Model Predictive Control — Lecture 4
Integral Action in MPC

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Integral Action in MPC

To get offset free tracking of a constant set-point in the presence of an unknown but constant disturbance, we need:

1. In steady state, the minimum of the MPC cost function must be consistent with zero tracking errors
2. The predictions must be unbiased, i.e. the prediction model should give, in steady state, $\hat{\mathbf{Y}} = \begin{bmatrix} I, & I, & \ldots, & I \end{bmatrix}^T y_{\text{SS}}^{\text{REAL}}$, regardless of any differences between the model and the process due to uncertainty and disturbances.

MPC Cost Function, 1/2

The use of incremental control in the cost function, i.e.

$$J = \sum_{i=N_1}^{N_2} (\hat{y}(k+i|k) - w(k+i|k))^2 + \lambda \sum_{i=1}^{N_u} (\Delta\hat{u}(k+i-1|k))^2$$

satisfies the first criteria since $J = 0$ is achieved when $y - w = 0$ and $\Delta u = 0$.

This cost function, however,

$$J = \sum_{i=N_1}^{N_2} (\hat{y}(k+i|k) - w(k+i|k))^2 + \lambda \sum_{i=1}^{N_u} (\hat{u}(k+i-1|k))^2$$

will not give offset-free tracking as $y - w = 0$ and $u = 0$ will be inconsistent most of the time. The best minimum in the steady state would be a compromise between the norms of $y - w$ and u and hence $y - w \neq 0$ in general.

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MPC Cost Function, 2/2

If one wants to avoid the use of input increments, then one alternative is to include weights on the distance of the inputs from their steady state values u_{ss}, e.g.

$$J = \sum_{i=N_1}^{N_2} (\hat{y}(k+i|k) - w(k+i|k))^2 + \lambda \sum_{i=1}^{N_u} (\hat{u}(k+i-1|k) - u_{ss})^2$$

as then $y - w = 0$ and $u = u_{ss}$ are consistent and the minimum $J = 0$ occurs with no offset.

However, one then needs a model that gives unbiased predictions for the pair $\hat{y} = w$, u_{ss}.

Unbiased Predictions , 1/5

The basic idea of this criteria can be illustrated using the prediction model given by

$$\hat{\mathbf{Y}} = \underbrace{\Phi\xi(k)}_{\text{past}} + \underbrace{G\hat{\mathbf{U}}}_{\text{future}} \tag{1}$$

and the MPC control law

$$\hat{\mathbf{U}} = (G^T G + \lambda I)^{-1} G^T (\hat{\mathbf{W}} - \Phi\xi(k)) \tag{2}$$

We assume that the state vector $x(k)$ and the actual plant output $y(k)^{\text{REAL}}$ are measured.
At steady state, $\hat{U} = 0$. This implies that, from Eq 2, we must have

$$\hat{\mathbf{W}}_{\text{SS}} - \Phi\xi_{\text{SS}} = 0.$$

where $\hat{\mathbf{W}}_{\text{SS}} = \begin{bmatrix} I, & I, & \ldots, & I \end{bmatrix}^T w_{\text{SS}}$.
Thus, for offset free tracking, we require

$$\Phi\xi_{\text{ss}} = \begin{bmatrix} I, & I, & \ldots, & I \end{bmatrix}^T y_{\text{SS}}^{\text{REAL}} \tag{3}$$

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Unbiased Predictions, 2/5

With state space model given by

$$\begin{aligned}\begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} &= \begin{bmatrix} A_p & B_p \\ 0 & I \end{bmatrix}\begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} B_p \\ I \end{bmatrix}\Delta u(k) \\ y(k) &= \begin{bmatrix} C_p & 0 \end{bmatrix}\begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}\end{aligned} \quad (4)$$

we have

$$A \leftarrow \begin{bmatrix} A_p & B_p \\ 0 & I \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} B_p \\ I \end{bmatrix}, \quad C \leftarrow \begin{bmatrix} C_p & 0 \end{bmatrix}$$

giving

$$CA^i \leftarrow \begin{bmatrix} C_p A_p^i & \sum_{i=0}^{N-1} C_p A_p^i B_p \end{bmatrix}$$

and

$$\xi(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} \quad \Rightarrow \quad \xi_{ss} = \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix}$$

Unbiased Predictions, 3/5

The unbiased preduction condition (Eq 3) would only be satisfied if

$$(C_p A_p^i) x_{\mathrm{SS}} + (\sum_{i=0}^{N-1} C_p A_p^i B_p) u_{\mathrm{SS}} = y_{\mathrm{SS}}^{\mathrm{REAL}}$$

and this is only possible if (i) the model is accurate, and (ii) there is no disturbance on the process.

Unbiased Predictions, 4/5

On the other hand, with the state space model given by

$$
\begin{aligned}
\begin{bmatrix} \Delta x(k+1) \\ y(k) \end{bmatrix} &= \begin{bmatrix} A_p & 0 \\ C_p & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k-1) \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} \Delta u(k) \\
y(k) &= \begin{bmatrix} C_p & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k-1) \end{bmatrix}
\end{aligned} \tag{5}
$$

we have

$$
A \leftarrow \begin{bmatrix} A_p & 0 \\ C_p & I \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad C \leftarrow \begin{bmatrix} C_p & I \end{bmatrix}
$$

giving

$$
CA^i \leftarrow \begin{bmatrix} \sum_{i=0}^{N} C_p A_p^i & I \end{bmatrix}
$$

and

$$
\xi(k) = \begin{bmatrix} \Delta x(k) \\ y(k-1) \end{bmatrix} \quad \Rightarrow \quad \xi_{\mathrm{SS}} = \begin{bmatrix} 0 \\ y_{\mathrm{SS}}^{\mathrm{REAL}} \end{bmatrix}
$$

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Unbiased Predictions, 5/5

The condition (Eq 3) is thus satisfied even if the model is inaccurate and/or disturbance is present.

Similarly, the state space model given by

$$\begin{aligned} \begin{bmatrix} \Delta x(k+1) \\ y(k+1) \end{bmatrix} &= \begin{bmatrix} A_p & 0 \\ C_p A_p & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B_p \\ C_p B_p \end{bmatrix} \Delta u(t) \\ y(k) &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k) \end{bmatrix} \end{aligned} \tag{6}$$

also gives offset free tracking.

Example

Does the control law has integral action?

Ex 3.3: Swimming Pool Example

The water temperature in a heated swimming pool, θ, is related to the heater input power, q, and the ambient air temperature, θ_a, according to the equation

$$T\frac{d\theta}{dt} = kq + \theta_a - \theta$$

where $T = 1$ hour and $k = 0.2^{\circ}\mathrm{C}/kW$. Predictive control is to be applied to keep the water at a desired temperature, and a sampling interval $T_s = 0.25$ hour is to be used. The control update interval is to be the same as T_s.

1. Use `MATLAB` to show that the corresponding discrete-time model is

$$\theta(k+1) = 0.7788\theta(k) + 0.0442q(k) + 0.2212\theta_a(k)$$

2. Verify that if the horizons $N_1 = 1$, $N_2 = 10$ and $N_u = 3$ are used, then

$$\Delta q(k) = 22.604w(k) - 17.604\Delta\theta(k) - 22.604\theta(k)$$

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