Model Predictive Control — Lecture 3
Tuning of MPC

Ling KV (ekvling@ntu.edu.sg)

January 2025

School of Electrical and Electronic Engineering

What are we trying to do?

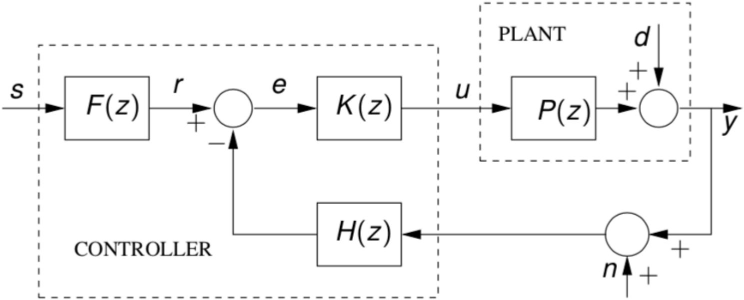
- Feedback is dangerous.
- The *only* purpose of using feedback is to reduce the effects of uncertainty.

Simulating response to step *set-point* change is not very meaningful except in special cases:

- If step response $\Rightarrow$ something about the robustness to modelling errors.
- If set-point response $\Rightarrow$ disturbance response.
- If step response $\Rightarrow$ response to other signals.

The canonical feedback loop

Two 'degrees of freedom' feedback system



$$y(z) = S(z)d(z) + S(z)P(z)K(z)F(z)s(z) - T(z)n(z)$$

where:
$$S(z) = [I + P(z)K(z)H(z)]^{-1}$$
$$T(z) = [I + P(z)K(z)H(z)]^{-1} P(z)K(z)H(z)$$

The canonical feedback loop

The Sensitivity and Complementary Sensitivity functions

$$S(z) = [I + P(z)K(z)H(z)]^{-1}$$
$$T(z) = [I + P(z)K(z)H(z)]^{-1} P(z)K(z)H(z)$$

- $S(z)$ — The Sensitivity function. Small so that effect of output disturbance is kept small and closed-loop performance less sensitive to open loop changes.
- $T(z)$ — The Complementary Sensitivity function. Small so that the effect of measurement noise is kept small,. larger stability margins
- $S(z) + T(z) = I$
  — $S(z)$ and $T(z)$ cannot be small simultaneously
- Response to set-point changes: $S(z)P(z)K(z)$
- Response to output disturbances: $S(z)$

Feedback properties of MPC

- Optimisation formulation easily related to tracking requirements and to real constraints
- But not easily related to good feedback properties
- No/few tools to analyse feedback properties when constraints are active
- So analyse unconstrained situation
- And rely on simulation for constrained case

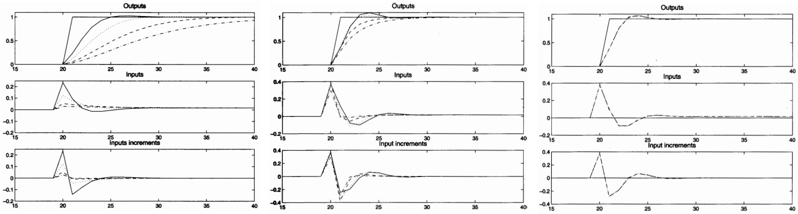
Tuning MPC, What can we do?

- Adjust the horizons
- Adjust the weights
- Disturbance model and observer dynamics
- Reference trajectory
- Reverse engineering of existing controllers

The next few slides gives a series of examples and demonstrates how MPC might be tuned quickly and effectively.
The example used is

$$y(k) = \frac{z^{-1} + 0.2z^{-2}}{(1 - 0.9z^{-1})(1 - 0.8z^{-1})} u(k)$$

Effect of varying the output horizon, N_2

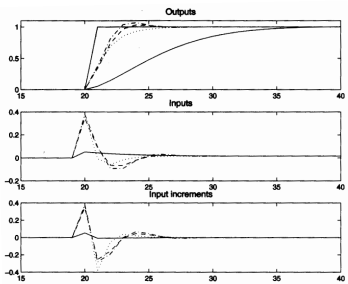


$N_u = 1$ $N_u = 2$ $N_u = 5$

$N_2 = 3$	$N_2 = 5$	$N_2 = 10$	$N_2 = 20$
Solid	Dotted	Dashed	Dash-dot

- If N_u is small, increasing N_2 causes the loop dynamics to slow down.
- If N_u is large, increasing N_2 does not improve performance.

Effect of varying the control horizon, N_u

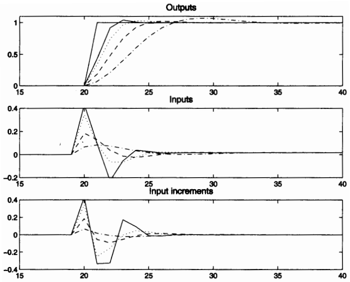


$N_2 = 10$ for different N_u

$N_u = 1$	$N_u = 2$	$N_u = 3$	$N_u = 5$
Solid	Dotted	Dashed	Dash-dot

- If N_2 is large, then increasing N_u improves performance.

Effect of varying the control weighting, λ



$N_2 = 10$, $N_u = 3$ for different λ

$\lambda = 0.1$	$\lambda = 1$	$\lambda = 10$	$\lambda = 100$
Solid	Dotted	Dashed	Dash-dot

- Increasing λ slows down the responses.

Tuning of MPC

Some special (extreme) cases

Recall the MPC cost funtion

$$J = \sum_{i=N_1}^{N_2} \| r(k+i) - \hat{y}(k+i|k) \|^2_{Q(i)} + \sum_{i=1}^{N_u-1} \| \Delta u(k+i|k) \|^2_{R(i)}$$

- Effects of control weighting
- Mean-level control
- Deadbeat control
- ‘Perfect’ control

Effects of control weighting

The matrix R

- $R(i)$ — 'move supression factors'
- $R(i)$ large $\Rightarrow$ feedback 'switched off'
  - With stable plant expect stability (not always true) and slow response
  - With unstable plant expect instability
- We have better ways of ensuring stability
- Large R can improve conditioning of least-squares problem

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‘Mean-Level’ Control

Stable plant, $N_u = 1$, $R(i) = 0$

Assume a constant reference trajectory: $r(k+i) = r_1$.
Then MPC minimises

$$\sum_{i=1}^{N_2} ||\hat{y}(k+i|k) - r_1||^2_{Q(i)}$$

using one control move.
If $N_2 \to \infty$ then optimal action is to set $u(k)$ to that value which gives $\hat{y} = r_1$ in the steady-state.
Hence perfect model $\Rightarrow$ control trajectory is a step
$\Rightarrow$ get **open-loop response** of the output — *‘natural response’*.

'Mean-Level' Control

Stable plant, $N_u = 1$, $R(i) = 0$

Hence (for square plant, ie equal numbers of inputs and outputs):

$$\bar{y}(z) = P(1)^{-1}P(z)\frac{r_1}{1 - z^{-1}}$$

Comparing with earlier expression:

$$S(z)P(z)K(z) = P(1)^{-1}P(z)$$

so output-disturbance response is (SISO case):

$$S(z) = \frac{1}{P(1)K(z)}$$

Can still adjust osberver independently, *eg* deadbeat response to disturbances:

- 'Natural' response to set-point changes
- Agressive correction of disturbances
- Optimal practical combination?

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Deadbeat control

Suppose $P(z)$ is n-th order (ie n states).
Let $N_1 = N_u = n$, $R(i) = 0$, $r(k+i) = r_1$, $N_2 \geq 2n$.
So now cost is:

$$\sum_{i=n}^{2n} ||\hat{y}(k+i|k) - r_1||^2_{Q(i)}$$

The controller has time to drive output to r_1 and leave it there.
(It needs n steps and n moves in general.)
$N_1 = n \Rightarrow$ zero cost, so this is the optimal strategy.
$N_2 \geq 2n \Rightarrow$ no hidden modes emerge later.

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Deadbeat control

Deadbeat set-point response $\Rightarrow S(z)P(z)K(z) = z^{-n}N(z)$
SISO case:

$$S(z) = \frac{N(z)}{z^n P(z)K(z)}$$

But zeros of $N(z)$ include zeros of $P(z)$
So OK even if non-minimum-phase plant.
Two ‘opposite extremes’:

- Mean-level control $N_2 \to \infty$, $H_u = 1$
- Deadbeat control $N_2 \geq 2n$, $N_1 = N_u = n$

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‘Perfect’ control

An ironic misnomer — it is very poor

$N_u = N_2 = 1, \quad R(i) = 0$
Cost — very short-term:

$$||\hat{y}(k+1|k) - r(k+1)||^2_{Q(i)}$$

Suppose (SISO)

$$\bar{y}(z) = P(z)\bar{u}(z) = \frac{B(z^{-1})}{A(z^{-1})}\bar{u}(z)$$

or

$$b_0 u(k) = y(k+1) + \ldots + a_n y(k+1-n) - b_1 u(k-1) - \ldots - b_n u(k-n)$$

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‘Perfect’ control

An ironic misnomer — it inverts the plant

We get $y(k+1) = r(k+1)$ by setting:

$$b_0 u(k) = r(k+1) + a_1 y(k) + \ldots + a_n y(k+1-n) - \ldots - b_n u(k-n)$$

so eventually:

$$b_0 u(k) = r(k+1) + a_1 r(k) + \ldots + a_n r(k+1-n) - \ldots - b_n u(k-n)$$

or

$$B(z^{-1}) u(k) = A(z^{-1}) r(k+1)$$

So the controller is the inverse of the plant.
No good if plant is non-minimum-phase (zeros outside the unit circle) or has zeros close to the unit circle.

Tuning of MPC

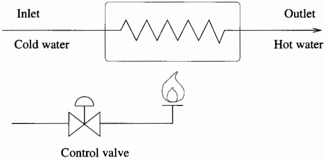
Summary

- Tuning parameters:
  - weights
  - horizons
  - disturbance [not discussed]
  - observer [not discussed]
  - reference trajectory [not discussed]
- Rules of thumb, little theory [stability theory beyond syllabus]

Case Study: A Water Heater, 1/3

Consider a water heater where the cold water is heated by means of a gas burner. The outlet temperature depends on the energy added to the water through the gas burner. Therefore this temperature can be control by the valve which manipulates the gas flow to the heater.

Design a MPC to control the outlet temperature of a water heater.



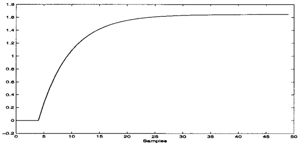
Case Study: A Water Heater, 2/3

To perform a MPC design, we need a model of the process. The step response model of this process can used to design the controller.

g_1 0	g_2 0	g_3 0.271	g_4 0.498	g_5 0.687	g_6 0.845	g_7 0.977	g_8 1.087	g_9 1.179	g_{10} 1.256
g_{11} 1.320	g_{12} 1.374	g_{13} 1.419	g_{14} 1.456	g_{15} 1.487	g_{16} 1.513	g_{17} 1.535	g_{18} 1.553	g_{19} 1.565	g_{20} 1.581
g_{21} 1.592	g_{22} 1.600	g_{23} 1.608	g_{24} 1.614	g_{25} 1.619	g_{26} 1.623	g_{27} 1.627	g_{28} 1.630	g_{29} 1.633	g_{30} 1.635

The step response can be approximated by a transfer function

$$G(z) = \frac{y(k)}{u(k)} = \frac{0.2713z^{-3}}{1 - 0.8351z^{-1}}$$



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Case Study: A Water Heater, 3/3

A state space model

$$x(k+1) = A_p x(k) + B_p u(k), \quad y(k) = C_p x(k)$$

can be formed with

$$A_p = \begin{bmatrix} 0.8351 & 0 & 0.2713 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

with

$$x(k) = \begin{bmatrix} y(k) & u(k-1) & u(k-2) \end{bmatrix}^T$$

Additional Exercises

Design a MPC controller for the processes given below. Choose your own values for the prediction and control horizons, etc and experiment with them.

- Camacho, p.46, Q3.7

$$G(s) = \frac{225(s-1)}{(s+1)(s^2+30s+225)}$$

  The MPC will be implemented with a sampling time of 0.2 second.

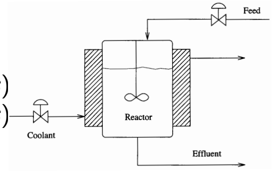
- Camacho, p.78, Q4.1

$$y(k) - 1.5y(k-1) + 0.56y(k-2) = 0.9u(k-1) - 0.6u(k-2)$$

MIMO MPC

Consider the linearized model of a stirred tank reactor.

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+0.7s} & \frac{5}{1+0.3s} \\ \frac{1}{1+0.5s} & \frac{2}{1+0.4s} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$



where U_1 and U_2 are the feed flow rate and the flow of coolant in the jacket, respectively. The controlled variables Y_1 and Y_2 are the effluent concentration and the reactor temperature respectively.

The model is discretised for a sampling time of 0.03 minutes.