1 Hamiltonian of QLM as a spin system

We define the Square Ice Hamiltonian

$$H = \sum_{\square}(-f_{\square} + \lambda f_{\square}^2)\,, \tag{1}$$

where we some over all plaquettes. The plaquette operator is defined as:

$$f_{\square} = \sigma^+_{\mu_1}\sigma^+_{\mu_2}\sigma^-_{\mu_3}\sigma^-_{\mu_4} \ + \sigma^-_{\mu_1}\sigma^-_{\mu_2}\sigma^+_{\mu_3}\sigma^+_{\mu_4} \tag{2}$$

$$f_{\square}^2 = \sigma^+_{\mu_1}\sigma^-_{\mu_1}\ \sigma^+_{\mu_2}\sigma^-_{\mu_2}\ \sigma^-_{\mu_3}\sigma^+_{\mu_3}\ \sigma^-_{\mu_4}\sigma^+_{\mu_4} \ + hc \tag{3}$$

If we define $p+$ and $p+$ as:

$$p+ = \frac{\mathbb{1} + \sigma^z}{2}\ ;\ p- = \frac{\mathbb{1} - \sigma^z}{2} \tag{4}$$

I have:

$$f_{\square}^2 = p^+_{\mu_1}p^+_{\mu_2}p^-_{\mu_3}p^-_{\mu_4} \ + p^-_{\mu_1}p^-_{\mu_2}p^+_{\mu_3}p^+_{\mu_4} \tag{5}$$