EE 6225 – Multivariable Control System

Part I – Advanced Process Control

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Course Overview

- Dynamic Model, MIMO, Limits of Performance
- PID Controller, Direct Synthesis Method
- IMC, IMC-Based PID, Smith Predictor
- Cascade and Feedforward Control
- Ratio Control, Split-Range Control
- Interaction Dynamics, RGA, Decoupling, Multivariable Feedback Control
- System Identification, Least Squares Estimation, Recursive Least Squares, State Space-Based System Identification

References:

- B. Wayne Beguette, ``Process Control Modelling, Design and Simulation", Prentice Hall, 2002.
- Sigurd Skogestad, Ian Postlethwaite, "Multivariable Feedback Control Analysis and Design", John Wiley, 2001.
- Goodwin, Graebe, Salgado, "Control System Design", Pearson, 2000

Course Outline

1.Dynamic Model, MIMO, Poles and Zeros, Trade-Offs and Limits of Performance

2.SISO - PID, DSM, IMC, IMC-Based PID, Smith Predictor

3.SISO - Cascade, Feedforward, Ratio, Split Range, Selective Control

4,5. MIMO – Multiloop Interaction Dynamics, RGA, Decoupling, Multivariable Feedback Control

6.System Identification, Linear Regression, RLS, 4SID

1. Chemical Process System

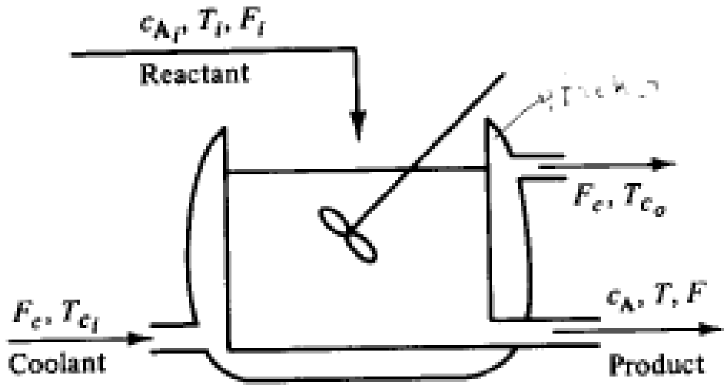


Fig. 1.1 Continuous Stirred Tank Reactor (CSTR)

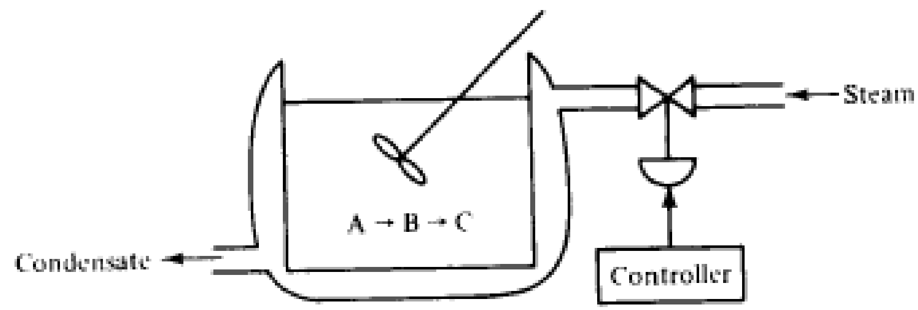


Fig. 1.2 Batch Reactor

A) Dynamic System Characteristics:

1. Multiple Input Multiple Output (e.g. for CSTR: inputs - flow and temperature of coolant; output - volume and temperature of product)
2. May be Unstable (e.g. CSTR uses an exothermic reaction where heat energy is released and may exceed the heat absorption at certain equilibrium points)
3. MIMO,Dead Time, Slow Dynamics (e.g. Large Time Constant vs Drone)

B) Control Objectives:

1. Ensuring *Stability* (e.g. Energy stability of CSTR)
2. *Performance* Goal (e.g. Maximize production of desirable output B and minimize production of waste C from given input raw material A by control of Steam flow rate)
3. Minimize Effects of *External Disturbances* (e.g. Temperature or Flow of the Feed Reactant in CSTR)

Outline of Lecture 1

1.1 Dynamic Model

1.2 Multi-Input Multi-Output (MIMO) Systems

- 1.2.1 State Space Model
- 1.2.2 Transfer Function Matrix
- 1.2.3 Poles and Zeros

1.3 Design Requirements, Trade-Offs, Performance and Bandwidth Limits due to *Delay, RHP Zeros and Poles*

- 1.3.1 Design Requirements and Trade-Offs
- 1.3.2 Bode's Integral Formula and Performance Limits
- 1.3.3 Poisson's Integral Formula and Bandwidth Limits

1. Learning Objectives

- Development and use of models for control systems engineering.
- Steady-state solution and linearization to form state space models
- Dynamic behavior of linear systems, starting with state space models and then covering transfer function-based models
- Factors that limit the achievable performance of a control system under feedback control

1.1 Dynamic Models

Models

Model ($\mathcal{M}$): A description of the system. The model should capture the essential information about the system.

Systems	**Models**
Complex	Approximative (idealization). Should capture the relevant information.
Building/Examine systems is expensive, dangerous, time consuming, etc.	Models can answer many questions about the system.

1.1 Mathematical Models

- Models
  - Set of Equations that allow us to predict the behaviour of a System

  (a) *Analytical* Models are derived from first principles physical-chemical relationships

  (b) *Experimental* Models are derived based on least squares fit to some measurement data (Lecture 6)

- Mathematical models are described by the following types of equations (including combinations)
  - Algebraic Equations
  - Ordinary Differential Equations (ODE)
  - Partial Differential Equations (PDE)

  In this course, we shall emphasize on models based on ODE and its discretized form for digital computer simulation

1.1 Applications

Applications

- Process design. Ex. Designing new cars, new airplanes.
- Control design. Simple regulators $\Rightarrow$ simple models, and optimal regulators $\Rightarrow$ sophisticated models.
- Prediction. Forecast the weather, predict the stock market, etc.
- Signal processing. Ex. Communication, echo cancellation.
- Simulation. Ex. Train nuclear plant operators, try new operation strategies.
- Fault detection.

1.1 Model of 2 Stirred Tank Reactor System

- **Two tanks in series (Ex3.7)**
  - **No reaction**

$$V_1 \frac{dc_1}{dt} + qc_1 = qc_i$$

$$V_2 \frac{dc_2}{dt} + qc_2 = qc_1$$

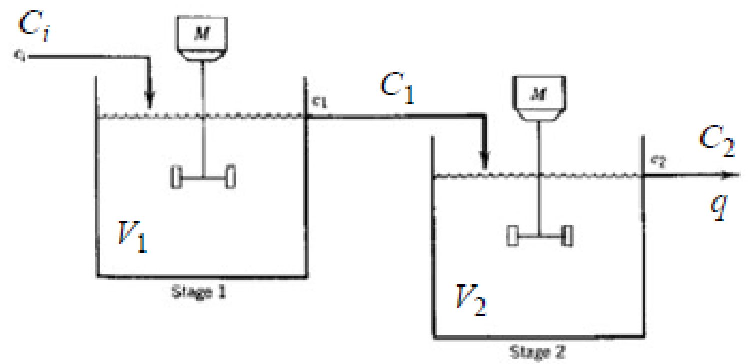


Figure 3.4. Two-stage stirred-tank reactor system.

  - **Initial condition:** $c_1(0)= c_2(0)=1$ kg mol/m^3 **(Use deviation var.)**
  - **Parameters:** V_1/q**=2 min.,** V_2/q**=1.5 min.**
  - **Transfer functions**

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{(V_1/q)s+1} \qquad \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)} = \frac{1}{(V_2/q)s+1}$$

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)} = \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{((V_2/q)s+1)((V_1/q)s+1)}$$

1.1 Model of Isothermal Stirred Tank Reactor

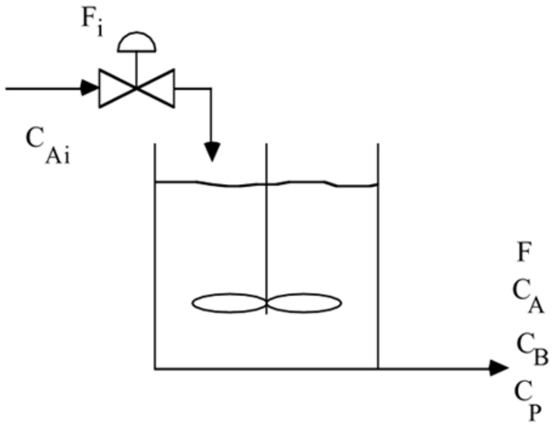
- Model
  - Overall Material Balance

$$\frac{dV\rho}{dt} = F_i\rho_i - F\rho$$

If liquid phase density is not a function of concentration,

$\rho_i = \rho$, we have

$$\frac{dV}{dt} = F_i - F$$



  - Component Material Balance

$$\frac{dC_A}{dt} = \frac{F_i}{V}\left(C_{Ai} - C_A\right) - kC_A$$

$$\frac{dC_P}{dt} = -\frac{F_i}{V}C_P + kC_A$$

  - where C_A and C_P represent the molar concentrations of *A* and *P*

1.1 Model of Isothermal Stirred Tank Reactor

- Resulting Ordinary Differential Equation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V \\ C_A \\ C_P \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_i \\ F \\ C_{Ai} \end{bmatrix} \qquad p = [p_1] = k$$

  - Using State Variable notation, we have

$$\dot{x}_1 = dx_1/dt = u_1 - u_2 = f_1(x, u, p)$$

$$\dot{x}_2 = dx_2/dt = \frac{u_1}{x_1}(u_3 - x_2) - p_1 x_2 = f_2(x, u, p)$$

$$\dot{x}_3 = dx_3/dt = -\frac{u_1}{x_1} x_3 + p_1 x_2 = f_3(x, u, p)$$

  - Or

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix} = \begin{bmatrix} f_1(x,u,p) \\ f_2(x,u,p) \\ f_3(x,u,p) \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ (u_1/x_1)(u_3 - x_2) - p_1 x_2 \\ -(u_1/x_1)x_3 + p_1 x_2 \end{bmatrix}$$ Equation (1.1)

1.1 Examples of Mathematical Models

$\boxed{\text{Ex. Models}}$

- Nonlinear versus linear
- Time continuous versus time discrete (RLS, MPC)
- Deterministic versus stochastic.

1.2 Multi-Input Multi-Output (MIMO) System

Example: Distillation Column

- **Outputs:** $x_D, x_B, P, h_D, \text{and } h_B$
- **Control Inputs :** $\mathrm{D, B, R, Q_D, and\ Q_B}$

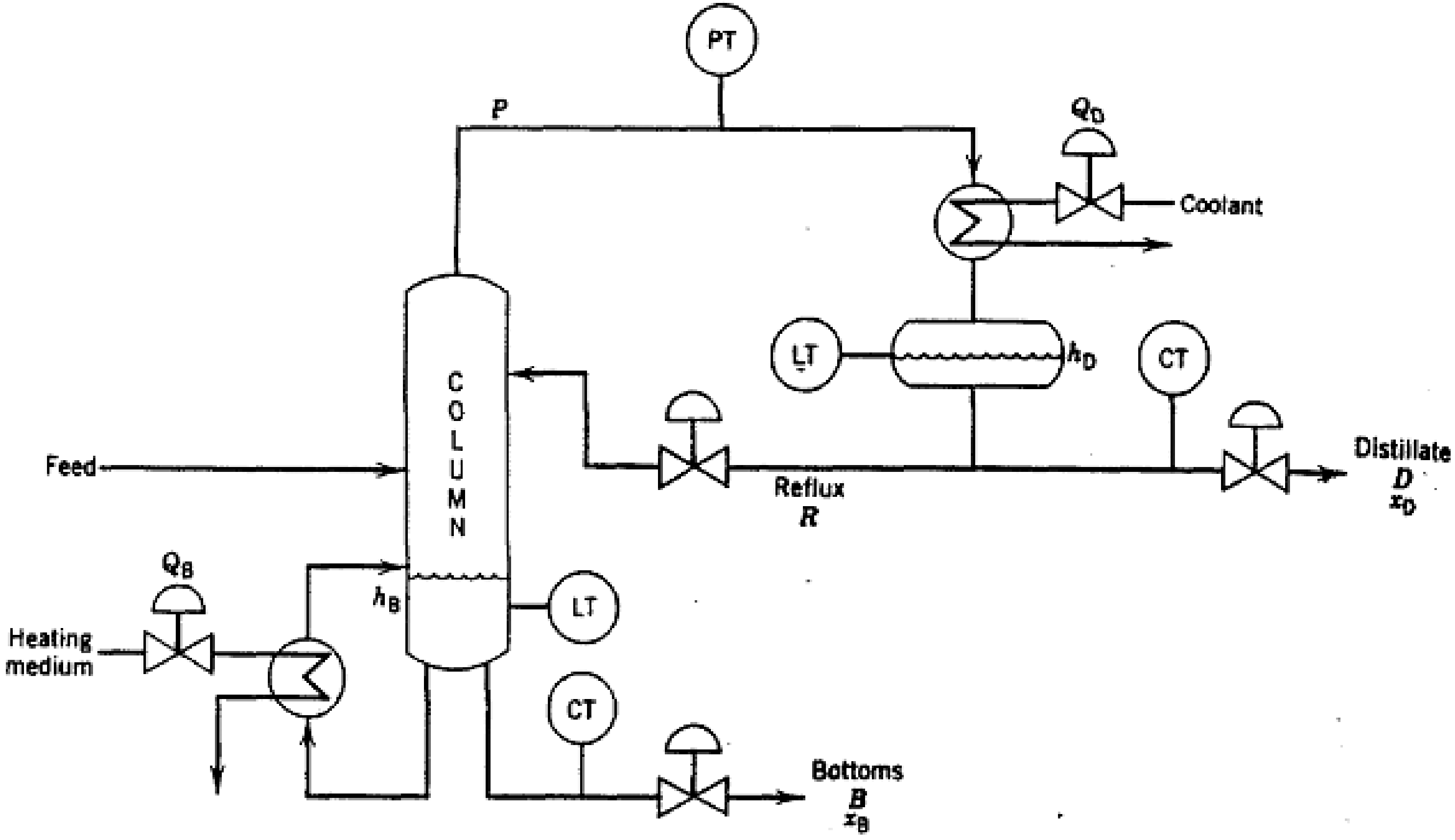


Figure 18.8. Controlled and manipulated variables for a typical distillation column.

1.2.1 MIMO Model Linearization

- Refer to Equation (1.1)
- Nonlinear model: $\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{v})$

    $\boldsymbol{z}$: state variable $(n \times 1)$

    $\boldsymbol{v}$: control variable $(r \times 1)$

- Linearization→ define deviation variables:

$$\boldsymbol{x} = \boldsymbol{z} - \boldsymbol{z}_{ss}$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_{ss}$$

(Requires iterative solution of $\boldsymbol{f}(\boldsymbol{z}_{ss}, \boldsymbol{v}_{ss}) = 0$)

1.2.1 MIMO State Space Model

- Use 1st order Taylor Series (invalid for large $\boldsymbol{x}$, $\boldsymbol{u}$)
- Scalar case:

$$\dot{z} = \dot{x} = f(z_{ss}, v_{ss}) + \left.\frac{\partial f}{\partial z}\right|_{ss} (z - z_{ss}) + \left.\frac{\partial f}{\partial v}\right|_{ss} (v - v_{ss})$$

$$= \left.\frac{\partial f}{\partial z}\right|_{ss} (x) + \left.\frac{\partial f}{\partial v}\right|_{ss} (u)$$

- Vector case:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{A}_{n\times n} = \left\{ \left.\frac{\partial f_i}{\partial z_j}\right|_{ss} \right\} (Jacobian), \boldsymbol{B}_{n\times r} = \left\{ \left.\frac{\partial f_i}{\partial v_j}\right|_{ss} \right\}$$

1.2.1 Non-Uniqueness of State Space Representation

Consider a MIMO system described by the state space model

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad , x \in R^{n\times n}, u \in R^{r\times 1} \\ y(t) &= Cx(t) + Du(t) \qquad , y \in R^{m\times 1} \end{aligned}\right\} \quad (1.2)$$

Define a new state vector by

$x(t) = T\hat{x}(t)$ known as similarity transformation

where T is a non-singular $n \times n$ matrix

Then from Equation (1.2), we have

$$T\dot{\hat{x}}(t) = AT\hat{x}(t) + Bu(t)$$

i.e.,

$$\dot{\hat{x}}(t) = T^{-1}AT\hat{x}(t) + T^{-1}Bu(t)$$

1.2.1 Non-Uniqueness of State Space Representation

Define

$$T^{-1}AT = \hat{A}, \quad T^{-1}B = \hat{B}, \; CT = \hat{C}, \; D = \hat{D}$$

We obtain

$$\dot{\hat{x}}(t) = \hat{A}x(t) + \hat{B}u(t)$$

$$y(t) = \hat{C}x(t) + \hat{D}u(t)$$

Hence, there are infinitely many state space representations for a given system since T can be any non-singular $n \times n$ matrix

In some applications, we may want to diagonalize the state matrix A. This may be done by properly choosing a matrix T such that

$T^{-1}AT =$ diagonal form or Jordan canonical form

1.2.1 Controllability and Observability

Consider a system described by the state-space model:

$\dot{x}(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

where the state $x(t)$ is a nth-vector.

Controllability: The above system, or equivalently the pair (A,B), is said to be state controllable if, for any initial state $x(0) = x_0$, any time $t_1 > 0$ and any final state x_1, there exists an input $u(t)$ such that $x(t_1) = x_1$. Otherwise, the system is said to be state uncontrollable.

The state space system is CONTROLLABLE if and only if the rank of

$\Sigma = \left[B\ AB\ \cdots A^{n-1}B \right]$ is n.

Observability: The above system (or the pair (A,C)) is said to be state observable if, for any time $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_1]$. Otherwise the system, or (A,C), is said to be state unobservable..

The state space system is OBSERVABLE if and only if the rank of

$\Theta = \left[C^T\ A^T C^T\ \cdots (A^{n-1})^T C^T \right]^{\mathrm{T}}$ is n.

1.2.1 Minimal Realization

A state space realization of a general linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is called a MINIMAL realization if the state-space system is both CONTROLLABLE and OBSERVABLE.

NOTE: If the state space model is NOT a MINIMAL realization, then the poles of the system $G(s)$ are a subset of the eigenvalues of A.

1.2.1 State Space to Transfer Function Matrix

- Given the Linear Time-Invariant (LTI) state dynamics

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \qquad & x \in R^{n\times n}, u \in R^{r\times 1} \\ y(t) &= C\mathbf{x}(t) + Du(t) \qquad & y \in R^{m\times 1} \end{aligned}\right\} \text{Equation (1.1)}$$

  what is the corresponding transfer function?

- Start by taking the Laplace Transform of these equations

$$\begin{aligned} \mathcal{L}\{\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t)\} \\ sX(s) - \mathbf{x}(0^-) &= AX(s) + BU(s) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{y(t) &= C\mathbf{x}(t) + Du(t)\} \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

  which gives

$$\begin{aligned} (sI - A)X(s) &= BU(s) + \mathbf{x}(0^-) \\ \Rightarrow X(s) &= (sI - A)^{-1}BU(s) + (sI - A)^{-1}\mathbf{x}(0^-) \end{aligned}$$

  and

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) + C(sI - A)^{-1}\mathbf{x}(0^-)$$

- By definition $G(s) = C(sI - A)^{-1}B + D$ is called the **Transfer Function** of the system.

- And $C(sI - A)^{-1}\mathbf{x}(0^-)$ is the initial condition response.
  - It is part of the response, but not part of the transfer function.

1.2.1 Example: Controllability, Observability, Minimal Realization

$$\dot{x}_1 = -5x_1 - 6x_2 + u$$

$$\dot{x}_2 = x_1$$

$$y = x_1 + 2x_2$$

That is,

$$A = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \end{bmatrix},\ D = 0$$

$$G(s) = C(sI - A)^{-1} B + D = \frac{s+2}{(s+2)(s+3)}$$ (Pole-Zero Cancellation)

Controllability Matrix $\Sigma = [B \mid AB] = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

Rank of $\Sigma = 2 \Rightarrow (A, B)$ is Controllable

Observability Matrix $\Theta = \left[C^T \mid C^T A^T\right]^T = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$

Rank of $\Theta = 1 \Rightarrow (A, C)$ is Unobservable

Hence, (A, B, C, D) is not a minimal realization of $G(s)$

1.2.1 Example: Non-Unique State Space Representation

Alternatively, consider

$$\dot{x}_1' = -6x_2' + u$$

$$\dot{x}_2' = x_1' - 5x_2'$$

$$y = x_1' - 4x_2'$$

That is,

$$A' = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix},\ B' = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C' = \begin{bmatrix} 1 & -4 \end{bmatrix},\ D' = 0$$

Controllability Matrix $\Sigma = \left[B' \mid A'B' \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Rank of $\Sigma = 2 \Rightarrow \left(A', B' \right)$ is Controllable

Observability Matrix $\Theta = \left[C'^T \mid C'^T A'^T \right]^T = \begin{bmatrix} 1 & -4 \\ -4 & 14 \end{bmatrix}$

Rank of $\Theta = 2 \Rightarrow \left(A', C' \right)$ is Observable

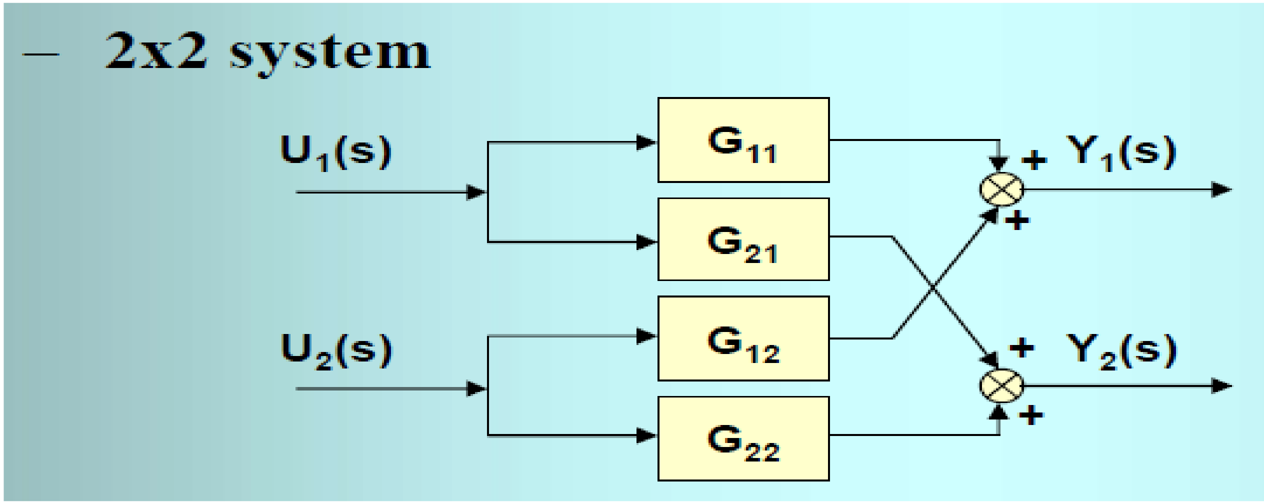
$$G(s) = C'\left(sI - A' \right)^{-1} B' + D' = \frac{s+1}{(s+2)(s+3)}$$

Hence, $\left(A', B', C', D' \right)$ is another minimal realization of $G(s)$

1.2.2 MIMO Transfer Function Matrix

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_m(s) \end{bmatrix} = \underbrace{\begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1r}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & & G_{mr}(s) \end{bmatrix}}_{G(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_r(s) \end{bmatrix}$$

- **SISO (r=1,m=1) MISO (m=1) and SIMO (r=1) are possible**
- **Note that G(s) is equivalent to a Smith-McMillan form and can also be expressed as Matrix Fraction Description (MFD)**



1.2.2 Transfer Function Matrix to State Space

$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \qquad \boldsymbol{x} = \boldsymbol{V}\boldsymbol{z} \quad \rightarrow \qquad \boldsymbol{V}\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{V}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}$

$\dot{\boldsymbol{z}} = \boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V}\boldsymbol{z} + \boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{u} = \boldsymbol{\Lambda}\boldsymbol{z} + \boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{u}$

$(s\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{Z}(s) = \boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{U}(s)$

$\boldsymbol{Z}(s) = (s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{U}(s), \quad \boldsymbol{V}^{-1}\boldsymbol{X}(s) = \boldsymbol{Z}(s)$

$\boldsymbol{V}^{-1}\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{U}(s)$

$\boldsymbol{X}(s) = \boldsymbol{V}(s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{U}(s)$

$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) = \boldsymbol{C}\boldsymbol{V}(s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V}^{-1}\boldsymbol{B}\boldsymbol{U}(s)$

Make $\boldsymbol{C}\boldsymbol{V} = \boldsymbol{\gamma}$, $\boldsymbol{V}^{-1}\boldsymbol{B} = \boldsymbol{E}$, then the transfer function is

$\boldsymbol{G}_p(s) = \boldsymbol{C}\boldsymbol{V}(s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V}^{-1}\boldsymbol{B} = \boldsymbol{\gamma}(s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{E}$

Question:

Given $\boldsymbol{G}_p(s)$, how do we find $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ which gives a model of minimal order? (or given step response for all inputs/outputs)

- Example: 2×2 (Second order)

$$\boldsymbol{G}_p(s) = \begin{bmatrix} \dfrac{2s+11}{(s+4)(s+2)} & \dfrac{s+6}{(s+4)(s+2)} \\ \dfrac{s-5}{(s+4)(s+2)} & \dfrac{s-2}{(s+4)(s+2)} \end{bmatrix} = \boldsymbol{\gamma}(s\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{E}$$

(2 poles => 2nd order, 2×2)

$$= \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} \dfrac{1}{s-p_1} & 0 \\ 0 & \dfrac{1}{s-p_2} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{E_{11}\gamma_{11}}{s-p_1} + \dfrac{E_{21}\gamma_{12}}{s-p_2} & \dfrac{E_{12}\gamma_{11}}{s-p_1} + \dfrac{E_{22}\gamma_{12}}{s-p_2} \\ \dfrac{E_{11}\gamma_{21}}{s-p_1} + \dfrac{E_{21}\gamma_{22}}{s-p_2} & \dfrac{E_{12}\gamma_{21}}{s-p_1} + \dfrac{E_{22}\gamma_{22}}{s-p_2} \end{bmatrix}$$

1.2.2 Example: Transfer Function Matrix to State Space

Expand $\boldsymbol{G}_p(s)$ by partial fraction expansion

$$\boldsymbol{G}_p(s) = \begin{bmatrix} \frac{-{}^{3}/_{2}}{(s+4)} + \frac{{}^{7}/_{2}}{(s+2)} & \frac{-1}{(s+4)} + \frac{2}{(s+2)} \\ \frac{{}^{9}/_{2}}{(s+4)} - \frac{{}^{7}/_{2}}{(s+2)} & \frac{3}{(s+4)} - \frac{2}{(s+2)} \end{bmatrix}$$

Matching coefficients,

$$\begin{bmatrix} E_{11}\gamma_{11} & E_{12}\gamma_{11} \\ E_{11}\gamma_{21} & E_{12}\gamma_{21} \end{bmatrix} = \begin{bmatrix} -3/2 & -1 \\ 9/2 & 3 \end{bmatrix}$$

Let $\gamma_{11} = 1$, then $E_{11} = -3/2$, $E_{12} = -1$, $\gamma_{21} = -3$

$$\begin{bmatrix} E_{21}\gamma_{12} & E_{22}\gamma_{12} \\ E_{21}\gamma_{22} & E_{22}\gamma_{22} \end{bmatrix} = \begin{bmatrix} 7/2 & 2 \\ -7/2 & -2 \end{bmatrix}$$

Let $\gamma_{12} = 1$, then $E_{21} = 7/2$, $E_{22} = 2$, $\gamma_{22} = -1$

1.2.2 Example: Transfer Function Matrix to State Space

Therefore

$$\boldsymbol{E} = \boldsymbol{V}^{-1}\boldsymbol{B} = \begin{bmatrix} -3/2 & -1 \\ 7/2 & -1 \end{bmatrix} \qquad (1)$$

$$\boldsymbol{\gamma} = \boldsymbol{C}\boldsymbol{V} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix} \qquad (2)$$

i. Assume $\boldsymbol{C}$, calculate $\boldsymbol{V}$ in Eq. (2);

ii. Using $\boldsymbol{V}^{-1}$, find $\boldsymbol{B}$ in Eq.(1);

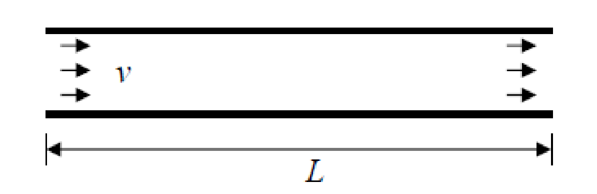
iii. $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$.

Note that there is an infinite number of realizations to yield $\boldsymbol{G}_p(s)$

1.2.2 Time Delay

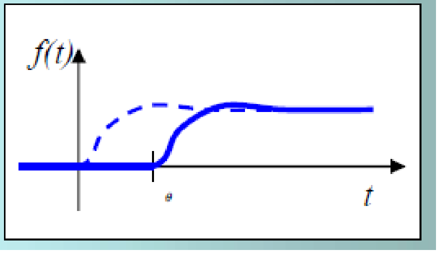
- **Fluid transportation through a pipe**
  - Also, called distance-velocity lag, transportation lag, dead time

$$\theta = \frac{\text{length of pipe}}{\text{fluid velocity}} = \frac{\text{volume of pipe}}{\text{volumetric flowrate}}$$



- **Transfer function**

$$y(t) = \begin{cases} 0 & \text{for } t < \theta \\ x(t-\theta) & \text{for } t \geq \theta \end{cases} \Rightarrow \boxed{\frac{Y(s)}{X(s)} = G(s) = e^{-\theta s}}$$



- Note that there is no exact representation of a pure time delay using ordinary differential equations - this would require an infinite number of states. Instead, the time delay is instead introduced explicitly in the argument when representing the control variable $u(t-\theta)$ as a function of time.

1.2.2 Continuous to Discrete-Time Model

- Fundamental models based on first principles modelling will typically result in continuous-time models.
- Often, control design is performed with a continuous-time model. The continuous-time controller is thereafter converted to a discrete-time controller for implementation in a computer.
- Consider a continuous-time linear model

$$\dot{x} = A_c x(t) + B_c u(t)$$

- Assuming zero order hold and a timestep of length *h*, integration over one timestep (from *t* = *kh* to *t* = *kh* + *h*) gives

$$x(kh+h) = e^{A_c h} x(kh) + \int_{kh}^{kh+h} e^{A_c(kh+h-r)} B_c u(r) dr$$

- This is commonly expressed as the following discrete-time model

$$x_{k+1} = A_d x_k + B_d u_k$$

where the matrices A_d and B_d are given by

$$\begin{aligned} A_d &= e^{A_c h} \\ B_d &= \int_{kh}^{kh+h} e^{A_c(kh+h-r)} B_c u(r) dr = A_c^{-1} \left(e^{A_c h} - I \right) B_c \end{aligned}$$

1.2.3 Rank of a Rational Matrix

Given a $q \times p$ rational (polynomial matrix) $G(s)$. The normal rank of $G(s)$ is n when the number of linearly independent rows (columns) over the field of rational functions $\alpha_i, i = 1, \ldots, n$ is n, i.e.

$\alpha_1 G_{1:q,1}(s) + \ldots \alpha_n G_{1:q,n}(s) = 0 \Rightarrow \alpha_1 = \ldots = \alpha_n = 0.$

Normal rank is the rank of s at "almost all" values of s.

Examples:

(a) $G(s) = \begin{bmatrix} s & 1 \\ s^2 + s & s+1 \end{bmatrix}$ has normal rank = 1 because

$$\alpha_1 \begin{bmatrix} s \\ s^2 + s \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ s+1 \end{bmatrix} = 0 \text{ for all } s \text{ with } \alpha_1 = \frac{1}{s}, \alpha_2 = -1$$

(b) $G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s+2}{(s+1)(s+3)} \\ \dfrac{1}{s+2} & \dfrac{1}{s+3} \end{bmatrix}$ has normal rank = 1 because

$$\alpha_1 \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s+2} \end{bmatrix} + \alpha_2 \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} \\ \dfrac{1}{s+3} \end{bmatrix} = 0 \text{ for all } s \text{ with } \alpha_1 = -1, \alpha_2 = \frac{s+3}{s+2}$$

(c) $G(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{s+1}{s} \\ 0 & \dfrac{s+1}{s+2} \end{bmatrix}$ has normal rank = 2 because

$$\alpha_1 \begin{bmatrix} \dfrac{1}{s} \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} \dfrac{s+1}{s} \\ \dfrac{s+1}{s+2} \end{bmatrix} = 0 \text{ for all almost all } s \text{ with } \alpha_1 = \alpha_2 = 0. \text{ Rank } G = 1 \text{ only at } s = -1.$$

1.2.3 Poles and Zeros of MIMO Systems – Transfer Function Matrix

- Given $m\times r$ $G(s)=N(s)D^{-1}(s)$ where $(N(s),D(s))$ is a right co-prime Matrix Factor Description (MFD) (Lecture 5). Suppose the normal rank $G(s)=\min(m,r)$.
  (i) p is a Smith McMillan pole of $G(s)\Leftrightarrow r\times r$ $D(p)$ is singular
  (ii) z is a Smith McMillan zero of $G(s)\Leftrightarrow m\times r$ $N(z)$ has rank $<$ $\min(m,r)$

- Definition: Given any *m* x *r* constant rational matrix *M*, the minors of order *n* is the set of determinants of all possible *n* x *n* submatrices obtained by deleting rows and columns in *M*.

- The pole polynomial of $G(s)$ is the Least Common Multiple (LCM) of all the minors of $G(s)$

- Generically, <u>NON-SQUARE</u> transfer function matrix has <u>NO</u> zero as it is not typical for several of its minors to become simultaneously singular at specific values of *s*.

1.2.3 Poles and Zeros of MIMO Systems – SQUARE Transfer Function Matrix

1) Consider special case when $G(s)$ is a SQUARE $n \times n$ matrix, i.e. number of inputs = number of outputs:

The inverse of the square transfer function matrix $G(s)$ is given by:

$$G^{-1}(s) = \frac{adj\left[G(s)\right]}{\left|G(s)\right|}$$

2) For SISO, the zeros of transfer function $g(s) = N(s)/D(s)$ are the poles of $1/g(s) = D(s)/N(s)$

3) By extension, the zeros of a SQUARE transfer function matrix $G(s)$ is defined as the poles of the inverse transfer function matrix, i.e. roots of the equation:

$$\left|G(s)\right| = 0$$

4) If the system has no time-delay, the determinant of a SQUARE transfer function matrix is a ratio of two polynomials; the roots of the numerator polynomial are the zeros, and the roots of the denominator polynomial are the poles.

1.2.3 Poles and Zeros of MIMO Systems – State Space

Since the inverse of matrix $(sI - A)$ can be written as:

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

Then the transfer function matrix $G(s)$ is given by:

$$G(s) = \frac{Cadj(sI - A)B}{|sI - A|} + D$$

<u>POLES of $G(s)$</u>: The poles of $G(s)$ are the zeros of $|sI - A|$, i.e. eigenvalues of A, given by:

$$|sI - A| = 0$$

or

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

<u>INVARIANT ZEROS of $G(s)$</u>: The invariant zeros of the system are the values of s that makes rank $E(s) < n + \min(m, r)$

where

Rosenbrock's system matrix $E(s) = \begin{bmatrix} sI - A & B \\ C & D \end{bmatrix}$

Assuming that $G(s)$ has no Smith McMillan poles and zeros at same frequency s

$$\begin{bmatrix} I & 0 \\ C(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sI - A & B \\ 0 & G(s) \end{bmatrix},$$

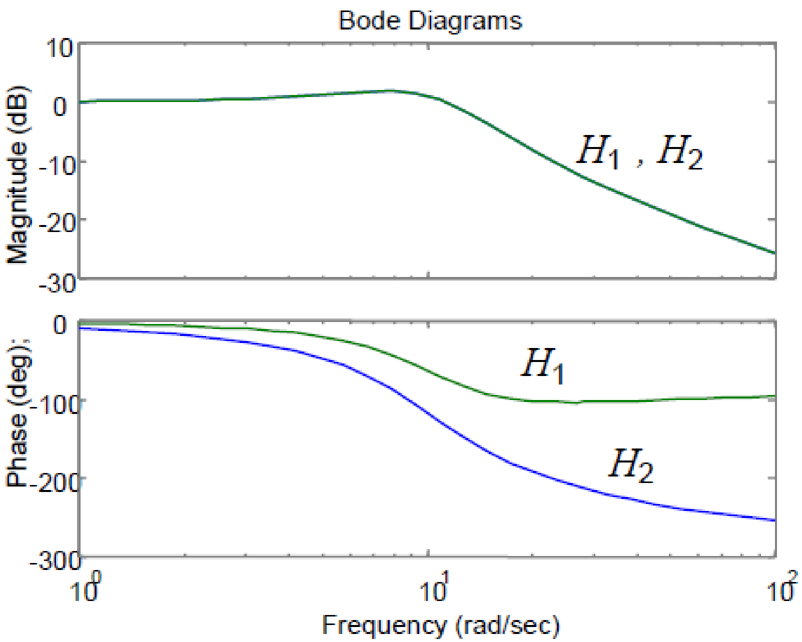
So, rank of $\begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = n + \min(m, r)$

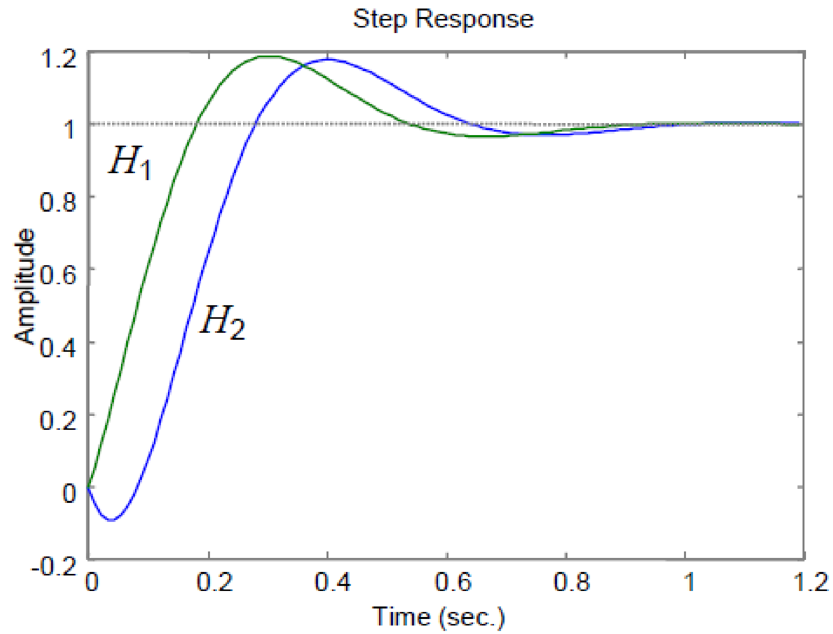
Note: INVARIANT ZERO is the SAME as TRANSMISSION ZERO for system in <u>minimal</u> form. TRANSMISSION ZERO is same as Smith McMillan zero.

1.2.3 Systems with RHP Zeros

- Right half plane zeros produce "non-minimum phase" behaviour
  - Phase of frequency response has additional phase lag for given magnitude
  - Can cause output to move *opposite* from input for a short period of time
- Example: $H_1 = \dfrac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ vs $H_2 = \dfrac{-(s-a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$= H_1(s) \times \left(-\frac{s-a}{s+a}\right)$$





1.2.3 System Poles and Stability

- The poles of a transfer function matrix, and the eigenvalues of the equivalent system matrix A in the state-space form, are one and the same

<u>Stability Theorem</u>

- A MIMO system is stable or Hurwitz if all the poles of the transfer function matrix lie in the left-half plane (LHP); otherwise it is unstable

- Stability requires the roots of the equation all lie in the LHP. It is identical whether we determine the stability of a multivariable system in terms of its transfer function matrix or its state-space model.

1.2.3 Interpretation of Transmission Zero

- Given a minimal state-space realization, we can evaluate $G(s) = C(sI - A)^{-1} B + D$. Let $G(s)$ have a transmission zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there will exist non-zero vectors u_z and y_z such that $G(z)u_z = 0$ and $k_z^H G(z)=0$
- Here u_z is defined as the input zero direction, and k_z is defined as the output zero direction. We usually normalize the direction vectors to have unit length,

  $u_z^H u_z = 1$; $k_z^H k_z = 1$

- (a) $r \le m$ There exists an intial state x_0 and an input $u_z(t) = u_0 e^{zt}$, $t \ge 0, u_0 \in C^r \ne 0,$ such that $y(t) = 0$ for all $t \ge 0$ where

  $$\begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  (b) $m \le r$ There exists a linear combination of outputs

  $k_z y(t) = k_1 y_1(t) + k_2 y_2(t) + \ldots + k_m y_m(t)$ such that $ky(t) = 0$

  for all input of the form $u(t) = u_0 e^{zt}, t \ge 0$ with arbitrary u_0

1.2.3 Example: Poles, Stability, Transmission Zero

$$G(s)=\begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & -\dfrac{1}{s+1} \\ \dfrac{2}{s+1} & -\dfrac{1}{2(s+1)(s+2)} \end{bmatrix}=\frac{1}{(s+1)(s+2)}\begin{bmatrix} 4 & -(s+2) \\ 2(s+2) & -\dfrac{1}{2} \end{bmatrix}$$

1) Since $G(s)$ is a SQUARE transfer function matrix

$$|G(s)|=\frac{-2}{(s+1)^2(s+2)^2}+\frac{2}{(s+1)^2}=\frac{-2+2(s+2)^2}{(s+1)^2(s+2)^2}=\frac{2(s+1)(s+3)}{(s+1)^2(s+2)^2}$$

The poles of $G(s)$ are at $s=-1, s=-1, s=-2, s=-2.$

So $G(s)$ is stable (Hurwitz).

The zeros of $G(s)$ are at $s=-1, s=-3.$

2) It is not obvious what is the zero of $G(s)$ by inspection.

However, note that at $s=-3$,

$$G(s=-3)=\begin{bmatrix} 2 & \dfrac{1}{2} \\ -1 & -\dfrac{1}{4} \end{bmatrix}\Rightarrow \operatorname{rank}\big(G(s=-3)\big)=1 \text{ (loses rank)}$$

So, $s=-3$ is a transmission zero of $G(s)$

1.2.3 Example: Poles, Stability, Transmission Zero

3) For info.,

Minors of Order 1 : $\dfrac{4}{(s+1)(s+2)}, -\dfrac{1}{s+1}, \dfrac{2}{s+1}, -\dfrac{1}{2(s+1)(s+2)}$

have denominator $(s+1)(s+2)$

Minors of Order 2 : $\dfrac{2(s+1)(s+3)}{(s+1)^2(s+2)^2}$ has denonimator $(s+1)^2(s+2)^2$

Hence, pole polynomial is the LCM of all minors of $G(s) = (s+1)^2(s+2)^2$

Poles of $G(s)$ are the roots of the pole polynomial

4) Using MatLab function :

```
s = tf('s');
P = [2/(s^2+3*s+2) -1/(s+1); 2/(s+1) -0.5/(s^2+3*s+2)]
S = ss(P)
```

A =

	x1	x2	x3	x4	x5	x6
x1	-3	-2	0	0	0	0
x2	1	0	0	0	0	0
x3	0	0	-1	0	0	0
x4	0	0	0	-1	0	0
x5	0	0	0	0	-3	-2
x6	0	0	0	0	1	0

B =

	u1	u2
x1	2	0
x2	0	0
x3	2	0
x4	0	1
x5	0	0.5
x6	0	0

C =

	x1	x2	x3	x4	x5	x6
y1	0	1	0	-1	0	0
y2	0	0	1	0	0	-1

D =

	u1	u2
y1	0	0
y2	0	0

1.2.3 Example: Transmission Zero

Example: Consider the transfer function

$$P(s)=\begin{bmatrix}\dfrac{2}{s^2+3s+2} & \dfrac{2s}{s^2+3s+2}\\ \dfrac{-2s}{s^2+3s+2} & \dfrac{-2}{s^2+3s+2}\end{bmatrix}$$

A minimal realization of $P(s)$ is given by

$$A=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix},\ B=\begin{bmatrix}2 & -2\\ -2 & 4\\ -4 & 2\end{bmatrix}\ C=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},\ D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Let's now use MATLAB to calculate transmission zeros:

```
»tzero(A,B,C,D)
ans =
    1.0000
```

Let's verify that $z=1$ is a transmission zero by calculating the rank of the system matrix:

```
»RSM_1 = [1*eye(3)-A  B;-C  D] ;
»rank(RSM_1)
ans =
    4
```

1.2.3 Example: Blocking of Input Signals

let's now find the input zero and state zero directions by looking at the nullspace of $RSM(1)$:

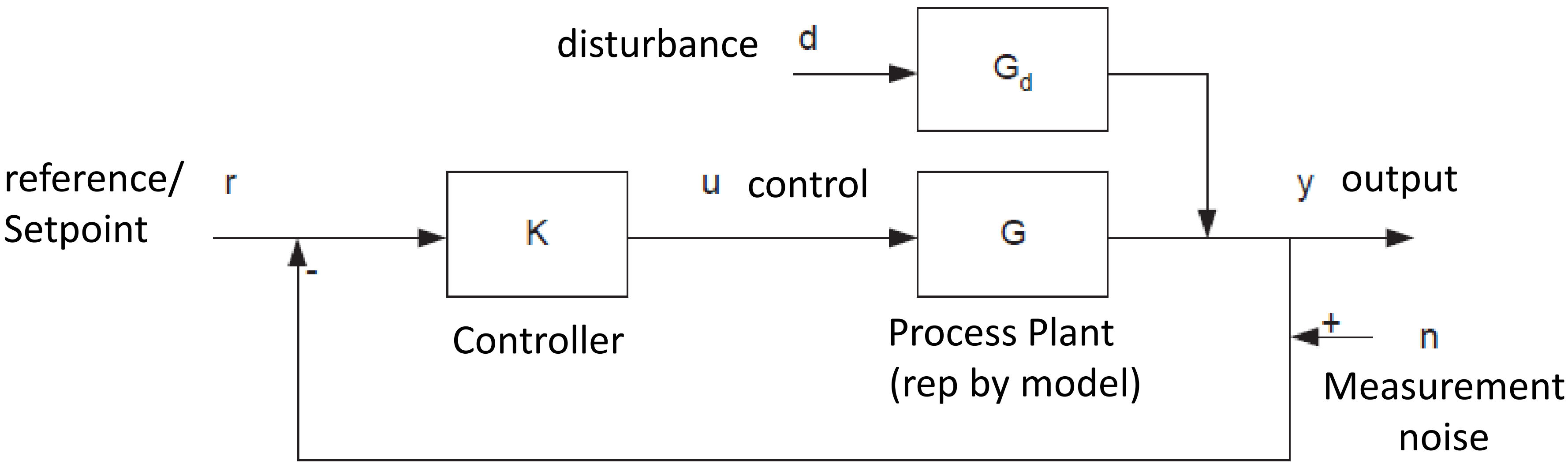
```
»null(RSM_1)
ans =

   0.5345
  -0.5345
  -0.5345
  -0.2673
   0.2673
```

It follows that the zero state direction is $x_0 = \begin{bmatrix} 0.5345 \\ -0.5345 \\ -0.5345 \end{bmatrix}$, and that the input zero direction is $u_0 = \begin{bmatrix} 0.2673 \\ -0.2673 \end{bmatrix}$.

Let's now simulate the response of the system to initial condition x_0 and an input $u(t) = u_0 e^t$:

```
»x0 = [0.5345; -0.5345; -0.5345];
»u0 = [0.2673; -0.2673];
»t = linspace(0,5);
»u = exp(t);
»[y,x] = lsim(A,B*u0,C,D*u0,u,t,x0);
»plot(t,y)
»axis([0 5 -1 1])
»xlabel('time, seconds')
»stitle('response to x_{0} and u(t)=u_{0}e^{t}')
```

1.3 Feedback Control System



State Space Representation:

$$\dot{x} = Ax + Bu + Ed$$

$$y = Cx + Du + Fd$$

Transfer Function Representation:

$$y(s) = \begin{bmatrix} G(s) & G_d(s) \end{bmatrix} \begin{bmatrix} u(s) \\ d(s) \end{bmatrix}$$

where

$$G(s) = C(sI - A)^{-1} B + D \quad , \quad G_d(s) = C(sI - A)^{-1} E + F$$

1.3.1 Closed Loop Transfer Function

- Output *y* of Closed Loop System

$$
\begin{aligned}
y(s) &= \left(I + G(s)K(s)\right)^{-1} G_d(s)d(s) + \left(I + G(s)K(s)\right)^{-1} G(s)K(s)\left(r(s) - n(s)\right) \\
&= S(s)G_d(s)d(s) + T(s)\left(r(s) - n(s)\right) \quad , \quad s = j\omega
\end{aligned}
$$

where

~low frequency ~low frequency ~high frequency

(output) Sensitivity Function $S(s) = \left(I + G(s)K(s)\right)^{-1}$

(output) Complementary Sensitivity Function $T(s) = \left(I + G(s)K(s)\right)^{-1} G(s)K(s)$

$$= G(s)K(s)\left(I + G(s)K(s)\right)^{-1}$$

$S(s) + T(s) = I \quad$ (Complementary)

1.3.1 Closed Loop Control

- Control *u*

$$
\begin{aligned}
u(s) &= K(s)\left(I+G(s)K(s)\right)^{-1}\left(r(s)-n(s)\right)-K(s)\left(I+G(s)K(s)\right)^{-1}G_d d(s) \\
&= \left(I+K(s)G(s)\right)^{-1}K(s)\left(r(s)-n(s)\right)-\left(I+K(s)G(s)\right)^{-1}K(s)G_d d(s) \\
&= S_I(s)K(s)\left(r(s)-n(s)\right)-S_I(s)K(s)G_d d(s)
\end{aligned}
$$

where

(input) Sensitivity Function $S_I(s)=\left(I+K(s)G(s)\right)^{-1}$

In general, for MIMO systems, $S_I(s)\neq\left(I+G(s)K(s)\right)^{-1}=S(s)$

1.3.1 Closed Loop Design Requirements

Requirements:

- Want sensitivity function S to be small to reject disturbance d
- When S is small, complementary sensitivity function $T \approx \mathrm{I}$, so closed loop system will track setpoint r very accurately
- However, measurement noise n will then affect the output
- We also want the loop gain GK to be large (S to be small) to handle model uncertainty

Considerations:

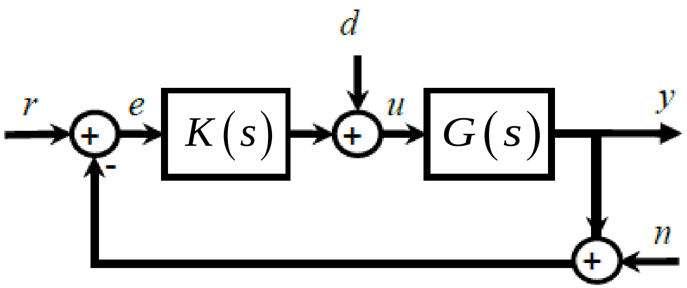
- Setpoint and disturbance are typically low frequency
- Measurement noise is high frequency
- Model uncertainty usually occurs at high frequency

Solution:

- Design S to be small (so $T \approx \mathrm{I}$) at low frequency
- Design T to be small (so $S \approx \mathrm{I}$) at high frequency
- Performance (T small at large frequency) vs Robustness (S small at high frequency) Trade-Offs

1.3.1 Design Trade-Offs

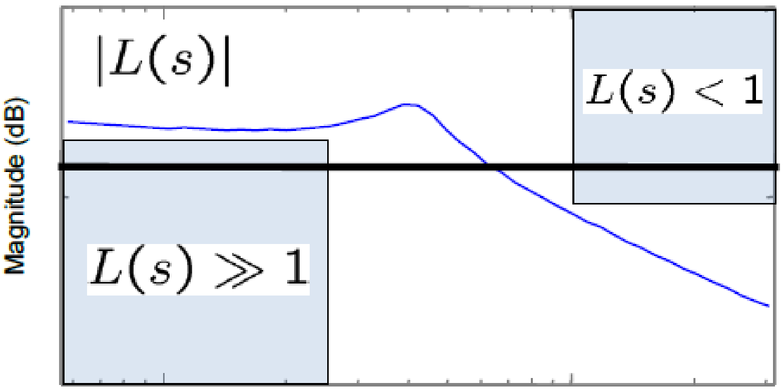
First consider Single Input Single Output (SISO) Process



$$H_{er} = \frac{1}{1+GK} =: S$$ Sensitivity function

$$H_{yn} = \frac{GK}{1+GK} =: T$$ Complementary sensitivity function

- Goal: keep S & T small
  - S small: low tracking error
  - T small: good noise rejection
- Problem: S + T = 1
  - Can't make *both* S & T small at the same frequency
  - Solution: keep S small at low frequency and T small at high frequency
  - Loop gain interpretation: keep L large at low frequency, small at high freq.



1.3.2 Bode's Integral Constraints on Sensitivity

- Consider a one d.o.f. stable control loop with open loop transfer function and delay τ

$$G_o(s)C(s) = e^{-s\tau}H_{ol}(s) \qquad \qquad \tau \geq 0$$

where $H_{0l}(s)$ is a rational transfer function of relative degree $n_r > 0$ and define

$$\kappa \stackrel{\triangle}{=} \lim_{s \to \infty} sH_{ol}(s)$$

- Assume that $H_{0l}(s)$ has no open loop poles in the open RHP. Then the nominal sensitivity function satisfies:

$$\int_0^{\infty} \ln|S_o(j\omega)|d\omega = \begin{cases} 0 & \text{for } \tau > 0 \\ -\kappa\frac{\pi}{2} & \text{for } \tau = 0 \end{cases}$$

1.3.2 Bode's Integral with RHP Poles and Delay

- The extension to *open loop unstable systems* is as follows:
- Consider a feedback control look with open loop transfer function having unstable poles located at p_1, …, p_N, pure time delay τ, and relative degree $n_r \geq 1$. Then, the nominal sensitivity satisfies:

$$\int_0^\infty \ln |S_o(j\omega)| \, d\omega = \pi \sum_{i=1}^{N} \mathcal{R}\{p_i\} \qquad \text{for} \quad n_r > 1$$

$$\int_0^\infty \ln |S_o(j\omega)| \, d\omega = -\kappa \frac{\pi}{2} + \pi \sum_{i=1}^{N} \mathcal{R}\{p_i\} \qquad \text{for} \quad n_r = 1$$

where $\kappa = \lim_{s\to\infty} sH_{0l}(s)$

1.3.2 Bode's Integral with RHP Zeros and Delay

- Assume that $H_{0l}(s)$ has open loop zeros in the open RHP, located at c_1, c_2, …, c_M, then

$$\int_0^{\infty} \frac{1}{w^2} \ln |T_0(jw)| \, dw = \pi\tau + \pi \sum_{i=1}^{M} \frac{1}{c_i} - \frac{\pi}{2k_v}$$

1.3.2 Bode's Integral Formula: Waterbed Effect

- Bode's integral formula for S = 1/(1+GK)= 1/(1+L):
  - Let p_k be the *unstable* poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
  - Theorem: the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1+L(j\omega)|} d\omega = \pi \sum_{p_k \in \mathrm{RHP}} \mathrm{Re}\, p_k$$

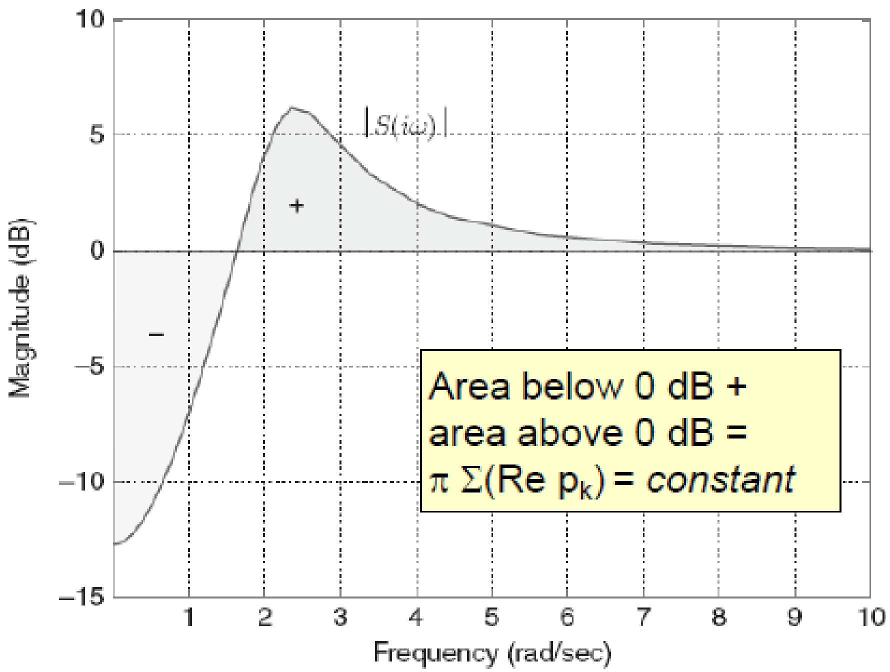


Fig. 5 Magnitude of $S(i\omega)$, illustrating the area rule (31): for this system, the area of attenuation (denoted −) must equal the area of amplification (denoted +), no matter what controller $C(s)$ is used.

Waterbed effect:

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Reading: Chandra, Buzi, Doyle, Glycolytic Oscillations and limits on robust efficiency, Science, 33, July 2011.

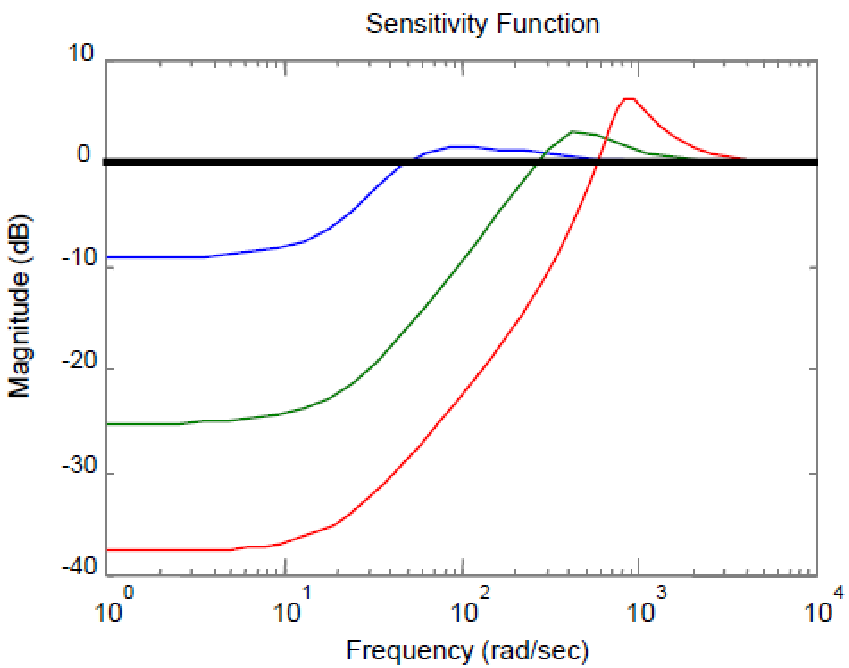
1.3.2 Example: Performance Limits

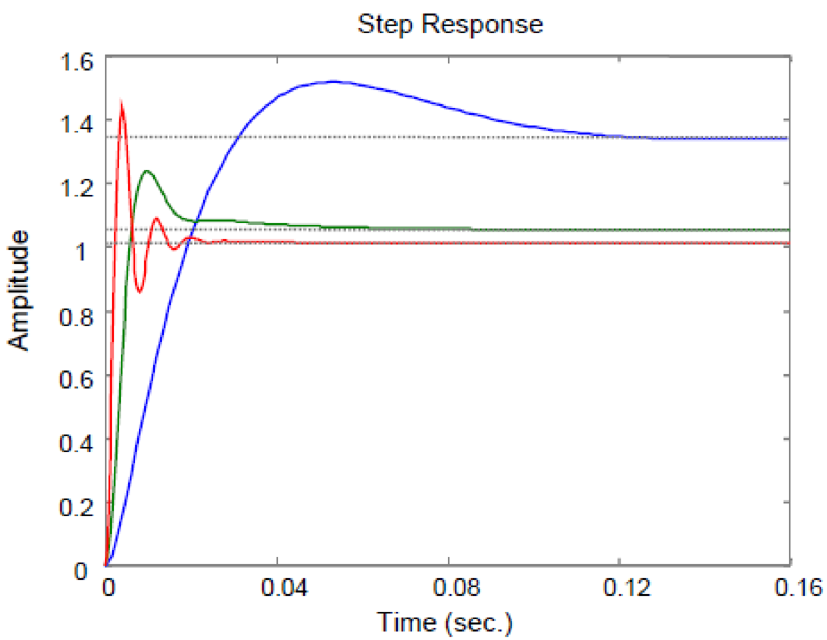
- Nominal design gives low perf
  - Not enough gain at low frequency
  - Try to adjust overall gain to improve low frequency response
  - Works well at moderate gain, but notice waterbed effect

- Bode integral limits improvement

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi r$$

  - Must increase sensitivity at some point





1.3.3 Poisson's Integral for Sensitivity

- Consider a feedback control loop with open loop RHP zeros located at $c_1, c_2, \ldots, c_M$, where $c_k = \gamma_k + j\delta_k$ and open loop unstable poles located at $p_1, p_2, .., p_N$ where $p_i = \alpha_i + j\beta_i$. Then the nominal sensitivity satisfies

$$\int_{-\infty}^{\infty} \ln |S_o(j\omega)| \frac{\gamma_k}{\gamma_k^2 + (\delta_k - \omega)^2} d\omega = -\pi \ln |B_p(c_k)| \qquad \text{for} \quad k = 1, 2, \ldots M$$

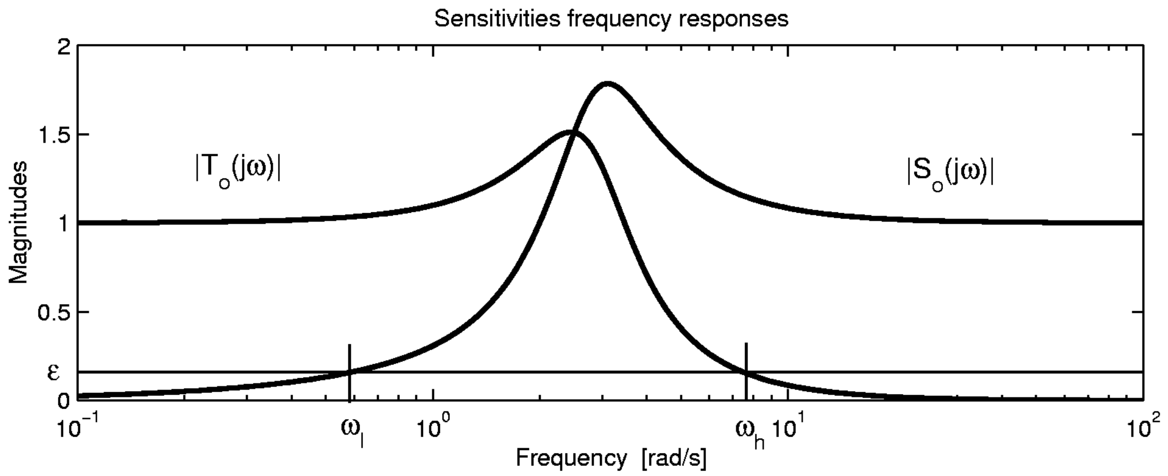
where the Blaschke products are defined as

$$B_z(s) = \prod_{k=1}^{M} \frac{s - c_k}{s + c_k^*} \qquad B_p(s) = \prod_{i=1}^{N} \frac{s - p_i}{s + p_i^*}$$

1.3.3 Design Constraints on both |S| and |T|

- Design constraints can be imposed on both $|S_0|$ and $|T_0|$. For example, say that we require

$$|S_0(jw)| < \varepsilon \quad \text{for} \quad w < w_l$$
$$|T_0(jw)| < \varepsilon \quad \text{for} \quad w > w_h$$



1.3.3 Peak S and T in the Presence of Both RHP Pole and Zero

- Using Poisson's Integral, we can develop bounds on the sensitivity

$$\ln S_{max} > \frac{1}{\Omega(c_k,\omega_h)-\Omega(c_k,\omega_l)}\left[|\pi \ln|B_p(c_k)|| + |(\ln\epsilon)\Omega(c_k,\omega_l)| - (\pi-\Omega(c_k,\omega_h))\ln(1+\epsilon)\right]$$

$$\Omega(c_k,\omega_c) = 2\arctan\left(\frac{\omega_c}{\gamma_k}\right)$$

with

$$\Omega(c_k,\infty) = 2\lim_{\omega_c\to\infty}\arctan\left(\frac{\omega_c}{\gamma_k}\right) = \pi$$

- If we require that $|T_0| < \varepsilon$ for $w > w_h$. Then it follows from the above result that the peak value of the complementary sensitivity will be bounded from below as follows:

$$\ln T_{max} > \frac{1}{\Omega(\alpha_i,\omega_h)}\left[\pi|\ln|B_z(\alpha_i)|| + \tau\alpha_i + |\ln\epsilon|(\pi-\Omega(\alpha_i,\omega_h)\right]$$

1.3.3 Bandwidth Limitations due to RHP Zeros

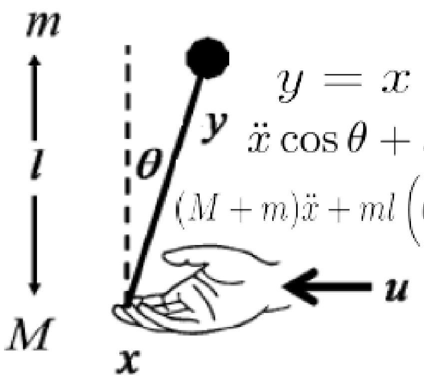
- Say we were to require the closed loop bandwidth to be greater than the magnitude of a right half plane (*real*) zero. In terms of the notation used in the figure, this would imply $\omega_l > \gamma_k$. We can then show using the Poisson formula that there is necessarily a very large sensitivity peak occurring beyond ω_l since $\Omega(c_k, \omega_l) \approx \Omega(c_k, \omega_h)$.
- The conclusion is that the <u>closed loop bandwidth should not exceed the magnitude of the smallest RHP open loop zero</u>. The penalty for not following this guideline is that a very large sensitivity peak will occur, leading to fragile loops (*non robust*) and large undershoots and overshoots.

1.3.3 Bandwidth Limitations due to RHP Poles

- We see that the lower bound on the complementary sensitivity peak is larger for systems with pure delays, and the influence of a delay increases for unstable poles which are far away from the imaginary axis, i.e. large α_i.
- The peak, T_{max}, <u>grows unbounded when a RHP zero approaches an unstable pole</u>, since then $|\ln|B_z(p_i)||$ grows unbounded.
- Say that we ask that the closed loop bandwidth be much smaller than the magnitude of a right half plane (*real*) pole. In terms of the notation used above, we would then have $\omega_h << \alpha_i$. Under these conditions, $\Omega(p_i, \omega_h)$ will be very small, leading to a very large complementary sensitivity peak. This is an unacceptable result. Thus we conclude that the <u>closed loop bandwidth should be greater than any RHP open loop poles.</u>

1.3.3 Example: Balancing an Inverted Pendulum

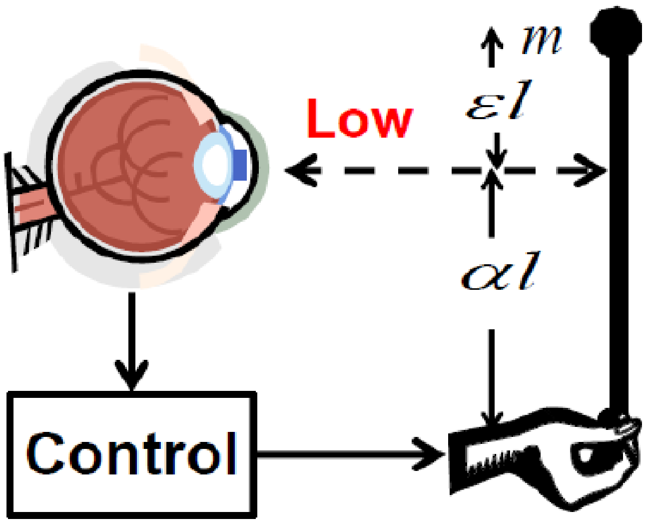
- Linearize the system
- Transfer function from u to y
- Calculate zeros and poles



$$y = x - \alpha l \sin\theta$$

$$\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta = 0$$

$$(M+m)\ddot{x} + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u$$



1.3.3 Example: Balancing an Inverted Pendulum

Dynamics (transfer function):

$$\frac{(1+\alpha l)\ell s^2 - g}{s^2(M\ell s^2 - (M+m)g)}$$

- RHP pole at:

$$p = \sqrt{g(M+m)/M}\frac{1}{\sqrt{\ell}}$$

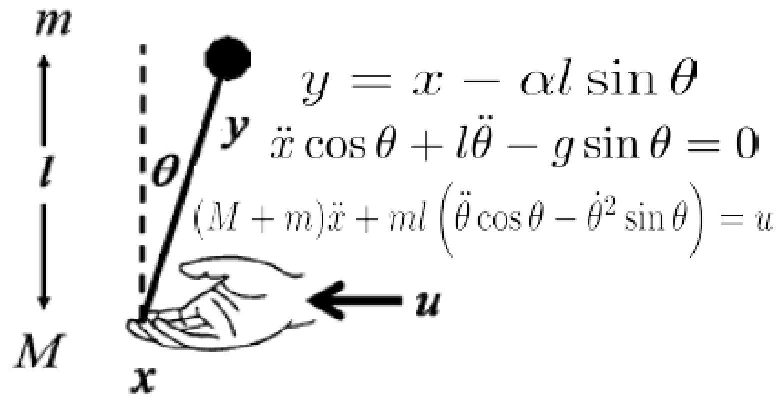
- A longer pendulum is easier to stabilize;

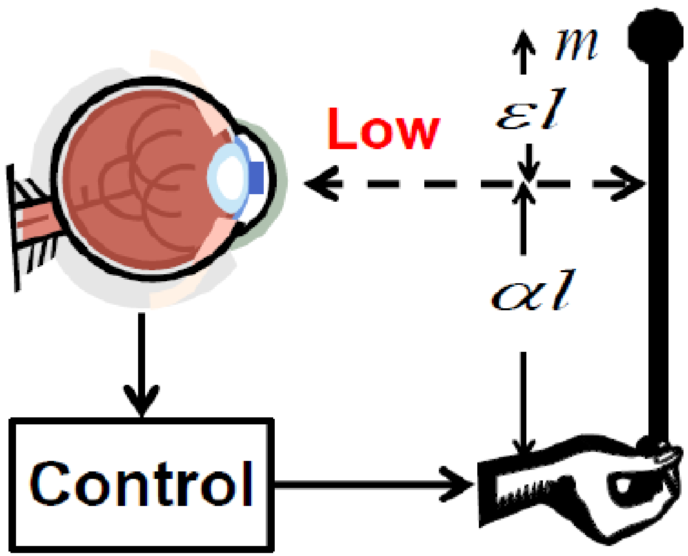
- RHP zero at

$$z = \sqrt{g/((1+\alpha)\ell)}$$

This is why if we want to move the pendulum to a new location, initially the pendulum will move to the opposite direction.

- RHP pole *and* RHP zero is *very* hard

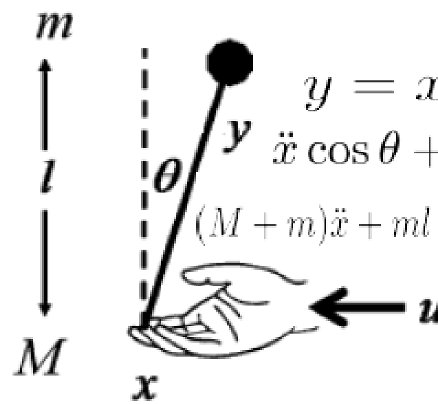




*Example from John Doyle

1.3.3 Example: Balancing an Inverted Pendulum

- $$p = \sqrt{g(M+m)/M}\frac{1}{\sqrt{\ell}}$$
  $$z = \sqrt{g/((1+\alpha)\ell)}$$



$$y = x - \alpha l \sin\theta$$
$$\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta = 0$$
$$(M+m)\ddot{x} + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u$$

$$\boxed{\begin{gathered} z = \sqrt{\frac{g}{l(1+\alpha)}},\ p = z\sqrt{1+r}\sqrt{1+\alpha},\ r = \frac{m}{M} \\ \frac{p+z}{p-z} = \frac{\sqrt{1+r}\sqrt{1+\alpha}+1}{\sqrt{1+r}\sqrt{1+\alpha}-1} \end{gathered}}$$

Want r and α large (but p small).

Want $p << z$ (not possible) with p to be small (less unstable)!

1.3.3 Extension to MIMO

- For a multivariable system, *S* must be small *in all directions* for $T \approx I$ to hold. One may state that the equality $S + T = I$ implies that the gain of $T \approx I$ *in the directions where S* is small.

- Freudenberg and Looze, show that Bode's and Poisson's Integral Formula can be generalized to multivariable systems

- We shall discuss it more in the MIMO module in Lecture 5