Lecture 9: Polynomial Fraction Applications

Three main applications of polynomial fraction:

Minimal realization

Notions of poles and zeros

State feedback.

Minimal Realization

Consider the following $p \times m$ strictly-proper rational transfer function $G(s)$ specified by a CRPFD

$$G(s) = N(s)D^{-1}(s) \tag{1}$$

with $D(s)$ column reduced. Then $c_j[N] < c_j[D]$, $j = 1, \cdots, m$. Assume $c_1[D], \cdots, c_m[D] \geq 1$

As $D(s)$ is column reduced, the degree of the polynomial fraction description (1) is $c_1[D] + \cdots + c_m[D]$.

Want to construct for (1) minimal realization of the following form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

Definition 8.1: Suppose $N(s)D^{-1}(s)$ is a CRPFD for $G(s)$. Then the degree of this polynomial fraction description is call the *McMillan degree* of $G(s)$. □

Lemma 8.1: The dimension of any realization of a strictly-proper rational transfer function $G(s)$ is at least the McMillan degree of $G(s)$. □

Proof: Suppose (2) is a dimension-n minimal realization for $G(s)$. Then (2) is both controllable and observable, and

$$G(s) = C(sI - A)^{-1}B$$

Define a $n \times m$ strictly-proper transfer function $H(s)$ as

$$H(s) = D_L^{-1}(s)N_L(s) \triangleq (sI - A)^{-1}B \tag{3}$$

Clearly, this LPFD has degree n. Since (2) is controllable, by Theorem 5.2 (iv),

$$\operatorname{rank}\begin{bmatrix} D_L(s_0), & N_L(s_0) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} (s_0 I - A), & B \end{bmatrix} = n, \qquad \forall s_0 \in \mathbf{C}$$

Then, by Theorem 7.7, (3) is coprime.

Now suppose $N_a(s)D_a^{-1}(s)$ is a CRPFD for $H(s)$, then it also has degree n (same as $D_L^{-1}(s)N_L(s)$), and

$$G(s) = \begin{bmatrix} CN_a(s) \end{bmatrix} D_a^{-1}(s)$$

is a degree-n RPFD for $G(s)$, though not necessarily coprime.

Therefore, the McMillan degree of $G(s)$ is no greater than n. $\square$

Given m positive integers $\alpha_1, \cdots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = n$, introduce the following notations in (13.22)

Integrator Coefficient Matrices

$$A_o = \text{block diagonal} \left\{ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{\alpha_i \times \alpha_i}, \qquad i = 1, \cdots, k \right\}$$

$$B_o = \text{block diagonal} \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\alpha_i \times \alpha_i}, \qquad i = 1, \cdots, k \right\} \qquad (13.22)$$

and define the corresponding *integrator polynomial matrices* by

$$\Psi(s) = \text{block diagonal} \left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\alpha_i - 1} \end{bmatrix}, \qquad i = 1, \cdots, k \right\}$$

$$\Delta(s) = \text{diagonal} \left\{ s^{\alpha_1}, \ \cdots, \ s^{\alpha_m} \right\} \qquad (4)$$

Lemma 8.2: The integrator polynomial matrices provide a right polynomial fraction description for the corresponding integrator state equation. That is

$$(sI - A_o)^{-1}B_o = \Psi(s)\Delta^{-1}(s) \tag{5}$$

Proof: To verify (5), first left-multiply by $(sI - A_o)$ and right-multiply by $\Delta(s)$ to obtain

$$B_o\Delta(s) = s\Psi(s) - A_o\Psi(s) \tag{6}$$

This expression is easy to check. For example, the first column of (6) is

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ s^{\alpha_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ \vdots \\ s^{\alpha_1 - 1} \\ s^{\alpha_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} s \\ \vdots \\ s^{\alpha_1 - 1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The other columns can be verified in a similar way. The proof is complete. $\square$

Theorem 8.1: Suppose that a strictly-proper rational transfer function $G(s)$ is described by a CRPFD (1), where $D(s)$ is column reduced with column degrees $c_1[D], \cdots, c_m[D] \geq 1$. Then the McMillan degree of $G(s)$ is given by $n = c_1[D] + \cdots + c_m[D]$, and minimal realizations of $G(s)$ have dimension n. Furthermore, write

$$\begin{aligned} N(s) &= N_l \Psi(s) \\ D(s) &= D_{hc}\Delta(s) + D_l \Psi(s) \end{aligned} \tag{7}$$

where $\Psi(s)$ and $\Delta(s)$ are the integrator polynomial matrices corresponding to $c_1[D], \cdots, c_m[D]$ and a minimal realization for $G(s)$ is

$$\begin{aligned} \dot{x}(t) &= (A_o - B_o D_{hc}^{-1} D_l)x(t) + B_o D_{hc}^{-1} u(t) \\ y(t) &= N_l x(t) \end{aligned} \tag{8}$$

where A_o and B_o are the integrator polynomial matrics corresponding to $c_1[D], \cdots, c_m[D]$. $\square$

Proof: First, verify that (8) is a realization for $G(s)$. Solving (7) for $\Delta(s)$ and substituting into (6) gives

$$\begin{aligned} & B_o D_{hc}^{-1} D(s) - B_o D_{hc}^{-1} D_l \Psi(s) = s\Psi(s) - A_o \Psi(s) \\ \Longrightarrow \quad & B_o D_{hc}^{-1} D(s) = (sI - A_o + B_o D_{hc}^{-1} D_l)\Psi(s) \\ \Longrightarrow \quad & (sI - A_o + B_o D_{hc}^{-1} D_l)^{-1} B_o D_{hc}^{-1} = \Psi(s) D^{-1}(s) \end{aligned} \tag{9}$$

from which the transfer function for (8) is

$$N_l(sI - A_o + B_o D_{hc}^{-1} D_l)^{-1} B_o D_{hc}^{-1} = N_l \Psi(s) D^{-1}(s) = N(s) D^{-1}(s)$$

Thus (8) is a realization of $G(s)$ with dimension $c_1[D] + \cdots + c_m[D]$, which is the McMillan degree of $G(s)$. Then by invoking Lemma 7.1 we conclude that the McMillan degree of $G(s)$ is the dimension of minimal realizations of $G(s)$. □

Example 8.1 Given

$$N(s) = \begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 0 & -(s^3 + 4s^2 + 5s + 2) \\ (s+2)^2 & s+2 \end{bmatrix}$$

find a minimal realization of $N(s)D^{-1}(s)$.

It is easy to check that $N(s)$ and $D(s)$ are right coprime and $N(s)D^{-1}(s)$ is strictly proper. Also, note that

$$c_1[D] = 2, \quad c_2[D] = 3$$

$$
\begin{aligned}
D_{hc} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D_l = \begin{bmatrix} 0 & 0 & -2 & -5 & -4 \\ 4 & 4 & 2 & 1 & 0 \end{bmatrix} \\
N_l &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Then,

$$
D_{hc}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad D_{hc}^{-1} D_l = \begin{bmatrix} -4 & -4 & -2 & -1 & 0 \\ 0 & 0 & -2 & -5 & -4 \end{bmatrix}
$$

$$
A_0 = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & \\ 1 & \\ & 0 \\ & 0 \\ & 1 \end{bmatrix}
$$

Hence, a minimal realization is given by

$$
\begin{aligned}
A &= A_0 - B_0 D_{hc}^{-1} D_l \\
&= \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \\ 1 & \\ & 0 \\ & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} -4 & -4 & -2 & -1 & 0 \\ 0 & 0 & -2 & -5 & -4 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & -4 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{bmatrix} \\
B_0 &= B_0 D_{hc}^{-1} = \begin{bmatrix} 0 & \\ 1 & \\ & 0 \\ & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \\
C &= N_l = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

It is easy to see that the second row of A is the first row of $-D_{hc}^{-1}D_l$ and the fifth row of A is the second row of $-D_{hc}^{-1}D_l$. The second row of B is the first row of D_{hc}^{-1} and the fifth row of B is the second row of D_{hc}^{-1}.

Similar results exist for left polynomial fraction description.

Theorem 8.2: Suppose that a strictly-proper rational transfer function $G(s)$ is described by a CLPFD $D_L^{-1}(s)N_L(s)$, where $D_L(s)$ is row reduced with row degrees $r_1[D_L], \cdots, r_p[D_L] \geq 1$. Then the McMillan degree of $G(s)$ is $n = r_1[D_L] + \cdots + r_p[D_L] \geq 1$, and minimal realization of $G(s)$ have dimension n. Furthermore, write

$$\begin{aligned} N_L(s) &= \Psi^T(s)N_l \\ D_L(s) &= \Delta(s)D_{hr} + \Psi^T(s)D_l \end{aligned} \tag{11}$$

where $\Psi(s)$ and $\Delta(s)$ are the integrator polynomial matrices corresponding to $r_1[D_L], \cdots, r_p[D_L] \geq 1$. Then a minimal realization of $G(s)$ is

$$\begin{aligned} \dot{x}(t) &= (A_o^T - D_l D_{hr}^{-1} B_o^T)x(t) N_l u \\ y(t) &= D_{hr}^{-1} B_o^T x(t) \end{aligned}$$

where A_o and B_o are the integrator coefficient matrices corresponding to $r_1[D_L], \cdots, r_p[D_L] \geq 1$. $\square$

Poles and Zeros

Consider coprime polynomial fraction descriptions

$$G(s) = N(s)D^{-1}(s) = D_L^{-1}(s)N_L(s) \tag{12}$$

Obviously $\det D(s)$ and $\det D_L(s)$ have the same roots.
Furthermore, these roots are the same for all coprime polynomial fraction descriptions (Theorem 7.9).

Hence, the following definition, is also compatible with the SISO case.

Definition 8.2: Suppose $G(s)$ is a strictly-proper rational transfer function, and $G(s) = N(s)D^{-1}(s)$ is coprime and $D(s)$ columne reduced. Then a complex number s_0 is call a *pole* of $G(s)$ if

$$\det D(s_0) = 0$$

Theorem 8.3: Suppose (2) is controllable and observable. Then the complex number s_0 is a pole of

$$G(s) = C(sI - A)^{-1}B$$

iff $\exists x_0 \in \mathbf{C}^{n\times 1}$ and $\exists y_0 \in \mathbf{C}^{n\times 1}$ with $y_0 \neq 0$ such that

$$Ce^{At}x_0 = y_0 e^{s_0 t}, \qquad t \geq 0 \tag{14}$$

(i.e., the initial condition x_0 excites only the s_0 "mode".)

Proof: If s_0 is a pole of $G(s)$, then s_0 is an eigenvalue of A. With x_0 an eigenvector of A corresponding to s_0, we have

$$Ax_0 = s_0 x_0, \quad A^i x_0 = s_0^i x_0, \;\; i = 1, 2, \cdots$$

and the zero-input response is

$$x(t) = e^{At}x_0 = x_0 e^{s_0 t}, \qquad y(t) = Cx_0 e^{s_0 t}$$

(because $e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^i}{i!}t^i + \cdots$).
Since (2) is observable, we have (14) with $y_0 = Cx_0 \neq 0$ (Theorem 5.6(v)).

For sufficiency, assume (14) holds. Taking Laplace transform gives

$$C(sI - A)^{-1}x_0 = y_0(s - s_0)^{-1}$$

or
$$(s - s_0)C\Big[\mathrm{adj}(sI - A)\Big]x_0 = y_0 \cdot \det(sI - A) \tag{15}$$

Since $y_0 \neq 0$, (15) implies that $\det(s_0 I - A) = 0$. Therefore, s_0 is an eigenvalue of A, and, by minimality of the state equation (2), a pole of $G(s)$. The proof is complete. $\square$

The proof is constructive, and can be used to find initial condition x_0 such that (14) holds.

The concept of a zero of a transfer function is more delicate. The definition of zeros of $G(s)$ is through the loss of rank of $G(s)$ at values of $s \in \mathbf{C}$.

Note that, given $G(s)$, we can write a CRPFD and a CLPFD

$$G(s) = N(s)D^{-1}(s) = D_L^{-1}(s)N_L(s) \tag{16}$$

Since $\det D^{-1}(s) = 1/\det[D(s)] \neq 0$, $\forall s \in \mathbf{C}$, we have that $D^{-1}(s)$ is of full rank (i.e. nonsingular) $\forall s \in \mathbf{C}$. Similarly for $D_L^{-1}(s)$. Hence

$$\text{rank} G(s) = \text{rank} N(s) = \text{rank} N_L(s), \qquad \forall s \in \mathbf{C}$$

The definition of “zero” can be equivalently defined through $N(s)$ (a CRPFD) or $N_L(s)$ (a CLPFD).

Definition 8.3: Suppose $G(s)$ is a strictly-proper rational transfer function with rank $G(s) = \min[m, p]$ for almost all complex numbers s. A complex number s_0 is called a *transmission zero* of $G(s)$ if $\text{rank} N(s_0) < \min[m, p]$, where $N(s)D^{-1}(s)$ is any CRPFD for $G(s)$.

This is compatible with the SISO definition.

Example 8.2: Consider the following CRPFD

$$G(s) = \begin{bmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+1}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} (s+1)^2 & 0 \\ 0 & (s+2)^2 \end{bmatrix}^{-1} \tag{17}$$

There are 2 poles of multiplicity 2 : $s = -1$ and $s = -2$.
There are 2 transmission zeros: $s = -1$ and $s = -2$.
Thus a MIMO transfer function can have coincident poles and transmission zeros — something that cannot happen in SISO.

Example 8.3: Consider the following CLPFD

$$G(s) = \begin{bmatrix} \frac{s+1}{(s+3)^3} & 0 \\ 0 & \frac{s+2}{(s+4)^2} \\ \frac{s+2}{(s+5)^2} & \frac{s+1}{(s+5)^2} \end{bmatrix} = \begin{bmatrix} (s+3)^2 & 0 & 0 \\ 0 & (s+4)^2 & 0 \\ 0 & 0 & (s+5)^2 \end{bmatrix}^{-1} \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \\ s+2 & s+1 \end{bmatrix} \tag{18}$$

This system has no transmission zeros, eventhough various entries of $G(s)$, viewed as SISO transfer funtions, have transmission zeros at $s = -1$ or $s = -2$.

Another characterization of transmission zeros is in terms of making the output of a minimal realization of $G(s)$ identically zero for a particular intial state and particular input signal.

We must assume $m \leq p$ for this development (i.e., $G(s)$, $p \times m$, is a tall matrix, not a fat one).

Theorem 8.4: Suppose (2) is controllable and observable, and

$$G(s) = C(sI - A)^{-1}B \tag{19}$$

has rank m for almost all $s \in \mathbf{C}$. If $s_0 \in \mathbf{C}$ is not a pole of $G(s)$, then it is a transmission zero of $G(s)$ iff $\exists\, u_0 \in \mathbf{C}^{m\times 1}$ with $u_0 \neq 0$ and $x_0 \in \mathbf{C}^{n\times 1}$ such that

$$Ce^{At}x_0 + \int_0^t Ce^{A(t-\sigma)}Bu_0e^{s_0\sigma}d\sigma = 0, \qquad t \geq 0 \tag{20}$$

Proof: Suppose $N(s)D^{-1}(s)$ is a CRPFD for (19).

If $s_0 \in \mathbf{C}$ is not a pole of $G(s)$, then $D(s_0)$ is invertible and s_0 is not an eigenvalue of A.

If x_0 and $u_0 \neq 0$ are such that (20) holds, then we have

$$C(sI - A)^{-1}x_0 + N(s)D^{-1}(s)u_0 \times (s - s_0)^{-1} = 0$$

or

$$(s - s_0)C(sI - A)^{-1}x_0 + N(s)D^{-1}(s)u_0 = 0$$

Hence,

$$N(s_0)D^{-1}(s_0)u_0 = 0$$

and this implies that rank$N(s_0) < m$. That is, s_0 is a transmission zero of $G(s)$.

One the other hand, suppose s_0 is not a pole of $G(s)$. It is easy to verify

$$(s_0I - A)^{-1}(s - s_0)^{-1} = (sI - A)^{-1}(s_0I - A)^{-1} + (sI - A)^{-1}(s - s_0)^{-1} \quad (21)$$

We can write, for any $u_0 \in \mathbf{C}^{m\times 1}$ and $x_0 = (s_0 I - A)^{-1} B u_0$,

$$\begin{aligned}\mathbf{L}&\left(Ce^{At}x_0 + \int_0^t Ce^{A(t-\sigma)} B u_0 e^{s_0\sigma} d\sigma \right)\\ &= C(sI-A)^{-1}x_0 + C(sI-A)^{-1}Bu_0(s-s_0)^{-1}\\ &= C\left[(sI-A)^{-1}(s_0I-A)^{-1} + (sI-A)^{-1}(s-s_0)^{-1}\right]Bu_0\\ &= C(s_0I-A)^{-1}(s-s_0)^{-1}Bu_0 = G(s_0)u_0(s-s_0)^{-1}\\ &= N(s_0)D^{-1}(s_0)u_0(s-s_0)^{-1}\end{aligned}$$

Taking the inverse Laplace transform gives

$$Ce^{At}x_0 + \int_0^t Ce^{A(t-\sigma)} B u_0 e^{s_0\sigma} d\sigma = N(s_0)D^{-1}(s_0)u_0 e^{s_0 t} \tag{22}$$

Clearly we can choose u_0 so that (22) is zero if $\operatorname{rank} N(s_0) < m$, i.e., if s_0 is a transmission zero of $G(s)$. □

The above proof is constructive and can be used to find x_0 and $u(t) = u_0 e^{s_0 t}$ to achieved (20).

We can also find x_0 and u_0 by letting $u(t) = u_0 e^{s_0 t}$ and $y(t) = 0$, i.e.,

$$C(sI - A)^{-1} x_0 + C(sI - A)^{-1} B u_0 (s - s_0)^{-1} \;=\; 0$$

and solve for x_0 and u_0.

For the case of $m > p$ (a fat matrix, more output than input), a different characterization can be obtained.

Suppose the linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \qquad x(t) \in \mathbf{R}^n, \;\; u(t) \in \mathbf{R}^m, \;\; y(t) \in \mathbf{R}^p \end{aligned}$$

with $p < m$ is controllable and observable, and

$$G(s) = C(sI - A)^{-1} B$$

has rank p for almost all complex values of s.

Suppose the complex number s_0 is not a pole of $G(s)$. Then it can be shown that s_0 is a transmission zero of $G(s)$ iff there is nonzero complex $1 \times p$ vector h with the property that, for ANY complex $m \times 1$ vector u_0, there is a complex $n \times 1$ vector x_0 such that

$$h\,y(t) \;=\; hCe^{At}x_0 + \int_0^t hCe^{A(t-\sigma)}Bu_0e^{s_0\sigma}d\sigma \;=\; 0, \quad t \geq 0$$

State Feedback

Linear state feedback of the form

$$u(t) = Kx(t) + Mr(t)$$

can be recast in the polynomial fraction description formulation.

Consider a strictly-proper rational transfer function $G(s)$ in a CRPFD

$$G(s) = N(s)D^{-1}(s)$$

where $D(s)$ is column reduced. Let the input be $U(s)$, then the output $Y(s)$ is given by

$$Y(s) = N(s)\underbrace{D^{-1}(s)U(s)}_{\xi(s)} \tag{23}$$

Rewriting this as a pair of equations with polynomial matrix coefficients,

$$\begin{aligned} D(s)\xi(s) &= U(s) \\ Y(s) &= N(s)\xi(s) \end{aligned} \tag{24}$$

The $m \times 1$ vector $\xi(s)$ is called the *pseudo-state* of the plant, motivated by the minimal realization of the form (8) for $G(s)$.

From (9) we write

$$\begin{aligned} \Psi(s)\xi(s) &= \Psi(s)D^{-1}(s)U(s) \\ &= (sI - A_o + B_o D_{hc}^{-1} D_l)^{-1} B_o D_{hc}^{-1} U(s) \end{aligned}$$

or

$$s\Psi(s)\xi(s) = (A_o - B_o D_{hc}^{-1} D_l)\Psi(s)\xi(s) + B_o D_{hc}^{-1} U(s) \tag{25}$$

Defining the $n \times 1$ vector $x(t)$ as

$$x(t) = \mathcal{L}^{-1}\Big[\Psi(s)\xi(s)\Big]$$

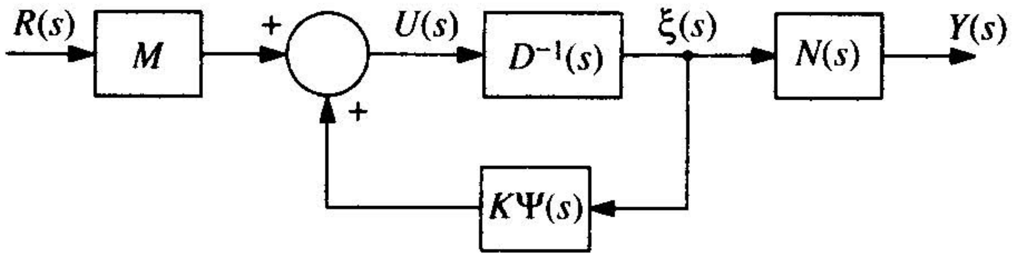
then (25) is just the linear state equation (8) with zero initial condition.

Hence, linear state feedback for (2) corresponds to feedback of $\Psi(s)\xi(s)$ in the associated pseudo-state representation (24).

So linear state feedback for (24) is represented by

$$U(s) = K\Psi(s)\xi(s) + MR(s) \tag{26}$$

where K and M are real matrices of $m \times n$ and $m \times m$, resp. This is depicted in Figure 17.14.



17.14 Figure Transfer function diagram for state feedback.

Assume M is invertible. We now develop a PFD for the resulting closed-loop transfer function. Sutituting (26) into (24),

$$\Big[D(s) - K\Psi(s)\Big]\xi(s) = MR(s)$$
$$Y(s) = N(s)\xi(s)$$

Note that the column degree coefficient matrix for the polynomial matrix $D(s) - K\Psi(s)$ is the same as that for $D(s)$ (which is assumed column reduced), and hence invertible. Hence, $D(s) - K\Psi(s)$ is invertible. Therefore,

$$\begin{aligned} \xi(s) &= \Big[D(s) - K\Psi(s)\Big]^{-1} MR(s) \\ Y(s) &= N(s)\xi(s) \end{aligned} \tag{27}$$

Since M is invertible, (27) gives a RPFD for the closed-loop transfer function

$$N(s)\hat{D}^{-1}(s) = N(s)\Big[M^{-1}D(s) - M^{-1}K\Psi(s)\Big]^{-1} \tag{28}$$

This is not necessarily coprime, though $\hat{D}(s)$ is column reduced. Write $D(s)$ as

$$D(s) = D_{hc}\Delta(s) + D_l\Psi(s)$$

Then, $\hat{D}(s)$ in (28) becomes

$$\hat{D}(s) = M^{-1}D_{hc}\Delta(s) + M^{-1}[D_l - K]\Psi(s)$$

By selecting K and M as

$$M = D_{hc}\hat{D}_{hc}^{-1}, \qquad K = -M\hat{D}_l + D_l$$

the closed-loop $\hat{D}(s)$ matrix becomes

$$\hat{D}(s) = \hat{D}_{hc}\Delta(s) + \hat{D}_l\Psi(s) \qquad (*)$$

Hence, the closed-loop $\hat{D}(s)$ matrix can be assigned to any polynomial of the form $(*)$.

The choices of K and M do not directly affect $N(s)$ (i.e. the transmission zeros of the system). However, there do exist an indirect effect in that (28) may not be coprime, hence resulting in the cancellation of zeros of the system.

Feedback does not affect the controllability of a system, but it does affect observability of the system in some special cases.