

Tensor Study of Quantum Link Model

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1 Hamiltonian of QLM as a spin system

We want to study the Square Ice Hamiltonian

$$\mathcal{H} = \sum_{\square} (-f_{\square} + \lambda f_{\square}^2), \quad (1)$$

as the sum over all plaquettes:

$$f_{\square} = \sigma_{\mu_1}^+ \sigma_{\mu_2}^+ \sigma_{\mu_3}^- \sigma_{\mu_4}^- + \sigma_{\mu_1}^- \sigma_{\mu_2}^- \sigma_{\mu_3}^+ \sigma_{\mu_4}^+. \quad (2)$$

on a long cylindrical lattice

$$\Omega = \{\mu = (n, m) | n \in \{1, \dots, L_x\}, m \in \{1, \dots, L_y\}\} \quad (3)$$

displayed in fig. 1.

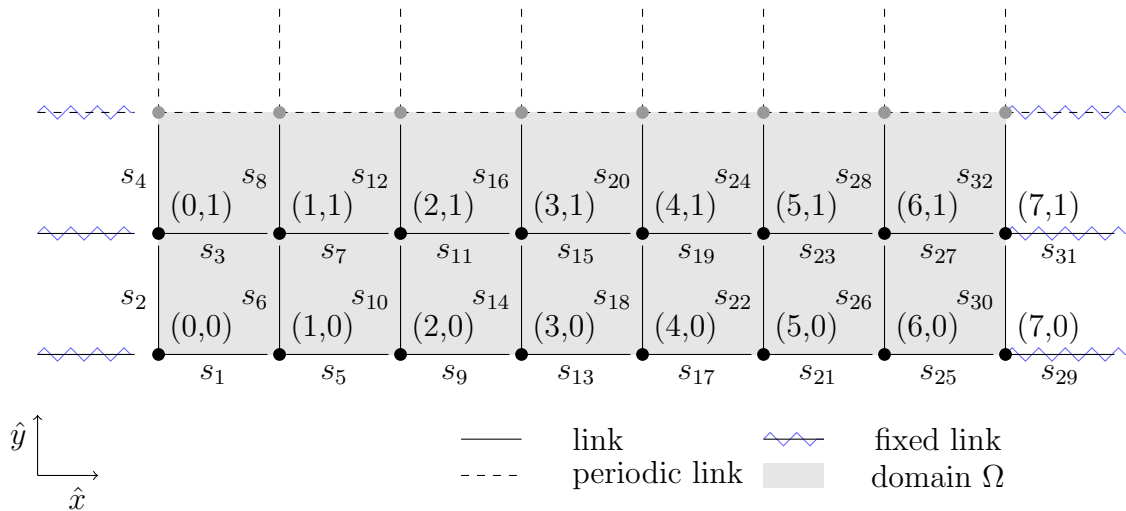


Figure 1: Definition of the computational mesh

In the conventional up/down $s^\pm \in \mathbb{R}^2$ ($[1\ 0]/[0\ 1]$) basis the link operators σ^\pm are pauli matrices :

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4)$$

In all our lattices $L_y \ll L_x$. For now we fix $L_y = 2$ as in fig. 1. Before explaining the computations we want to point out some properties of the Hamiltonian.

1.1 Mathematical Properties of the Quantum Link Model

Gauss Law: The Hamiltonian \mathcal{H} in eq. (1) commutes with the *vertex operator* G_μ , which counts the number of in and outgoing arrows at vertex μ . We can therefore fix the total charge at each vertex with the *Gauss law constraint*:

$$G_\mu = 0 \quad \text{for all} \quad \mu \in \Omega, \quad (5)$$

$$G_\mu := \sum_{\hat{i} \in \{\hat{x}, \hat{y}\}} (s_{\nu - \hat{i}/2} - s_{\nu + \hat{i}/2}). \quad (6)$$

Winding Numbers

$$W_y = \frac{1}{2L_y} \sum_{\mu} E_{\mu, x} \quad (7)$$

Fluxes

1.2 Hilbert-Space

In the absence of the ice rule eq.(5) the hilbertspace becomes $2^{2 \cdot L_x L_y}$ dimensional and the linear combination of every state is given by:

$$|\psi\rangle = \sum_{i_1, i_2, \dots, i_{L_x}} A_{i_1, i_2, \dots, i_{L_x}} |i_1\rangle |i_2\rangle \dots |i_{L_x}\rangle \quad (8)$$

where $i_n = 1, 2, \dots, 2^{2L_y}$ labels the corresponding quantum state at site n . For the $L_y = 2$ we thus have 16 different quantum states at each site $|i_n\rangle = |(s_1, s_2, s_3, s_4)\rangle$, where $s_i \in \{0, 1\}$ labels the i th spin in the local basis drawn in fig. 2. The 16 different combinations in the set can be explicitly written down:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \quad (9)$$

The number of elements in the set will be also referred to as local hilbertspace dimension D .

2 Computational Basis

For the MPS we have to rewrite the Hamiltonian of the system in the nearest neighbour setting. The local Hamiltonian $H_{n,n+1}$ thus defines the interaction between the states at site $|i_n\rangle$ and $|i_{n+1}\rangle$. The Hamilton operator (1) consists of 4 terms. Where on each site we have $m = 1, \dots, L_y$ possible interactions. Thus the hamiltonian consists of $4L_y$ Kronecker products:

$$H_{n,n+1} = \sum_{j=1}^4 \sum_{m=1}^{L_y} h_{\sqsubset,n,m}^{(j)} \otimes h_{\sqsupset,n+1,m}^{(j)} \quad (10)$$

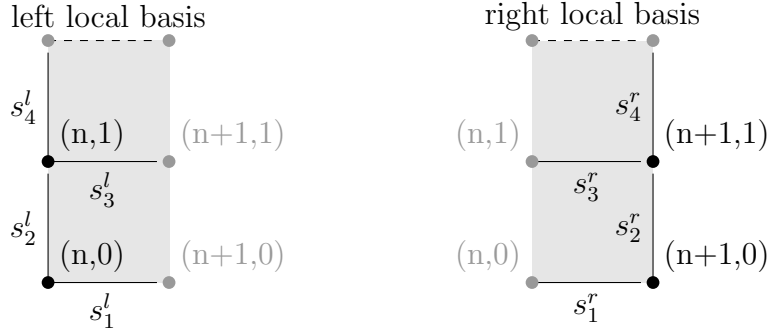


Figure 2: Definition of the computational mesh

To identify the different local interaction terms in the Hamilton operator (10) with (1) we rewrite the plaquette-operator into our computational basis $|i_n\rangle$. A plaquette operator defines our nearest neighbor interaction between state $|i_n\rangle$ and $|i_{n+1}\rangle$

$$f_{\square} = f_{\sqsubset,n,m} \otimes f_{\sqsupset,n,m} + h.c. \quad (11)$$

$$f_{\sqsubset,n,m} = \sigma_{r,n,m+1}^- \sigma_{v,n,m+1}^- \sigma_{r,n,m}^+ \quad (12)$$

$$f_{\sqsupset,n+1,m} = \sigma_{l,n+1,m+1}^+ \sigma_{v,n+1,m+1}^+ \sigma_{l,n+1,m}^- \quad (13)$$

Comparing this to (10) yields:

$$h_{\sqsubset,n,m}^{(1)} = -f_{\sqsubset,n,m} \quad h_{\sqsupset,n+1,m}^{(1)} = f_{\sqsupset,n+1,m} \quad (14)$$

$$h_{\sqsubset,n,m}^{(2)} = -f_{\sqsubset,n,m}^\dagger \quad h_{\sqsupset,n+1,m}^{(2)} = f_{\sqsupset,n+1,m}^\dagger \quad (15)$$

$$h_{\sqsubset,n,m}^{(3)} = \lambda f_{\sqsubset,n,m}^\dagger f_{\sqsubset,n,m} \quad h_{\sqsupset,n+1,m}^{(3)} = f_{\sqsupset,n+1,m}^\dagger f_{\sqsupset,n+1,m} \quad (16)$$

$$h_{\sqsubset,n,m}^{(4)} = \lambda f_{\sqsubset,n,m} f_{\sqsubset,n,m}^\dagger \quad h_{\sqsupset,n+1,m}^{(4)} = f_{\sqsupset,n+1,m} f_{\sqsupset,n+1,m}^\dagger \quad (17)$$

$$(18)$$

For example in our $L_y = 2$ system we get 64×64 size Operators :

$$h_{\sqsubset,n,m}^{(1)} = -\sigma^+ \otimes \sigma^- \otimes \sigma^+ \otimes I_2 \otimes I_2 \otimes I_2 \in \mathbb{R}^{2^6, 2^6} \quad (19)$$

$$h_{\sqsupset,n,m}^{(1)} = I_2 \otimes I_2 \otimes \sigma^+ \otimes I_2 \otimes \sigma^- \otimes \sigma^+ \in \mathbb{R}^{2^6, 2^6} \quad (20)$$

$$(21)$$

Note that this already inherits the periodicity in \hat{y} . For the chosen up/down ($[1\ 0]/[0\ 1]$) basis the link operators are given by:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (22)$$

$$f_{\square}^2 = \sigma_{\mu_1}^+ \sigma_{\mu_1}^- \sigma_{\mu_2}^+ \sigma_{\mu_2}^- \sigma_{\mu_3}^+ \sigma_{\mu_3}^- \sigma_{\mu_4}^+ \sigma_{\mu_4}^- + hc \quad (23)$$

If we define p_+ and p_- as:

$$p_+ = \frac{1 + \sigma^z}{2} ; p_- = \frac{1 - \sigma^z}{2} \quad (24)$$

I have:

$$f_{\square}^2 = p_{\mu_1}^+ p_{\mu_2}^+ p_{\mu_3}^- p_{\mu_4}^- + p_{\mu_1}^- p_{\mu_2}^- p_{\mu_3}^+ p_{\mu_4}^+ \quad (25)$$

2.1 Todos

- Hamiltonian in external magnetic field, $\phi_{\square} \in \mathbb{R}$. Therefore we define the generalized plaquette operator

$$f(\phi_{\square}) := u_{\square} e^{i\phi_{\square}} + u_{\square}^{\dagger} e^{-i\phi_{\square}} \quad (26)$$

and plug it in (1)

- Winding number operators

$$W_y = \quad (27)$$

3 Order Parameters

To detect the phase transitions, we study the so-called sublattice magnetization ($\mathcal{M}_A, \mathcal{M}_B$) which are defined as follows:

$$\mathcal{M}_A(x) = \mathbb{P}_{x,\mu}^+ \mathbb{P}_{x+\mu,\nu}^+ \mathbb{P}_{x+\nu,\mu}^- \mathbb{P}_{x,\nu}^- - \mathbb{P}_{x,\mu}^- \mathbb{P}_{x+\mu,\nu}^- \mathbb{P}_{x+\nu,\mu}^+ \mathbb{P}_{x,\nu}^+ \quad (28)$$

where $\mathbb{P}_{x,\mu}^+$ and $\mathbb{P}_{x,\mu}^-$ are the projection operators on the spin components $S^z = \pm \frac{1}{2}$ respectively.

4 Numerical Results and Simulation Parameters

Table 1: Parameter sets for all Simulations

Parameters	Simulation 1	Simulation 2
Vertical grid size L_x	20,40,60,100,200	60
Horizontal grid size L_y	2	2
Coupling λ	$[-4.0, -3.5, -3.0, \dots, -1.0]$	-1.0
Magnetic field angle θ	0	$\theta_k = \frac{\pi}{4}k, k = 0, 1, \dots, 8$

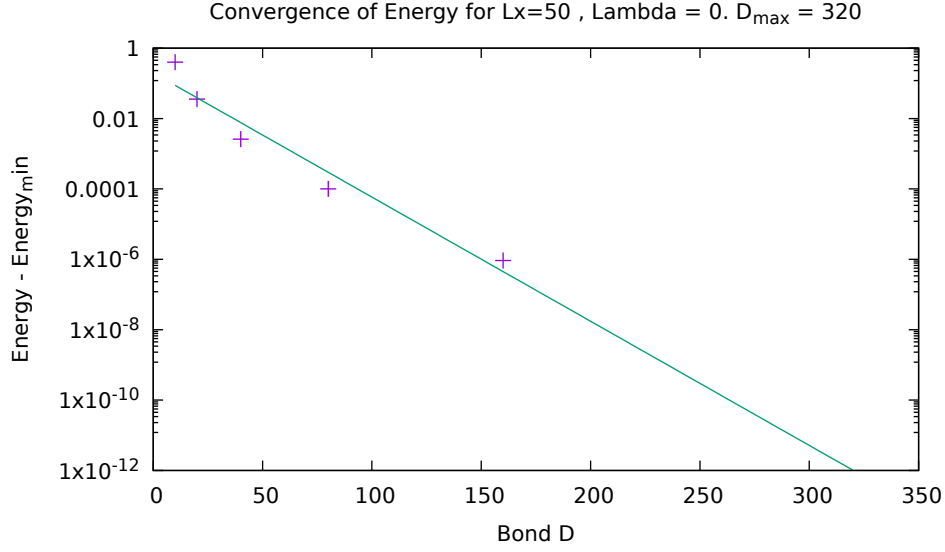


Figure 3: Extrapolation of the bond dimension

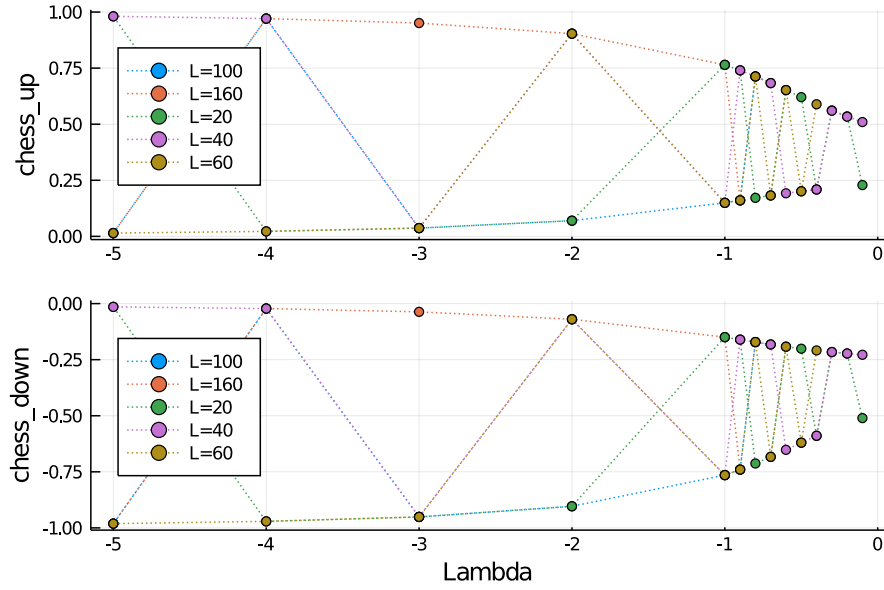


Figure 4: Chess operators