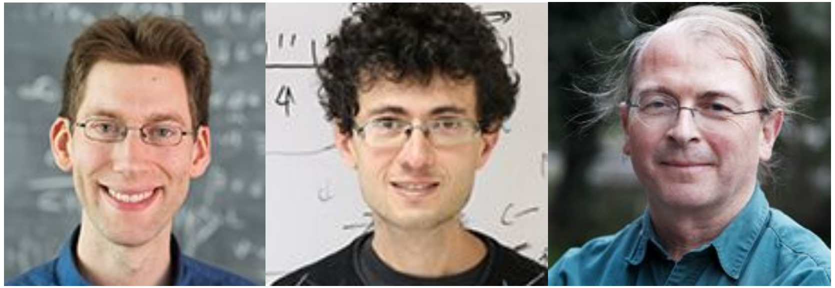
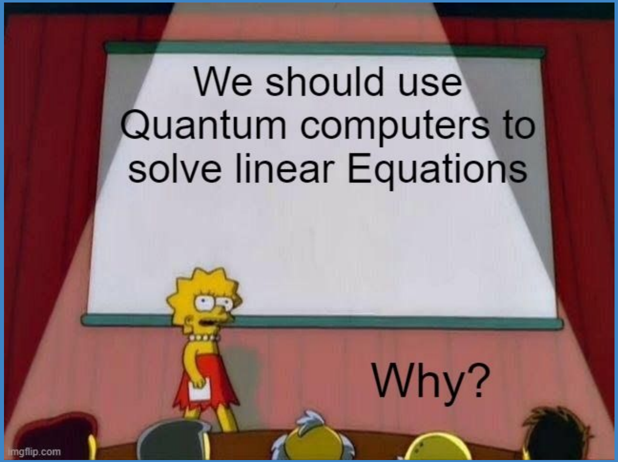


Harrow, Hassidim, and Lloyd: Quantum algorithm for linear systems of equations (2009)

HHL Algorithm

Max Kern, Philipp Krüger, Simone Spedicato

1. Introduction
2. Theory
3. Implementation
4. Results
5. The Fine Print
6. Summary and Outlook



1. Introduction

Example: Solving Discretized PDEs numerically

Finance: Black-Scholes Equation

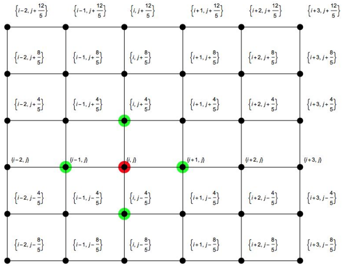
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Mechanics: Navier Stokes Equation

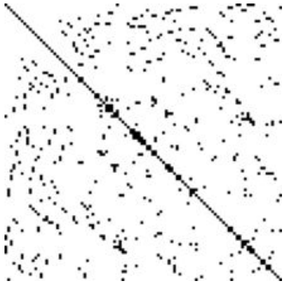
$$\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \nabla\cdot(\rho\,\mathbf{u}\otimes\mathbf{u}) = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho\,\mathbf{g}$$

… and many more!

Finite Difference/Element Method



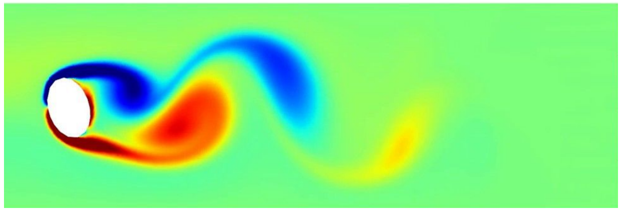
⇒ Sparse matrix A



Numerical Problem as a System
linear equations

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$



Linear System Problems: Classical vs. Quantum

Classical Vector:

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Conjugate gradient method:
minimizes $|A\vec{x} - \vec{b}|^2$

Complexity: $O(Ns\kappa \log(1/\epsilon))$

- Best classical algorithm: Conjugate gradient method
- returns full solution
- superior error scaling

QM Vector:

$$\hat{A}\,|x\rangle = |b\rangle$$ A is Hermitian, sparse

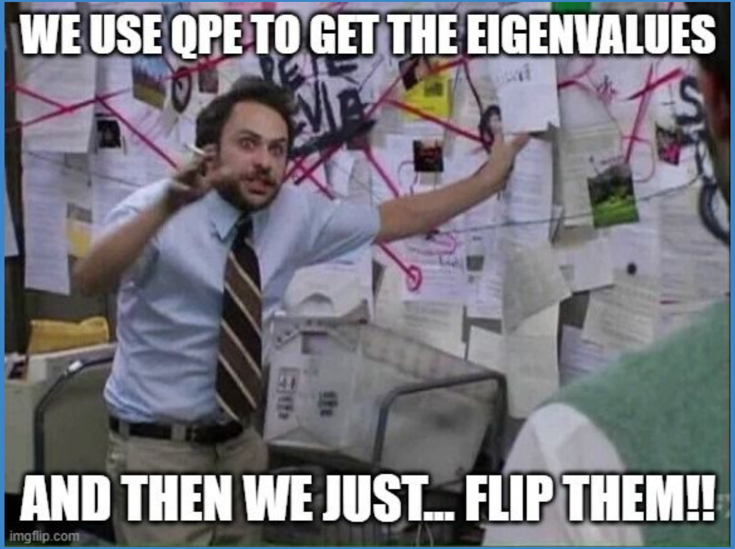
$$|x\rangle = \boxed{\hat{A}^{-1}}\,|b\rangle$$ Solution with precision $\epsilon \geq 0$

HHL method:
inverts A using QPE

Complexity: $O(\boxed{\log(N)}s^2\kappa^2/\epsilon)$ $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$

- **Exponential speed up**: log(N)
- we don't obtain the vector, only expectation values $\langle x| M |x\rangle$
- Obtaining complete x would scale linearly in N again

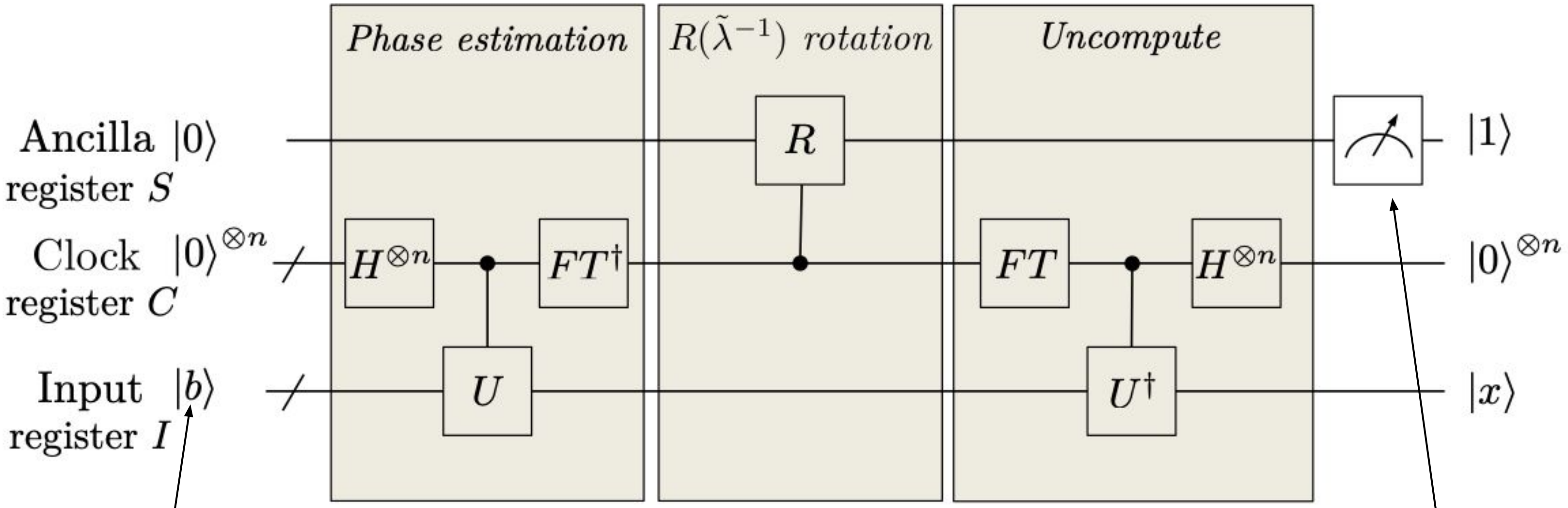
Harrow et al. arXiv:0811.3171 (2009)


WE USE QPE TO GET THE EIGENVALUES
AND THEN WE JUST... FLIP THEM!!
imgflip.com

2. Theory

Theory: The Algorithm

Dervovic et al. arXiv:1802.08227v1 (2018)



1. load data

2. apply QPE

3. rotate ancilla conditioned on $|\lambda_j\rangle$

4. apply $\mathrm{QPE}^{\dagger}$

5. measure

Theory: Inverting Hermitian Matrices

Problem: $\hat{A}\left|x\right\rangle = \left|b\right\rangle \quad \Rightarrow \quad \left|x\right\rangle = \hat{A}^{-1}\left|b\right\rangle$

$$\hat{A} = \hat{A}^{\dagger} \;\Rightarrow\; \hat{A} = \sum_{j=0}^{N-1} \lambda_j \left|u_j\right\rangle\left\langle u_j\right|,\; \lambda_j \in \mathbb{R} \quad \text{with} \quad \hat{A}\left|u_j\right\rangle = \lambda_j \left|u_j\right\rangle$$

$$\Rightarrow \boxed{\hat{A}^{-1} = \sum_{j=0}^{N-1} \lambda_j^{-1} \left|u_j\right\rangle\left\langle u_j\right|}$$

$\left|b\right\rangle$ in eigenbasis of $\hat{A}$: $\boxed{\left|b\right\rangle = \sum_{j=0}^{N-1} b_j \left|u_j\right\rangle,\; b_j \in \mathbb{C}}$

$$\boxed{\left|x\right\rangle = \hat{A}^{-1}\left|b\right\rangle = \sum_{j=0}^{N-1} \frac{b_j}{\lambda_j} \left|u_j\right\rangle}$$

Theory: Exact Quantum Phase Estimation (QPE)

Project an initial state onto the eigenbasis of $\hat{A}$

QPE: - inputs: $\hat{U}$ and $|0\rangle_n |\psi\rangle_m$, with $\hat{U}|\psi_j\rangle_m = e^{2\pi i\theta_j}|\psi_j\rangle_m$

- output: state $|\tilde{\theta}\rangle_n |\psi\rangle_m$, $\tilde{\theta}$ binary approx. to $2^n\theta$, truncated to n digits

$$\mathrm{QPE}(\hat{U}, |0\rangle_n |\psi\rangle_m) = |\tilde{\theta}\rangle_n |\psi\rangle_m \qquad \hat{U} := e^{i\hat{A}t} = \sum_{j=0}^{N-1} e^{i\lambda_j t} |u_j\rangle \langle u_j|$$

$$\boxed{\mathrm{QPE}\left(e^{i\hat{A}t}, \sum_{j=0}^{N-1} b_j |0\rangle_n |u_j\rangle_m\right) = \sum_{j=0}^{N-1} b_j |\lambda_j\rangle_n |u_j\rangle_m}$$

Theory: Conditioned Rotation

eventual goal

$$\boxed{|x\rangle = \hat{A}^{-1}|b\rangle = \sum_{j=0}^{N-1} \frac{b_j}{\lambda_j}|u_j\rangle}$$

$$\sum_{j=0}^{N-1} \underbrace{b_j|\lambda_j\rangle|u_j\rangle}_{\text{initial state}} \otimes \underbrace{|0\rangle}_{\text{added ancilla}} \longrightarrow \sum_{j=0}^{N-1} b_j|\lambda_j\rangle|u_j\rangle|0\rangle$$

apply $e^{-i\theta Y}$ on ancilla $|0\rangle$,

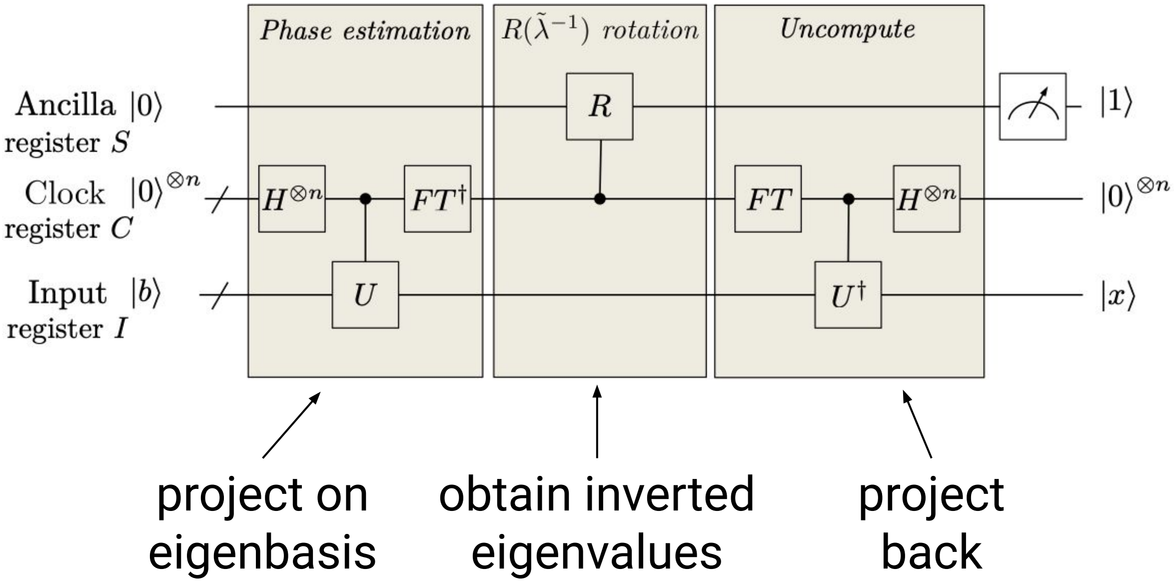
conditioned on $|\lambda_j\rangle$

result after conditioned rotation:

$$\boxed{\boxed{\sum_{j=0}^{N-1} b_j|\lambda_j\rangle|u_j\rangle}\left(\sqrt{1-\frac{C^2}{\lambda_j^2}}|0\rangle + \boxed{\frac{C}{\lambda_j}|1\rangle}\right)}$$

Theory: Summary

Dervovic et al. arXiv:1802.08227v1 (2018)

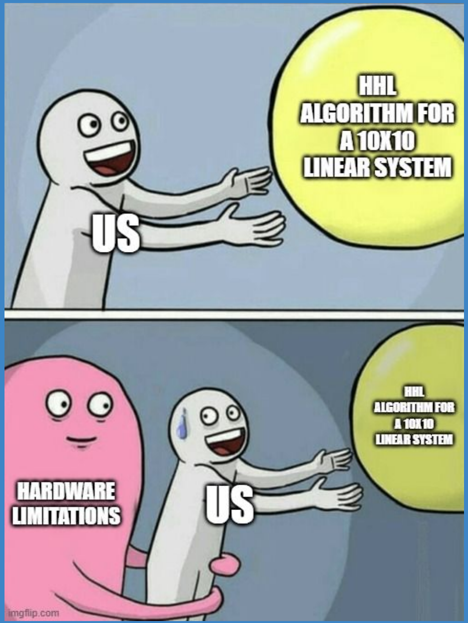


- Map linear problem to QM: $\hat{A}\,|x\rangle = |b\rangle$
- Exponentially faster than any known classical algorithm

Danial Dervovic et al., 2018, Quantum linear systems algorithms: a primer

Qiskit: An Open-source Framework for Quantum Computing, 2019

Qiskit tutorial: https://qiskit.org/textbook/ch-applications/hhl_tutorial.html



3. Implementation

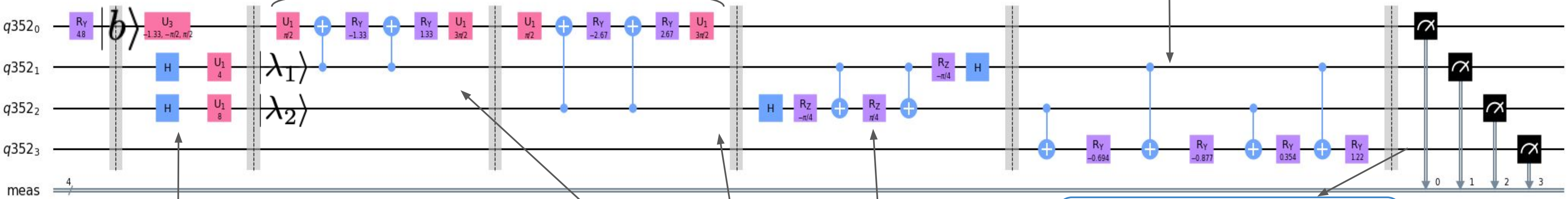
HHL: Optimized Circuit

$$A\vec{x} = \vec{b}(\theta)$$

state initialization through rotation by 2θ around y

Conditioned rotation to rescale eigenvalues

Eigenvalue rotation to measure the ancilla qubit in the computational basis



Rotation conditioned on $|\lambda_1\rangle$ and $|\lambda_2\rangle$

QPE

$\mathrm{QPE}^\dagger$

$$|x\rangle = \hat{A}^{-1}|b\rangle = \sum_{j=0}^{N-1} \frac{b_j}{\lambda_j}|u_j\rangle$$

Ancilla measurement of prob to be in $|1\rangle$, which corresponds to C/λ_j, as can be see from the state

$$\sum_{j=0}^{N-1} b_j |u_j\rangle \left(\sqrt{1 - \frac{C^2}{\lambda_j^2}}\,|0\rangle + \frac{C}{\lambda_j}|1\rangle\right)$$

A Tailored Problem for an Optimized Circuit

$$A\vec{x} = \vec{b}(\theta)$$

$$\begin{pmatrix} 1 & -1/3 \\ -1/3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$\rightarrow$ Analytical solution:

$$|x|^2 = 0.625 + 0.375 \sin(2\theta)$$

General vs. Optimized algorithm

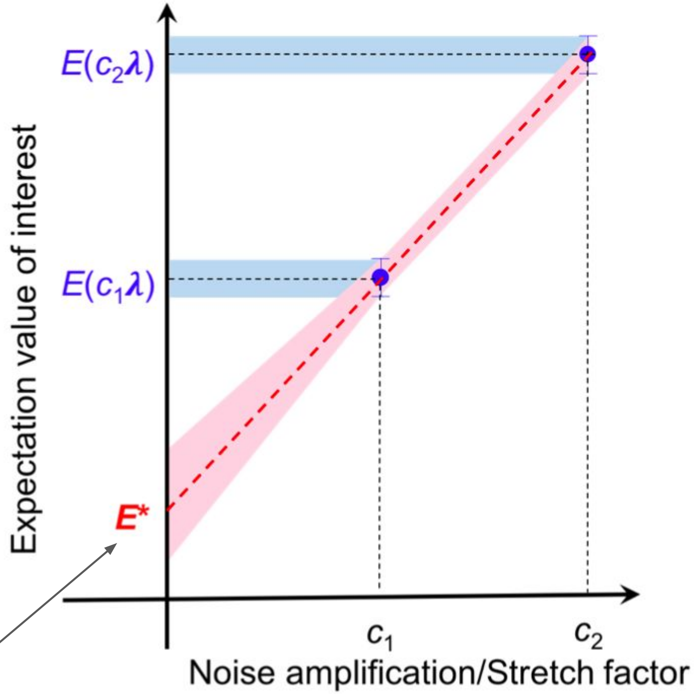
Width:	7	vs. 4 (1 ancilla instead of 3)
Depth:	102	vs. 26
CNOTs:	54	vs. 10 (factor of 5)

$\rightarrow$ Mitigating readout error with *Richardson extrapolation* of error amplification

Qiskit: An Open-source Framework for Quantum Computing, 2019

Qiskit tutorial: https://qiskit.org/textbook/ch-applications/hhl_tutorial.html

Richardson Extrapolation



zero noise
expectation value

$\propto$ #CNOTs

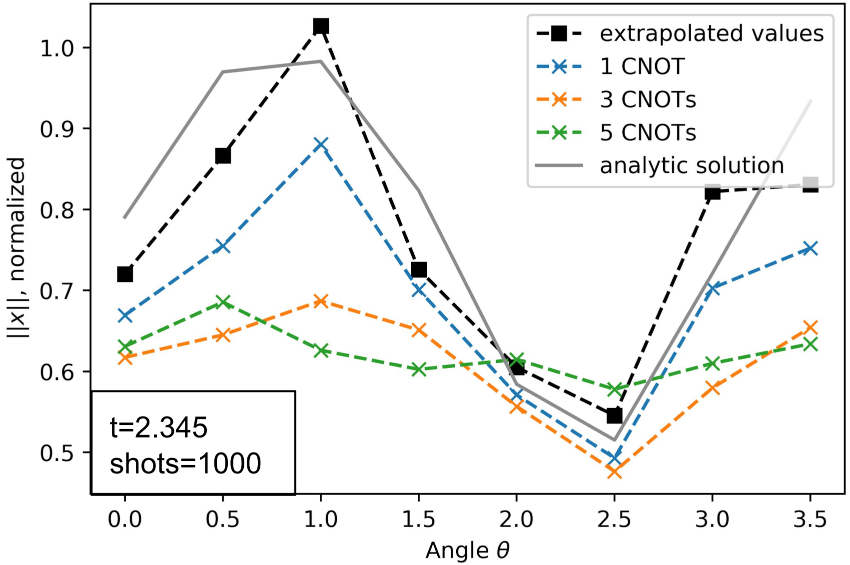
The expectation values of the observables scale quadratically with the error:

$$\boxed{\langle \mathcal{O}(r) \rangle = Ar^2 + Br + \langle \mathcal{O}(0) \rangle}$$

Abhinav Kandala et al. arXiv:1805.04492 (2019)
Kübra Yeter-Aydeniz et al. arXiv:1912.06226 (2019)

4. Results

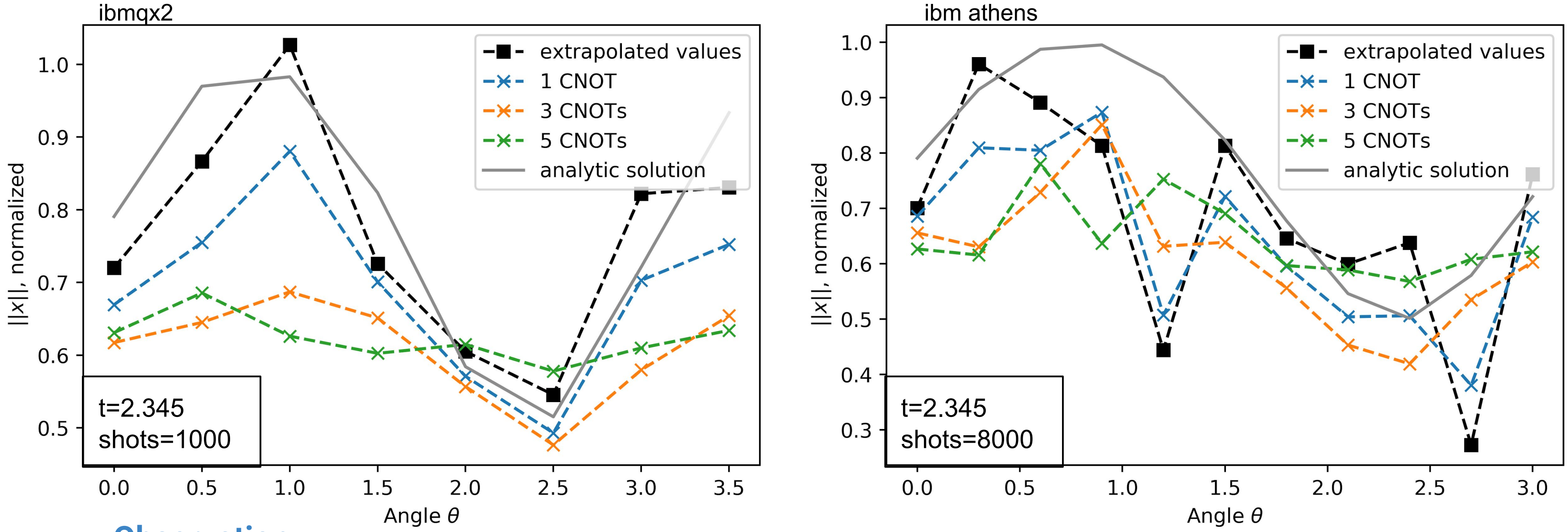
Results: Richardson Extrapolation



Conclusions

- An increasing amount of CNOTs amplified the error.
- We were able to improve our results with Richardson extrapolation and received results that were very similar to the analytic solution.

Results for IBMQX2 vs IBM Athens in Comparison



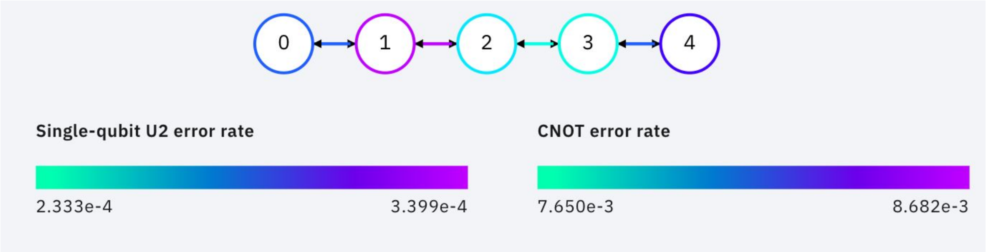
Observation

- The amount of CNOTs used impacted the error on both quantum computers.
- **IBMQX2:** results improved with Richardson extrapolation using 1000 shots.
- **IBM ATHENS:** results worst, even with 8000 of shots.

Testing the Circuit on a different Quantum Computer

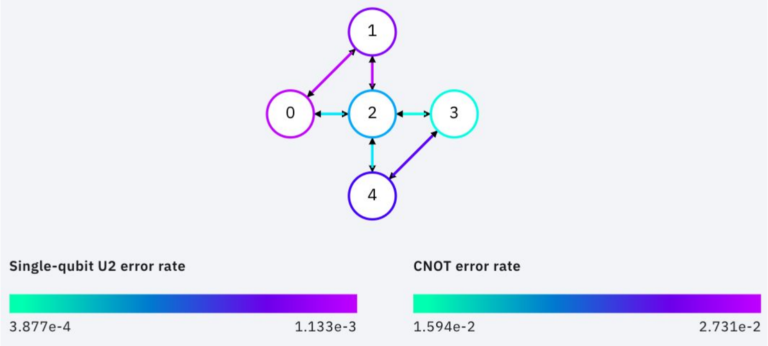
IBM Athens

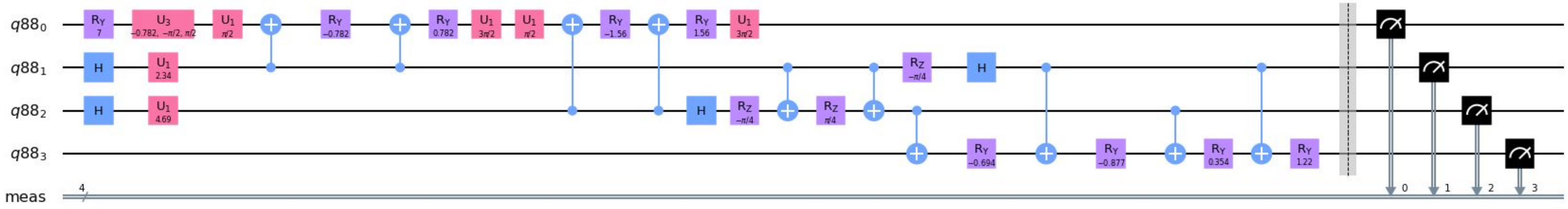
- Better CNOT error rates
- better readout errors



IBMQX2

- Better Qubit Connectivity



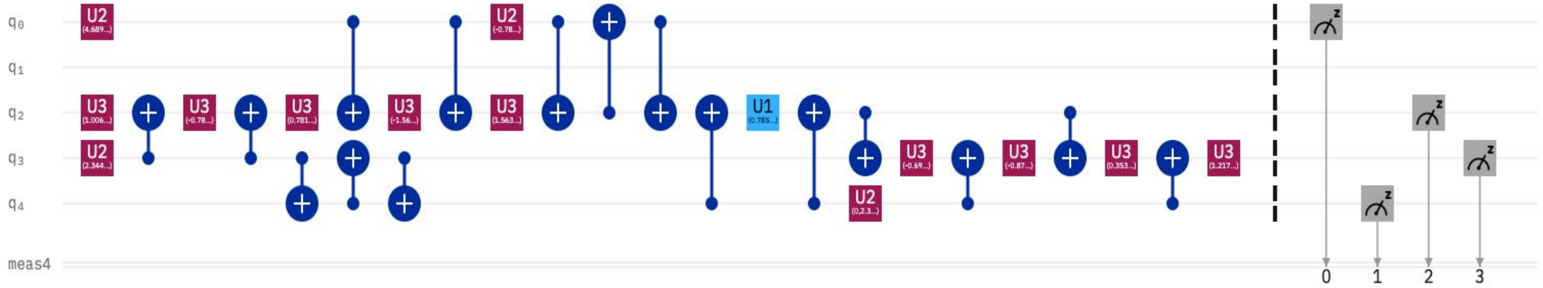


Transpiled Circuits

ibm qx2

depth: 23 vs 26

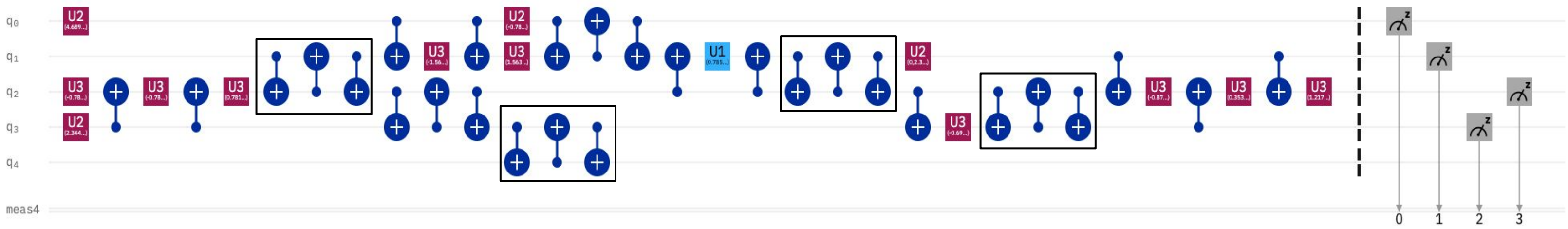
CNOTs: 16 vs 10



athens

depth: 32 vs 26

CNOTs: 28 vs 10



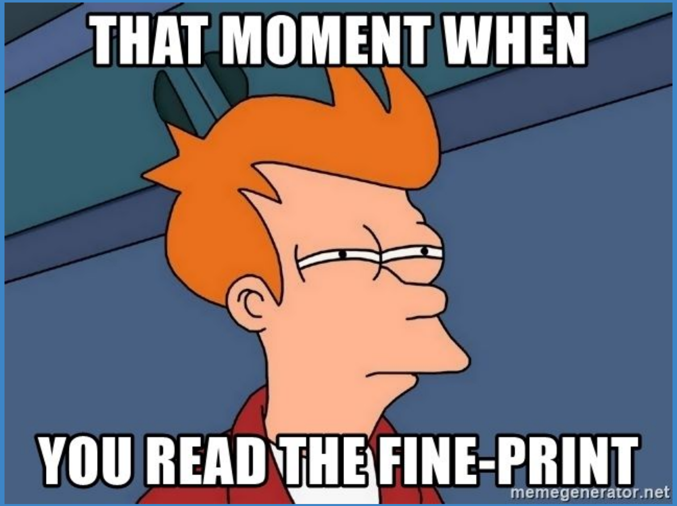
Conclusion: Additional SWAP gates used in IBM Athens generate additional noise.

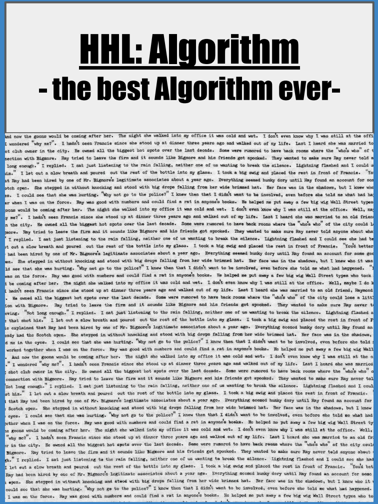
Results: Summary

1. **Richardson extrapolation**: Can work very well using 2 additional CNOT configuration to amplify the noise.
2. **Quantum Architecture**: The Qubit connectivity is crucial to obtain high quality results.
3. **IBM quantum experience**: Good platform for having an idea of the potentiality but limited for results of more complex linear systems

Qiskit: An Open-source Framework for Quantum Computing, 2019

Qiskit tutorial: https://qiskit.org/textbook/ch-applications/hhl_tutorial.html





5. The Fine Print

The Fine Print

Four general critique points by Aaronson: “Read the fine print” (2015) $O(\log(N)s^2\kappa^2/\epsilon)$

- Load vector b quickly into QC memory. Most methods would already scale with n.
- QCs architecture needs to be able to efficiently apply QPE $e^{i\hat{A}t}$ for various t, which scales quadratically with the size of non-sparse matrices A
- A needs to be robustly invertible/ “well-conditioned”: small condition number $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$
- Writing the output vector x would require n steps. By default only the state itself or an expectation value is generated.

Aaronson 10.1038/nphys3272 (2015)

6. Summary and Outlook

Outlook: Applications

Similar Applications

2010: linear differential equations. Berry arXiv:1010.2745 (2010)

2012: least-squares fitting. Wiebe et al. arXiv:1204.5242 (2012)

2013: radar cross-section of a complex shape using a preconditioner.

Claser et al. arXiv:1301.2340 (2013)

Road towards Quantum Machine Learning

2013: Qubits can represent large tensor spaces.

Lloyd et al. arXiv:1307.0411 (2013)

2013: Quantum support vector machine for classifying big data.

Rebentrost et al. arXiv:1307.0471v2 (2013)

2018: Bayesian training of deep neural networks “General Purpose Algorithm”

Zhao et al. arXiv:1806.11463 (2018)

⇒ HHL became more a template for other quantum algorithms than a quantum algorithm on in its own

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$$

$$|\boldsymbol{\lambda}\rangle = \mathbf{A}^{-1}\mathbf{I}(\mathbf{F}^{\dagger})|\mathbf{y}\rangle$$

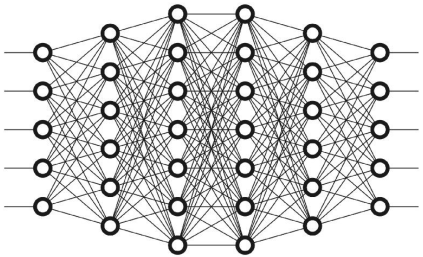
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$L(\vec{\alpha}) = \sum_{j=1}^{M} y_j \alpha_j - \frac{1}{2}\sum_{j,k=1}^{M} \alpha_j K_{jk} \alpha_k$$



Outlook: Experimental Realizations

2013: 2x2 system with photonic Qubits

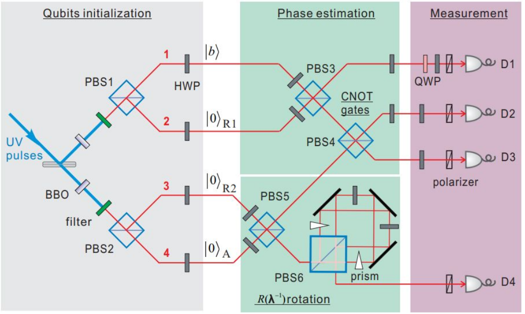
Cai et al. arXiv:1302.4310 (2013)

2013: 2x2 system using NMR Quantum Computer (96% fidelity)

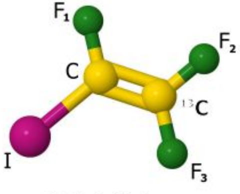
Pan et al. arXiv:1302.1946 (2013)

2018: 8x8 system using NMR and Adiabatic Quantum Computing

Wen et al. 10.1103/PhysRevA.99.012320 (2018)



	^{13}C	F_1	F_2	F_3
^{13}C	15479.7 Hz			
F_1	-297.7 Hz	-33122.4 Hz		
F_2	-275.7 Hz	64.6 Hz	-42677.7 Hz	
F_3	39.1 Hz	51.5 Hz	129.0 Hz	-56445.8 Hz
T_2^*	1.22 s	0.66 s	0.63 s	0.61 s
T_2	7.9 s	4.4 s	6.8 s	4.8 s

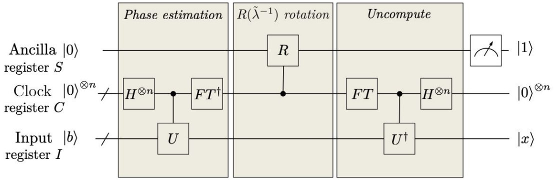


^{13}C-labelled iodotrifluoroethylene

No Quantum Advantage with NISQ technology

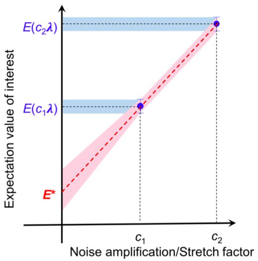
Summary: Lessons Learned

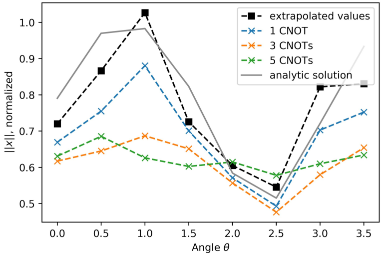
$$|x\rangle = \hat{A}^{-1}\,|b\rangle$$



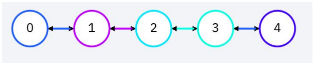
$$O(\log(N)s^2\kappa^2/\epsilon)$$

Exponential speed up





Error mitigation with Richardson Extrapolation



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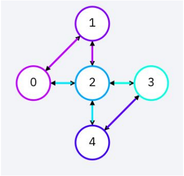
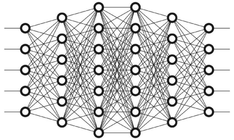


For practical implementations Qubit Connectivity is more important than gate errors

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$



No Experimental quantum advantage yet but promising general applications

7. Appendix

Appendix: Complexity

QPE is Hamiltonian simulation scaling with $\mathcal{O}(\log(N))$

Complexity of HHL dominated by QPE

$\Rightarrow$ HHL scales with $\mathcal{O}(\log(N))$ as well!

Appendix: Non-exact Quantum Phase Estimation

QPE only exact, if $\frac{\lambda_j t}{2\pi}$ can be represented exactly with n_l binary bits

$\Rightarrow$ If one knows the eigenvalues, we can rescale them such that $\frac{\lambda_j t}{2\pi}$ can be represented exactly!

$\rightarrow$ Optimized version of algorithm possible

$\rightarrow$ Important for working with real hardware

$\rightarrow$ HHL would also work without prior knowledge of the eigenvalues of $\hat{A}$

Appendix: Different Times t

time t [ms]	1	2	2.3	3	analytical solution
result	0.896	0.603	0.506	0.669	0.5014

Observation

- random t values more distant from the analytical value give worse results.
- by varying t and running the algorithm several times we can get closer to the wanted value.

Conclusion

- We picked t to the analytically best value to represent the eigenvalues binary.
- Due to hardware noise it was not possible to obtain similar results with non-optimal times.
- Here, we would lose our quantum advantage, since for general NxN hermitian matrices, we would need O(N) time to optimize.

Appendix: Optimal Parameters Setting

- **Time t = 2.3:** Binary representation exact if $$2^{n_l} = \frac{\lambda_j t}{2\pi}$$

- **1000/8000 shots:**

  - Improve statistics of the final probabilities at ancilla value = 1

  - Accuracy did not improve above 1000 shots

- **{1, 3, 5} CNOTs:** Enables error quantification via Richardson extrapolation

Appendix: Advances in Quantum Algorithms for Linear Equations

Ambainis (2010, 2012): $\mathcal{O}(\kappa \log(N)/\epsilon^3)$

- Variable-time amplitude amplification: Branches of the computation stop earlier than other branches.

Childs et al. (2017): $\mathcal{O}(\kappa \log(N) \text{poly} \log(1/\epsilon))$

- Replaced Phase Estimation algorithm with linear combination of Fourier or Chebyshev series operator representation.

Subasi et al. (2019): $\mathcal{O}(\kappa \log(N)/\epsilon)$

- “Adiabatically Inspired” simplification of the algorithm that connects $|b\rangle$and $|x\rangle$ by continuous evolution of the Hamiltonian.