Chapter 7 Time-domain Synthesis of Linear Feedback System

7.1 Introduction

- Given the system model, determine a control input so that the system possesses some desired behaviors. This is called *control synthesis* which is opposite to system analysis.
- Feedback is often adopted to achieve the above goal. Feedback has the capability of mitigating model uncertainties and external disturbances.
- Synthesis and design are two different concepts. Synthesis is a more 'theoretical' term while design is an engineering concept and needs to consider constraints, including the realizability and choices of components/devices to realize the proposed control.
- We focus on SISO linear time-invariant systems and their stabilization problem.
- The behaviors of the system can be described by the locations of the closed-loop poles as well as steady-state tracking error or some criterion on transient error, e.g. minimizing the energy of the tracking error.

7.2 State Feedback

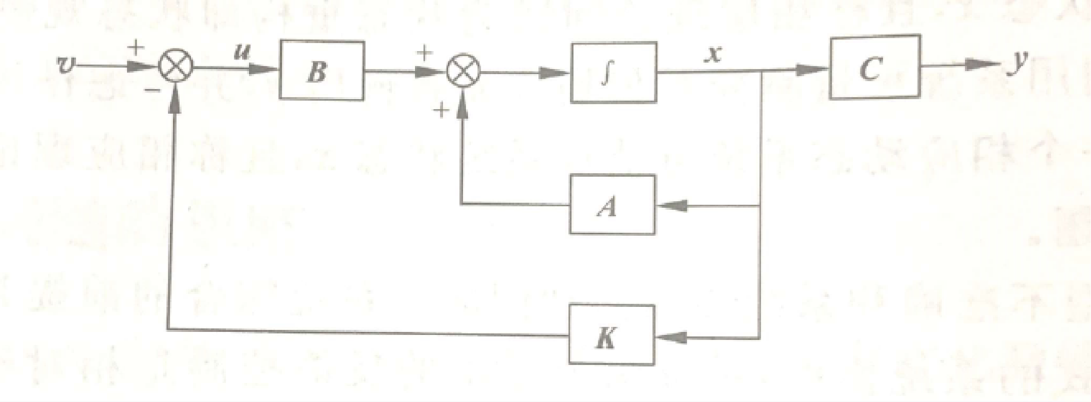
- Consider SISO LTI system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \ \ x(0) = x_0, \ t \geq 0 \\ y &= Cx \end{aligned}$$

- If the state is available for feedback, we can introduce state feedback

$$u(t) = -Kx(t) + v(t) \tag{1}$$

where $K \in R^{1 \times n}$ is the state feedback gain matrix and v is the reference input.



- Under the state feedback, the closed-loop system becomes

$$
\begin{aligned}
\dot{x} &= (A - BK)x + Bv(t) &\quad (2)\\
y &= Cx &\quad (3)
\end{aligned}
$$

and the corresponding closed-loop transfer function is

$$G_c(s) = C(sI - A + BK)^{-1}B \tag{4}$$

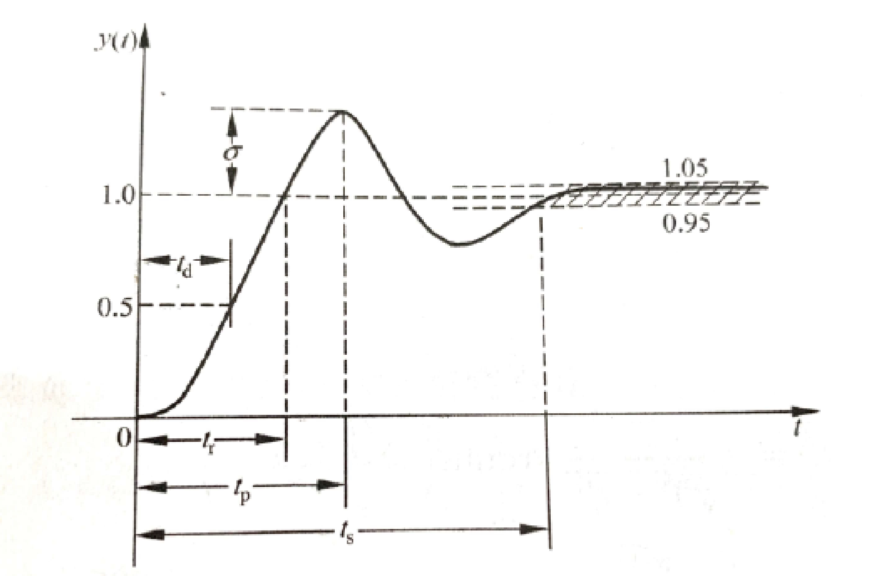
- Observe from (4), the closed-loop poles are the eigenvalues of the closed-loop matrix $A - BK$.
- The time-domain synthesis problem is to determine K so that the closed-loop system possesses desired behaviours.
- If the objective is to place the closed-loop poles at desired locations

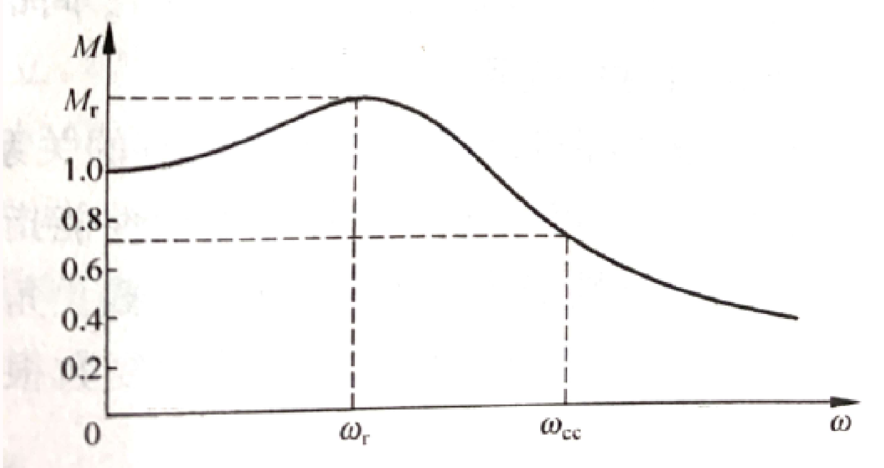
$$\lambda_i(A - BK) = \lambda_i^*, \;\; i = 1, 2, \cdots, n \tag{5}$$

the problem is called *pole placement* problem.

Relationship between locations of poles and system performance

Time-domain and frequency-domain specifications defined by step response:





Consider the standard second order system

$$T^2\ddot{y} + 2\zeta T\dot{y} + y = u,$$

where T is the time constant and ζ is the damping ratio ($\zeta \leq 0.707$). The poles of the system are

$$s_1,\ s_2 = -\frac{1}{T}\zeta \pm j\frac{1}{T}\sqrt{1-\zeta^2} = -\omega_n\zeta \pm j\omega_n\sqrt{1-\zeta^2}$$

where $\omega_n = \frac{1}{T}$ is the natural frequency. Then, time-domain specs are:

- Maximum overshoot: $\sigma = e^{-\zeta/\sqrt{1-\zeta^2}\pi}$
- Rise time: $t_r = \frac{T}{\sqrt{1-\zeta^2}} arctan\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right)$
- Peak time: $t_p = \frac{\pi T}{\sqrt{1-\zeta^2}}$
- Settling time: $t_s \approx \frac{3-ln\sqrt{1-\zeta^2}}{\zeta\omega_n}$ (5% error band)

and the frequency domain specs are:

- Resonance peak value: $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$
- Resonance frequency: $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
- Cut-off frequency (bandwidth): $\omega_{cc} = \omega_n\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}$

For a general $n-th$ order system ($n > 2$), we can use the dominant poles concept to place other remaining $n-2$ poles 4-6 times away from the dominant poles, i.e. setting $s_i = -m\omega_n\zeta,\ 4 \le m \le 6,\ i = 3, 4, \cdots, n$.

Pole-placement:

(a) Consider the system under the controllable canonical form:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6}$$

$$= \left[\begin{array}{c:cccc} 0 & \vdots & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & \vdots & & & 1 \\ \cdots & & \cdots & \cdots & \cdots \\ -a_0 & \vdots & -a_1 & \cdots & -a_{n-1} \end{array}\right] \bar{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{7}$$

Then, the characteristic polynomial (CP) is

$$\Delta(s) = |sI - \bar{A}| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{8}$$

Assume the desired closed-loop poles are $\lambda_i^*,\ i = 1, 2, \cdots, n$. Then, the desired CP is

$$\Delta^*(s) = (s-\lambda_1^*)(s-\lambda_2^*)\cdots(s-\lambda_n^*) = s^n + a_{n-1}^* s^{n-1} + \cdots + a_1^* s + a_0^* \quad (9)$$

Note that with the state feedback $u = -\bar{K}\bar{x} + v$, the closed-loop matrix is

$$\begin{aligned}\bar{A} - \bar{b}\bar{K} &= \begin{bmatrix} 0 & \vdots & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & \vdots & & & 1 \\ \cdots & & \cdots & \cdots & \cdots \\ -a_0 & \vdots & -a_1 & \cdots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & \vdots & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & \vdots & & & 1 \\ \cdots & & \cdots & \cdots & \cdots \\ -a_0 - \bar{k}_1 & \vdots & -a_1 - \bar{k}_2 & \cdots & -a_{n-1} - \bar{k}_n \end{bmatrix}\end{aligned}$$

and the CP is

$$\Delta_{\bar{K}}(s) = s^n + (a_{n-1} + \bar{k}_n)s^{n-1} + \cdots + (a_1 + \bar{k}_2)s + (a_0 + \bar{k}_1) \tag{10}$$

By setting $\Delta_{\bar{K}}(s) = \Delta^*(s)$ and comparing the coefficients of (9) and (10), we obtain

$$\bar{k}_i = a^*_{i-1} - a_{i-1}, \;\; i = 1, 2, \cdots, n$$

(b) Next, consider system with a general state-space model

$$\dot{x} = Ax + bu \tag{11}$$

If (A, b) is controllable, we can place the closed-loop poles at any locations and we can introduce the linear transform $x = P\bar{x}$, where

$$P = \begin{bmatrix} A^{n-1}b & \cdots & Ab & b \end{bmatrix} \begin{bmatrix} 1 & & & \\ a_{n-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_1 & \cdots & a_{n-1} & 1 \end{bmatrix} \tag{12}$$

to transform the system to the controllable canonical form. Then, the desired state feedback is

$$u = -\bar{K}\bar{x} = -\bar{K}P^{-1}x$$

i.e.

$$K = \bar{K}P^{-1}$$

Therefore, the procedure for pole-placement is as follows:

- Check the controllability of (A, b). If controllable, continue. If not, stop.
- Compute the CP of A:

$$\Delta(s) = |sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- Compute the desired CP:

$$\Delta^*(s) = \Pi_{i=1}^n (s - \lambda_i^*) = s^n + a_{n-1}^* s^{n-1} + \cdots + a_1^* s + a_0^*$$

- Compute

$$\bar{K} = [a_0^* - a_0, \ \cdots \ , \ a_{n-1}^* - a_{n-1}]$$

- Compute transformation matrix in (12).
- Compute: $K = \bar{K} P^{-1}$

Example 7.1 *Consider LTI system*

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -6 & 0 \\ 0 & 1 & -12 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Design a state feedback $u = -Kx + v$ so that the closed-loop system poles are located at

$$\lambda_1^* = -2,\ \lambda_2^* = -1 + j,\ \lambda_3^* = -1 - j$$

- *Construct the controllability matrix*

$$\mathcal{C} = [b\ \ Ab\ \ A^2 b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Since rank($\mathcal{C}$)=3, the system is controllable.

- *Compute CP*

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & 0 & 0 \\ -1 & s+6 & 0 \\ 0 & -1 & s+12 \end{vmatrix} = s^3 + 18s^2 + 72s$$

- Compute desired CP:

$$\Delta^*(s) = (s+2)(s+1-j)(s+1+j) = s^3 + 4s^2 + 6s + 4$$

- Compute: $\bar{K} = [a_0^* - a_0 \;\; a_1^* - a_1 \;\; a_2^* - a_2] = [4 \;\; -66 \;\; -14]$
- Compute

$$\begin{aligned} P &= [A^2b \;\; Ab \;\; b] \begin{bmatrix} 1 & & \\ a_2 & 1 & \\ a_1 & a_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 18 & 1 & 0 \\ 72 & 18 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 72 & 18 & 1 \\ 12 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

- Compute

$$K = \bar{K}P^{-1} = [4 \;\; -66 \;\; -14] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -18 & 144 \end{bmatrix} = [-14 \;\; 186 \;\; -1220]$$

Alternatively, we can determine K by directly comparing the coefficients

$$|sI - (A - BK)| = \Pi_{i=1}^{n}(s - \lambda_i^*)$$

Note that

$$A - BK = \begin{bmatrix} -k_1 & -k_2 & -k_3 \\ 1 & -6 & 0 \\ 0 & 1 & -12 \end{bmatrix}$$

and

$$\begin{aligned} |sI - (A - BK)| &= \left| \begin{array}{ccc} s + k_1 & k_2 & k_3 \\ -1 & s + 6 & 0 \\ 0 & -1 & s + 12 \end{array} \right] \\ &= (s + k_1) \begin{vmatrix} s + 6 & 0 \\ -1 & s + 12 \end{vmatrix} + \begin{vmatrix} k_2 & k_3 \\ -1 & s + 12 \end{vmatrix} \\ &= (s + k_1)(s^2 + 18s + 72) + k_2(s + 12) + k_3 \\ &= s^3 + (18 + k_1)s^2 + (72 + 18k_1 + k_2)s + (72k_1 + 12k_2 + k_3) \\ &= s^3 + 4s^2 + 6s + 4 \end{aligned}$$

$$\Longrightarrow 18 + k_1 = 4, \;\; 72 + 18k_1 + k_2 = 6, \;\; 72k_1 + 12k_2 + k_3 = 4$$

$$k_1 = -14, \; k_2 = 186, \; k_3 = -1220$$

7.3 Stabilization

If a system is uncontroallable, arbitrary closed-loop eigenvalues placement is not possible.

For an uncontralloable system, Kalman decomposition gives

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \mathbf{u} \tag{13}$$

By applying the state feedback control law

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \mathbf{v},$$

one obtains the closed-loop system

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} - \bar{B}_1 K_1 & \bar{A}_{12} - \bar{B}_1 K_2 \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \mathbf{v} \tag{14}$$

The closed-loop state matrix is

$$A_c = \begin{bmatrix} \bar{A}_{11} - \bar{B}_1 K_1 & \bar{A}_{12} - \bar{B}_1 K_2 \\ 0 & \bar{A}_{22} \end{bmatrix}$$

The eigenvalues of the closed-loop system are given by

$$|\lambda I - A_c| = |\lambda I - (A_{11} - \bar{B}_1 K_1)| \cdot |\lambda I - \bar{A}_{22}|$$

Thus, the system can be made stable by state feedback if and only of the uncontrollable part of the system does not have unstable eigenvalues, i.e. the system is stabilizable.

7.4 State Observer (Luenberger Observer)

- In the state feedback case, we assume that the state is available (measurable). However, this may not be the case in practice as some internal state variables may not be accessible or it is too costly to measure them.
- When the state is not available, we need to estimate the state based on measurable output.
- Consider the system

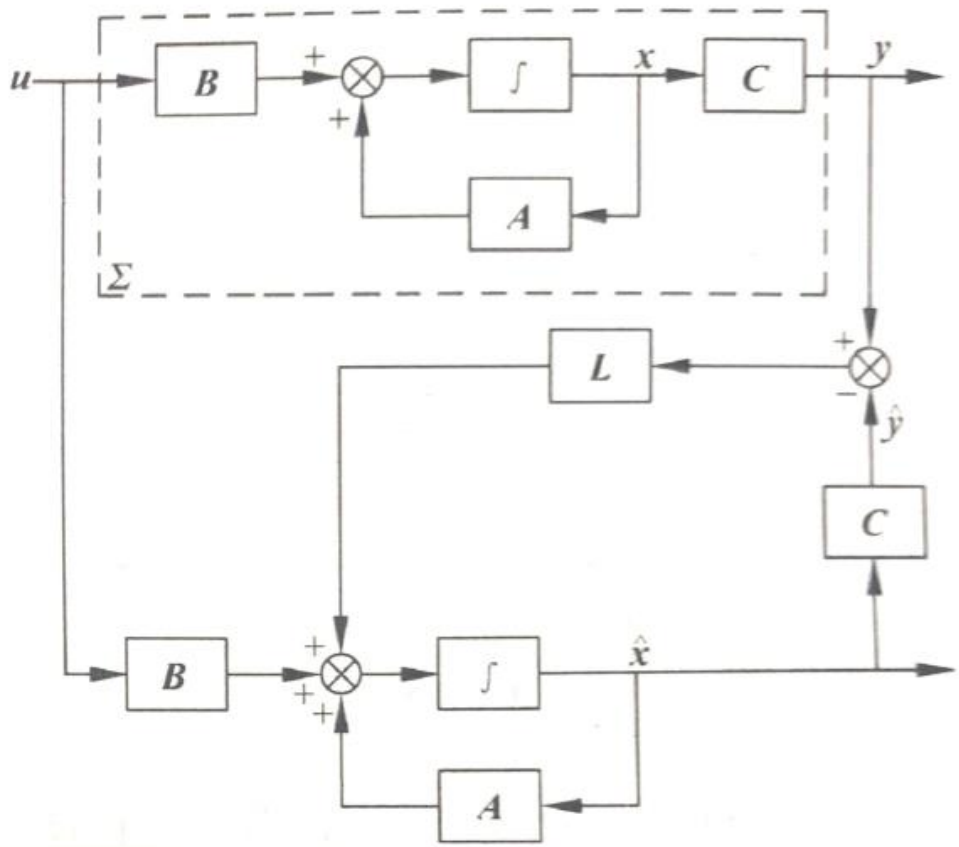
$$\begin{aligned} \dot{x} &= Ax + Bu, \ \ x(0) = x_0, \ \ t \geq 0 & (15) \\ y &= Cx & (16) \end{aligned}$$

- Introduce the state observer as follows

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L(y - \hat{y}) + Bu, \ \ \hat{x}(0) = \hat{x}_0, \ \ t \geq 0 & (17) \\ \hat{y} &= C\hat{x} & (18) \end{aligned}$$

- Define the state observer error as $e = x - \hat{x}$. Then, from (15) and (24), we have

$$\dot{e} = (A - LC)e, \ \ e(0) = x_0 - \hat{x}_0 \qquad (19)$$



- From the error equation (19), it can be seen that if the matrix $A - LC$ is asymptotically stable, for any initial state estimation error $e(0)$, $e(t) \to 0$ when $t \to \infty$.
- Note the importance of the feedback term $L(y - \hat{y}) = L(y - C\hat{x})$. Without the feedback (open-loop),

$$\dot{\hat{x}} = A\hat{x} + Bu, \;\; \hat{x}(0) = \hat{x}_0, \;\; t \geq 0$$

the error dynamics becomes

$$\dot{e} = Ae, \; e(0) = x_0 - \hat{x}_0$$

If A is unstable, the error will become unbounded.

The matrix $A - LC$ can be made asymptotically stable if (A, C) is detectable. Moreover, its eigenvalues (observer poles) can be placed at any locations if and only if (A, C) is observable.

Note that the observability of (A, C) is equivalent to the controllability of (A^T, C^T). Thus, we can place the observer poles using a similar algorithm as the pole placement in state feedback design:

- Check the observability of (A, C). If yes, proceed the following steps; otherwise stop.
- Let $\bar{A} = A^T$ and $\bar{B} = C^T$.
- Given the desired observer poles $\mu_1^*, \; \mu_2^*, \cdots, \mu_n^*$, follow the pole placement algorithm in the state feedback design to design K such that

$$\lambda_i(\bar{A} - \bar{B}K) = \mu_i^*, \;\; i = 1, 2, \cdots, n$$

- Set $L = K^T$.

Alternatively, we can determine L by directly comparing the coefficients

$$|s - (A - LC)| = \Pi_{i=1}^{n}(s - \mu_i^*)$$

Example 7.2 *Consider the following system*

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x\end{aligned}$$

Design a state observer so that the observer poles are located at $\mu_{1,2}^* = -1 \pm j, \;\mu_2^* = -2.$

- Check the observability of the system

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

Clearly, $rank(\mathcal{O}) = 3$, the system is observable.

- Calculate $|sI - (A - LC)|$, where $L = [l_1 \;\; l_2 \;\; l_3]^T$:

$$A - LC = \begin{bmatrix} 1 - l_1 & 1 & 0 \\ l_2 & s + 2 & -1 \\ l_3 & 0 & s - 1 \end{bmatrix}$$

Then,

$$|sI - (A - LC)| = \begin{vmatrix} s - 1 + l_1 & 1 & 0 \\ l_2 & s + 2 & -1 \\ l_3 & 0 & s - 1 \end{vmatrix}$$

$$\begin{aligned} &= (s + l_1 - 1)(s + 2)(s - 1) - (l_2(s - 1) + l_3) \\ &= s^3 + l_1 s^2 + (-3 + l_1 - l_2)s + 2 - 2l_1 + l_2 - l_3 \qquad (20) \end{aligned}$$

- Compute the desired CP

$$\Pi_{i=1}^{3}(s-\mu_i^*) = (s+1-j)(s+1+j)(s+2) = s^3+4s^2+6s+4 \quad (21)$$

- Compare (20) with (21) to get

$$l_1 = 4,\;\; l_1 - l_2 - 3 = 6,\;\; 2 - 2l_1 + l_2 - l_3 = 4$$

or yet

$$l_1 = 4,\; l_2 = -5,\; l_3 = -15$$

7.5 Observer-based Feedback Control

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \;\; x(0) = x_0, \; t \geq 0 && (22) \\ y &= Cx && (23) \end{aligned}$$

where the state is not available for feedback. We introduce the observer based feedback control:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L(y - C\hat{x}) + Bu, \;\; \hat{x}(0) = \hat{x}_0, \;\; t \geq 0 && (24) \\ u &= -K\hat{x} + v && (25) \end{aligned}$$

where v is the reference input. Define $e = x - \hat{x}$ or $\hat{x} = x - e$. Then,

$$\begin{aligned} \dot{x} &= Ax - BK\hat{x} + Bv = (A - BK)x + BKe + Bv \\ \dot{e} &= (A - LC)e \end{aligned}$$

By defining $\xi = [x^T \;\; e^T]^T$, we have the closed-loop system

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \xi + \begin{bmatrix} B \\ 0 \end{bmatrix} v && (26) \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \xi && (27) \end{aligned}$$

The closed-loop matrix is

$$A_c = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$

Thus, the closed-loop poles will be the eigenvalues of $A - BK$ and $A - LC$. That is, we can design the state feedback gain K and the observer gain L separately. This is the so-called *separation principle*.

Overview of Part I

- Description of linear systems: ODE, impulse response, transfer function, state-space representation
- From ODE, transfer function to state-space representation
- Linearization
  - Linearization around operating point: $\dot{x} = f(x, u) \Longrightarrow \dot{\bar{x}} = A\bar{x} + B\bar{u}$
  - Linearization for autonomous systems $\dot{x} = f(x)$: equilibrium point $f(x_e) = 0$, linearization at equilibrium
- Solution of state-space equation:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Key: find e^{At}

- Stability
  - BIBO and internal stability
  - Lyapunov stability
  - Stability of LTI system $\dot{x} = Ax$: check $\lambda(A)$; Lyapunov equation $A^T P + PA = -M$
  - Stability of nonlinear system $\dot{x} = f(x)$ in relation to that of linearzied system at equilibrium
  - Domain of attraction
  - Global stability
- Controllability and observability; stabilizability and detectability
- Kalman decomposition
- Feedback system design: Pole-placement