Lecture 10: Pole Placement and Robust Tracking

10.1 Control Configuration

(a) Open Loop, as shown in Figure 9.1(a).

(b) Unity Feedback Configuration, as shown in Figure 9.1(b).

$$\hat{u}(s) = C(s)[p\hat{r}(s) - \hat{y}(s)] \tag{1}$$

(c) Controller-Estimator or Plant-Input-Output-Feedback Configuration, as shown in Figure 9.1(c).

$$\hat{u}(s) = \frac{1}{1+C_1(s)}\hat{r}(s) - \frac{C_2(s)}{1+C_1(s)}\hat{y}(s) \tag{2}$$

(d) Two-Parameter Configuration, as shown in Figure 9.1(d).

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) \tag{3}$$

Terminology:

Minimum-phase zeros: Zeros with negative real parts

Nonminimum-phase zeros: Zeros with zero or positive real parts

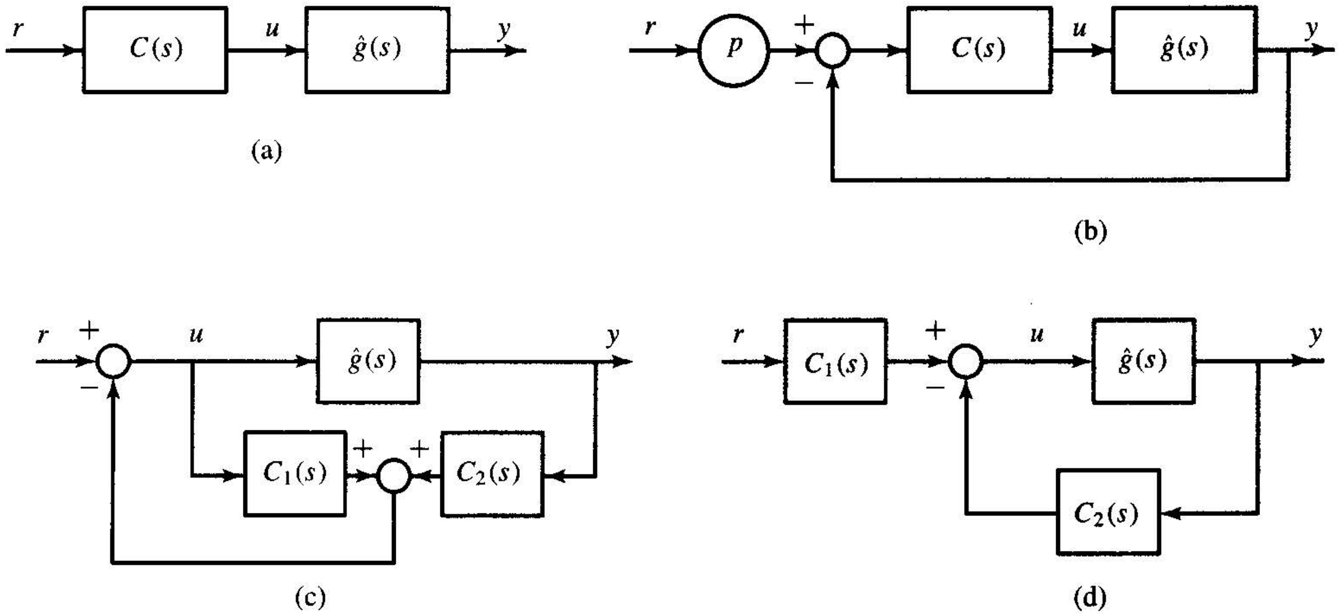

Figure 9.1 Control configurations.

Compensator Equation

$$A(s)D(s) + B(s)N(s) = F(s) \tag{4}$$

where $D(s)$, $N(s)$ and $F(s)$ are given polynomials and $A(s)$ and $B(s)$ are unknown polynomials.
This is known as *Diophantine Equation*.

Theorem 10.1 Given polynimials $D(s)$ and $N(s)$, polynomial solutions $A(s)$ and $B(s)$ exist in (4) for *any* polynomial $F(s)$ iff $D(s)$ and $N(s)$ are coprime.

Proof. Necessity. Suppose $D(s)$ and $N(s)$ are not coprime and contain the same factor $(s + a)$. Then $(s + a)$ will appear in $F(s)$. Thus, if $F(s)$ does not contain this factor, then no solution in (4).
Sufficiency. If $D(s)$ and $N(s)$ are coprime, then $\exists\ \bar{A}(s)$ and $\bar{B}(s)$ such that

$$\bar{A}(s)D(s) + \bar{B}(s)N(s) = 1 \tag{5}$$

For any $F(s)$, this implies

$$\left[F(s)\bar{A}(s)\right] D(s) + \left[F(s)\bar{B}(s)\right] N(s) = F(s) \tag{6}$$

This completes the proof. $\square$

10.2 Unity-Feedback Configuration

Consider Fig 9.1(b), unity feedback configuration. Let $\hat{g}(s) = N(s)/D(s)$ and $C(s) = B(s)/A(s)$.

Then the transfer function from r to y is

$$\hat{g}_c(s) = \frac{pC(s)\hat{g}(s)}{1 + C(s)\hat{g}(s)} = \frac{p\frac{B(s)}{A(s)}\frac{N(s)}{D(s)}}{1 + \frac{B(s)}{A(s)}\frac{N(s)}{D(s)}} = \frac{pB(s)N(s)}{A(s)D(s) + B(s)N(s)} \tag{11}$$

Let

$$A(s)D(s) + B(s)N(s) = F(s) \tag{12}$$

Then, the closed-loop system has poles as in $F(s) = 0$ and zeros as in $B(s)N(s) = 0$. (Note: extra zeros as in $B(s) = 0$.)

If $D(s)$ and $N(s)$ are coprime, then, according to Theorem 10.1, we can solve (12) for $A(s)$ and $B(s)$ for any given $F(s)$

Of course, no guarantee that $C(s) = B(s)/A(s)$ be proper!

How to solve (12)? Let

$$
\begin{aligned}
D(s) &= D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n, \qquad D_n \neq 0 \\
N(s) &= N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n \\
A(s) &= A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m \\
B(s) &= B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m \\
F(s) &= F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m}
\end{aligned}
$$

Substituting the above into (12), we have

$$
\begin{aligned}
A_0 D_0 + B_0 N_0 &= F_0 \\
A_0 D_1 + B_0 N_1 + A_1 D_0 + B_1 N_0 &= F_1 \\
A_0 D_2 + B_0 N_2 + A_1 D_1 + B_1 N_1 + A_2 D_0 + B_2 N_0 &= F_2 \\
&\vdots \\
A_{m-1} D_n + B_{m-1} N_n + A_m D_{n-1} + B_m N_{n-1} &= F_{n+m-1} \\
A_m D_n + B_m N_n &= F_{n+m}
\end{aligned}
$$

i.e.

$$
\begin{bmatrix} A_0 & B_0 & A_1 & B_1 & \cdots & A_m & B_m \end{bmatrix} \mathbf{S}_m = \begin{bmatrix} F_0 & F_1 & \cdots & F_{n+m} \end{bmatrix} \tag{13}
$$

with $[2(m+1)] \times (n+m+1)$ matrix

$$
\mathbf{S}_m = \left[\begin{array}{cccccccc}
D_0 & D_1 & \cdots & D_m & \cdots & D_n & 0 & \cdots & 0 \\
N_0 & N_1 & \cdots & N_m & \cdots & N_n & 0 & \cdots & 0 \\
\hline
0 & D_0 & \cdots & D_{m-1} & \cdots & D_{n-1} & D_n & \cdots & 0 \\
0 & N_0 & \cdots & N_{m-1} & \cdots & N_{n-1} & N_n & \cdots & 0 \\
\hline
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\hline
0 & 0 & \cdots & D_0 & \cdots & D_{n-m} & D_{n-m+1} & \cdots & D_n \\
0 & 0 & \cdots & N_0 & \cdots & N_{n-m} & N_{n-m+1} & \cdots & N_n
\end{array}\right] \tag{14}
$$

Note that (13) has a solution for *any* $F(s)$ iff $\mathbf{S}_m$ is of full column rank (but the solution may not be unique).

A necessary condition for $\mathbf{S}_m$ to have full column rank is that $\mathbf{S}_m$ is square or has more rows than columns:

$$
2(m+1) \geq n+m+1, \quad \text{i.e.} \quad m \geq n-1
$$

If $m < n-1$, (13) may have solution for some $F(s)$, but not for all $F(s)$. Hence, arbitrary pole assignment is not possible if the compensator $C(s)$ is of degree less than $n-1$.

If $m = n-1$, then $\mathbf{S}_{n-1}$ is $(2n) \times (2n)$, and is just the transpose of the Sylvester resultant in Chapter 6. Recall that $\mathbf{S}_{n-1}$ is nonsingular iff $D(s)$ and $N(s)$ are coprime.

Note: if $D(s)$ and $N(s)$ are coprime, then $\mathbf{S}_n$ is $2(n+1) \times (2n+1)$ and is also of full column rank $2n+1$, because $D_n \neq 0$.
In fact, if $D(s)$ and $N(s)$ are coprime, then $\mathbf{S}_m$ is of full column rank $n+m+1$ for all $m \geq n-1$.

Theorem 10.2 Consider the unity-feedback system in Fig 9.1(b). The plant is described by

$$\hat{g}(s) = \frac{N(s)}{D(s)}, \quad \text{with} \begin{cases} N(s) \text{ and } D(s) \text{ coprime,} \\ \deg N(s) < \deg D(s) = n \end{cases}$$

Let $m \geq n-1$. Then for any polynomial $F(s)$ of degree $(n+m)$, there exists a proper compensator

$$C(s) = \frac{B(s)}{A(s)}, \quad \text{with } \deg B(s) \leq \deg A(s) = m$$

such that the overall transfer function is

$$\hat{g}_c(s) = \frac{pB(s)N(s)}{A(s)D(s) + B(s)N(s)} = \frac{pB(s)N(s)}{F(s)}$$

Furthermore, $C(s)$ can be obtained by solving (13), and the solution $A(s)$ and $B(s)$ is unique if $m = n-1$, and not unique if $m > n-1$.

Proof. Existence of solution follows from the fact that $\mathbf{S}_m$ is of full column rank $n + m + 1$ if $m \geq n - 1$.

Only need to show $C(s)$ is proper, i.e., $A_m \neq 0$.

Note that $\deg N(s) < \deg D(s) = n$. So $N_n = 0$.

Then the last eqn of (13) becomes

$$A_m D_n = F_{n+m}$$

As $\deg F(s) = m$, i.e., $F_{n+m} \neq 0$, hence, $A_m \neq 0$.

When $m = n - 1$, $\mathbf{S}_{n-1}$ is square and nonsingular, so the solution exists and is unique.

When $m > n - 1$, the solution is not unique. $\square$

Regulation and Tracking

Regulation is achieved if all poles of $\hat{g}_c(s)$ (i.e., roots of $F(s) = 0$) have negative real parts. Then, for step reference $\hat{r}(t) = a/s$, we have

$$\hat{y}(s) = \hat{g}_c(s)\hat{r}(s) = \hat{g}_c(s)\frac{a}{s}$$

and

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = \hat{g}(0)a$$

As $\hat{g}(0) = p\frac{B_0 N_0}{F_0}$, so tracking of a step input can be achieved by letting

$$p = \frac{F_0}{B_0 N_0} \tag{15}$$

Robust Tracking and Disturbance Rejection

The tracking solution in (15) is affected by variation of plant and controller parameters N_0, B_0 and p.

Robust tracking and disturbance rejection problem: to make output y track asymptotically a class of reference signal r even with the presence of disturbance and plant parameter variations.

Consider the system shown in Fig. 9.2.

First, let $r(\infty) \neq 0$ and $w(\infty) \neq 0$. Otherwise, closed-loop stability implies asymptotic tracking — a trivial case.

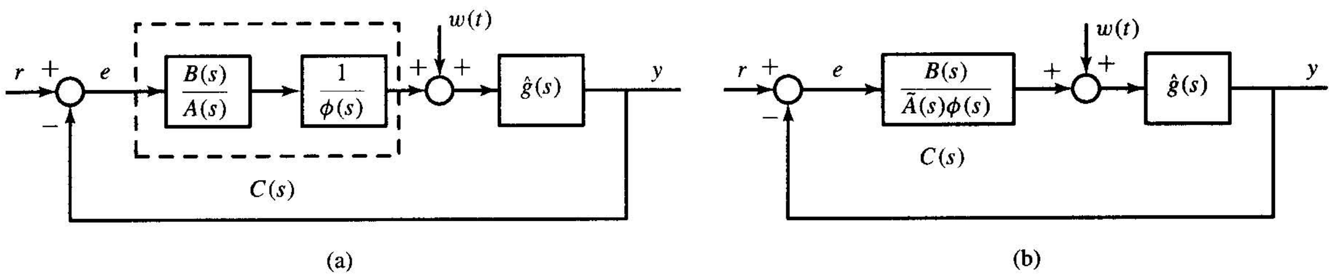

Figure 9.2 Robust tracking and disturbance rejection.

We must have some knowledge on $r(t)$ and $w(t)$. Assume that

$$\hat{r}(s) = \mathcal{L}[r(t)] = \frac{N_r(s)}{D_r(s)}, \qquad \hat{w}(s) = \mathcal{L}[w(t)] = \frac{N_w(s)}{D_w(s)} \tag{20}$$

where $D_r(s)$ and $D_w(s)$ are known, but $N_r(s)$ and $N_w(s)$ unknown.
For example, $r(t) = a$ $(t \geq 0)$ (step input) and $w(t) = b + c\sin(\omega_0 t + d)$ with $a,\ b,\ c,\ d$ unknown but ω_0 known. Note that $\hat{r}(s) = a/s$ and $\hat{\omega}_0(s) = \frac{N_w(s)}{s^2+\omega_0^2}$.

Theorem 10.3 Consider the unity-feedback system in Fig. 9.2(a) with a strictly proper plant transfer function $\hat{g}(s) = N(s)/D(s)$, with $D(s)$ and $N(s)$ coprime. The reference signal $r(t)$ and disturbance $w(t)$ are modeled as $\hat{r}(s) = N_r(s)/D_r(s)$ and $\hat{w}(s) = N_w(s)/D_w(s)$. Let $\phi(s)$ be the least common denominator of the unstable poles of $\hat{r}(s)$ and $\hat{w}(s)$. If no root of $\phi(s)$ is a zero of $\hat{g}(s)$, then there exists a proper compensator such that the overall system will track $r(t)$ and reject $w(t)$, both asymptotically and robustly.

Proof. If no root of $\phi(s)$ is a zero of $\hat{g}(s) = N(s)/D(s)$, then $D(s)\phi(s)$ and $N(s)$ are coprime. Thus, $\exists\, C(s) = B(s)/A(s)$ such that

$$A(s)D(s)\phi(s) + B(s)N(s) = F(s)$$

with $F(s)$ having any desired roots (in particular, stable roots).

Claim: The following compensator, as in Fig 9.2(a), achieves the design.

$$C(s) = \frac{B(s)}{A(s)\phi(s)}$$

First,

$$\hat{g}_{yw}(s) = \frac{\frac{N(s)}{D(s)}}{1 + \frac{B(s)}{A(s)\phi(s)}\frac{N(s)}{D(s)}} = \frac{N(s)A(s)\phi(s)}{F(s)}$$

Thus, output due to $w(t)$ is

$$\hat{y}_w(s) = \hat{g}_{yw}(s)\hat{w}(s) = \frac{N(s)A(s)\phi(s)}{F(s)}\frac{N_w(s)}{D_w(s)} \tag{21}$$

Note: All unstable roots of $D_w(s)$ are canceled by $\phi(s)$, all poles of $\hat{y}_w(s)$ are stable. Hence,

$$\lim_{t\to\infty} y_w(t) = 0, \qquad \text{i.e., disturbance rejection.}$$

Now look at $y(t)$ due to $r(t)$:

$$\begin{aligned}\hat{y}_r(s) = \hat{g}_{yr}(s)\hat{r}(s) &= \frac{B(s)N(s)}{A(s)D(s)\phi(s) + B(s)N(s)}\hat{r}(s) \\ \Longrightarrow \quad \hat{e}(s) &\stackrel{\triangle}{=} \hat{r}(s) - \hat{y}_r(s) = [1 - \hat{g}_{yr}(s)]\,\hat{r}(s) \\ &= \frac{A(s)D(s)\phi(s)}{F(s)} \frac{N_r(s)}{D_r(s)} \end{aligned} \tag{22}$$

Again all unstable roots of $D_r(s)$ are canceled by $\phi(s)$, we have,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\hat{e}(s) = 0, \text{ i.e., } \lim_{t\to\infty}[y_r(t) - r(t)] = 0$$

Hence

$$y(t) = y_w(t) + y_r(t), \quad \text{and} \quad \lim_{t\to\infty}[y(t) - r(t)] = 0$$

i.e., tracking and disturbance rejection. This is achieved so long as the system is BIBO, i.e., $A(s)D(s)\phi(s) + B(s)N(s)$ is stable. $\square$

No pole-zero cancelation in system transfer function! Hence, can be implemented in practice.

Note: $1/\phi(s)$ captures all the unstable poles of $\hat{r}(s)$ and $\hat{w}(s)$. The insertion of $1/\phi(s)$ in the loop is called the *internal model principle*.

Since $B(s)/A(s)$ is proper, the compensator

$$C(s) = \frac{B(s)}{A(s)\phi(s)}$$

is always strictly proper.

When $\deg A(s) = m = n - 1$, the solution $A(s)$ and $B(s)$ is unique.

However, when $\deg A(s) = m \geq n$, the solution is not unique. We can use the extra parameter to embed the internal model $\phi(s)$ into $C(s) = B(s)/A(s)$ directly. See Section 9.2.3 of CT Chen for details.

Example 10.1 Consider the plant $\hat{g}(s) = \frac{s-2}{s^2-1}$. Design a unity-feedback system with a set of desired poles to track robustly any step reference input.

Introduce the internal model $1/\phi(s) = 1/s$. Then,

$$A(s)D(s)\phi(s) + B(s)N(s) = F(s)$$

Note that $\tilde{D}(s) = D(s)\phi(s) = (s^2-1)s = 0 - s + 0s^2 + s^3$ and $N(s) = -2 + s + 0s^2 + 0s^3$.
Since $\tilde{D}(s)$ is of degree 3, select $A(s)$ and $B(s)$ to have degree 2. Hence, $F(s)$ has degree 5. Select the five desired poles at $-2, \ -2 \pm j1$, and $-1 \pm j2$. Then,

$$F(s) = (s+2)(s^2+4s+5)(s^2+2s+5) = S^5+8s^4+30s^3+66s^2+85s+50$$

Form

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 & 0 \end{bmatrix} = [50\ 85\ 66\ 30\ 8\ 1]$$

which gives the solution

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2] = [127.3\ \ -25\ \ 0\ \ -118.7\ 1\ \ -96.3]$$

Thus, the compensator is

$$C(s) = \frac{B(s)}{A(s)\phi(s)} = \frac{-96.3s^2 - 118.7s - 25}{(s^2 + 127.3)s}$$

Embedding Internal Models

Consider

$$A(s)D(s) + B(s)N(s) = F(s)$$

If $deg[D(s)] = n$ and $deg[A(s)] = n - 1$, then the solution $A(s)$ and $B(s)$ is unique. If we increase $deg[A(s)]$ to n, then there is one free parameter we can select to include an internal model in the compensator.

Example 10.2 Consider the plant in Example 10.1. Since $deg[D(s)] = 2$, select $deg[A(s)] = 2$. Then, $F(s)$ has degree 4 and can be selected as

$$F(s) = (s^2 + 4s + 5)(s^2 + 2s + 5) = s^4 + 6s^3 + 18s^2 + 30s + 25$$

Form

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 \end{bmatrix} = [25\ 30\ 18\ 6\ 1]$$

For the proper compensator

$$C(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2}$$

to contain $1/s$ as a factor, require $A_0 = 0$. Solving the above equation, we get

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2] = [0\ \ -12.5\ 34.8\ \ -38.7\ 1\ \ -28.8]$$

and the compensator

$$C(s) = \frac{B(s)}{A(s)} = \frac{-28.8s^2 - 38.7s - 12.5}{s^2 + 34.8s}$$

which can achieve robust tracking and has one degree less than the one obtained in Example 10.1.

Example 10.3 Consider the unity-feedback system in Fig. 9.2(b) with $\hat{g}(s) = 1/s$. Design a proper compensator $C(s) = B(s)/A(s)$ so that the system will track asymptotically any step reference input and reject disturbance $w(t) = asin(2t + \theta)$ with unknown a and θ.

To reject the disturbance, $A(s)$ must contain the disturbance model (s^2+4). For reference input, since the plant already contains s, we do not need to include it in $A(s)$. Consider

$$A(s)D(s) + B(s)N(s) = F(s)$$

Let

$$A(s) = \tilde{A}_0(s^2 + 4), \ \ B(s) = B_0 + B_1 s + B_2 s^2$$

Define

$$\tilde{D}(s) = D(s)(s^2 + 4) = s(s^2 + 4) = \tilde{D}_0 + \tilde{D}_1 s + \tilde{D}_2 s^2 + \tilde{D}_3 s^3$$

Then, $\tilde{A}_0\tilde{D}(s) + B(s)N(s) = F(s)$ leads to

$$\left[\tilde{A}_0\ B_0\ B_1\ B_2\right]\begin{bmatrix} \tilde{D}_0 & \tilde{D}_1 & \tilde{D}_2 & \tilde{D}_3 \\ N_0 & N_1 & 0 & 0 \\ 0 & N_0 & N_1 & 0 \\ 0 & 0 & N_0 & N_1 \end{bmatrix} = [F_0\ F_1\ F_2\ F_3]$$

If we set

$$F(s) = (s+2)(s^2+2s+2) = s^3 + 4s^2 + 6s + 4$$

Then,

$$\left[\tilde{A}_0\ B_0\ B_1\ B_2\right]\begin{bmatrix} 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [4\ 6\ 4\ 1]$$

leading to the solution $[1\ 4\ 2\ 4]$. The compensator is

$$C(s) = \frac{4s^2 + 2s + 4}{s^2 + 4}$$

A Servomechanism Problem: A State Space Analysis

Consider

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{E}w(t), \qquad x(t_0) = x_0 \\ y(t) &= \mathbf{C}x(t) + \mathbf{F}w(t) \end{aligned} \tag{23}$$

where $u(t) \in \mathbf{R}^m$ is the control input, $w(t) \in \mathbf{R}^{q\times 1}$ is the disturbance signal and $y(t) \in \mathbf{R}^p$ is the output.
Assume $p = m$ (i.e. same # of input and output).

Objective: Use output feedback to make output track a reference input with zero steady-state error in face of unknown constant disturbance, and the coefficients of the characteristic polynomial are arbitrarily assignable. This is called a *servomechanism problem*.

Strategy: Use observer to get an output feedback control law.

The disturbance signal $w(t)$ is described by

$$\dot{w}(t) = \mathbf{A}_w w(t)$$

where $\mathbf{A}_w$ has only eigenvalues with non-negative real parts.

Augmented system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{E} \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} \mathbf{C} & \mathbf{F} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{24}$$

Construct an full-order observer for (24):

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{E} \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} [y(t) - \hat{y}(t)]$$
$$\hat{y}(t) = \begin{bmatrix} \mathbf{C} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} \tag{25}$$

Let the error signals be $e_x(t) = x(t) - \hat{x}(t)$ and $e_w(t) = w(t) - \hat{w}$, and the error dynamics is given by

$$\begin{bmatrix} \dot{e}_x(t) \\ \dot{e}_w(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{H}_1\mathbf{C} & \mathbf{E} - \mathbf{H}_1\mathbf{F} \\ -\mathbf{H}_2\mathbf{C} & \mathbf{A}_w - \mathbf{H}_2\mathbf{F} \end{bmatrix} \begin{bmatrix} e_x(t) \\ e_w(t) \end{bmatrix} \tag{26}$$

Consider the feedback using the estimated state

$$u(t) = \mathbf{K}_1\hat{x}(t) + \mathbf{K}_2\hat{w}(t) + \mathbf{N}r(t) \tag{27}$$

The closed-loop system is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{BK}_1 & \mathbf{BK}_2 \\ \mathbf{H}_1\mathbf{C} & \mathbf{A} + \mathbf{BK}_1 - \mathbf{H}_1\mathbf{C} & \mathbf{E} + \mathbf{BK}_2 - \mathbf{H}_1\mathbf{F} \\ \mathbf{H}_2\mathbf{C} & -\mathbf{H}_2 C & \mathbf{A}_w - \mathbf{H}_2\mathbf{F} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{BN} \\ \mathbf{BN} \\ \mathbf{0}_{q\times m} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{E} \\ \mathbf{H}_1\mathbf{F} \\ \mathbf{H}_2\mathbf{F} \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \mathbf{F}w(t) \end{aligned} \tag{28}$$

Apply the following variable change

$$\begin{bmatrix} x(t) \\ e_x(t) \\ -\hat{w}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times q} \\ \mathbf{I}_n & -\mathbf{I}_n & \mathbf{0}_{n\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times n} & -\mathbf{I}_q \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix}$$

and the closed-loop state equation becomes

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{e}_x(t) \\ -\dot{\hat{w}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}+\mathbf{B}\mathbf{K}_1 & -\mathbf{B}\mathbf{K}_1 & -\mathbf{B}\mathbf{K}_2 \\ \mathbf{0} & \mathbf{A}-\mathbf{H}_1\mathbf{C} & \mathbf{E}-\mathbf{H}_1\mathbf{F} \\ \mathbf{0} & -\mathbf{H}_2\mathbf{C} & \mathbf{A}_w-\mathbf{H}_2\mathbf{F} \end{bmatrix} \begin{bmatrix} x(t) \\ e_x(t) \\ -\hat{w}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{N} \\ \mathbf{0}_{n\times m} \\ \mathbf{0}_{q\times m} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{E} \\ \mathbf{E}-\mathbf{H}_1\mathbf{F} \\ -\mathbf{H}_2\mathbf{F} \end{bmatrix} w(t) \tag{29}
$$

$$
y(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \mathbf{F}w(t)
$$

Because of the block-diagonal structure in (29), the closed-loop characteristic polynomial is given by

$$
\det\left[s\mathbf{I} - (\mathbf{A}+\mathbf{B}\mathbf{K}_1)\right] \det \begin{bmatrix} s\mathbf{I} - (\mathbf{A}-\mathbf{H}_1\mathbf{C}) & \mathbf{E}-\mathbf{H}_1\mathbf{F} \\ -\mathbf{H}_2\mathbf{C} & s\mathbf{I}-\mathbf{A}_w+\mathbf{H}_2\mathbf{F} \end{bmatrix}
$$

which depends only on $\mathbf{K}_1$, $\mathbf{H}_1$ and $\mathbf{H}_2$. Notice the controller poles and the observer poles are separated!

Next, we use the additional freedom K_2 and N in the controller to achieve input-output objectives of asymptotic tracking and disturbance rejection. Straightforward calculations show that

$$
\begin{aligned}
Y(s) = {} & C\Big[sI-(A+BK_1)\Big]^{-1} BNR(s) \\
& + C\Big[sI-(A+BK_1)\Big]^{-1} EW(s) \\
& - \Bigg[C\Big[sI-(A+BK_1)\Big]^{-1} BK_1\,, \\
& \qquad - C\Big[sI-(A+BK_1)\Big]^{-1} BK_2\Bigg] \\
& \times \begin{bmatrix} sI-(A-H_1C) & -E+H_1F \\ H_2C & sI+H_2F \end{bmatrix}^{-1} \\
& \times \begin{bmatrix} E-H_1F \\ -H_2F \end{bmatrix} W(s) + FW(s)
\end{aligned}
\tag{30}
$$

For step reference and disturbance inputs

$$R(s) = \mathbf{r}_0 \frac{1}{s}, \qquad W(s) = \mathbf{w}_0 \frac{1}{s}$$

applying the final value theorem of Laplace Transform and noting that

$$\begin{bmatrix} -(A - H_1 C) & -E + H_1 F \\ H_2 C & H_2 F \end{bmatrix}^{-1} \begin{bmatrix} E - H_1 F \\ -H_2 F \end{bmatrix} = \begin{bmatrix} 0 \\ -I_q \end{bmatrix}$$

we have the following steady-state output

$$\begin{aligned} \lim_{t \to \infty} y(t) = & -C(A + BK_1)^{-1} BN\mathbf{r}_0 \\ & + \left[F - C(A + BK_1)^{-1} E \right. \\ & \left. -C(A + BK_1)^{-1} BK_2\right] \mathbf{w}_0 \end{aligned} \tag{31}$$

Then, if $C(A + BK_1)^{-1}B$ is invertible, then we can choose

$$\begin{aligned} N = & -[C(A + BK_1)^{-1} B]^{-1} \\ K_2 = & NC(A + BK_1)^{-1} E - NF \end{aligned} \tag{33}$$

simplifies (31) to

$$\lim_{t \to \infty} y(t) = \mathbf{r}_0 \tag{34}$$

Finally, we establish the equivalence between the invertibility of the following matrices (assuming $A + BK_1$ nonsingular):

$$C(A + BK_1)^{-1}B \text{ is invertible} \tag{$*$}$$

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \text{ is invertible} \tag{$**$}$$

$$\begin{bmatrix} A + BK_1 & BN \\ C & 0 \end{bmatrix} \text{ is invertible} \tag{$***$}$$

The equivalence between $(**)$ and $(***)$ is easily established by noting that N is assumed nonsingular and that

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ K_1 & N \end{bmatrix} = \begin{bmatrix} A + BK_1 & BN \\ C & 0 \end{bmatrix}$$

and the equivalence between $(*)$ and $(***)$ can be established by noting that

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ -C(A + BK_1)^{-1} & I \end{bmatrix} \begin{bmatrix} A + BK_1 & BN \\ C & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} A + BK_1 & BN \\ 0 & -C(A + BK_1)^{-1}BN \end{bmatrix} \end{aligned}$$

and the assumption that $A + BK_1$ is invertible, which can be deduced from its stability.

The above development is summarized in the following Theorem 15.9.

Theorem 10.4: Suppose the plant (23) is controllable for $E = 0$, the augmented plant (24) is observable, and the following $(n+m)\times(n+m)$ matrix is invertible

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

Then a linear dynamic output feedback of the form (27) and (25) has the following properties. The gains K_1, H_1 and H_2 can be chosen such that the closed-loop state equation (28) is exponentially stable with any desired characteristic polynomial coefficients.

Furthermore, the gains in (33) are such that for any constant reference input $r(t) = \mathbf{r}_0$ and constant disturbance $w(t) = \mathbf{w}_0$, the response of the closed-loop state equation satisfies (34). $\square$