Solutions to Exercises

Chapter 1

1. Find a state-space realization for each of the following systems
(i) $y^{(3)} + 2y^{(2)} + 6y^{(1)} + 3y = 5u$
(ii) $y^{(3)} + 8y^{(2)} + 5y^{(1)} + 13y = 4u^{(1)} + 7u$

Sol: (i) The system is strictly proper. One state-space realization can be easily derived as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} x$$

(ii)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -5 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 7 & 4 & 0 \end{bmatrix} x$$

2. Find a state-space realization for each of the following systems
(i) $\hat{g}(s) = \frac{2s^2+18s+40}{s^3+6s^2+11s+6}$
(ii) $\hat{g}(s) = \frac{3(s+5)}{(s+3)^2(s+1)}$

Sol: (i)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 40 & 18 & 2 \end{bmatrix} x$$

(ii) $\hat{g}(s) = \frac{3(s+5)}{(s+3)^2(s+1)} = \frac{3s+15}{s^3+7s^2+15s+9}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -15 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 15 & 3 & 0 \end{bmatrix} x$$

3. Find the transfer function of the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -3 & 5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} u \\
y &= x_2 + 3x_3
\end{aligned}
$$

Sol:

$$
\begin{aligned}
|sI - A| &= \begin{vmatrix} s & -2 & 0 \\ 0 & s & -2 \\ -1 & 3 & s-5 \end{vmatrix} = s^3 - 5s^2 + 6s - 4 \\
adj(sI - A) &= adj \begin{bmatrix} s & -2 & 0 \\ 0 & s & -2 \\ -1 & 3 & s-5 \end{bmatrix} = \begin{bmatrix} s^2 - 5s + 6 & 2s - 10 & 4 \\ 2 & s^2 - 5s & 2s \\ s & -3s + 2 & s^2 \end{bmatrix}
\end{aligned}
$$

Then, the transfer function is given by

$$
\begin{aligned}
\hat{g}(s) &= C(sI - A)^{-1}B = \frac{C adj(sI - A)B}{|sI - A|} \\
&= \frac{\begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} s^2 - 5s + 6 & 2s - 10 & 4 \\ 2 & s^2 - 5s & 2s \\ s & -3s + 2 & s^2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}{s^3 - 5s^2 + 6s - 4} = \frac{18s^2 - 26s + 22}{s^3 - 5s^2 + 6s - 4}
\end{aligned}
$$

Chapter 2

2.1 Consider the equation

$$
\mathbf{x}(n) = A^n \mathbf{x}(0) + A^{n-1} bu(0) + A^{n-2} bu(1) + \cdots + Abu(n-2) + bu(n-1)
$$

where A is an $n \times n$ matrix and b is an $n \times 1$ column vector. Under what conditions on A and b will there exist $u(0), u(1), \cdots, u(n-1)$ to meet the equation for any $\mathbf{x}(n)$ and $\mathbf{x}(0)$?

Sol:

$$\mathbf{x}(n) - A^n \mathbf{x}(0) = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(u-2) \\ \vdots \\ u(0) \end{bmatrix}$$

The above equation has a solution for any $x(0)$ and $x(n)$ if and only if the matrix $\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$ has full row rank.

2.2 Find the Jordan forms of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Note that A_1 is upper triangular and the eigenvalues are 1, 2, and 3 which are distinct. Then, we can easily find their corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

and thus $T = [v_1 \;\; v_2 \;\; v_3]$ and

$$\hat{A}_1 = T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For A_2, it is also upper triangular with eigenvalues 1, 1 and 2. For $\lambda_{1,2} = 1$,

$$rank(A - \lambda_{1,2} I) = rank \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 = 3 - 2$$

So, there are two independent eigenvectors. Solving $(A - \lambda_{1,2} I)v = 0$, we can get

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = 2$,

$$(A - \lambda_3 I)v = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0, \quad \Longrightarrow v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.3 Consider the companion-form matrix:

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Show that the characteristic polynomial is given by

$$\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Show also that if λ_i is an eigenvalue of A, then $[\lambda_i^3 \ \ \lambda_i^2 \ \ \lambda_i \ \ 1]^T$ is an eigenvector of A associated with λ_i.

Sol:

$$\begin{aligned} \Delta(\lambda) &= |\lambda I - A| = \begin{vmatrix} \lambda + \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} \\ &= (\lambda + \alpha_1) \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix} \\ &= \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 \end{aligned}$$

If λ_i is an eigenvalue of A, then $\Delta(\lambda_i) = 0$, i.e.

$$\lambda_i^4 = -\alpha_1\lambda_i^3 - \alpha_2\lambda_i^2 - \alpha_3\lambda_i - \alpha_4$$

Using this equation, it can be easily checked that

$$A\begin{bmatrix}\lambda_i^3\\ \lambda_i^2\\ \lambda_i\\ 1\end{bmatrix} = \begin{bmatrix}\lambda_i^4\\ \lambda_i^3\\ \lambda_i^2\\ \lambda_i\end{bmatrix} = \lambda_i\begin{bmatrix}\lambda_i^3\\ \lambda_i^2\\ \lambda_i\\ 1\end{bmatrix}$$

2.4 Compute A^{100}, where

$$A = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$$

Sol: Let $f(\lambda) = \lambda^{100}$. First, find the eigenvalues of A:

$$\Delta(\lambda) = |sI - A| = \lambda^2 + 2\lambda + 1$$

The eigenvalues are -1 and -1. Let $h(\lambda) = \beta_0 + \beta_1\lambda$. Then,

$$f(-1) = h(-1) \implies (-1)^{100} = \beta_0 - \beta_1$$
$$f'(-1) = h'(-1) \implies 100 \times (-1)^{99} = \beta_1$$

Hence, $\beta_1 = -100,\ \beta_0 = -99$. Then, $h(\lambda) = -99 - 100\lambda$ and

$$A^{100} = h(A) = -99I - 100A = \begin{bmatrix}-199 & -100\\ 100 & 101\end{bmatrix}$$

Chapter 3

1. An oscillation can be generated by

$$\dot{\mathbf{x}} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{x}$$

Show that its solution is

$$\mathbf{x}(t) = \begin{bmatrix}cos(t) & sin(t)\\ -sin(t) & cos(t)\end{bmatrix}\mathbf{x}(0)$$

Sol:

$$(sI - A)^{-1} = \begin{bmatrix}s & -1\\ 1 & s\end{bmatrix}^{-1} = \begin{bmatrix}\frac{s}{s^2+1} & \frac{1}{s^2+1}\\ -\frac{1}{s^2+1} & \frac{s}{s^2+1}\end{bmatrix}$$

Hence,

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix}$$

and

$$x(t) = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix} x(0)$$

2. Use two different methods to find the unit-step response of

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [2 \;\; 3]x \end{aligned}$$

Sol: Method 1 - Laplace transform

$$\begin{aligned} \hat{g}(s) &= [2 \;\; 3] \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [2 \;\; 3] \frac{1}{s^2+2s+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{5}{s^2+2s+2} \end{aligned}$$

Then,

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s) = \frac{5s}{(s+1)^2+1} \frac{1}{s}$$

Hence,

$$y(t) = 5e^{-t} sin(t)$$

Method 2:

$$(sI - A)^{-1} = \begin{bmatrix} \frac{(s+1)+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-2}{(s+1)^2+1} & \frac{s+1-1}{(s+1)^2+1} \end{bmatrix}$$

Then,

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} e^{-t}(sin(t) + cos(t)) & e^{-t} sin(t) \\ -2e^{-t} sin(t) & e^{-t}(cos(t) - sin(t)) \end{bmatrix}$$

For $u(t) = 1, \; t \geq 0$,

$$y(t) \;\; = \;\; [2 \;\; 3] \int_0^t e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 d\tau$$

$$
\begin{aligned}
&= \int_0^t \left(5e^{-(t-\tau)}cos(t-\tau) - 5e^{-(t-\tau)}sin(t-\tau)\right) d\tau \\
&= 5e^{-t}sin(t)
\end{aligned}
$$

3. Discretize the system

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
y &= [2 \quad 3]\mathbf{x}
\end{aligned}
$$

with $T = 1$ and $T = \pi$.

Sol: From question 3, for $T = 1$,

$$
A_d = e^{AT} = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix}
$$

Since A is nonsingular, we can compute

$$
B_d = A^{-1}(A_d - I)B = \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}
$$

and $C_d = C$.

For $T = \pi$, it can be obtained that

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix} x(k) + \begin{bmatrix} 1.5648 \\ -1.0432 \end{bmatrix} u(k) \\
y(k) &= [2 \quad 3]x(k)
\end{aligned}
$$

4. Find a fundamental matrix and the state transition matrix of

$$
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} \mathbf{x}
$$

Sol:

$$
\dot{x}_2 = tx_2 \longrightarrow x_2(t) = x_2(0)e^{0.5t^2}
$$

$$
\dot{x}_1 = x_2 \longrightarrow x_1(t) = \left(\int_0^t e^{0.5\tau^2} d\tau\right) x_2(0) + x_1(0)
$$

Let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then, $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then, $x(t) = \begin{bmatrix} \int_0^t e^{0.5\tau^2} d\tau \\ e^{0.5t^2} \end{bmatrix}$. Thus, a fundamental matrix is

$$X(t) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5t^2} \end{bmatrix}$$

and

$$X^{-1}(t) = \begin{bmatrix} 1 & -e^{-0.5t^2} \int_0^t e^{0.5\tau^2} d\tau \\ 0 & e^{-0.5t^2} \end{bmatrix}$$

The state transition matrix is

$$\Phi(t, t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} 1 & -e^{0.5t^2} \int_{t_0}^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5(t^2 - t_0^2)} \end{bmatrix}$$

Chapter 4

1. Is a system with impulse response $g(t) = 1/(t+1)$ BIBO stable? How about $g(t) = te^{-t}$ for $t \geq 0$.

Sol: $\int_0^\infty |g(t)|dt = \int_0^\infty \frac{1}{t+1} dt = ln(1+t)|_0^\infty = \infty$. Thus the system is not BIBO stable.

For the system with $g(t) = te^{-t}$, we have

$$\hat{g}(s) = \mathcal{L}[g(t)] = \frac{1}{(s+1)^2}$$

It is clear that the system is BIBO stable.

2. Consider a discrete-time system with impulse response sequence:

$$g(k) = k(0.8)^k, \quad k \geq 0$$

Is the system BIBO stable?

Sol: $hatg(z) = \mathcal{Z}[g(k)] = \frac{0.8z}{(z-0.8)^2}$. Since the poles of the system lie inside the unit circle, the system is BIBO stable.

3. Show that all eigenvalues of A have real parts less than $-\mu < 0$ if and only if, for any given positive definite matrix M,

$$A^T Q + QA + 2\mu Q = -M$$

has a unique positive definite solution Q.

Sol. Note that

$$(A + \mu I)^T Q + Q(A + \mu I) = -M$$

for positive definite matrices M and Q. Thus,

$$Re[\lambda_i(A + \mu I)] < 0, \;\; i = 1, 2, \cdots, n$$

Note that

$$\lambda_i(A + \mu I) = \lambda_i(A) + \mu$$

Thus,

$$Re[\lambda_i(A)] < -\mu$$

4. Show that all eigenvalues of A have magnitudes less than ρ if and only if, for any given positive definite matrix M,

$$A^T QA - \rho^2 Q = \rho^2 M$$

has a positive definite solution Q.

Sol:

$$\left(\frac{1}{\rho}A^T\right) Q \left(\frac{1}{\rho}A\right) - Q = M$$

Since $M > 0$ and $Q > 0$,

$$\left|\lambda_i\left(\frac{1}{\rho}A\right)\right| < 1$$

That is,

$$|\lambda_i(A)| < \rho$$

5. Determine if the following system is: (a) asymptotically stable; (b) BIBO stable.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u, \;\; y = \begin{bmatrix} -25 & 5 & 0 \end{bmatrix} x$$

Sol: (a) Note that

$$|sI - A| = s^3 + 5s^2 - 250$$

By Routh criterion, it can be easily known that not all the roots have negative real part. Thus, the system is not asymptotically stable.

(b) Compute the transfer function of the system:

$$\begin{aligned}
\hat{g}(s) &= C(sI-A)^{-1}B = \frac{1}{|sI-A|}Cadj(sI-A)B \\
&= \frac{1}{s^3+5s^2-250}\begin{bmatrix} -25 & 5 & 0 \end{bmatrix} adj \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -250 & 0 & s+5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\
&= \frac{1}{s^3+5s^2-250}\begin{bmatrix} -25 & 5 & 0 \end{bmatrix} \begin{bmatrix} * & * & 1 \\ * & * & s \\ * & * & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\
&= \frac{50(s-5)}{(s^2+10s+50)(s-5)} = \frac{50}{s^2+10s+50}
\end{aligned}$$

Both the poles of $\hat{g}(s)$ have negative real part. So, the system is BIBO stable.

Chapter 5

1. Check the controllability and observability of the system with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = [1 \;\; 0 \;\; 1]$$

Sol: The controllability matrix is

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad rank(\mathcal{C}) = 3$$

The system is controllable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}, \quad rank(\mathcal{O}) = 3$$

The system is observable.

2. Show that the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \mathbf{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \mathbf{u}$$

is controllable only if the pair (A_{22}, A_{21}) is controllable.

Sol: (A, B) is controllable $\Longrightarrow$

$$rank \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} - sI & 0 \end{bmatrix} = n$$

for every s. This implies that the matrix $[A_{21} \ \ A_{22} - sI]$ has full row rank for every s. That is, (A_{22}, A_{21}) is controllable.

3. Reduce the state equation

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u} \\ y &= [0 \ \ 1 \ \ 1 \ \ 1 \ \ 0 \ \ 1]\mathbf{x} \end{aligned}$$

to a controllable and observable equation.

Sol: Rearrange the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \end{bmatrix} \bar{x}$$

Thus the system can be reduced as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \tilde{x}$$

It can be concluded that the above reduced system is controllable. However, further rearrange the system as

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \\ \dot{x}_1 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 1 & 0 & \lambda_1 & 0 \\ 0 & 1 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \\ x_1 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \hat{x}$$

It can be seen that x_1 and x_4 are not observable. Thus, the system can be further reduced to

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$$

which is both controllable and observable.