EE7204 Linear Systems-Assignment #1

Instructions:

(a) This is a take home assignment. Student is to work on the assignment himself/herself. Penalty shall be imposed for any plagiarism in the assignment.
(b) Student should submit hard copy of the assignment to Professor Xie Lihua located at S2-B2c-94. The submission deadline is Monday, 21 October 2024.

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1. Figure 1 shows a schematic diagram of a Translational Oscillator with Rotating Actuator system. The system consists of a platform of mass M connected to a fixed frame of reference by a linear spring, with spring constant k. The platform can only move in the horizontal plane, parallel to the spring axis. On the platform, a rotating proof mass is actuated by a DC motor. It has mass m and moment of inertial I around its center of mass, located at a distance L from its rotational axis. The control torque applied to the proof mass is denoted by u. The rotating proof mass creates a force which can be controlled to dampen the translational motion of the platform. We will derive a model for the system, neglecting friction. Figure 1a shows that the proof mass is subject to forces F_x and F_y and a torque u. The governing equations are

$$m\frac{d^2}{dt^2}(x_c + L\sin\theta) = F_x, \quad m\frac{d^2}{dt^2}(L\cos\theta) = F_y$$

$$I\frac{d^2\theta}{dt^2} = u + F_y L\sin\theta - F_x L\cos\theta$$

where θ is the angular position of the proof mass (measured counter clockwise). The platform is subject to the forces F_x and F_y, in the opposite directions, as well as the restoring force of the spring. Newton's law for the platform yields

$$M\frac{d^2 x_c}{dt^2} = -F_x - kx_c$$

where x_c is the translational position of the platform.

(a) Carrying out the indicated differentiation and eliminating F_x and F_y, show that the equations of motion reduce to

$$\begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & M + m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

and thus obtain

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \begin{bmatrix} u \\ mL\dot{\theta}^2 \sin\theta - kx_c \end{bmatrix}$$

where

$$\Delta(\theta) = (I+mL^2)(m+M) - m^2L^2\cos^2\theta \geq (I+mL^2)M + mI > 0$$

(b) Using $x_1 = \theta, x_2 = \dot{\theta}, x_3 = x_c$, and $x_4 = \dot{x}_c$ as the state variables and u as the control input, write down the state equation.

(c) Fine all equilibrium points of the system and linearize the system at one of the equilibrium points.

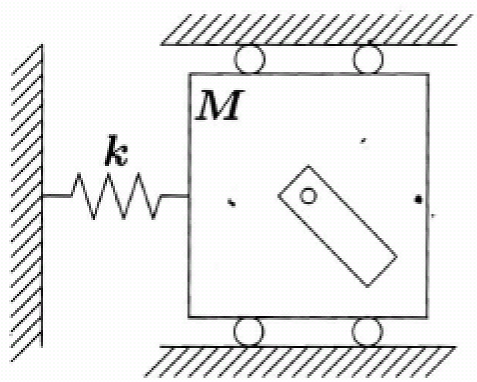

Figure 1

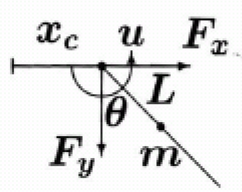

Figure 1a

2 (a) Find a linear transformation that transforms the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

into the diagonal or Jordan form, respectively, and thus compute $e^{A_1 t}$ and $e^{A_2 t}$.

(b) Consider the system

$$\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where $\omega > 0$ is a positive scalar, $x \in R^2$ is the state, u is the input and y is the output. Given the initial state $x(0) = [1 \quad 0]^T$, find the output of the system when the input $u(t) = 1, t \geq 0$.

3. (a) Consider the linear time-varying system

$$\dot{x}_1 = -\alpha x_1 + x_2 \cos(\omega t)$$
$$\dot{x}_2 = x_1 \sin(\omega t) - \beta x_2$$

where $\alpha, \beta, \omega > 0$ are constants. Show that the system is uniformly asymptotically stable if $2\alpha\beta > \sqrt{\alpha^2 + \beta^2}$. (Hint: You may apply the Lyapunov function candidate $V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$)

(b) Consider the system

$$\dot{x}_1 = -2x_1 + x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_2 + x_2(x_1^2 + x_2^2)$$

Find the equilibriums and show that the equilibrium at the origin is exponentially stable. Further, determine an estimate for the domain of attraction of the equilibrium at the origin.

4. (a) Find the ranges of a and b if exists such that the system

$$\dot{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} u, \quad y = [1 \quad -1 \quad b] x$$

is controllable and observable.

(b) Verify the controllability of the system:

$$\dot{x} = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 0 & 2 & -2 \\ -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u$$

and find the Kalman canonical decomposition of the system.

5. Consider the linear system

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

where x is the state, u is the control input and y is the output. Can one find a state feedback $u = -Kx + r$, where r is the reference input, so that the closed-loop poles can be at any desired locations? Design, if possible, a feedback gain K so that the closed-loop poles are located at -2, $-1 \pm j1$.