

# Tensor Study of Quantum Link Model

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## 1 Hamiltonian of QLM as a spin system

We want to study the Square Ice Hamiltonian

$$\mathcal{H} = \sum_{\square} (-f_{\square} + \lambda f_{\square}^2), \quad (1)$$

as the sum over all plaquettes:

$$f_{\square} = \sigma_{\mu_1}^+ \sigma_{\mu_2}^+ \sigma_{\mu_3}^- \sigma_{\mu_4}^- + \sigma_{\mu_1}^- \sigma_{\mu_2}^- \sigma_{\mu_3}^+ \sigma_{\mu_4}^+. \quad (2)$$

on a long cylindrical lattice

$$\Omega = \{\mu = (n, m) | n \in \{1, \dots, L_x\}, m \in \{1, \dots, L_y\}\} \quad (3)$$

displayed in fig. 1.

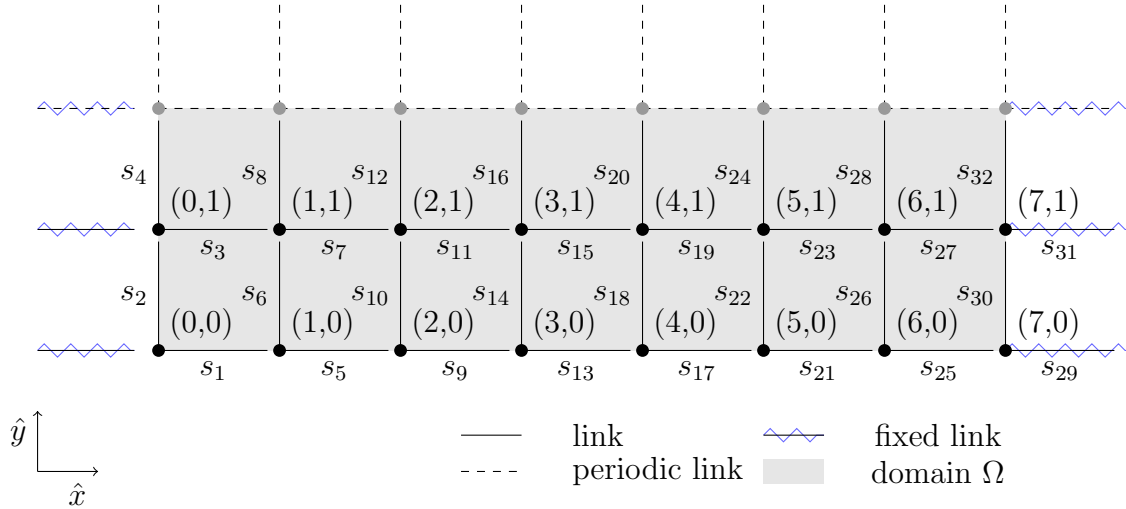


Figure 1: Definition of the computational mesh

In the conventional up/down  $s^{\pm} \in \mathbb{R}^2$  ( $[1 \ 0]/[0 \ 1]$ ) basis the link operators  $\sigma^{\pm}$  are pauli matrices :

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4)$$

In all our lattices  $L_y \ll L_x$ . For now we fix  $L_y = 2$  as in fig. 1. Before explaining the computations we want to point out some properties of the Hamiltonian.

## 1.1 Mathematical Properties of the Quantum Link Model

**Gauss Law:** The Hamiltonian  $\mathcal{H}$  in eq. (1) commutes with the *vertex operator*  $G_\mu$ , which counts the number of in and outgoing arrows at vertex  $\mu$ . We can therefore fix the total charge at each vertex with the *Gauss law constraint*:

$$G_\mu = 0 \quad \text{for all} \quad \mu \in \Omega, \quad (5)$$

$$G_\mu := \sum_{\hat{i} \in \{\hat{x}, \hat{y}\}} (s_{\nu - \hat{i}/2} - s_{\nu + \hat{i}/2}). \quad (6)$$

### Winding Numbers

$$W_y = \frac{1}{2L_y} \sum_{\mu} E_{\mu, x} \quad (7)$$

### Fluxes

## 1.2 Hilbert-Space

In the absence of the ice rule eq.(5) the hilbertspace becomes  $2^{2 \cdot L_x L_y}$  dimensional and the linear combination of every state is given by:

$$|\psi\rangle = \sum_{i_1, i_2, \dots, i_{L_x}} A_{i_1, i_2, \dots, i_{L_x}} |i_1\rangle |i_2\rangle \dots |i_{L_x}\rangle \quad (8)$$

where  $i_n = 1, 2, \dots, 2^{2L_y}$  labels the corresponding quantum state at site  $n$ . For the  $L_y = 2$  we thus have 16 different quantum states at each site  $|i_n\rangle = |(s_1, s_2, s_3, s_4)\rangle$ , where  $s_i \in \{0, 1\}$  labels the  $i$ th spin in the local basis drawn in fig. 2. The 16 different combinations in the set can be explicitly written down:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \quad (9)$$

The number of elements in the set will be also referred to as local hilbertspace dimension  $D$ .

## 2 Computational Basis

For the MPS we have to rewrite the Hamiltonian of the system in the nearest neighbour setting. The local Hamiltonian  $H_{n,n+1}$  thus defines the interaction between the states at site  $|i_n\rangle$  and  $|i_{n+1}\rangle$ . The Hamilton operator (1) consists of 4 terms. Where on each site we have  $m = 1, \dots, L_y$  possible interactions. Thus the hamiltonian consists of  $4L_y$  Kronecker products:

$$H_{n,n+1} = \sum_{j=1}^4 \sum_{m=1}^{L_y} h_{\sqsubset,n,m}^{(j)} \otimes h_{\sqsupset,n+1,m}^{(j)} \quad (10)$$

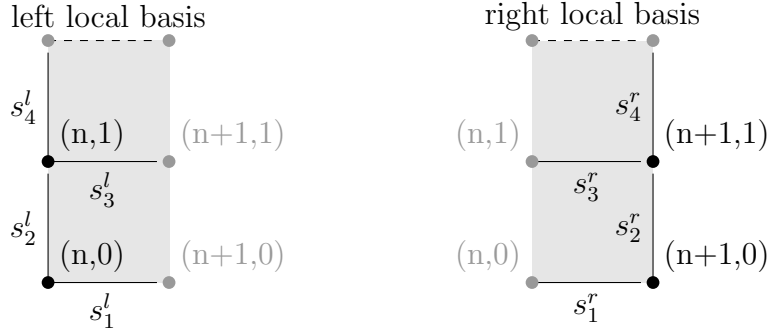


Figure 2: Definition of the computational mesh

To identify the different local interaction terms in the Hamilton operator (10) with (1) we rewrite the plaquette-operator into our computational basis  $|i_n\rangle$ . A plaquette operator defines our nearest neighbor interaction between state  $|i_n\rangle$  and  $|i_{n+1}\rangle$

$$f_{\square} = f_{\sqsubset,n,m} \otimes f_{\sqsupset,n,m} + h.c. \quad (11)$$

$$f_{\sqsubset,n,m} = \sigma_{r,n,m+1}^- \sigma_{v,n,m+1}^- \sigma_{r,n,m}^+ \quad (12)$$

$$f_{\sqsupset,n+1,m} = \sigma_{l,n+1,m+1}^+ \sigma_{v,n+1,m+1}^+ \sigma_{l,n+1,m}^- \quad (13)$$

Comparing this to (10) yields:

$$h_{\sqsubset,n,m}^{(1)} = -f_{\sqsubset,n,m} \quad h_{\sqsupset,n+1,m}^{(1)} = f_{\sqsupset,n+1,m} \quad (14)$$

$$h_{\sqsubset,n,m}^{(2)} = -f_{\sqsubset,n,m}^\dagger \quad h_{\sqsupset,n+1,m}^{(2)} = f_{\sqsupset,n+1,m}^\dagger \quad (15)$$

$$h_{\sqsubset,n,m}^{(3)} = \lambda f_{\sqsubset,n,m}^\dagger f_{\sqsubset,n,m} \quad h_{\sqsupset,n+1,m}^{(3)} = f_{\sqsupset,n+1,m}^\dagger f_{\sqsupset,n+1,m} \quad (16)$$

$$h_{\sqsubset,n,m}^{(4)} = \lambda f_{\sqsubset,n,m} f_{\sqsubset,n,m}^\dagger \quad h_{\sqsupset,n+1,m}^{(4)} = f_{\sqsupset,n+1,m} f_{\sqsupset,n+1,m}^\dagger \quad (17)$$

$$(18)$$

For example in our  $L_y = 2$  system we get  $64 \times 64$  size Operators :

$$h_{\sqsubset,n,m}^{(1)} = -\sigma^+ \otimes \sigma^- \otimes \sigma^+ \otimes I_2 \otimes I_2 \otimes I_2 \in \mathbb{R}^{2^6, 2^6} \quad (19)$$

$$h_{\sqsupset,n,m}^{(1)} = I_2 \otimes I_2 \otimes \sigma^+ \otimes I_2 \otimes \sigma^- \otimes \sigma^+ \in \mathbb{R}^{2^6, 2^6} \quad (20)$$

$$(21)$$

Note that this already inherits the periodicity in  $\hat{y}$ . For the chosen up/down ( $[1\ 0]/[0\ 1]$ ) basis the link operators are given by:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (22)$$

$$f_{\square}^2 = \sigma_{\mu_1}^+ \sigma_{\mu_1}^- \sigma_{\mu_2}^+ \sigma_{\mu_2}^- \sigma_{\mu_3}^+ \sigma_{\mu_3}^- \sigma_{\mu_4}^+ \sigma_{\mu_4}^- + hc \quad (23)$$

If we define  $p_+$  and  $p_-$  as:

$$p_+ = \frac{1 + \sigma^z}{2} ; p_- = \frac{1 - \sigma^z}{2} \quad (24)$$

I have:

$$f_{\square}^2 = p_{\mu_1}^+ p_{\mu_2}^+ p_{\mu_3}^- p_{\mu_4}^- + p_{\mu_1}^- p_{\mu_2}^- p_{\mu_3}^+ p_{\mu_4}^+ \quad (25)$$

## 2.1 Todos

- Hamiltonian in external magnetic field,  $\phi_{\square} \in \mathbb{R}$ . Therefore we define the generalized plaquette operator

$$f(\phi_{\square}) := u_{\square} e^{i\phi_{\square}} + u_{\square}^{\dagger} e^{-i\phi_{\square}} \quad (26)$$

and plug it in (1)

- Winding number operators

$$W_y = \quad (27)$$

## 3 Order Parameters

To detect the phase transitions, we study the so-called sublattice magnetization ( $\mathcal{M}_A, \mathcal{M}_B$ ) which are defined as follows:

$$\mathcal{M}_A(x) = \mathbb{P}_{x,\mu}^+ \mathbb{P}_{x+\mu,\nu}^+ \mathbb{P}_{x+\nu,\mu}^- \mathbb{P}_{x,\nu}^- - \mathbb{P}_{x,\mu}^- \mathbb{P}_{x+\mu,\nu}^- \mathbb{P}_{x+\nu,\mu}^+ \mathbb{P}_{x,\nu}^+ \quad (28)$$

where  $\mathbb{P}_{x,\mu}^+$  and  $\mathbb{P}_{x,\mu}^-$  are the projection operators on the spin components  $S^z = \pm \frac{1}{2}$  respectively.

## 4 Numerical Results and Simulation Parameters

Table 1: Parameter sets for all Simulations

Parameters	Simulation 1	Simulation 2
Vertical grid size $L_x$	20,40,60,100,200	60
Horizontal grid size $L_y$	2	2
Coupling $\lambda$	$[-4.0, -3.5, -3.0, \dots, -1.0]$	-1.0
Magnetic field angle $\theta$	0	$\theta_k = \frac{\pi}{4}k, k = \{0, 1, \dots, 8\}$

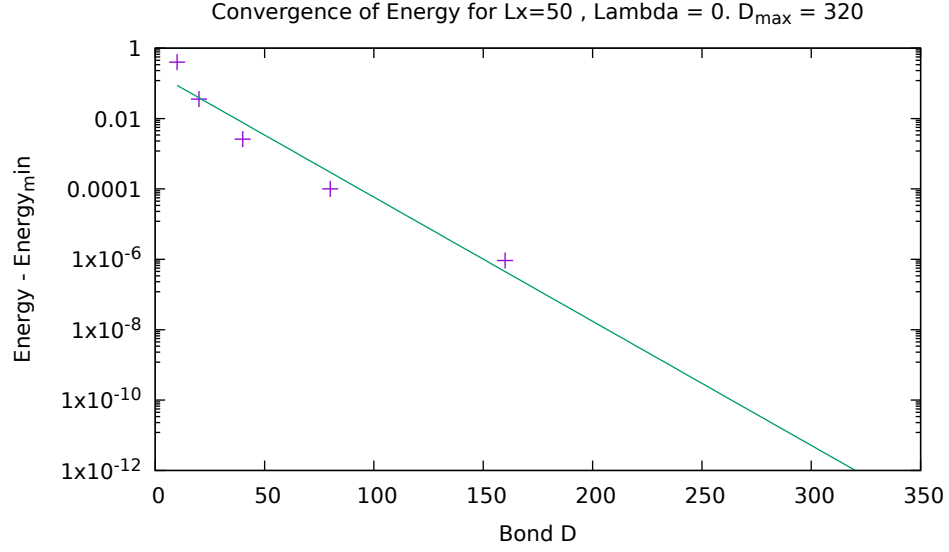


Figure 3: Extrapolation of the bond dimension