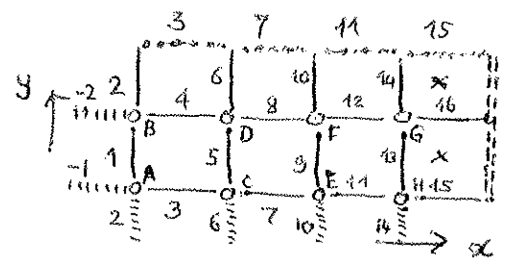
∘ Quantum Dynamics with Tensor Networks.

We aim to study the real-time dynamics of quantum gauge theories - realized as Quantum link models with tensor networks. To start with we consider the case of pure Abelian gauge theories, on a ladder system which we denote as:



The system is considered elongated in one direction than the other. The dotted horizontal links are the periodic copies of the links while the dashed vertical links are not in the system - the aim being to consider open boundary conditions in the x-direction.

As we know the Hamiltonian of the system is as follows.

$$\mathcal{H} = -J\Big[\big(S_3^+ S_5^+ S_4^- S_1^- + S_4^+ S_6^+ S_3^- S_2^- + S_7^+ S_9^+ S_8^- S_5^- + S_8^+ S_{10}^+ S_7^- S_6^- + h.c.\big) + S_{11}^+ S_{13}^+ S_{12}^- S_9^- + S_{12}^+ S_{14}^+ S_{11}^- S_{10}^- + h.c.\Big] \quad -(1)$$

where S_n^+, S_n^- are the raising and lowering operators at ~~site~~ bond n. The Gauss' Law, which commutes with the Hamiltonian is defined on the site M.

$$\left.\begin{aligned}
G_A &= S_3^z + S_1^z - S_{-1}^z - S_2^z ; & G_B &= S_4^z + S_2^z - S_{-2}^z - S_1^z ; \\
G_C &= S_7^z + S_5^z - S_3^z - S_6^z ; & G_D &= S_8^z + S_6^z - S_4^z - S_5^z ; \\
G_E &= S_{11}^z + S_9^z - S_7^z - S_{10}^z ; & G_F &= S_{12}^z + S_{10}^z - S_8^z - S_9^z ; \\
G_G &= S_{16}^z + S_{14}^z - S_{12}^z - S_{13}^z ; & G_H &= S_{15}^z + S_{13}^z - S_{11}^z - S_{14}^z ;
\end{aligned}\right\} \quad -(2)$$

In this example, the links -1, -2, 15 and 16 are spectator links, which enable a proper definition of the gauss law operator; and take values that do not change.

We anticipate that the explicit choice of the boundary links can also affect the bulk physics, and we choose them such that (in the z-basis) the links -1 and -2 are pointing in opposite directions - such that the net electric flux pointing into the bulk is zero. This ~~chooses~~ chooses the zero winding sector in the ~~y~~ y-direction. Also to enable an exact comparison and benchmark with Exact diagonalization we choose the following values:

$$\left.\begin{array}{ll} E_{-2} = 1\,; & E_{-1} = -1. \\ E_{15} = -1\,; & E_{16} = 1. \end{array}\right\} \quad -(3)$$

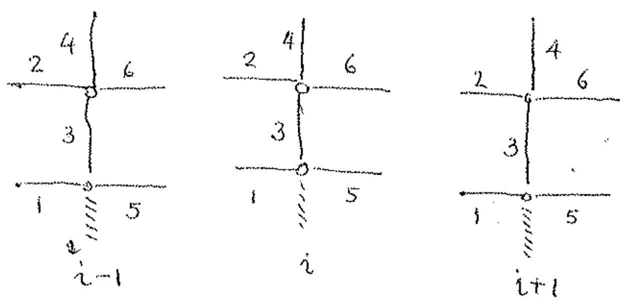
These can be hardcoded in the code using the operators

$$\frac{1}{2^4}(1-\sigma^z_{-2})(1+\sigma^z_{-1})(1+\sigma^z_{15})(1-\sigma^z_{16}) \quad -(4)$$

etc.

Further we need to have OBC, ~~which we ensure~~ with frozen links which we ensure via the use of projection operators as shown above. The further implication of this is that the last two plaquettes marked by a × in their centres are "~~switch~~ switched off" — there are effectively only 6 active plaquettes in the geometry drawn on the other side.

As the next major implementation step for the MPS states, we rewrite the hamiltonian into as local degrees of freedom as possible. We think of a "thick" chain with the following sub-structure:



The spin σ^+ operator on the link 5, site i is then denoted by

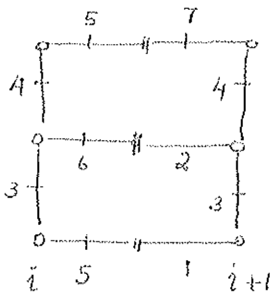
$$\sigma^+_{i,5}$$

Now we write the hamiltonian in the local form.

$$\mathcal{H} = -J\sum_{\square}\left(f_i^{\square} + f_i^{\square\dagger}\right) + \lambda\sum_{\square}\left(f_i^{\square} + f_i^{\square\dagger}\right)^2 \qquad \leftarrow (5)$$

For the moment $\lambda = 0$.

$$\left.\begin{aligned} f_i^{\square} = {} & \mathcal{P}^1_{i,i+1}\left(\sigma^+_{i,5}\,\sigma^+_{i+1,1}\,\sigma^+_{i+1,3}\,\sigma^-_{i+1,2}\,\sigma^-_{i,6}\,\sigma^-_{i,3}\right)\mathcal{P}^2_{i,i+1} \\ & + \mathcal{P}^2_{i,i+1}\left(\sigma^+_{i,6}\,\sigma^+_{i+1,2}\,\sigma^+_{i+1,4}\,\sigma^-_{i+1,1}\,\sigma^-_{i,5}\,\sigma^-_{i,4}\right)\mathcal{P}^1_{i,i+1} \end{aligned}\right\} \quad (6)$$



Clearly these two are the two vertically stacked plaquettes at each value of i. The projection operators are.

$$\left.\begin{aligned} \mathcal{P}^1_{i,i+1} &= \frac{1}{2}\left(1 + \sigma^z_{i,5}\,\sigma^z_{i+1,1}\right) \\ \mathcal{P}^2_{i,i+1} &= \frac{1}{2}\left(1 + \sigma^z_{i,6}\,\sigma^z_{i+1,2}\right) \end{aligned}\right\} \quad (7)$$

The physical effect of the projection operators is to ensure that the spins at (5 and 1) as well as (6 and 2) are physically kept the same. In reality the 6-spin interaction in eq (6) is actually a 4-spin interaction.

Finally, let us ~~talk~~ discuss about the MPS state, and the imposition of the Gauss Law. In the absense of the Gauss Law constant, each local spin-agglomeration has a dimension $2^6 = 64$ states. With a lattice of spatial extent of N_s, this gives a wavefunction which is $(2^6)^{N_s} = 64^{N_s}$ dimensions big. Thus $|\Psi\rangle \sim (64)^{N_s}$

In the MPS form, the wavefn is stored as

$$|\Psi\rangle \sim N_s \cdot \left(\frac{D^2}{2}\right) \cdot d = N_s \left(\frac{D^2}{2}\right) \cdot 64$$

where the symbol $\sim$ denotes the Hilbert space dimension.

$d = 2^6 = 64$ the physical dimensions for each site.

Thus at each site, there is the tensor: $[A^x_{\alpha\beta}]_m$ of dimension $D \times D$.

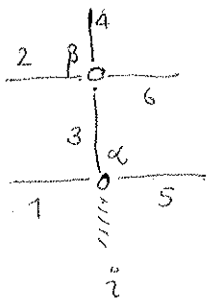
$x \rightarrow$ site of the lattice

$\alpha\beta \rightarrow$ indices of the $D \times D$ matrix; runs from $\alpha \in \{1, \ldots, D\}$, $\beta \in \{1, \ldots, D\}$

$m \rightarrow$ corresponds to the physical index.

Clearly with enough ~~tensors~~ $D \times D$ tensors at each site the full Hilbert space will be covered.

However, we will impose the Gauss Law at each vertex (there are two vertices in each spin-structure at i).



Thus at the physical site i, we have the two vertices α & β, for which we impose

$$\left.\begin{aligned} G_{i,\alpha} &= \sigma^z_{i,3} + \sigma^z_{i,5} - \sigma^z_{i,1} - \sigma^z_{i,4} \\ G_{i,\beta} &= \sigma^z_{i,4} + \sigma^z_{i,6} - \sigma^z_{i,2} - \sigma^z_{i,3} \end{aligned}\right\} \text{— (8)}$$

Symbolically,

$$|\Psi_P\rangle = P_G |\Psi\rangle \quad \text{— (9)}$$

The initial state created as an MPS can then be immediately projected onto the gauge invariant subspace — one since the Hamiltonian commutes with the Gauss law:

$$[\mathcal{H}, P_G] = 0 \quad \Rightarrow \quad [H, G_\alpha] = 0 \ \& \ [H, G_\beta] = 0$$

for all i.