Stability of Higgs Bundles

Definition 0.1. *Let M compact Riemann surface. A GL_n-Higgs bundle is pair (V, φ) where V is a holomorphic rank n vector bundle and φ is a holomorphic section of $\mathrm{End}(V) \otimes K$. That is $\varphi \in H^0(\mathrm{End}(V) \otimes K)$.*

Definition 0.2. *Say two Higgs bundles (V, φ) and (V', φ') are isomorphic if there exists $\psi : V \to V'$ inducing the commutative diagram*

Recall that for P a principal G-bundle, and connection A on P, then $\varphi \in \omega^1(M, \mathrm{ad}(P) \otimes \mathbb{C})$ such that $F_A = [\varphi, \varphi^*]$ and $\bar{\partial}_A \varphi = 0$

Claim: $V = (P \otimes \mathbb{C}^n)/G$ then V, φ is a higgs bundle. Note that $\Omega^1(M, \mathrm{ad}(P) \otimes \mathbb{C} = H^0(M, (\mathrm{ad}P \otimes \mathbb{C}) \otimes K)$

Commutative diagram defining that map

$$(\mathrm{ad}P \otimes \mathbb{C})_x \to \mathrm{End}(V)_x \tag{1}$$

$$[p, m] \mapsto ([pg, v] \mapsto [pg, g^{-1}mgv]) \tag{2}$$

1 Stability

Definition 1.1. *Let (V, φ) be higgs bundle. Say $E \subset L$ is φ invariant if $\varphi(E) \subset E \otimes K$. In such cases, say E is a Higgs subbundle.*

A Higgs bundle is stable if $\mu(E) < \mu(V)$ for all Higgs subbundles. Recall slope $\mu(E) := \deg(\det(E))/\mathrm{rank}(E)$

Example: $(V, 0)$ is stable if and only if V is stable as a vector bundle. In fact, if V is stable then any φ gives rise to stable Higgs bundle. Furthermore, if (V, φ) and (V, φ') are isomorphic, then $\varphi = \varphi'$.

$$\mathcal{M}_{\mathrm{Higgs}} \subset \{(V, \varphi) |\ V \text{ stable}\} \to T^*\mathcal{M}_{\mathrm{Vect}} \tag{3}$$

$$(V, \varphi) \mapsto \varphi \tag{4}$$

There is a 1-1 correspondence between self dual Hitchin Pair (F_A, E) upto Gauge and (semi?) stable Higgs bundles up to isomorphism

Example:

Let $G = \mathrm{SU}(2)$. Then

$$\mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}) = \{m \in \mathrm{Mat}(2, \mathbb{C}) |\ \mathrm{tr}(m) = 0\} \tag{5}$$

$\varphi \in H^0(\mathrm{End}_0(V) \otimes K)$ V of rank 2.

Remark:

$$\mathrm{Hom}(L_1, L_2) \to H^0(L_1^* \otimes L_2) \tag{6}$$

$M = \mathbb{P}^1$, $V = \mathcal{O}(m) \oplus \mathcal{O}(n)$, $K = \mathcal{O}(-2)$. Let $\varphi : \mathcal{O}(m) \oplus \mathcal{O}(n) \to \mathcal{O}(m-2) \oplus \mathcal{O}(n-2)$ then may regard φ as a matrix with elements forms in the relevant bundles.

Then $\det(\mathcal{O}(m) \oplus \mathcal{O}(n) = \mathcal{O}(m+n)$ As $\deg(\mathcal{O}(m) = m$, there are no stable Higgs bundles in $\mathbb{P}^1$.

Moduli space.

Fact: If V is unstable vector bundle of rank 2, then there exists a unique $L \subset V$ such taht $\deg(L) \geq \frac{1}{2}\deg(\Lambda^2 V)$

$$0 \to L \to V \to L^* \otimes \det(V) \otimes 0 \tag{7}$$

Proposition 1.2. *Let $g > 1$. A rank 2 vector bundle V occurs in a stble Higgs bundle (V, φ) iff one of the folloing holds.*

(i) V is stable,

(ii) V is semistable and $g > 2$,

(iii) V is semistable and $g = 2$, and $V = U \otimes L$, and U decomposable or $0 \to \mathcal{O} \to U \to \mathcal{O} \to 0$

(iv) V is not semistable and $h^0(L^{-2} \otimes \det(V) \otimes K) > 1$

(v) V is not semistable and $h^0(L^{-2} \otimes \det(V) \otimes K) = 1$ and $V = L \oplus (L^* \otimes \det(V))$

Suppose V is not stable. Let L be the unstable (most ?) subspace of V. Have

$$0 \to L \to V \to L^* \otimes \det V \to 0 \tag{8}$$

$$0 \to F \to \mathrm{End}_0(V \otimes K) \to L^{-2} \otimes \det(V) \otimes K \to 0 \tag{9}$$

$H^0(L^{-2} \otimes \mathrm{Det}(V) \otimes K) = \mathrm{Hom}(L...)$

$H^0(F) = H^0(\mathrm{End}_0(V) \otimes K)$ if and only if V does not occur in a stable Higgs bundle.