Overview talk

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Higgs bundles were introduced in 1987 Nigel Hitchen. They are at the core of many mathematical and physical objects and theories. In physics they appear when working with Yang-Mills equations, mirror symmetry, topological quantum field theory, supergravity. In mathematics they appear in Teichmuller theory, integrable systems, and representations of $\pi_1(X)$.

Definition 0.1. *A Higgs bundle (E, φ) where $E \to X$ is a holomorphic bundle over a compact Riemann Surface. $\varphi \in H^0(X, \mathrm{End}(E \otimes K_X))$ where K_X is the canonical bundle of X.*

Let X be an oriented Riemannian manifold, and $P \to X$ a principal G-bundle. Yang - Mills functional YM is a functional on the space of connections of the associated adjoint bundle, defined by

$$YM(A) = \int_X \mathrm{tr}(F_A \wedge *F_A) \tag{1}$$

where $F_A = dA + [A, A]$ is the curvature of connection A, and $*$ is the Hodge star. $d_A F_A = d_A(*F_A)$ critical points of $D_A(F_A)$ are

where d_A is the covariant derivative and (1) gives the so called Yang Mills equations.

This comes from dimensional reduction on the Yang mills. In 1 dim, this leads to magnetic monopoles studied by Atiyah and Hitchen in 2 dim, get Higgs bundles via the Higgs equations.

$$\bar{\partial}\varphi = 0 \tag{2}$$

$$F_A + [\varphi, \varphi^*] = 0 \tag{3}$$

In the case that $F_A = *F_A$ is an instanton

Let (E, φ) where $E \to X$ holomorphic, and $E = H^{\mathbb{C}}$ bundle. Let G be a non-maximal component. Then $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}} \otimes K)$ $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$

Then $G = U(p, q)$ and $E = V \oplus W$, $\varphi = \begin{matrix} 0 & \gamma \\ \beta & 0 \end{matrix}$

Upcoming: Non-abelian Hodge theory Symplectic structure Hitchin map Singular Higgs bundles (parabolic bundles) Topology of moduli spaces Mirror symmetry for the character varieties.

1 Non abelian Hodge

Let (E, φ) be our Higgs bundle over Riemann surface X.

$\varphi = 0$. $M(r, d)$ is the moduli space of stable vector bundles of rank r and degree d. Narashimhain-Sashadin theorem.

$$M(r, 0) \cong \{\pi_1(X) \to \mathrm{U}(n)\}/\mathrm{U}(n) \tag{4}$$

Donaldson generalises representation 3 ingredients, v, b, connections $\rho : \pi_1(X) \to G$

$$H^1(X, G) \cong H_1(X, G) \cong \mathrm{Hom}(\pi/[\pi, \pi], G) \cong \mathrm{Hom}(\pi, G) \tag{5}$$

Non-abelian hodge theory consists of the following technique

$$H^1_\Delta(X, G) \cong H^1_{dR}(X, G) \cong H^1_{\partial}(X, G) \tag{6}$$

Relates the singular, d'Rham, and Dolbeaut cohomology. The topological, the differential, and the algebraic

2 Symplectic Geometry

Suppose $H^1(X,\mathbb{R})=\mathbb{R}$, for X compact Riemann surface of genus g. The cup product provides $H^1(X,\mathbb{R})$ with symplectic structure. Actually, $\mathcal{M}^s(r,d)$ the moduli of stable Higgs bundles is symplectic.

$$T_E M = H^1(X, \mathrm{End}(E)) \tag{7}$$

by Serre duality

$$T_E M^* = H^0(X, \mathrm{E}\otimes K_X \tag{8}$$

M^s is the moduli space of stable vector bundles Say that vector bundle $E\to X$ is stable if $\forall F\subset E$

$$\deg(F)/\mathrm{rk}(F) < \deg(E)/\mathrm{rk}(E) \tag{9}$$

slope condition. In the case of Higgs bundles require stronger condition

$$\forall F\subset E, \quad \varphi(F)\subset F\otimes K \tag{10}$$

3 Symplectic geometry

$$\mathrm{dim} M = r^2(g-1)+1 \tag{11}$$

4 Hitchin map

$(G=\mathrm{GL}(r,\mathbb{C}))$

$$\mathcal{M}^s \supset T^*M^s \to M^s \tag{12}$$

The hitchin map

$$\mathcal{M}^s \to h\bigoplus_{i>0} H^{)}(X,K^i) = H \tag{13}$$

$$(E,\varphi)\mapsto \mathrm{tr}(\bigwedge^i\varphi) \tag{14}$$

Proper map, with $\mathrm{dim}H=\frac{1}{2}\mathrm{dim}\mathcal{M}$ (cf spectral curves, mirror symmetry, and integral systems)
SYZ- mirror symmetry

5 Parabolic

[NO IDEA]

6 Topology of moduli of stable Higgs

Define a perfect morse function

$$\mu : \mathcal{M}\to\mathbb{R} \tag{15}$$
$$(E,\varphi)\mapsto \|\varphi\|^2 \tag{16}$$

$$P_t(\mathcal{M}) = \sum t^{\mathrm{ind}} P_t(\mathcal{N}) \tag{17}$$

where $\mathcal{N}$ are some critical somethings of something.