Chapter 6: Controllability and Observability

- Consider the linear time-varying system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \quad (1)$$

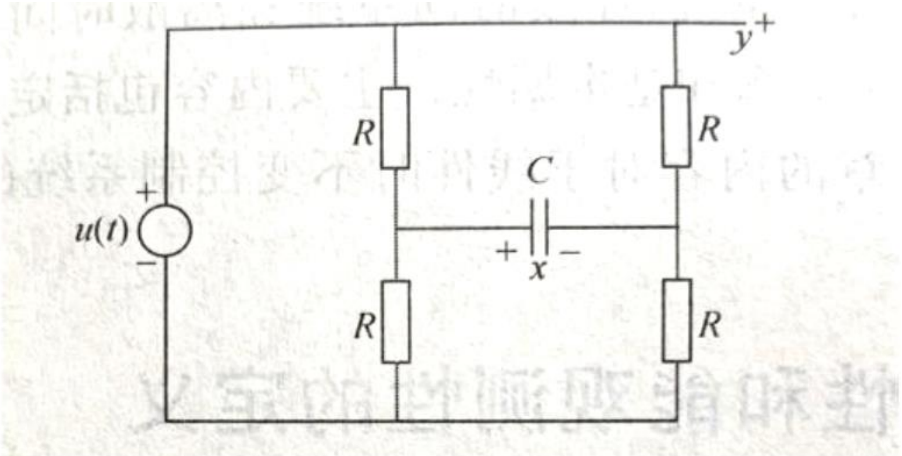
$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \quad (2)$$

  where $\mathbf{x} \in \mathcal{R}^n$ is the state vector, $\mathbf{u} \in \mathcal{R}^m$ is the input vector and $\mathbf{y} \in \mathcal{R}^p$ is the output vector.

- The controllability is concerned with the influence of input on the state whereas the observability examines the influence of state on the output.

- Since term $D(t)\mathbf{u}(t)$ does not affect the above link, it can be excluded when discussing controllability and observability.

Consider the following example:



It is clear that $x = 0$ *no matter what is the input. So, the state is not controllable. On the other hand, if* $u(t) = 0$*, then* $y(t) = 0$ *regardless of the initial charge of the capacitor. That is,* x *cannot be reflected from* y*. So, the system is not observable.*

Controllability

- The system (1)-(2) is called controllable on $[t_0, t_f]$ if given any initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and any final state $\mathbf{x}(t_f) = \mathbf{x}_f$, there exists a continuous input signal $\mathbf{u}(t)$ that transfers $\mathbf{x}_0$ to $\mathbf{x}_f$ in a finite time.
- Note that there is no requirement for the system to stay at the origin after t_f.

Theorem 6.1 *The linear system (1)-(2) is controllable on $[t_0, t_f]$ if and only if the $n \times n$ matrix*

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, t) B(t) B^T(t) \Phi(t_f, t) dt \qquad (3)$$

is invertible. (Observe that $W(t_0, t_f)$ is non-negative definite and is called controllability Gramian).

Proof. '$\Longleftarrow$': Suppose $W(t_0, t_f)$ is invertible. Choose

$$\mathbf{u}(t) = -B^T(t)\Phi^T(t_f, t)W^{-1}(t_0, t_f)[\Phi(t_f, t_0)\mathbf{x}_0 - \mathbf{x}_f], \quad t \in [t_0, t_f] \tag{4}$$

Then,

$$\begin{aligned}
\mathbf{x}(t_f) &= \Phi(t_f, t_0)\mathbf{x}_0 + \int_{t_0}^{t_f} \Phi(t_f, \sigma)B(\sigma)\mathbf{u}(\sigma)d\sigma \\
&= \Phi(t_f, t_0)\mathbf{x}_0 - \int_{t_0}^{t_f} \Phi(t_f, \sigma)B(\sigma)B^T(\sigma)\Phi^T(t_f, \sigma)W^{-1}(t_0, t_f) \\
&\qquad \times [\Phi(t_f, t_0)\mathbf{x}_0 - \mathbf{x}_f]d\sigma \\
&= \Phi(t_f, t_0)\mathbf{x}_0 - W(t_0, t_f)W^{-1}(t_0, t_f)[\Phi(t_f, t_0)\mathbf{x}_0 - \mathbf{x}_f] \\
&= \mathbf{x}_f
\end{aligned}$$

That is, the system is controllable on $[t_0, t_f]$.

'$\Longrightarrow$': By contradiction, suppose the system (1)-(2) is controllable but $W(t_0, t_f)$ is not invertible. Then, there exists a non-zero $\mathbf{x}_f$ such that

$$0 = \mathbf{x}_f^T W(t_0, t_f)\mathbf{x}_f = \int_{t_0}^{t_f} \mathbf{x}_f^T \Phi(t_f, t) B(t) B^T(t) \Phi^T(t_f, t)\mathbf{x}_f dt \qquad (5)$$

which implies

$$\mathbf{x}_f^T \Phi(t_f, t) B(t) = 0, \quad t \in [t_0, t_f] \qquad (6)$$

On the other hand, since the system is controllable on $[t_0, t_f]$, choosing $\mathbf{x}_0 = 0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$, there exists an input $u(t)$ such that

$$\mathbf{x}_f = \int_{t_0}^{t_f} \Phi(t_f, \sigma) B(\sigma) u(\sigma) d\sigma$$

Multiplying on the left by $\mathbf{x}_f^T$ and applying (6), we obtain

$$\mathbf{x}_f^T \mathbf{x}_f = \int_{t_0}^{t_f} \mathbf{x}_f^T \Phi(t_0, \sigma) B(\sigma) u(\sigma) d\sigma = 0$$

which contradicts $\mathbf{x}_f \neq 0$.

Linear Time-invariant Case:

$$A(t) = A, \;\; B(t) = B, \;\; C(t) = C, \;\; D(t) = D$$

Theorem 6.2 *The following are equivalent:*
(i) The system or pair (A, B) *is controllable;*
(ii) The matrix

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

is positive definite for any $t > 0$*;*
(iii) The controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has full row rank n*;*
(iv) The matrix $[A - \lambda I, \;\; B]$ *has full row rank (rank* n*) for all* λ *in complex domain or rank[*$\lambda_i I - A, B$*]=*n*,* $i = 1, 2, \cdots, n$*, where* λ_i *is an eigenvalue of* A*.*
(v) Let λ *and* $\mathbf{x}$ *be any eigenvalue and any corresponding left eigenvector of* A*, i.e.* $\mathbf{x}^* A = \mathbf{x}^* \lambda$*, then* $\mathbf{x}^* B \neq 0$*; (iv)-(v) Popov-Belevitch-Hautus (PHB) tests*
(vi) The eigenvalues of $A + BF$ *can be freely assigned by a suitable choice of* F*.*

Proof. (i) $\Longleftrightarrow$ (ii) is similar to the time-varying case.

(ii) $\Longleftrightarrow$ (iii): Suppose $W_c(t) > 0$ but the matrix $\mathcal{C}$ does not have full row rank. Then, there exists a $\mathbf{v} \in \mathcal{R}^n$ such that

$$\mathbf{v}^* A^i B = 0, \quad i = 0, 1, \cdots, n-1$$

In fact, the above equality holds for all $i \geq 0$ by the Cayley-Hamilton Theorem. Hence,

$$\mathbf{v}^* e^{At} B = 0$$

for any t, then $\mathbf{v}^* W_c(t) = 0$, which is a contradiction as $W_c(t) > 0$. Conversely, suppose $\mathcal{C}$ has full row rank but $W_c(t)$ is singular for some t_1. Then, there exists a non-zero vector $\mathbf{v}$ such that $\mathbf{v}^* e^{At} B = 0$ for all $t \in [0, t_1]$. By setting $t = 0$, we know $\mathbf{v}^* B = 0$. Further, by evaluating the i-th derivative of $\mathbf{v}^* e^{At} B = 0$ at $t = 0$ gives

$$\mathbf{v}^* A^i B = 0, \;\; i > 0$$

Hence,

$$\mathbf{v}^* [B \;\; AB \;\; A^2 B \;\cdots\; A^{n-1} B] = 0$$

which contradicts with the full rank of $\mathcal{C}$.

(iii) $\Longrightarrow$ (iv): Suppose, on the contrary, that the matrix

$$[A - \lambda I \quad B]$$

does not have full row rank for some λ. Then, there exists a vector $\mathbf{x}$ such that

$$\mathbf{x}^*[A - \lambda I \quad B] = 0$$

i.e.

$$\mathbf{x}^* A = \lambda \mathbf{x}^*, \quad \mathbf{x}^* B = 0$$

This results in

$$\mathbf{x}^*[B \;\; AB \;\; \cdots \;\; A^{n-1}B] = [\mathbf{x}^* B \;\; \lambda \mathbf{x}^* B \;\; \cdots \;\; \lambda^{n-1}\mathbf{x}^* B] = 0$$

which contradicts with the fact that $\mathcal{C}$ has full row rank.

(iv) $\Longrightarrow$ (v): It is obvious from the above proof.

The rest of the proof can be known later.

Remark 6.1 *For time-invariant systems, the controllability matrix $\mathcal{C}$ is independent of t_0 and t_f. Thus, the controllability of the system is not tied up to a particular time interval.*

Remark 6.2 *If A is asymptotically stable and (A, B) is controllable, then the unique solution of*

$$AW_c + W_cA^T = -BB^T$$

is positive definite and the solution is the controllability Gramian*:*

$$W_c = \int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau$$

Example 6.1

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Then, the controllability matrix

$$\mathcal{C} = [B \;\; AB \;\; A^2B] = \left[\begin{array}{cc|cc|cc} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & -4 & -7 \\ 1 & 1 & -4 & -7 & 13 & 25 \end{array}\right]$$

It is easy to check that $rank(\mathcal{C}) = 3$. *Hence, the system is controllable.*

Example 6.2

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The controllability matrix

$$\mathcal{C} = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \lambda_2 + \lambda_2 \\ 1 & \lambda_2 & \lambda_2^2 \end{bmatrix}$$

It is easy to see that $rank(\mathcal{C}) = 2$*. The system is not controllable.*

Alternatively,

$$[A - \lambda_1 I, \quad B] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 & 1 \end{bmatrix}$$

does not have full row rank. So, the system is not controllable. In fact, $\dot{x}_1 = \lambda_1 x_1$ *is not influenced by the control input.*

Example 6.3

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

where $a_1 \neq a_2$.

In the case of $b_1(t) = b_1$ *and* $b_2(t) = b_2$*, the system is controllable if and only if* $b_1 \neq 0$ *and* $b_2 \neq 0$.

However, for time-varying case

$$b_1(t) = e^{a_1 t}, \;\; b_2(t) = e^{a_2 t}$$

we have

$$W(t_0, t_f) = (t_f - t_0) \begin{bmatrix} e^{2a_1 t_0} & e^{(a_1+a_2)t_0} \\ e^{(a_1+a_2)t_0} & e^{2a_2 t_0} \end{bmatrix}$$

Since $\det W(t_0, t_f) = 0$*, the system is not controllable.*

Hence, point-wise interpretation of controllability property can be misleading.

Example 6.4

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

If $x_1(0) = 10$ *and* $x_2(0) = -1$, *can we apply a control input to bring the system state to* 0 *in 2 seconds?*

Note that

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix} \implies rank(\mathcal{C}) = 2$$

Now,

$$W_c(2) = \int_0^2 \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} [0.5 \;\; 1] \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau = \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$$\begin{aligned} u(t) &= -[0.5 \;\; 1] \begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} W_c^{-1}(2) \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix} \\ &= -58.82e^{0.5t} + 27.96e^{t} \end{aligned}$$

Remark 6.3 *If you allow a longer time duration, say 4 seconds, to transfer the state from* $[10 \quad -1]^T$ *to 0, the control signal would be smaller.*

Example 6.5 *Controllability of CCF form*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

has full row rank. So, the system in CCF form is controllable.

Controllability Indices

- Let $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times p}$ with B full column rank. Note that if B is not of full column rank, there exist redundant inputs.
- Denote $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$. Then, the controllability matrix:

$$\mathcal{C} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p \vdots A\mathbf{b}_1 \ \cdots \ A\mathbf{b}_p \vdots \ \cdots \ \vdots A^{n-1}\mathbf{b}_1 \ \cdots \ A^{n-1}\mathbf{b}_p]$$

- To search linearly independent columns of $\mathcal{C}$ from left to right, note that if $A^i\mathbf{b}_m$ is linearly dependent on its left hand side columns, so is $A^{i+k}\mathbf{b}_m$ $(k > 0)$.
- Let $\mu_m,\ m = 1, 2, \cdots, p$ be the number of linearly independent columns in $\mathcal{C}$ associated with $\mathbf{b}_m$. Then, if (A, B) is controllable,

$$\mu_1 + \mu_2 + \cdots + \mu_p = n$$

- The set $\{\mu_1, \mu_2, \cdots, \mu_p\}$ is called *the controllability indices* and

$$\mu = \max\{\mu_1, \mu_2, \cdots, \mu_p\}$$

  is the *controllability index* of (A, B).

- μ is the least integer such that

$$rank(\mathcal{C}_1) = rank[B\ AB\ \cdots\ A^{\mu-1}B] = n$$

- It is clear that

$$n/p \leq \mu \leq n - p + 1$$

Corollary 6.1 *Let B be of full column rank. (A, B) is controllable if and only if*

$$rank[B\ AB\ \cdots\ A^{n-p}B] = n$$

Example 6.6

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $rank(B) = 2 = p$ *and* $n - p = 2$,

$$[B\ AB\ A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

is of rank 4. *The system is controllable.*

Theorem 6.3 *The controllability property is invariant under any equivalence transformation.*

Proof Consider (A, B) and its equivalent pair $(\bar{A}, \bar{B})$, where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$. Note that $\bar{A}^k = TA^kT^{-1}$ and $\bar{A}^k\bar{B} = TA^kB$. Then, the controllability matrix of $(\bar{A}, \bar{B})$ is

$$\begin{aligned}
\bar{\mathcal{C}} &= [\bar{B}\ \bar{A}\bar{B}\ \cdots\ \bar{A}^{n-1}\bar{B}] \\
&= [TB\ TAB\ \cdots\ TA^{n-1}B] \\
&= T[B\ AB\ \cdots\ A^{n-1}B] \\
&= T\mathcal{C}
\end{aligned}$$

Since T is non-singular,

$$rank(\bar{\mathcal{C}}) = rank(\mathcal{C})$$

The result follows.

Theorem 6.4 *The set of the controllability indices of (A, B) is invariant under any equivalence transformation and any reordering of the columns of B.*

Proof Define

$$\mathcal{C}_k = [B\ AB\ \cdots\ A^{k-1}B]$$

Then, $rank(\mathcal{C}_k) = rank(\bar{\mathcal{C}}_k)$. Hence, the controllability indices is invariant under any equivalence transformation.
Next, the reordering of the columns of B can be expressed as

$$\hat{B} = BM$$

where $M \in \mathcal{R}^{p\times p}$ is an elementary matrix which is non-singular. Then,

$$\hat{\mathcal{C}}_k = [\hat{B}\ A\hat{B}\ \cdots\ A^{k-1}\hat{B}] = \mathcal{C}_k diag\{M, M, M, M\}$$

Since M is non-singular,

$$rank(\hat{\mathcal{C}}_k) = rank(\mathcal{C}_k)$$

That is, the reordering of the columns of B does not change the controllability indices

Observability

- The observability studies the effect of the state on the output. Hence, WLOG, we consider the system (1)-(2) with zero-input, i.e.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 && (7) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) && (8) \end{aligned}$$

- **Definition.** The above system is called *observable* on $[t_0, t_f]$ if any initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is uniquely determined by the output $\mathbf{y}(t)$ for $t \in [t_0, t_f]$.

Theorem 6.5 *The above system is observable on* $[t_0, t_f]$ *if and only if the* $n \times n$ *matrix*

$$M(t_0, t_f) = \int_{t_0}^{t_f} \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) dt \tag{9}$$

is invertible.

Proof. Note the solution

$$\mathbf{y}(t) = C(t)\Phi(t, t_0)\mathbf{x}_0$$

Multiplying the above from the left by $\Phi^T(t, t_0)C^T(t)$ and integrating yields

$$\int_{t_0}^{t_f} \Phi^T(t, t_0)C^T(t)\mathbf{y}(t)dt = M(t_0, t_f)\mathbf{x}_0 \tag{10}$$

It is clear from (10) that if $M(t_0, t_f)$ is invertible, $\mathbf{x}_0$ is uniquely determined. On the other hand, if $M(t_0, t_f)$ is not invertible, then there exists a non-zero vector $\mathbf{x}_a$ such that $M(t_0, t_f)\mathbf{x}_a = 0$, implying $\mathbf{x}_a^T M(t_0, t_f)\mathbf{x}_a = 0$. Using (10), we know

$$\int_{t_0}^{t_f} \mathbf{x}_a^T \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)\mathbf{x}_a dt = 0$$

which implies

$$C(t)\Phi(t, t_0)\mathbf{x}_a = 0, \quad t \in [t_0, t_f]$$

Thus, $\mathbf{x}(t_0) = \mathbf{x}_0 + \mathbf{x}_a$ gives the same zero input response on $[t_0, t_f]$ as $\mathbf{x}_0$. Hence, the system is not observable.

Remark 6.4 *The matrix* $M(t_0, t_f)$ *is called* observable Gramian. *It is always symmetric and positive semidefinite and it is positive definite if and only if the system is observable.*

Observability has important applications in the design of observer and estimator. That system is observable allows us to design an observer/estimator to estimate the state $\mathbf{x}(t)$ based on the output $\mathbf{y}(t)$.

Linear Time-invariant Systems

$$A(t) = A, \quad C(t) = C$$

Theorem 6.6 *The following are equivalent:*
(i) The system or pair (C, A) *is observable;*
(ii) The matrix

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is positive definite for any $t > 0$*;*
(iii) The observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank n*;*
(iv) The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full column rank (rank* n*) for all* λ *in complex domain;*

(v) Let λ *and* $\mathbf{y}$ *be any eigenvalue and any corresponding right eigenvector of* A*, i.e.* $A\mathbf{y} = \lambda\mathbf{y}$*, then* $C\mathbf{y} \neq 0$*;*
(vi) The eigenvalues of $A + LC$ *can be freely assigned by a suitable choice of* F*.*
(vi) (A^T, C^T) *is controllable.*
Proof. (i) $\Longleftarrow$ (iii): Under zero input,

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(0), \quad t \in [0, t_f]$$

from which we have

$$\begin{bmatrix} \mathbf{y}(0) \\ \dot{\mathbf{y}}(0) \\ \vdots \\ \mathbf{y}^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \mathbf{x}(0)$$

Since $\mathcal{O}$ is of full column rank, $\mathbf{x}(0)$ can be uniquely determined. Hence, the system is observable.

(iii) $\Longleftarrow$ (i): By contradiction, assume that (C, A) is observable but $\mathcal{O}$ does not have full column rank. Then, there exists a vector $\mathbf{x}_0$ such that $\mathcal{O}\mathbf{x}_0 = 0$ or $CA^i\mathbf{x}_0 = 0, \forall i \geq 0$ (by Cayley-Hamilton Theorem). Choose $\mathbf{x}(t_0) = \mathbf{x}_0$. Then

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(t_0) = 0$$

This implies that the system is not observable as $\mathbf{x}(t_0)$ cannot be determined from $\mathbf{y}(t) = 0$.

The rest of the proof follows by the duality.

Remark 6.5 *If A is asymptotically stable and (C, A) is observable, then the unique solution of*

$$W_oA + A^TW_o = -C^TC$$

is positive definite and the solution is the observability Gramian*:*

$$W_o = \int_0^\infty e^{A^T\tau}C^TCe^{A\tau}d\tau$$

Example 6.7 *Observability of OCF form*

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

Observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2^2 - a_1 & -a_2 & 1 \end{bmatrix}, \quad rank(\mathcal{O}) = 3$$

So, OCF form is always observable.

Example 6.8 *Consider the system*

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 1]$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}, \quad rank(\mathcal{O}) = 2$$

The system is not observable.

In fact, by similarity transformation $\bar{x} = Tx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x$*, we obtain*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \bar{C} = CT = [1 \quad 1 \quad 0]$$

It can be seen that $\bar{x}_3$ *is not observable.*

Example 6.9 *Verify if the system below is observable.*

$$\dot{x} = \begin{bmatrix} -1 & -4 & -2 \\ 0 & 6 & -1 \\ 1 & 7 & -1 \end{bmatrix} x, \quad n = 3$$

$$y = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$$

The observability matrix is

$$rank(\mathcal{O}) = rank \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 19 & -3 \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} = 3 = n$$

The system is observable.

Example 6.10 *Determine if the following system is observable using PBH test.*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x, \quad n = 4$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} x$$

Note that

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -5 & \lambda \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of A: $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = \sqrt{5},\ \lambda_4 = -\sqrt{5}.$

$$rank\begin{bmatrix}\lambda_{1,2}I - A\\ C\end{bmatrix} = rank\begin{bmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -5 & 0\\ 0 & 1 & 0 & -2\\ 1 & 0 & 1 & 0\end{bmatrix} = 4 = n$$

$$rank\begin{bmatrix}\lambda_{3}I - A\\ C\end{bmatrix} = rank\begin{bmatrix}\sqrt{5} & -1 & 0 & 0\\ 0 & \sqrt{5} & 1 & 0\\ 0 & 0 & \sqrt{5} & -1\\ 0 & 0 & -5 & \sqrt{5}\\ 0 & 1 & 0 & -2\\ 1 & 0 & 1 & 0\end{bmatrix} = 4 = n$$

$$rank\begin{bmatrix}\lambda_{3}I - A\\ C\end{bmatrix} = rank\begin{bmatrix}-\sqrt{5} & -1 & 0 & 0\\ 0 & -\sqrt{5} & 1 & 0\\ 0 & 0 & -\sqrt{5} & -1\\ 0 & 0 & -5 & -\sqrt{5}\\ 0 & 1 & 0 & -2\\ 1 & 0 & 1 & 0\end{bmatrix} = 4 = n$$

So, the system is observable.

Observability Indices

- $C \in \mathcal{R}^{q \times n}$ with C full row rank.
- Let $\mathbf{c}_i$ be the ith row of C. To search linearly independent rows of $\mathcal{O}$ from top to bottom, note that if $\mathbf{c}_m A^i$ is linearly dependent on its rows above, so is $\mathbf{c}_m A^{i+k}$ $(k > 0)$.
- Let $\nu_m,\ m = 1, 2, \cdots, q$ be the number of linearly independent rows in $\mathcal{O}$ associated with $\mathbf{c}_m$. Then, if (A, C) is observable,

$$\nu_1 + \nu_2 + \cdots + \nu_q = n$$

- The set $\{\nu_1, \nu_2, \cdots, \nu_q\}$ is called *the observability indices* and

$$\nu = \max\{\nu_1, \nu_2, \cdots, \nu_q\}$$

is the *observability index* of (A, C).

- ν is the least integer such that

$$rank(\mathcal{O}_1) = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix} = n$$

- It is clear that

$$n/q \leq \nu \leq n - q + 1$$

Corollary 6.2 *Let C be of full row rank. (A, C) is observable if and only if*

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-q} \end{bmatrix} = n$$

Theorem 6.7 *The observability property is invariant under any equivalence transformation.*

Theorem 6.8 *The set of the observability indices of (A, C) is invariant under any equivalence transformation and any reordering of the rows of C.*

Definition. Let λ be an eigenvalue of A, or equivalently, a mode of the system.

The mode λ is said to be controllable (observable) if for all left (right) eigenvectors associated with λ, i.e. $\mathbf{x}^*A = \lambda\mathbf{x}^*$ ($A\mathbf{x} = \lambda\mathbf{x}$), $\mathbf{x}^*B \neq 0$ ($C\mathbf{x} \neq 0$).

Otherwise, the mode is said to be uncontrollable (unobservable).

Example 6.9

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \quad C = [1 \;\; \beta]$$

If $\alpha = 0$, *the mode* λ_2 *is not controllable; if* $\beta = 0$, *the mode* λ_2 *is not observable.*

Stabilizability (Time-invariant Case)

- **Definition.** The system (1) or the pair (A, B) is said to be stabilizable if there exists a state feedback $\mathbf{u} = F\mathbf{x}$ such that the system is stable, i.e. $A + BF$ is stable.
- Obviously, that a system is controllable implies that it is stabilizable. The converse is not necessarily true. Stabilizability basically implies that the unstable modes of the system are controllable.

Theorem 6.9 *The following are equivalent:*
(i) (A, B) *is stabilizable;*
(ii) The matrix $[A - \lambda I,\ B]$ *has full row rank for all* $Re\lambda \geq 0$*;*
(iii) For all λ *and* $\mathbf{x}$ *such that* $\mathbf{x}^*A = \mathbf{x}^*\lambda$ *and* $Re\lambda \geq 0$, $\mathbf{x}^*B \neq 0$*;*
(iv) There exists a matrix F *such that* $A + BF$ *is Hurwitz.*

Detectability (Time-invariant Case)

- **Definition.** The system (1)-(2) or the pair (C, A) is said to be *detectable* if $A + LC$ is is stable for some L.
- Obviously, that a system is observable implies that it is detectable. The converse is not necessarily true. Detectability basically implies that the unstable modes of the system are observable.

Theorem 6.10 *The following are equivalent:*
(i) (C, A) *is detectable;*
(ii) The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full column rank for all* $Re\lambda \geq 0$*;*
(iii) For all λ *and* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$ *and* $Re\lambda \geq 0$, $C\mathbf{x} \neq 0$*;*
(iv) There exists a matrix L *such that* $A + LC$ *is Hurwitz;*
(v) (A^T, C^T) *is stabilizable.*

Kalman Canonical Decomposition

- For a given system, there are infinite number of state-space realizations. In fact, let a state-space representation of a system be

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \quad (11)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \quad (12)\end{aligned}$$

  A nonsingular transformation $\mathbf{z} = T\mathbf{x}$ will define another state-space realization of the same system:

$$\begin{aligned}\dot{\mathbf{z}}(t) &= TAT^{-1}\mathbf{z}(t) + TB\mathbf{u}(t) \quad (13)\\ \mathbf{y}(t) &= CT^{-1}\mathbf{x}(t) + D\mathbf{u}(t) \quad (14)\end{aligned}$$

- It is easy to check that they have the same transfer function.

$$G(s) = C(sI - A)^{-1}B + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D$$

- Although in many applications, state-space variables will be chosen to have some physical meanings such as position, velocity, voltage, current and etc., there are some state-space realizations which make system analysis and synthesis much easier.

- The controllability and observability matrices of the systems (11)-(12) and (13)-(14) are related by

$$\mathcal{C}_z = T\mathcal{C}, \quad \mathcal{O}_z = \mathcal{O}T^{-1}$$

- The controllability (or stabilizability) and observability (or detectablility) are invariant under similarity transformations.

- **Theorem.** If $rank(\mathcal{C}) = k_1 < n$, then there exists a similarity transformation

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = T\mathbf{x}$$

  such that

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \mathbf{u} \qquad (15)$$

$$\mathbf{y} = [\bar{C}_1 \ \bar{C}_2] \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + D\mathbf{u} \qquad (16)$$

  with $\bar{A}_{11} \in \mathcal{R}^{k_1 \times k_1}$ and $(\bar{A}_{11}, \bar{B}_1)$ controllable. Moreover,

$$G(s) = C(sI - A)^{-1}B + D = \bar{C}_1(sI - \bar{A}_{11})^{-1}\bar{B}_1 + D$$

Proof. Since $rank(\mathcal{C}) = k_1 < n$, there exist k_1 linearly independent columns $q_i,\ i = 1, 2, \cdots, k_1$ in $\mathcal{C}$. Let $q_i,\ i = k_1 + 1, \cdots, n$ be any $n - k_1$ linearly independent vectors such that the matrix

$$Q = [q_1 \ \cdots \ q_{k_1} \ \ q_{k_1+1} \ \cdots \ q_n]$$

is nonsingular. Then, the transformation $T = Q^{-1}$ will be the desired one.

To show this, since $Aq_i,\ i = 1, 2, \cdots, k_1$ is a linear combination of the columns of $\mathcal{C}$ by Cayley-Hamilton Theorem, Aq_i can be written as a linear combination of $q_i,\ i = 1, 2, \cdots, k_1$. Hence,

$$\begin{aligned} AT^{-1} &= [Aq_1 \ \cdots \ Aq_k \ Aq_{k_1+1} \ \cdots \ Aq_n] \\ &= [q_1 \ \cdots \ q_{k_1} \ q_{k_1+1} \ \cdots \ q_n] \left[\begin{array}{cc} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{array} \right] \\ &= T^{-1} \left[\begin{array}{cc} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{array} \right] \end{aligned}$$

Similarly, each column of the matrix B is a linear combination of $q_i,\ i = 1, 2, \cdots, k_1$, hence

$$B = Q \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = T^{-1} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

To show $(\bar{A}_{11}, \bar{B}_1)$ is controllable, note that

$$T\mathcal{C} = \begin{bmatrix} \bar{B}_1 & \bar{A}_{11}\bar{B}_1 & \cdots & \bar{A}_{11}^{k_1-1}\bar{B}_1 & \cdots & \bar{A}_{11}^{n-1}\bar{B}_1 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Again, by Cayley Hamilton theorem, $\bar{A}_{11}^j,\ j \geq k_1$ is a linear combination of $\bar{A}_{11}^i,\ i = 0, 1, \cdots, k_1 - 1$. Hence,

$$rank[\bar{B}_1\ \ \bar{A}_{11}\bar{B}_1\ \cdots\ \bar{A}_{11}^{k_1-1}\bar{B}_1] = rank(\mathcal{C}) = k_1$$

i.e. $(\bar{A}_{11}, \bar{B}_1)$ is controllable.

- The above decomposition clearly gives the controllable and uncontrollable modes of the system. The uncontrollable modes, i.e. the eigenvalues of $\bar{A}_{22}$ will be cancelled with zeros when computing the transfer function $G(s)$.
- The space $\mathbf{z}$ is partitioned into the controllable subspace $\begin{bmatrix} \mathbf{z}_1 \\ 0 \end{bmatrix}$ and uncontrollable subspace $\begin{bmatrix} 0 \\ \mathbf{z}_2 \end{bmatrix}$.
- A numerical reliable way of computing T is the QR factorization. In fact, let $\mathcal{C} = QR$, then $T = Q^{-1}$.
- If the system is stabilizable and $rank(\mathcal{C}) = k_1 < n$, then the decomposed system will have stable $\bar{A}_{22}$.

Example 6.10

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= [1 \;\; 1 \;\; 1] \mathbf{x}
\end{aligned}
$$

Since $rank(B) = 2$ *and*

$$
[B \; AB] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}
$$

is of rank $2 < 3$*, the system is uncontrollable.*

Let $T^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$*, where the first two columns are the two first linearly independent columns of* $[B \;\; AB]$ *and the last column is arbitrarily chosen to make* T *non-singular.*

Then,

$$
\begin{aligned}
\bar{A} &= TAT^{-1} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix} \\
\bar{B} &= TB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix} \\
\bar{C} &= CT^{-1} = [1 \;\; 2 \vdots 1]
\end{aligned}
$$

Note that A_{12} happens to be zero in this example. Thus, the system can be reduced to

$$
\dot{\mathbf{z}}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{z}_1 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}, \;\; \mathbf{y} = [1 \;\; 2]\mathbf{z}_1
$$

Theorem 6.11 *If the observability matrix with* $rank(\mathcal{O}) = k_2 < n$*, then there exists a similarity transformation*

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = T\mathbf{x}$$

such that

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \mathbf{u} \tag{17}$$

$$\mathbf{y} = [\bar{C}_1 \;\; 0] \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + Du \tag{18}$$

with $\bar{A}_{11} \in \mathcal{R}^{k_2 \times k_2}$ *and* $(\bar{C}_1, \bar{A}_{11})$ *observable. Moreover,*

$$G(s) = C(sI - A)^{-1}B + D = \bar{C}_1(sI - \bar{A}_{11})^{-1}\bar{B}_1 + D$$

If the system is detectable and $rank(\mathcal{O}) = k_2 < n$*, then the decomposed system will have stable* $\bar{A}_{22}$*.*

Since $rank(\mathcal{O}) = k_2 < n$, there exist k_2 linearly independent rows $p_1, \cdots, p_{k_2}$ in $\mathcal{O}$. Let $p_i,\ i = k_2 + 1, \cdots, n$ be any $n - k_2$ linearly independent row vectors such that the matrix

$$T = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

is invertible. Then, $\bar{A} = TAT^{-1},\ \bar{B} = TB,\ \bar{C} = CT^{-1}$ are of the form in the theorem.

Example 6.11

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x} \end{aligned}$$

The observability matrix

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 4 & 0 & 4 \end{bmatrix}, \quad rank(\mathcal{O}) = 1 < 3$$

The system is not observable.

Let $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then,

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$\bar{A} = TAT^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \bar{B} = TB = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Kalman Canonical Decomposition

Theorem 6.12 *For a LTI system (11)-(12), there exists a nonsingular transformation* $\mathbf{z} = T\mathbf{x}$ *such that*

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \\ \dot{\mathbf{z}}_3 \\ \dot{\mathbf{z}}_4 \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \\ 0 \end{bmatrix} \mathbf{u} \tag{19}$$

$$\mathbf{y} = [\bar{C}_1 \;\; 0 \;\; \bar{C}_3 \;\; 0] \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} + D\mathbf{u} \tag{20}$$

where $\mathbf{z}_1$ *is both controllable and observable,* $\mathbf{z}_2$ *is controllable but unobservable,* $\mathbf{z}_3$ *is observable but uncontrollable and* $\mathbf{z}_4$ *is both uncontrollable and unobservable. Further,* $G(s) = \bar{C}_1(sI - \bar{A}_{co})^{-1}\bar{B}_1 + D$.

Remark 6.6 *Although the input and output relationship only depends on the subsystem which is both controllable and observable, the internal behaviour including stability depends on all the subsystems.*

Discrete-time Systems

Consider the system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k), \;\; \mathbf{x}(k_0) = \mathbf{x}_0 && (21) \\ \mathbf{y}(k) &= C\mathbf{x}(k) && (22) \end{aligned}$$

where $A, \; B, \; C$ are respectively $n \times n, \; n \times p$ and $q \times n$ real constant matrices.

Definition 6.1 *The discrete-time system (21)-(22) or the pair* (A, B) *is said to be controllable if for any initial state* $\mathbf{x}(0) = \mathbf{x}_0$ *and any final state* $\mathbf{x}(k_f) = \mathbf{x}_f$*, there exists an input sequence of finite length that transfers* $\mathbf{x}_0$ *to* $\mathbf{x}_f$.

Theorem 6.13 *The following are equivalent:*

- *The pair* (A, B) *is controllable.*
- *The matrix*

$$W_{dc}(n-1) = \sum_{m=0}^{n-1} A^m B B^T (A^T)^m$$

  is non-singular.
- *The* $n \times np$ *controllability matrix*

$$\mathcal{C}_d = [B \; AB \; \cdots \; A^{n-1}B]$$

  has rank n *(full row rank).*
- $[A - \lambda I \;\; B]$ *has full row rank at every eigenvalue of* A.
- *If, in addition, all eigenvalues of* A *have magnitudes less than 1, then the unique solution of*

$$W_{dc} - A W_{dc} A^T = B B^T$$

  is positive definite. W_{dc} *is the discrete Controllability Gramian and is given by*

$$W_{dc} = \sum_{m=0}^{\infty} A^m B B^T (A^T)^m$$

Note that the solution at $k = n$,

$$\mathbf{x}(n) = A^n \mathbf{x}(0) + \sum_{m=0}^{n-1} A^{n-1-m} B \mathbf{u}(m)$$

which can be expressed as

$$\mathbf{x}(n) - A^n \mathbf{x}(0) = [B \; AB \; \cdots \; A^{n-1}B] \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

It is well known that there exists a solution $\{\mathbf{u}(0), \mathbf{u}(1), \cdots, \mathbf{u}(n-1)\}$ to the above linear system of equations if and only if the controllability matrix is of full row rank (n).

Remark 6.7 *A linear time-invariant discrete-time system might fail to be reachable simply because the time-interval* $[0, k_f]$ *is too short - This does not happen for continuous LTI systems. A SISO n-dimensional system may require* n *steps to reach a specified state.*

Observability

Definition 6.2 *The discrete-time system (21)-(22) or the pair (A, C) is said to be observable if for any unknown initial state $\mathbf{x}(0) = \mathbf{x}_0$, there exists a finite integer k_f such that the knowledge of the input sequence $\mathbf{u}(k)$ and the output sequence $\mathbf{y}(k)$ from $k = 0$ to $k = k_f$ suffices to determine uniquely the initial state $\mathbf{x}(0)$.*

Duality The pair (A, C) is observable if and only if (A^T, C^T) is controllable.

Theorem 6.14 *The following are equivalent:*

- *The pair (A, C) is observable.*
- *The matrix*

$$W_{do}(n-1) = \sum_{m=0}^{n-1} (A^T)^m C^T C A^m$$

is non-singular.

- The $nq \times n$ observability matrix

$$\mathcal{O}_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

  has rank n (full column rank).

- The $(n+q) \times n$ matrix

$$\begin{bmatrix} A - \lambda I \\ B \end{bmatrix}$$

  has full column rank at every eigenvalue of A.

- If, in addition, all eigenvalues of A have magnitudes less than 1, then the unique solution of

$$W_{do} - A^T W_{do} A = C^T C$$

  is positive definite. W_{do} is the discrete Observability Gramian and is given by

$$W_{do} = \sum_{m=0}^{\infty} (A^T)^m C^T C A^m$$

- In literature, there are three different controllability definitions:

  1 Transfer any state to any other state.

  2 Transfer any state to the origin, called controllability to the origin.

  3 Transfer zero state to any state, called controllability from the origin, more often, *reachability*.

Remark 6.8 • *For continuous-time systems, since* e^{At} *is non-singular, the above definitions are equivalent.*

- *For discrete-time systems, if* A *is non-singular, the three definitions are again equivalent.*

- *However, if* A *is singular, (1) and (3) are equivalent but not (2). For example,*

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(k)$$

  whose controllability matrix has rank 0. But for any $k \geq 3$, $A^k = 0$. *Thus,* $\mathbf{x}(3) = A^3\mathbf{x}(0) = 0$.

Controllability after Sampling

Given a continuous-time system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

and piecewise constant input:

$$\mathbf{u}(k) = \mathbf{u}(kT) = \mathbf{u}(t), \quad kT \leq t \leq (k+1)T$$

the discretized system is

$$\bar{\mathbf{x}}(k+1) = \bar{A}\bar{\mathbf{x}}(k) + \bar{B}\mathbf{u}(k)$$

where

$$\bar{A} = e^{AT}, \;\; \bar{B} = \left(\int_0^T e^{At} dt\right) B := MB$$

Question: What is the relationship of controllability between the original and discretized systems?

Apparently, if the discretized system is controllable, the original continuous-time system must be controllable. The converse is not necessarily true.

Theorem 6.15 *Suppose* (A, B) *is controllable. A sufficient condition for* $(\bar{A}, \bar{B})$ *with sampling period* T *is controllable is that*

$$|Im(\lambda_i - \lambda_j)| \neq 2\pi m/T, \quad m = 1, 2, \cdots$$

whenever $Re(\lambda_i - \lambda_j) = 0$. *For the single-input case, the condition is also necessary.*

Remark 6.9
- *If* A *has only real eigenvalues, then the discretized system with any sampling period* T *is always controllable if the original continuous-time system is.*
- *Suppose* A *has complex conjugate eigenvalues* $\alpha \pm j\beta$. *If* $T \neq m\pi/\beta$, $m = 1, 2, \cdots$, *then the discretized system is controllable. If* $T = m\pi/\beta$, $m = 1, 2, \cdots$, *then the discretized system may be uncontrollable. This is because for* $\lambda_1 = \alpha + j\beta$ *and* $\lambda_2 = \alpha - j\beta$, *the corresponding eigenvalues of* $\bar{A}$ *become a repeated eigenvalue of* $-e^{\alpha T}$ *or* $e^{\alpha T}$ *which causes the discretized system uncontrollable.*

Example 6.12 *Consider the continuous-time system*

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -3 & -7 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\mathbf{u} \\ y &= [0 \;\; 1 \;\; 2]\mathbf{x}\end{aligned}$$

The eigenvalues of A are -1, $-1 \pm j2$. All three have the same real part. Thus, the discretized system is controllable if and only if

$$T \neq \frac{2\pi m}{2} = \pi m \quad \text{and} \quad T \neq \frac{2\pi m}{4} = 0.5\pi m$$

for $m = 1, 2, \cdots$. For example, if setting $T = 0.5\pi$ ($m = 1$). Then, by
Matlab:[ad,bd]=c2d(a,b,pi/2)
we get

$$\bar{A} = \begin{bmatrix} -0.1039 & 0.2079 & 0.5197 \\ -0.1390 & -0.4158 & -0.5197 \\ 0.1039 & 0.2079 & 0.3118 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} -0.1039 \\ 0.1039 \\ 0.1376 \end{bmatrix}$$

It is easy to check that the controllability matrix is of rank 2. Hence, the discretized system is uncontrollable.

Summary

- Concepts of controllability, observability, stabilizability, and detectability
- Determine if a LTI system is controllable and observable by checking the rank of controllability and observability
- Kalman decompsition
- Contollability of sampled systems

Exercises

1. Check the controllability and observability of the system with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = [1 \ \ 0 \ \ 1]$$

2. Show that the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \mathbf{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \mathbf{u}$$

is controllable if and only if the pair (A_{22}, A_{21}) is controllable.

3. Reduce the state equation

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u} \\ y &= [0 \ \ 1 \ \ 1 \ \ 1 \ \ 0 \ \ 1]\mathbf{x} \end{aligned}$$

to a controllable and observable equation.

4. Consider the system

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} \mathbf{u} \\
y &= [c_1 \;\; c_{11} \;\; c_{12} \;\; c_{21} \;\; c_{22}]\mathbf{x}
\end{aligned}$$

The system has one real eigenvalue and two pairs of complex conjugate eigenvalues. It is assumed that they are distinct. Show that the system is controllable if and only if $b_1 \neq 0$; $b_{i1} \neq 0$ or $b_{i2} \neq 0$ for $i = 1, 2$. It is observable if and only if $c_1 \neq 0$; $c_{i1} \neq 0$ or $c_{i2} \neq 0$ for $i = 1, 2$.