Polynomials with infinite solutions

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1 Die Transformation des Dreiecks

Ausgangslage des Integrationsproblems ist ein Dreieck im $\mathbb{R}^3$, und eine Funktion f:

$$\tau(i,j,k) = \left\{ \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \right\}$$

$$f(x) : \mathbb{R}^2 \to \mathbb{R}$$

Um die zweidimensionale Gauss-Quadratur anzuwenden, muss das Dreieck affin auf das Einheitsquadrat in $\mathbb{R}^2$ abgebildet werden.

2 Transformation des Dreiecks ins Einheitsquadrat in $\mathbb{R}^2$

Sei $\tau \subset \mathbb{R}^3$ ein beliebiges Dreieck (A,B,C)

$$\tau = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\}$$

und $\tau_r \subset \mathbb{R}^2$ ein Referenzdreieck mit

$$\tau_r = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Gesucht wird eine affine Funktion χ_i von τ nach τ_r so dass: $\chi(\tau) = \tau_r$

Die Funktion ist gegeben durch:

$$\chi_I(x) = I + M \cdot x, \quad M := \big[\big(B - A\big) \mid \big(B - C\big)\big] \in Mat(3 \times 2, \mathbb{R})$$

wobei I der Eckpunkt von τ ist, der auf 0 abgebildet wird.

Ausgehend von diesem Referenzdreieck wird die Transformation ins Einheitsquadrat sehr einfach. Sie ist durch folgende Funktion gegeben:

$$\rho : \mathbb{R}^2 \to \mathbb{R}^2, \big(\mu, \nu\big) \mapsto \begin{pmatrix} \mu \\ \mu \cdot \nu \end{pmatrix}$$

3 A polynomial of degree infinity

Most of the time when we consider polynomials, we consider the degree to be finite. However, there are instances when a polynomial can be thought of as having an infinite degree. Like a power series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor series is defined to be infinite. It is called a Maclaurin series when $a = 0$.

4 Roots of a degree infinity polynomial

By the fundamental theorem of algebra, we know that a degree n polynomial's equation has n roots. Therefore, a polynomial of degree ∞ has ∞ solutions.

Given a polynomial $P(x)$, let us assume that it has a solution set S. Let us also assume that the solution set is finite, i.e., there are a finite number of solutions for $P(x)$. Now, let us take n to be the largest root in S. Since it is a root,

$$P(n) = 0$$
$$a_0 + a_1 n + a_2 n^2 + ... = 0 \tag{1}$$

Let us also take any other arbitrary m. Now, let us see if $m + n$ is a root of $P(x)$.

$$\begin{aligned} a_0 + a_1(n+m) + a_2(n+m)^2 + ... &= 0 \\ a_0 + a_1 n + a_1 m + a_2(n^2 + 2mn + m^2) + ... &= 0 \\ a_0 + a_1 n + a_1 m + a_2 n^2 + a_2 m^2 + 2a_2 mn + ... &= 0 \end{aligned}$$

We see that simplifying produces $a_i n^i + a_i m^i$ along with all the intermediate terms of a bionomial expansion of the form $(a \pm b)^n$. Let the sum of all these intermediate terms be T_{m+n}.

Now,

$$\begin{aligned} a_1 n + a_1 m + a_2 n^2 + a_2 m^2 + ... + T_{m+n} &= 0 \\ (a_1 n + a_2 n^2 + ...) + (a_1 m + a_2 m^2 + ...) + T_{m+n} &= 0 \end{aligned}$$

But from 1,

$$(a_1 m + a_2 m^2 + ...) + T_{m+n} = 0 \tag{2}$$

Since it is possible for the expression 2,

$$P(m + n) = 0$$

But n is by definition, the largest root, then, by contradiction, the solution set S has to be infinite. Therefore, in this case,there are infinite solutions to the polynomial $P(x)$.

But what if there is no value of m that satisfies $T_{m+n} = 0$? Then, in that case, the solution set can be thought of as not having as many elements as the degree of the polynomial. Just like how

$$x^2 + 1 - 2x = 0$$

Has one solution $x = 1$. The solution set for this quadratic would be $\{1\}$. Here, there are two roots - but they are both equal. The case when m does not exist would also be similar. In other words, the infinity of the solution set (∞_S) would be smaller than or equal to the infinity of the degree of the polynomial (∞_P), for any given infinite polynomial. Or,

$$\infty_S \leqslant \infty_P \tag{3}$$

5 Such equations in action

Here are a few examples :

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

This is the Taylor series for the $sin(x)$ function. It is how your calculator get the value when you feed x to it.

A note on other fields

A field is a set on which the binary operations $+$, $-$, $\times$ and $\div$ are defined. Besides integral fields, there are other fields where polynomials will behave differently and a polynomial with finite terms and of a finite degree can also have infinite solutions.

Bibliography