

Polynomials with infinite solutions

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1 Die Transformation des Dreiecks

Ausgangslage des Integrationsproblems ist ein Dreieck im \mathbb{R}^3 , und eine Funktion f :

$$\tau(i, j, k) = \left\{ \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \right\}$$
$$f(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Um die zweidimensionale Gauss-Quadratur anzuwenden, muss das Dreieck affin auf das Einheitsquadrat in \mathbb{R}^2 abgebildet werden.

2 Transformation des Dreiecks in ein Referenzdreieck im \mathbb{R}^2

Sei $\tau \subset \mathbb{R}^3$ ein beliebiges Dreieck (A,B,C)

$$\tau = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\}$$

und $\tau_r \subset \mathbb{R}^2$ ein Referenzdreieck mit

$$\tau_r = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Gesucht wird eine affine Funktion χ_i von τ nach τ_r so dass: $\chi(\tau) = \tau_r$

Die Funktion ist gegeben durch:

$$\chi_I(x) = I + Mx, \quad M := [(B - A) \mid (B - C)] \in \text{Mat}(3 \times 2, \mathbb{R})$$

wobei I der Eckpunkt von τ ist, der auf 0 abgebildet wird.

3 A polynomial of degree infinity

Most of the time when we consider polynomials, we consider the degree to be finite. However, there are instances when a polynomial can be thought of as having an infinite degree. Like a power series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

A Taylor series is defined to be infinite. It is called a Maclaurin series when $a = 0$.

4 Roots of a degree infinity polynomial

By the fundamental theorem of algebra, we know that a degree n polynomial's equation has n roots. Therefore, a polynomial of degree ∞ has ∞ solutions.

Given a polynomial $P(x)$, let us assume that it has a solution set S . Let us also assume that the solution set is finite, i.e., there are a finite number of solutions for $P(x)$. Now, let us take n to be the largest root in S . Since it is a root,

$$\begin{aligned} P(n) &= 0 \\ a_0 + a_1n + a_2n^2 + \dots &= 0 \end{aligned} \tag{1}$$

Let us also take any other arbitrary m . Now, let us see if $m + n$ is a root of $P(x)$.

$$\begin{aligned} a_0 + a_1(n + m) + a_2(n + m)^2 + \dots &= 0 \\ a_0 + a_1n + a_1m + a_2(n^2 + 2mn + m^2) + \dots &= 0 \\ a_0 + a_1n + a_1m + a_2n^2 + a_2m^2 + 2a_2mn + \dots &= 0 \end{aligned}$$

We see that simplifying produces $a_in^i + a_im^i$ along with all the intermediate terms of a binomial expansion of the form $(a \pm b)^n$. Let the sum of all these intermediate terms be T_{m+n} .

Now,

$$\begin{aligned} a_1n + a_1m + a_2n^2 + a_2m^2 + \dots + T_{m+n} &= 0 \\ (a_1n + a_2n^2 + \dots) + (a_1m + a_2m^2 + \dots) + T_{m+n} &= 0 \end{aligned}$$

But from 1,

$$(a_1m + a_2m^2 + \dots) + T_{m+n} = 0 \quad (2)$$

Since it is possible for the expression 2,

$$P(m+n) = 0$$

But n is by definition, the largest root, then, by contradiction, the solution set S has to be infinite. Therefore, in this case, there are infinite solutions to the polynomial $P(x)$.

But what if there is no value of m that satisfies $T_{m+n} = 0$? Then, in that case, the solution set can be thought of as not having as many elements as the degree of the polynomial. Just like how

$$x^2 + 1 - 2x = 0$$

Has one solution $x = 1$. The solution set for this quadratic would be $\{1\}$. Here, there are two roots - but they are both equal. The case when m does not exist would also be similar. In other words, the infinity of the solution set (∞_S) would be smaller than or equal to the infinity of the degree of the polynomial (∞_P), for any given infinite polynomial. Or,

$$\infty_S \leq \infty_P \quad (3)$$

5 Such equations in action

Here are a few examples :

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

This is the Taylor series for the $\sin(x)$ function. It is how your calculator get the value when you feed x to it.

A note on other fields

A field is a set on which the binary operations $+$, $-$, \times and \div are defined. Besides integral fields, there are other fields where polynomials will behave differently and a polynomial with finite terms and of a finite degree can also have infinite solutions.

Bibliography