EE7204 Linear Systems

Lihua Xie

School of Electrical and Electronic Engineering, BLK S2
Nanyang Technological University, Singapore 639798
S2-B2c-94
https://personal.ntu.edu.sg/elhxie

August 8, 2024

Course Overview:

- 39 Lecturing hours
- Part I (21 hours)

– Chapter 1: Introduction to Dynamical Systems
– Chapter 2: Mathematical Representation of Linear Systems
– Chapter 3: Review of Basic Linear Algebra
– Chapter 4: Solution of State-space Equation
– Chapter 5: Stability
– Chapter 6: Controllability and Observability
– Chapter 7: Time-Domain Synthesis of Linear Feedback Systems

- Continuous Assessments (2 quizzes: 20%; 2 take home assignments: 20%)
- Final Exam: 60%

Textbook:

C. T. Chen, *Linear System Theory and Design,* 3rd Edition, Oxford University Press, 1999.

References:

P.J. Antsaklis and A.N. Michel, *A Linear Systems Primer,* Birkhauser, 2007.
W. L. Brogan, *Modern Control Theory,* Prentice-Hall, Englewood Cliffs, 1991.

1. Introduction to Dynamical Systems

1.1 Introduction

- Linear Systems theory is essential in various engineering and non-engineering fields. For example, in electrical engineering, it is fundamental and indispensable for studying control engineering, signal processing, communication, electrical circuits, etc.
- There are empirical and analytical methods for the study and design of physical systems. The empirical method relies on error-and-trial and is difficult to be applied to complex systems. In this case, analytical methods become indispensable.
- The analytical study of physical systems consists in: modeling, analysis, and design.
- There are various approaches to Linear Systems: State-space method, polynomial method, geometric approach, abstract algebraic approach, rational fractions, etc.

In this course, we shall focus on the State-space method and the polynomial method.

- State-space representation of linear systems is fundamental to the analysis and design of dynamical systems.
- Wide range of topics on state-space approach to linear systems include stability, controllability, observability, redundancy and minimality, finite and infinite zero structures, invertibility, and geometric subspace, etc.
- State-space approach facilitates the development of modern control and system theory.
- Though most practical systems are nonlinear in nature, their behavior under certain operating conditions can be well approximated by linear systems (linearized) for which there are various tools for analysis and design.
- The idea of feedback which uses the current and past system information to influence system behavior lies in the heart of control engineering.

1.2 Basics of Dynamical Systems

- Systems under study have some inputs (causes) and outputs (effects). When an input is applied, an unique output (response) can be measured.

- The input-output relationship of a dynamic system is governed by differential equation(s). For example, a mass-spring system can be described by

$$m\ddot{x} + k_1\dot{x} + k_2 x = F(t)$$

  where F is the force applied, x is the position of the mass, k_1 is the friction constant and k_2 is the spring constant.

- Depending on different analytical methods used or different applications, different mathematical models are used.

- For example, if one is only interested in the terminal properties of a network, she/he may use a transfer function or impulse response to describe the system.

- If one would like to know the voltages and/or currents of nodes of the network, he will need to find a set of differential equations or state-space representation.

- Depending on the numbers of inputs and outputs, we have the cases of SISO (single-input single-output), SIMO (single-input multiple-output), MISO and MIMO.

- A system is said to be a *single-variable system* if it has one input and one output only. Otherwise, it will be called *multivariable system*.

- Unlike a static system, a dynamic system has memory. That is, its output at current time instant will not only depend on the current input but also past or future input input.

- A system that accepts and generates continuous-time signals ($u(t)$, $y(t)$, $-\infty \leq t \leq \infty$) is called a continuous-time system.

- A system that accepts and generates discrete-time signals ($u(kT)$, $y(kT)$, T is the sampling period and $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$) is called a discrete-time system.

Causality

- A system is called a memoryless system if its output $y(t)$ is only dependent on its input at t.
- A dynamical system usually has memory, i.e. its output $y(t)$ depends on its past, current and possibly future inputs.
- A system is called causal if its output depends only on its past and current inputs, i.e. $y(t)$ depends only on $u(\tau),\ \tau \leq t$.
- The state $x(t_0)$ is the information of the system at t_0 which together with the input $u(\tau),\ t_0 \leq \tau \leq t$ uniquely determines the output $y(t)$ for all $t \geq t_0$.
- The implication is that once $x(t_0)$ is known, there is no need to know the input before t_0 when determining the future state.

1.3 Linear Systems

- A system is called a linear system if it satisfies *superposition*. Assuming that a linear system is mathematically described by $\mathcal{L}$, then given any inputs u_1, u_2 and any non-zero constants c_1 and c_2,

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)$$

  For example, the system described by ODE (e.g. RC circuit)

$$\dot{y} + 2y = u \tag{1}$$

  is a linear system since if $\dot{y}_1 + 2y_1 = u_1,\ \dot{y}_2 + 2y_2 = u_2$, then letting $y = c_1y_1 + c_2y_2$ and $u = c_1u_1 + c_2u_2$, (1) is satisfied.

- There are linear time-invariant (LTI) systems and linear time-varying (LTV) systems. For example, (1) is LTI whereas $\dot{y} + ty = u$ is LTV.

- Strictly speaking, all practical systems are nonlinear. However, for most nonlinear systems, without certain operating range, their behavior can be approximated by linear systems.

2. Mathematical Descriptions of Linear Systems

2.1 Modelling

Why modelling ?

Modeling of physical systems: a vital component of modern engineering

- often consists of complex coupled differential equations
- only when we have good understanding of a system can we control it so that the system can behave as what we want.
- can simulate and predict actual system response.
- design model-based controllers

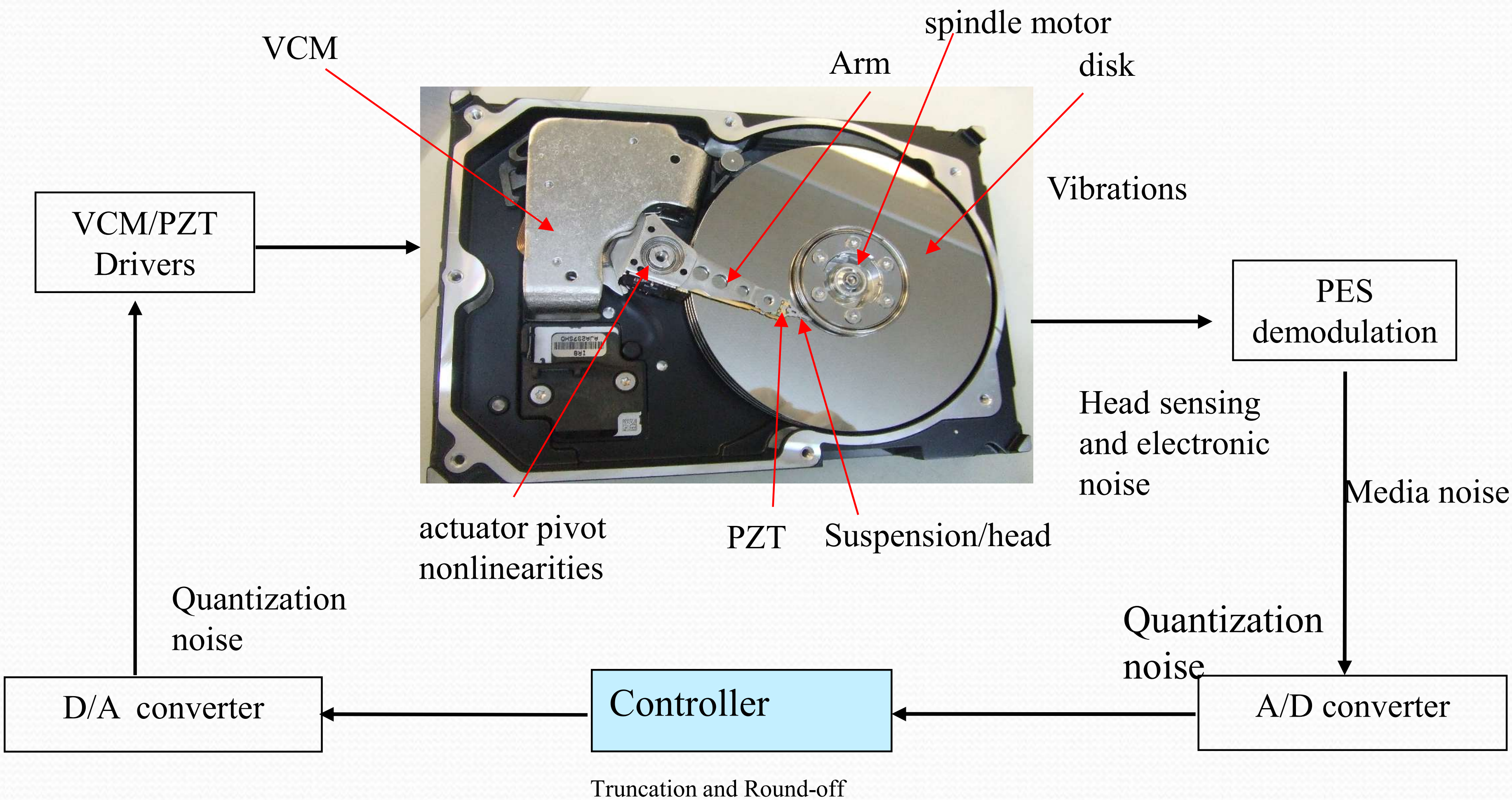
examples: flight control, nanometer precision control

Two general approaches for modeling

- First principle approach: based on physics, using fundamental engineering principles such as Newton's laws, energy conservation, etc.
- Based on measurement data: using input-output response of the system, a field itself known as system identification.

Popular recent developments include data-driven control, learning based control such as reinforcement learning based control.

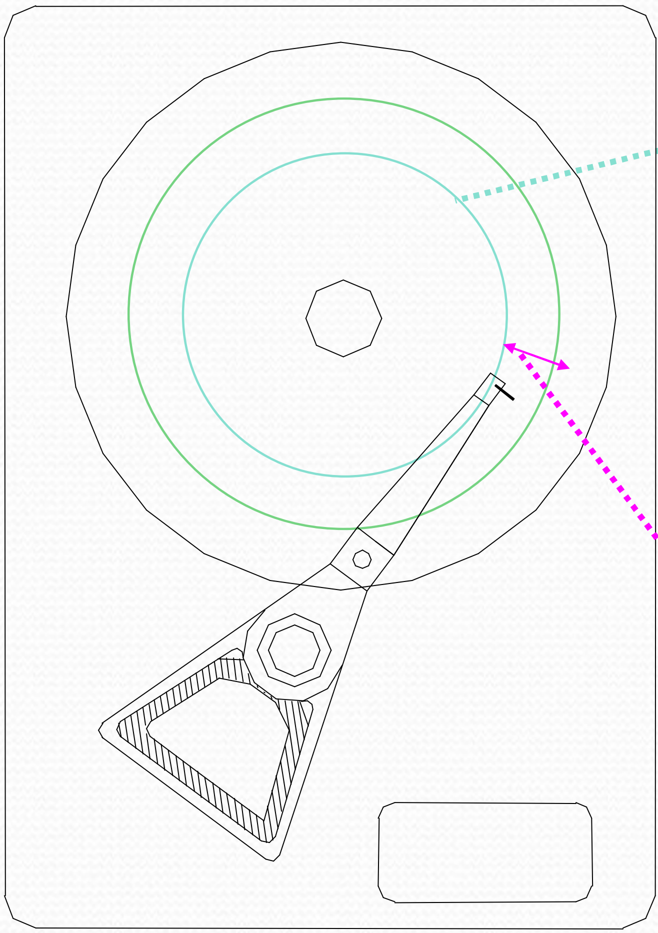
HDD Servo Control System



VCM: voice-coil-motor

PES: position error signal

Servo-Control in Hard Disk Drives



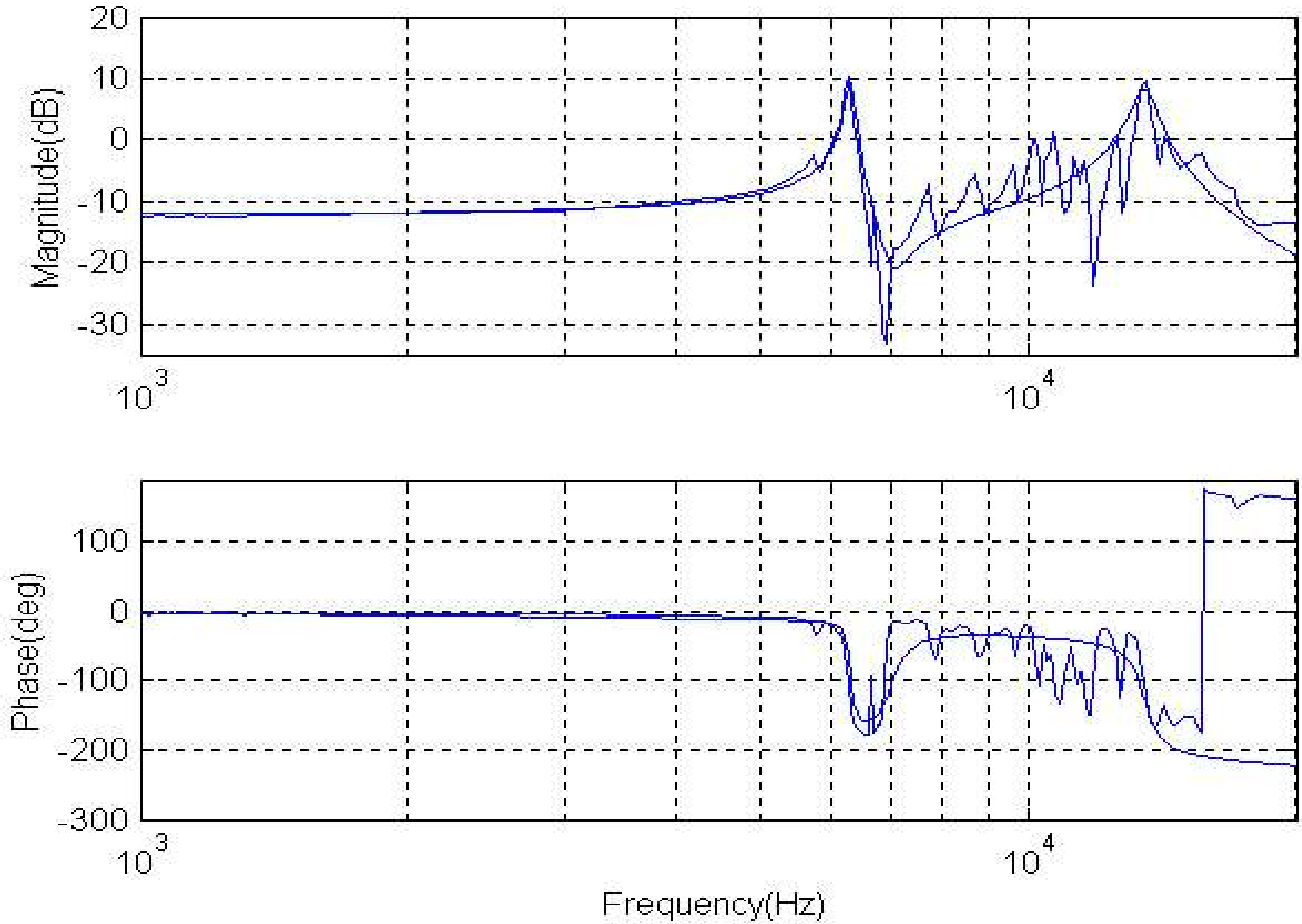
Tasks in Servo-Control of HDDs

1) Track following: The head has to follow a concentric data track so that data can be read or written on the track. Linear high bandwidth controllers are usually used for this to achieve good error and disturbance rejection.

Settling: Often, a intermediate control step is introduced to achieve fast settling with low overshoot

2) Track seeking: The head has to jump from one concentric data track to another to be able to access data in other parts of a hard disk.
The input signal into the VCM-driver has a voltage limit for actuator and voltage protection.

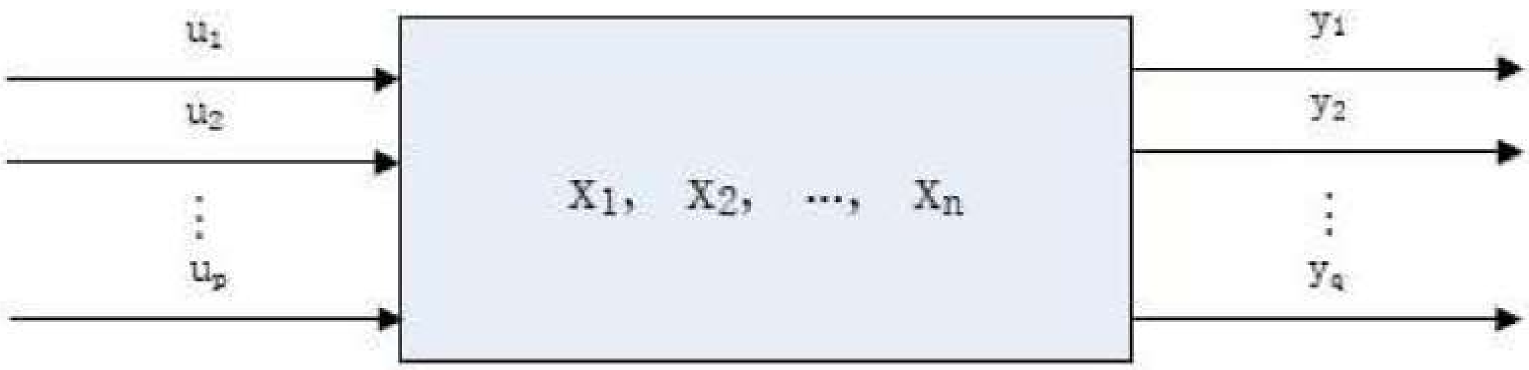
Obtain a transfer function model based on the frequency response:




NANYANG TECHNOLOGICAL UNIVERSITY
SINGAPORE
DoubleBee: A Hybrid Aerial-Ground Robot with Two Active Wheels
Muqing Cao*, Xinhang Xu*, Shenghai Yuan, Kun Cao, Kangcheng Liu, and Lihua Xie
* Equal Contribution

2.2 Mathematical Descriptions of Linear Systems

- A dynamical system can be descried by



- Input: $u_1,\ u_2, \cdots, u_p$
- Output: $y_1,\ y_2, \cdots, y_q$
- State: $x_1,\ x_2,\ \cdots, x_n$
- If $p = q = 1$, the system is called *single variable system* or SISO.

- If the input is zero, the response due to the initial state $\mathbf{x}(t_0)$ (e.g. the initial charge of R-C circuit) is called the zero input response, denoted by $\mathbf{y}_{zi}(t)$.
- If the initial state is zero, the response due to the input is called zero state response, denoted as $\mathbf{y}_{zs}(t)$.
- Then, the response due to both the initial state and input is $\mathbf{y}(t) = \mathbf{y}_{zi}(t) + \mathbf{y}_{zs}(t)$.

(a) Input-output description

- Input-output description treats system as a 'blackbox', i.e. internal structure and information are not available.
- Describe a linear SISO system by ODE

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y^{(1)}+a_0y = b_mu^{(m)}+b_{m-1}u^{(m-1)}+\cdots+b_1u^{(1)}+b_0u \tag{2}$$

where $y^{(i)} = \frac{d^iy}{dt^i}$, $i = 1, 2, \cdots, n$ and $u^{(j)} = \frac{d^ju}{dt^j}$, $j = 1, 2, \cdots, m$. If a_i and b_i are constants, the system is linear time-invariant (LTI); otherwise, linear time-varying (LTV).

- **Impulse response:** Let $g(t,\tau)$ be the impulse response of the system (zero state response due to the impulse $\delta(t-\tau)$). Then, the output due to an input $\{u(t),\ -\infty < t < \infty\}$ is

$$y(t) = \int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau \tag{3}$$

- For a causal system, $g(t,\tau) = 0$ when $\tau > t$. Also, assume that the system is at rest when $t = t_0$ ($\mathbf{x}(t_0) = 0$). Then,

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \tag{4}$$

- For LTI systems, since the system characteristics do not change with time, we can set $t_0 = 0$. Taking $T = -\tau$ leads to $g(t,\tau) = g(t+T,\tau+T) = g(t-\tau,0) = g(t-\tau)$. Hence, zero state response is given by

$$y(t) = \int_{0}^{t} g(t-\tau)u(\tau)d\tau = \int_{0}^{t} g(\tau)u(t-\tau)d\tau$$

- For an MIMO system, $g(t,\tau)$ will be replaced by the impulse response matrix $\mathbf{G}(t,\tau)$ where

$$\mathbf{G}(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \cdots & g_{2p}(t,\tau) \\ \ddots & \ddots & & \ddots \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

where $g_{ij}(t,\tau)$ is the impulse response of the i-th output due to the impulse applied at the j-th input at time τ.

(b) State-space description

- State-space describes the internal behavior of a system. It consists of two equations: state equation (link state to input) and output equation (link output to state and input).

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \tag{5}$$

$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \tag{6}$$

where $\mathbf{x} = [x_1,\ x_2, \cdots,\ x_n]^T$ is the state vector, $\mathbf{u} = [u_1,\ u_2, \cdots,\ u_p]^T$ is the control input vector and $\mathbf{y} = [y_1,\ y_2, \cdots,\ y_q]^T$ is the output vector.

(c) Linear Time-Invariant (LTI) Systems

- To define time-invariant precisely, we introduce a shifting operator Q_α, $Q_\alpha y(t)\mathbf{1}(t) = y(t-\tau)\mathbf{1}(t-\tau)$ where $\mathbf{1}(t)$ is the unit step function.
- A system is said to be time-invariant if and only if

$$HQ_\alpha u = Q_\alpha H u$$

  for any input u and any real α. Examples include RLC circuits.
- In other words, if the initial state and the input are the same, no matter at what time they are applied, the output will always be the same. Hence, for LTI systems, we can assume, WLOG, that $t_0 = 0$.
- By taking Laplace transform, the SISO LTI system (2) can be described by transfer function

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{7}$$

- For a linear time-invariant system, the state-space representation is

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $A,\ B,\ C$ and D are constant matrices.

- Some systems are to be modelled as time-varying systems, for example, a burning rocket whose mass decreases rapidly with time.

2.3 State-space Representation and Canonical Forms

Example 2.1 *Consider a simple example*

$$\ddot{y} + 2\dot{y} + 3y = 2u$$

Let $x_1 = y$ *and* $x_2 = \dot{y}$. *Then,*

$$\begin{aligned}\dot{x}_1 &= \dot{y} = x_2\\ \dot{x}_2 &= \ddot{y} = -3y - 2\dot{y} + 2u = -3x_1 - 2x_2 + 2u\end{aligned}$$

Denote $x = [x_1 \quad x_2]^T$. *Then, we obtain a state-space model:*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\
y &= [1 \quad 0]\, x
\end{aligned}
$$

• **Controllable Canonical Form (CCF)**

Consider, from (7) with $m < n$,

$$
\hat{y}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \hat{u}(s) \tag{8}
$$

Let

$$
\begin{aligned}
\hat{\bar{y}}(s) &= \frac{1}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \hat{u}(s) \\
\hat{y}(s) &= \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\right) \hat{\bar{y}}(s)
\end{aligned}
$$

Note that $\mathcal{L}^{-1}\{s^i \bar{y}(s)\} = \bar{y}^{(i)}(t)$ and $\mathcal{L}^{-1}\{s^j \hat{\bar{y}}(s)\} = \bar{y}^{(j)}(t)$. Then,

$$
\begin{aligned}
&\bar{y}^{(n)} + a_{n-1} \bar{y}^{(n-1)} + \cdots + a_1 \bar{y}^{(1)} + a_0 \bar{y} = u \\
&y = b_m \bar{y}^{(m)} + b_{m-1} \bar{y}^{(m-1)} + \cdots + b_1 \bar{y}^{(1)} + b_0 \bar{y}
\end{aligned}
$$

Introduce state variables: $x_1 = \bar{y},\ x_2 = \bar{y}^{(1)}, \cdots, x_n = \bar{y}^{(n-1)}$. Then,

$$\begin{aligned}
\dot{x}_1 &= \bar{y}^{(1)} = x_2 \\
\dot{x}_2 &= \bar{y}^{(2)} = x_3 \\
&\vdots \\
\dot{x}_{n-1} &= \bar{y}^{(n-1)} = x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u
\end{aligned}$$

and

$$y = b_0 x_1 + b_1 x_2 + \cdots + b_m x_{m+1}$$

We obtain a state-space model (Controllable Canonical Form or CCF form):

$$\dot{x} = \begin{bmatrix} 0 & \vdots & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & \vdots & & & 1 \\ \cdots & & \cdots & \cdots & \cdots \\ -a_0 & \vdots & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{9}$$

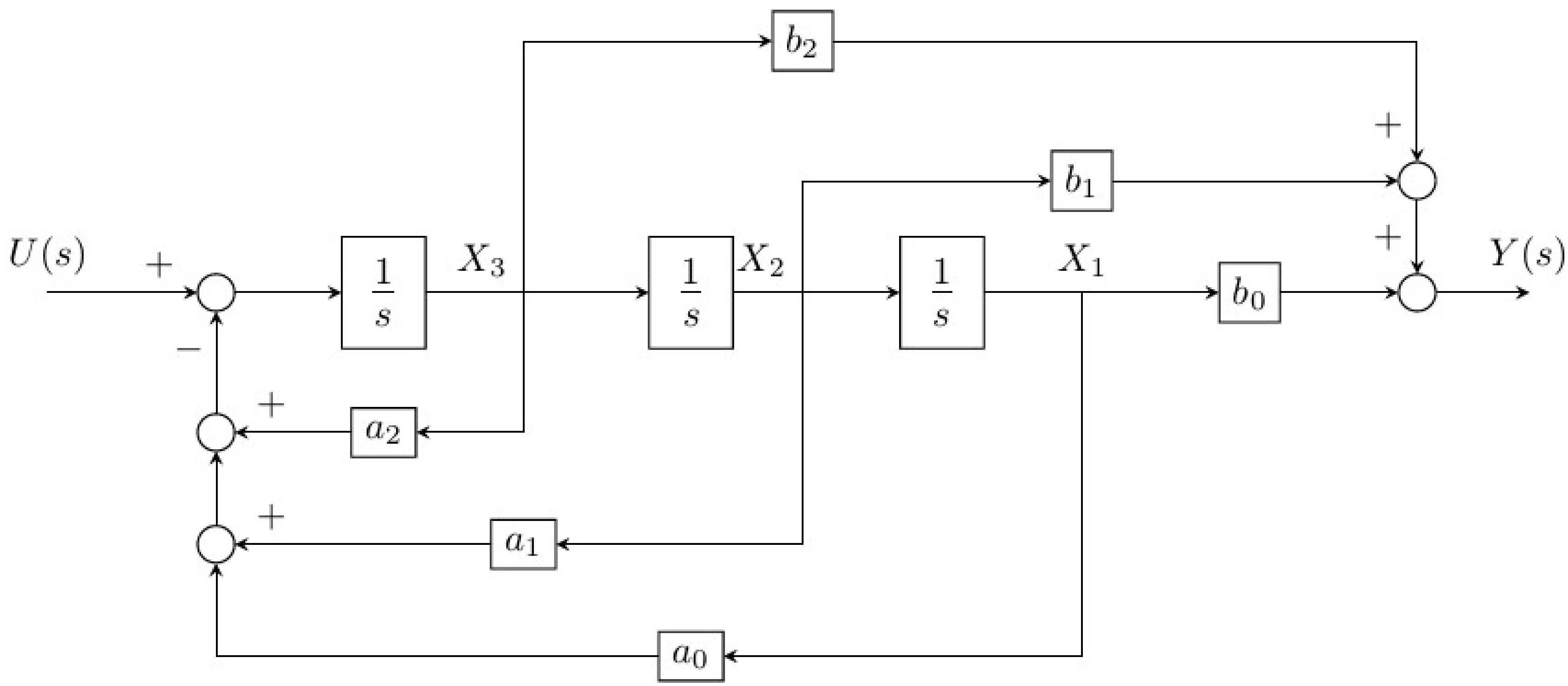
$$y = [b_0, \cdots, b_m, 0, \cdots, 0]\, x \tag{10}$$

For $3rd$ order system: $Y(s) = \frac{b_2 s^2 + b_1 s + b0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$, its CCF form is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{11}$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} x \tag{12}$$

The block diagram realization of the CCF form is given by



- When $m = n$, (8) can be rewritten as

$$\hat{y}(s) = \left[b_n + \frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + \cdots + (b_0 - b_n a_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \right] \hat{u}(s)$$

  Then, the state equation remains the same as (9) but output equation becomes

$$y = [b_0 - b_n a_0, \cdots, b_{n-1} - b_n a_{n-1}]\, x + b_n u \qquad (13)$$

Example 2.2 *Consider SISO system described by transfer function*

$$\hat{g}(s) = \frac{4s^3 + 160s + 720}{s^3 + 16s^2 + 194s + 640}$$

Then,

$$\begin{aligned} b_0 - b_3 a_0 &= 720 - 4 \times 640 = -1840 \\ b_1 - b_3 a_1 &= 160 - 4 \times 194 = -616 \\ b_2 - b_3 a_2 &= 0 - 4 \times 16 = -64 \end{aligned}$$

and a state-space description can be obtained as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -194 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [-1840 \quad -616 \quad -64] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 4u$$

- **Observable Canonical Form (OCF form)**

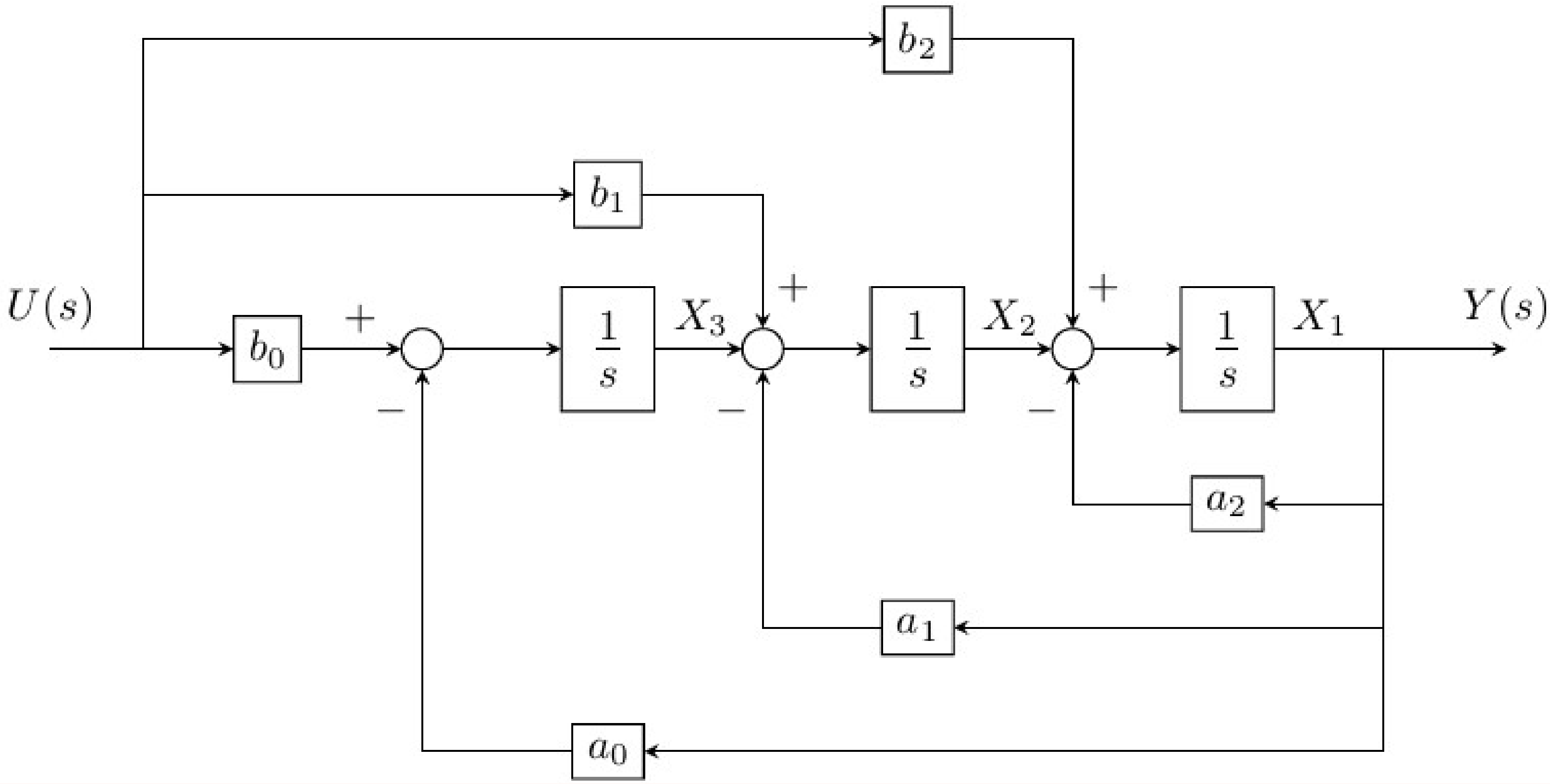
Consider the 3rd order system

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

which can be rewritten as

$$Y(s) = -\frac{a_2}{s} Y(s) - \frac{a_1}{s^2} Y(s) - \frac{a_0}{s^3} Y(s) + \frac{b_2}{s} U(s) + \frac{b_1}{s^2} U(s) + \frac{b_0}{s^3} U(s)$$

The block diagram is given by



By assigning the state variables as shown in the figure, we have

$$\begin{aligned} Y(s) &= X_1(s) \\ sX_1(s) &= -a_2X_1(s) + X_2(s) + b_2U(s) \\ sX_2(s) &= -a_1X_1(s) + X_3(s) + b_1U(s) \\ sX_3(s) &= -a_0X_1(s) + b_0U(s) \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} y(t) &= x_1(t) \\ \dot{x}_1(t) &= -a_2x_1(t) + x_2(t) + b_2u(t) \\ \dot{x}_2(t) &= -a_1x_1(t) + x_3(t) + b_1u(t) \\ \dot{x}_3(t) &= -a_0x_1(t) + b_0u(t) \end{aligned}$$

Observable Canonical Form (OCF):

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \end{aligned}$$

- **Diagonal Form**

If the poles of the transfer function (7)λ_i, $i = 1, 2, \cdots, n$ are real and distinct. Then, when $m < n$, by partial fraction, (7) can be expressed as

$$\hat{g}(s) = \frac{k_1}{s-\lambda_1} + \frac{k_2}{s-\lambda_2} + \cdots + \frac{k_n}{s-\lambda_n}$$

where

$$k_i = \lim_{s\to\lambda_i}(s-\lambda_i)\hat{g}(s)$$

- By denoting $\hat{x}_i(s) = \frac{k_i}{s-\lambda_i}\hat{u}(s)$, i.e. $\dot{x}_i = \lambda_i x_i + k_i u$, we can obtain a state-space model

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} u \\
y &= [1,\ 1, \cdots, 1]\, x
\end{aligned}
$$

The case of $m = n$ can be dealt with similarly.

Example 2.3 *Consider SISO system described by*

$$
\hat{g}(s) = \frac{7s^2 + 2s + 1}{s^3 + 6s^2 + 11s + 6}
$$

The poles (roots of denominator) are: $\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$ *which are distinct. Note that*

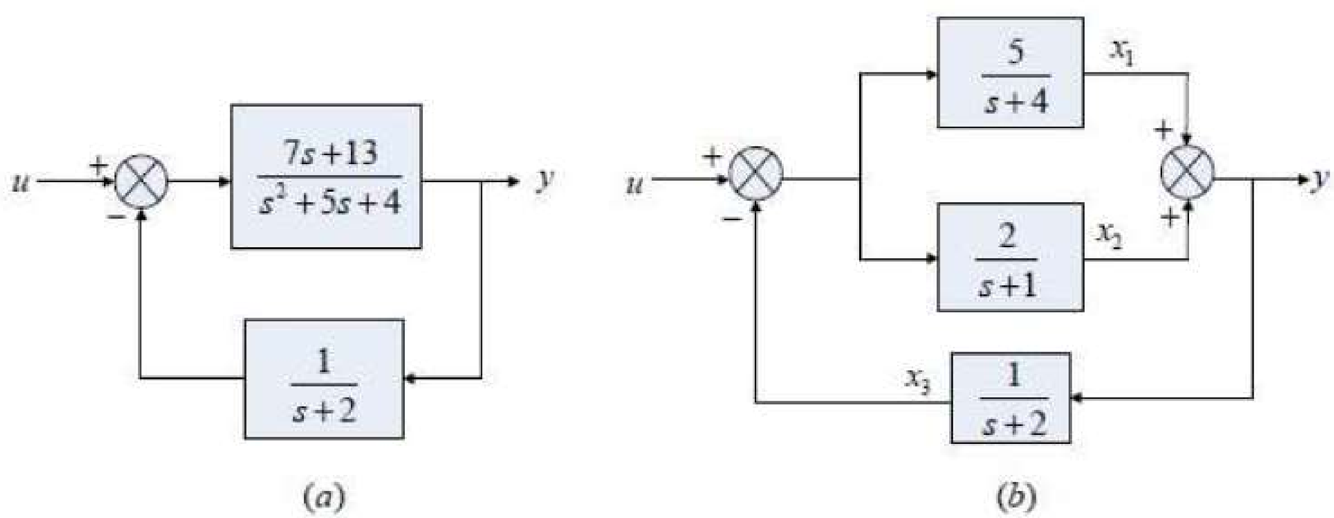
$$
\begin{aligned}
k_1 &= \lim_{s \to -1} (s+1)\hat{g}(s) = 3 \\
k_2 &= \lim_{s \to -2} (s+2)\hat{g}(s) = -25 \\
k_3 &= \lim_{s \to -3} (s+3)\hat{g}(s) = 29
\end{aligned}
$$

Then, a state-space model can be obtained as

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 3 \\ -25 \\ 29 \end{bmatrix} u$$
$$y = [1 \ \ 1 \ \ 1]\, x$$

2.4 Obtain state-space model from system block diagram

Example 2.4 *Consider the system in Fig. (a).*



Convert second order block to two first order: (a)$\rightarrow$ (b)

$$\frac{7s+13}{s^2+5s+4} = \frac{5}{s+4} + \frac{2}{s+1}$$

Define state variables as shown in (b).

$$\begin{aligned}
x_1 &= \frac{5}{s+4}(u - x_3) \\
x_2 &= \frac{2}{s+1}(u - x_3) \\
x_3 &= \frac{1}{s+2}(x_1 + x_2)
\end{aligned}$$

and $y = x_1 + x_2$.

The corresponding time-domain relations are

$$\begin{aligned}
\dot{x}_1 &= -4x_1 - 5x_3 + 5u \\
\dot{x}_2 &= -x_2 - 2x_3 + 2u \\
\dot{x}_3 &= x_1 + x_2 - 2x_3
\end{aligned}$$

or yet

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -4 & 0 & -5 \\ 0 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} u \\ y &= [1 \;\; 1 \;\; 0]\, x \end{aligned}$$

2.5 Obtain Transfer Function of System from State-space Description

Consider a system represented by the state-space equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{14}$$

$$y(t) = Cx(t) + Du(t) \tag{15}$$

By taking Laplace transform of (14) under zero initial condition, we have

$$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s)$$

i.e.

$$\hat{x}(s) = (sI - A)^{-1} B\hat{u}(s)$$

Also, taking Laplace transform on (15) and substituting the above, we can get

$$\hat{\mathbf{G}}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = C(sI - A)^{-1}B + D$$

Example 2.5 *Given the state-space description of a system:*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \end{aligned}$$

Find the transfer function matrix of the system.

First, note that $(sI - A)^{-1} = \frac{adj(sI-A)^T}{|sI-A|}$. Compute:

$$\begin{aligned} |sI - A| &= s^3 + 2s^2 + s + 3 \\ adj(sI - A) &= adj \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 1 & s+2 \end{bmatrix} = \begin{bmatrix} s^2 + 2s + 1 & s + 2 & 1 \\ -3 & s^2 + 2s & s \\ -3s & -s - 3 & s^2 \end{bmatrix} \end{aligned}$$

The transfer function matrix is calculated as

$$
\begin{aligned}
\hat{\mathbf{G}}(s) &= C(sI-A)^{-1}B = \frac{C\,adj(sI-A)B}{|sI-A|} \\
&= \frac{[1\ \ 1\ \ 1]\begin{bmatrix} s^2+2s+1 & s+2 & 1 \\ -3 & s^2+2s & s \\ -3s & -s-3 & s^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}{s^3+2s^2+s+3} \\
&= \begin{bmatrix} \dfrac{2s^2-1}{s^3+2s^2+s+3} & \dfrac{2s^2+3s}{s^3+2s^2+s+3} \end{bmatrix}
\end{aligned}
$$

- Observe for MIMO (p-input q-output) systems,

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \\ \vdots \\ \hat{y}_q(s) \end{bmatrix} = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \\ \vdots \\ \hat{u}_p(s) \end{bmatrix}$$

  or

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

  where $\hat{g}_{ij}(s)$ is the transfer function from the j-th input to i-th output.

- Every rational transfer function can be expressed as $\hat{g}(s) = N(s)/D(s)$
- $\hat{g}(s)$ is proper if $degD(s) \geq degN(s)$ or $\hat{g}(\infty) = 0$ or nonzero constant
- $\hat{g}(s)$ is strictly proper if $degD(s) > degN(s)$ or $\hat{g}(\infty) = 0$
- $\hat{g}(s)$ is biproper if $degD(s) = degN(s)$ or $\hat{g}(\infty)$ is nonzero constant
- $\hat{g}(s)$ is improper if $degD(s) < degN(s)$ or $\hat{g}(\infty) = \infty$
- Improper rational transfer functions rarely exist in real world as they amplify high-frequency noise

- In terms of poles and zeros, a transfer function can be expressed as (zero-pole-gain form):

$$\hat{g}(s) = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where $p_i,\ i = 1, 2, \cdots, n$ and $z_i,\ i = 1, 2, \cdots, m$ are called poles and zeros, respectively.

- The zero-pole-gain form can be obtained from the Matlab function:

$$[z, p, k] = tf2zp(num, den)$$

- For MIMO case, a transfer function matrix $\hat{\mathbf{G}}(s)$ is proper if its every entry is proper. We call p is a pole of $\hat{\mathbf{G}}(s)$ if it is a pole of one of the entries

- We call z a blocking zero if it is a zero of every nonzero entry of $\hat{\mathbf{G}}(s)$. Transmission zero will be defined later.

- Conversion from state-space to transfer function using Matlab:

$$[num, den] = ss2tf(A, B, C, D, i)$$

where i means i-th column (or i input)

- Conversion from transfer function to state-space for SISO using Matlab:

$$(A, B, C, D) = tf2ss(num, den)$$

2.6 Linearization

- Consider general nonlinear time-varying systems

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) & (16) \\ \mathbf{y}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) & (17) \end{aligned}$$

- Suppose for some input $\mathbf{u}_0(t)$ and some initial state, $\mathbf{x}_0(t)$ is the solution of (16), i.e.

$$\dot{\mathbf{x}}_0(t) = \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) \qquad (18)$$

- Under some perturbation, $\mathbf{u}_0(t) \to \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)$ and $\mathbf{x}_0(t_0) \to \mathbf{x}_0(t_0) + \bar{\mathbf{x}}(t_0)$,

$$\mathbf{x}_0(t) \to \mathbf{x}_0(t) + \bar{\mathbf{x}}(t)$$

- By Taylor expansion,

$$\begin{aligned} \dot{\mathbf{x}}_0(t) + \dot{\bar{\mathbf{x}}}(t) &= \mathbf{h}(\mathbf{x}_0(t) + \bar{\mathbf{x}}(t), \mathbf{u}_0(t) + \bar{\mathbf{u}}(t), t) \\ &\approx \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \bar{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \bar{\mathbf{u}} + \cdots \end{aligned}$$

- By ignoring the higher order terms, it follows from (16) that

$$\dot{\bar{\mathbf{x}}}(t) = A(t)\bar{\mathbf{x}}(t) + B(t)\bar{\mathbf{u}}(t), \;\; \bar{\mathbf{x}}(t_0) \tag{17}$$

where

$$A(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t), \;\; B(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)$$

- Similarly, the output equation

$$\mathbf{y}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

can be linearized at $(\mathbf{x}_0(t), \mathbf{u}_0(t))$ to obtain

$$\bar{\mathbf{y}}(t) = C(t)\bar{\mathbf{x}}(t) + D(t)\bar{\mathbf{u}}(t) \tag{18}$$

where

$$C(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t), \;\; D(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)$$

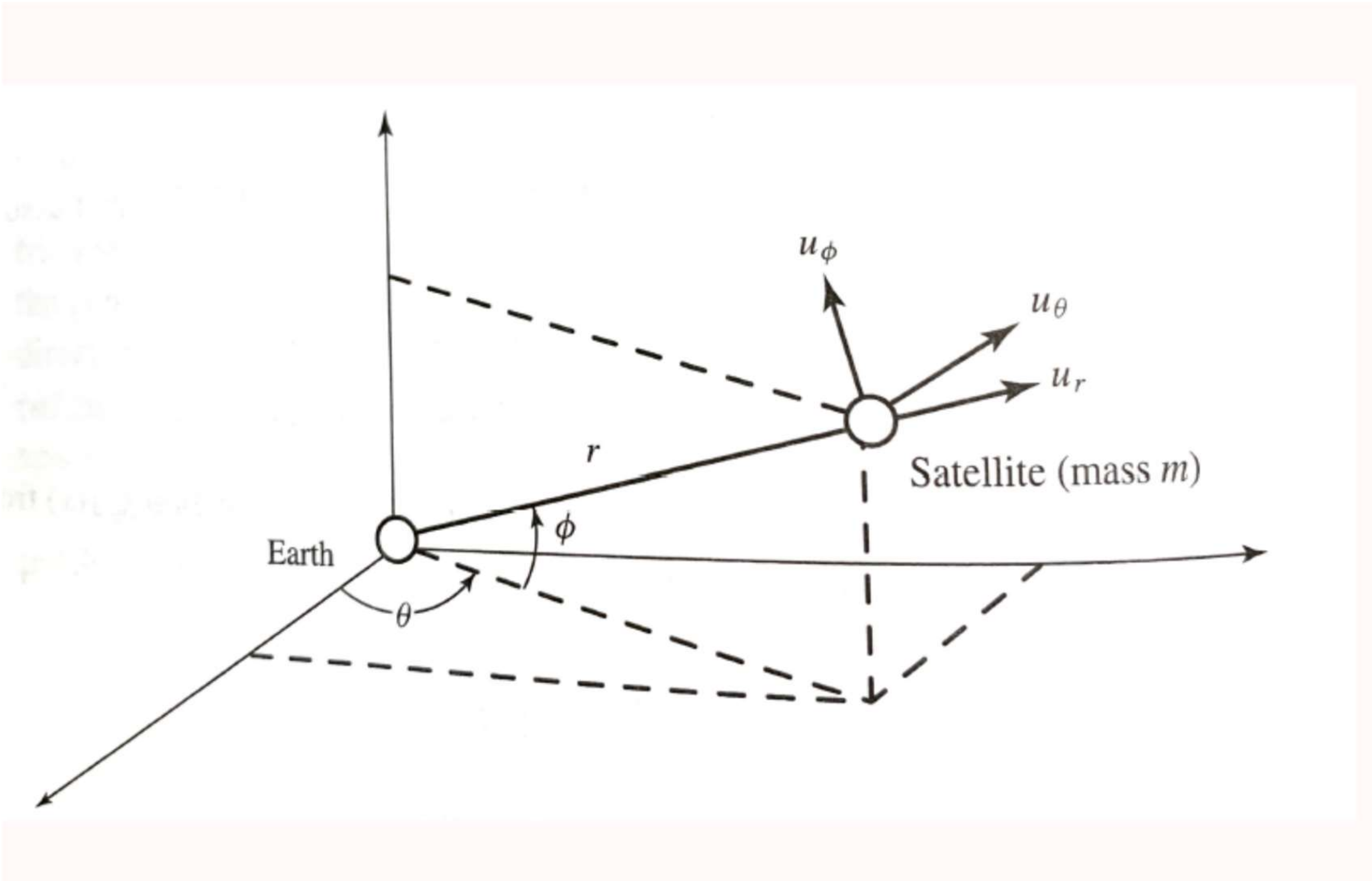
Examples of Linearization

Example 2.6 *The altitude of a communication satellite is specified by* $r(t)$, $\theta(t)$ *and* $\phi(t)$. *Choose the state variables, control inputs and system outputs to be:*

$$\mathbf{x}(t) = \begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_r(t) \\ u_\theta(t) \\ u_\phi(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} r(t) \\ \theta(t) \\ \phi(t) \end{bmatrix}$$

Then the system can be described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 cos^2\phi + r\dot{\phi}^2 - k/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi}sin\phi/cos\phi + u_\theta/mrcos\phi \\ \dot{\phi} \\ -\dot{\theta}^2 cos\phi sin\phi - 2\dot{r}\dot{\phi}/r + u_\phi/mr \end{bmatrix} \tag{21}$$



u_ϕ
u_θ
u_r
r
Satellite (mass m)
ϕ
Earth
θ

- A circular equatorial orbit: $\mathbf{x}_0(t) = [r_0 \;\; 0 \;\; \omega_0 t \;\; \omega \;\; 0 \;\; 0]'$, $\mathbf{u}_0 \equiv \mathbf{0}$ with $r_0^3 \omega_0^2 = k$.
- Some perturbation makes the state deviate from the orbit:

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \bar{\mathbf{x}}(t), \;\; \mathbf{u}(t) = \mathbf{u}_0(t) + \bar{\mathbf{u}}(t), \;\; \mathbf{y}(t) = \mathbf{y}_0(t) + \bar{\mathbf{y}}(t)$$

- The linearized system:

$$\dot{\bar{\mathbf{x}}}(t) = A\bar{\mathbf{x}}(t) + B\bar{\mathbf{u}}(t), \quad \bar{\mathbf{y}}(t) = C\bar{\mathbf{x}}(t)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 r_0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ 0 & -2\omega_0/r_0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \vdots & -\omega_0^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \vdots & 0 \\ 1/m & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \\ 0 & 1/(mr_0) & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 1/(mr_0) \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}$$

Example 2.7 *Consider the RLC circuit shown in next page, where T is a tunnel diode with the following characteristics.*

- *By KVL and KCL, it can be easily derived that*

$$\dot{x}_1(t) = -\frac{h(x_1(t))}{C} + \frac{x_2(t)}{C} \quad (22)$$

$$\dot{x}_2(t) = \frac{-x_1(t) - Rx_2(t)}{L} + \frac{E}{L} \quad (23)$$

  which involves nonlinear differential equations.

- *If x_1 lies only inside (a, b), then $h_1(x_1(t)) \approx x_1(t)/R_1$ (see Fig. (b)). The network is reduced to Fig. (c) and (22)-(23) can be approximated by linear ODEs.*

- *If $x_1 \in (c, d)$, by introducing variables $\bar{x}_1(t) = x_1(t) - v_0$, $\bar{x}_2(t) = x_2(t) - i_0$, $h_1(x_1(t)) \approx i_0 - \bar{x_1}(t)/R_2$, and $\bar{E} = E - v_0 - Ri_0$. Then, the network is reduced to Fig. (d) and the approximate state equations are:*

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 1/(CR_2) & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \bar{E}$$

08/02 16:03:15 ScreenHunter

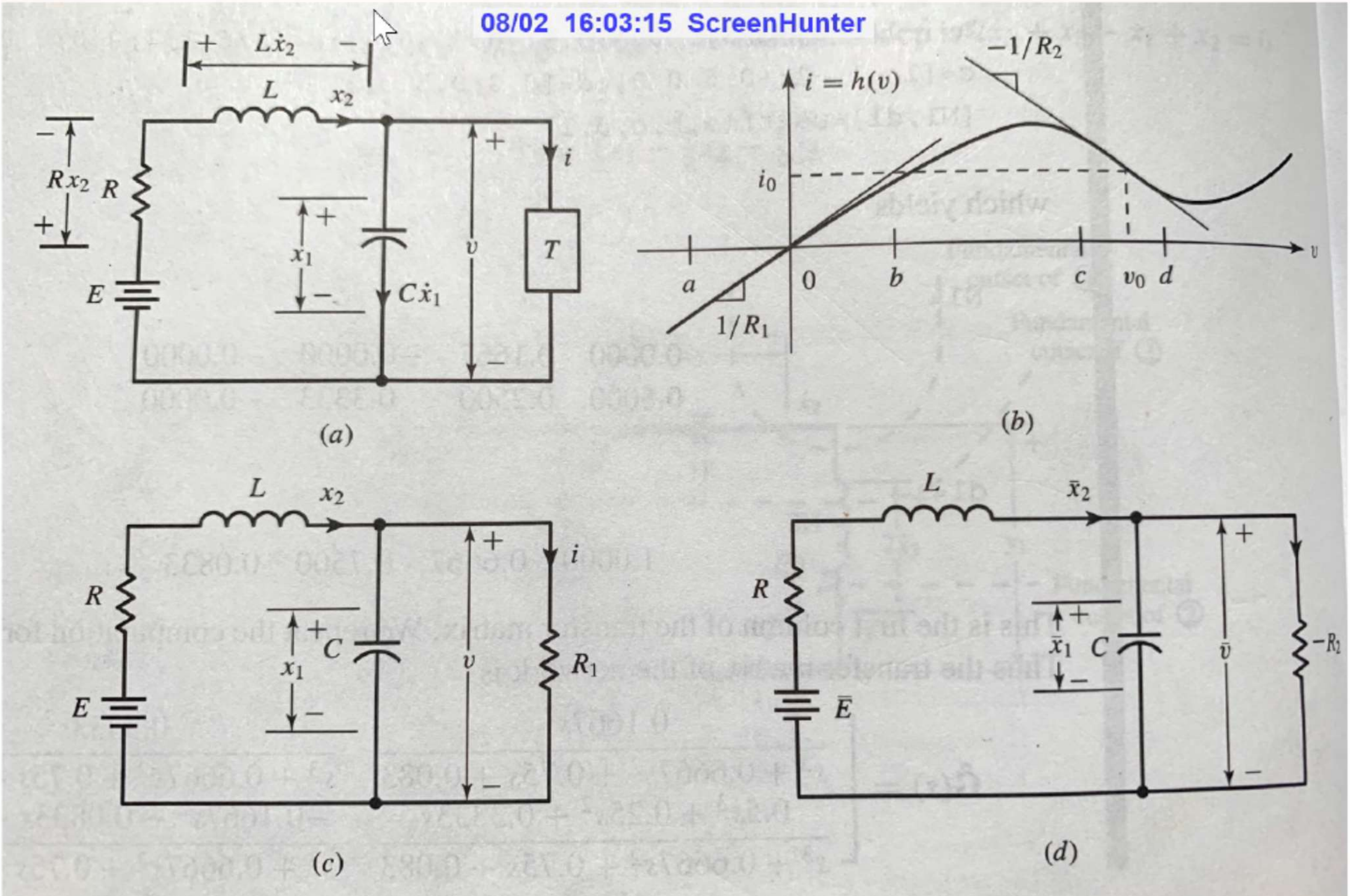


Figure 2.17 Network with a tunnel diode.

Recall Linearization

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

Linearize it around the operating point (x_0, u_0) to obtain the linear system

$$\dot{\overline{x}} = A\overline{x} + B\overline{u}$$
$$\overline{y} = C\overline{x} + D\overline{u}$$

Note that x_0, u_0 can be time-varying

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=x_0, u=u_0}; \quad B = \frac{\partial f}{\partial u}\Big|_{x=x_0, u=u_0}$$
$$C = \frac{\partial h}{\partial x}\Big|_{x=x_0, u=u_0}; \quad D = \frac{\partial h}{\partial u}\Big|_{x=x_0, u=u_0}$$

Example 2.8 • *Consider a rocket ascending from the surface of the Earth. Let $m(t)$ be the mass of the rocket at time t and $h(t)$ the altitude. The thrust force is $v_e u_0$ with v_e the constant relative exhaust velocity and u_0 the constant rate of change of mass. Then,*

$$\begin{aligned} m(t)\dot{v}(t) &= -m(t)g + v_e u_0 && (24) \\ \dot{h}(t) &= v(t) && (25) \\ m(t) &= m_0 + u_0 t && (26) \end{aligned}$$

• *Denote $x_1(t) = h(t)$ and $x_2(t) = v(t)$. Then,*

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -g + v_e u_0/(m_0 + u_0 t) \end{bmatrix} && (27) \\ y(t) &= [1 \quad 0]\mathbf{x}(t) && (28) \end{aligned}$$

The input is now a given function.

We now assume that the rate of mass expulsion can be varied:

$$u(t) = \dot{m}(t) \tag{27}$$

Letting $x_3(t) = m(t)$, we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g + v_e u(t)/x_3(t) \\ u(t) \end{bmatrix} \tag{28}$$

$$y(t) = x_1(t) \tag{29}$$

Suppose that the nominal input is $\tilde{u}(t) = u_0 < 0$. We now find the linearized model of the system. Note that

$$\frac{\partial \mathbf{h}(\mathbf{x}, u)}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -v_e u_0/x_3^2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{h}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} 0 \\ v_e/x_3 \\ 1 \end{bmatrix}$$

The linearized model of the system at the nominal data is

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{v_e u_0}{(m_0+u_0 t)^2} \\ 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}}(t) + \begin{bmatrix} 0 \\ \frac{v_e}{m_0+u_0 t} \\ 1 \end{bmatrix} \bar{u}(t) && (30) \\ \bar{y}(t) &= [1 \ \ 0 \ \ 0]\bar{\mathbf{x}}(t) && (31) \end{aligned}$$

2.7 Discrete-Time Systems

- Denote $u(k) = u(kT)$ and $y(k) = y(kT)$ where T is sampling period. Also, denote the impulse sequence:

$$\delta(k-m) = \begin{cases} 1 & if\ k = m \\ 0 & if\ k \neq m \end{cases}$$

- An input sequence can then be expressed as

$$u(k) = \sum_{-\infty}^{\infty} u(m)\delta(k-m)$$

- Let $g(k, m)$ be the impulse response applied at time instant m. Then, the output of a causal system is

$$y(k) = \sum_{k_0}^{k} g(k, m)u(m)$$

- For LTI systems, WLOG, set $k_0 = 0$. $g(k, m) = g(k - m)$ and

$$y(k) = \sum_{m=0}^{k} g(k-m)u(m) = \sum_{m=0}^{k} g(m)u(k-m)$$

- The z-transform of the output of a LTI system is

$$\hat{y}(z) = \sum_{k=0}^{\infty} y(k)z^{-k}$$

- Similar to the continuous-time case, it can be easily shown that

$$\hat{y}(z) = \hat{g}(z)\hat{u}(z)$$

where $\hat{g}(z) = \sum_{l=0}^{\infty} g(l)z^{-l}$.

Example 2.9 *For the unit time delay system*

$$y(k) = u(k-1)$$

the impulse response is

$$g(k) = \delta(k-1)$$

Thus the transfer function of the system is

$$\hat{g}(z) = \mathcal{Z}[\delta(k-1)] = z^{-1}$$

Example 2.10 *For the unity feedback system consisting of a delay device, the impulse response is*

$$g(k) = a\delta(k-1) + a^2\delta(k-2) + \cdots$$

The transfer function is then

$$\hat{g}(z) = \mathcal{Z}[g(k)] = az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots = \frac{az^{-1}}{1 - az^{-1}}$$

which is a rational function of z. Hence, unlike the continuous-time counterpart, it is a lumped system.

- State-space equations:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \tag{32}$$

$$\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k) \tag{33}$$

- Relationship between transfer function and state-space:

$$\hat{\mathbf{G}}(z) = C(zI - A)^{-1}B + D$$

Exercises

1. Find a state-space realization for each of the following systems
(i) $y^{(3)} + 2y^{(2)} + 6y^{(1)} + 3y = 5u$
(ii) $y^{(3)} + 8y^{(2)} + 5y^{(1)} + 13y = 4u^{(1)} + 7u$

2. Find a state-space realization for each of the following system-s
(i) $\hat{g}(s) = \frac{2s^2+18s+40}{s^3+6s^2+11s+6}$
(ii) $\hat{g}(s) = \frac{3(s+5)}{(s+3)^2(s+1)}$

3. Find the transfer function of the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -3 & 5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} u \\
y &= x_2 + 3x_3
\end{aligned}
$$

Further, obtain the CCF and OCF forms of the transfer function.