Chapter 4
State-space Solutions

Existence and Uniqueness of Solution

Consider linear time-varying state equation

$$\dot{x} = A(t)x + B(t)u, \;\; x(t_0) = x_0, \; t \in [t_0, \; t_a] \tag{1}$$

- If all elements of $A(t)$ and $B(t)$ and the input $u(t)$ are real and continuous on $[t_0, \; t_a]$, then solution of state equation (1) exists and is unique. These conditions can be satisfied by many physical systems.
- The above conditions can be relaxed to:

$$(a) \;\; \int_{t_0}^{t_a} |a_{ij}(t)| dt < \infty, \;\; i, j = 1, 2, \cdots, n;$$

$$(b) \;\; \int_{t_0}^{t_a} [b_{ij}(t)]^2 \, dt < \infty, \;\; i, j = 1, 2, \cdots, n, \;\; j = 1, 2, \cdots, p;$$

$$(c) \;\; \int_{t_0}^{t_a} [u_j(t)]^2 \, dt < \infty, \;\; j = 1, 2, \cdots, p$$

- For LTI systems, (a) and (b) are automatically satisfied, so condition (c) suffices to guarantee the existence and uniqueness of solution.

Solution of State Equation:

- LTI system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \ \ \mathbf{x}(t_0) = x_0 \tag{2}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{3}$$

- Premultiplying e^{-At} on both sides of (2),

$$e^{-At}\dot{\mathbf{x}}(t) - e^{-At}A\mathbf{x}(t) = e^{-At}B\mathbf{u}(t)$$

  Applying the fact that $Ae^{At} = e^{At}A$,

$$\frac{d}{dt}(e^{-At}\mathbf{x}(t)) = e^{-At}B\mathbf{u}(t)$$

- Integrating from t_0 to t yields

$$e^{-At}\mathbf{x}(t) - e^{-At_0}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{-A\tau}B\mathbf{u}(\tau)d\tau$$

  or yet

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{4}$$

Observe from (4) that the state response is the sum of the zero input response and zero state response (superposition). The output is then

$$\mathbf{y}(t) = Ce^{A(t-t_0)}\mathbf{x}(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + D\mathbf{u}(t) \tag{5}$$

- The key is to compute the state transition matrix e^{At}.

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

- Method 1: Apply the method of Chapter 3: (1) Compute the eigenvalues of A; (2) Find $h(\lambda)$ of degree $n-1$ such that $h(\lambda_i) = e^{\lambda_i t}$ (or $h^{(i)}(\lambda_i) = (e^{\lambda t})^{(i)}|_{\lambda=\lambda_i}$) where $\lambda_i,\ i = 1, 2, \cdots, n$ are the eigenvalues of A; and (3) $e^{At} = h(A)$.

- Method 2: Use diagonal or Jordan form: (1) Compute T such that $A = T\hat{A}T^{-1}$, where $\hat{A}$ is a Jordan form; (2) $e^{At} = Te^{\hat{A}t}T^{-1}$. Note that $e^{\hat{A}t}$ is computed in Chapter 3.

- Method 3: Use Laplace inverse transform:

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

Example 4.1 *Consider the state equation with $t_0 = 0$:*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Since

$$(sI - A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} (s+2)/(s+1)^2 & -1/(s+1)^2 \\ 1/(s+1)^2 & s/(s+1)^2 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Hence,

$$\mathbf{x}(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)}u(\tau)d\tau \\ \int_0^t \left[1-(t-\tau)\right] e^{-(t-\tau)}u(\tau)d\tau \end{bmatrix}$$

Example 4.2 *Consider the LTI system*

$$\dot{x} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} x + \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} u$$

To find e^{At}*, we first calculate the eigenvalues of* A*:*

$$|\lambda I - A| = (\lambda - 2)(\lambda + 1)(\lambda - 1) = 0$$

Thus,

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$$

Since the eigenvalues are distinct, there are three independent eigenvectors. We now find their corresponding eigenvectors by solving:

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix} = \lambda_i \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix}$$

We have

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Note that $A = T\Sigma T^{-1}$, *where* $T = [v_1 \;\; v_2 \;\; v_3]$, *i.e.*

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$\Sigma = diag\{2, 1, -1\}$$

Thus,

$$\begin{aligned} e^{At} &= Te^{\Sigma t}T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & -e^{2t} + e^{t} & -e^{2t} + e^{t} \\ 0 & e^{-t} & 0 \\ 0 & e^{t} - e^{-t} & e^{t} \end{bmatrix} \end{aligned}$$

The above showed the case where A is diagonalizable. In general, for a given matrix A, you may only be able to transform it to the Jordan form. For example, if there exists a similarity transformation matrix T such that $T^{-1}AT = J$, where

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

Then,

$$e^{At} = Te^{Jt}T^{-1} = T \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2!}t^2 e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1}$$

Let us have a more detailed look at the 3rd order case. For general 3×3 with $\det(\lambda I - A) = (\lambda - \lambda_0)^3$, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_0$, we want to find T so that $T^{-1}AT = J$ or $A = TJT^{-1}$, where J has the following 3 canonical forms:

$$1) \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix}, \quad 2) \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

$$3) \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix} \; or \; \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

The first canonical form happens

- when A has three linearly independent eigenvectors, i.e. $(\lambda_0 I - A)v = 0$ yields $v_1,\ v_2,\ v_3$ that spans $\mathcal{R}^3$.
- mathematically, when $nullity(\lambda_0 I - A) = 3$, namely,

$$rank(\lambda_0 I - A) = 3 - nullity(\lambda_0 I - A) = 0$$

- So, if $rank(\lambda_0 I - A) = 0$, we know there are 3 independent eigenvectors to diagonalize A.

For the case 2), $J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$.

- this happens when $(\lambda_0 I - A)v$ yields only one linearly independent solution, i.e. $nullity(\lambda_0 I - A) = 1$ and $rank(\lambda_0 I - A) = 3 - 1 = 2$
-

$$
\begin{aligned}
A[v_1 \;\; v_2 \;\; v_3] &= [v_1 \;\; v_2 \;\; v_3] \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix} \\
\Longrightarrow [\lambda_0 v_1, v_1 + \lambda_0 v_2, v_2 + \lambda_0 v_3] &= [Av_1, Av_2, Av_3]
\end{aligned}
$$

That is,

$$
\begin{aligned}
(A - \lambda_0 I)v_1 &= 0 \\
(A - \lambda_0 I)v_2 &= v_1, \;\; (find\, the\, generalized\, eigenvector\, v_2) \\
(A - \lambda_0 I)v_3 &= v_2, \;\; (find\, the\, generalized\, eigenvector\, v_3)
\end{aligned}
$$

For Case 3), $J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix}$ or $J = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$

- this happens when $(\lambda_0 I - A)v = 0$ yields two linearly independent eigenvectors, i.e. $nullity(\lambda_0 I - A) = 2$ and $rank(\lambda_0 I - A) = 3 - 2 = 1$
-

$$\begin{aligned} A[v_1 \;\; v_2 \;\; v_3] &= [v_1 \;\; v_2 \;\; v_3] \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix} \\ \Longrightarrow [\lambda_0 v_1, v_1 + \lambda_0 v_2, \lambda_0 v_3] &= [Av_1, Av_2, Av_3] \end{aligned}$$

That is,

$$\begin{aligned} (A - \lambda_0 I)v_1 &= 0 \\ (A - \lambda_0 I)v_2 &= v_1, \quad (find\; the\; generalized\; eigenvector\; v_2) \\ (A - \lambda_0 I)v_3 &= 0 \end{aligned}$$

So, v_1 and v_3 are directly computed eigenvectors while v_2 is the generalized eigenvector.

Example 4.3

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \ \det(\lambda I - A) = \lambda^2 \Longrightarrow \lambda_1 = \lambda_2 = 0$$

- *Two repeated eigenvalues with* $rank(0 \times I - A) = 1$. *Hence, there is only one independent eigenvector:*

$$(A - 0 \times I)v_1 = 0 \Longrightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- *Generalized eigenvector:* $(A - 0 \times I)v_2 = v_1 \Longrightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $T = [v_1 \ \ v_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$

$$\begin{aligned} A &= TJT^{-1} = T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T^{-1}, \\ e^{At} &= Te^{Jt}T^{-1} = T \begin{bmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{bmatrix} T^{-1} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \end{aligned}$$

Modal form

- Suppose A has complex eigenvalues. Since A has only real coefficients, complex eigenvalues must be in the form complex conjugate.
- For example, A has two real and two complex eigenvalues, then,

$$J = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix}$$

which is not useful in practice.

- It can be further transformed to the modal form:

$$\begin{aligned} T_1^{-1} J T_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & j & -j \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix} = \bar{A} \end{aligned}$$

- Matlab: [am,bm,cm,dm,T]=canon(a,b,c,d,'modal')

Let us consider the zero-input response $e^{At}\mathbf{x}(0)$. Consider an example of Jordan block, we have

$$e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2 & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1}$$

It is clear that every entry of e^{At} is a linear combination of $\{e^{\lambda_1 t},\ te^{\lambda_1 t}, t^2 e^{\lambda_1 t},\ e^{\lambda_2 t}\}$.

- If every eigenvalue has negative real part, then any zero-input response will approach 0 as $t \to \infty$.
- If A has an eigenvalue with a positive real part, then most zero input responses will go unbounded as $t \to \infty$.
- If A has some eigenvalues with zero real part and all with index 1 and the remaining eigenvalues all have negative real parts, then zero-input response will not go unbounded. However, if the index is 2 or higher, some zero-input responses may go unbounded, for example, if A has eigenvalue 0 with index 2, e^{At} will contain $\{1, t\}$ and hence some zero-input responses may go unbounded.

State Transition Matrix of LTI system $\dot{x} = Ax + Bu$

Definition 4.1 *The state transition matrix is defined as the $n \times n$ matrix solution $\Phi(t - t_0)$ of the equation*

$$\dot{\Phi}(t - t_0) = A\Phi(t - t_0), \;\; \Phi(0) = I, \; t \geq t_0 \tag{6}$$

Definition 4.2 *A fundamental matrix of the LTI system is the solution to the matrix equation*

$$\dot{\Psi}(t) = A\Psi(t), \;\; \Psi(t_0) = H, \; t \geq t_0, \tag{7}$$

where H is any nonsingular matrix.

Clearly, fundamental matrix is not unique (different H results in different solution).

Lemma 4.1 *One fundamental matrix can be derived by n linearly independent solutions of the state equation*

$$\dot{x}(t) = Ax(t), \;\; x(t_0) = x_0$$

Proof Let the n linearly independent solutions of the state equation be given by $x_i(t),\ i = 1, 2, \cdots, n$ and $\mathbf{X}(t) = [x_1(t)\ \ x_2(t)\ \ \cdots\ \ x_n(t)]$. Then, $\dot{x}_i(t) = Ax_i(t)$ and

$$\begin{aligned}\dot{\mathbf{X}}(t) &= [\dot{x}_1(t)\ \ \dot{x}_2(t)\ \ \cdots\ \ \dot{x}_n(t)] \\ &= [Ax_1(t)\ \ Ax_2(t)\ \ \cdots\ \ Ax_n(t)] = AX(t)\end{aligned}$$

$\mathbf{X}(t_0) = [x_1(t_0)\ \ x_2(t_0)\ \ \cdots\ \ x_n(t_0)] = H$ is nonsingular. Hence, $\mathbf{x}(t)$ is a fundamental matrix.

Lemma 4.2 *A fundamental matrix of the LTI system can be given by*

$$\Psi(t) = e^{At},\ \ t \geq t_0 \tag{8}$$

Proof Note that $\dot{\Psi}(t) = \frac{de^{At}}{dt} = Ae^{At} = A\Psi(t),\ \ t \geq t_0$. Also, since e^{At_0} is nonsingular for any t_0, set $H = e^{At_0}$. The result follows from Definition 4.2.

Lemma 4.3 *Given a fundamental matrix of the LTI system, the state transition matrix* $\Phi(t - t_0)$ *can be given by*

$$\Phi(t - t_0) = \Psi(t)\Psi^{-1}(t_0),\ \ t \geq t_0 \tag{9}$$

Proof It follows from (9) that

$$\dot{\Phi}(t-t_0) = \dot{\Psi}(t)\Psi^{-1}(t_0) = A\Psi(t)\Psi^{-1}(t_0) = A\Phi(t-t_0)$$

Also, when $t = t_0$,

$$\Phi(0) = \Phi(t_0 - t_0) = \Psi(t_0)\Psi^{-1}(t_0) = I$$

Lemma 4.4 *The state transition matrix is unique and independent of* $\Psi(t)$.

Proof By the ODE theory, given the initial condition, the solution of (6) is unique. Further, if $\Psi_1(t)$ and $\Psi_2(t)$ are two fundamental matrices, then there exists a nonsingular matrix P such as $\Psi_2(t) = \Psi_1(t)P$. Hence,

$$\Phi(t-t_0) = \Psi_2(t)\Psi_2^{-1}(t_0) = \Psi_1(t)PP^{-1}\Psi_1^{-1}(t_0) = \Psi_1(t)\Psi_1^{-1}(t_0)$$

That is, $\Phi(t-t_0)$ is independent of the choice $\Psi(t)$.

From (8) and (9), the state transition matrix is given by $\Phi(t-t_0) = e^{A(t-t_0)}$. Thus, the solution of the state equation can be expressed as

$$x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)d\tau, \;\; t \geq t_0 \tag{10}$$

Properties of State Transition Matrix

- $\Phi(0) = I$.
- $\Phi^{-1}(t - t_0) = \left[\Psi(t)\Psi^{-1}(t_0)\right]^{-1} = \Psi(t_0)\Psi^{-1}(t) = \Phi(t_0 - t)$.
- $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Psi(t_2)\Psi^{-1}(t_1)\Psi(t_1)\Psi^{-1}(t_0) = \Psi(t_2)\Psi^{-1}(t_0) = \Phi(t_2 - t_0)$.
- $\dot{\Phi}(t - t_0) = A\Phi(t - t_0) = \Phi(t - t_0)A$.
- $\frac{d}{dt}\Phi^{-1}(t - t_0) = \frac{d}{dt}\Phi(t_0 - t) = -A\Phi(t_0 - t) = -\Phi(t_0 - t)A$.

Discretization

- Consider the continuous-time system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \qquad (11) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \qquad (12) \end{aligned}$$

- For computation or controller design, the system is to be discretized.
- Assuming sampling period of T and considering zero order hold for control input, i.e. $u(t) = u(kT) := u(k)$, $kT \le t \le (k+1)T$ for $k = 0, 1, 2, \cdots$. Then,

$$\begin{aligned} \mathbf{x}(k+1) &= e^{AT}\mathbf{x}(k) + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]} B\mathbf{u}(\tau)d\tau \\ &= e^{AT}\mathbf{x}(k) + \int_{0}^{T} e^{\alpha A} d\alpha B\mathbf{u}(k) \qquad (13) \end{aligned}$$

Hence, if we compute only the response at $t = kT$, (11)-(12) becomes

$$\begin{aligned} \mathbf{x}(k+1) &= A_d\mathbf{x}(k) + B_d\mathbf{u}(k) \qquad (14) \\ \mathbf{y}(k) &= C_d\mathbf{x}(k) + D_d\mathbf{u}(k) \qquad (15) \end{aligned}$$

- where

$$A_d = e^{AT}, \;\; B_d = \int_0^T e^{A\tau} d\tau B, \;\; C_d = C, \;\; D_d = D$$

- Note that if A is non-singular,

$$\begin{aligned}
\int_0^T e^{A\tau} d\tau &= \int_0^T \left(I + A\tau + A^2 \frac{\tau^2}{2!} + \cdots \right) d\tau \\
&= TI + \frac{T^2}{2!} A + \frac{T^3}{3!} A^2 + \cdots \\
&= A^{-1} \left(TA + \frac{T^2}{2!} A^2 + \frac{T^3}{3!} A^3 + \cdots + I - I \right) \\
&= A^{-1}(e^{AT} - I) = A^{-1}(A_d - I)
\end{aligned}$$

- Thus,

$$B_d = A^{-1}(A_d - I)B$$

- Matlab: [Ad,Bd]=c2d(A,B,T)
  Conversion from continuous-time state equation to discrete-time one under sampling period T

Example 4.4 *Consider LTI system*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \;\; t \geq 0$$

Given sampling period $T = 0.1s$, find the discretized state-space model.

- *Find e^{At}:*

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

By taking inverse Laplace transform,

$$e^{At} = \begin{bmatrix} 1 & 0.5(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

- *Compute:*

$$A_d = e^{AT} = \begin{bmatrix} 1 & 0.5(1 - e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} 1 & 0.091 \\ 0 & 0.819 \end{bmatrix}$$

$$B_d = \left(\int_0^T e^{At} dt \right) B = \int_0^T \begin{bmatrix} 1 & 0.5(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} dt \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.005 \\ 0.091 \end{bmatrix}$$

Solution of Discrete-time State Equation

- From the state equation

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$

by recursion, we have

$$\mathbf{x}(k) = A^k\mathbf{x}(0) + \sum_{m=0}^{k-1} A^{k-1-m}B\mathbf{u}(m) \tag{16}$$

and the output is

$$\mathbf{y}(k) = CA^k\mathbf{x}(0) + \sum_{m=0}^{k-1} CA^{k-1-m}B\mathbf{u}(m) + D\mathbf{u}(k)$$

- A^k is called *transition matrix* like e^{At} in continuous-time case.
- For Jordan block

$$\hat{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \Longrightarrow A^k = T \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix} T^{-1}$$

Let us consider the zero-input response $A^k\mathbf{x}(0)$. Consider an example of Jordan block, we have

$$A^k = T \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix} T^{-1}$$

It is clear that every entry of A^k is a linear combination of $\{\lambda_1^k,\ k\lambda_1^{k-1},\ k^2\lambda_1^{k-2},\ \lambda_2^k\}$.

- If every eigenvalue has its magnitude less than 1, then any zero-input response will approach 0 as $k \to \infty$.
- If A has an eigenvalue with magnitude greater than 1, then most zero input responses will go unbounded as $k \to \infty$.
- If A has some eigenvalues with magnitude 1 and all with index 1 and the remaining eigenvalues all have magnitudes less than 1, then zero-input response will not go unbounded. However, if the index is 2 or higher, some zero-input responses may go unbounded, for example, if A has eigenvalue 1 with index 2, A^k contains $\{1, k\}$ and hence some zero-input responses may go unbounded.

Equivalent State Equations: Consider the n-dimensional state equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{17}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{18}$$

Suppose we now introduce a state transformation $\bar{\mathbf{x}} = P\mathbf{x}$, where P is non-singular. Then, from (17)-(18) we have

$$\dot{\bar{\mathbf{x}}}(t) = \bar{A}\bar{\mathbf{x}}(t) + \bar{B}\mathbf{u}(t) \tag{19}$$
$$\mathbf{y}(t) = \bar{C}\mathbf{x}(t) + \bar{D}\mathbf{u}(t) \tag{20}$$

where

$$\bar{A} = PAP^{-1}, \;\; \bar{B} = PB, \;\; \bar{C} = CP^{-1}, \;\; \bar{D} = D.$$

- (19-(20) is said to be (algebraically) equivalent to (17)-(18).
- The matrix D, called direct transmission part between input and output, is independent of the state transformation.
- It can be easily shown that

$$\bar{\Delta}(\lambda) = det(\lambda I - \bar{A}) = det(\lambda I - A) = \Delta(\lambda)$$

$$\hat{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C(sI - A)^{-1}B + D = G(s)$$

Two state equations are said to be *zero state equivalent* if they have the same transfer matrix,

$$C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

i.e.

$$D+CBs^{-1}+CABs^{-2}+CA^2Bs^{-3}+\cdots = \bar{D}+\bar{C}\bar{B}s^{-1}+\bar{C}\bar{A}\bar{B}s^{-2}+\bar{C}\bar{A}^2\bar{B}s^{-3}-$$

Theorem 4.1 *Two linear time-invariant state equations* $\{A, B, C, D\}$ *and* $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ *are zero-state equivalent or have the same transfer matrix if and only if* $D = \bar{D}$ *and*

$$CA^mB = \bar{C}\bar{A}^m\bar{B}, \;\; m = 0, 1, 2, \cdots$$

Clearly, algebraic equivalence implies zero-state equivalence but the converse is generally not true.
See Example 4.4 of page 96 (C.T. Chen) where the state equations for system 1:

$$\dot{x} = 0x + 0u, \;\; y = 0.5u$$

and for system 2: $\dot{x} = x, \;\; y = 0.5x + 0.5u$. The zero-state responses of the above systems are the same but they are not algebraically equivalent.

Linear Time-Varying Systems

- State equations:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \quad (21)$$

$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \quad (22)$$

  where $\mathbf{x} \in \mathcal{R}^n$ is the state, $\mathbf{u} \in \mathcal{R}^m$ is the input and $\mathbf{y} \in \mathcal{R}^p$ is the output.

- $A(t)$, $B(t)$, $C(t)$, $D(t)$ are continuous, real-valued functions on $[t_0, \infty)$.

- Input $\mathbf{u}(t)$ is piecewise continuous.

- Scalar ODE: $\dot{x} = ax \implies x(t) = e^{at}x(0)$.
- Matrix case: $\dot{\mathbf{x}}(t) = A\mathbf{x} \implies \mathbf{x}(t) = e^{At}\mathbf{x}(0)$.
- Time-varying scalar ODE: $\dot{x} = a(t)x \implies x(t) = e^{\int_0^t a(\tau)d\tau}x(0)$ since

$$\frac{d}{dt}e^{\int_0^t a(\tau)d\tau} = a(t)e^{\int_0^t a(\tau)d\tau} = e^{\int_0^t a(\tau)d\tau}a(t)$$

- For time-varying matrix case: $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, is

$$\mathbf{x}(t) = e^{\int_0^t A(\tau)d\tau}\mathbf{x}(0)$$

  a solution?

- Answer is no in general. Since

$$e^{\int_0^t A(\tau)d\tau} = I + \int_0^t A(\tau)d\tau + \frac{1}{2}\left(\int_0^t A(\tau)d\tau\right)\left(\int_0^t A(s)ds\right) + \cdots$$

  but

$$\begin{aligned}\frac{d}{dt}\left(e^{\int_0^t A(\tau)d\tau}\right) &= A(t) + \frac{1}{2}A(t)\left(\int_0^t A(s)ds\right) + \frac{1}{2}\left(\int_0^t A(\tau)d\tau\right)A(t) + \cdots \\ &\neq A(t)e^{\int_0^t A(\tau)d\tau}\end{aligned}$$

Fundamental matrix:

- Consider $\dot{\mathbf{x}} = A(t)\mathbf{x}$.
- Suppose for every initial state $\mathbf{x}_i(t_0),\ i = 1, 2, \cdots, n$, the solution is $\mathbf{x}_i(t)$.
- Let $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]$. If $\mathbf{X}(t_0)$ is non-singular, $\mathbf{X}(t)$ is called *fundamental matrix* which is non-singular.
- Fundamental matrix is not unique.

Example 4.5 *Consider*

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \mathbf{x}(t)$$

i.e.

$$\dot{x}_1(t) = 0, \quad \dot{x}_2(t) = t x_1(t)$$

Then,

$$x_1(t) = x_1(0), \quad x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0)$$

Hence,

$$\begin{aligned}
\mathbf{x}(0) &= \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \mathbf{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix} \\
\mathbf{x}(0) &= \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \mathbf{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix}
\end{aligned}$$

Since the two initial states are independent,

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

is a fundamental matrix.

State Transition Matrix:

- $\mathbf{\Phi}(t, t_0) := \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$ is called *the state transition matrix*.
- The state transition matrix is the unique solution of

$$\frac{\partial}{\partial t}\mathbf{\Phi}(t, t_0) = A(t)\mathbf{\Phi}(t, t_0) \tag{23}$$

with initial condition $\mathbf{\Phi}(t_0, t_0) = I$.

- Properties:

$$\begin{aligned} \mathbf{\Phi}(t, t) &= I \\ \mathbf{\Phi}^{-1}(t, t_0) &= \left[\mathbf{X}(t)\mathbf{X}^{-1}(t_0)\right]^{-1} = \mathbf{X}(t_0)\mathbf{X}^{-1}(t) = \mathbf{\Phi}(t_0, t) \\ \mathbf{\Phi}(t, t_0) &= \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0) \end{aligned}$$

for every t, t_0 and t_1.

Solution of State-space Equation of LTV system

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau, \quad t \geq t_0 \tag{24}$$

Hence, the output is

$$\begin{aligned} \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \\ &= C(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} C(t)\mathbf{\Phi}(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau + D(t)\mathbf{u}(t) \end{aligned} \tag{25}$$

Example 4.6 *Consider*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -g + v_e u_0/(m_0 + u_0 t) \end{bmatrix}, \quad \mathbf{x}(0) = 0$$

where $x_1(t)$ and $x_2(t)$ are respectively the altitude and velocity of an ascending rocket. Assuming that the mass supply is exhausted at time $t_e > 0$. Calculate the state trajectory on $[0, t_e]$.

It is easy to obtain

$$\Phi(t, \tau) = \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix}$$

With $\mathbf{x}(0) = 0$, the zero-state response is

$$\begin{aligned} \mathbf{x}(t) &= \int_0^t \begin{bmatrix} 1 & t - \sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -g + v_e u_0/(m_0 + u_0 \sigma) \end{bmatrix} d\sigma \\ &= \begin{bmatrix} -gt^2/2 - (v_e m_0/u_0)(1 + u_0 t/m_0) ln(1 + u_0 t/m_0) \\ -gt + v_e ln(1 + u_0 t/m_0) \end{bmatrix}, \quad t \in [0, t_e] \end{aligned}$$

At t_e, the thrust becomes zero but the rocket will not stop. For $t > t_e$,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

and the state trajectory needs to be calculated.
Denote the remaining mass of the rocket by

$$m_e = m_0 + u_0 t_e$$

Then,

$$\mathbf{x}(t_e) = \begin{bmatrix} -gt_e^2/2 - v_e t_e + (v_e m_e/u_0) ln(m_e/m_0) \\ -gt_e + v_e ln(m_e/m_0) \end{bmatrix}$$

Hence, the state response is

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, t_e)\mathbf{x}(t_e) + \int_{t_e}^{t} \Phi(t,\sigma) \begin{bmatrix} 0 \\ -g \end{bmatrix} d\sigma \\ &= \begin{bmatrix} -v_e t_e + v_e(t + m_0/u_0) ln(m_e/m_0) - gt^2/2 \\ v_e ln(m_e/m_0) - gt \end{bmatrix}, \quad t \geq t_e \end{aligned}$$

The above is valid until the altitude reaches zero again.

Exercise:

1. An oscillation can be generated by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

Show that its solution is

$$\mathbf{x}(t) = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix} \mathbf{x}(0)$$

2. Use two different methods to find the unit-step response of

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [2 \quad 3] x \end{aligned}$$

3. Given

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

find e^{At}.

3. Discretize the system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ y &= [2 \;\; 3]\mathbf{x} \end{aligned}$$

with $T = 1$ and $T = \pi$.

4. Find a fundamental matrix and the state transition matrix of

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} \mathbf{x}$$