

# Polynomials with infinite solutions

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## Abstract

A solution to any polynomial  $P(x)$  is a value of  $x$  that satisfies  $P(x) = 0$ . A polynomial of degree 1 forms an equation called a "linear" equation. A linear equation can be expressed as  $ax + b = 0$ . Its solution is then :

$$x = \frac{-b}{a}$$

For polynomials of degree 2, things go a little complicated :

$$P(x) = ax^2 + bx + c$$

Nonetheless, a quadratic equation can be solved rather easily with the help of the quadratic formula :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubics and quartics each have 3 and 4 roots respectively. Overall, any polynomial of degree  $n$  has exactly  $n$  solutions or roots. But can there be a polynomial equation which has infinite solutions? As it turns out - yes.

*Keywords:* Polynomial, solution, degree of a polynomial, linear, quadratic, quadratic formula, roots, cubics, quartics

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## 1. The null polynomial

In general, any polynomial can be expressed as :

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_ix^i$$

The null polynomial is a polynomial that returns 0 for any value of  $x$ .  $P(x) = 0$  It can be understood better as a polynomial with every coefficient equal to 0:

$$P(x) = 0 + 0x + 0x^2 + \dots + 0x^i$$

It will return 0 no matter what. Since it returns 0 for any given value of  $x$ , it has infinite solutions.[? ]

## 2. A polynomial of degree infinity

Most of the time when we consider polynomials, we consider the degree to be finite. However, there are instances when a polynomial can be thought of as having an infinite degree. Like a power series. One prime example is of a Taylor series.[? ]

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

A Taylor series is defined to be infinite. It is called a Maclaurin series when  $a = 0$ .

## 3. Roots of a degree infinity polynomial

By the fundamental theorem of algebra, we know that a degree  $n$  polynomial's equation has  $n$  roots. Therefore, a polynomial of degree  $\infty$  has  $\infty$  solutions.

Given a polynomial  $P(x)$ , let us assume that it has a solution set  $S$ . Let us also assume that the solution set is finite, i.e., there are a finite number of solutions for  $P(x)$ . Now, let us take  $n$  to be the largest root in  $S$ . Since it is a root,

$$\begin{aligned} P(n) &= 0 \\ a_0 + a_1n + a_2n^2 + \dots &= 0 \end{aligned} \tag{1}$$

Let us also take any other arbitrary  $m$ . Now, let us see if  $m + n$  is a root of  $P(x)$ .

$$\begin{aligned} a_0 + a_1(n + m) + a_2(n + m)^2 + \dots &= 0 \\ a_0 + a_1n + a_1m + a_2(n^2 + 2mn + m^2) + \dots &= 0 \\ a_0 + a_1n + a_1m + a_2n^2 + a_2m^2 + 2a_2mn + \dots &= 0 \end{aligned}$$

We see that simplifying produces  $a_i n^i + a_i m^i$  along with all the intermediate terms of a binomial expansion of the form  $(a \pm b)^n$ . Let the sum of all these intermediate terms be  $T_{m+n}$ .

Now,

$$\begin{aligned} a_1 n + a_1 m + a_2 n^2 + a_2 m^2 + \dots + T_{m+n} &= 0 \\ (a_1 n + a_2 n^2 + \dots) + (a_1 m + a_2 m^2 + \dots) + T_{m+n} &= 0 \end{aligned}$$

But from 1,

$$(a_1 m + a_2 m^2 + \dots) + T_{m+n} = 0 \quad (2)$$

Since it is possible for the expression 2,

$$P(m+n) = 0$$

But  $n$  is by definition, the largest root, then, by contradiction, the solution set  $S$  has to be infinite. Therefore, in this case, there are infinite solutions to the polynomial  $P(x)$ .

But what if there is no value of  $m$  that satisfies  $T_{m+n} = 0$ ? Then, in that case, the solution set can be thought of as not having as many elements as the degree of the polynomial. Just like how

$$x^2 + 1 - 2x = 0$$

Has one solution  $x = 1$ . The solution set for this quadratic would be  $\{1\}$ . Here, there are two roots - but they are both equal. The case when  $m$  does not exist would also be similar. In other words, the infinity of the solution set  $(\infty_S)$  would be smaller than or equal to the infinity of the degree of the polynomial  $(\infty_P)$ , for any given infinite polynomial. Or,

$$\infty_S \leq \infty_P \quad (3)$$

#### 4. Such equations in action

Here are a few examples :

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

This is the Taylor series for the  $\sin(x)$  function. It is how your calculator get the value when you feed  $x$  to it.

### **A note on other fields**

A field is a set on which the binary operations  $+$ ,  $-$ ,  $\times$  and  $\div$  are defined. Besides integral fields, there are other fields where polynomials will behave differently and a polynomial with finite terms and of a finite degree can also have infinite solutions.

### **Bibliography**