NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2023-2024

EE6225 – MULTIVARIABLE CONTROL SYSTEMS ANALYSIS & DESIGN

April / May 2024 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 5 pages.
2. Answer all 4 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.

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1. (a) In Figure 1, consider the first-order process with time delay:

$$G(s) = \frac{K}{\tau s + 1} e^{-\theta s}.$$

We shall use the model

$$\tilde{G}(s) = \frac{K}{(\tau + \theta)s + 1}$$

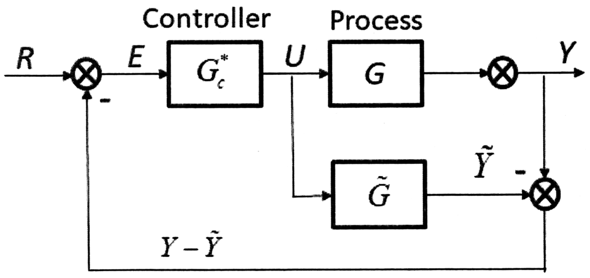

Figure 1.

Note: Question No. 1 continues on page 2.

(i) Factor $\tilde{G}(s)$ into $\tilde{G}(s) = \tilde{G}_+(s)\tilde{G}_-(s)$ where $\tilde{G}_+(s)$ contains all right half plane zeros and $\tilde{G}_+(s=0) = 1$. Find the IMC controller $G_c^* = \frac{1}{\tilde{G}_-}f$ where $f = \frac{1}{\tau_c s+1}$ is a low pass filter.

(5 Marks)

(ii) Find the equivalent standard feedback controller $G_c(s) = \frac{G_c^*(s)}{1-G_c^*(s)\tilde{G}(s)}$.

(5 Marks)

(iii) Re-write $G_c(s)$ in part a(ii) in the PID form shown below and determine the corresponding PID parameters

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_I s} + s\tau_D\right).$$

(5 Marks)

(b) Consider the following 2×2 transfer function matrix (TFM)

$$H(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 4 \\ -2 & -1 \end{bmatrix}.$$

(i) Compute the poles and zeros of $H(s)$.

(5 Marks)

(ii) A pre-compensator/decoupler, $W(s)$, is to be designed such that

$$H(S)W(s) = \begin{bmatrix} \frac{c-s}{(c+s)^2} & 0 \\ 0 & \frac{c-s}{(c+s)^2} \end{bmatrix}.$$

What is an appropriate value for the variable c? Use this value to determine $W(s)$.

(5 Marks)

2. (a) Consider the following 2×2 Multiple Input Multiple Output (MIMO) system

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}.$$

(i) Derive the relative gain array (RGA), $\Lambda(s) = G(s) \times (G^{-1}(s))^T$.

(5 Marks)

(ii) Determine $\Lambda(s=0)$, given that

$$G_{11}(s) = \frac{1}{s^2+3s+2}, \quad G_{12}(s) = \frac{2}{s+1},$$

$$G_{21}(s) = \frac{-0.5}{s^2+2s+1}, \quad G_{22}(s) = \frac{3}{s^2+5s+6}.$$

What is the best pairing of the inputs and outputs to minimize loop interactions?

(5 Marks)

Note: Question No. 2 continues on page 3.

(b) Consider the plant $G(s)$ with controller $K(s)$ as shown in Figure 2.

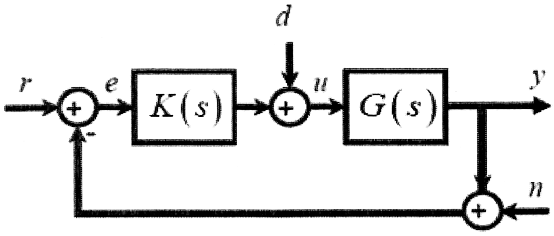

Figure 2.

(i) Show that the output $y(s)$ is given by

$$y(s) = [I + G(s)K(s)]^{-1}G(s)[K(s)r(s) - K(s)n(s) + d(s)]$$

What are the desired frequency characteristics of the output complementary sensitivity function

$$[I + G(s)K(s)]^{-1}G(s)K(s)?$$

(5 Marks)

(ii) Assume that the plant $G(s)$ and controller $K(s)$ are given as follows:

$$G(s) = \frac{4(s+1)}{s+3}, \quad K(s) = \frac{s+5}{2(s+8)}.$$

Compute $[I + G(s)K(s)]^{-1}G(s)K(s)$ and determine whether the closed loop system is robustly stable to unstructured output multiplicative uncertainty of the form $G_{\text{true}} = (1 + \Delta(s))G(s)$ where $\Delta(s)$ is a stable transfer function and $\|\Delta(s)\|_\infty \leq 1$.

(5 Marks)

(iii) Consider the discrete time system $G(z)$ as shown in Figure 3.

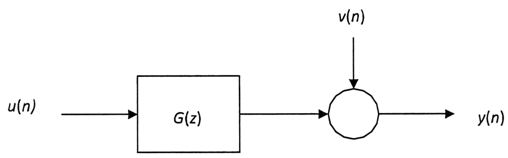

Figure 3.

Assume that the input $u(n)$ is provided for $n = 0, 1, \ldots, N-1$ and the output $y(n)$, corrupted by noise $v(n)$, is observed for the same interval, can be expressed as

$$y(n) = \sum_{k=1}^{m} h(k)u(n-k) + v(n), \quad n = 0, 1, \ldots, N-1.$$

Derive a linear regression method to estimate the impulse response $h(k)$, $k = 1, \ldots, m$ for $G(z)$. What is the condition to ensure that the solution exists?

(5 Marks)

3. (a) Let k_p and k_i be the P and I gains of the continuous PI control law

$$u(t) = k_p e(t) + k_i \int_0^t e(t)\, dt$$

where $e(t) = w(t) - y(t)$ is the error between the reference $w(t)$ and plant output $y(t)$. Assume a unit time sampling interval and denote u_k and e_k as the values of $u(t)$ and $e(t)$ at sampling instance k. Show that the discrete version of the above PI control law can be represented as

$$\Delta u_k = k_p(e_k - e_{k-1}) + k_i e_k$$

where Δ denotes the difference operator, i.e., $\Delta(\cdot)_k = (\cdot)_k - (\cdot)_{k-1}$.
(Hint: use the approximation $\int_0^t e(t)\, dt \approx \sum_{i=0}^k e_i$.)

(5 Marks)

(b) Consider a single-input single-output discrete-time model with parameters a, b:

$$y_k = a y_{k-1} + b u_{k-1}.$$

where y_k and u_k are the output and input of the model at sampling instance k respectively.

(i) Convert the model into an equivalent state space model of the form

$$x_{k+1} = A x_k + B \Delta u_k, \quad y_k = C x_k$$

where $x_k = \begin{bmatrix} y_k & \Delta y_k \end{bmatrix}^T$. State clearly the matrices A, B, and C.

(5 Marks)

(ii) With w as a constant reference, derive the Model Predictive Control (MPC) law which minimises the cost function

$$J = \sum_{j=1}^{2} (w - y_{k+j})^2 + \Delta u_k^2 \text{ with } \Delta u_{k+j} = 0, j = 1, 2, \ldots.$$

Write the MPC law in the form $\Delta u_k = K_1 w + K_2 x_k$. Show that the discrete-time PI controller in part (a) can be written in the same form as the MPC law. Express the equivalent P and I gains of your MPC law in terms of the model parameters a and b.

(10 Marks)

(iii) Generalise the MPC law of part b(ii) to include **future** references, i.e., the MPC cost function is now

$$J = \sum_{j=1}^{2} (w_{k+j} - y_{k+j})^2 + \Delta u_k^2 \text{ with } \Delta u_{k+j} = 0, j = 1, 2, \ldots.$$

Comment on the structure of the new MPC law. Does it give zero offset at the steady state? Justify your answer.

(5 Marks)

4. (a) MPC solves online Quadratic Programs (QPs) whose standard form is

$$\min_{\theta} \frac{1}{2}\theta^T H\theta + f^T\theta, \text{subject to } \Omega\theta \leq \omega.$$

Such QPs can be solved by calling the function `qp(H,f,Ω,ω)`. Derive the required H, f, Ω, and ω for the MPC problem formulation. For your derivation, use the single-input/single-output model

$$x_{k+1} = Ax_k + B\Delta u_k, \quad y_k = Cx_k.$$

The MPC cost function is

$$J = \sum_{j=1}^{3}(w - y_{k+j})^2 + \sum_{j=0}^{2}\Delta u_{k+j}^2$$

and the constraints are

$$-4 \leq \Delta u_{k+i} \leq 5, \quad i = 0, 1, 2; \quad -6 \leq y_{k+i} \leq 7, \quad i = 1, 2, 3.$$

(15 Marks)

(b) (i) Given a two-output system

$$x_{k+1} = Ax_k, \quad y_{1,k} = c_1 x_k, \quad y_{2,k} = c_2 x_k$$

The output y_1 is available at every sampling instance while y_2 is only available every alternate sampling instance. In other words, at time k, one would have

$$y_{1,k},\ y_{1,k-1},\ y_{1,k-2}, \cdots, \quad y_{2,k},\ y_{2,k-2},\ y_{2,k-4}, \cdots$$

Derive a Moving Horizon Estimator (MHE) to estimate x_k using the current and past measurements over a horizon of $2N$, i.e., $y_{1,k-j},\ y_{2,k-j}, j = 0, 1, \ldots, 2N$.

(5 Marks)

(ii) Repeat part b(i) with the model

$$x_{k+1} = Ax_k + Bu_k, \quad y_{1,k} = c_1 x_k, \quad y_{2,k} = c_2 x_k$$

Again, the output y_1 is available at every sampling instance while y_2 is only available every alternate sampling instance.

(5 Marks)

END OF PAPER

EE6225 MULTIVARIABLE CONTROL SYSTEMS ANALYSIS & DESIGN

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.