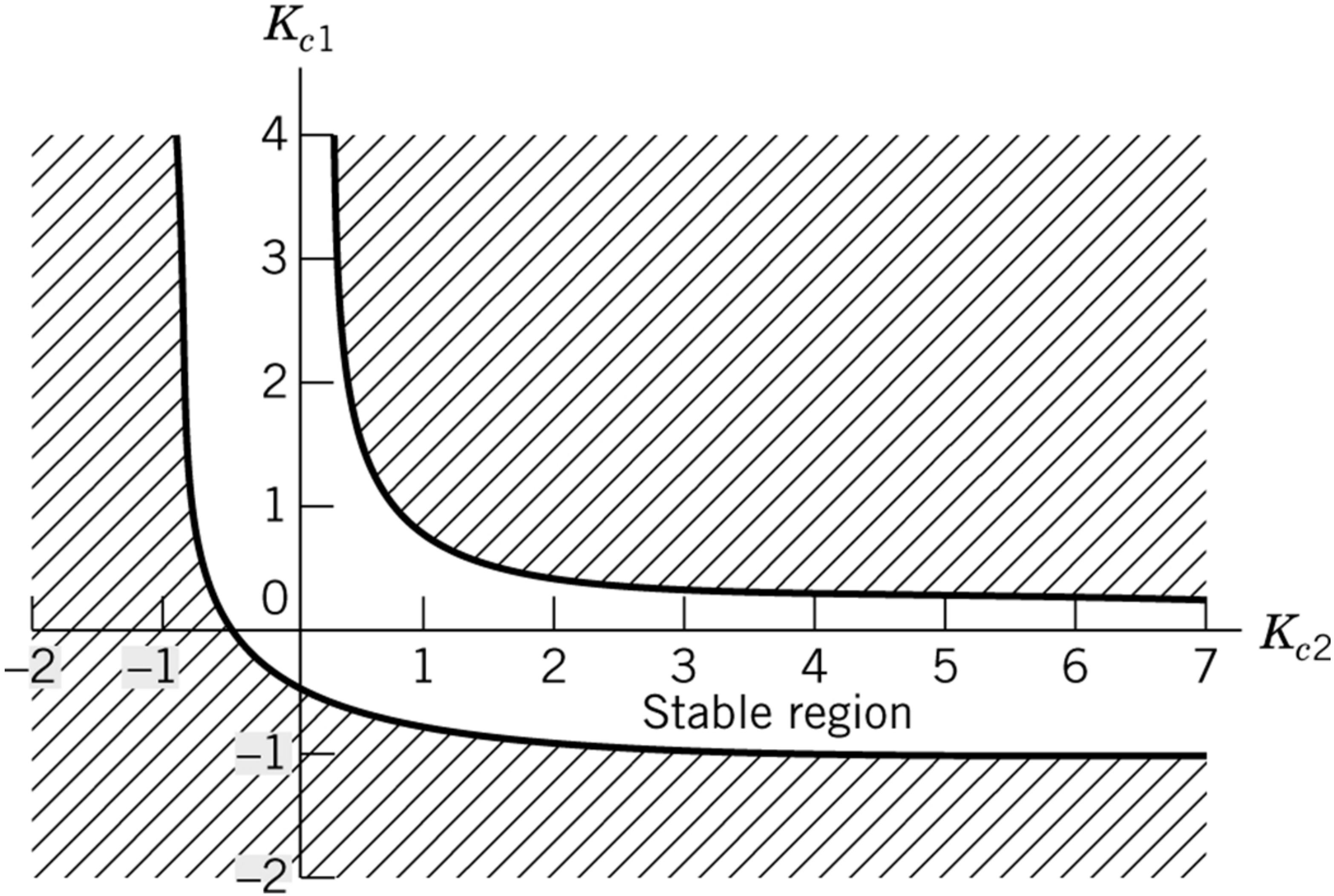


Figure 18.6 Stability region for Example 18.2 with 1-1/2-2 controller pairing

4.1 Example: Interaction Effects

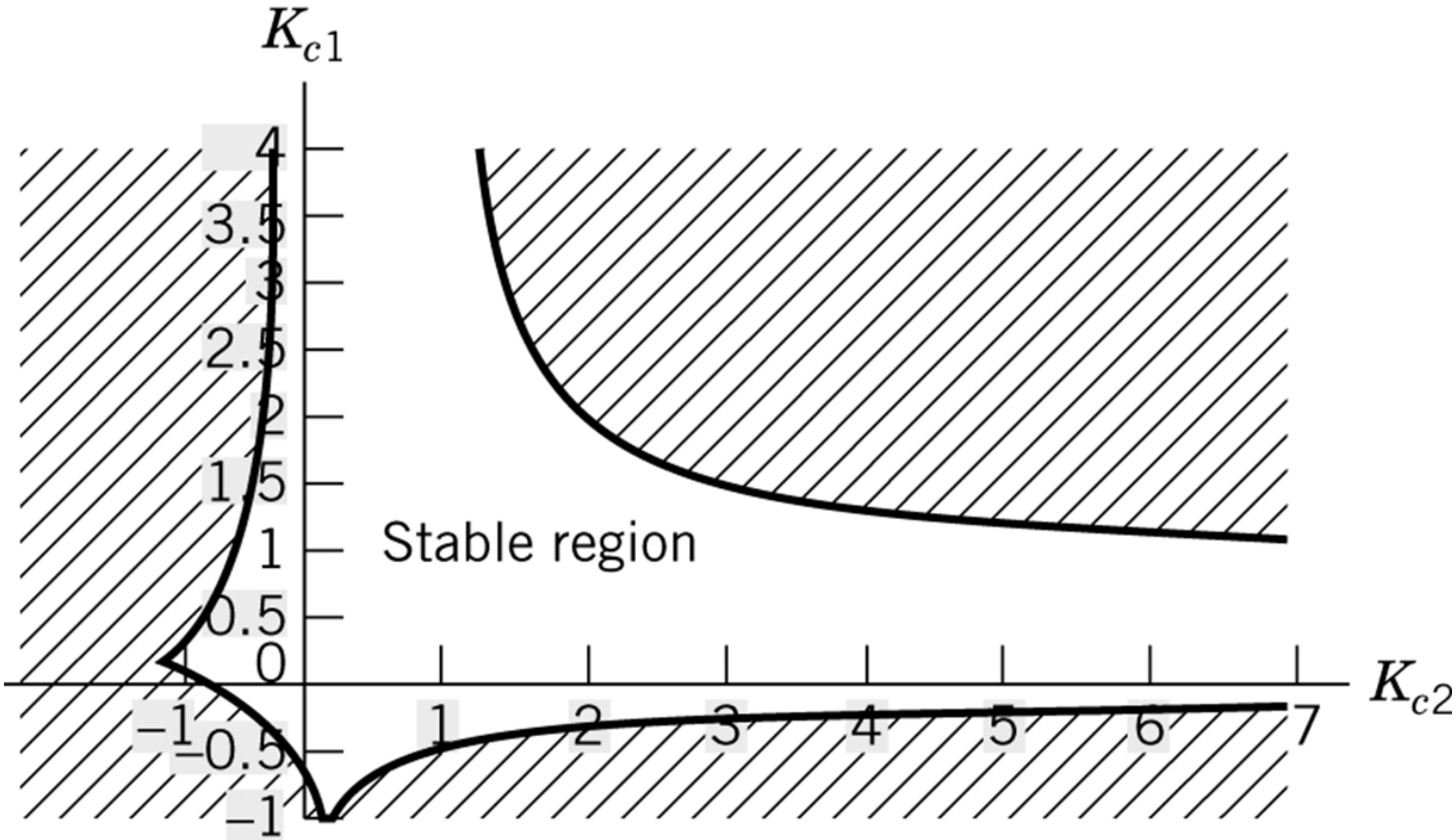


Figure 18.7 Stability region for Example 18.2 with 1-2/2-1 controller pairing

4.2 Relative Gain Array (RGA)

- RGA was introduced by Bristol in 1966. Although extensions of the RGA to non-square systems have been proposed, we will focus on the use of the RGA for square plants, i.e. number of controlled variables (outputs) equal the number of manipulated variables (controller inputs).
- Provides two useful types of information:
  1) Measure of process interactions
  2) Recommendation about best pairing of controlled and manipulated variables.
- Requires knowledge of steady state gains but <u>NOT</u> process dynamics.

4.2.1 RGA

Consider a $n \times n$ plant $G(s)$.

$$y(s) = G(s)u(s) \qquad (1)$$

i) All other loops open: $u_k = 0, \forall k \neq j$

The open loop gain from $u_j(s)$ to $y_i(s)$ is $g_{ij}(s)$ since $y = Gu \Rightarrow \left(\frac{\partial y_i}{\partial u_j}\right)_{u_k = 0, k \neq j} = [G]_{ij}$

ii) All other loops closed with perfect control: $y_k = 0, \forall k \neq i$

Rewriting Equation (1) as

$$u(s) = G^{-1}(s)y(s) \qquad (2)$$

$$\Rightarrow \left(\frac{\partial u_j}{\partial y_i}\right)_{y_k = 0, k \neq i} = \left[G^{-1}\right]_{ji}$$

It can be seen that the closed loop gain from $u_j(s)$ to $y_i(s)$ is

$1/\left[G^{-1}(s)\right]_{ji}$ when all the other y's are perfectly controlled.

iii) The relative gain matrix consists of the ratios of these open and closed loop gain. Thus, a matrix of relative gains can be computed from

$$\Lambda(s) = G(s) \times \left(G^{-1}(s)\right)^T \qquad (3)$$

where the symbol '$\times$' denotes the element-by-element product.

4.2.1 RGA for 2 x 2 System

- Consider the 2 x 2 system. Suppose that we want to control y_1 using u_1

$$G(s)=\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

Open Loop: $u_2=0 \rightarrow y_1(s)=G_{11}(s)u_1(s)$

Closed Loop: $y_2=0 \rightarrow y_2(s)=G_{21}(s)u_1(s)+G_{22}(s)u_2(s)=0$

$$u_2(s)=-\frac{G_{21}(s)}{G_{22}(s)}u_1(s)$$

$$y_1(s)=G_{11}(s)u_1(s)-\frac{G_{12}(s)G_{21}(s)}{G_{22}(s)}u_1(s)=\left(\frac{G_{11}(s)G_{22}(s)-G_{12}(s)G_{21}(s)}{G_{22}(s)}\right)u_1(s)$$

$$\Lambda_{11}(s)=\frac{\text{open loop gain from } u_1 \text{ to } y_1}{\text{closed loop gain from } u_1 \text{ to } y_1}=\frac{G_{11}(s)G_{22}(s)}{G_{11}(s)G_{22}(s)-G_{12}(s)G_{21}(s)}$$

4.2.1 Steady State RGA (s=0) for 2 x 2 System

- Steady-state (i.e. s=0) process model,

$$Y_1 = K_{11}U_1 + K_{12}U_2$$

$$Y_2 = K_{21}U_1 + K_{22}U_2$$

The steady-state RGA (s=0) is defined as:

$$RGA = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

where the relative gain, λ_{ij}, relates the i^{th} controlled variable and the j^{th} manipulated variable

$$\lambda_{ij} = \frac{\text{open-loop gain}}{\text{closed-loop gain}}$$

4.2.1 Steady State RGA (*s*=0) for Higher-Order System

In general, for and *n x n* system, steady-state RGA (i.e. $K = G(\underline{s=0})$) is

$$\Lambda = \begin{array}{c} \\ Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{array} \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & \cdots & U_n \end{array} \\ \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n1} & \cdots & \lambda_{nn} \end{bmatrix} \end{array}$$

Each λ_{ij} can be calculated from the relation

$$\lambda_{ij} = K_{ij} H_{ij}$$

Where K_{ij} is *the (i,j)* -element in the <u>steady-state</u> gain matrix, K :

$$Y = KU$$

And H_{ij} is the *(i,j)* -element of the $H = \left(K^{-1}\right)^T$.

Note that,

$$\Lambda \neq KH$$

4.2.1 Properties of RGA

i) λ_{ij} is dimensionless.

ii) It is scaling independent (independent of the units of measure used for *u* and *y*). Mathematically,

$$\Lambda\left(D_1 G D_2\right) = \Lambda\left(G\right) \text{ where } D_1 \text{ are } D_2 \text{ diagonal matrices}$$

iii) All rows and columns sums equal 1. $\sum_i \lambda_{ij} = \sum_j \lambda_{ij} = 1.0$

For 2 x 2 system, $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad K^{-1} = \dfrac{1}{K_{11}K_{22} - K_{12}K_{21}} \begin{bmatrix} K_{22} & -K_{12} \\ -K_{21} & K_{11} \end{bmatrix}$

$$\lambda_{11} = \frac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}}, \quad \lambda_{12} = 1 - \lambda_{11}, \quad \lambda_{21} = 1 - \lambda_{11}, \quad \lambda_{22} = \lambda_{11}$$

4.2.1 Additional Properties of RGA

iv. If G is triangular (and hence also if it is diagonal), $\Lambda(G) = I$.

<u>Examples for 2 x 2 Process:</u>

Process Gain Matrix, K:		Relative Gain Array, Λ:
$\begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$	$\Rightarrow$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & K_{12} \\ K_{21} & 0 \end{bmatrix}$	$\Rightarrow$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix}$	$\Rightarrow$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}$	$\Rightarrow$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4.2.2 RGA as Interaction Measure

- Since the elements of the RGA can be interpreted as the ratio of the open loop process gain to the gain when all other outputs are perfectly controlled, it should be intuitively clear that the RGA can serve as an interaction measure (i.e., a measure of to what extent the control of output *i* will be affected by the control of other outputs).
- If an element of the RGA differs significantly from one, the use of the corresponding input-output pair for control will imply that the control of that output will be affected by the control actions in the other loops to a significant degree.
- It is of most interest to consider the interactions in the (expected) bandwidth region for control.

4.2.3 RGA for Controller Pairing

- **<u>In General:</u>**
  1. Pairings which correspond to negative pairings should not be selected as it may cause instability.
  2. Otherwise, choose the pairing which has λ_{ij} closest to one.

4.2.3 Example RGA (*s*=0) for 2x2 Systems

Recall, for 2X2 systems...

$$Y_1 = K_{11}U_1 + K_{12}U_2$$

$$Y_2 = K_{21}U_1 + K_{22}U_2$$

$$\lambda_{11} = \frac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}}, \qquad \lambda_{12} = 1 - \lambda_{11} = \lambda_{21}$$

EXAMPLE:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1.5 \\ 1.5 & 2 \end{bmatrix}$$

$$\therefore \lambda = \begin{bmatrix} 2.29 & -1.29 \\ -1.29 & 2.29 \end{bmatrix}$$

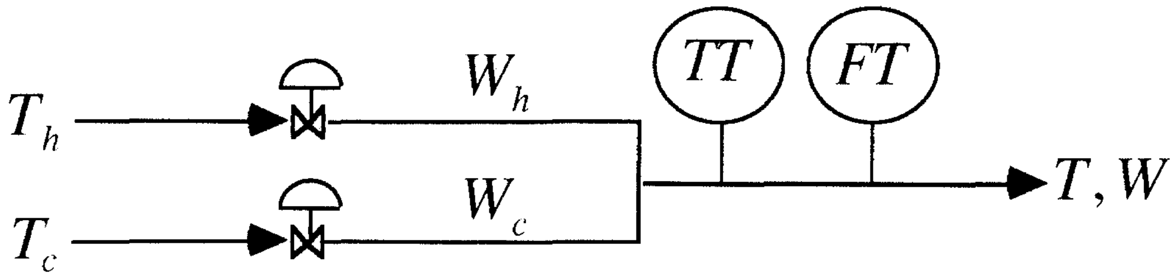
$\therefore$ Recommended pairing is Y_1 and U_1, Y_2 and U_2. (1-1/2-2 Pairing)

EXAMPLE:

$$K = \begin{bmatrix} -2 & 1.5 \\ 1.5 & 2 \end{bmatrix} \Rightarrow \lambda = \begin{bmatrix} 0.64 & 0.36 \\ 0.36 & 0.64 \end{bmatrix}$$

$\therefore$Recommended pairing is Y_1 with U_1, Y_2 with U_2.

4.2.3 Example Thermal Mixing System



Linearize system and express RGA in terms of the manipulated variables:

$$\underline{\Lambda} = \begin{array}{c c} & \begin{array}{cc} W_h & W_c \end{array} \\ \begin{array}{c} W \\ T \end{array} & \begin{bmatrix} \dfrac{W_c}{W_c + W_h} & \dfrac{W_h}{W_c + W_h} \\ \dfrac{W_h}{W_c + W_h} & \dfrac{W_c}{W_c + W_h} \end{bmatrix} \end{array}$$

since

$$\left\{ \begin{bmatrix} dW \\ dT \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ \dfrac{W_h}{W_c + W_h} & -\dfrac{W_c}{W_c + W_h} \end{bmatrix}}_{K} \Bigg|_{W, T, T_h, T_c} \begin{bmatrix} dW_h \\ dW_c \end{bmatrix} \right\}$$

Note that each relative gain is between 0 and 1. Recommended controller pairing depends on values of W_h and W_c which depends on nominal values of W,T, T_h, and T_c.

4.2.3 Example Hydrocracker

The RGA for a hydrocracker has been reported as,

$$\Lambda = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{array} \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 0.931 & 0.150 & 0.080 & -0.164 \\ -0.011 & -0.429 & 0.286 & 1.154 \\ -0.135 & 3.314 & -0.270 & -1.910 \\ 0.215 & -2.030 & 0.900 & 1.919 \end{bmatrix} \end{array}$$

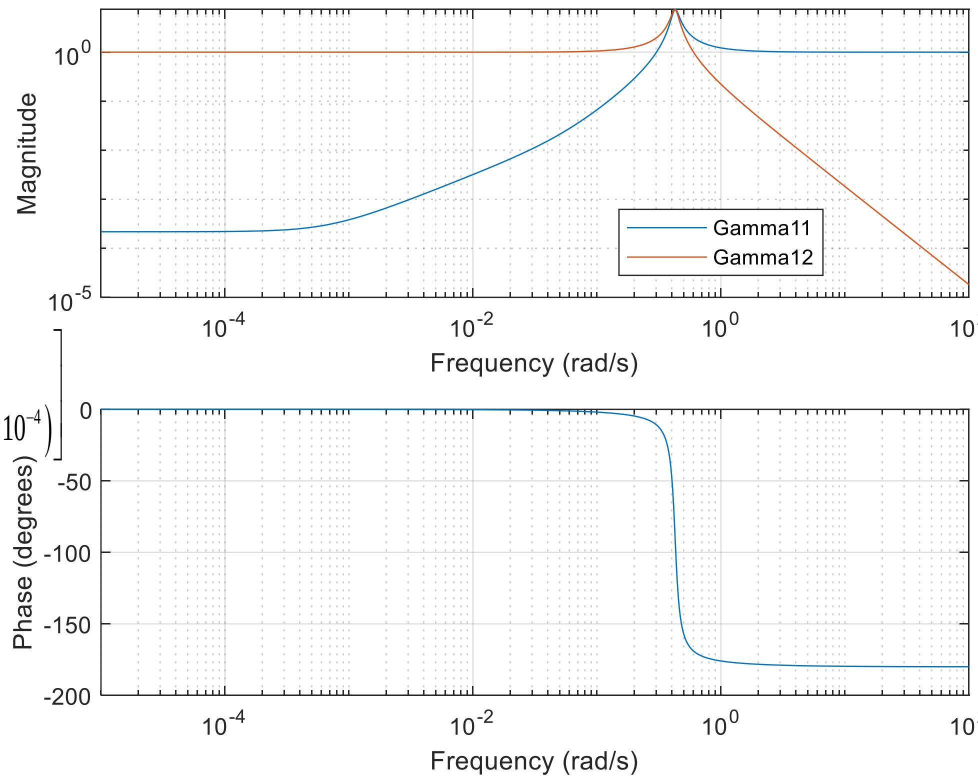
Recommended controller pairing?

4.2.3 Example RGA ($s \neq 0$) for Controller Pairing

- Frequency Dependent RGA. In general, since $\Lambda(G)$ is frequency dependent, different pairing might be suitable for different frequency ranges.

$$G(s)=\frac{0.01e^{-5s}}{\left(s+1.72\times10^{-4}\right)\left(4.32s+1\right)}\begin{bmatrix}-34.54\left(s+0.0572\right) & 1.913\\ -30.225 & -9.188\left(s+6.95\times10^{-4}\right)\end{bmatrix}$$

$$\Lambda(G)=\frac{1}{34.54\times9.188\left(s+0.0572\right)\left(s+6.95\times10^{-4}\right)+1.913\times30.225}\begin{bmatrix}34.54\times9.188\left(s+0.0572\right)\left(s+6.95\times10^{-4}\right) & 1.913\times30.225\\ 1.913\times30.225 & 34.54\times9.188\left(s+0.0572\right)\left(s+6.95\times10^{-4}\right)\end{bmatrix}$$



- For above example, at high frequency, diagonal pairing is the best. At low frequency range, off diagonal pairing is the best.

4.2.3 Niederlinski's Theorem for Stability

Consider an $n \times n$ multivariable system whose input and output have been paired as follows: $y_1 \leftrightarrow u_1, y_2 \leftrightarrow u_2, \ \ldots, y_n \leftrightarrow u_n,$ resulting in a transfer function model of the form

$$y(s) = G(s)u(s)$$

with $g_{ij}(s)$ to be rational (no delay) and open-loop stable and let n individual feedback controllers (with integral action) be designed for each loop so that each of the resulting n feedback control loops is stable when all the other $n-1$ loops are open.

<u>Sufficient Condition</u>: Under closed-loop condition in all n loops, the system will be unstable for all possible values of controller parameters if the Niederlinski index NI defined below is negative, i.e. :

$$NI = \frac{|G(0)|}{\prod_{i=1}^{n} g_{ii}(0)} < 0$$

4.2.3 Niederlinski's Index (*NI*)

1. Given *G(s),* obtain steady-state gain matrix $K = G(0)$, obtain the RGA, $\Lambda(G)$; obtain the determinant of *K*, and the product of the elements on its main diagonal.
2. Use Rule #1 to obtain tentative loop pairing suggestions from the RGA, by pairing on positive elements which are closest to 1.0.
3. Use Niederlinski's condition to verify the stability status of the control configuration resulting from 2; if the pairing is unacceptable, select another.
4. Variables should be paired in such a way that the resulting pairing corresponds to an *NI* closest to 1.0. The *NI* interaction rule is based on *empirical* observations of the definition of the *NI* and is justified largely from the relationship between the *size* of the RGA and that of the *NI*. The *NI* interaction rule has been found to be capable of avoiding ambiguities in using the RGA interaction rule.

4.2.3 Example RGA and *NI*

Calculate RGA for system with steady-sate gain matrix:

$$K = G(0) = \begin{bmatrix} \frac{5}{3} & 1 & 1 \\ 1 & \frac{1}{3} & 1 \\ 1 & 1 & \frac{1}{3} \end{bmatrix}$$

By taking the inverse, and then calculate the RGA, we have

$$\Lambda = \begin{bmatrix} 10 & -4.5 & -4.5 \\ -4.5 & 1 & 4.5 \\ -4.5 & 4.5 & 1 \end{bmatrix}$$

Using the RGA guidelines, the 1-1/2-2/3-3 pairing is recommended.

Adopting this pairing, the steady-state gain determinant is given by

$$\left|G(0)\right| = -0.148$$

and the product of the diagonal elements of K is

$$\prod_{i=1}^{3} g_{ii}(0) = \left(\frac{5}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{5}{27}$$

$$NI = -0.148 \Big/ \frac{5}{27} < 0$$

Hence, this pairing leads to an unstable configuration.

4.2.3 Example RGA and NI

According to NI, the 1-1/2-2/3-3 configuration is unstable.

Let us consider the 1-1/2-3/3-2 pairing which will have the following RGA

$$\Lambda = \begin{bmatrix} 10 & -4.5 & -4.5 \\ -4.5 & 4.5 & 1 \\ -4.5 & 1 & 4.5 \end{bmatrix}$$

(i.e. swap the (2,2) element with (2,3) element and the (3,3) element with the (3,2) element)

The corresponding steady state gain (again after swapping elements) is given by

$$K = G(0) = \begin{bmatrix} \frac{5}{3} & 1 & 1 \\ 1 & 1 & \frac{1}{3} \\ 1 & \frac{1}{3} & 1 \end{bmatrix}$$

with the determinant $|K| = \frac{4}{27}$

The Niederlinkski Index NI is given by

$$NI = \frac{4/27}{5/3} = \frac{4}{45} > 0$$

which may not be structurally unstable (due to sufficient condition)

4.2.3 Example Ill-Conditioned Gain Matrix

$$\boldsymbol{y} = \boldsymbol{K}\boldsymbol{u}$$

2 x 2 process $\quad y_1 = 5\,u_1 + 8\,u_2$

$$y_2 = 10\,u_1 + 15.8\,u_2$$

specify operating point ***y***, solve for ***u***

$$\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{y} = \frac{\text{Adj } K}{\det K} \cdot \boldsymbol{y}$$

$$\text{RGA}: \; \lambda_{11} = \frac{K_{11}K_{22}}{K_{11}K_{22} - K_{12}K_{21}} = \frac{K_{11}K_{22}}{\det K}$$

effect of det $K \rightarrow 0$? … Next Section

4.2.4 Singular Value Analysis

$K = W \Sigma V^T$

Σ is diagonal matrix of singular values

$(\sigma_1, \sigma_2, \ldots, \sigma_r)$

The singular values are the positive square roots of the eigenvalues of

$K^T K$ (r = rank of $K^T K$)

W,*V* are input and output singular vectors Columns of *W* and *V* are orthonormal. Also

$WW^T = I$

$VV^T = I$

Calculate Σ, *W*, *V* using MATLAB (svd = singular value decomposition)

Condition number (CN) is the ratio of the largest to the smallest singular value and indicates if *K* is ill-conditioned.

4.2.4 Singular Value Analysis

CN is a measure of sensitivity of the matrix properties to changes in a specific element.

Consider

$$\boldsymbol{K} = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}$$

λ (RGA) = 1.0

If K_{12} changes from 0 to 0.1, then K becomes a singular matrix, which corresponds to a process that is hard to control.

RGA and SVA used together can indicate whether a process is easy (or hard) to control.

$$\Sigma(K) = \begin{bmatrix} 10.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad \text{CN} = 101$$

K is poorly conditioned when CN is a large number (e.g., > 10). Hence small changes in the model for this process can make it very difficult to control.

4.2.4 Example: RGN and CN

- Arrange the singular values in order of largest to smallest and look for any $\sigma_i/\sigma_{i-1} > 10$; then one or more inputs (or outputs) can be deleted.
- Delete one row and one column of *K* at a time and evaluate the properties of the reduced gain matrix.
- Example:

$$K = \begin{bmatrix} 0.48 & 0.90 & -0.006 \\ 0.52 & 0.95 & 0.008 \\ 0.90 & -0.95 & 0.020 \end{bmatrix}$$

4.2.4 Example: CN and RGA

$$W = \begin{bmatrix} 0.5714 & 0.3766 & 0.7292 \\ 0.6035 & 0.4093 & -0.6843 \\ -0.5561 & 0.8311 & 0.0066 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1.618 & 0 & 0 \\ 0 & 1.143 & 0 \\ 0 & 0 & 0.0097 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.0541 & 0.9984 & 0.0151 \\ 0.9985 & -0.0540 & -0.0068 \\ -0.0060 & 0.0154 & -0.9999 \end{bmatrix}$$

CN = 166.5 (σ_1/σ_3)

The RGA is as follows:

$$\begin{bmatrix} -2.4376 & 3.0241 & 0.4135 \\ 1.2211 & -0.7617 & 0.5407 \\ 2.2165 & -1.2623 & 0.0458 \end{bmatrix}$$

Preliminary pairing: *y*1-*u*2, *y*2-*u*3,*y*3-*u*1. (1-2/2-3/3-1 pairing)

CN suggests only two output variables can be controlled. Eliminate one input and one output (3x3→2x2).

4.3 Decentralized Control

- After the control configuration is fixed, we focus on fully decentralized control. That is, it is assumed that the overall controller consists of multiple single-input, single-output controllers, and the pairing of manipulated and controlled variables has been determined.
- Despite the prevalence of decentralized controllers in industry, the tuning (determination of controller parameters) of decentralized controllers is not a solved problem in mathematical terms.
- We will briefly discuss 4 different tuning approaches

4.3.1 BLT

- Naive independent design. The most well known of tuning methods in this category, is the so-called Biggest Log Modulus Tuning 'BLT'.
- It essentially consists of tuning each loop individually (typically with the Ziegler-Nichols closed loop tuning), and then to check the infinity norm of the multivariable complementary sensitivity function (the transfer function from reference to controlled variable), $T = GK\left(I + GK\right)^{-1}$.
- If the 'peak value' of this transfer function

$$\|T\|_\infty = \max_\omega \bar{\sigma}\left(T\left(j\omega\right)\right)$$

is too large (> 2), a common detuning factor is applied to the proportional gain for all loops.

4.3.2 Sequential

- This is the most common approach in industry for designing decentralized controllers.
- The controllers are designed and put into operation one at a time, and the controllers that have been designed are kept in operation when new controllers are designed.
- The conventional rule of thumb is to close the fast loops first (similar to cascade control approach). This is intuitively reasonable, as it is often the case that the faster loops are comparatively unaffected by the tuning in slower loops.

4.3.3 Parallel

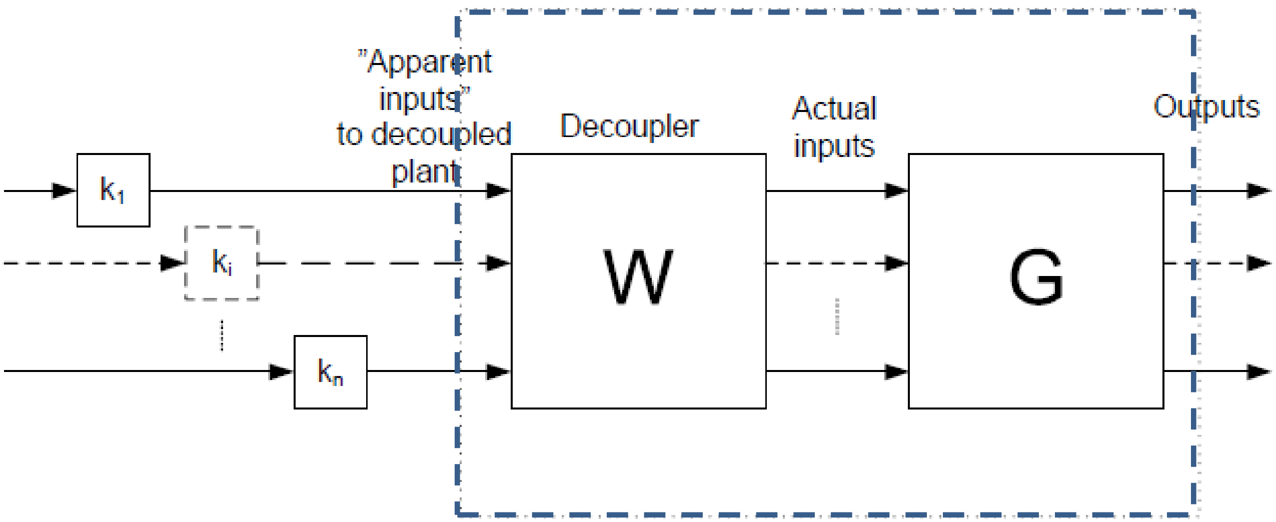
- Simultaneous design implies that the tuning parameters for all loops are determined/tuned simultaneously.
- Simultaneous design is done by choosing a particular controller parametrization (e.g., PID control), and using optimization to find the controller parameters which optimizes some measure of system performance.
- Although such simultaneous design appears feasible in theory, the optimization problems are typically non-convex, and convergence to a global optimum cannot be guaranteed.
- If the optimization fails to find acceptable controller parameters, it is not obvious whether this is because no acceptable parameters exist (for the particular choice of controller parametrization), or whether it is simply due to an unfortunate initial guess of parameter values.

4.3.3 Independent Design

- Rigorous independent design was introduced by Skogestad and Morari[91]. They approach the tuning problem from a robust control viewpoint, using the *structured singular value* framework.
- They effectively include the controller in the uncertainty description for the robust stability (or robust performance) problem. This is done in two different ways, by
  - Expressing the controller as an LFT of the sensitivity functions for the individual loops, and
  - Expressing the controller as an LFT of the complementary sensitivity functions for the individual loops.
- The 2 different LFTs will result in 2 different robust stability problems with a diagonal, complex-valued ‘uncertainty’. Iteration is used to find the bounds for these uncertainties. Robust stability/performance will be guaranteed provided all individual loops at all frequencies fulfil the magnitude bound on either the sensitivity function or the complementary sensitivity function.
- The bounds are *conservative*. Also theoretically and numerically *complex*. Robust stability/performance will be explained in Lecture 5.

4.4 Decoupling

- The use of decouplers have long been a popular way of converting multivariable control problems into (what appears to be) a number of monovariable control problems.
- This popularity of decoupling seem to continue, despite the more general and easily applicable multivariable control design methods that have been developed over the last several decades.
- The basic idea behind the use of a decoupler can be illustrated in below figure.



4.4 Decoupling

- A precompensator $W(s)$ is used, in order to make the precompensated plant GW diagonal, thus allowing for simple monovariable control design of the individual loop controllers k_i.
- Assume that the desired precompensated plant is given by $G_{des}(s)$. It is then simple to find the corresponding precompensator, by solving the equation

$$G(s)W(s) = G_{des}(s)$$

- Note that
  - Typically $G_{des}(s)$ is diagonal (which will be assumed henceforth), but occasionally 'one way decouplers' are used, corresponding to $G_{des}(s)$ being upper or lower triangular.
  - $G_{des}(s)$ must contain all RHP poles and (multivariable) RHP zeros of $G(s)$ - otherwise the system will be internally unstable.
  - The precompensator obviously cannot remove time delays.
  - A popular choice is $G_{des}(s) = g_{des}(s) \bullet I$, with $g_{des}(s)$ scalar. Any multivariable RHP zeros in $G(s)$ must then also be present in $g_{des}(s)$. This means that all loops for the precompensated system will be affected by the RHP zero, even if only a few inputs or outputs in $G(s)$ are affected by the multivariable zero.

4.4 Example Decoupling

- Consider the following G(s) which does not have RHP Zeros and is Bi-Proper

$$G(s)=\frac{1}{s+1}\begin{bmatrix} s+2 & s+3 \\ s & s+2 \end{bmatrix}$$

Assume $G_{des}(s)=I$, then

$$W(s)=G^{-1}(s)=\frac{s+1}{(s+2)^2-s(s+3)}\begin{bmatrix} s+2 & -(s+3) \\ -s & s+2 \end{bmatrix}$$

Then, the controller

$$K(s)=\begin{bmatrix} K_1(s) & 0 \\ 0 & K_2(s) \end{bmatrix}$$

where $K_1(s)$ and $K_2(s)$ are individual controllers which can be designed separately for a unit constant gain plant.

4.4 Example Decoupling for Plant with Delay

- Consider the following G(s) with time delay

$$G(s)=\frac{1}{s+1}\begin{bmatrix}(s+2)e^{-3s} & (s+3)e^{-2s}\\ s^{-2s} & (s+2)e^{-s}\end{bmatrix}$$

Assume $G_{des}(s)=e^{-3s}I$, then

$$W(s)=G^{-1}(s)e^{-3s}=\frac{(s+1)e^{-3s}}{(s+2)^2e^{-4s}-s(s+3)e^{-4s}}\begin{bmatrix}(s+2)e^{-s} & -(s+3)e^{-2s}\\ -s^{-2s} & (s+2)e^{-3s}\end{bmatrix}$$

$$=\frac{s+1}{(s+2)^2-s(s+3)}\begin{bmatrix}(s+2) & -(s+3)e^{-s}\\ -s^{-s} & (s+2)e^{-2s}\end{bmatrix}\text{ is realizable}$$

Then, the controller can be chosen as

$$K(s)=\begin{bmatrix}K_1(s) & 0\\ 0 & K_2(s)\end{bmatrix}$$

where $K_1(s)$ and $K_2(s)$ are individual controllers which can be designed separately for a unit constant gain plant.

4.4 Summary on Decoupling

- Decouplers are prone to robustness problems, especially for highly interactive and ill-conditioned plants - which is exactly the type of plant for which one would like to use decoupling.

- The robustness problems can be further aggravated by input saturation.