Chapter 5

Design of Discrete-time Controllers Based on Transfer Functions

5.1 Introduction

- Recall the discrete-time control system shown in Fig 5.1

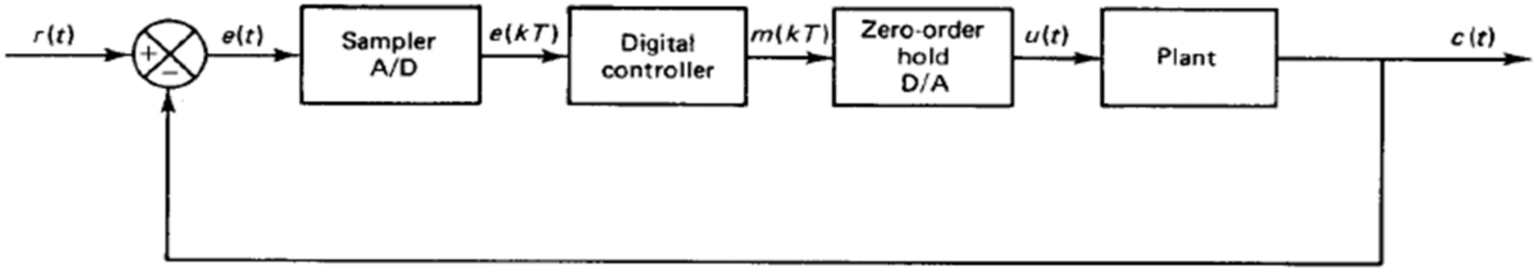


Figure 5.1

- It can be represented in Z-domain as in Fig 5.2

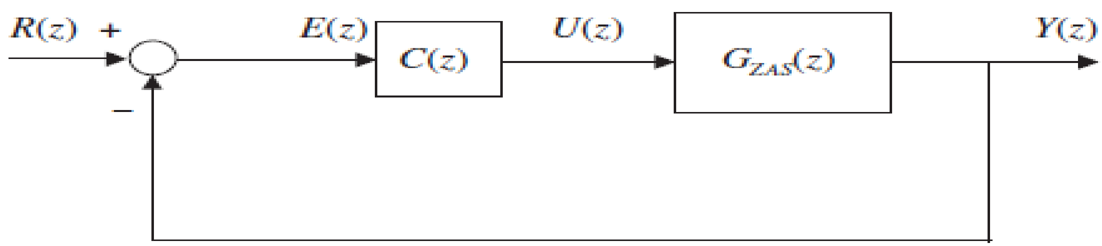


Figure 5.2

The system includes

1) a pulse transfer function model of the ZOH and the analog subsystem (plant); and
2) A cascade controller.

In this chapter, we first study how to deign digital controllers in the above figures based on transfer functions with the following approaches:

- Digital Implementation of Analog Controller Design
- Direct Z-domain Digital Controller Design - Frequency Response Design Approach
- Direct Control Design Based on Pulse Transfer Function

5.2 Digital Implementation of Analog Controller Design

- This section introduces an indirect approach to digital controller design.
- The approach is based on

1) designing an *analog controller* for the *analog plant;*
2) and then obtaining an *equivalent digital controller* and using it to digitally implement the desired control.

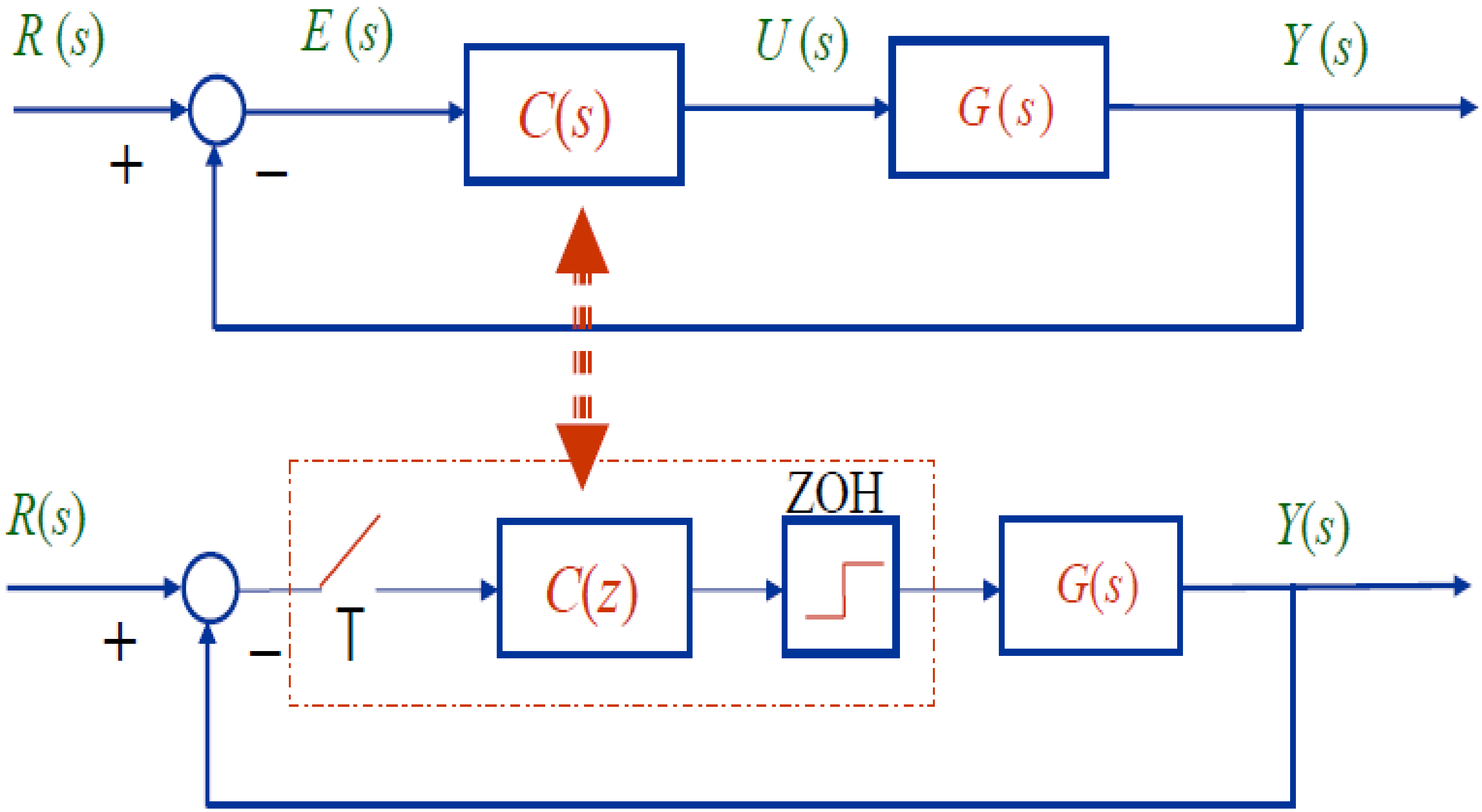
5.2.1 General Procedure

1. Design a controller *C(s)* for the analog subsystem to meet the desired design specifications.

   Use whatever methods we have learnt to design a continuous time controller

2. Map the analog controller to a digital controller $G_D(z)$ (i.e. C(z) in Figure 5.2) using a ***suitable transformation***.

3. Check the sampled time response of the digital control system and repeat steps 1 to 2, if necessary, until the design specifications are met.



The above design works very well if sampling period T is sufficiently small

Step 2 must satisfy the following requirements:

1. A stable analog filter (poles in the left half plane LHP) must transform to a stable digital filter.

2. The frequency response of the digital controller must closely resemble the frequency response of the analog filter in the frequency range $0 \rightarrow \frac{\omega_s}{2}$ where ω_s is the sampling frequency.

5.2.2 Methods for mapping from G(s) *to* $G_D(z)$ *:*

Consider the following 1st order controller

$$\frac{dy(t)}{dt} + ay(t) = ax(t) \qquad \text{Or} \qquad \frac{dy(t)}{dt} = -ay(t) + ax(t) \tag{5.1}$$

Its transfer function is given by:

$$\frac{Y(s)}{X(s)} = G(s) = \frac{a}{s+a} \tag{5.2}$$

Integrating (5.1) from $(k-1)T$ to kT gives

$$\int_{(k-1)T}^{kT} \frac{dy(t)}{dt}dt = -a\int_{(k-1)T}^{kT} y(t)dt + a\int_{(k-1)T}^{kT} x(t)dt \tag{5.3}$$

- **Backward Difference Method**

Integration by the backward difference method means that we approximate the areas

$$\int_{(k-1)T}^{kT} y(t)dt \approx y(kT)T \quad \text{and} \quad \int_{(k-1)T}^{kT} x(t)dt \approx x(kT)T$$

Then (5.3) becomes

$$y(kT) - y((k-1)T) \approx -ay(kT)T + ax(kT)T$$

Taking z-Transform and simplifying :

$$Y(z) = z^{-1}Y(z) - aT\left[Y(z) - X(z)\right]$$

So

$$G_D(z) = \frac{Y(z)}{X(z)} = \frac{a}{\frac{1-z^{-1}}{T} + a}$$

Notes:

- The mapping between *s* and *z* is

$$s = \frac{1-z^{-1}}{T} = \frac{z-1}{Tz} \qquad (5.4)$$

- The digital system is stable if

$$\mathrm{Re}\left(\frac{z-1}{Tz}\right) < 0$$

- **Forward Difference Method**

Consider the approximation that

$$\int_{(k-1)T}^{kT} y(t)dt \approx Ty((k-1)T), \qquad \int_{(k-1)T}^{kT} x(t)dt \approx Tx((k-1)T)$$

$$\Rightarrow \quad y(kT) - y((k-1)T) \approx -aT\left[y((k-1)T + aTx((k-1)T)\right]$$

Taking z-Transform and simplifying yield:

$$Y(z) = (1 - aT)z^{-1}Y(z) + aTz^{-1}X(z)$$

$$G_D(z) = \frac{Y(z)}{X(z)} = \frac{a}{\dfrac{1-z^{-1}}{Tz^{-1}} + a}$$

The mapping between *s* and *z* is $\quad s = \dfrac{1-z^{-1}}{Tz^{-1}} = \dfrac{z-1}{T}$ (5.5)

• Bilinear Transformation Method

Approximating

$$\int_{(k-1)T}^{kT} y(t)dt \approx \frac{1}{2}\left[y(kT)+y((k-1)T)\right]T, \qquad \int_{(k-1)T}^{kT} x(t)dt \approx \frac{1}{2}\left[x(kT)+x((k-1)T)\right]T$$

$$\Rightarrow\ y(kT)-y((k-1)T) \approx -\frac{aT}{2}[y(kT)+y((k-1)T)]+\frac{aT}{2}[x(kT)+x((k-1)T)]$$

Taking z-Transform and simplifying :

$$Y(z)=z^{-1}Y(z)-\frac{aT}{2}\left[Y(z)+z^{-1}Y(z)\right]\ +\frac{aT}{2}\left[X(z)+z^{-1}X(z)\right]$$

$$G_D(z)=\frac{Y(z)}{X(z)}=\frac{a}{\dfrac{2(1-z^{-1})}{T(1+z^{-1})}+a}$$

The mapping between *s* and *z* is $s=\dfrac{2(1-z^{-1})}{T(1+z^{-1})}=\dfrac{2(z-1)}{T(z+1)}$ (5.6)

- **Pole-Zero Matching**

In pole-zero matching, a discrete approximation is obtained from analog transfer function by mapping both poles and zeros using the following relationship:

$$z = e^{sT}$$

If the analog filter has *n* poles and *m* zeros, then there are *n-m* zeros at infinity. Thus add *n-m* or *n-m-1* zeros at -1 to obtain proper or strictly proper digital filter, respectively. That is, setting

$$z = e^{(j\omega T)} = -1 \;\; (\text{i.e., } \omega T = \pi)$$

This corresponds to selecting the folding frequency $\frac{\omega_s}{2}$, the highest frequency allowable without aliasing.

Finally, adjust the gain of the digital filter equal to that of the analog filter at a critical frequency

For an analog filter with transfer function

$$G_a(s) = K\frac{\prod_{i=1}^{m}(s-a_i)}{\prod_{i=1}^{n}(s-b_i)}$$

we have the following strictly proper digital filter

$$G(z) = \alpha\frac{(z+1)^{n-m-1}\prod_{i=1}^{m}(z-e^{a_iT})}{\prod_{i=1}^{n}(z-e^{b_iT})}$$

where α is a constant selected for equal filter gains at a critical frequency. For example, for a low-pass filter, α is selected to match the DC gains using

$$G(1)=G_a(0)$$

Example 5.1 Find a pole-zero matched digital filter approximation for the analog filter

$$G_a(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If the damping ratio is equal to *0.5* and the undamped natural frequency is *5 rad/s*, determine the transfer function of the digital filter for a sampling period of *0.1 s*. Check your answer using the frequency response of digital filters.

The filter has two zeros at infinity and two complex conjugate poles at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

where $$\omega_d = \omega_n\sqrt{1-\varsigma^2}$$

Then

$$G(z) = \frac{\alpha(z+1)}{z^2 - 2e^{-\zeta\omega_n T}\cos(\omega_d T)z + e^{-2\zeta\omega_n T}}$$ (Please see next page for derivations)

For the given numerical values, we have

$$\omega_d = 5\sqrt{1-0.5^2} = 4.33 \text{ rad/s}$$

Thus

$$G(z) = \frac{0.09625(z+1)}{z^2 - 1.414z + 0.6065}$$

$$(z-e^{s_1T})(z-e^{s_2T})$$

$$= z^2-(e^{s_1T}+e^{s_2T})z+e^{s_1T}e^{s_2T}$$

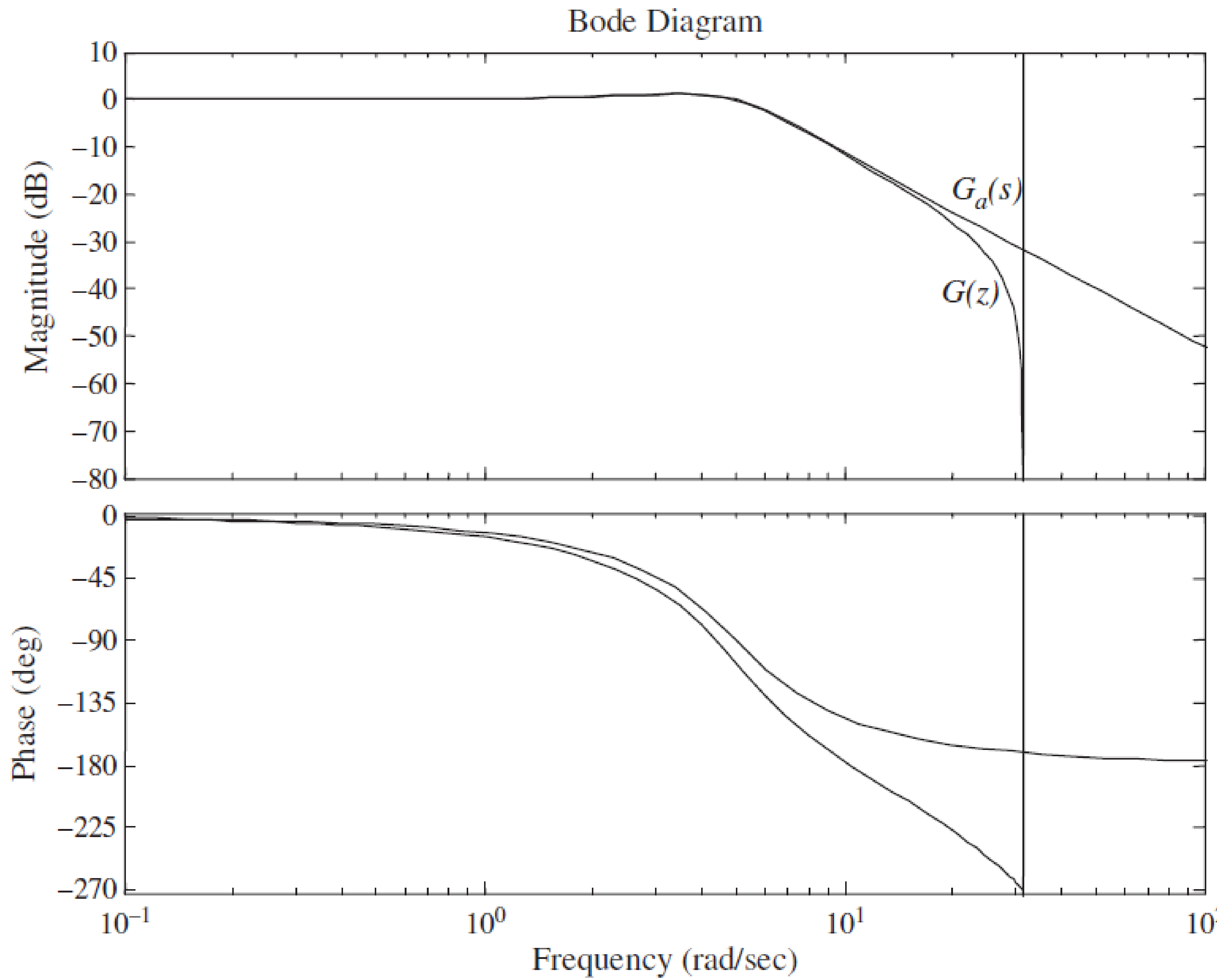
$$e^{s_1T}+e^{s_2T} = e^{-\zeta\omega_nT}e^{j\omega_dT}+e^{-\zeta\omega_nT}e^{-j\omega_dT}$$

$$= e^{-\zeta\omega_nT}(e^{j\omega_dT}+e^{-j\omega_dT})$$

$$= 2e^{-\zeta\omega_nT}\cos(\omega_dT)$$

$$e^{s_1T}e^{s_2T} = e^{(s_1+s_2)T}$$

$$= e^{-2\zeta\omega_nT}$$


Bode Diagram
Magnitude (dB)
10
0
−10
−20
−30
−40
−50
−60
−70
−80
$G_a(s)$
$G(z)$
Phase (deg)
0
−45
−90
−135
−180
−225
−270
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Example 5.2 A continuous-time transfer function with damping ratio 0.88, undamped natural frequency 1.15 rad/s and unity DC gain is given as

$$G_{cl}(s) = \frac{1.322}{s^2 + 2.024s + 1.322}$$

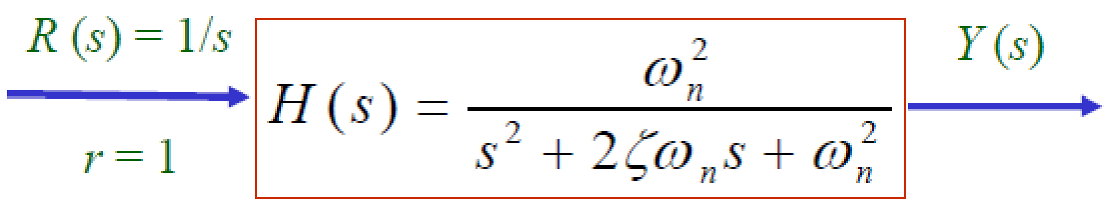
Suppose that the sampling time is chosen as *T= 0.02s*. Using pole-zero matching discussed above, the digital transfer function $G_{cl}(z)$ is obtained as

$$G_{cl}(z) = 0.25921 \cdot 10^{-3} \frac{z+1}{z^2 - 1.96z + 0.9603}$$

Example 5.3

Consider a car (BMW), which has a weight m = 1000 kg. Assuming the average friction coefficient b = 100, design a cruise control system such that the car can reach 100 km/h from 0 km/h in 8 s with an overshoot less 20%.

Suppose designed closed-loop system is



To meet the specifications

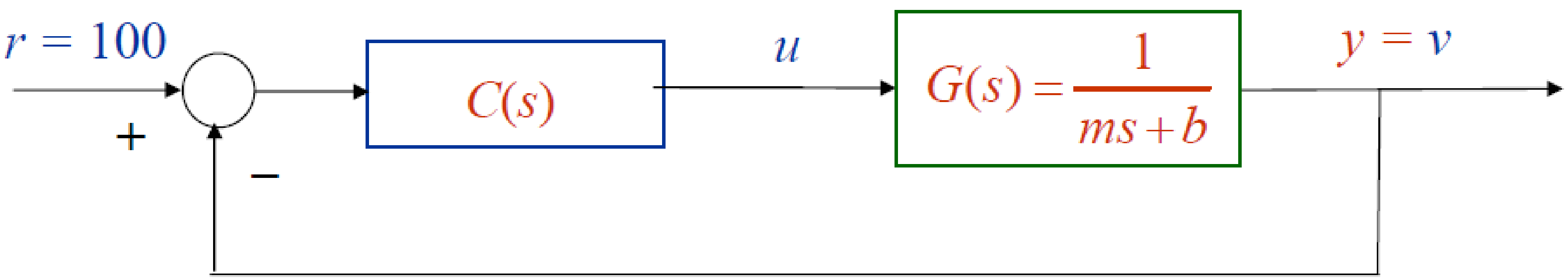
For overshoot: $\zeta \geq 0.46 \Rightarrow \zeta = 0.7$

For settling time (1%): ω_n =0.82 from $t_s = \dfrac{4.6}{\zeta\omega_n}$

The desired closed-loop transfer function

$$H_{\text{desired}}(s) = \frac{0.67}{s^2 + 1.15s + 0.67}$$

The car is modelled as *G(s)* as shown below



Let us try a PI controller, i.e. $C(s) = k_p + \frac{k_i}{s}$.

Then the actual closed loop system transfer function is

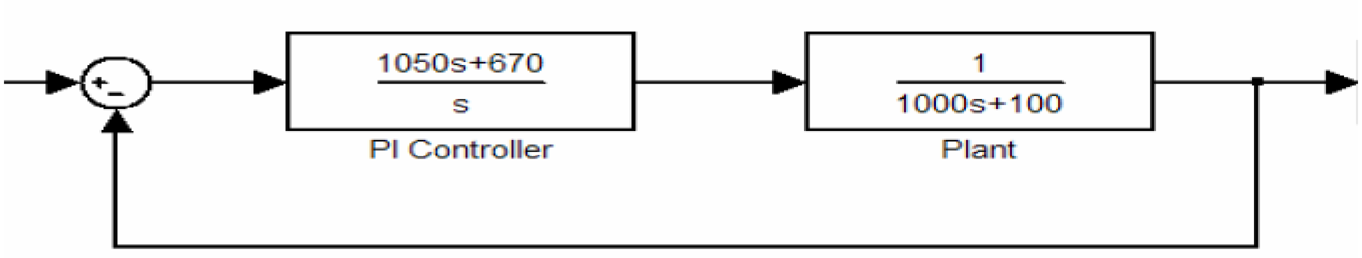
$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{0.001k_p s + 0.001k_i}{s^2 + (0.1 + 0.001k_p)s + 0.001k_i}$$

Comparing the coefficients on the denominators, we have

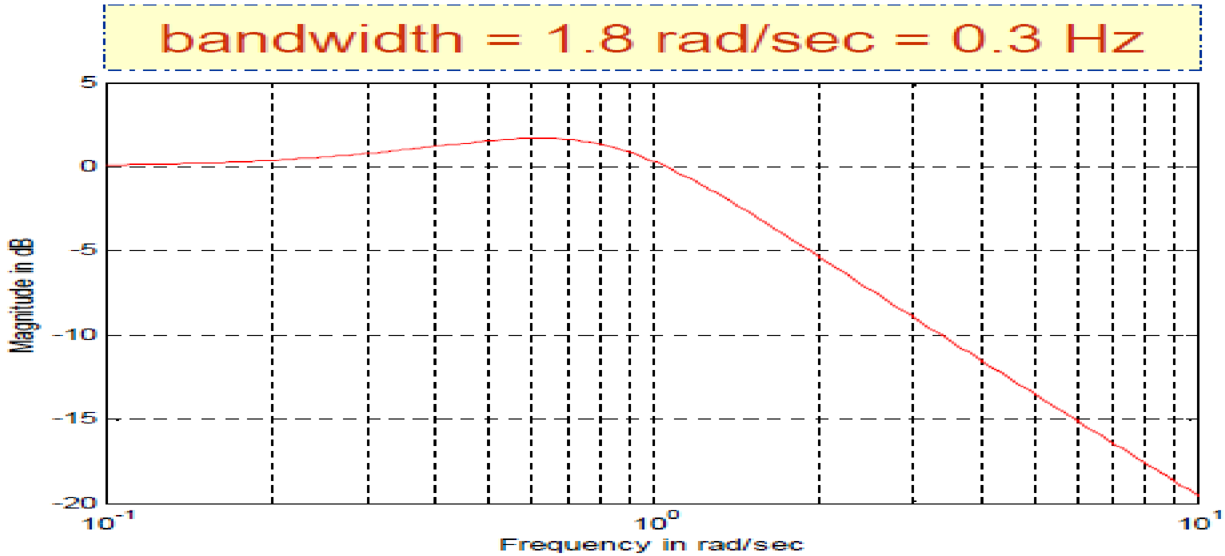
$$0.1 + 0.001k_p = 1.15 \quad \Rightarrow \quad k_p = 1050$$

$$0.001k_i = 0.67 \quad \Rightarrow \quad k_i = 670$$

The closed loop system is



From its frequency response, the bandwidth is 0.3Hz

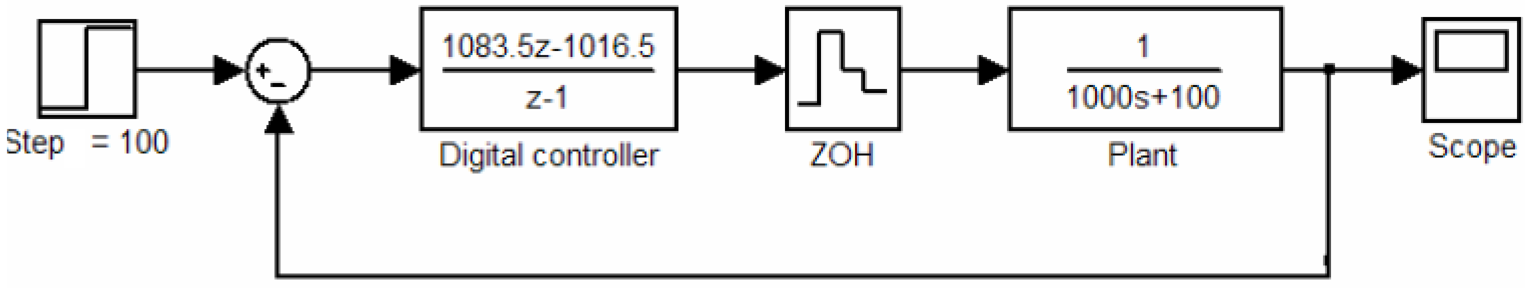


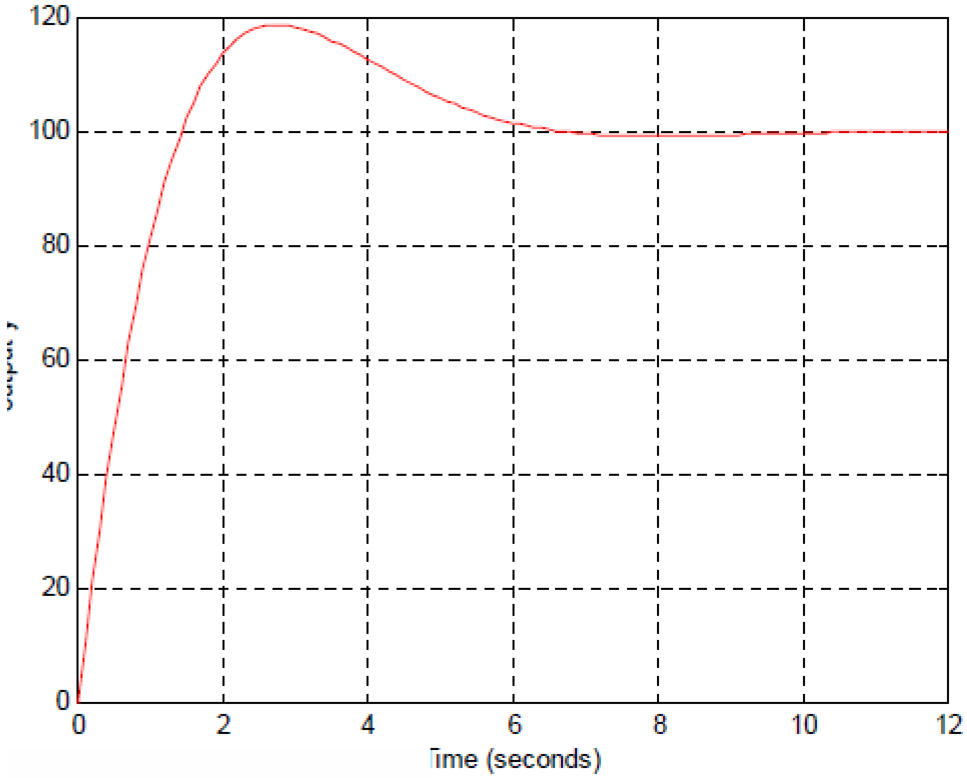
We first discretize the continuous-time PI control law with T = 1/(30×0.3) ≈ 0.1 seconds, using a bilinear transformation method, i.e.,

$$C(z) = C(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} = \frac{1050\,s + 670}{s}\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{1050\,\frac{2}{T}\left(\frac{z-1}{z+1}\right) + 670}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{1083.5z - 1016.5}{z-1}$$


Step = 100
1083.5z-1016.5
z-1
Digital controller
ZOH
1
1000s+100
Plant
Scope


120
100
80
60
40
20
0
0
2
4
6
8
10
12
Time (seconds)

5.3 Direct Z-domain Digital Controller Design
- *Frequency Response Design*

Obtaining digital controllers from analog designs involves approximation that may result in significant controller distortion.

5.3.1 Frequency Response Design Approach

- Essentially an iterative trial-and error method.
- Based on Bode plots (i.e. logarithmic plots) of frequency response .
- Frequency response of discrete transfer functions *G(z)* is obtained by substituting $z = e^{j\omega T}$ to get $G\left(e^{j\omega T}\right)$
- Thus $G\left(e^{j\omega T}\right)$ is not a rational function of $j\omega$
- To resolve the problem, a bilinear transformation is used:

- **Bilinear Transformation and the w-plane**

We apply the *w* transformation, a bilinear transformation (or Tustin transformation), defined by

$$z = \frac{1 + \frac{wT}{2}}{1 - \frac{wT}{2}} \qquad (5.9)$$

where *T* is the sampling period and *w* is a complex variable. Note the difference between *w* and ω . From (5.9), we have

$$w = \frac{2}{T}\frac{z-1}{z+1} \qquad (5.10)$$

Eqn (5.10) maps the inside of the unit circle in the z-plane into the entire left half plane in the w-plane

The w-plane is a complex plane whose imaginary part is denoted by *v,* i.e. $\mathrm{Im}\left[w\right]=v$

Suppose *G(z)* is transformed to *G(w)* through (5.9). Then the frequency response of *G(w)* can be obtained as

$$G\left(w\right)\big|_{w=jv}=\left|G\left(jv\right)\right|\angle G\left(jv\right)$$

where *v* is the fictitious frequency in the w-plane.

Example 5.4

Consider the cruise control system with the following transfer function

$$G(s) = \frac{1}{(s+1)}$$

Transform the corresponding $G_{zAS}(z)$ to the *w*-plane by considering both T =0.1s and T =0.01s. Evaluate the role of the sampling period by analyzing the corresponding Bode plots.

Solution

When T = 0.1s, we have $G_{ZAS}(z) = \dfrac{0.09516}{z - 0.9048}$

Applying (5.9), we obtain, $G_1(w) = \dfrac{-0.05w + 1}{w + 1}$

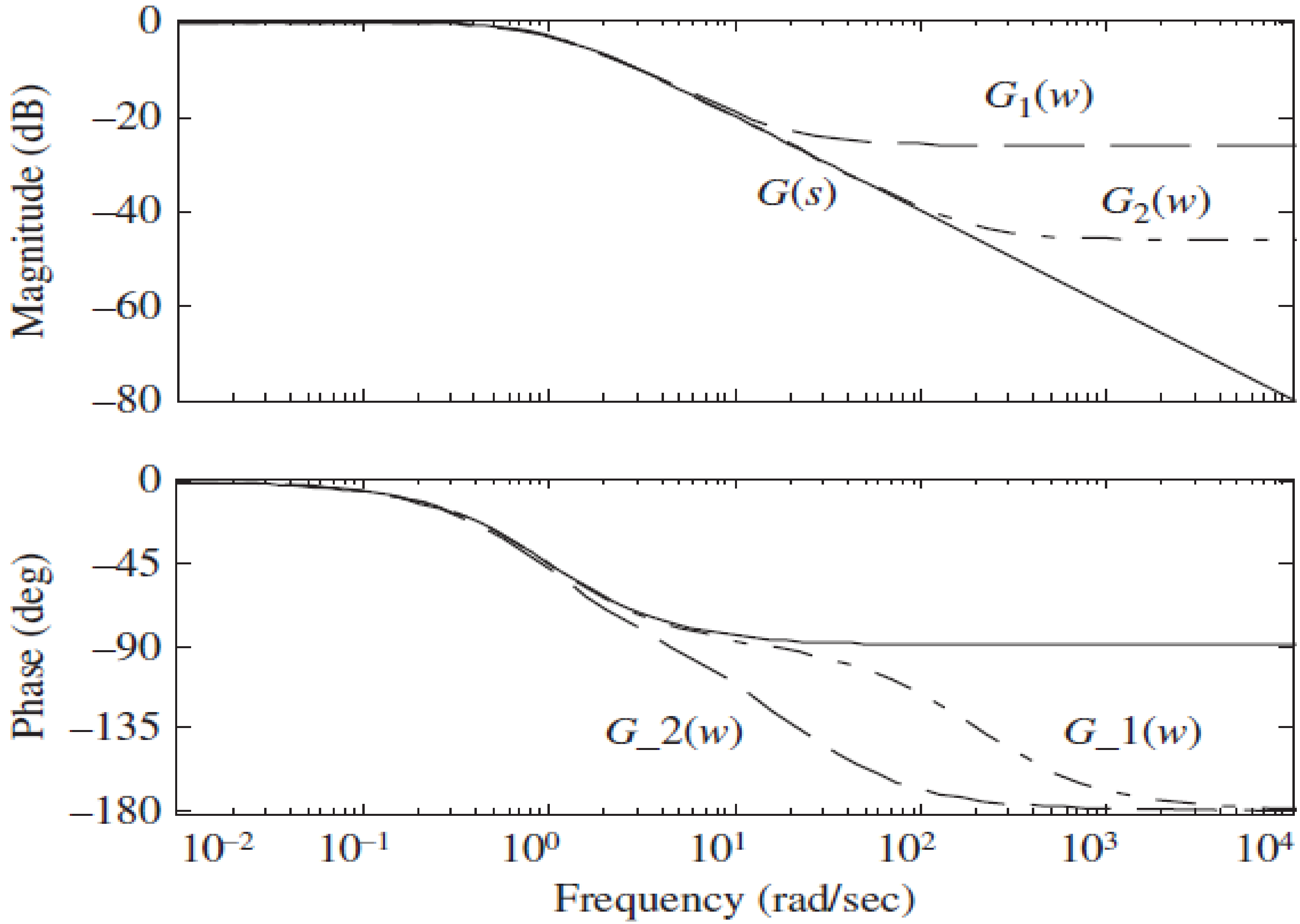
When T=0.01s, we have

$$G_{ZAS}(z) = \frac{0.00995}{z - 0.99} \qquad G_2(w) = \frac{-0.005w + 1}{w + 1}$$

- Both cases have the same pole in the w- and s-plane.
- Both $G_1(w)$ and $G_2(w)$ have a zero, whereas G(s) does not.

→ difference between frequency response of analog & digital systems at high frequencies, as shown in Figure 5.4.

- Influence of zero on system dynamics is more significant when the sampling period is larger.



0
–20
–40
–60
–80
Magnitude (dB)
G1(w)
G(s)
G2(w)
0
–45
–90
–135
–180
Phase (deg)
G_2(w)
G_1(w)
10–2
10–1
100
101
102
103
104
Frequency (rad/sec)

Figure 5.4.

- **Design Procedure**

1. Select a sampling period and obtain a transfer function $G_{zAS}(z)$ of the discretized process (please refer to Figures 5.1 and 5.2).
2. Transform $G_{zAS}(z)$ into *G(w)* using (5.9).
3. Draw the Bode plot of *G(jv),* and use analog frequency response methods to design a controller *C(w)* (or $G_D(w)$) that satisfies the frequency domain specifications.
4. Transform the controller back into the z-plane by means of (5.10), thus determining *C(z)* (or $G_D(z)$) .
5. Verify that the performance obtained is satisfactory.

5.3.2 A Review of Compensators

- **Phase-lead compensator** (including PD controllers):

- improves stability margins
- increases system bandwidth and hence faster response
- subject to high-frequency noise problems

- **Phase-lag compensator** (including PI Controllers):

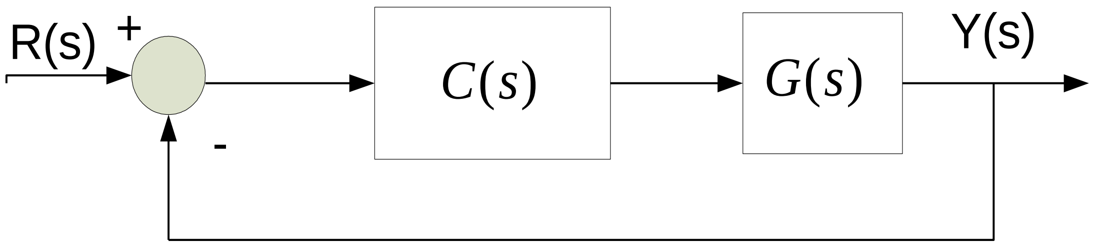
- reduces system gain at high-frequencies
- reduces system bandwidth and hence slower response
- increases low-frequency gain and hence improves steady-state accuracy
- attenuates high-frequency noise

- **Phase Lag-lead compensator** (including PID controllers):

- increases low-frequency gain while increases bandwidth and stability margins

5.3.3 Revision – Frequency Domain Design for Continuous Systems

Consider the control system



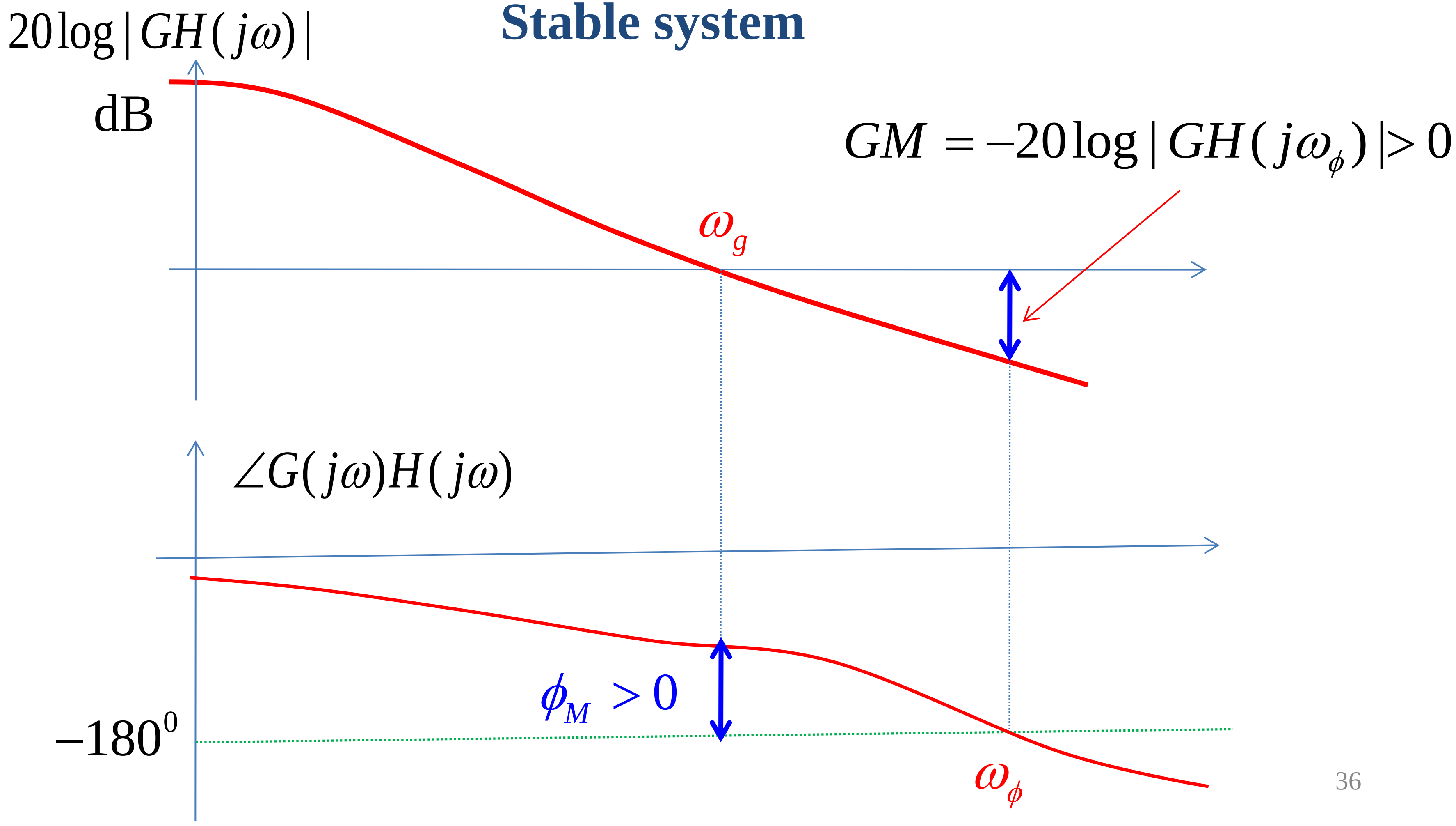
- **Our objective is to design a compensator $C(s)$ such that the closed-loop system has desired performance.**
- **Performance specifications can be given in either the time-domain or the frequency domain.**

- **The time-domain specifications include rise time, settling time, maximum overshoot and steady state error.**
- **The frequency-domain specifications are given by bandwidth or gain crossover frequency (speed of response), phase margin, gain margin, (rough estimate of damping ratio), and steady state error.**
- **Note that phase margin is closely related to the damping ratio and the gain crossover frequency or bandwidth is related to the rise time. Steady state error can be determined by the low frequency gain.**
- **The frequency domain design is essentially concerned with reshaping the frequency response (Bode magnitude and phase plots) of the system such that certain desired specifications can be met by adding compensator .**

- **In the Bode plots, the phase margin is the difference between $\angle G(j\omega_g)H(j\omega_g)$ and -180^0 since**

$$\phi_M = \angle G(j\omega_g)H(j\omega_g) - (-180^0)$$

Stable system



➢ Design of Lead Compensator

- **Lead compensators are used to improve the transient performance of systems.**
- **This is usually done by increasing the phase margin of the system (increasing the damping ratio) without changing much in steady state accuracy.**
- **The transfer function of a lead compensator is**

Basic factors:

$$C(s) = KC_0(s) = K\frac{Ts+1}{\alpha Ts+1}, \quad 0<\alpha<1$$

$Ts+1, 1/T$

$\dfrac{1}{\alpha Ts+1}, 1/(\alpha T)$

- **Since** $\alpha<1$, **the corner frequency for the pole is larger than that of the zero.**
- **The Bode plots of** $C_0(s)$ **are**

$$\frac{jT\omega+1}{j\alpha T\omega+1}$$

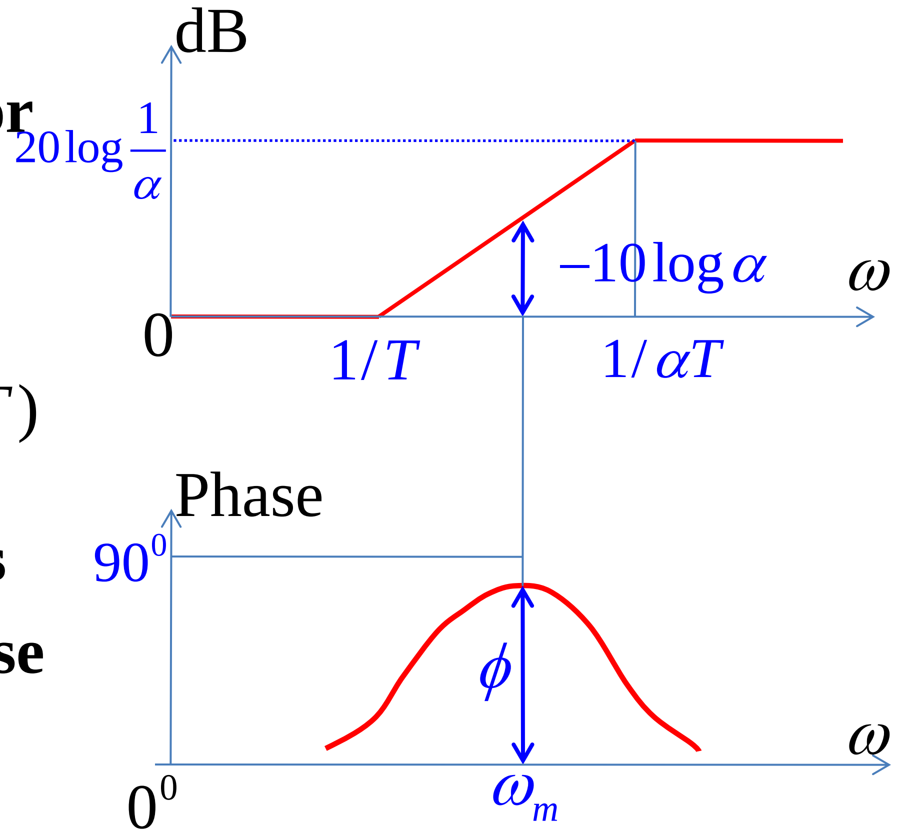
- **The phase of a lead compensator is between 0^0 and 90^0. That is, it provides a phase lead.**

$$\angle C(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

- **By setting** $\dfrac{d\angle C(j\omega)}{d\omega} = 0$ **, it is known that the maximum phase happens at** $\boxed{\omega_m = \dfrac{1}{\sqrt{\alpha T}}}$

and the maximum phase is

$$\phi = \sin^{-1}\frac{1-\alpha}{1+\alpha} \Rightarrow \sin\phi = \frac{1-\alpha}{1+\alpha} \Rightarrow \boxed{\alpha = \frac{1-\sin\phi}{1+\sin\phi}}$$



At $\omega = \omega_m = \dfrac{1}{\sqrt{\alpha T}}$, $\boxed{20\log|C_0(j\omega)| = -10\log\alpha}$ (Please verify!).

A design procedure for lead compensator is stated as

1) **Determine the compensator gain K to satisfy the steady state error requirement.** $e_{ss} = \frac{1}{1+K_P} \leq 0.1 \Rightarrow K_P = \lim_{s \to 0} KG(s)$

2) **Draw the Bode plots of $KG(s)$. See the figure below.**

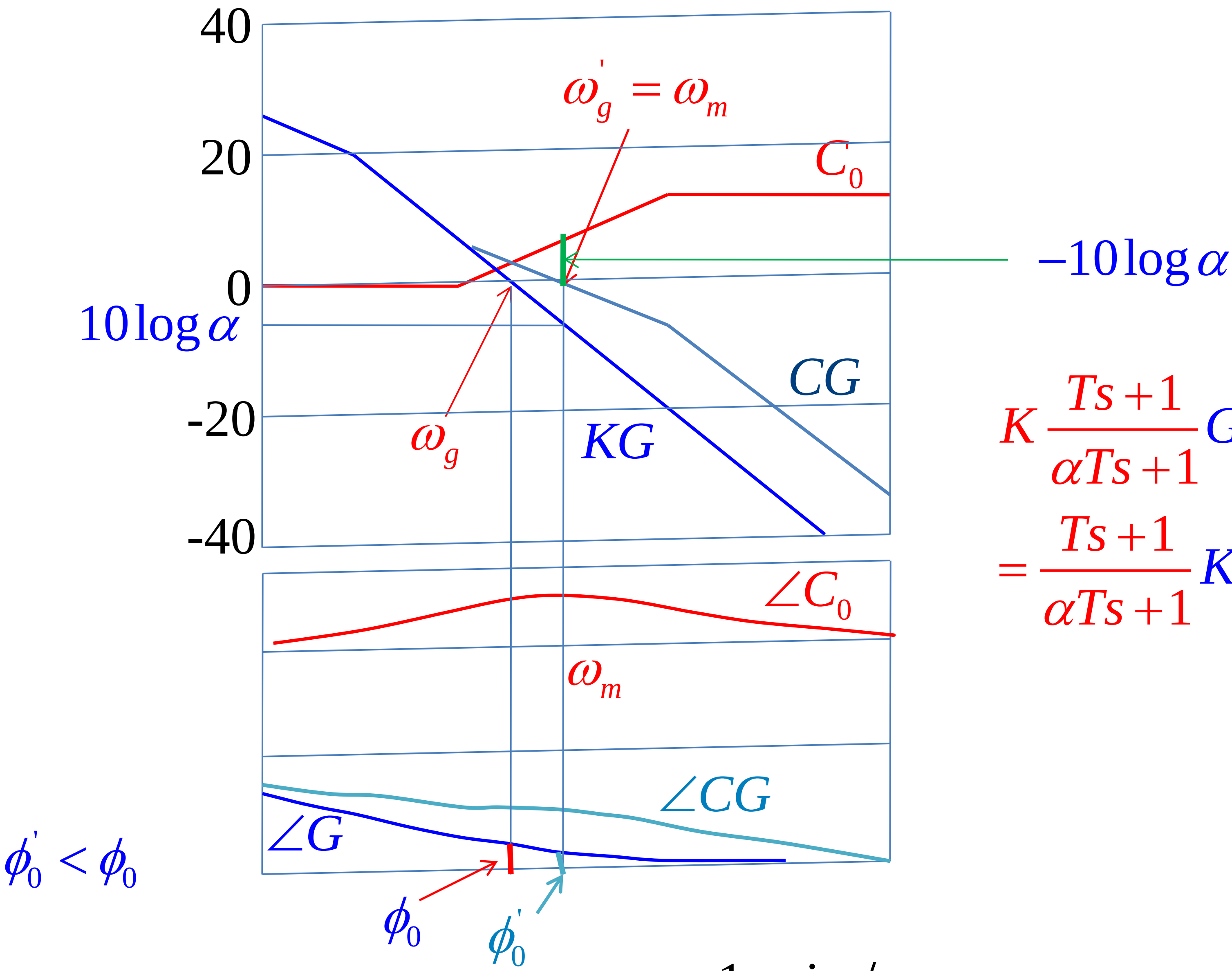
3) **From the phase margin of $KG(s)$ and the required phase margin, determine the phase lead ϕ to be added.**

$$\phi = PM - \phi_0 + (5^0 \text{ to } 10^0)$$

Desired PM

PM of KG(s)

Added to compensate the difference between ϕ_0 and $\phi_0^{'}$



$$K\frac{Ts+1}{\alpha Ts+1}G(s) = \frac{Ts+1}{\alpha Ts+1}KG(s)$$

4) From ϕ, compute α using $\alpha = \dfrac{1-\sin\phi}{1+\sin\phi}$

5) To achieve the maximum phase lead, place $\omega_m = \dfrac{1}{\sqrt{\alpha T}}$ **at the new gain crossover frequency. Note that**

$$20\log|C_0(j\omega_m)| = -10\log\alpha$$

Hence, the new gain crossover frequency ω_g' **should be chosen such that**

$$20\log|KG(j\omega_g')| = 10\log\alpha$$

This tells how to choose ω_g'

Therefore,

$$\omega_g' = \omega_m = \frac{1}{\sqrt{\alpha T}} \Rightarrow T = \frac{1}{\sqrt{\alpha}\,\omega_g'}$$

6) Form the lead compensator

$$C(s) = K\frac{Ts+1}{\alpha TS+1}$$

and verify the results by plotting the Bode plots of $C(s)G(s)$ **.**

Remarks:

- **The lead compensator improves the phase margin and thus the transient performance of the system.**
- **The gain crossover frequency is increased, which means a larger bandwidth. Thus, the speed of the system response is improved.**
- **However, the lead compensator increases the high frequency gain of the system. This makes the system more susceptible to noise signals.**

5.3.4 Frequency Domain Design for Digital Controller – An Example

Example 5.5: Consider the digital control system shown in Figure 5.5. Design a digital controller in the *w* plane such that the phase margin is 50°, the gain margin is at least 10 dB, and the static velocity error constant K_v is 2 sec^{-1}. Assume that the sampling period is 0.2 sec'.

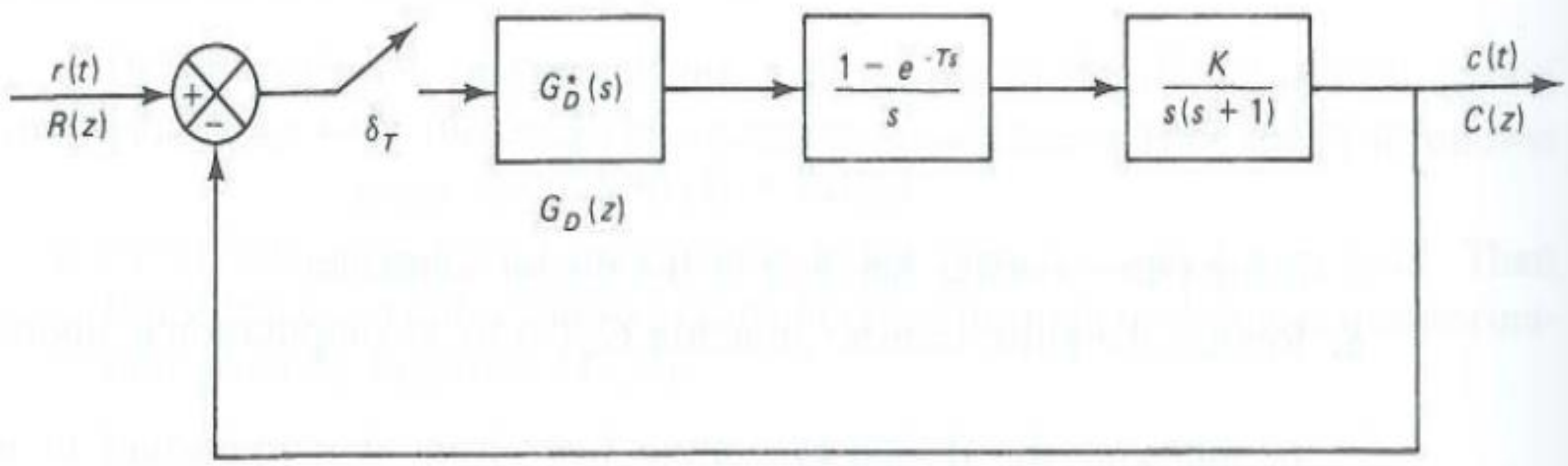


Figure 5.5

Solution :

$$G_{ZAS}(z) = Z\left\{\left[\frac{1-e^{-0.2s}}{\mathrm{s}}\,\frac{K}{s(s+1)}\right]\right\}$$

$$=\left(1-z^{-1}\right)\ Z\left\{\left[\frac{K}{s^{2}\left(s+1\right)}\right]\right\}$$

$$=0.01873\left[\frac{K(z+0.9356)}{(z-1)(z-0.8187)}\right]$$

Using bilinear transformation

$$z=\ \frac{1+\ \frac{wT}{2}}{1-\ \frac{wT}{2}}=\ \frac{1+\ 0.1w}{1-\ \ 0.1w}$$

$$G_{ZAS}(w) = G_{ZAS}(z) \Big|_{z = \frac{1+0.1w}{1-0.1w}}$$

$$\approx \frac{K\left(\frac{w}{300}+1\right)\left(1-\frac{w}{10}\right)}{w(w+1)}$$

Let the digital controller be a phase lead compensator

$$G_D(w) = \frac{Tw+1}{\alpha Tw+1}, \ 0 < \alpha < 1$$

Then

$$G_D(w)G_{ZAS}(w) = \frac{Tw+1}{\alpha Tw+1} \frac{K\left(\frac{w}{300}+1\right)\left(1-\frac{w}{10}\right)}{w(w+1)}$$

It is required that the static velocity error constant $K_v = 2$.

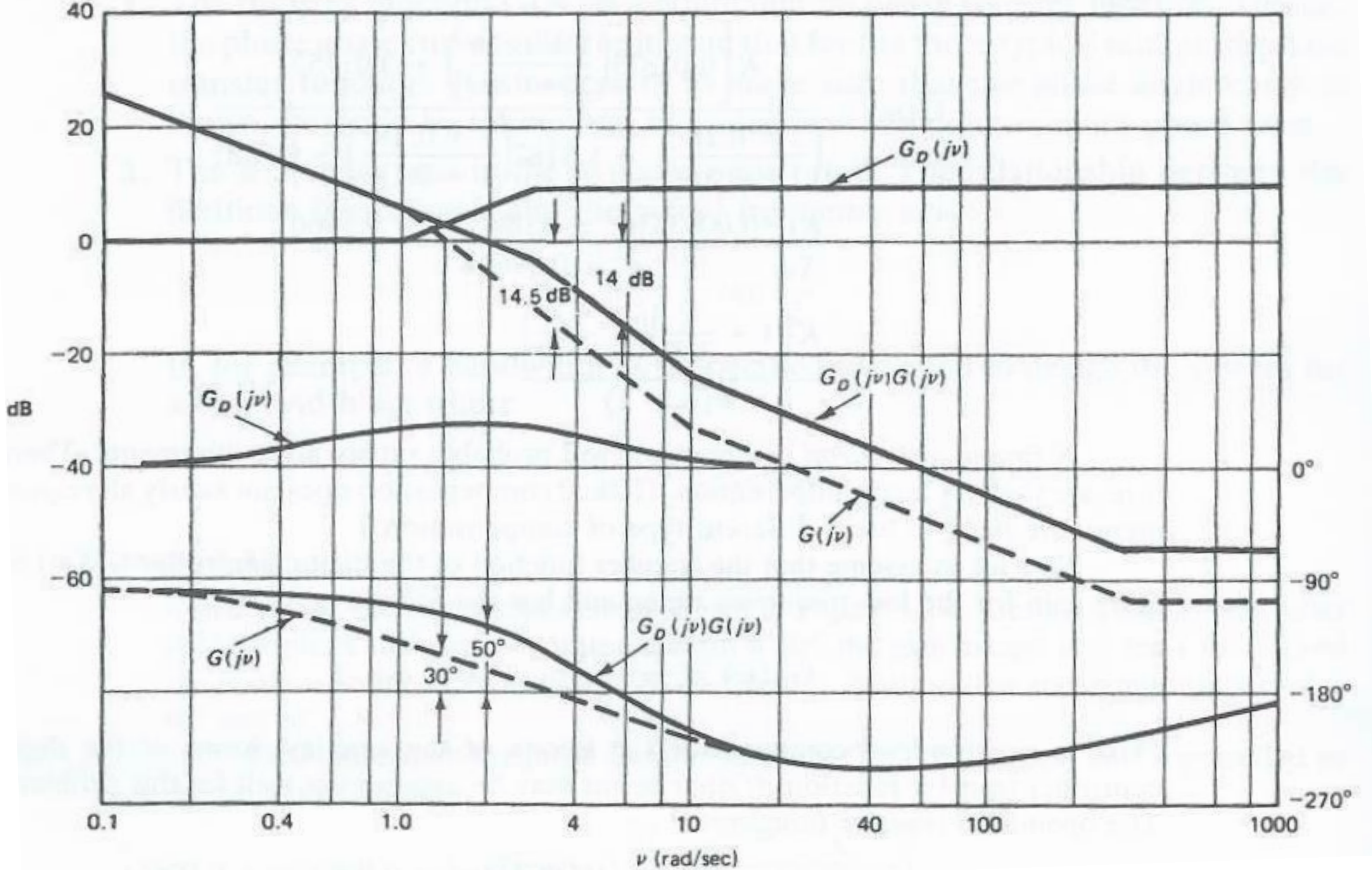
Note that

$$K_v = \lim_{w \to 0} \{ w[G_D(w) G_{ZAS}(w)] \quad \} = K$$

So $K = 2$. Then let

$$G(w) = 2G_{ZAS}(w) = \frac{2\left(\frac{w}{300} + 1\right)\left(1 - \frac{w}{10}\right)}{w(w+1)}$$

The Bode diagram of *G(jv)* (dashed curve) is plotted in Fig 5. 4



40
20
0
-20
-40
-60
dB
0°
-90°
-180°
-270°
$G_D(j\nu)$
14 dB
14.5 dB
$G_D(j\nu)G(j\nu)$
$G_D(j\nu)$
$G(j\nu)$
50°
30°
$G(j\nu)$
$G_D(j\nu)G(j\nu)$
0.1
0.4
1.0
4
10
40
100
1000
ν (rad/sec)

Figure 5. 6

From the Bode plots, the phase margin is 30^0.

To achieve the desired phase margin of 50^0, we need to add 20^0 .

Further, by considering the fact that the gain crossover frequency will be shifted to the right after adding the lead compensator, set .

$$\phi = 50^0 - 30^0 + 8^0 = 28^0$$

Hence, $$\alpha = \frac{1-\sin\phi}{1+\sin\phi} = \frac{1-\sin 28^0}{1+\sin 28^0} = 0.361$$

As $20\log|G(j\omega)| = 10\log\alpha = -4.425dB \Rightarrow v_g^{'} = 1.7$ rad/s

Then $v_g^{'} = \dfrac{1}{\sqrt{\alpha T}} = 1.7 \Rightarrow T = 0.9790$

Thus the compensator is

$$G_D(w) = \frac{Tw+1}{\alpha Tw+1} = \frac{0.9790w+1}{0.3534w+1}$$

The Bode plots of $G_D(jv)G(jv)$ are the solid lines in Fig 5.4

It can be seen that the phase margin is about 50^0 and the gain margin is 14dB.

Thus, the designed compensator meets the specifications requirement.

Now transform the controller back into the z-plane by means of (5.10), thus determining $G_D(z)$.

$$G_D(z) = G_D(w)\Big|_{w=\frac{2}{T}\frac{z-1}{z+1}}$$

$$= \frac{0.9790\left(\dfrac{2}{0.2}\dfrac{z-1}{z+1}\right)+1}{0.3534\left(\dfrac{2}{0.2}\dfrac{z-1}{z+1}\right)+1}$$

$$= \frac{2.3798z - 1.9378}{z - 0.5589}$$

To check the performance of the designed system, its unit step response using MATLAB is obtained as shown in Figure 5.7.

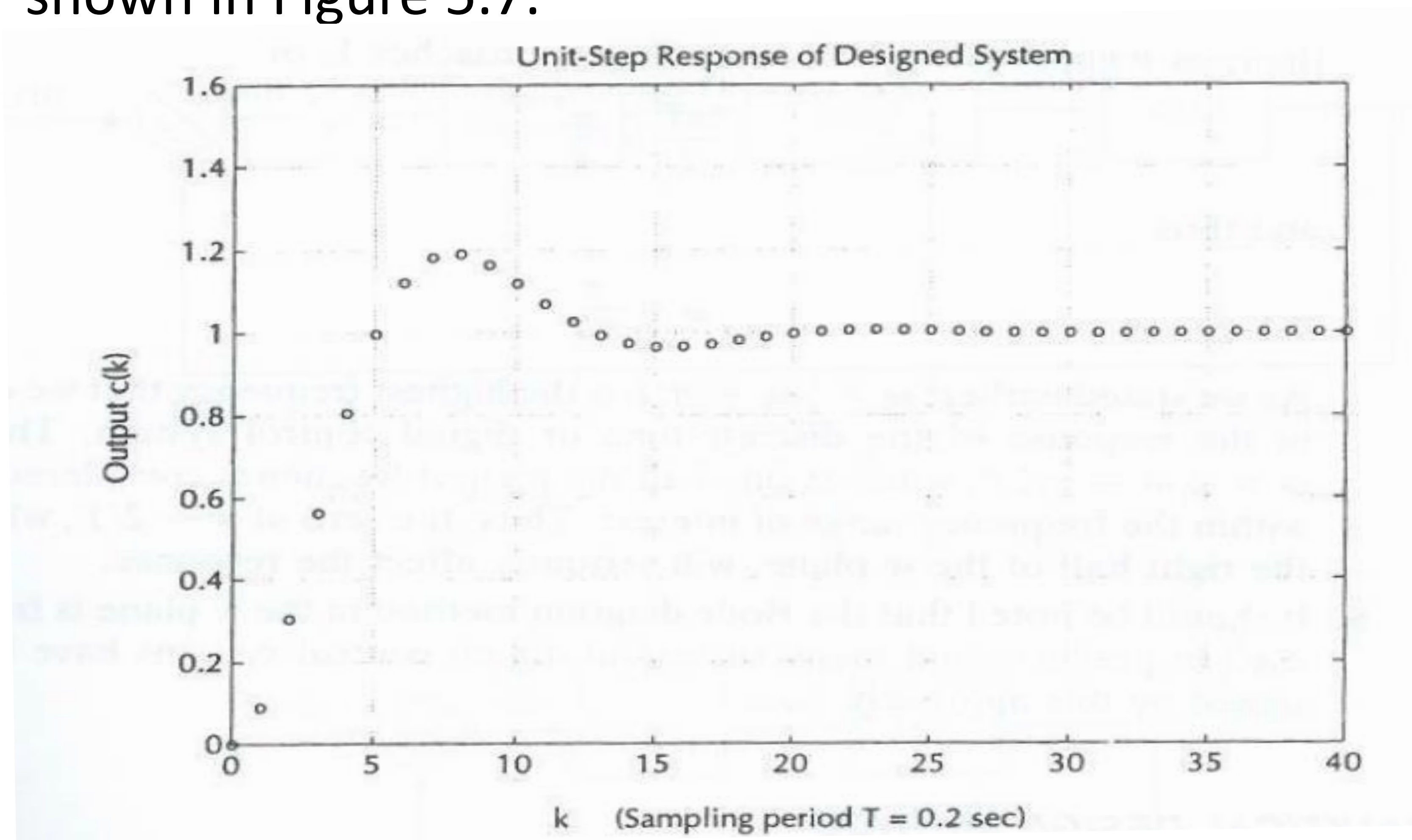


Figure 5.7.

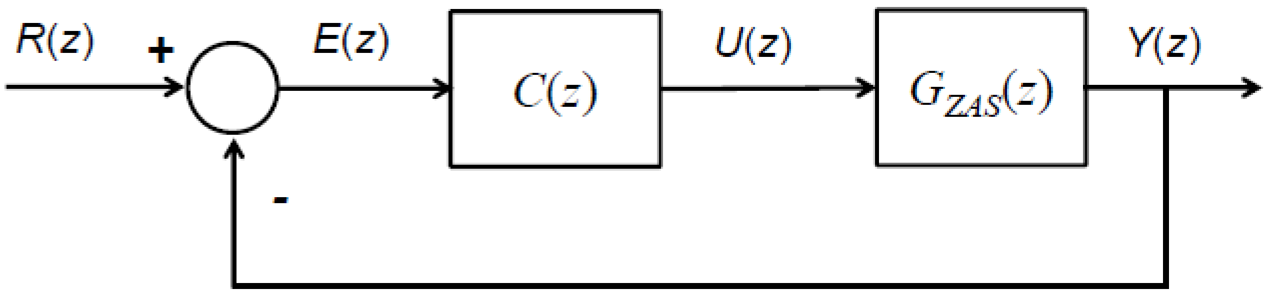
5.4 Direct Control Design

- In certain applications, the desired closed-loop transfer function is known from the design specification.

- Thus, it is possible to calculate the controller transfer function for a given plant from the desired closed-loop transfer function. This approach to design is known as synthesis.

- Note: the resulting controller must be realizable for this approach to yield a useful design

Consider the following closed loop system and suppose the desired closed loop transfer function $G_{cl}(z)$.



Then

$$G_{cl}(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

Solving for the controller, we have

$$C(z) = \frac{1}{G_{ZAS}(z)} \frac{G_{cl}(z)}{1 - G_{cl}(z)} \qquad (5.11)$$

- The controller must be **causal** and must ensure the **asymptotic stability** of the closed-loop control system.

  Whether a controller is causal or not can be examined by checking the degrees of the numerator and the denominator of *C(z)*.

  Causal controller must have

  1) poles not less than zeros;
  2) no time advance

  To ensure this, the relative degree of $G_{cl}(z)$ must not be less than that of $G_{ZAS}(z)$.

- If unstable pole-zero cancellation occurs, the system is input-output stable, but not asymptotically stable.

→ the set of zeros of $G_{cl}(z)$ must include all the zeros of $G_{ZAS}(z)$ that are outside the unit circle.

→ The zeros of *1* - $G_{cl}(z)$ must include all the unstable poles of $G_{ZAS}(z)$ (stability);

For example, suppose that the process has an unstable pole, $z = \bar{z}, |\bar{z}| > 1$ i.e

$$G_{ZAS}(z) = \frac{G_1(z)}{z - \bar{z}}$$

The from (5.11), to avoid unstable pole-zero cancellation, we need

$$1 - G_{cl}(z) = \frac{1}{1 + C(z)\dfrac{G_1(z)}{z - \bar{z}}} = \frac{z - \bar{z}}{z - \bar{z} + C(z)G_1(z)}$$

→ i.e. $\mathrm{z} = \bar{z}$ must be a zero of $1 - G_{cl}(z)$

→If zero steady-state error due to a step input is required, based on the Final Value Theorem, an additional condition must be

$$G_{cl}(1) = 1$$

In summary, necessary conditions required for the choice of $G_{cl}(z)$ are as follows:

- The relative degree of $G_{cl}(z)$ must not be less than that of $G_{ZAS}(z)$. (causality);
- $G_{cl}(z)$ must contain all the unstable zeros of $G_{ZAS}(z)$ as its zeros (stability);
- The zeros of *1* - $G_{cl}(z)$ must include all the unstable poles of $G_{ZAS}(z)$ (stability);
- $G_{cl}(1)$ =1 (zero steady-state error).

5.4.1 A Suggested Procedure for Choosing $G_{cl}(z)$

1) Select the desired settling time T_s (and other specifications e.g the desired maximum overshoot);
2) Select a suitable continuous-time closed-loop first-order or second-order closed-loop system with unit gain;
3) Obtain $G_{cl}(z)$ by converting the s-plane pole location to the z-plane pole location using pole-zero matching, $z_i = e^{s_i T}$ where z_i and s_i are discrete and continuous poles, respectively;
4) Verify that $G_{cl}(z)$ meets the conditions for causality, stability, and steady-state error. If not, modify $G_{cl}(z)$ until the conditions are met.

Example 5.6: Design a digital controller with its output to a zero-order-hold for the DC motor speed control system with the following analog transfer function

$$G(s) = \frac{1}{(s+1)(s+10)}$$

to obtain 1) zero steady-state error due to a unit step and 2) a settling time of about 4 s.

The sampling time is chosen as T= 0.02 s.

Solution: The discretized process transfer function is

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = 1.8604\times10^{-4}\,\frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}$$

Note: there are no poles and zeros outside the unit circle.

Based on the specifications, a desired continuous-time closed-loop transfer function with damping ratio 0.88, undamped natural frequency 1.15 rad/s ($\zeta\omega_n = 1$), and with unity gain can be chosen as

$$G_{cl}(s) = \frac{1.322}{s^2 + 2.024s + 1.322}$$

The desired closed-loop transfer function $G_{cl}(z)$ is obtained using pole-zero matching [see **Example 5.2**]

$$G_{cl}(z) = 0.25921\cdot10^{-3}\,\frac{z+1}{z^2 - 1.96z + 0.9603}$$

Applying (5.11), we have

$$C(z) = \frac{1.3932(z-0.8187)(z-0.9802)(z+1)}{(z-1)(z+0.9293)(z-0.9601)}$$

The closed-loop step response is shown in Figure 5.8.

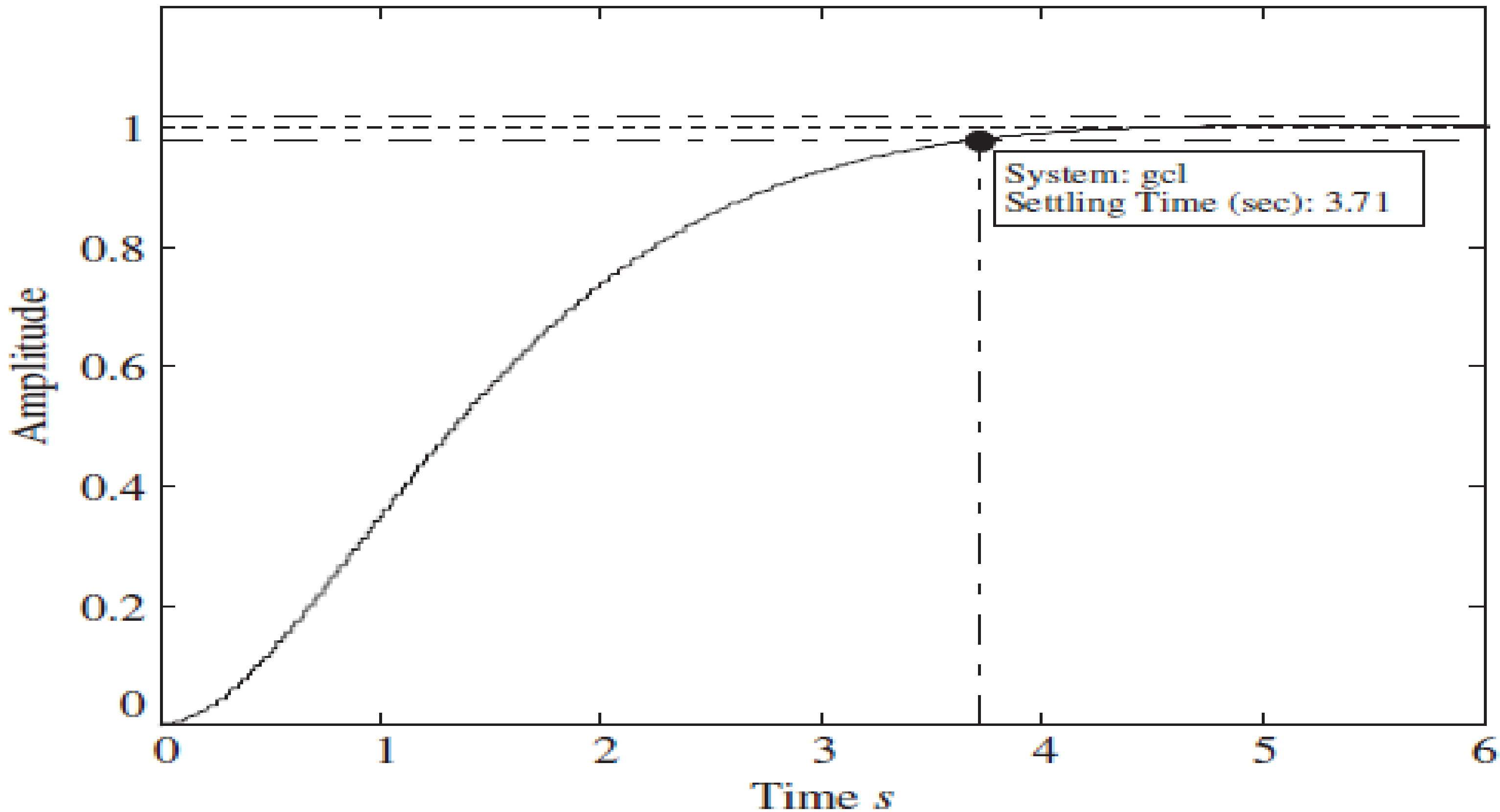


Figure 5.8

Example 5.7: Design a digital controller for the type 0 analog plant

$$G(s) = \frac{1}{10s+1} e^{-5s}$$

to obtain 1) zero steady-state error due to a unit step and 2) a settling time of about 10s (5% error tolerance) with no overshoot.

The sampling time is chosen as T= 1s.

Solution: The discretized process transfer function is

$$G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{0.09516}{(z - 0.9048)} z^{-5}$$

To meet the *causality requirements*, a delay of 5 sampling periods must be included in the desired closed-loop transfer function.

A settling time of 10s (including the time delay) is achieved by considering a closed-loop transfer function with a pole in z =0.5 (mapped from the continuous pole at -0.7) as follows

$$G_{cl}(z) = \frac{K}{z-0.5} z^{-5}$$

Setting $G_{cl}(1) = 1$ yields *K = 0.5,* applying (5.11) we have

$$C(z) = \frac{5.2543(z-0.9048)z^5}{z^6 - 0.5z^5 - 0.5}$$

The resulting closed-loop step response is shown Figure 5.9 below.

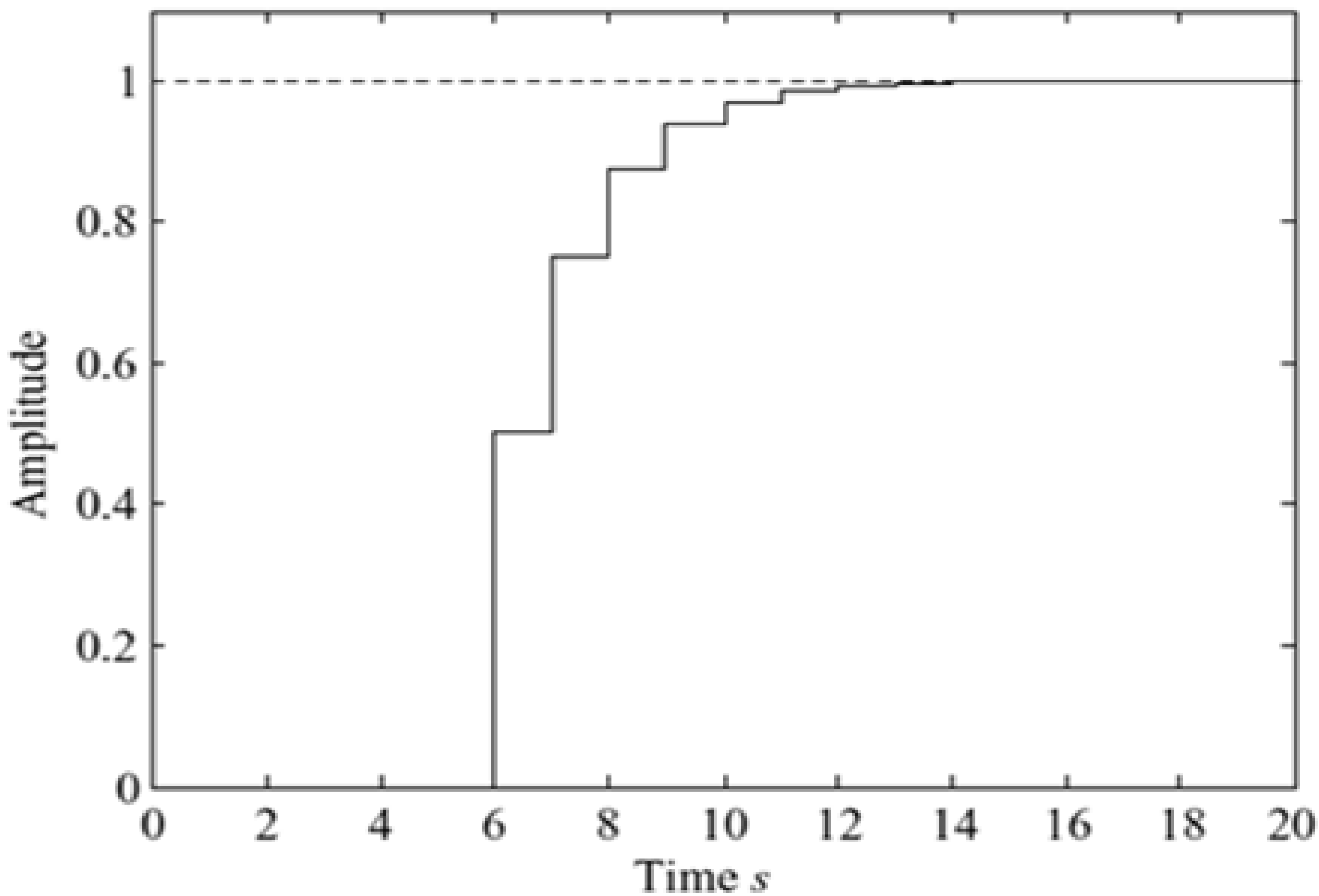


Figure 5.9

5.4.2 Finite Settling Time Design

- Continuous-time systems can only reach the desired output asymptotically after an infinite time period.
- In contrast, digital control systems can be designed to settle at the reference output after a finite time period and follow it exactly thereafter.
- Following the direct control design method, if all the poles and zeros of the discrete-time process are inside the unit circle, an attractive choice is:

$$G_{cl}(z) = z^{-k}$$

where $k \geq$ the intrinsic delay + relative degree of the discretized process.

Disregarding the time delay, the definition implies that a unit step is tracked perfectly starting at the first sampling point.

Deadbeat control is to bring the output to the steady state in the smallest number of time steps.

From (5.11) we have the **deadbeat** controller

$$C(z) = \frac{1}{G_{ZAS}(z)}\left[\frac{z^{-k}}{1-z^{-k}}\right] \qquad (5.12)$$

In this case, the only design parameter is the sampling period *T*. Thus the overall control system design is very simple.

Example 5.8: Design a deadbeat controller for the following system

$$G_{ZAS}(z) = \frac{z^{-2}}{1 - z^{-1}}$$

Solution: $$G_{ZAS}(z) = \frac{1}{z^2 - z}$$

→ no zeros outside the unit circle
→ one pole is at z=1, and thus it needs to be included as the zero of $1 - G_{cl}(z)$
→ a deadbeat controller can be designed by setting

$$G_{cl}(z) = z^{-2}$$

Using (5.12), we get the dead-beat controller to be

$$C(z) = \frac{z^{-2}}{(1 - z^{-2})z^{-2}/(1 - z^{-1})} = \frac{1 - z^{-1}}{1 - z^{-2}} = \frac{1}{1 + z^{-1}}$$

Example 5.9: Design a deadbeat controller with its output to a zero-order-hold for the DC motor speed control system with an analog transfer function

$$G(s)=\frac{1}{(s+1)(s+10)}$$

and the sampling time is initially chosen as T = 0.02 s. Redesign the controller with T = 0.1 s.

Solution: The discretized process transfer function is

$$G_{ZAS}(z)=(1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}=1.8604\times10^{-4}\,\frac{z+0.9293}{(z-0.8187)(z-0.9802)}$$

→ no poles and zeros outside or on the unit circle

→ a deadbeat controller can be designed by setting

$$G_{cl}(z)=z^{-1}$$

Applying (5.12), we have

$$C(z) = \frac{5375.0533(z-0.8187)(z-0.9802)}{(z-1)(z+0.9293)}$$

- The resulting sampled and analog closed-loop step response is shown in Figure 5.10, the corresponding control variable is shown in Figure 5.11.
- Clearly, the sampled process output attains its steady state value after just one sample—i.e. t = T = 0.02 s, but between samples the output oscillates wildly and the control variable has very high magnitude.
- In other words, the oscillatory behavior of the control variable causes an unacceptable intersample oscillation

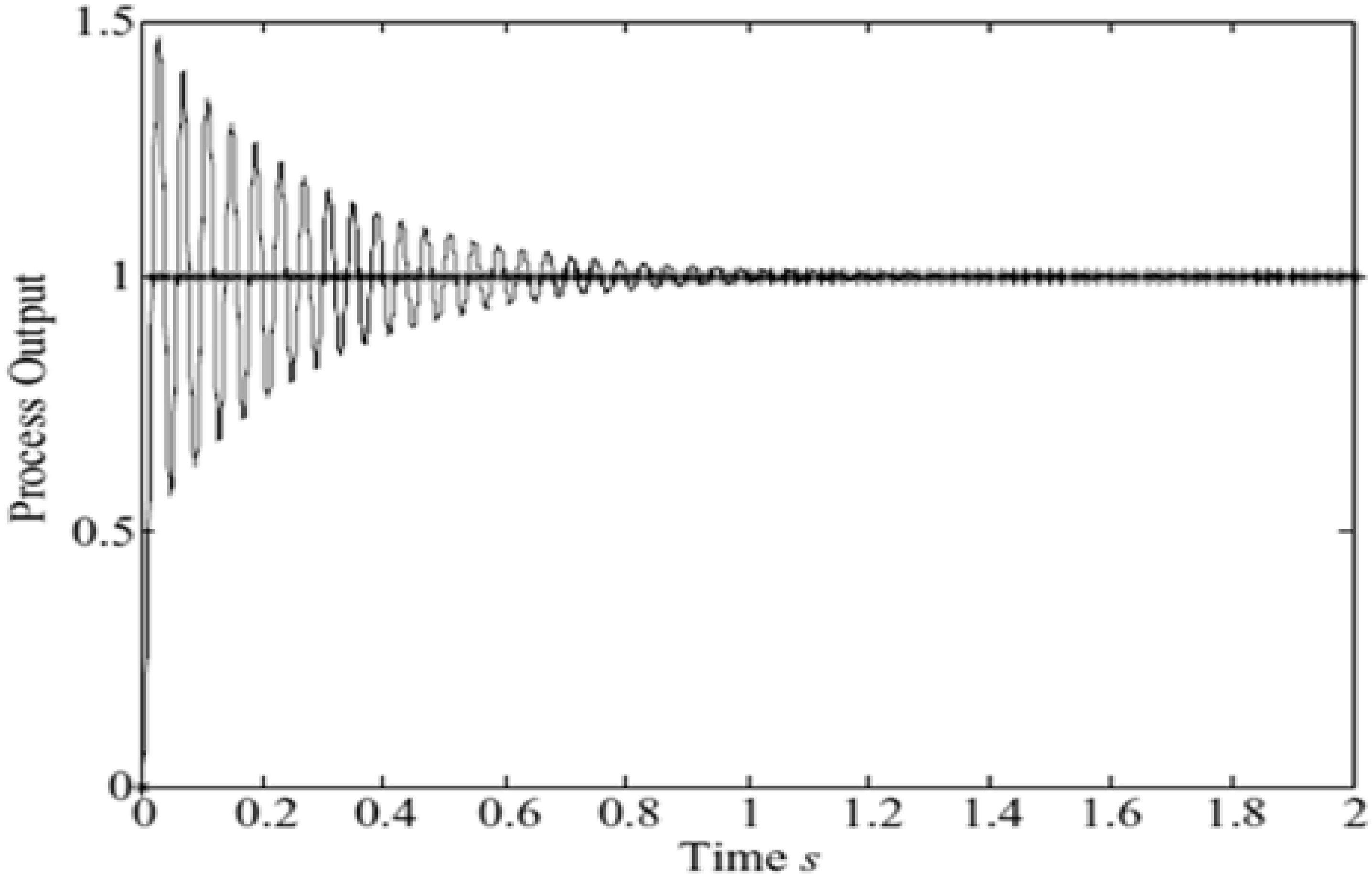


Figure 5.10 Step response

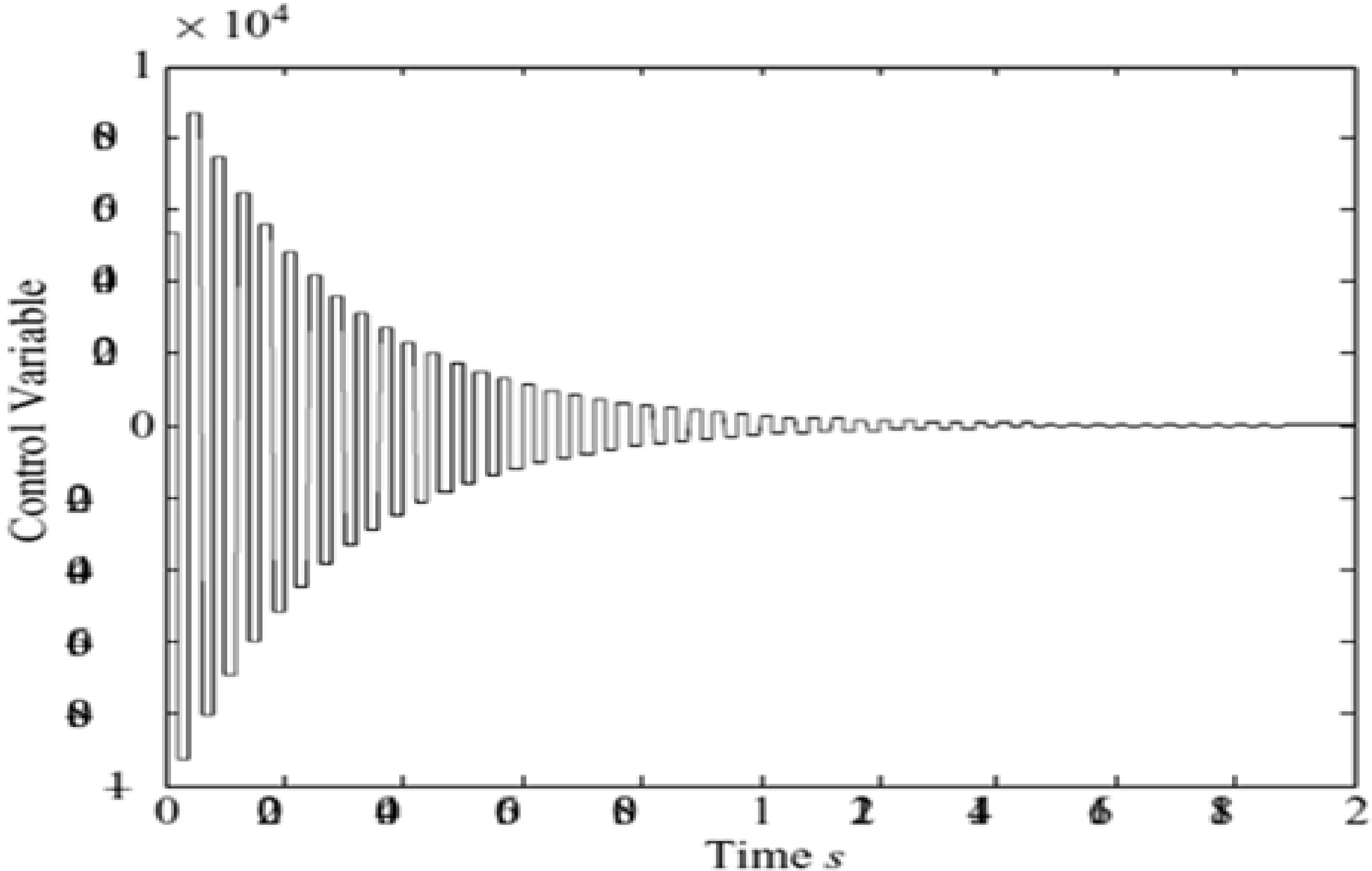


Figure 5.11 Control variable

Remarks

- Finite settling time designs may exhibit undesirable intersample behavior (oscillations) because the control is unchanged between two consecutive sampling points.
- The control variable can easily assume values that may cause saturation of the DAC or the actuator, resulting in unacceptable system behavior.
- The behavior of finite settling time designs such as deadbeat controller must be carefully checked before implementation

Example 5.10

To reduce intersample oscillations in ***Example 5.9***, we use *T=0.1s* and the discretized process transfer function is

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = 35.501 \times 10^{-4}\,\frac{z + 0.6945}{(z - 0.9048)(z - 0.3679)}$$

For $G_{cl}(z) = z^{-1}$, we have

$$C(z) = \frac{281.6855(z - 0.9048)(z - 0.3679)}{(z - 1)(z + 0.6945)}$$

The simulation results are shown Figures 5.12 and 5.13

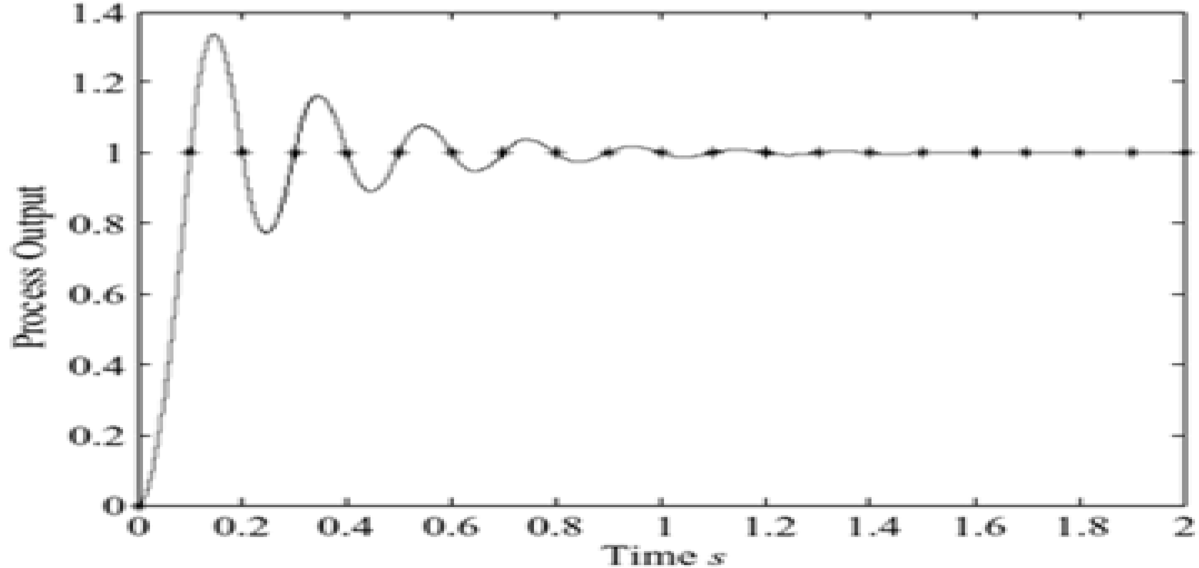


Figure 5.12 Step response

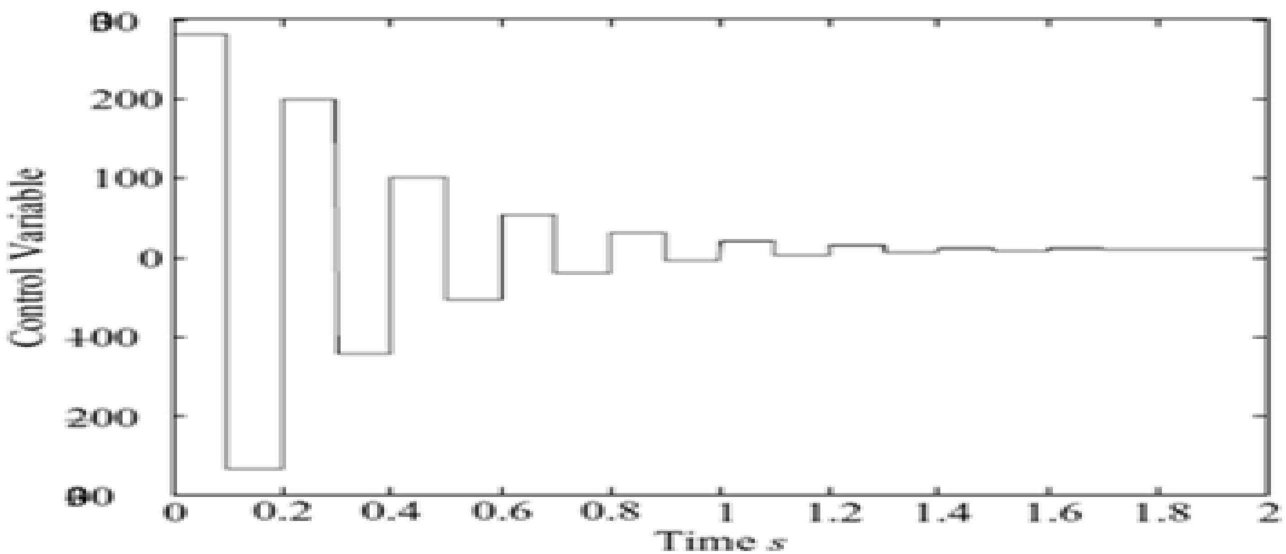


Figure 5.13 Control variable

5.4.3 Ripple-free Controller

- To avoid intersample oscillations, we maintain the control variable and output constants after *n* samples, where *n* is the degree of the denominator of the discretized process.
- Considering Figure 5.2, we have

$$U(z) = \frac{Y(z)}{G_{ZAS}(z)} = \frac{Y(z)}{R(z)}\frac{R(z)}{G_{ZAS}(z)} = G_{cl}(z)\frac{R(z)}{G_{ZAS}(z)} \quad (5.13)$$

- Obtain $G_{cl}(z)$ based on *U(z)* from (5.13) or *E*(*z*), the constraints that output is constant and $G_{cl}(1)$ =1 (zero steady-state error)
- Obtain *C*(*z*) from (5.11)

Example 5.11: Design a ripple-free deadbeat controller with ZOH for the type 1 vehicle positioning system whose transfer function is

$$G(s) = \frac{1}{s(s+1)}$$

The sampling time is chosen as T = 0.1 second.

Solution:

The discretized process transfer function is

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{0.0048374(1 + 0.9672z^{-1})z^{-1}}{(1 - z^{-1})(1 - 0.9048z^{-1})}$$

The z-transform of the step reference signal is

$$R(z) = \frac{1}{1 - z^{-1}}$$

From (5.13), we have

$$U(z) = G_{cl}(z)\frac{R(z)}{G_{ZAS}(z)} = G_{cl}(z)\frac{206.7218(1 - 0.9048z^{-1})}{z^{-1}(1 + 0.9672z^{-1})}$$

Because the process is of type 1, we require that the control variable be zero after two samples (note that n = 2), i.e.

$$U(z) = a_0 + a_1 z^{-1}$$

$$G_{cl}(z) = K \times z^{-1}(1 + 0.9672z^{-1})$$

$$U(z) = K \times 206.7218(1 - 0.9048z^{-1})$$

By imposing $G_{cl}(1) = 1$, $K = 0.5083$. Thus, by applying (5.11) we have

$$C(z) = \frac{105.1z - 95.08}{z + 0.4917}$$

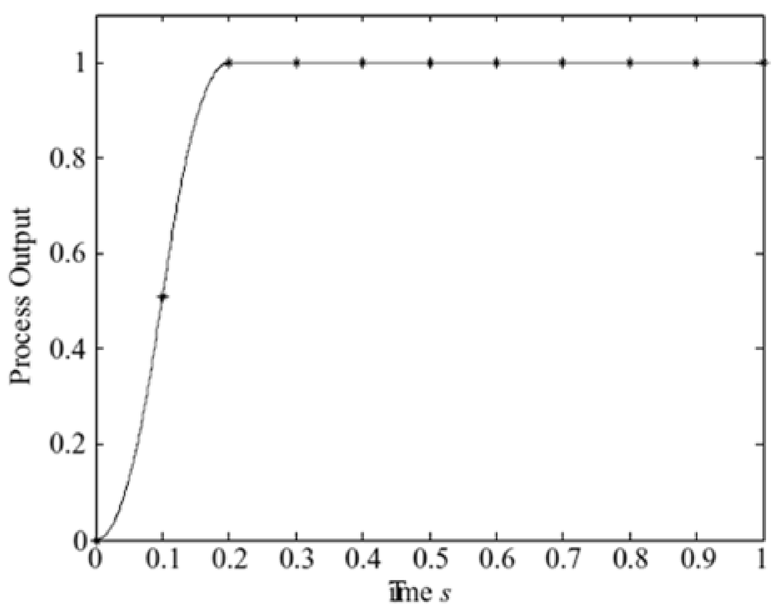


Fig. 6.44 Step response

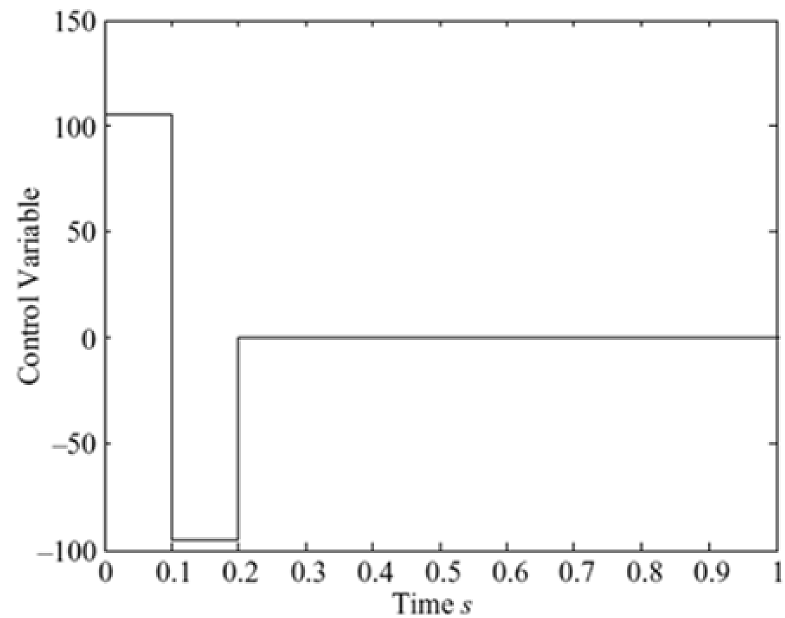


Fig. 6.45 Control variables

Example 5.12: Design a ripple-free deadbeat controller with ZOH for the DC motor speed control system whose transfer function is

$$G(s) = \frac{1}{(s+1)(s+10)}$$

The sampling period is chosen as $T = 0.1$ s.

Solution

The discretized process transfer function is

$$G_{ZAS}(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} = \frac{0.0035501(1 + 0.6945z^{-1})z^{-1}}{(1 - 0.9048z^{-1})(1 - 0.3679z^{-1})}$$

For a sampled unit step input,

$$R(z) = \frac{1}{1 - z^{-1}}$$

We have the control input

$$U(z) = G_{cl}(z)\frac{R(z)}{G_{ZAS}(z)} = G_{cl}(z)\frac{281.6855(1-0.9048z^{-1})(1-0.3679z^{-1})}{z^{-1}(1-z^{-1})(1+0.6945z^{-1})}$$

Note that $u(k)$ should be maintained as a **nonzero** constant for $k \geq n$.

But $e(k)=0$ for $k \geq n$. For $n=2$, $E(z) = e_0 + e_1 z^{-1}$ (5.14)

Then, in a similar way of considering *U(z)* in ***Example 5.11***,

$$E(z)=\frac{1}{C(z)}U(z)=G_{cl}(z)\frac{281.6855(1-0.9048z^{-1})(1-0.3679z^{-1})}{C(z)z^{-1}(1-z^{-1})(1+0.6945z^{-1})} \quad (5.15)$$

To achieve (5.14) , *C(z)* must have an integrator to cancel the factor $(1-z^{-1})$, while $G_{cl}(z)$ should cancel the other factors in the denominator in (5.15). With such a $G_{cl}(z)$, it can be shown that the numerator of *C(z)* calculated from (5.11) can cancel the factors in the numerator in (5.15).

So $G_{cl}(z)$ is chosen as

$$G_{cl}(z)=K\times z^{-1}(1+0.6945z^{-1})$$

By imposing $G_{cl}(1)$ = 1, *K = 0.5901* . Thus, by applying (5.11) we have

$$C(z) = \frac{166.2352(z-0.9048)(z-0.3679)}{(z-1)(z+0.4099)}$$

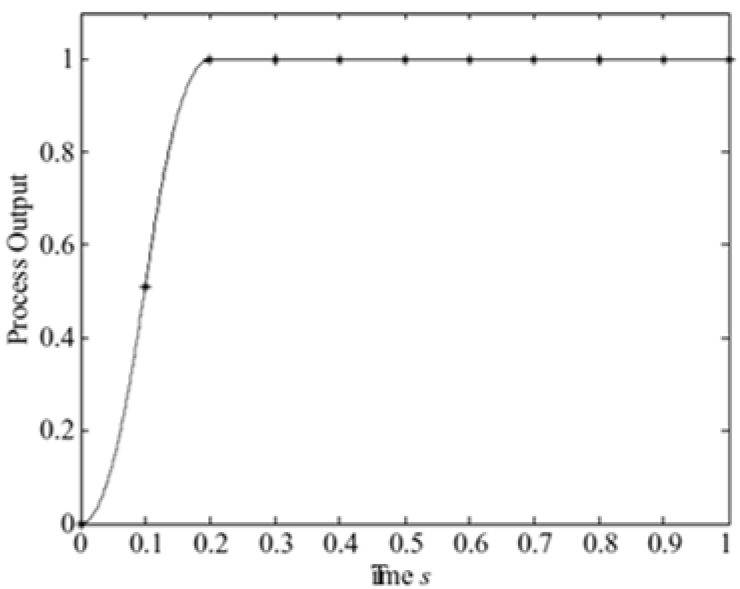


Fig. 5.14 Step response

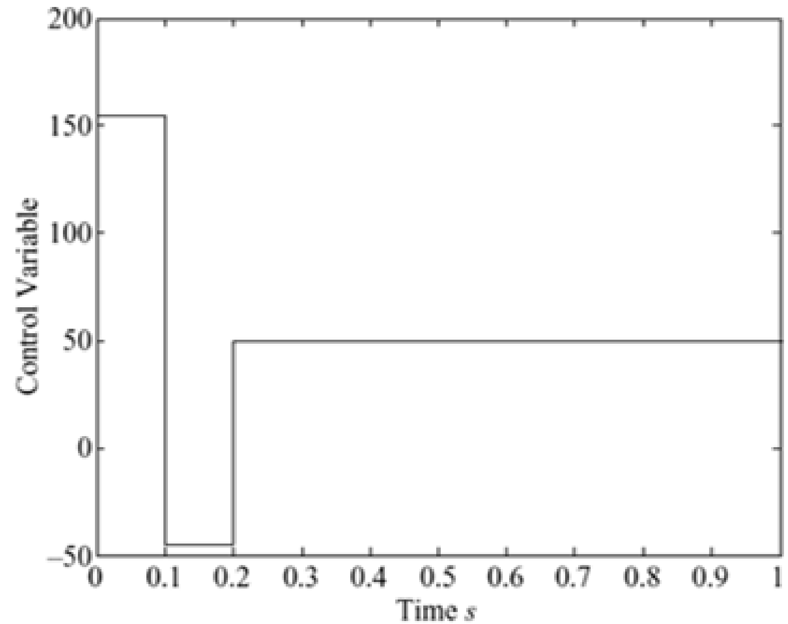


Fig. 5.15 Control variables

5.5 One Degree of Freedom Feedback Controller

Consider the feedback system in Figure 5.16.

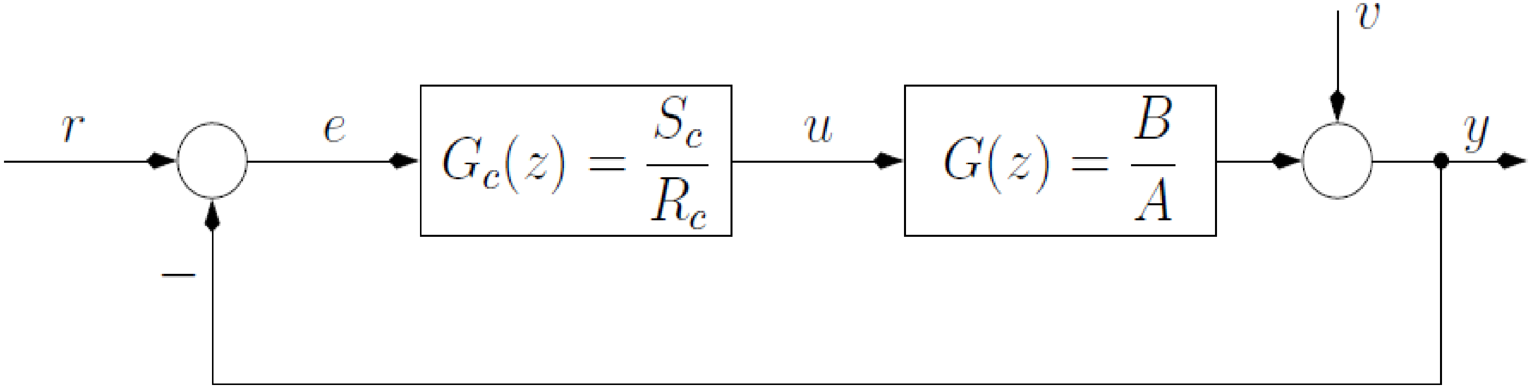


Figure 5.16: Schematic of a one degree of freedom feedback controller

We obtain the following expression for y:

$$y(n) = \frac{G(z)G_c(z)}{1 + G(z)G_c(z)} r(n) + \frac{1}{1 + G(z)G_c(z)} v(n)$$

$$= T(z)r(n) + S(z)v(n)$$

where the transfer functions T and S respectively are:

$$T(z) = \frac{G(z)G_c(z)}{1 + G(z)G_c(z)} \quad \text{and} \quad S(z) = \frac{1}{1 + G(z)G_c(z)}$$

Two requirements:

1. To make $y(n)$ follow $r(n)$ in an acceptable manner.
2. To remove the effect of $v(n)$ on $y(n)$.

To achieve this, we should be able to modify *S(z)* and *T(z) independently. However,*

$$S(z) + T(z) = 1$$

As a result, once *S(z)* is specified, *T(z)* is fixed and vice versa. This is known as the *one degree of freedom* controller, which is abbreviated as *1-DOF* controller.

5.6 Two Degrees of Freedom Feedback Controller

There are many different 2-DOF structures. We will discuss one of them in Figure 5.17.

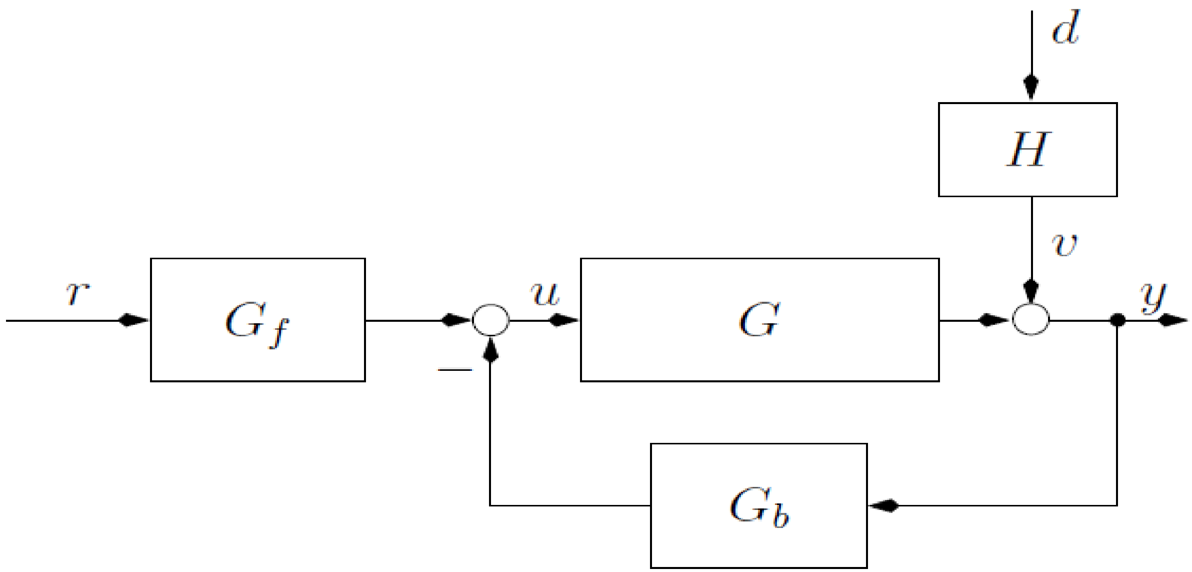


Figure 5.17: Two degrees of freedom feedback control structure *where* G_b and G_f together constitute the controller.

Denote

$$G_f = \frac{T_c}{R_c} \qquad\qquad G_b = \frac{S_c}{R_c}$$

where R_c, S_c and T_c are polynomials in $z^{-1}. Then$

$$R_c(z)u(n) = T_c(z)r(n) - S_c(z)y(n)$$

For a plant with the model

$$A(z)y(n) = z^{-k}B(z)u(n) + v(n)$$

where A and *B* are also polynomials in powers of z^{-1}. Then

$$Ay(n) = z^{-k}\frac{B}{R_c}[T_c r(n) - S_c y(n)] + v(n)$$

$$\left(\frac{R_c A + z^{-k}BS_c}{R_c}\right) y(n) = z^{-k}\frac{BT_c}{R_c} r(n) + v(n)$$

$$y(n) = z^{-k}\frac{BT_c}{\phi_{cl}}r(n) + \frac{R_c}{\phi_{cl}}v(n)$$

where $\emptyset_{cl}$ is the closed loop characteristic polynomial

We want:

1. The zeros of $\emptyset_{cl}$ to be inside the unit circle, so that the closed loop system is stable.
2. $\frac{R_c}{\emptyset_{cl}}$ to be made small, so that we achieve disturbance rejection.
3. $\frac{BT_c}{\emptyset_{cl}}$ to be made close to 1, so that we achieve set-point tracking.

There are two degree of freedoms for controller design.

5.7 Pole Placement Controller

Consider the 2-DOF control structure presented in Fig 5.18

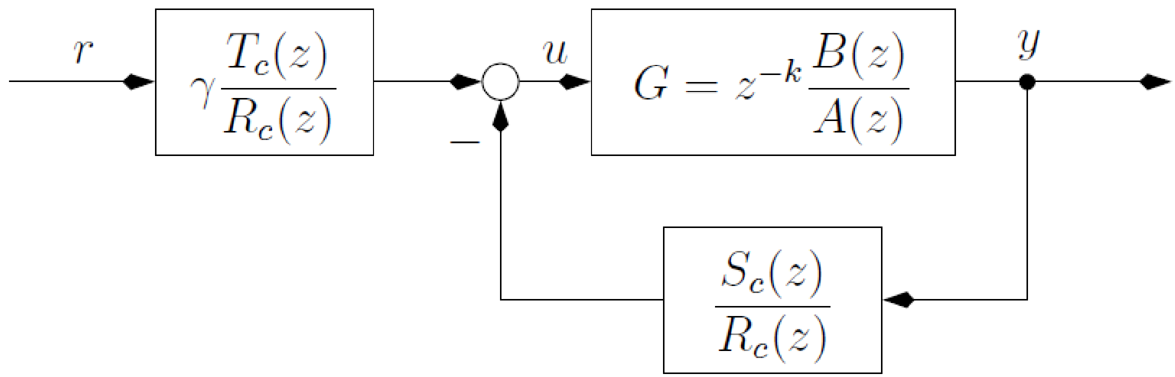


Figure 5.18: Two degrees of freedom feedback control

The plant has pulse transfer function

$$G(z) = z^{-k}\frac{B(z)}{A(z)} \tag{5.16}$$

where *B(z)* and *A(z)* are coprime. Then

$$Y(z) = G(z)U(z) \tag{5.17}$$

Design a controller such that the plant output *y* is related to the setpoint signal *r* as follows

$$Y_m(z) = \gamma z^{-k} \frac{B_r}{\phi_{cl}} R(z) \qquad (5.18)$$

where $\emptyset_{cl}$ is the characteristic polynomial obtained by the desired poles and

$$\gamma = \frac{\phi_{cl}(1)}{B_r(1)} \qquad (5.19)$$

so that at steady state $Y(\infty) = R(\infty)$.

Towards this end, we look for a controller of the form

$$R_c(z)U(z) = \gamma T_c(z)R(z) - S_c(z)Y(z) \qquad (5.20)$$

where R_c, S_c and T_c are polynomials in z^{-1}, to be determined

The controller has two components:

1. A feedback component $\frac{R_c}{\emptyset_{cl}}$ that helps ensure internal stability and reject disturbances.
2. A feed forward component $\frac{T_c}{\emptyset_{cl}}$ that helps *Y* track *R*.

Substituting (5.16) and (5.20) to (5.17) gives

$$Y = \gamma z^{-k} \frac{T_c B}{AR_c + z^{-k} BS_c} R$$

Equating this Y to the variable Y_m in (5.18) and cancelling common terms arrive at

$$\frac{BT_c}{AR_c + z^{-k} BS_c} = \frac{B_r}{\phi_{cl}} \qquad (5.19)$$

In general,

$$\deg\ B_r < \deg\ B$$

so that the desired closed loop transfer function is of lower order than that of BT_c. This is achieved by cancelling common terms between the numerator and denominator. In view of this, we factorize *B* and *A* as good and bad factors:

$$B = B^g B^b \qquad\qquad A = A^g A^b$$

Let

$$R_c = B^g R_1 \qquad S_c = A^g S_1 \qquad T_c = A^g T_1 \qquad (5.22)$$

Then (5.19) becomes

$$\frac{B^g B^b A^g T_1}{A^g A^b B^g R_1 + z^{-k} B^g B^b A^g S_1} = \frac{B_r}{\phi_{cl}}$$

This gives

$$\frac{B^b T_1}{A^b R_1 + z^{-k} B^b S_1} = \frac{B_r}{\phi_{cl}}$$

On equating the numerator, we obtain

$$B_r = B^b T_1 \tag{5.23}$$

Equating the denominator results in

$$A^b R_1 + z^{-k} B^b S_1 = \phi_{cl} \tag{5.24}$$

which can be solved for R_1 and S_1.

There are many options to choose T_1. For example, choosing T_1 to S_1, the 2-DOF controller is reduced to the 1-DOF configuration.

We can also make the following choice

$$T_1 = 1 \qquad (5.25)$$

From (5.23) and (5.25), (5.17) becomes

$$\gamma = \frac{\phi_{cl}(1)}{B^b(1)}$$

The closed loop transfer function is now obtained from (5.23) and (5.25) as

$$G_Y = \gamma z^{-k} \frac{B^b}{\phi_{cl}}$$

This shows that the bad zeros of the original transfer function cannot be changed by feedback.

Example 5.13: Consider the magnetically suspended ball in Fig 5.18

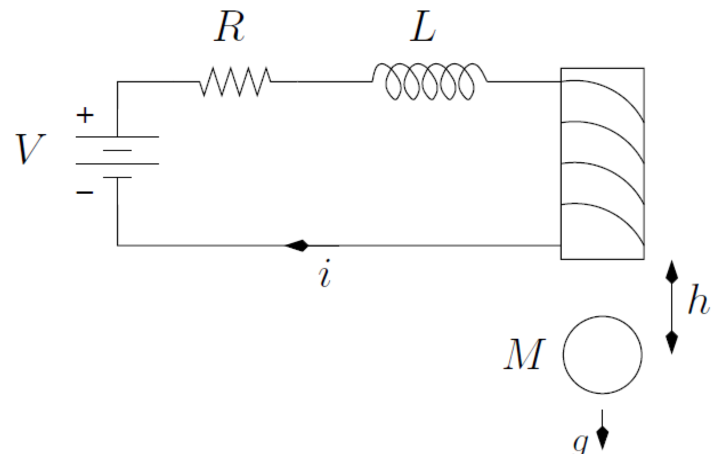


Figure 5.18: Magnetically suspended steel ball. The current through the coil creates a magnetic force, which counter balances the force due to gravity.

The input is current i and output is the distance between the ball and the armature, h.

Suppose that the ball is stationary at a chosen distance h_s with current i_s. Linearizing the system is around (i_s, h_s) gives the following model:

$$G(s) = \frac{-280.14}{s^3 + 100s^2 - 981s - 98100}$$

which has poles at 31.32, −31.32 and −100. It is unstable.

With sampling period Ts = 0.01 s and ZOH, we have

$$G(z) = z^{-1}\frac{(-3.7209\times10^{-5} - 1.1873\times10^{-4}z^{-1} - 2.2597\times10^{-5}z^{-2})}{1-2.4668z^{-1}+1.7721z^{-2}-0.3679z^{-3}}$$

$$= z^{-1}\frac{-3.7209\times10^{-5}(1+2.9877z^{-1})(1+0.2033z^{-1})}{(1-1.3678z^{-1})(1-0.7311z^{-1})(1-0.3679z^{-1})} = z^{-k}\frac{B}{A}$$

where k = 1. We factor A and B into good and bad parts:

$$A = A^g A^b$$

$$A^g = (1-0.7311z^{-1})(1-0.3679z^{-1})$$

$$A^b = (1-1.3678z^{-1})$$

$$B = B^g B^b$$

$$B^g = -3.7209\times10^{-5}(1+0.2033z^{-1})$$

$$B^b = (1+2.9877z^{-1})$$

The controller is designed to have the following the desired closed loop polynomials

$$\phi_{cl}(z) = 1 - 1.8z^{-1} + 0.819z^{-2}$$

From (5.24), we have

$$(1 - 1.3678z^{-1})R_1 + z^{-1}(1 + 2.9877z^{-1})S_1$$

$$= 1 - 1.8z^{-1} + 0.819z^{-2}$$

Solving the equation gives

$$S_1 = 0.0523$$
$$R_1 = 1 - 0.4845z^{-1}$$

Using (5.22), we obtain

$$R_c = -3.7209 \times 10^{-5}(1 - 0.2812z^{-1} - 0.0985z^{-2})$$
$$S_c = 0.0523 - 0.0575z^{-1} + 0.0141z^{-2}$$
$$T_c = A^g = (1 - 0.7311z^{-1})(1 - 0.3679z^{-1})$$

Simulation is carried out using the Simulink. The resulting profiles are shown in Fig. 5.19.

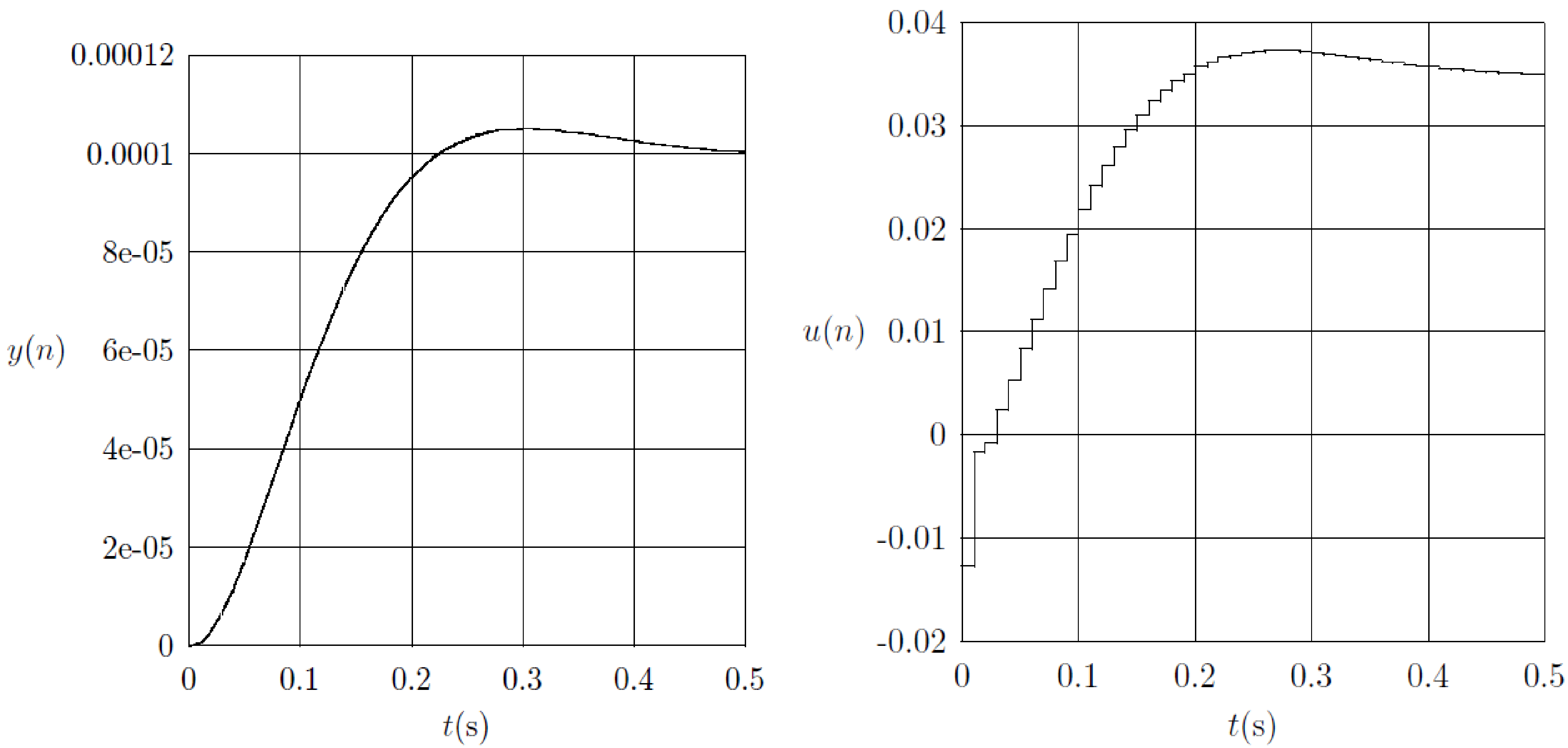
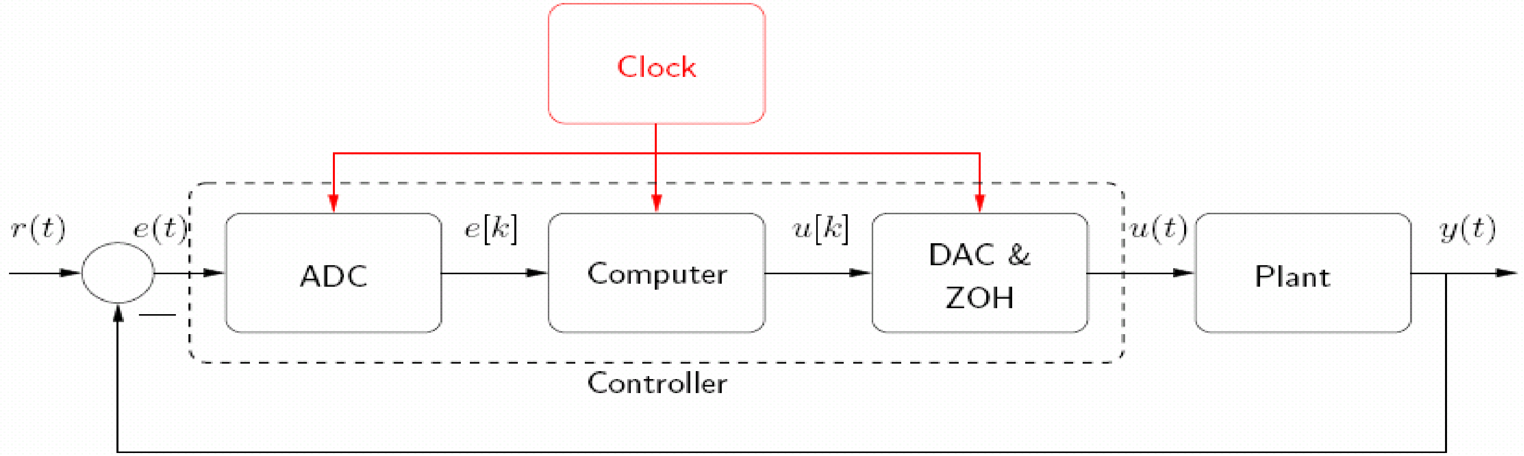


Fig 5.19, Results of Example 5.13. Note that the output y is measured in m. The equilibrium value is 1 cm or 0.01 m.

Chapter 6
Implementation of Digital Controllers

6.1 Introduction

Recall that the following block diagram shows a general feedback control system with a digital controller



ADC - Analog-Digital-Converter (includes sampler), DAC - Digital Analogue Converter, ZOH - Zero Order Hold

Questions

After designing the transfer function of a digital controller, how to implement it in practice?

- Software Realization
  - Write computer programs

- Hardware Realization
  - Construct circuitry using
    - digital adders,
    - multipliers,
    - delay elements (i.e. shift registers)

6.2 General form of transfer function

For any given digital controller, we can express it in the general form as,

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \quad (6.1)$$

- $n \geq m$ (causality)
- a_i and b_i are real coefficients (some can be zero).

***Example 6.1*:**

Represent the following PID controller in the form of (6.1)

$$G(z) = K_p + \frac{K_I}{1-z^{-1}} + K_D(1-z^{-1})$$

Solution

$$G(z) = K_p + \frac{K_I}{1-z^{-1}} + K_D(1-z^{-1})$$

$$= \frac{K_p(1-z^{-1}) + K_I + K_D(1-z^{-1})^2}{1-z^{-1}}$$

$$= \frac{(K_p + K_I + K_D) - (K_p + 2K_D)z^{-1} + K_D z^{-2}}{1-z^{-1}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where

$$a_1 = -1, \qquad a_2 = 0$$

$$b_0 = K_p + K_I + K_D$$

$$b_1 = -(K_p + 2K_D)$$

$$b_2 = K_D$$

Let the PID controller output be *Y(z)* and the input be *X(z),* then

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

Its difference equation is,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

The new output y(n) depends on:

- the previous outputs *y(n-1), y(n-2),*
- the previous inputs *x(n-1), x(n-2)*
- the new input *x(n)*.

6.3 Direct Programming

Consider (6.1) which has *n* poles and *m* zeros. To realize the transfer function, we have

$$(1 + a_1 z^{-1} + \ldots + a_N z^{-N})Y(z) =$$

$$(b_0 + b_1 z^{-1} + \ldots + b_m z^{-m})X(z)$$

i.e. $$Y(z) = -a_1 z^{-1}Y(z) - a_2 z^{-2}Y(z) - \ldots - a_n z^{-n}Y(z)$$

$$+ b_0 X(z) + b_1 z^{-1}X(z) + b_2 z^{-2}X(z) + \ldots + b_m z^{-m}X(z)$$

Taking inverse Z-transform, we get:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) +$$

$$b_0 x(k) + b_1 x(k-1) + b_2 x(k-2) + \ldots + b_m x(k-m) \qquad (6.2)$$

The controller in (6.2) can be realized in Figure 6.1

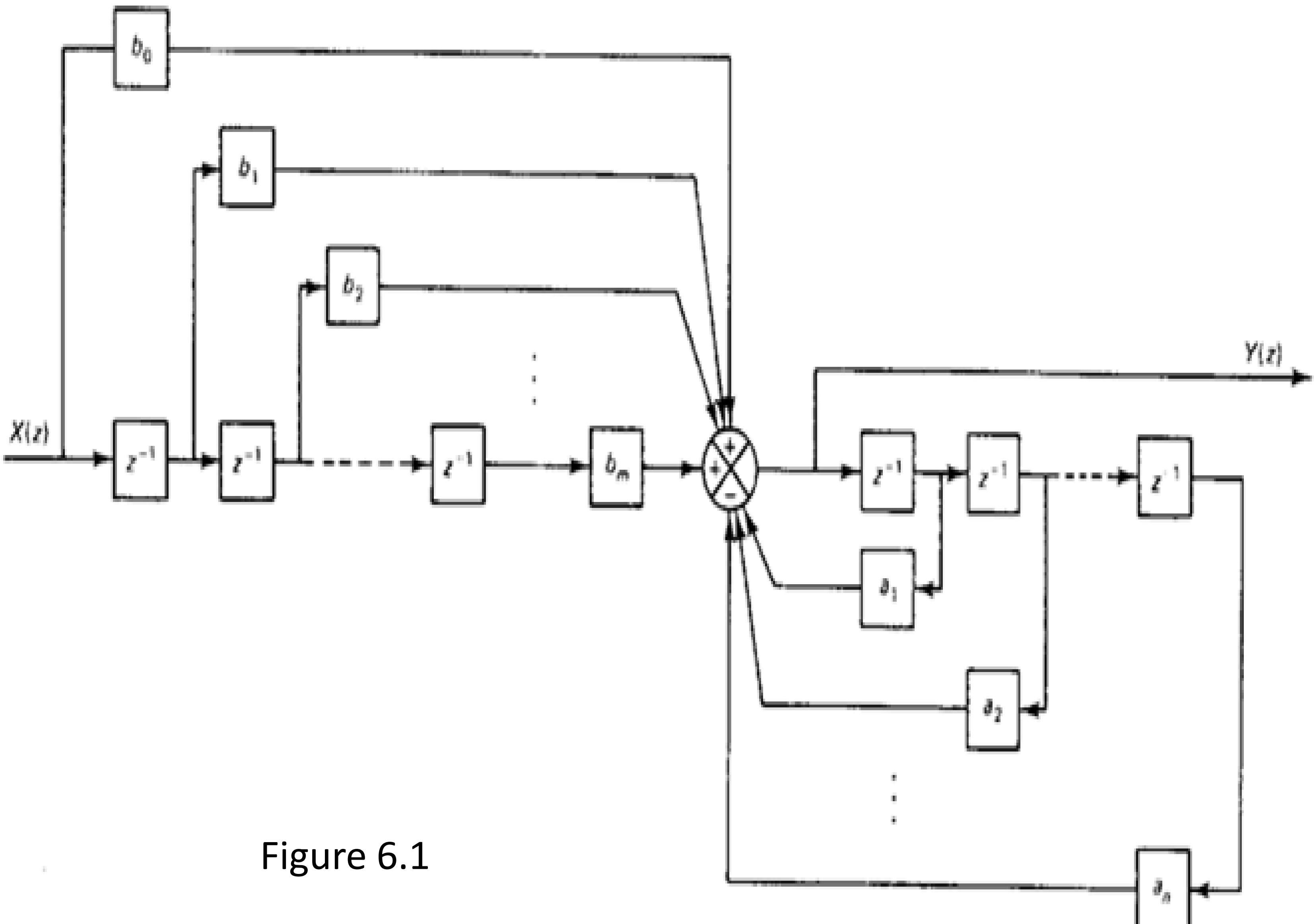


Figure 6.1

- The realization uses separate sets of delay elements for the numerator and denominator.
- The numerator uses m delay elements.
- The denominator uses n delay elements.
  →Total delay elements is $n+m$.
- In practice, we try to minimise the number of delay elements.

Questions

1. Can we reduce the number of delay elements?
2. If we can, what is the minimum number?

6.4 Standard Programming

In this approach, the delay elements can be reduced to *n*. Rearranging (6.1) as,

$$\frac{Y(z)}{X(z)}=\frac{Y(z)}{H(z)}\frac{H(z)}{X(z)}=(b_0+b_1z^{-1}+b_2z^{-2}+\ldots+b_mz^{-m})\frac{1}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_nz^{-n}}$$

Hence,

$$\frac{Y(z)}{H(z)}=b_0+b_1z^{-1}+b_2z^{-2}+\ldots+b_mz^{-m} \qquad (6.3)$$

$$\frac{H(z)}{X(z)}=\frac{1}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_nz^{-n}} \qquad (6.4)$$

(6.3) and (6.4) are rewritten as,

$$Y(z)=b_0H(z)+b_1z^{-1}H(z)+b_2z^{-2}H(z)+\ldots+b_mz^{-m}H(z) \qquad (6.5)$$

$$H(z)=X(z)-a_1z^{-1}H(z)-a_2z^{-2}H(z)-\ldots-a_nz^{-n}H(z) \qquad (6.6)$$

The controller in (6.5) and (6.6) can be realized in Figure 6.2

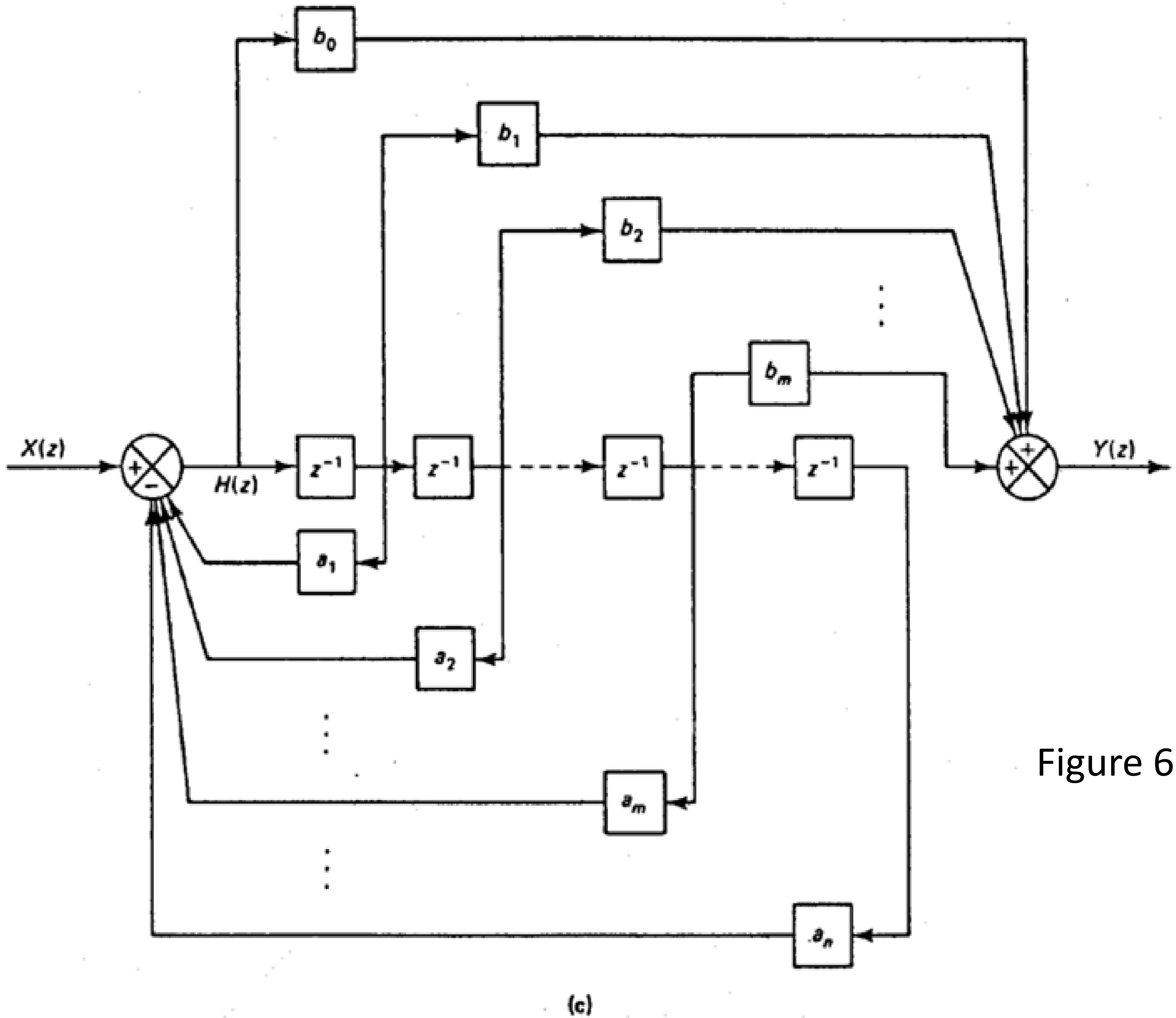


Figure 6.2

- Only *n* delay elements.
- $a_1, a_2, a_3, \ldots, a_n$ appear as feedback elements.
- $b_1, b_2, b_3, \ldots, b_m$ appear as feedforward elements.
- This implementation is equivalent to the direct programming. It is preferred since it uses less delay elements. Consequently, it saves memory space.

6.5 Sources of Errors

1) **Quantization of input signal into a finite number of discrete levels.**

2) **Accumulation of round-off errors in the arithmetic operations in the digital system.**

3) **Quantization errors of the controller coefficients.**

   – Cases 1) & 2), which can be found from text books and references, will not be covered here.

6.6 Analysis of Sensitivity to Quantization of Controller Coefficients

- In realizing a controller with a microprocessor, the accuracy with which the controller coefficients can be specified is limited by the word length of the processor's register that is provided to store the coefficients.
- Since the implemented controller is not exact, the poles and zeros are different from the desired values.
- Thus, the controller response will be different from the one designed.

Sensitivity Analysis

Consider the following general form of a controller

$$H(z) = \frac{\sum_{j=0}^{m} b_j z^{-j}}{1 + \sum_{j=1}^{n} a_j z^{-j}}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

$$= \frac{(1 - z_1 z^{-1}) \cdots (1 - z_m z^{-1})}{(1 - p_1 z^{-1}) \cdots (1 - p_n z^{-1})}$$

Suppose the controller is implemented with quantized coefficients, which is

$$\overline{H}(z) = \frac{\sum_{j=0}^{m} \overline{b}_j z^{-j}}{1 + \sum_{j=1}^{n} \overline{a}_j z^{-j}} = \frac{\overline{b}_0 + \overline{b}_1 z^{-1} + \cdots + \overline{b}_m z^{-m}}{1 + \overline{a}_1 z^{-1} + \cdots + \overline{a}_n z^{-n}}$$

$$= \frac{(1 - \overline{z}_1 z^{-1}) \cdots (1 - \overline{z}_m z^{-1})}{(1 - \overline{p}_1 z^{-1}) \cdots (1 - \overline{p}_n z^{-1})}$$

where the quantized coefficients are related to the unquantized coefficients as,

$$\overline{a}_j = a_j + \Delta a_j \qquad j = 1, 2, \cdots, n$$

$$\overline{b}_j = b_j + \Delta b_j \qquad j = 0, 1, 2, \cdots, m$$

Δa_j and Δb_j are the errors resulting from the quantization of the filter coefficients.

The denominator of *H(z)* is,

$$D(z) = 1 + \sum_{j=1}^{n} a_j z^{-j} = \prod_{i=1}^{n} (1 - p_i z^{-1})$$

where $\{p_i\}$ are the poles of *H(z)*. Similarly,

$$\overline{D}(z) = 1 + \sum_{j=1}^{n} \overline{a}_j z^{-j} = \prod_{i=1}^{n} (1 - \overline{p}_i z^{-1})$$

The quantization error causes the realized poles to be different from the desired poles:

$$\overline{p}_i = p_i + \Delta p_i \,, \qquad i = 1,2,\ldots,n$$

The magnitude of the perturbation error due to Δa_j can be expressed as,

$$|\Delta p_i|_{z=p_i} = \left| \sum_{j=1}^{n} \frac{\partial z}{\partial a_j}\bigg|_{z=p_i} \Delta a_j \right| \quad , \qquad i = 1,2,\ldots,n$$

Since $$\left(\frac{\partial D(z)}{\partial a_j}\right)_{z=p_i} = \left(\frac{\partial D(z)}{\partial z}\right)_{z=p_i} \left(\frac{\partial z}{\partial a_j}\right)_{z=p_i}$$

$$|\Delta p_i|_{z=p_i} = \left| \sum_{j=1}^{n} \frac{p_i^{n-j}}{\prod_{\substack{k=1 \\ k\neq i}}^{n} (p_i - p_k)} \Delta a_j \right|$$

$$D(z) = 1 + a_1 z^{-1} + \cdots + a_j z^{-j} + \cdots + a_n z^{-n} = \prod_{i=1}^{n} (1 - p_i z^{-1}) = z^{-n} \underbrace{(z-p_1)\cdots(z-p_i)\cdots(z-p_n)}_{D_1(z)}$$

$$\left.\frac{\partial D}{\partial a_j}\right|_{z=p_i} = \left. z^{-j} \right|_{z=p_i} = p_i^{-j}$$

$$\left.\frac{\partial D}{\partial z}\right|_{z=p_i} = \left. -n z^{-n-1} D_1(z) \right|_{z=p_i} + z^{-n} \left.\frac{\partial D_1(z)}{\partial z}\right|_{z=p_i} = p_i^{-n} \prod_{\substack{k=1 \\ k \neq i}}^{n} (p_i - p_k)$$

$$\left.\frac{\partial D_1(z)}{\partial z}\right|_{z=p_i} = \Big[(z-p_2)\cdots(z-p_i)\cdots(z-p_n) + \cdots + (z-p_1)\cdots(z-p_{i-1})(z-p_{i+1})\cdots(z-p_n) + \cdots + (z-p_1)\cdots(z-p_i)\cdots(z-p_{n-1}) \Big]\Big|_{z=p_i}$$

$$= \prod_{\substack{k=1 \\ k \neq i}}^{n} (p_i - p_k)$$

This expression provides a measure of the sensitivity of the ith pole to change in the coefficient a_j.

Similarly for the ith zero , we have

$$|\Delta z_i| = \left| \sum_{j=1}^{m} \frac{z_i^{m-j}}{\prod_{\substack{k=1 \\ k \neq i}}^{m} (z_i - z_k)} \Delta b_j \right|$$

- Hence, truncation or rounding of coefficients can cause major shifts in the realized poles and zeros.
- If the poles/zeros of the controller are tightly clustered together in the Z-plane, they will contribute large errors of the poles/zeros realized.
- The effects can be reduced by realizing higher order filter with interconnected second-order filter as discussed below.

6.7 Reducing Quantization Errors in Filter's Coefficient

By decomposing a higher-order pulse transfer function into a combination of low-order pulse transfer functions, the system can be made less sensitive to coefficient inaccuracies.

We consider two approaches :

1) Series programming
2) Parallel programming

6.7.1 Series programming

To implement the controller transfer function as a series connection of first-order and/or second-order transfer functions, as shown in Figure 6.3.

Let $G(z) = G_1(z)\ G_2(z) \ldots G_p(z)$

where $G_i(z)$s are either first- or second-order functions with real coefficients.

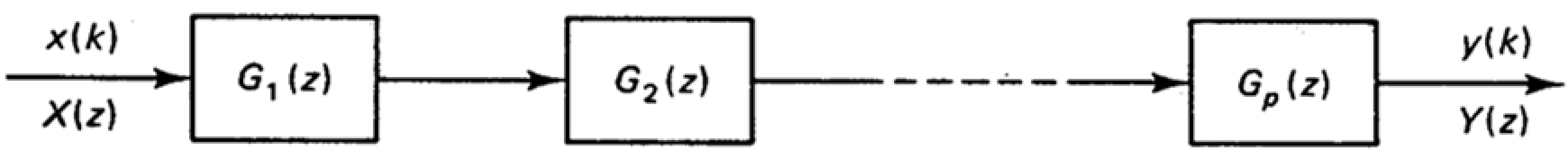


Figure 6.3

Question: How do we decide each of the $G_i(z)$?

- $G_i(z)$ may have real poles, real zeros, complex poles and complex zeros.
- We can group real poles and real zeros to produce either first- or second-order functions.
- Or group a pair of conjugate complex poles and a pair of conjugate complex zeros to produce a second order function.
- Or group two real zeros with a pair of conjugate complex poles, or vice versa to form a second-order function.

In summary, *G(z)* is decomposed as,

$$G(z) = G_1(z)\, G_2(z) \ldots G_p(z)$$

$$= \prod_{i=1}^{j} \frac{1 + b_i z^{-1}}{1 + a_i z^{-1}} \prod_{i=j+1}^{p} \frac{1 + e_i z^{-1} + f_i z^{-2}}{1 + c_i z^{-1} + d_i z^{-2}}$$

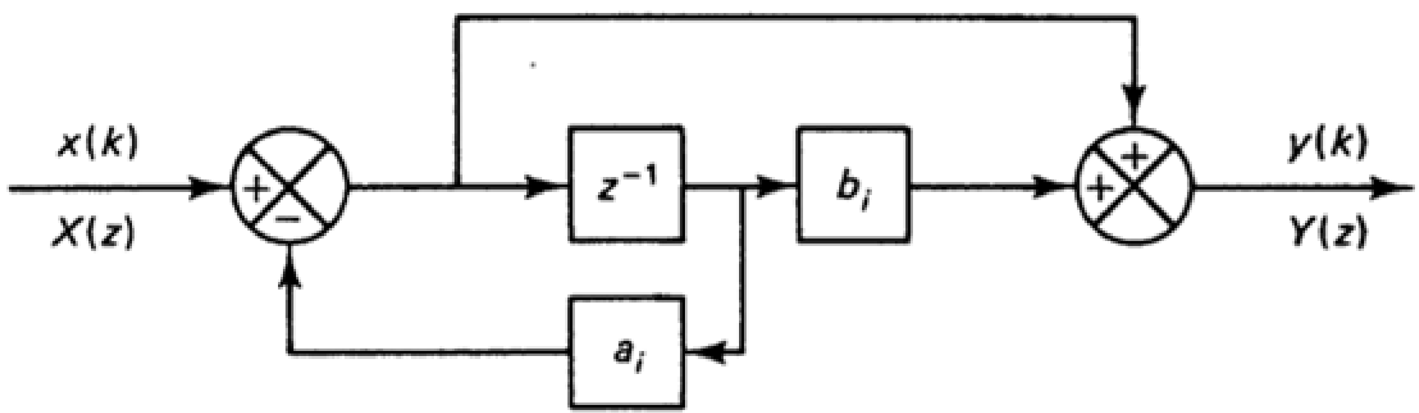


Figure 6.4: $\frac{Y(z)}{X(z)} = \frac{1 + b_i z^{-1}}{1 + a_i z^{-1}}$

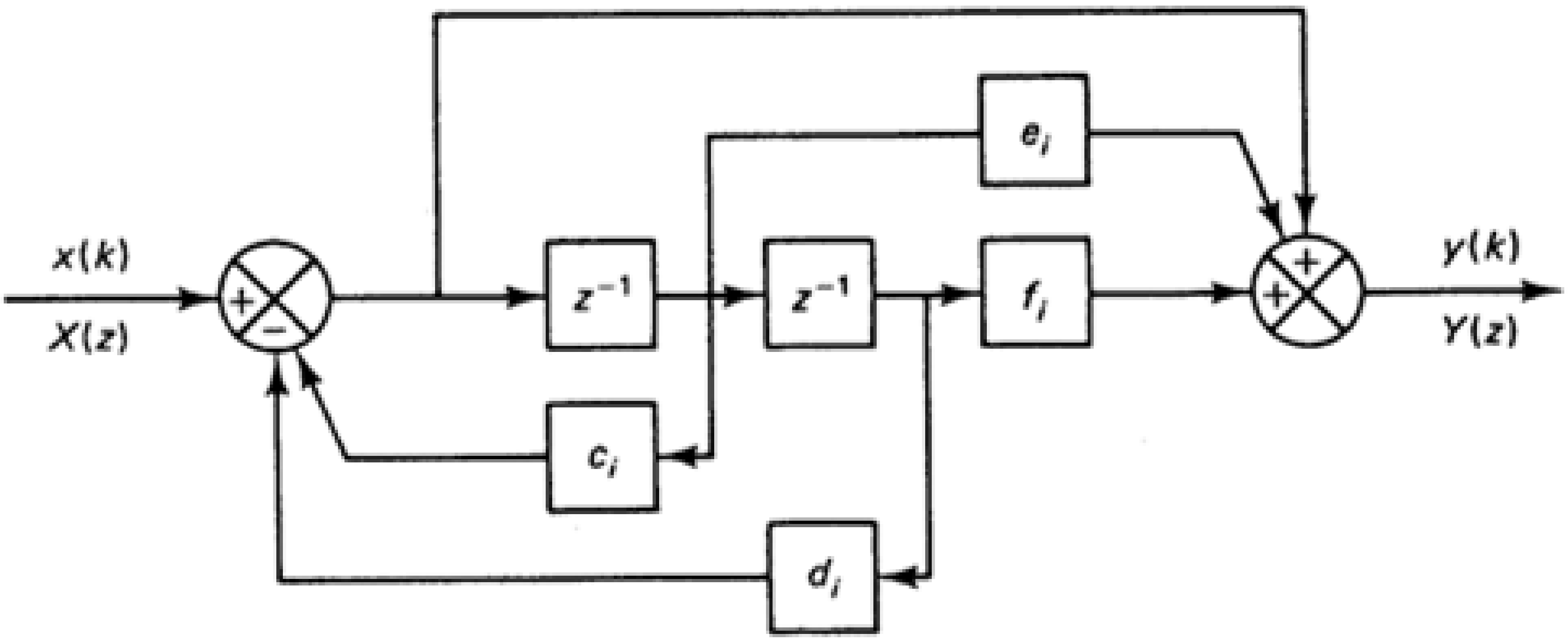


Figure 6.5: $$\frac{Y(z)}{X(z)} = \frac{1 + e_i z^{-1} + f_i z^{-2}}{1 + c_i z^{-1} + d_i z^{-2}}$$

6.7.2 Parallel Programming

Let $G(z) = A + G_1(z) + G_2(z) + \cdots + G_q(z)$

$$= A + \sum_{i=1}^{j} G_i(z) + \sum_{i=j+1}^{q} G_i(z)$$

$$= A + \sum_{i=1}^{j} \frac{b_i}{1 + a_i z^{-1}} + \sum_{i=j+1}^{q} \frac{e_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

- *A is* a simple constant.
- We have a parallel connection of *q+1* digital filters.
- The resultant first- and second-order filters are simpler than the series programming approach.

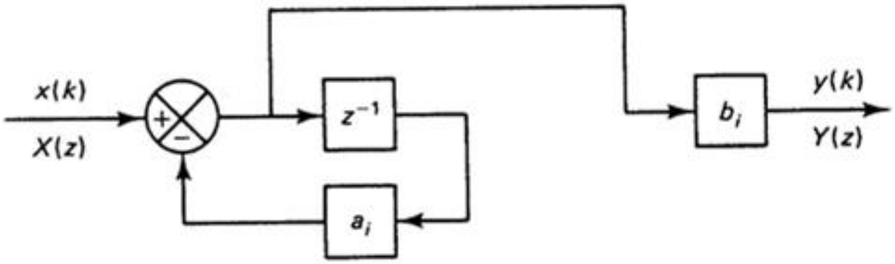


Figure 6.6: $$\frac{Y(z)}{X(z)} = \frac{b_i}{1 + a_i z^{-1}}$$

(a)

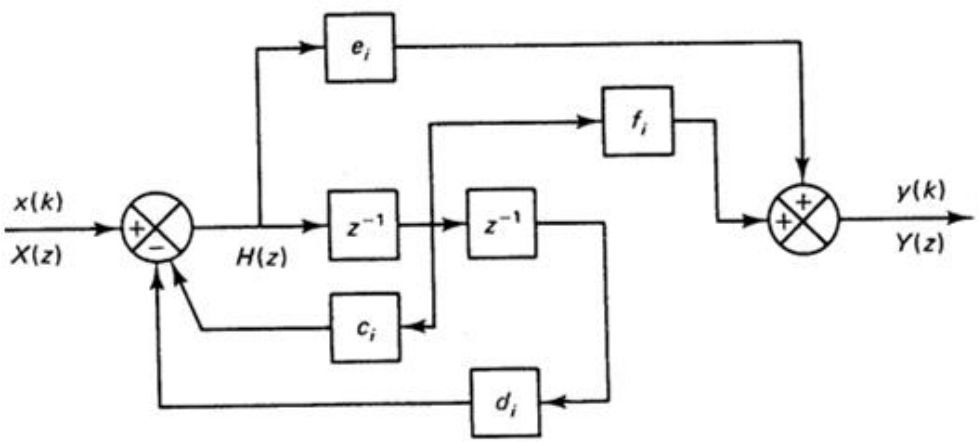


Figure 6.7: $$\frac{Y(z)}{X(z)} = \frac{e_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

(b)

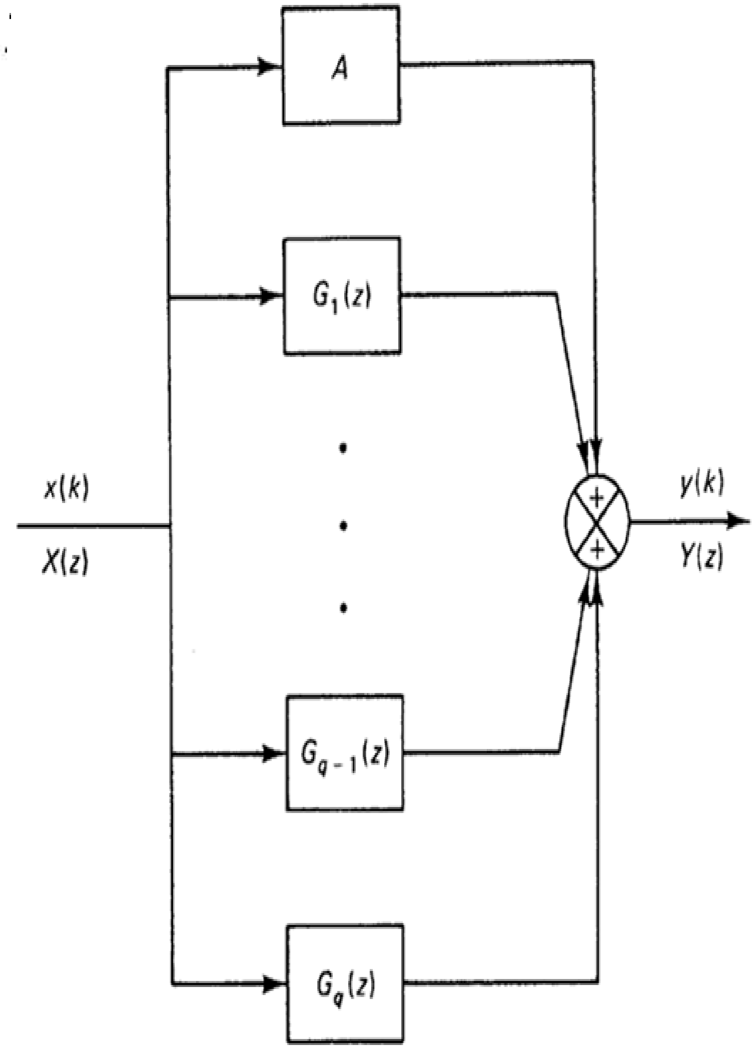
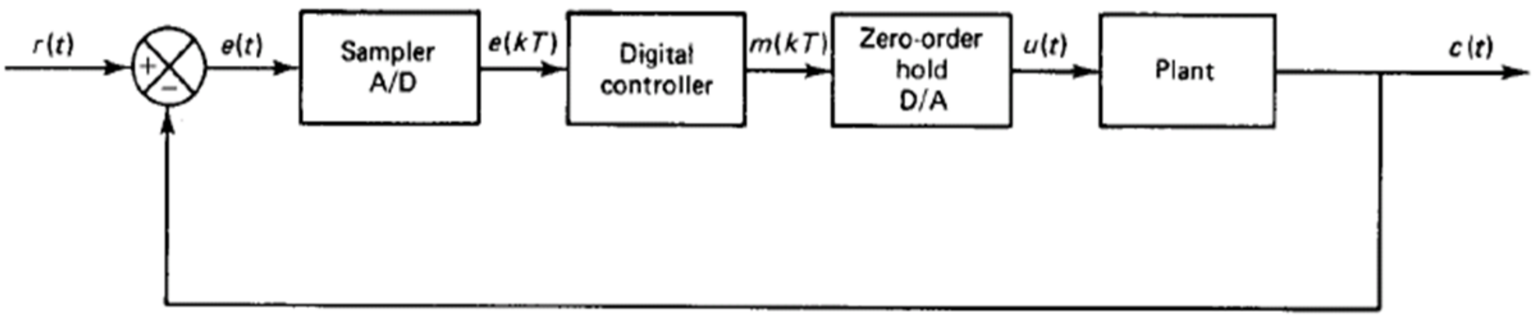
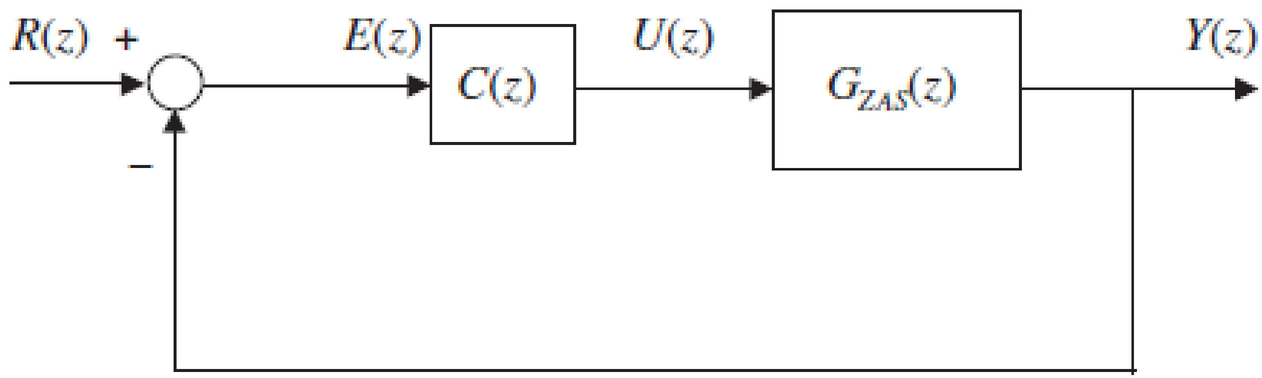


Figure 6.8

Summary
Chapters 5 and 6

Chap 5: Design of Discrete-time Controller





- Digital Implementation of Analog Controller Design
- Direct Z-domain Digital Controller Design - Frequency Response Design Approach
- Direct Control Design Based on Pulse Transfer Function

Digital Implementation of Analog Controller Design

General Procedure:

1. Design a controller *C(s)* for the analog subsystem to meet the desired design specifications.
2. Map the analog controller to a digital controller $G_D(z)$ (i.e. C(z) in Figure 5.2) using a ***suitable transformation***.
3. Check the sampled time response of the digital control system and repeat steps 1 to 2, if necessary, until the design specifications are met.

- **Backward Difference Method**

$$s = \frac{1 - z^{-1}}{T} = \frac{z - 1}{Tz}$$

- **Forward Difference Method**

$$s = \frac{1 - z^{-1}}{Tz^{-1}} = \frac{z - 1}{T}$$

- **Bilinear Transformation Method**

$$s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})} = \frac{2(z - 1)}{T(z + 1)}$$

- **Pole-Zero Matching**

  For an analog filter with transfer function

$$G_a(s) = K\frac{\prod_{i=1}^{m}(s-a_i)}{\prod_{i=1}^{n}(s-b_i)}$$

we have the following strictly proper digital filter

$$G(z) = \alpha\frac{(z+1)^{n-m-1}\prod_{i=1}^{m}(z-e^{a_iT})}{\prod_{i=1}^{n}(z-e^{b_iT})}$$

where α is a constant selected for equal filter gains at a critical frequency. For example, for a low-pass filter, α is selected to match the DC gains using

$$G(1)=G_a(0)$$

Frequency Response Design Approach

- **Design Procedure**

1. Select a sampling period and obtain a transfer function $G_{zAS}(z)$ of the discretized process
2. Transform $G_{zAS}(z)$ into *G(w)* using

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

3. Draw the Bode plot of *G(jv),* and use analog frequency response methods to design a controller *C(w)* (or $G_D(w)$) that satisfies the frequency domain specifications.
4. Transform *C(w)* (or $G_D(w)$) using

$$z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$$

   thus determining *C(z)* (or $G_D(z)$) .
5. Verify that the performance obtained is satisfactory.

Compensators

- **Phase-lead compensator** (including PD controllers):

- improves stability margins
- increases system bandwidth and hence faster response
- subject to high-frequency noise problems

- **Phase-lag compensator** (including PI Controllers):

- reduces system gain at high-frequencies
- reduces system bandwidth and hence slower response
- increases low-frequency gain and hence improves steady-state accuracy
- attenuates high-frequency noise

- **Phase Lag-lead compensator** (including PID controllers:

- increases low-frequency gain while increases bandwidth and stability margins

Direct Control Design the desired closed loop transfer function $G_{cl}(z)$.

$$C(z) = \frac{1}{G_{ZAS}(z)} \frac{G_{cl}(z)}{1 - G_{cl}(z)}$$

Necessary conditions required for the choice of $G_{cl}(z)$:

- The relative degree of $G_{cl}(z)$ must not be less than that of $G_{ZAS}(z)$. (causality);
- $G_{cl}(z)$ must contain all the unstable zeros of $G_{ZAS}(z)$ as its zeros (stability);
- The zeros of *1* - $G_{cl}(z)$ must include all the unstable poles of $G_{ZAS}(z)$ (stability);
- $G_{cl}(1)$ =1 (zero steady-state error).

A Suggested Procedure for Choosing $G_{cl}(z)$

1) Select the desired settling time T_s (and the desired maximum overshoot);
2) Select a suitable continuous-time closed-loop first-order or second-order closed-loop system with unit gain;
3) Obtain $G_{cl}(z)$ by converting the s-plane pole location to the z-plane pole location using pole-zero matching, $z_i = e^{s_i T}$ where z_i and s_i are discrete and continuous poles, respectively;
4) Verify that $G_{cl}(z)$ meets the conditions for causality, stability, and steady-state error. If not, modify $G_{cl}(z)$ until the conditions are met.

Finite Settling Time Design

- Digital control systems can be designed to settle at the reference output after a finite time period and follow it exactly thereafter.
- If all the poles and zeros of the discrete-time process are inside the unit circle,

$$G_{cl}(z) = z^{-k}$$

where $k \geq$ the intrinsic delay + relative degree of the discretized process. Then

$$C(z) = \frac{1}{G_{ZAS}(z)}\left[\frac{z^{-k}}{1-z^{-k}}\right]$$

The only design parameter is the sampling period T.

- Finite settling time designs may exhibit undesirable inter-sample behavior (oscillations)

Ripple-free Controller

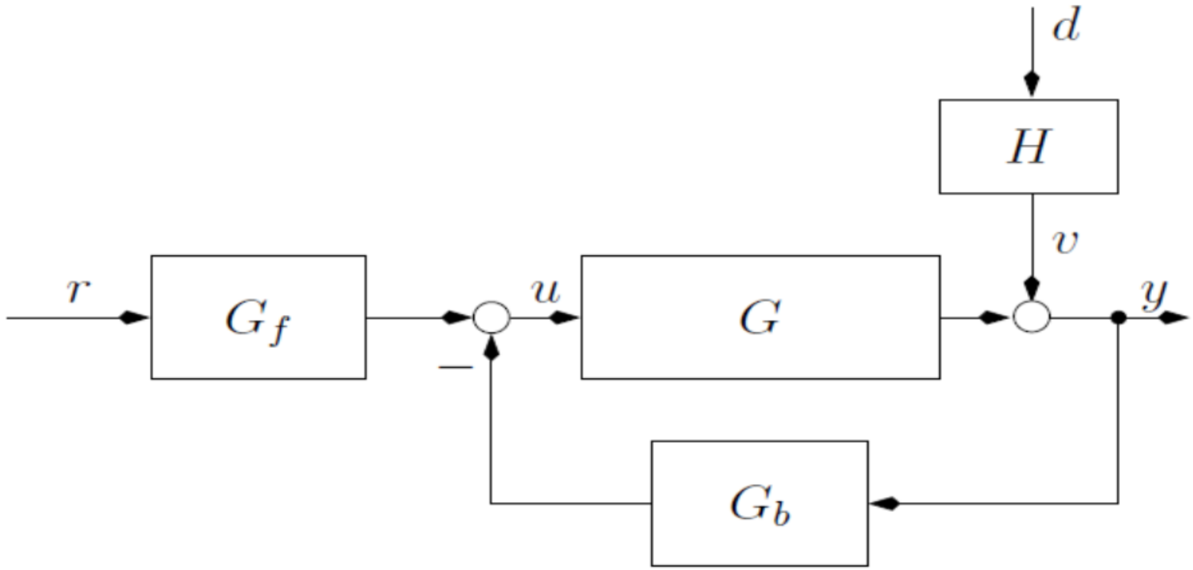
- To avoid intersample oscillations, we maintain the control variable constant after *n* samples, where *n* is the degree of the denominator of the discretized process.
- Considering Figure 5.2, we have

$$U(z) = \frac{Y(z)}{G_{ZAS}(z)} = \frac{Y(z)}{R(z)}\frac{R(z)}{G_{ZAS}(z)} = G_{cl}(z)\frac{R(z)}{G_{ZAS}(z)} \quad (5.13)$$

- We choose $G_{cl}(z)$ with the constraint that $G_{cl}(1)$ =1 (zero steady-state error) based on *U(z)* from (5.13) or *E(z) (see the examples).* Then

$$C(z) = \frac{1}{G_{ZAS}(z)}\frac{G_{cl}(z)}{1-G_{cl}(z)}$$

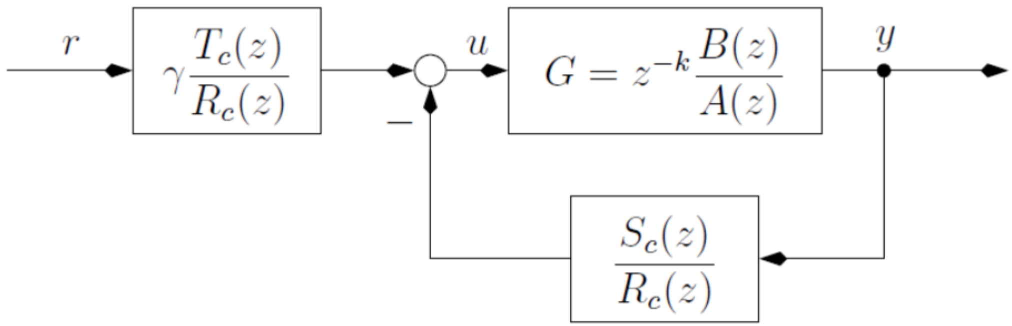
Two Degrees of Freedom Feedback Controller



$$G_f = \frac{T_c}{R_c} \qquad G_b = \frac{S_c}{R_c}$$

There are two degree of freedoms for controller design

Pole Placement Controller



$$B = B^g B^b$$
$$A = A^g A^b$$

Design a controller such that the plant output *y* is related to the setpoint signal *r* as follows

$$Y_m(z) = \gamma z^{-k}\frac{B_r}{\phi_{cl}}R(z), \quad \gamma = \frac{\phi_{cl}(1)}{B_r(1)}, \quad B_r = B^b T_1$$

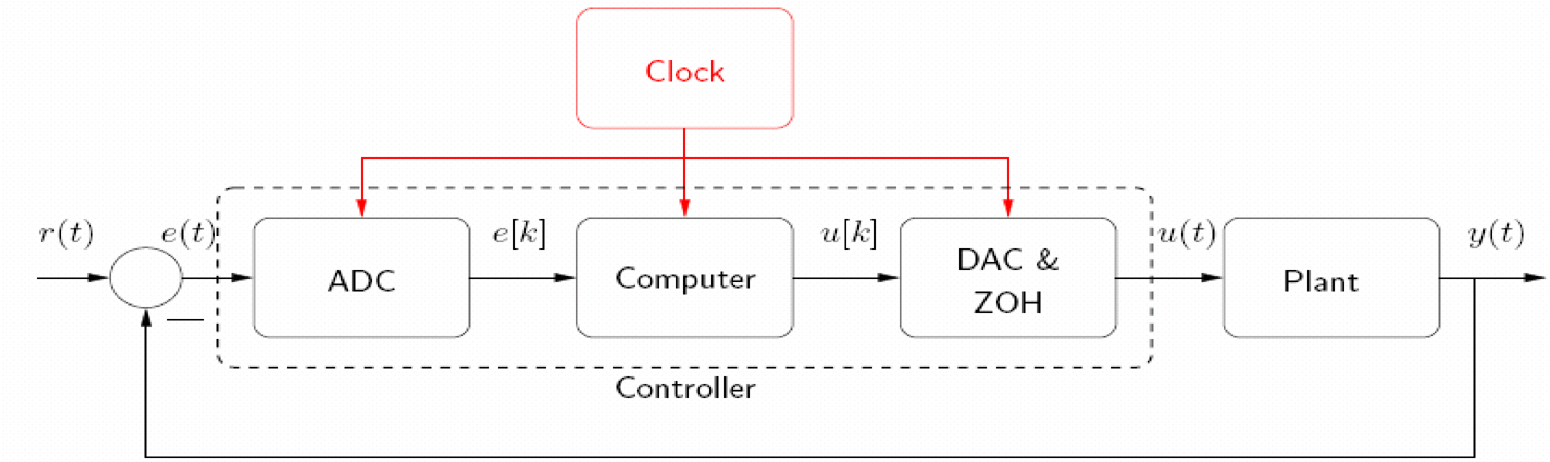
where ϕ_{cl} is the polynomial obtained by the desired poles

Let $R_c = B^g R_1, \quad S_c = A^g S_1, \quad T_c = A^g T_1$

Then $A^b R_1 + z^{-k} B^b S_1 = \phi_{cl}$

There are many options to choose T_1, e.g. $T_1 = S_1$ or 1.

Chapter 6 Controller Implementation



ADC - Analog-Digital-Converter (includes sampler), DAC - Digital Analogue Converter, ZOH - Zero Order Hold

Questions

After designing the transfer function of a digital controller, how to implement it in practice?

Answer: Software Realization (write computer programs)

6.2 General form of transfer function

For any given digital controller, we can express it in the general form as,

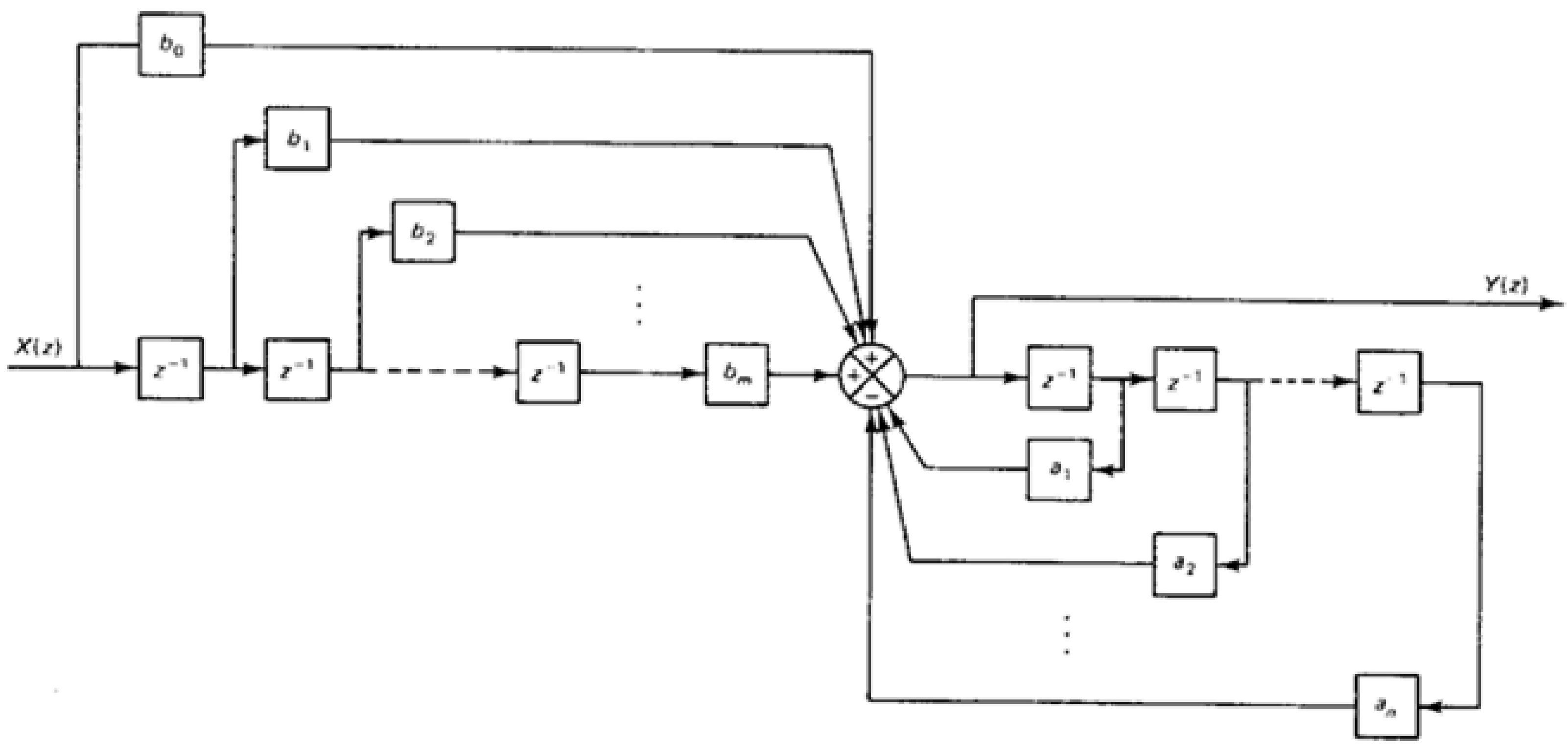
$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \quad (6.1)$$

- $n \geq m$ (causality)
- a_i and b_i are real coefficients (some can be zero).

6.3 Direct Programming

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) +$$

$$b_0 x(k) + b_1 x(k-1) + b_2 x(k-2) + \ldots + b_m x(k-m)$$



The controller in (6.2) can be realized in Figure 6.1

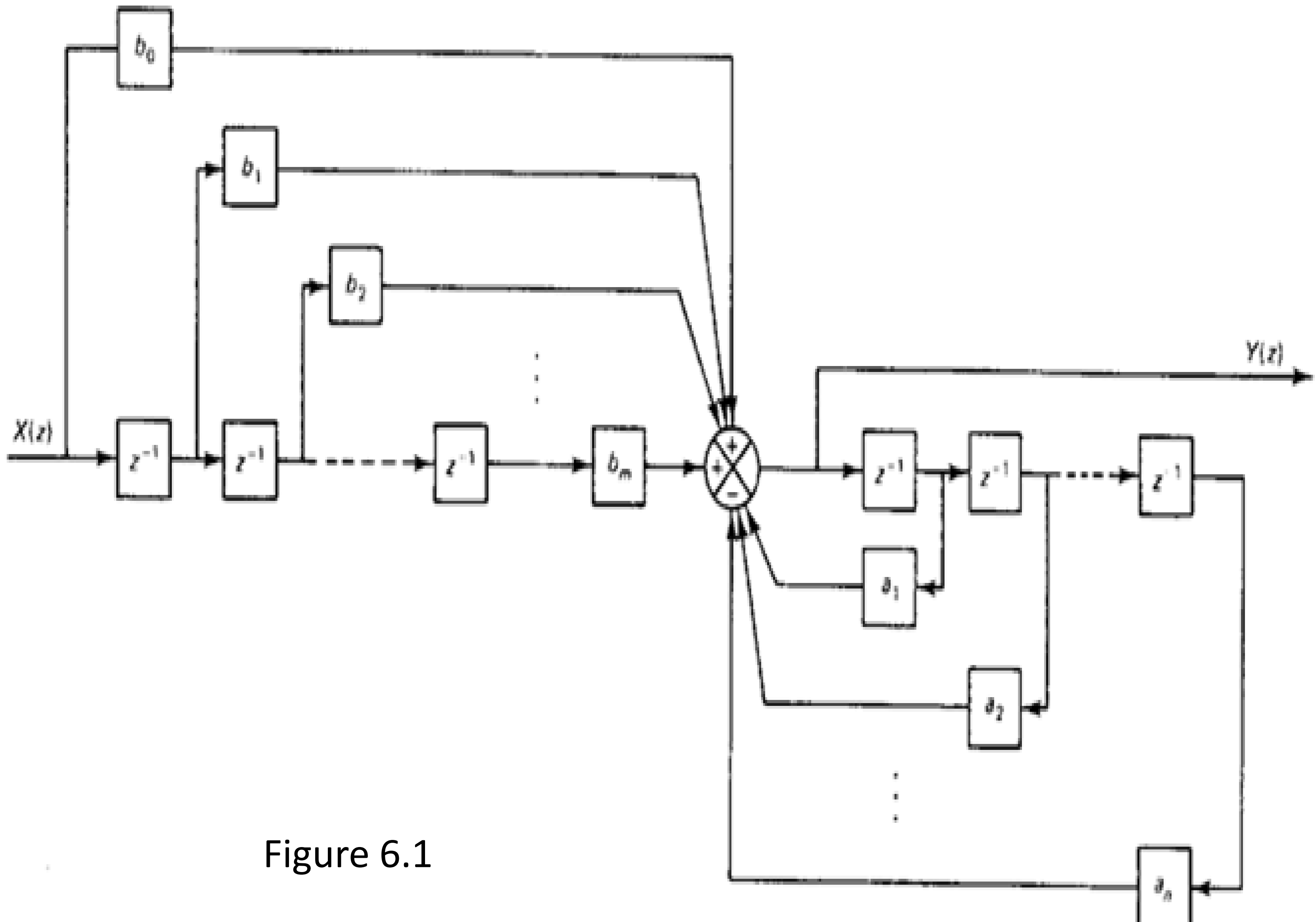


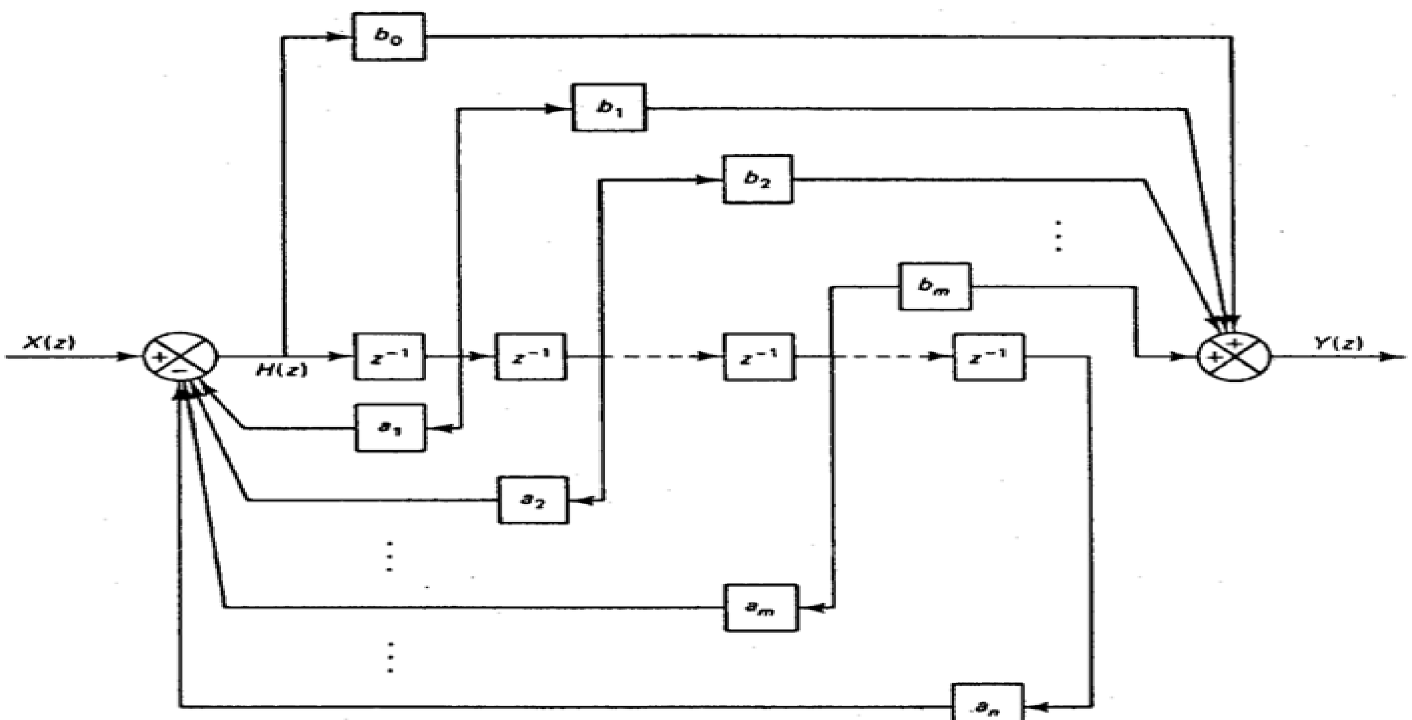
Figure 6.1

6.4 Standard Programming

$$\frac{Y(z)}{X(z)}=\frac{Y(z)}{H(z)}\frac{H(z)}{X(z)}=(b_0+b_1z^{-1}+b_2z^{-2}+\ldots+b_mz^{-m})\frac{1}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_nz^{-n}}$$

$$Y(z)=b_0H(z)+b_1z^{-1}H(z)+b_2z^{-2}H(z)+\ldots+b_mz^{-m}H(z)$$

$$H(z)=X(z)-a_1z^{-1}H(z)-a_2z^{-2}H(z)-\ldots-a_nz^{-n}H(z)$$



(c)

Analysis of Sensitivity to Quantization of Controller Coefficients

The sensitivity of the ith pole or the ith zero to change in the coefficient a_j:

$$|\Delta p_i| = \left| \sum_{j=1}^{n} \frac{p_i^{n-j}}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (p_i - p_k)} \Delta a_j \right|$$

$$|\Delta z_i| = \left| \sum_{j=1}^{m} \frac{z_i^{m-j}}{\prod_{\substack{k=1 \\ k \neq i}}^{m} (z_i - z_k)} \Delta b_j \right|$$

- Hence, truncation or rounding of coefficients can cause major shifts in the realized poles and zeros.
- If the poles/zeros of the controller are tightly clustered together in the Z-plane, they will contribute large errors of the poles/zeros realized.
- The effects can be reduced by realizing higher order filter with interconnected second-order filter as discussed below.

Reducing Quantization Errors in Filter's Coefficient

By decomposing a higher-order pulse transfer function into a combination of low-order pulse transfer functions, the system can be made less sensitive to coefficient inaccuracies.

We consider two approaches :

1) Series programming
2) Parallel programming

6.6.1 Series programming

To implement the controller transfer function as a series connection of first-order and/or second-order transfer functions, as shown in Figure 6.3.

Let $G(z) = G_1(z)\, G_2(z) \ldots G_p(z)$

where $G_i(z)$s are either first- or second-order functions with real coefficients.

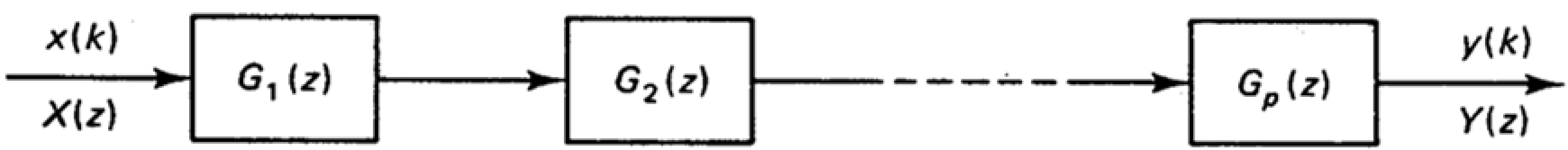


Figure 6.3

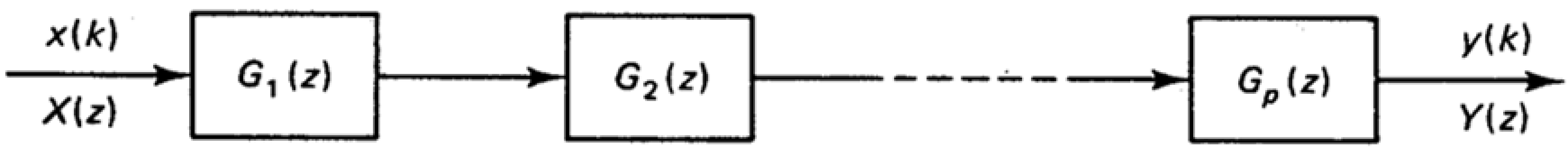
Question: How do we decide each of the $G_i(z)$?

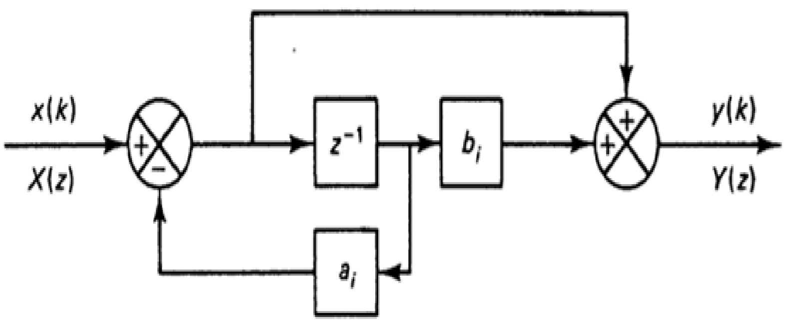
- $G_i(z)$ may have real poles, real zeros, complex poles and complex zeros.
- We can group real poles and real zeros to produce either first- or second-order functions.
- Or group a pair of conjugate complex poles and a pair of conjugate complex zeros to produce a second order function.
- Or group two real zeros with a pair of conjugate complex poles, or vice versa to form a second-order function.

Series programming: *G(z)* is decomposed as

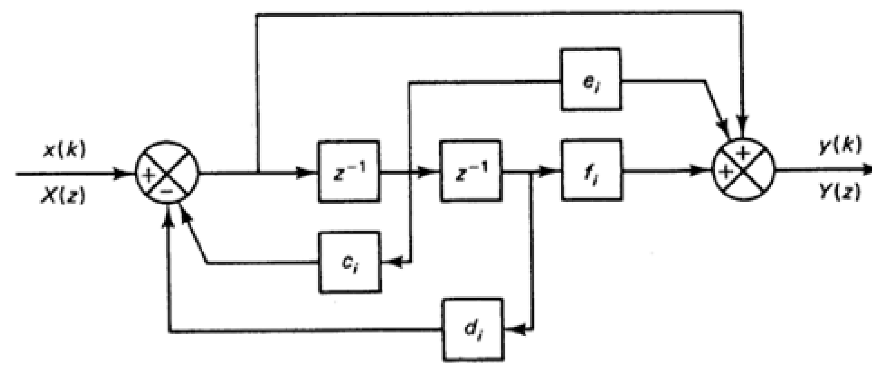
$$G(z)=G_1(z)\;G_2(z)\;\ldots G_p(z)$$

$$= \prod_{i=1}^{j} \frac{1+b_i z^{-1}}{1+a_i z^{-1}} \prod_{i=j+1}^{p} \frac{1+e_i z^{-1}+f_i z^{-2}}{1+c_i z^{-1}+d_i z^{-2}}$$





$$\frac{Y(z)}{X(z)} = \frac{1+b_i z^{-1}}{1+a_i z^{-1}}$$



$$\frac{Y(z)}{X(z)} = \frac{1+e_i z^{-1}+f_i z^{-2}}{1+c_i z^{-1}+d_i z^{-2}}$$

Parallel Programming:

$$G(z) = A + G_1(z) + G_2(z) + \cdots + G_q(z)$$

$$= A + \sum_{i=1}^{j} G_i(z) + \sum_{i=j+1}^{q} G_i(z)$$

$$= A + \sum_{i=1}^{j} \frac{b_i}{1 + a_i z^{-1}} + \sum_{i=j+1}^{q} \frac{e_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

- *A is* a simple constant.
- We have a parallel connection of *q+1* digital filters.
- The resultant first- and second-order filters are simpler than the series programming approach.

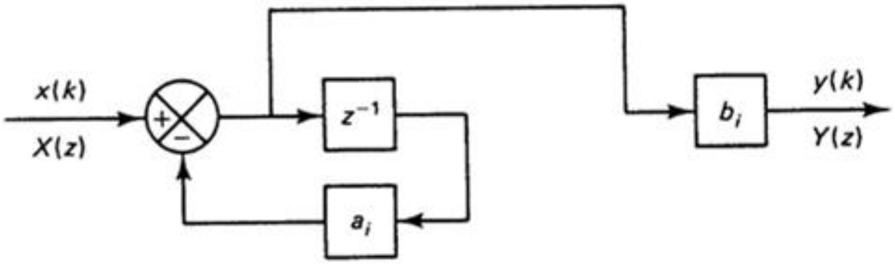


Figure 6.6: $$\frac{Y(z)}{X(z)} = \frac{b_i}{1 + a_i z^{-1}}$$

(a)

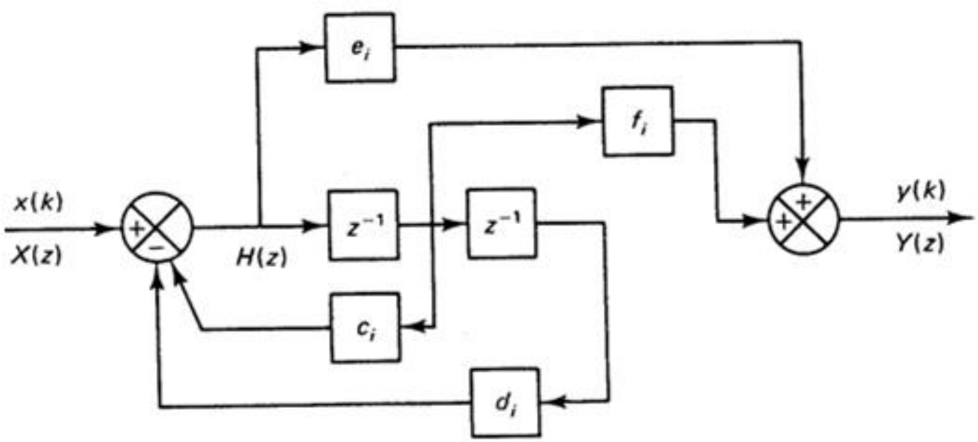


Figure 6.7: $$\frac{Y(z)}{X(z)} = \frac{e_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

(b)

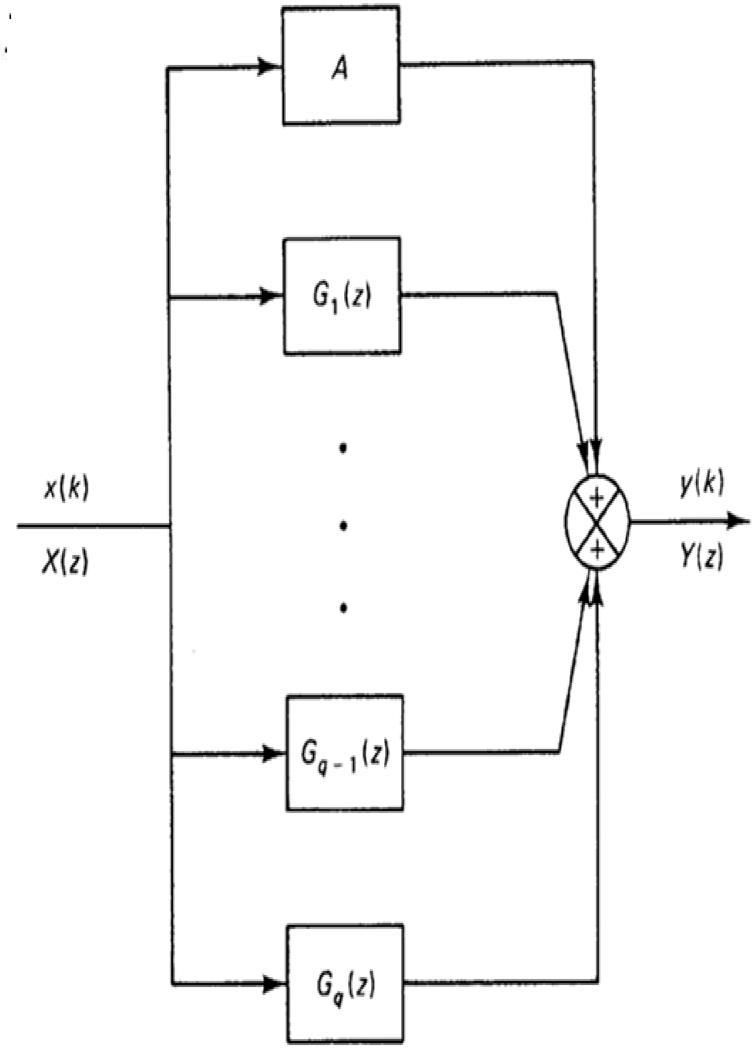


Figure 6.8