EE 6225 – Multivariable Control System

Part I – Advanced Process Control

Dr Poh Eng Kee

Professor (Adjunct)

e-mail: eekpoh@ntu.edu.sg

Course Outline

1.Dynamic Model, MIMO, Poles and Zeros, Trade-Offs and Limits of Performance

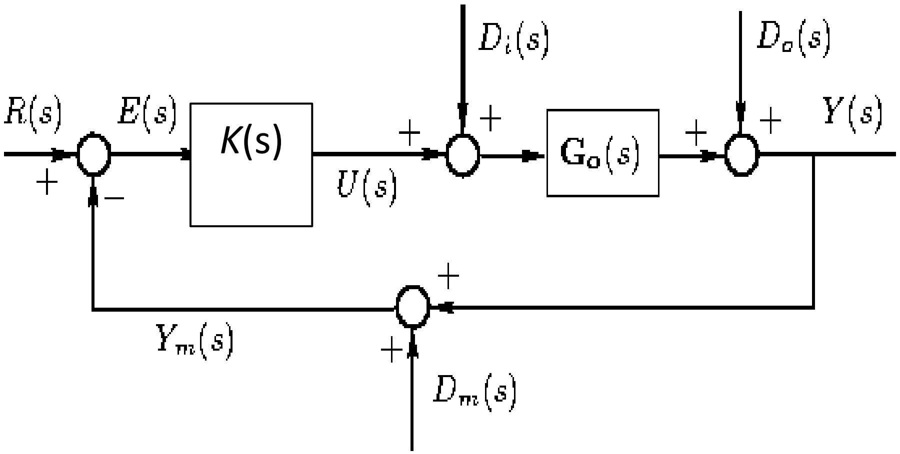
2.SISO - PID, DSM, IMC, IMC-Based PID, Smith Predictor

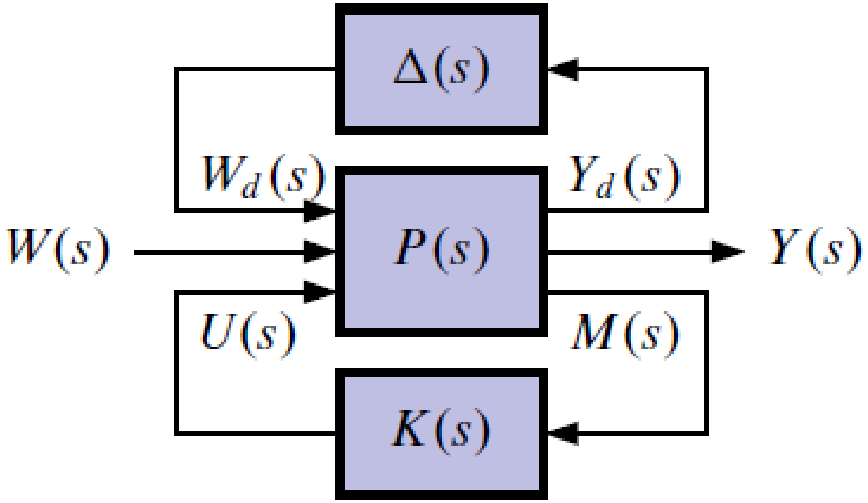
3.SISO - Cascade, Feedforward, Ratio, Split Range, Selective Control

4,5. MIMO – Multiloop Interaction Dynamics, RGA, Decoupling, Multivariable Feedback Control

6.System Identification, Linear Regression, RLS, 4SID

5. MIMO Control Problem Formulation





- Sensitivity Functions represent system response to reference, disturbances and noise

- The "plant" $P(s)$ is generally a combination of $G_0(s)$ and uncertainties added together.

5. Learning Objectives

- Performance Trade-Offs and Limitations in MIMO System Control Design
- Characterize Model Uncertainties for MIMO Systems
- Analysis of Robust Stability and Performance in the Presence of Model Uncertainties in MIMO Systems
- Introduction to Synthesis of Robust Multivariable (Centralized) Controller (Details to be covered by Prof Ling KV)

Outline of Lecture 5

5.1 Transfer Function Matrix (Poles and Zeros)

5.2 Limits on Performance for MIMO System

- 5.2.1 MIMO Sensitivity Functions and Trade-Off (Concept)
- 5.2.2 Matrix Fraction Description (MFD) and Bode's Integral
- 5.2.3 Poisson's Integral Constraints on Sensitivity Functions
- 5.2.4 Bandwidth Limitations due to RHP Poles and Zeros (Concept)

5.3 Analysis of MIMO Robust Stability and Performance

- 5.3.1 Nyquist Criteria
- 5.3.2 MIMO Unstructured Uncertainty
- 5.3.3 Robust Stability and Linear Fractional Transformation (LFT)
- 5.3.4 Robust Performance

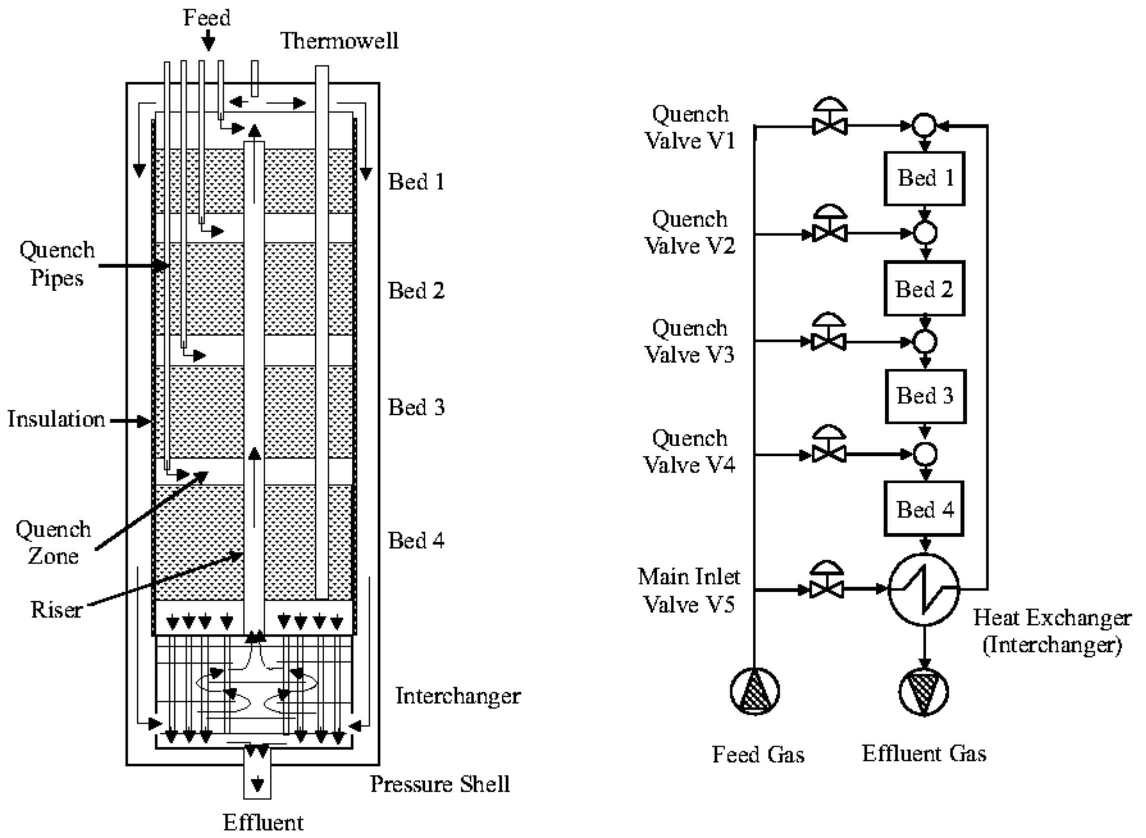
5.4 Advanced Topics (Optional – Not Included in Syllabus)

- 5.4.1 Proof of Robust Stability
- 5.4.2 Structure Uncertainty and Structured Singular Value (SSV)
- 5.4.3 Robust Performance Using SSV
- 5.4.4 Synthesis – Robust Multivariable Controller (Matlab Project)

5. MIMO Summary

- MIMO Transfer Function Matrix, $G(s)$ and State Space Representation (A,B,C,D) and the Poles and Zeros
- Performance Trade-Off and Limitations
  - MIMO Sensitivity Functions and Trade-Off (Concept)
  - Sensitivity Peaks and Bandwidth Limitations due to RHP Poles and Zeros (Concept)
- Design MIMO Controller (Lecture 4) for Stability and Performance (Steady State Error, Controller Effort, Disturbance and Noise Rejection)
- Presence of Model Uncertainties in $G(s)$ or equivalently (A,B,C,D)
- Analysis: Robust Stability against
  - Unstructured Uncertainty (Additive, Input and Output Multiplicative) using Singular Value $\left\| N_{y_d w_d} \right\|_\infty \le 1$
  - Structured Uncertainty using Structured Singular Value (X)
- Analysis: Robust Performance against
  - Unstructured Uncertainty using Structured Singular Value (X)
- Synthesis: Robust Multivariable Controller (X)

5. MIMO Processes – Ammonia-Synthesis Converter



A **B**

A typical industrial plant aimed at producing ammonia from natural gas is the Kellogg Process. In an integrated chemical plant of this type, there will be hundreds (*possibly thousands*) of variables that interact to some degree.

5. Transfer Function Matrix, Revisited

- It is straightforward to convert a state space model to a transfer-function model.
- The matrix transfer function **G**(*s*) corresponding to a state space model (**A**, **B**, **C**, **D**) is

$$\mathbf{G}(s) \triangleq \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

5. Transfer Function Matrix

- We will use $G_{ik}(s)$ to denote the transfer function from the k^{th} component of $U(s)$ to the i^{th} component of $Y(s)$. Then $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1k}(s) & \dots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2k}(s) & \dots & G_{2m}(s) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ G_{i1}(s) & G_{i2}(s) & \dots & G_{ik}(s) & \dots & G_{im}(s) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \dots & G_{mk}(s) & \dots & G_{mm}(s) \end{bmatrix}$$

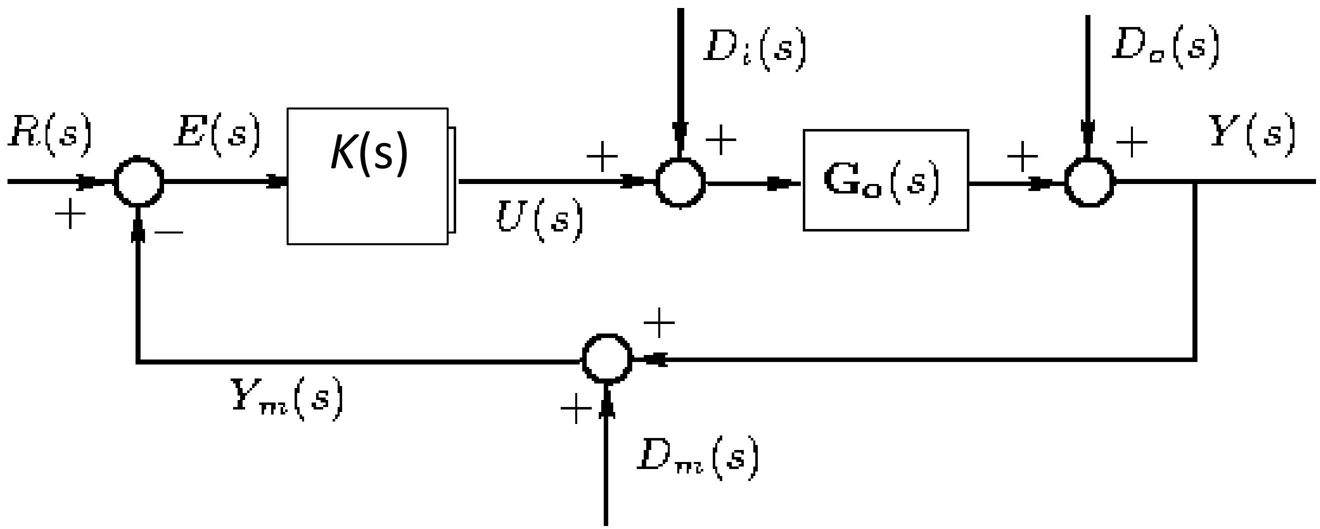
5. Impulse Response Matrix

- **Definition 20.2:** The *impulse response matrix* of the system, **g**(*t*), is the inverse Laplace transform of the transfer-function matrix **G**(*s*). For future reference, we express **g**(*t*) as

$$\mathbf{g}(t) = \begin{bmatrix} g_{11}(t) & g_{12}(t) & \dots & g_{1k}(t) & \dots & g_{1m}(t) \\ g_{21}(t) & g_{22}(t) & \dots & g_{2k}(t) & \dots & g_{2m}(t) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ g_{i1}(t) & g_{i2}(t) & \dots & g_{ik}(t) & \dots & g_{im}(t) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ g_{m1}(t) & g_{m2}(t) & \dots & g_{mk}(t) & \dots & g_{mm}(t) \end{bmatrix} = \mathcal{L}^{-1}\left[\mathbf{G}(s)\right]$$

5.1 General MIMO Feedback Loop

- Note:
  - 2 disturbances: $D_i(s)$ at process input and $D_o(s)$ at process output
  - 1 measurement noise: $D_m(s)$



5.2.1 MIMO Sensitivity Functions

- Now,

$$Y(s) = \mathbf{T_o}(s)R(s) - \mathbf{T_o}(s)D_m(s) + \mathbf{S_o}(s)D_o(s) + \mathbf{S_{io}}(s)D_i(s)$$
$$U(s) = \mathbf{S_{uo}}(s)R(s) - \mathbf{S_{uo}}(s)D_m(s) - \mathbf{S_{uo}}(s)D_o(s) - \mathbf{S_{uo}}(s)\mathbf{G_o}(s)D_i(s)$$
$$E(s) = \mathbf{S_o}(s)R(s) - \mathbf{S_o}(s)D_m(s) - \mathbf{S_o}(s)D_o(s) - \mathbf{S_{io}}(s)D_i(s)$$

where the sensitivity functions are defined by

$$S_o(s) = \left[I + G_o(s)K(s)\right]^{-1}$$

$$T_o(s) = G_o(s)K(s)\left[I + G_o(s)K(s)\right]^{-1} = \left[I + G_o(s)K(s)\right]^{-1} G_o(s)K(s)$$
$$= I - S_o(s)$$

$$S_{uo}(s) = K(s)\left[I + G_o(s)K(s)\right]^{-1}$$

$$S_{io}(s) = \left[I + G_o(s)K(s)\right]^{-1} G_o(s) = G_o(s)\left[I + K(s)G_o(s)\right]^{-1} = S_o(s)G_o(s)$$

5.2.1 Summary of Trade-Offs

- Analogously to the SISO case, MIMO performance specifications can generally not be addressed independently from another, because they are linked by a web of trade-offs.
- A number of the SISO fundamental algebraic laws of trade-off generalize rather directly to the MIMO case:
  - $\mathbf{S_o}(s) = \mathbf{I} - \mathbf{T_o}(s)$, implying a trade-off between speed of response to a change in reference or rejecting disturbances ($\mathbf{S_o}(s)$ *small*) versus necessary control effort, sensitivity to measurement noise, or modeling errors ($\mathbf{T_o}(s)$ *small*);
  - $Y_m(s) = -\mathbf{T_o}(s)D_m(s)$, implying a trade-off between the bandwidth of the complementary sensitivity and sensitivity to measurement noise.

5.2.1 Summary of Trade-Offs

- $\mathbf{S_{uo}}(s) = [\mathbf{G_o}(s)]^{-1}\mathbf{T}_o(s)$, implying that a complementary sensitivity with bandwidth significantly higher than the open loop will generate large control signals;
- $\mathbf{S_{io}}(s) = \mathbf{S_o}(s)\mathbf{G_o}(s)$, implying a trade-off between input and output disturbances; and

5.2.2 Matrix Fraction Description (MFD)

- Consider the state space model

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

We assume that the state space model is stabilizable.

- Let $u(t) = -\mathbf{K}x(t) + w(t)$ be stabilizing feedback. The system can then be written as follows, by adding and subtracting $\mathbf{BK}x(t)$:

$$\dot{x}(t) = (\mathbf{A} - \mathbf{BK})x(t) + \mathbf{B}w(t)$$
$$y(t) = \mathbf{C}x(t)$$
$$w(t) = u(t) + \mathbf{K}x(t)$$

5.2.2 Right Matrix Fraction Description (RMFD)

- We can express these equations, in the Laplace-transform domain with zero initial conditions, as

$$U(s) = (\mathbf{I} - \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B})W(s)$$
$$Y(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}W(s)$$

$$U(s) = \mathbf{G}_{\mathbf{D}}(s)W(s); \qquad Y(s) = \mathbf{G}_{\mathbf{N}}(s)W(s); \qquad Y(s) = \mathbf{G}_{\mathbf{N}}(s)[\mathbf{G}_{\mathbf{D}}(s)]^{-1}U(s)$$

where $\mathbf{G}_\mathbf{N}(s)$ and $\mathbf{G}_\mathbf{D}(s)$ are the following two stable transfer-function matrices:

$$\mathbf{G}_{\mathbf{N}}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}$$
$$\mathbf{G}_{\mathbf{D}}(s) = \mathbf{I} - \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}$$

- We see that ($\mathbf{G}_\mathbf{N}(s)$, $\mathbf{G}_\mathbf{D}(s)$) is a Right Matrix Fraction Description (RMFD). There is also a Left Matrix Fraction Description (LMFD) which can be derived using observer design.

5.2.2 Left Matrix Fraction Description (LMFD)

- Similarly, we can use an observer to develop a LMFD. We assume that the state space model is detectable. Consider the following observer

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \mathbf{B}u(t) + \mathbf{J}(y(t) - \mathbf{C}\hat{x}(t))$$
$$y(t) = \mathbf{C}\hat{x}(t) + \nu(t)$$

- We can express these equations in the Laplace domain as

$$\Phi(s) \stackrel{\triangle}{=} \mathcal{L}\left[\nu(t)\right] = (\mathbf{I} - \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{JC}]^{-1}\mathbf{J})Y(s) - \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{JC}]^{-1}\mathbf{B}U(s)$$

- We know that, for a stable observer, $v(t) \rightarrow 0$ exponentially fast, hence, in steady state, we can write

$$\overline{\mathbf{G}}_{\mathbf{D}}(s)Y(s) = \overline{\mathbf{G}}_{\mathbf{N}}(s)U(s)$$

where

$$\overline{\mathbf{G}}_{\mathbf{N}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{B}$$
$$\overline{\mathbf{G}}_{\mathbf{D}}(s) = \mathbf{I} - \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{J}$$

Hence $(\overline{\mathbf{G}}_{\mathbf{N}}(s), \overline{\mathbf{G}}_{\mathbf{D}}(s))$ is a LMFD for the system.

5.2.2 MIMO Poles and Zeros Revisited

- In the case of SISO plants, we found that performance limitations are intimately connected to the presence of open-loop RHP poles and zeros. We shall find that this is also true in the MIMO case.
- As a prelude to developing these results, we first review the appropriate definitions of poles and zeros.

5.2.2 MIMO Poles and Zeros Revisited

Consider the square plant model $\mathbf{G_o}(s)=\mathbf{G_{oN}}(s)[\mathbf{G_{oD}}(s)]^{-1}$ (Right Matrix Fraction Description – RMFD). We recall that z_0 is a <u>transmission</u> zero of $\mathbf{G_o}(s)$, with corresponding left directions $h_1, h_2, \ldots, h_{\mu z}$, if

$$\det(\mathbf{G_{oN}}(z_o)) = 0 \quad \text{and} \quad h_i^T(\mathbf{G_{oN}}(z_o)) = 0 \quad i = 1, 2, \ldots, \mu_z$$

Similarly, we say that η_0 is a pole of $\mathbf{G_o}(s)$, with corresponding right directions $g_1, g_2, \ldots, g_{\mu p}$, if

$$\det(\overline{\mathbf{G}}_{\mathbf{oD}}(\eta_o)) = 0 \quad \text{and} \quad (\overline{\mathbf{G}}_{\mathbf{oD}}(\eta_o))g_i = 0 \quad i = 1, 2, \ldots, \mu_p$$

5.2.2 Bode's Integral Constraint on MIMO System

- Consider a feedback control system with open loop transfer function having unstable poles located at $p_1, \ldots, p_{N_p}$, pure time delay τ, and relative degree $n_r > 1$. Then, the nominal sensitivity satisfies (Ref: Control System Design by Goodwin, et. al, Pg 244):

$$\int_0^\infty \ln\left|\det\left(S_o\left(j\omega\right)\right)\right| d\omega = \pi \cdot \sum_{i=1}^{N_p} \mathrm{Re}\left(p_i\right), \quad \text{where } p_i, i = 1, \ldots, N_p$$

Proof of Bode's Integral Constraint on Sensitivity (Optional)

Proof

We first treat the case $\tau = 0$.

We make the following changes in notation $s \to z$, $H_{ol}(s) \to l(z)$ *and* $g(z) = (1 + l(z))^{-1}$

We then observe that

$$S_o(z) = (1 + l(z))^{-1} = g(z) \tag{9.2.4}$$

By the assumptions on $H_{ol}(s)$ *we observe that* $\ln g(z)$ *is analytic in the closed RHP, then by* Theorem 1.7 on Slide 23

$$\oint_C \ln g(z) dz = 0 \tag{9.2.5}$$

where $C = C_i \cup C_\infty$ *is the contour defined in Figure* C.4

Then

$$\oint_C \ln g(z) dz = j \int_{-\infty}^{\infty} \ln g(j\omega) d\omega - \int_{C_\infty} \ln(1 + l(z)) dz \tag{9.2.6}$$

For the first integral on the right hand side of equation (9.2.6), we use the conjugate symmetry of $g(z)$ *to obtain*

$$\int_{-\infty}^{\infty} \ln g(j\omega) d\omega = 2 \int_{0}^{\infty} \ln |g(j\omega)| d\omega \tag{9.2.7}$$

For the second integral we notice that on C_∞, $l(z)$ *can be approximated by*

$$\frac{a}{z^{n_r}} \tag{9.2.8}$$

The result follows on using Example C.7 on Slide 24 and *on noticing that* $a = \kappa$ *for* $n_r = 1$. *The extension to the case* $\tau \neq 0$ *is similar using the results in* Example C.8 on Slide 25

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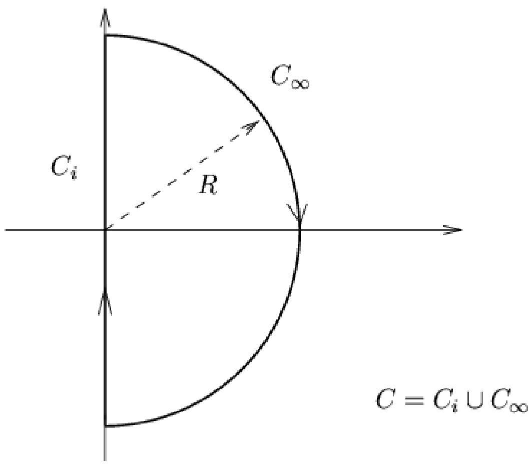

Figure C.4: RHP encircling contour

Independence of Path (Optional)

Theorem C.2 *If the integral* $\int Pdx + Qdy$ *is independent of the path in D, then*

$$\oint Pdx + Qdy = 0 \tag{C.2.13}$$

on every closed path in D. Conversely if (C.2.13) holds for every simple closed path in D, then $\int Pdx + Qdy$ *is independent of the path in D.*

Proof

Suppose the integral is independent of the path. Let C be a simple closed path in D and divide C into arcs $\vec{AB}$ and $\vec{BA}$ as in Figure C.2.

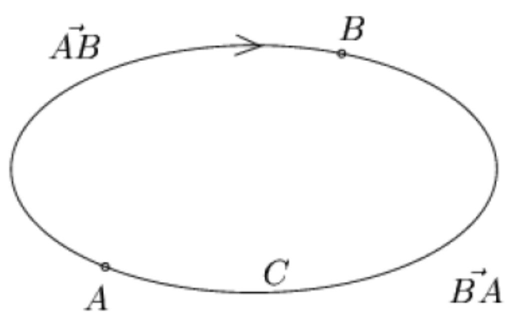

Figure C.2: Integration path

$$\oint_C (Pdx + Qdy) = \int_{AB} Pdx + Qdy + \int_{BA} Pdx + Qdy \tag{C.2.14}$$

$$= \int_{AB} Pdx + Qdy - \int_{AB} Pdx + Qdy \tag{C.2.15}$$

The converse result is established by reversing the above argument.

Cauchy Integral Theorem (Optional)

Theorem 1.7 (Cauchy Integral Theorem) *If $f(z)$ is analytic in some simply connected domain D, then $\int f(z)dz$ is independent of path in D and*

$$\oint_C f(z)dz = 0 \qquad \text{(C.7.1)}$$

where C is a simple closed path in D.

Proof

Follows from the Cauchy-Riemann conditions and Theorem C.2.

□□□

Analytic Function Theory (Optional)

Example C.7 *Consider the function*

$$f(z) = \ln\left(1 + \frac{a}{z^n}\right) \qquad n \geq 1 \tag{C.7.5}$$

and a semi circle, C, defined by $z = Re^{j\gamma}$ *for* $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *Then if C is followed clockwise*

$$I_R \triangleq \lim_{R\to\infty} \int_C f(z)dz = \begin{cases} 0 & \text{for } n > 1 \\ -j\pi a & \text{for } n = 1 \end{cases} \tag{C.7.6}$$

This is proved as follows.

On C we have that $z = Re^{j\gamma}$, *then*

$$I_R = \lim_{R\to\infty} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln\left(1 + \frac{a}{R^n} e^{-jn\gamma}\right) Re^{j\gamma} d\gamma \tag{C.7.7}$$

We also know that

$$\lim_{|x|\to 0} \ln(1 + x) = x \tag{C.7.8}$$

Then

$$I_R = \lim_{R\to\infty} \frac{a}{R^{n-1}} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} e^{-j(n-1)\gamma} d\gamma \tag{C.7.9}$$

From where, evaluating for n = 1 and for n > 1, the result follows.

Analytic Function Theory (Optional)

Example C.8 *Consider the function*

$$f(z) = \ln\left(1 + e^{-z\tau}\frac{a}{z^n}\right) \qquad n \geq 1, \quad \tau > 0 \tag{C.7.10}$$

and a semi circle, C, defined by $z = Re^{j\gamma}$ *for* $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *Then, for clockwise C,*

$$I_R \triangleq \lim_{R\to\infty}\int_C f(z)dz = 0 \tag{C.7.11}$$

This is proved as follows.

On C we have that $z = Re^{j\gamma}$, *then*

$$I_R = \lim_{R\to\infty} j\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\left[\ln\left(1 + \frac{a}{z^(n+1)}\frac{z}{e^{z\tau}}\right)z\right]_{z=Re^{j\gamma}} d\gamma \tag{C.7.12}$$

We recall that, if τ *is a positive real number, and* $\Re\{z\} > 0$, *then*

$$\lim_{|z|\to\infty}\frac{z}{e^{z\tau}} = 0 \tag{C.7.13}$$

And, for very large R, we have that,

$$\ln\left(1 + \frac{a}{z^{n+1}}\frac{z}{e^{z\tau}}\right)z\Big|_{z=Re^{j\gamma}} \approx \frac{1}{z^n}\frac{z}{e^{z\tau}}\Big|_{z=Re^{j\gamma}} \tag{C.7.14}$$

Thus, in the limit, this quantity goes to zero **for all positive n.** *The result then follows.*

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5.2.3 Poisson Integral Constraints on Complementary Sensitivity for MIMO System

- **Theorem** *Complementary sensitivity and unstable pole:*

Consider a MIMO system with an unstable pole located at $s = \eta_0 = \alpha + j\beta$ and having associated directions g_1, g_2, …, $g_{\mu p}$; then

(*i*)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \ln|[\mathbf{T_o}(j\omega)]_{r*}g_i|d\Omega(\eta_o,\omega) = \ln|B_{ir}(\eta_o)g_{ir}|; \qquad r \in \nabla_i; \qquad i = 1, 2, \ldots, \mu_p$$

(*ii*)

$$\int_{-\infty}^{\infty} \ln|[\mathbf{T_o}(j\omega)]_{rr}|\, d\Omega(\eta_o,\omega) \geq \int_{-\infty}^{\infty} \ln\left|\frac{[\mathbf{T_o}(j\omega)]_{rr}g_{ir}}{\sum_{k\in\nabla}[\mathbf{T_o}(j\omega)]_{rk}g_{ik}}\right| d\Omega(\eta_o,\omega)$$

where

$$d\Omega(\eta_o,\omega) = \frac{\alpha}{\alpha^2 + (\omega-\beta)^2}d\omega \Longrightarrow \int_{-\infty}^{\infty} d\Omega(\eta_o,\omega) = \pi$$

$$\nabla_i = \{r|g_{ir} \neq 0\}; \qquad i = 1, 2, \ldots, \mu_p$$

Poisson-Jensen Formula for Half Plane (Optional)

Lemma 1.1 *Consider a function $g(z)$ with the following properties*

(i) $g(z)$ *is analytic on the closed RHP.*
(ii) $g(z)$ *does not vanish on the imaginary axis.*
(iii) $g(z)$ *has zeros in the open RHP, located at* $a_1, a_2, \ldots, a_n$.
(iv) $g(z)$ *satisfies* $\lim_{|z|\to\infty} \frac{|\ln g(z)|}{|z|} = 0$

Consider also a point $z_0 = x_0 + jy_0$ *such that* $x_0 > 0$*, then*

$$\ln|g(z_0)| = \sum_{i=1}^{n} \ln\left|\frac{z_0 - a_i}{z_0 + a_i^*}\right| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_0}{x_0^2 + (\omega - y_0)^2} \ln|g(j\omega)|)d\omega \tag{C.8.13}$$

Proof

Let

$$\tilde{g}(z) \triangleq g(z) \prod_{i=1}^{n} \frac{z + a_i^*}{z - a_i} \tag{C.8.14}$$

Then, $\ln \tilde{g}(z)$ is analytic within the closed unit disk. If we now apply Theorem C.9 to $\ln \tilde{g}(z)$, we obtain

$$\ln \tilde{g}(z_0) = \ln g(z_0) + \sum_{i=1}^{n} \ln\left(\frac{z_0 + a_i^*}{z_0 - a_i}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_0}{x_0^2 + (\omega - y_0)^2} \ln \tilde{g}(j\omega) d\omega \tag{C.8.15}$$

We also recall that if x is any complex number, then $\Re\{\ln x\} = \Re\{\ln|x| + j\angle x\} = \ln|x|$. Thus the result follows on equating real parts in the equation above and on noting that

$$\ln|\tilde{g}(j\omega)| = \ln|g(j\omega)| \tag{C.8.16}$$

□□□

Proof of Poisson's Integral Constraint on Complementary Sensitivity (Optional)

Proof

The above result is an almost straightforward application of Lemma 1.1 on Slide 27 If the delay is non zero then, $\ln|T_o(j\omega)|$ does not satisfy the bounding condition (iv) in Lemma 1.1 Thus we first define

$$\bar{T}_o(s) = T_o(s)e^{s\tau} \Longrightarrow \ln|\bar{T}_o(j\omega)| = \ln|T_o(j\omega)| \tag{9.5.2}$$

The result then follows on applying Lemma 1.1 on Slide 27 to $\bar{T}_o(s)$ and on recalling that

$$\ln(T_o(p_i)) = 0 \qquad i = 1, 2, \dots, N \tag{9.5.3}$$

□□□

5.2.3 Poisson Integral Constraints on Sensitivity Function for MIMO System

Theorem *Sensitivity and NMP zero:*

Consider a MIMO plant having a NMP zero at $s = z_0 = \gamma + j\delta$, with associated directions $h_1^T, h_2^T, \ldots, h_{\mu_z}^T$; then the sensitivity in any control loop for that plant satisfies

(*i*)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \ln|h_i^T[\mathbf{S_o}(j\omega)]_{*r}|d\Omega(z_o,\omega) = \ln|h_{ir}[\mathbf{B_i'}(\mathbf{z_o})]_{rr}|; \qquad r \in \nabla_i'; \qquad i = 1, 2, \ldots, \mu_p$$

(*ii*)

$$\int_{-\infty}^{\infty} \ln|[\mathbf{S_o}(j\omega)]_{rr}|\, d\Omega(z_o,\omega) \geq \int_{-\infty}^{\infty} \ln\left|\frac{h_{ir}[\mathbf{S_o}(j\omega)]_{rr}}{\sum_{k\in\nabla'} h_{ik}[\mathbf{S_o}(j\omega)]_{kr}}\right| d\Omega(z_o,\omega)$$

where

$$d\Omega(z_o,\omega) = \frac{\gamma}{\gamma^2 + (\omega-\delta)^2}d\omega \Longrightarrow \int_{-\infty}^{\infty} d\Omega(z_o,\omega) = \pi$$

$$\nabla_i' = \{r|h_{ir} \neq 0\}; \qquad i = 1, 2, \ldots, \mu_z$$

5.2.3 Peaks on Sensitivity Functions

Theorem: Given a plant $G(s)$ with no delay. Suppose that $G(s)$ has $z_i, i=1, \ldots, N_z$ as RHP zeros with output directions $y_{z,i}$, $i=1, \ldots, N_z$. Also, suppose that $p_i, i=1, \ldots, N_p$ as RHP poles with output directions $y_{p,i}$, $i=1, \ldots, N_p$. Also suppose that all RHP poles and zeros are distinct,

$$\|S\|_\infty \geq \Lambda$$

$$\|T\|_\infty \geq \Lambda, \text{ where } \Lambda = \sqrt{1+\bar{\sigma}^2\left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2}\right)}$$

where

$$Q_z \in \Re^{N_z \times N_z}, \; [Q_z]_{ij} = \frac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j}$$

$$Q_p \in \Re^{N_p \times N_p}, \; [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j}$$

$$Q_{zp} \in \Re^{N_z \times N_p}, \; [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j}$$

5.2.4 Peak on Sensitivity Functions

- These results are similar to those derived for SISO control loops, because we also obtain lower bounds for sensitivity peaks. Furthermore, these bounds grow with *bandwidth* requirements.
- If closed-loop bandwidth is much smaller than the magnitude of a right half plane (real) pole, then there is be a very large $T_o(s)$ peak leading to large overshoots.
- If the closed-loop bandwidth is greater than the magnitude of a right half plane (real) zero, then there will be a very large $S_o(s)$ peak leading to large undershoots.
- However, a major difference is that in the MIMO case the bound refers to a linear combination of sensitivity peaks (since $S_o(s)$ and $T_o(s)$ are matrices). This combination is determined by the directions associated with the NMP zero under consideration.

5.3.1 SISO Nyquist Criterion

- Originally developed as a method to determine stability without having to solve for closed-loop poles (difficult in 1932!).
- Nyquist still useful for several reasons:
  - Gain margin and phase margin—measures of stability and robustness—readily determined from plot.
  - Can be applied to systems with time delays (Routh cannot).
  - Modifications to controller frequency response which improve gain- and phase-margin may be readily observed from Nyquist map.
- We primarily use Nyquist to observe GM and PM. Indicators of robustness.

5.3.1 SISO Nyquist Criterion

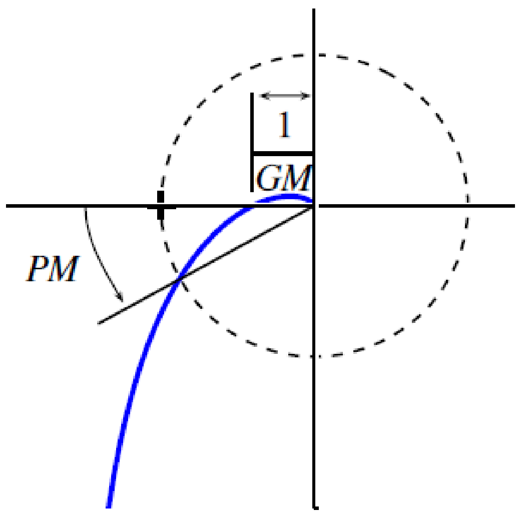
DEFINITION: The gain margin GM^+ is the minimum gain > 1 that results in an unstable closed-loop system.

DEFINITION: The downside gain margin GM^- is the maximum positive gain < 1 that results in an unstable closed-loop system.

DEFINITION: The phase margin PM is the minimum amount of phase shift added to $L(s)$ that results in an unstable closed-loop system.

- Phase margin is the amount of rotation required to cause instability ➠ Determined by intersection point between Nyquist map and unit circle.

- Many Nyquist plots are like this one. Increasing loop gain magnifies the plot.
- GM =1/(distance between origin and place where Nyquist map crosses real axis).
- If we increase gain, Nyquist map "stretches" and we may encircle -1.



5.3.1 Example SISO Nyquist Criterion

- Plant $G(s)$ with 2 unstable poles at $s = 1$

$$G(s) = \frac{g}{s^2 - 2s + 1}; \qquad K(s) = \frac{1500s - 100}{s^2 + 30s + 400}.$$

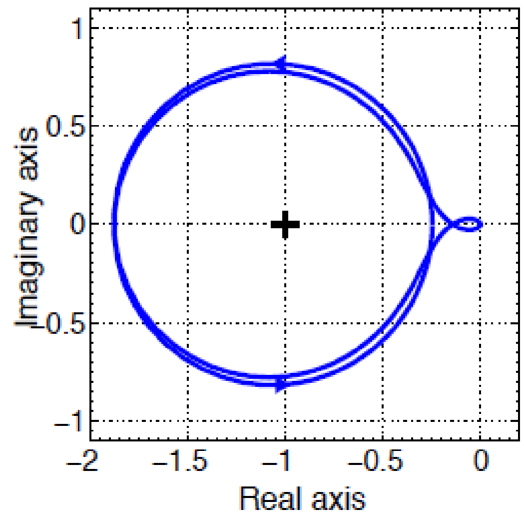
The gain g is uncertain but bounded $0.8 \leq g \leq 1.2$.

- The Nyquist map for this system is plotted (for $g = 1$).

- The map crosses the real axis at -0.25 and at -1.8.

- $GM^+ = 1/0.25 = 4.0.$
  $GM^- = 1/1.8 = 0.56.$

- The system is therefore "robustly stable."



5.3.2 Frequency Domain Analysis

- In the SISO case, the frequency domain gives valuable insights into the response of a closed loop to various inputs.
- Consider a MIMO system with m inputs and m outputs, having an $m \times m$ matrix transfer function $\mathbf{G}(s)$:

$$Y(s) = \mathbf{G}(s)U(s)$$

- We obtain the corresponding frequency response by setting $s = j\omega$. This leads to the question: *How can one define the* gain *(induced norm) of a MIMO system in the frequency domain ?*

5.3.2 Norms for MIMO Systems

Norms for MIMO systems: Given $\hat{G}(s)$ a multi-input multi-output system

2-Norm: This norm is defined as

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left[\hat{G}^*(j\omega)\hat{G}(j\omega) \right] d\omega \right)^{1/2}$$

∞-Norm: The $\mathcal{H}_\infty$ norm is defined as

$$\|\hat{G}\|_\infty = \sup_\omega \|\hat{G}(j\omega)\| = \sup_\omega \bar{\sigma}[\hat{G}(j\omega)]$$

Maximum Singular Value

$$= \sup \left\{ \frac{\|z\|_2}{\|w\|_2} \ : \ w \neq 0, \ w \in L_2[0,\infty) \right\}$$

Remark: The infinity norm has an important property (submultiplicative)

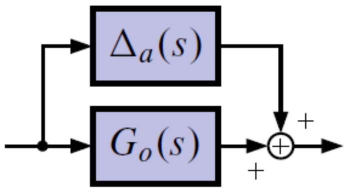
$$\|\hat{G}\hat{H}\|_\infty \le \|\hat{G}\|_\infty \, \|\hat{H}\|_\infty$$

5.3.2 Unstructured Uncertainty in MIMO System

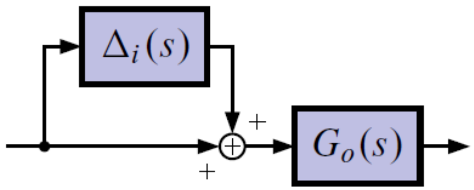
- GM and PM are special cases in modeling uncertainty in a plant.
- Many other types exist ➠ Here we consider perturbations to a nominal plant model.
- A perturbation is considered to be a bounded transfer function (with respect to its ∞-norm).
- This type of uncertainty is referred to as *unstructured* since no detailed model of perturbation is employed.

5.3.2 Types of Unstructured Uncertainty

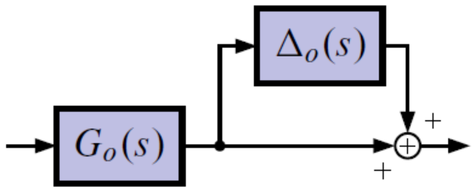
- Additive (unknown dynamics in parallel with the plant)
  $G(s) = G_o(s) + \Delta_a(s).$



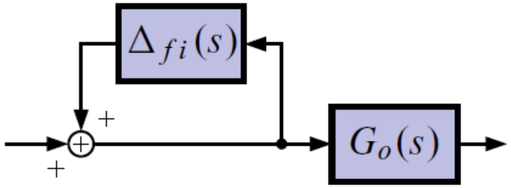
- Input multiplicative (unknown dynamics in series with the plant)
  $G(s) = G_o(s)[I + \Delta_i(s)].$



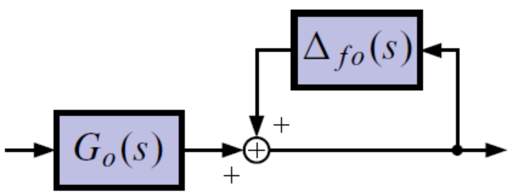
- Output multiplicative (unknown dynamics in series with the plant)
  $G(s) = [I + \Delta_o(s)]G_o(s).$



- Input feedback (uncertainty in gain/ phase/ pole locations of plant)
  $G(s) = G_o(s)[I - \Delta_{fi}(s)]^{-1}.$



- Output feedback (uncertainty in gain/ phase/ pole locations of plant)
  $G(s) = [I - \Delta_{fo}(s)]^{-1}G_o(s).$



5.3.2 Modelling Unstructured Uncertainty

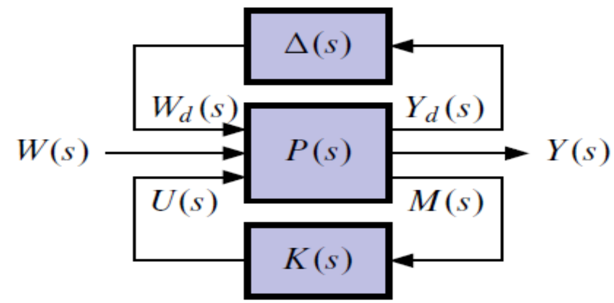
- Analysis may be performed when model perturbations are bounded

$$\bar{\sigma}\{\Delta'\} \leq \Delta_{\max}(j\omega)$$

where $\bar{\sigma}$ is the maximum singular value and Δ' is any of perturbation types considered. which is stable

- The bound $\Delta_{\max}$ is generally frequency-dependent, so uncertainty may vary over frequency.

- All the unstructured uncertainty models may be analyzed in a similar manner by placing them in a common framework.



- The perturbation is normalized so $\|\Delta(j\omega)\|_\infty \leq 1$ by defining

$$\Delta(j\omega) = \frac{1}{\Delta_{\max}(j\omega)}\Delta'(j\omega)$$

and

$$\begin{bmatrix} Y_d(s) \\ \hline Y(s) \\ \hline M(s) \end{bmatrix} = \left[\begin{array}{c|c|c} P_{y_d w_d}(s) & P_{y_d w}(s) & P_{y_d u}(s) \\ \hline P_{y w_d}(s) & P_{y w}(s) & P_{y u}(s) \\ \hline P_{m w_d}(s) & P_{m w}(s) & P_{m u}(s) \end{array}\right] \begin{bmatrix} W_d(s) \\ \hline W(s) \\ \hline U(s) \end{bmatrix} = P(s) \begin{bmatrix} W_d(s) \\ \hline W(s) \\ \hline U(s) \end{bmatrix}.$$

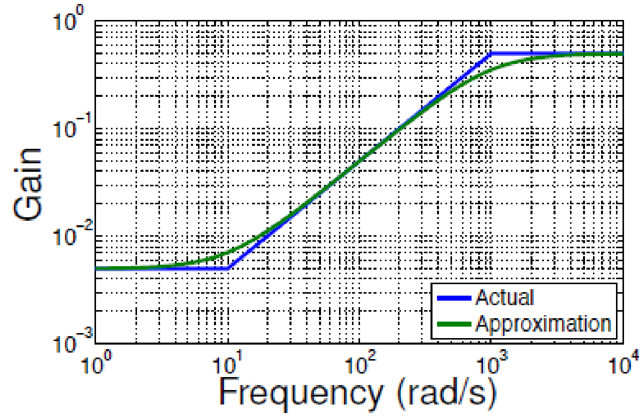
- The "plant" P(s) is generally a <u>combination of Go(s) and the plant uncertainties and 'weights' added together (see following example)</u>

5.3.2 Example Unstructured Input Multiplicative Uncertainty

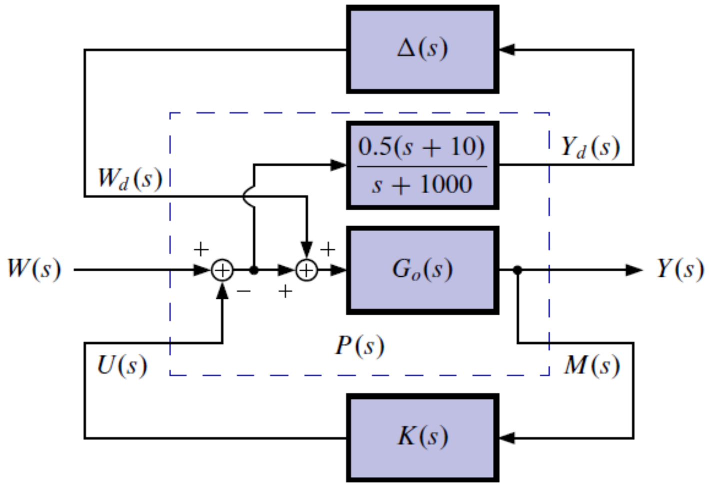
EXAMPLE: We consider a plant model to have input multiplicative uncertainty. The nominal model is accurate to within about $0.5\,\%$ at frequencies below $10\,\mathrm{rad\,s^{-1}}$ but is inaccurate to within about $50\,\%$ at frequencies above $1000\,\mathrm{rad\,s^{-1}}$. The accuracy should transition between these two extremes at intermediate frequencies.

- We can model the uncertainty as a first-order transfer function with a zero at $10\,\mathrm{rad\,s^{-1}}$ and a pole at $1000\,\mathrm{rad\,s^{-1}}$.

$$\Delta'_{\max}(j\omega) = 0.5\frac{(j\omega+10)}{(j\omega+1000)}.$$



- The uncertainty is coupled into standard form as:

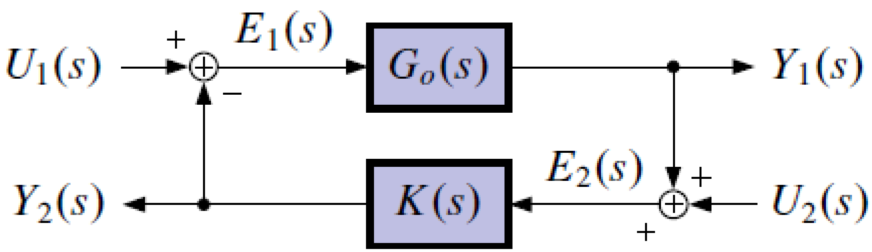


- From the diagram we see

$$P(s) = \left[\begin{array}{c:c:c} 0 & \dfrac{0.5(s+10)}{s+1000} & -\dfrac{0.5(s+10)}{s+1000} \\ \hdashline G_o(s) & G_o(s) & -G_o(s) \\ \hdashline G_o(s) & G_o(s) & -G_o(s) \end{array}\right].$$

5.3.3 Nominal Stability

- Controller stabilizes nominal system $G_o(s)$.
- Will generalize notion of stability to internal stability.
- Consider the following diagram.



- Consider

$$\left[\begin{array}{c} E_1(s) \\ \hdashline E_2(s) \end{array}\right] = \left[\begin{array}{c:c} (I+KG_o)^{-1} & -(I+KG_o)^{-1}K \\ \hdashline (I+G_oK)^{-1}G_o & (I+G_oK)^{-1} \end{array}\right] \left[\begin{array}{c} U_1(s) \\ \hdashline U_2(s) \end{array}\right],$$

which are the transfer functions from the inputs to the "errors". (equivalent to transfer functions from inputs to outputs).

- System is internally stable if all four transfer functions are stable.

5.3.3 Robust Stability Against Unstructured Uncertainty - Analysis

- We want to determine whether our controlled system is stable for all admissible plant perturbations $\Delta \in \bar{\Delta}$.
- Start with the 3-input 3-output standard form

$$\begin{bmatrix} Y_d(s) \\ \hdashline Y(s) \\ \hdashline M(s) \end{bmatrix} = \left[\begin{array}{c:c:c} P_{y_d w_d}(s) & P_{y_d w}(s) & P_{y_d u}(s) \\ \hdashline P_{y w_d}(s) & P_{yw}(s) & P_{yu}(s) \\ \hdashline P_{m w_d}(s) & P_{mw}(s) & P_{mu}(s) \end{array}\right] \begin{bmatrix} W_d(s) \\ \hdashline W(s) \\ \hdashline U(s) \end{bmatrix} = P(s) \begin{bmatrix} W_d(s) \\ \hdashline W(s) \\ \hdashline U(s) \end{bmatrix}.$$

- When a controller is added to the system,

$$\begin{aligned} U(s) &= K(s)M(s) \\ &= K(s)P_{mw_d}(s)W_d(s) + K(s)P_{mw}(s)W(s) + K(s)P_{mu}(s)U(s) \\ &= \{I - K(s)P_{mu}(s)\}^{-1}K(s)P_{mw_d}(s)W_d(s) + \\ &\qquad \{I - K(s)P_{mu}(s)\}^{-1}K(s)P_{mw}(s)W(s). \end{aligned}$$

5.3.3 Robust Stability Against Unstructured Uncertainty - Analysis

- We get the following closed-loop system (dropping "s" for clarity):

$$\left[\begin{array}{c} Y_d \\ \hdashline Y \end{array}\right] = \left[\begin{array}{c:c} N_{y_d w_d} & N_{y_d w} \\ \hdashline N_{y w_d} & N_{yw} \end{array}\right] \left[\begin{array}{c} W_d \\ \hdashline W \end{array}\right],$$

where

$$N_{y_d w_d} = P_{y_d w_d} + P_{y_d u}\{I - KP_{mu}\}^{-1} K P_{m w_d}$$

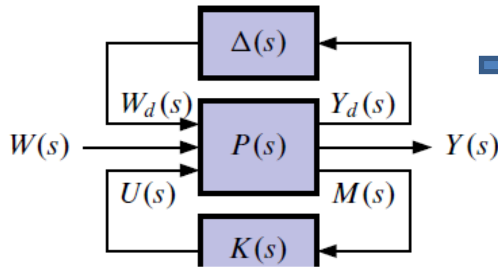
$$N_{y_d w} = P_{y_d w} + P_{y_d u}\{I - KP_{mu}\}^{-1} K P_{mw}$$

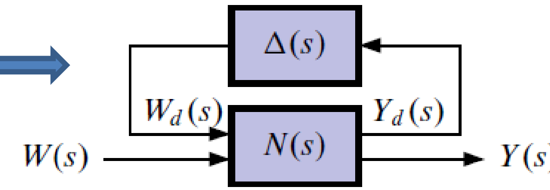
$$N_{y w_d} = P_{y w_d} + P_{yu}\{I - KP_{mu}\}^{-1} K P_{m w_d}$$

$$N_{yw} = P_{yw} + P_{yu}\{I - KP_{mu}\}^{-1} K P_{mw}.$$

5.3.3 Robust Stability and Linear Fractional Transformation - Analysis

- The modified standard form, incorporating the controller dynamics, is:





Upper LFT

$$Y_d = N_{y_d w_d} W_d + N_{y_d w} W$$

$$Y = N_{yw_d} W_d + N_{yw} W$$

$$W_d = \Delta Y_d$$

- The nominal system is assumed to be stable.
- The perturbation is also assumed to be stable.
- The combined system is then stable iff the feedback loop around the perturbation is internally stable.

After closing the upper loop to get TF from *w* to *y*, from upper LFT

$$T_{yw} = N_{yw}(s) + N_{yw_d}(s)\Delta(s)\left[I - N_{y_d w_d}(s)\Delta(s)\right]^{-1} N_{y_d w}(s)$$

- This is stable if the inverse is finite. Since $\|\Delta\|_\infty \le 1$ then the condition for stability is (Refer to Section 5.4.1 for proof)

$$\|N_{y_d w_d}\|_\infty < 1.$$

CONCLUSION: System is robustly stable iff

$$\|N_{y_d w_d}\|_\infty = \sup_\omega \{\bar{\sigma}[N_{y_d w_d}(j\omega)]\} < 1.$$

5.3.3 Example Robust Stability to Unstructured Additive Uncertainty - Analysis

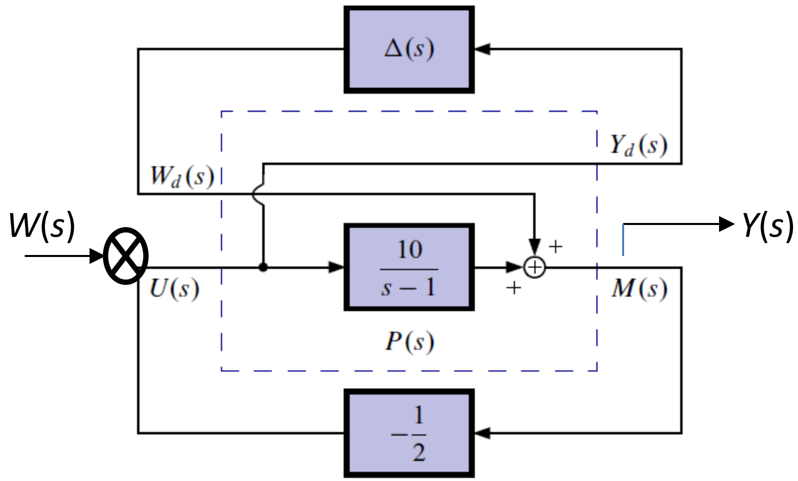
EXAMPLE: We are given the nominal plant

$$G_o(s) = \frac{10}{s-1},$$

with unstructured additive uncertainty

$$\|\Delta(s)\|_\infty \le 1,$$

and controller $K(s) = -\frac{1}{2}$.



From the diagram, we see

$$P = \begin{bmatrix} P_{y_d w_d} & P_{y_d w} & P_{y_d u} \\ P_{y w_d} & P_{yw} & P_{yu} \\ P_{m w_d} & P_{mw} & P_{mu} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & \frac{10}{s-1} & \frac{10}{s-1} \\ 1 & \frac{10}{s-1} & \frac{10}{s-1} \end{bmatrix}$$

- We can compute

$$\begin{aligned} N_{y_d w_d}(s) &= P_{y_d w_d} + P_{y_d u}(I - K P_{mu})^{-1} K P_{m w_d} \\ &= 0 + (1)\left(1 - \left(-\frac{1}{2}\right)\left(\frac{10}{s-1}\right)\right)^{-1}\left(-\frac{1}{2}\right)(1) \\ &= \left(\frac{s+4}{s-1}\right)^{-1}\left(-\frac{1}{2}\right) = \frac{-\frac{1}{2}(s-1)}{s+4}. \end{aligned}$$

- The nominal closed-loop system has a pole at $s = -4$ (stable). Is the system robustly stable?
- Compute

$$\bar{\sigma}\{N_{y_d w_d}(j\omega)\} = |N_{y_d w_d}(j\omega)| = \frac{1}{2}\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+16}}.$$

The ∞-norm is (the max is at $\omega \to \infty$)

$$\|N_{y_d w_d}(j\omega)\|_\infty = \sup_\omega \frac{1}{2}\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+16}} = \frac{1}{2}.$$

- Therefore the system robustly stable.

5.3.3 LFT and Robust Stability Against Unstructured Uncertainty - Analysis

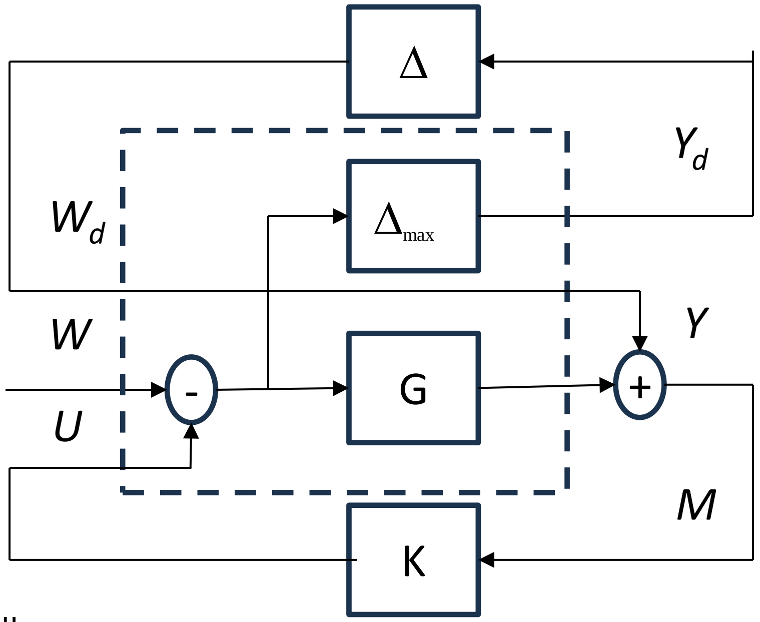
$$\sigma(\Delta') \le \Delta_{\max}(j\omega) \text{ so that } \|\Delta(j\omega)\|_\infty \le 1 \text{ where } \Delta(j\omega) = \frac{1}{\Delta_{\max}(j\omega)}\Delta'(j\omega)$$

where Δ' is any perturbation which is stable. Robust Stability requires

$$\left\| N_{y_d w_d} = P_{y_d w_d} + P_{y_d u}\left[I - KP_{mu}\right]^{-1} KP_{mw_d} \right\|_\infty \le 1$$

- Unstructured Additive Uncertainty

$$\begin{bmatrix} Y_d(s) \\ Y(s) \\ M(s) \end{bmatrix} = \begin{bmatrix} 0 & \Delta_{\max} & -\Delta_{\max} \\ I & G & -G \\ I & G & -G \end{bmatrix} \begin{bmatrix} W_d(s) \\ W(s) \\ U(s) \end{bmatrix}$$



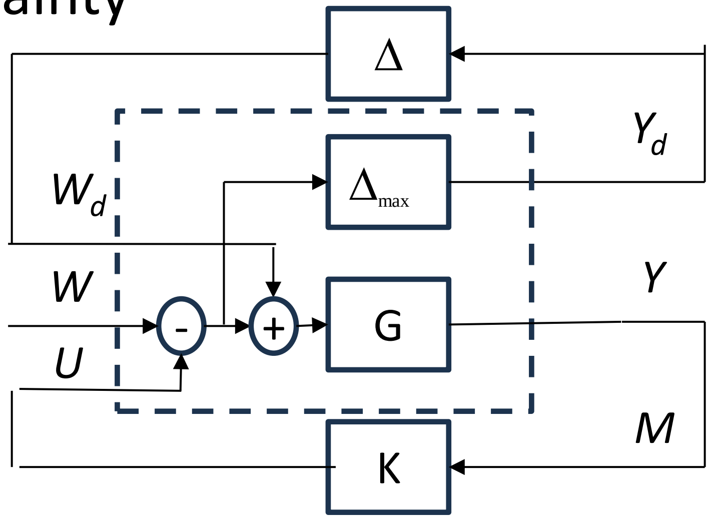
Show for robust stability

$$\left\| \Delta_{\max}\left[I + KG\right]^{-1} K \right\|_\infty = \left\| \Delta_{\max} K\left[I + GK\right]^{-1} \right\|_\infty \le 1$$

5.3.3 LFT and Robust Stability Against Unstructured Uncertainty - Analysis

- Unstructured I/P Multiplicative Uncertainty

$$\begin{bmatrix} Y_d(s) \\ Y(s) \\ M(s) \end{bmatrix} = \begin{bmatrix} 0 & \Delta_{\max} & -\Delta_{\max} \\ G & G & -G \\ G & G & -G \end{bmatrix} \begin{bmatrix} W_d(s) \\ W(s) \\ U(s) \end{bmatrix}$$

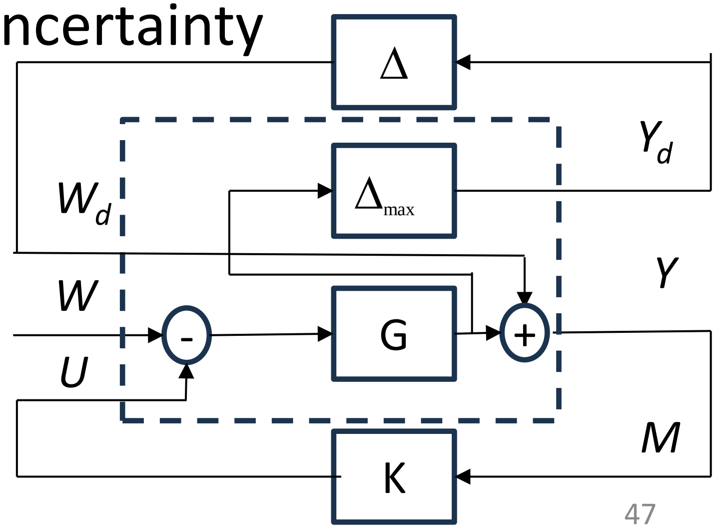


Show for robust stability

$$\left\| \Delta_{\max} [I + KG]^{-1} KG \right\|_\infty = \left\| \Delta_{\max} T_i \right\|_\infty \le 1$$

- Unstructured Output Multiplicative Uncertainty

$$\begin{bmatrix} Y_d(s) \\ Y(s) \\ M(s) \end{bmatrix} = \begin{bmatrix} 0 & \Delta_{\max} G & -\Delta_{\max} G \\ I & G & -G \\ I & G & -G \end{bmatrix} \begin{bmatrix} W_d(s) \\ W(s) \\ U(s) \end{bmatrix}$$



Show for robust stability

$$\left\| \Delta_{\max} G [I + KG]^{-1} K \right\|_\infty = \left\| \Delta_{\max} [I + GK]^{-1} GK \right\|_\infty \le 1$$

5.4.1 Structured Uncertainty (Optional)

- Sometimes, more information is available ➠ New constraints on uncertainty add "structure" to the set of admissible perturbations.
  - Plant subject to multiple perturbations.
  - Plant has multiple uncertain parameters.
  - Multiple unstructured but independent uncertainties.
- Structured uncertainty model similar to before

$$\Delta(s) = \left[\begin{array}{c:c:c:c} \Delta_1(s) & 0 & \cdots & 0 \\ \hdashline 0 & \Delta_2(s) & & 0 \\ \hdashline \vdots & & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & \Delta_n(s) \end{array}\right]$$

  is block diagonal.
- An individual $\Delta_k(s)$ may represent an uncertain parameter (scalar) or an unstructured uncertainty.
- All blocks have $\|\Delta_k\|_\infty \leq 1$.

5.4.1 Example Structured Uncertainty (Optional)

EXAMPLE: Consider the transfer function

$$G(s) = \frac{1}{(s+p_1)(s+p_2)}.$$

- The two poles are uncertain, as given by

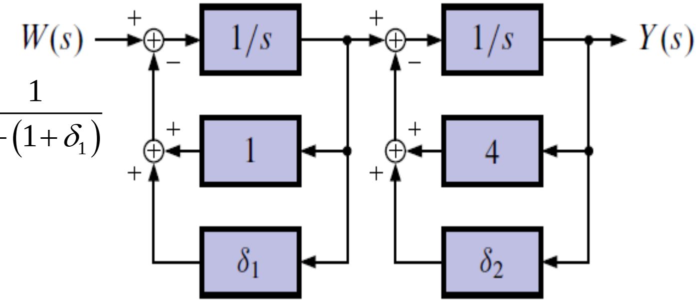
$$p_1 \in [0.9, 1.1]; \quad p_2 \in [3, 5].$$

- We can model the poles as nominal values plus perturbation

$$p_1 = 1 + \delta_1; \quad p_2 = 4 + \delta_2$$

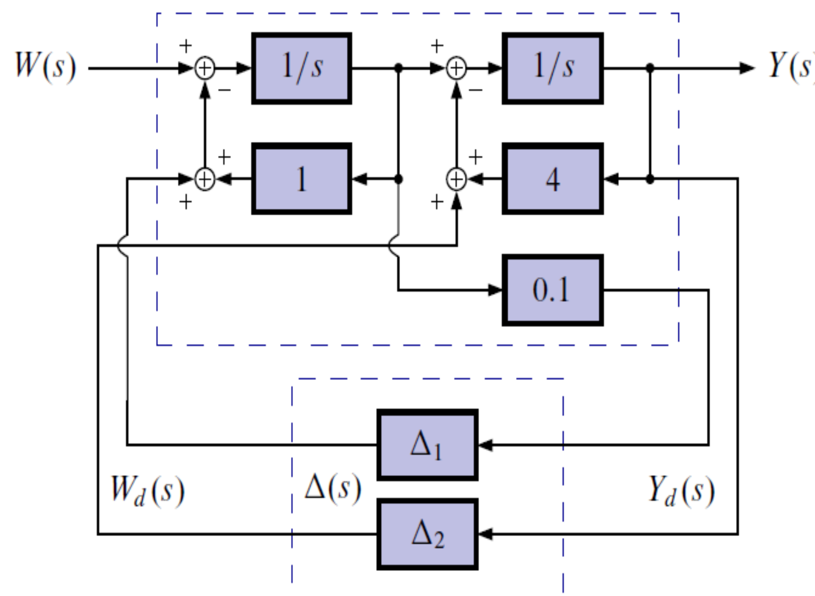
where $-0.1 \le \delta_1 \le 0.1$ and $-1 \le \delta_2 \le 1$. We can place the perturbations in a feedback loop around the nominal plant as

$$= \frac{\frac{1}{s}}{1+\frac{1}{s}(1+\delta_1)} = \frac{1}{s+(1+\delta_1)}$$



- The perturbations can be normalized and the system put into standard form such that

$$\|\Delta_1\|_\infty \le 1; \quad \|\Delta_2\|_\infty \le 1; \quad \|\Delta\|_\infty \le 1.$$



5.4.1 Robust Stability Against Structured Uncertainty (Optional)

- When the uncertainty has structure, the robust stability test $\|N_{y_d w_d}\|_\infty < 1$ is overly conservative.
- It assumes that Δ may have any structure subject to $\|\Delta\|_\infty \leq 1$. A prediction of instability may be based on a specific Δ that is not allowed given its structure.
- When there is structure, we revert back to the internal stability test and notice that the closed-loop system becomes unstable for Δ such that

$$\det\{I - N_{y_d w_d}(s)\Delta(s)\} = 0.$$

- To determine robust stability, we solve for the "smallest" delta that makes this true

$$\inf_{\omega}\left\{\min_{\Delta(j\omega)\in\bar{\Delta}} \{\bar{\sigma}[\Delta(j\omega)] : \det\{I - N_{y_d w_d}(j\omega)\Delta(j\omega) = 0\}\right\}.$$

- If this value is greater than 1 (the maximum size of Δ) then the system is robustly stable.
- Easier to solve if we define

$$\mu_{\bar{\Delta}}(N) = \frac{1}{\min_{\Delta\in\bar{\Delta}}\{\bar{\sigma}[\Delta] : \det\{I - N_{y_d w_d}\Delta\} = 0\}}.$$

- $\mu_{\bar{\Delta}}$ is called the "Structured Singular Value".
- Then, the system is robustly stable iff

$$\sup_{\omega}\{\mu_{\bar{\Delta}}[N_{y_d w_d}(j\omega)]\} < 1.$$

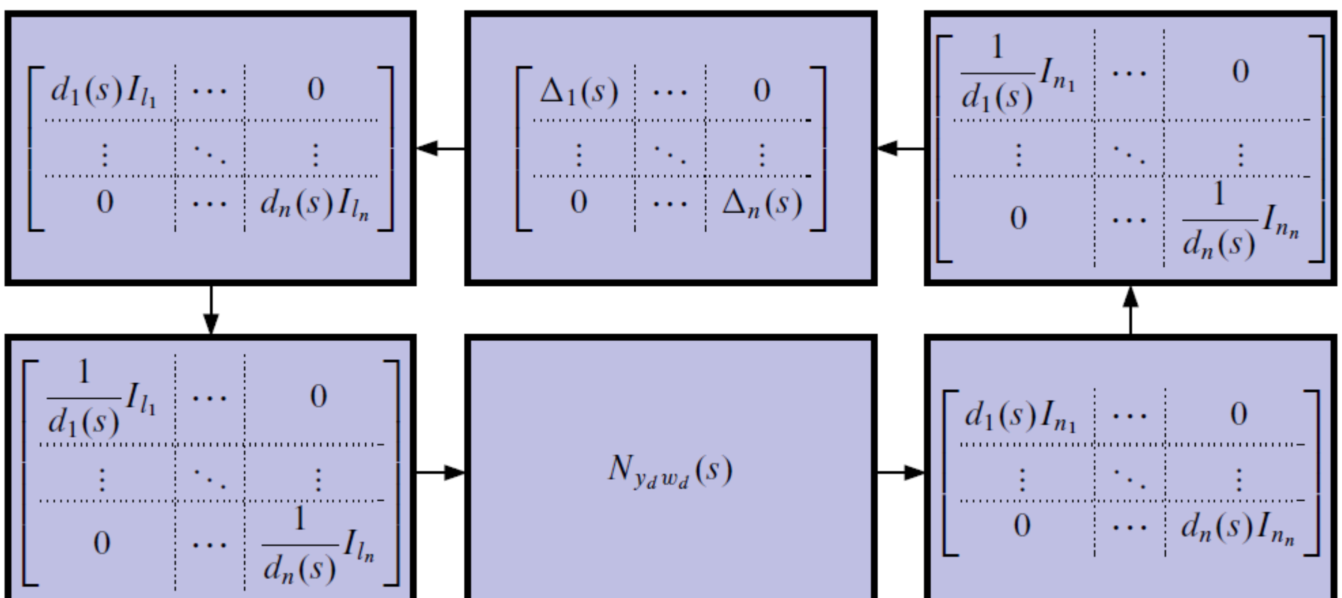
5.4.2 Structured Singular Value (Optional)

<u>*Computing* $\mu_{\bar{\Delta}}(N_{y_d w_d})$.</u>

- The bad news: There is no closed-form solution or numeric algorithm to compute $\mu_{\bar{\Delta}}(N_{y_d w_d})$ in the general case.
- The better news: We can often find quite good bounds on it. In particular, we know that

$$\mu_{\bar{\Delta}}(N_{y_d w_d}) \leq \bar{\sigma}(N_{y_d w_d}).$$

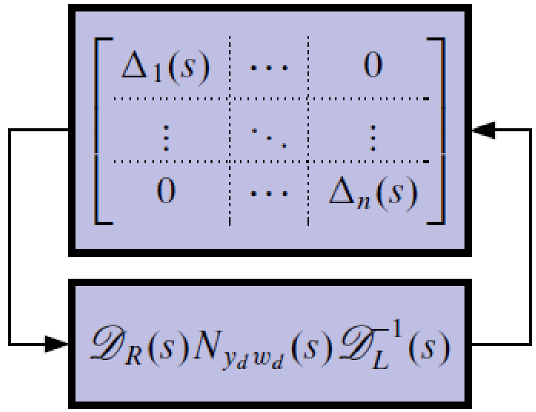
- This should be obvious since the structured singular value is a special case of the regular singular value.
- To make this bound better, consider:

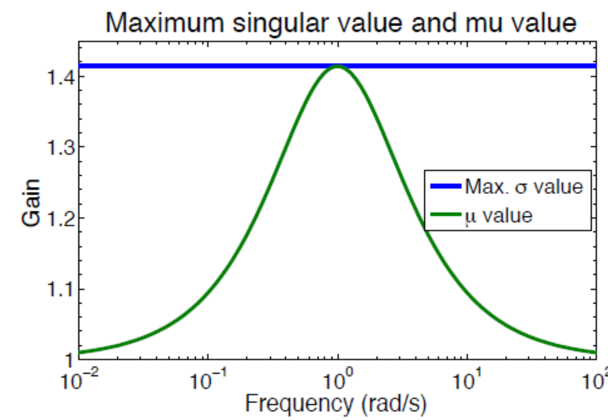


$D_L^{-1}(s) =$ $= D_R(s)$

5.4.2 Computing Structured Singular Value (Optional)

or





- We see that
$$\mu_{\bar{\Delta}}(N(s)) = \mu_{\bar{\Delta}}[\mathscr{D}_R(s)N_{y_d w_d}(s)\mathscr{D}_L^{-1}(s)] \leq \bar{\sigma}[\mathscr{D}_R(s)N_{y_d w_d}(s)\mathscr{D}_L^{-1}(s)],$$
for any $\mathscr{D}_R$ and $\mathscr{D}_L$. (called *D*-scaling)

- In particular,
$$\mu_{\bar{\Delta}}(N_{y_d w_d}(s)) \leq \min_{\substack{\{d_1,\ldots d_n\} \\ d_i \in (0,\infty)}} \bar{\sigma}[\mathscr{D}_R(s)N_{y_d w_d}(s)\mathscr{D}_L^{-1}(s)],$$

- This minimization is a convex-optimization problem for which a unique minima exists, and very efficient algorithms exist to solve it. The final bound is usually very close to the true structured-singular value.

5.4.3 Robust Performance against Unstructured Uncertainty – Analysis (Optional)

- Robust performance is defined in different ways than we are accustomed to.
- Bounds are placed on certain performance variables, such as steady-state error, disturbance in the output, control effort.
- These bounds are scaled in standard form, such that the transfer function between the reference- and disturbance inputs and the performance outputs must be bounded by infinity-norm 1:

$$\|T_{yw}\|_\infty < 1,$$

where T_{yw} is the closed-loop transfer function between the inputs and performance variables.

- The closed loop transfer function is

$$T_{yw} = N_{yw}(s) + N_{yw_d}(s)\Delta(s)\left[I - N_{y_d w_d}(s)\Delta(s)\right]^{-1} N_{y_d w}(s)$$

5.4.3 Robust Performance using SSV (Optional)

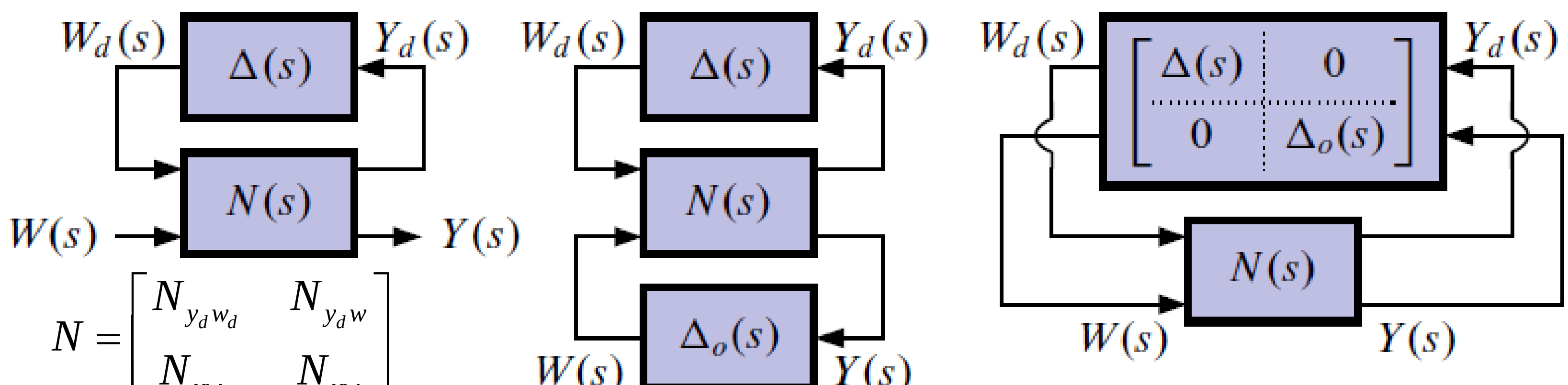
For Robust Performance, we must have robust stability

and $\left\|T_{yw}\right\|_{\infty} \leq 1$ for all stable $\left\|\Delta\right\|_{\infty} \leq 1$

- The closed loop transfer function is

$$T_{yw} = N_{yw}(s) + N_{yw_d}(s)\Delta(s)\left[I - N_{y_d w_d}(s)\Delta(s)\right]^{-1} N_{y_d w}(s)$$

- We can visualize this as a structured-singular-value stability problem, where we add an additional “uncertainty” block.



$$N = \begin{bmatrix} N_{y_d w_d} & N_{y_d w} \\ N_{y w_d} & N_{yw} \end{bmatrix}$$

Theorem (Robust Performance): The two conditions $\left\|N_{y_d w_d}\right\|_{\infty} \leq 1$ and $\left\|T_{yw}\right\|_{\infty} \leq 1$ for all $\left\|\Delta\right\|_{\infty} \leq 1$ are equivalent to $\mu_{\hat{\Delta}}(N) \leq 1$ for all $\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$, $\left\|\Delta_0\right\|_{\infty} \leq 1$, $\left\|\Delta\right\|_{\infty} \leq 1$

5.4.5 Example Robust Performance (Optional)

EXAMPLE: Consider a plant with uncertain pole location:

$$G(s) = \frac{1}{s+1+\delta}; \quad \delta \in \{-0.2, 0.2\}.$$

- A proportional-gain controller K is to be used for this system.
- The reference input is bandlimited to less than $10\,\mathrm{rad\,s^{-1}}$. Tracking error should be less than 0.1.
- We can then bound the closed-loop transfer function from referenc input to error output.

$$|T_{yw}(j\omega)| \le \begin{cases} 0.1, & \omega \le 10; \\ \infty, & \omega > 10. \end{cases}$$

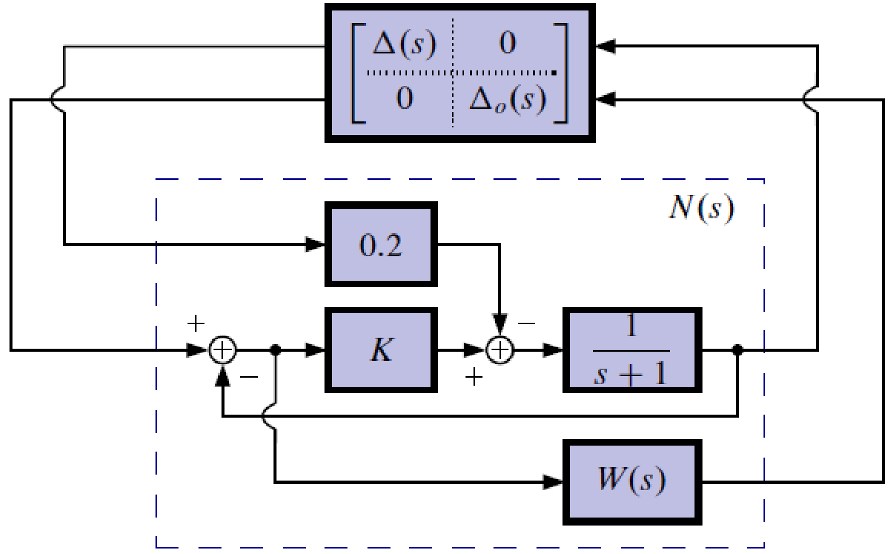
- The ∞ indicates that error in frequencies above $10\,\mathrm{rad\,s^{-1}}$ is unimportant.
- We normalize the performance goal using a weighting function

$$W(j\omega) = \begin{cases} 10, & \omega \le 10; \\ 0, & \omega > 10. \end{cases}$$

- A low-order rational approximation of this weighting function is

$$W(j\omega) = \frac{150}{j\omega + 10}.$$

5.4.5 Example Robust Performance (Optional)

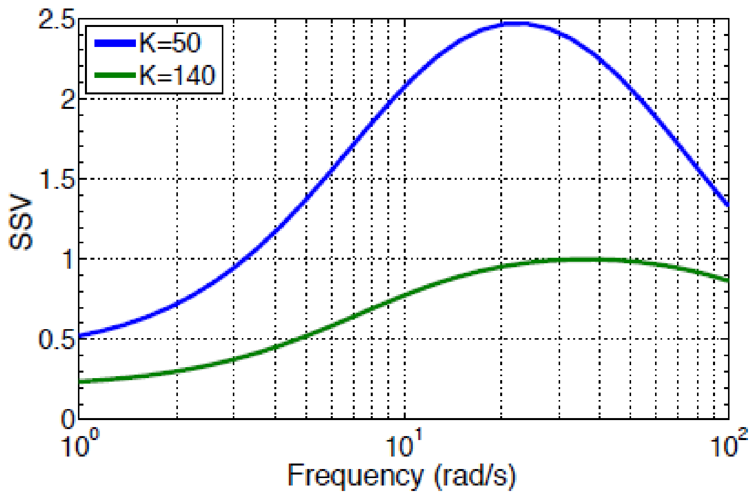


- Robust performance can be analyzed by using the SSV by appending the weighting function to the plant and adding the performance block Δ_o. The transfer function of the system is

$$N(s) = \begin{bmatrix} \dfrac{-0.2}{s+1+K} & \dfrac{K}{s+1+K} \\ \dfrac{30}{(s+1+K)(s+10)} & \dfrac{150(s+1)}{(s+1+K)(s+10)} \end{bmatrix}.$$

5.4.5 Example Robust Performance (Optional)

- Robust performance is tested by generating the SSV of $N(j\omega)$. The system is stable and meets the performance specifications for all admissible perturbations if SSV is less than 1 for all frequencies.



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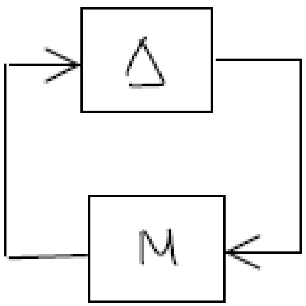
5.4.4 Proof of Robust Stability (Optional)

Assume $N(s)$ and $\Delta(s)$ are stable. Denote $N_{y_d w_d} = M$. Since every term in $T_{yw} = N_{yw}(s) + N_{yw_d}(s)\Delta(s)\left[I - M(s)\Delta(s)\right]^{-1} N_{y_d w}(s)$ other than $(I - M\Delta)^{-1}$ is known to be stable, we shall have stability of T_{yw}, and hence guaranteed stability of the actual closed-loop system if $(I - M\Delta)^{-1}$ is stable for all allowed $\Delta(s)$.



Theorem ('Unstructured" Small-Gain Theorem) Define the set of stable matrices $\Delta \square \{\Delta \mid \|\Delta\|_\infty \le 1\}$. If M is stable, then $(I - M\Delta)^{-1}$ is stable if and only if $\|M\|_\infty \le 1$.

Proof: Sufficiency. We need to show that if $\|M\|_\infty < 1$ then $(I - M\Delta)^{-1}$ has no poles in the closed right half plane for any $\|\Delta\|_\infty \le 1$, or equivalently that $I - M\Delta$ has no zeros there. For arbitrary $x \neq 0$ and any s_+ in the closed right half plane (CRHP), and since M and Δ are both stable and thus well-defined for any s_+

$$\|I - M(s_+)\Delta(s_+)\| \ge \|x\|_2 - \|M(s_+)\Delta(s_+)x\|_2 \qquad \text{(Triangular Inequality)}$$

$$\ge \|x\|_2 - \sigma_{\max}\left[M(s_+)\Delta(s_+)\right]\|x\|_2 \qquad \text{(Singular Value definition)}$$

$$\ge \|x\|_2 - \|M\|_\infty \|\Delta\|_\infty \|x\|_2 \qquad \text{(Maximum Modulus Theorem of Complex Variables)}$$

$$> 0 \qquad (\|M\|_\infty < 1,\ \|\Delta\|_\infty \le 1)$$

Hence, $I - M\Delta$ is non-singular and therefore has no zeros in the CRHP.

5.4.4 Proof of Robust Stability (Optional)

Necessity: We will show that if $\sigma_{\max}\left[M\left(j\omega_0\right)\right]>1$ for some ω_0, we can construct a $\|\Delta\|_\infty<1$ such that T_{yw} is unstable. Take SVD of $M\left(j\omega_0\right)$

$$M(j\omega_0) = U\Sigma V' = U\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} V'.$$

Since $\sigma_{max}[M(j\omega_0)] > 1$, $\sigma_1 > 1$. Then $\Delta(j\omega_0)$ can be constructed as:

$$\Delta(j\omega_0) = V\begin{bmatrix} 1/\sigma_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} U'$$

Clearly, $\sigma_{max}\Delta(j\omega_0) < 1$. We then have

$$(I - M\Delta)^{-1}(j\omega_0) = \quad I - U\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} V'V\begin{bmatrix} 1/\sigma_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} U'$$

$$= U\left[I - \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}\right] U'$$

$$= U\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U'$$

which is singular

5.4.4 Proof of Robust Stability (Optional)

which is singular. Only one problem remains, which is that $\Delta(s)$ must be legitimate as the transfer function of a *stable system*, evaluating to the proper value at $s = j\omega_0$, and having its maximum singular value over all ω bounded below 1. The value of the destabilizing perturbation at ω_0 is given by

$$\Delta_0(j\omega_0) = \frac{1}{\sigma_{max}(M(j\omega_0))} v_1 u_1'$$

Write the vectors v_1 and u_1' as

$$v_1 = \begin{bmatrix} \pm|a_1|e^{j\theta_1} \\ \pm|a_2|e^{j\theta_2} \\ \vdots \\ \pm|a_n|e^{j\theta_n} \end{bmatrix}, \quad u_1' = \begin{bmatrix} \pm|b_1|e^{j\phi_1} & \pm|b_2|e^{j\phi_2} & \cdots & \pm|b_n|e^{j\phi_n} \end{bmatrix},$$

where θ_i and ϕ_i belong to the interval $[0, \pi)$. Note that we used $\pm$ in the representation of the vectors v_1 and u_1' so that we can restrict the angles θ_i and ϕ_i to the interval $[0, \pi)$. Now we can choose the nonnegative constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that the phase of the function $\frac{s-\alpha_i}{s+\alpha_i}$ at $s = j\omega_0$ is θ_i, and the phase of the function $\frac{s-\beta_i}{s+\beta_i}$ at $s = j\omega_0$ is ϕ_i. Now the destabilizing $\Delta(s)$ is given by

$$\Delta(s) = \frac{1}{\sigma_{max}(M(j\omega_0))} g(s) h^T(s)$$

where

$$g(s) = \begin{bmatrix} \pm|a_1|\frac{s-\alpha_1}{s+\alpha_1} \\ \pm|a_2|\frac{s-\alpha_2}{s+\alpha_2} \\ \vdots \\ \pm|a_n|\frac{s-\alpha_n}{s+\alpha_n} \end{bmatrix}, \quad h(s) = \begin{bmatrix} \pm|b_1|\frac{s-\beta_1}{s+\beta_1} \\ \pm|b_2|\frac{s-\beta_2}{s+\beta_2} \\ \vdots \\ \pm|b_n|\frac{s-\beta_n}{s+\beta_n} \end{bmatrix}.$$

5.4.5 Proof of Robust Performance using SSV (Optional)

Proof (Main Loop Theorem): $N_{y_d w_d}(s) = M(s)$

Robust Stability implies $\|M(s)\|_\infty \le 1 \Leftrightarrow \underline{\det(I - M\Delta) \ne 0}$ (so that $(I-M\Delta)^{-1}$ exist) (5.4.1)

Robust Performance implies $\left\|N_{yw}(s) + N_{yw_d}(s)\Delta(s)\left[I - M(s)\Delta(s)\right]^{-1} N_{y_d w}(s)\right\|_\infty \le 1$ for all $\|\Delta(s)\|_\infty \le 1$ of $\dim\left(N_{y_d w_d}^T\right)$

Note that the condition

$$\left\|N_{yw} + N_{yw_d}\Delta\left[I - M\Delta\right]^{-1} N_{y_d w}\right\|_\infty = \|S(\Delta, N)\|_\infty \le 1 \Leftrightarrow \underline{\det\left(I - S(\Delta, N)\Delta_0\right) \ne 0} \text{ for all } \|\Delta_0(s)\|_\infty \le 1 \text{ of } \dim\left(N_{yw}^T\right) \quad (5.4.2)$$

Now from Equations (5.4.1) and (5.4.2), we have

$$\det(I - M\Delta) \ne 0 \text{ and } \det\left(I - S(\Delta, N)\Delta_0\right) \ne 0$$

$$\Leftrightarrow \det\begin{pmatrix} I - M\Delta & -N_{y_d w}\Delta_0 \\ -N_{yw_d}\Delta & I - N_{yw}\Delta_0 \end{pmatrix} \ne 0$$

$$\Leftrightarrow \det\left(I - \begin{bmatrix} M & N_{y_d w} \\ N_{yw_d} & N_{yw} \end{bmatrix} \underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}}_{\hat{\Delta}}\right) \ne 0$$

Note that $\hat{\Delta}$ is structured!

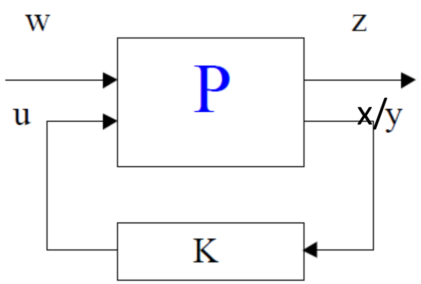
$$\text{(Note : } \det\left(I - S(\Delta, N)\Delta_0\right) = \det\left(I - \left[N_{yw} + N_{yw_d}\Delta\left[I - M\Delta\right]^{-1} N_{y_d w}\right]\Delta_0\right)$$

$$= \det\left(\left[I - N_{yw}\Delta_0\right] - \left[N_{yw_d}\Delta\left[I - M\Delta\right]^{-1} N_{y_d w}\Delta_0\right]\right) = \det\begin{pmatrix} I - M\Delta & -N_{y_d w}\Delta_0 \\ -N_{yw_d}\Delta & I - N_{yw}\Delta_0 \end{pmatrix}$$

from Schur formula for determinant)

5.5.1 Synthesis – Modern Multivariable Controller

- Mainly MIMO State-Based Design
  - State Feedback and Estimator
    - Eigenstructure Assignment
    - Bass-Gura Formula
  - Optimization-Based Design
    - H_2/Linear Quadratic Regulator (LQR), Kalman Filter, Linear Quadratic Gaussian (LQG)
    - H_∞ Full Information, Partial Information
    - Model Predictive Control (Prof Ling's Lectures)
  - MIMO Controller Affine Q-Parametrization (Advanced)
  - Decoupling Design (Not Robust)



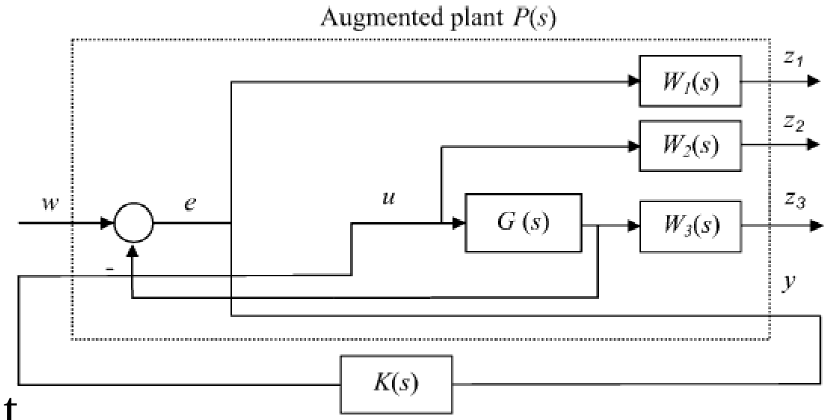
Standard Problem

5.5.2 Stability and Performance – H_∞ Mixed Sensitivity Controller Synthesis

$$Y(s) = \mathbf{T_o}(s)R(s) - \mathbf{T_o}(s)D_m(s) + \mathbf{S_o}(s)D_o(s) + \mathbf{S_{io}}(s)D_i(s)$$
$$U(s) = \mathbf{S_{uo}}(s)R(s) - \mathbf{S_{uo}}(s)D_m(s) - \mathbf{S_{uo}}(s)D_o(s) - \mathbf{S_{uo}}(s)\mathbf{G_o}(s)D_i(s)$$
$$E(s) = \mathbf{S_o}(s)R(s) - \mathbf{S_o}(s)D_m(s) - \mathbf{S_o}(s)D_o(s) - \mathbf{S_{io}}(s)D_i(s)$$

1) For performance (disturbance attenuation and steady state error specifications)

$$\bar{\sigma}\left(S_o\left(j\omega\right)\right) \le \left|W_1^{-1}\left(j\omega\right)\right| \quad , \qquad \forall \omega$$



2) Constraint on control input u

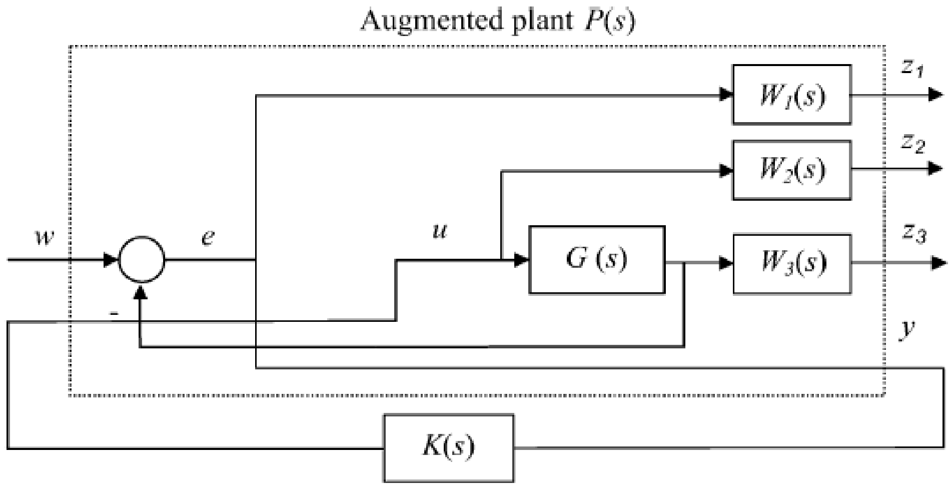
$$\bar{\sigma}\left(KS_o\left(j\omega\right)\right) \le \left|W_2^{-1}\left(j\omega\right)\right| \quad , \qquad \forall \omega$$

3) For stability (against unstructured output multiplicative uncertainty)

$$\bar{\sigma}\left(T_o\left(j\omega\right)\right) \le \left|W_3^{-1}\left(j\omega\right)\right| \quad , \qquad \forall \omega$$

- Note that all 3 weights on S_o, KS_o, T_o are used to shape 1 loop transfer function GK but they have to be judiciously selected to ensure that the fundamental identities $S_o + T_o = I$ and $T_o = GKS_o$ are NOT violated.

5.5.3 Stability and Performance – H_∞ Mixed Sensitivity Controller Synthesis



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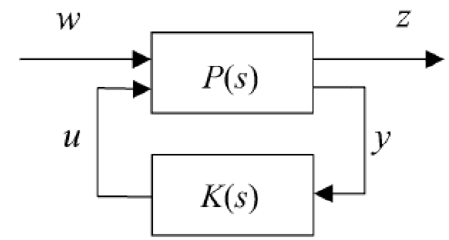

Lower LFT

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

$$u = K(s) y$$

$$\Rightarrow T_{zw} = P_{11} + P_{12} K \left(I - P_{22} K \right)^{-1} P_{21}$$

3 H_∞ Design Methods:

a) Direct H_∞ Controller Synthesis with Matlab commands
tss = augtf(G,W_1,W_2,W_3) and *hinfsyn(tss)*

b) H_∞ Mixed Sensitivity Controller Synthesis with Matlab commands
mixsyn(G,W_1,W_2,W_3)

b) H_∞ Loop Shaping Controller Synthesis with Matlab commands
mixsyn(G,W_1,W_2,W_3)