Hitchin to Higgs and back

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## 1 Hitchins to Higgs

Let $M$ be compact Riemann surface of genus $g > 1$ and group $G = \mathrm{U}(2)$. Let $V \to M$ be a unitary vector bundle of rank 2, that is, $V$ has a Hermitian metric, and admits action by $\mathrm{U}(2)$. Let $\{U_\alpha\}$ be a trivialisation of $V$ with transition functions $u_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{U}(2)$.

Suppose $(A, \varphi)$ be the solution to the Hitchin equation **??** for $A \in \Omega^1(V)$. In local coordinates we can express the connection $A$ as

$$\nabla_A s = ds + A_\alpha s \tag{1}$$

The transformation conditions dictate $A_j = u_{\alpha\beta}^{-1} A_\alpha u_{\alpha\beta} + u_{\alpha\beta}^{-1} du_{\alpha\beta}$. Over the cover then $F_A|_{U_\alpha} = dA_\alpha + 1/2[A_\alpha, A_\alpha]$ $\Phi \in \Omega^0(M, \mathrm{End}_0 V)$.

Hitchin's equations are

$$F_A + [\Phi, \Phi^*] = 0 \tag{2}$$

$$d_A^{0,1}\Phi = 0 \tag{3}$$

**Theorem 1.1.** *Let $(A, \Phi)$ be a solution to Hitchens equations. Let $L \subset V$ that is $\Phi$ invariant subbundle of rank 1. Then* $\deg(L) \leq \frac{1}{2}\deg(\det(V))$

Idea of proof.

Let $\omega$ be a 2form such that $\int 1/2\pi\omega = 1$

$s \in \Omega(M, L^* \otimes V) = \Omega^0(M, \mathrm{Hom}(L, V))$ Where $s$ is holomorphic since $L$ is holomorphic.

Construct a connection $B$ on $L^* \otimes V$ such that $F(B)s = F(A)s - \deg(L) \leq \omega + 1/2\deg(\det V) \leq \omega$ $-[\Phi, \Phi^*]s + ...$ and $\langle F(B)s, s\rangle_{L^2} \geq 0$

If $\deg(L) > 1/2\deg(\det V)$ then $\int \langle F(B)s, s\rangle < 0$ so contradiction.

**Lemma 1.2.** *$L \to M$ holomorphic line bundle. Then there exists connection $\vartheta$ on $L$ such that $F(\vartheta) = \deg(L)\omega$*

*Proof.* Let $\vartheta$ be a any connection on $L$. Then

$$\int_M \frac{1}{2\pi} F_\vartheta = \deg L = \int \frac{1}{2\pi} \deg L \cdot \omega \tag{4}$$

( Upto constant. ) This implies $[\frac{1}{2\pi} F_\vartheta] = [(\deg L \cdot \omega]$

Get a second connection $\vartheta' = \vartheta + \partial\rho$ for some $\rho \in C^\infty(M)$. Then $F(\vartheta') = F(\vartheta) - \partial\bar{\partial}\rho$.

Then $F(\vartheta') = \deg L \cdot \omega$ if and only if $\partial\bar{\partial} = F(\vartheta) - \deg L \cdot \omega$.

The latter is precisely the $d\bar{d}$ lemma . And since we have a Kahler manifold we can solve for $\rho$. $\square$

We have exact sequence of groups

$$0 \to \mathrm{U}(1) \to U(2) \to \mathrm{SO}(3) \to -0 \tag{5}$$

$SO(3)$ connection $A$ induces $\hat{A}$ on $V$ such that

$$F(\hat{A}) = F(A) + 1/2F(A_0)\mathrm{id} \tag{6}$$

$\deg\det V\omega$ connection on $L$ Choose $A_0$ such that curvature is $F(A_0) = \deg L \cdot \omega$

$\left\langle d_B^{0,1} s, s \right\rangle$, The first term lies in $\Omega^{1,0}(L^* \otimes V)$ while the latter term lies in $\Omega^0(L^* \otimes V)$.

$$d^{0,1} \left\langle d_B^{0,1} s, s \right\rangle = \left\langle d_B^{0,1} d_B^{0,1} s, s \right\rangle - \left\langle d_B^{0,1} s, d_B^{0,1} s \right\rangle \tag{7}$$

Have that $F(B) = d_B^2 = d_B^{0,1} d_B^{1,0} - d_B^{1,0} d_B^{0,1}$ As $s$ is holomorphic, ie $d_B^{0,1} s = 0$

$$\int_M \langle F(B)s, s\rangle \geq 0 \tag{8}$$

## 2 Stable Higgs to Hitchin

Let $\mathcal{A}$ be the space of all $C^\infty$ connections on stable Higgs bundle $(V, \Phi)$. $\mathcal{A}$ is an affine space that is modelled on $\Omega^1(\mathrm{End}(V))$. Thus $T_A\mathcal{A} = \Omega^1(\mathrm{End}(V))$. This has a natural symplectic structure

$$\omega(Y_1, Y_2)|_A := \int_M \mathrm{Tr}(Y_1(A) \wedge Y_2(A)) \tag{9}$$

where $Y_i$ vector fields on $\mathcal{A}$. Thus we have infinite dimensional symplectic manifold.

Recall that for symplectic manifold $(X, \omega)$, we have associated $G$ acting on $X$, where $L_x\omega = 0$. For $g \in \mathfrak{g}$ have the associated $z_g \in \Gamma(TX)$. $\omega(\cdot, z_g) = dH_g$ the action of the Hamiltonian if and only if there exists $H_g$ for $\forall g \in \mathfrak{g}$

Define the moment

$$\mu : M \to \mathfrak{g}^* \tag{10}$$

with $\langle \mu(m), g\rangle = H_g(m)$.

Let $\mathcal{G}$ be the group of gauge transformations. That is a collection of functions $g_\alpha : U_\alpha \to G$ defined on a trivialisations satisfying transformation rule $g_\alpha = u_{\alpha\beta} g_j u_{\beta\alpha}$. Such maps are called gauge transformations and are equivalently defined as sections $g \in \Gamma(\mathrm{Ad})$.

The general idea is that Hitchin's equations are the moment map on the space of connections $\mathcal{A} \times \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$.

For a connection $\nabla$, the gauge transformation $g$ induces the following transformation to $\nabla'$

$$\nabla' = \nabla - (\nabla g)g^{-1} \tag{11}$$

Now see $\varphi$ as an element of the Lie algebra of the Gauge group. $\varphi \in \mathrm{Lie}(\mathcal{G}) = \Omega^0(M, \mathrm{End}(V))$

For a connection $A$, have the associated vector field $z_\varphi(A) = \frac{d}{dt}|_{t=0}(\nabla_A - \nabla A(I + t\varphi)(I - t\varphi)) = -\nabla_A \varphi$

The associated hamiltonian to $z_\varphi(A)$ is $f_\varphi(A) = -\int_M \mathrm{Tr}(F_A \wedge \varphi)$

We see this by considering $df_\varphi|_A(a) = -\frac{d}{dt}|_{t=0} \int_M \mathrm{Tr}(F_{A+ta} \wedge \varphi)$ where $a \in T_A\mathcal{A}$ that is $a \in \Omega^1(\mathrm{End}(V))$.

We can write $F_{A+a} = F_A + d_A a + \frac{1}{2}[a, a]$. Commuting the differentiation and integration, and seeing that $\frac{d}{dt}|_{t=0} F_{A+ta} = d_A a$ we recover $\omega(a, z_\varphi) = -\int_M \mathrm{Tr}(d_A a \wedge \varphi)$ ie the Hamiltonian.

The moment map $\mu : \mathcal{A} \to \mathrm{Lie}^*(\mathcal{G})$

Note that since there exists a natural pairing between $\Omega^0(\mathrm{End}(V))$ and $\Omega^2(\mathrm{End}(V))$, so we can identify $\mathrm{Lie}^*(\mathcal{G}) = \Omega^2(\mathrm{End}(V))$

Now consider the action of $\mathrm{U}(n)$ on $\mathrm{C}^n$ $\mathrm{End}(\mathbb{C}^n)$ has hermitian product. $h(A, B) = \mathrm{tr}(AB^*)$, $U(n)$ acts on $\mathrm{End}(\mathbb{C}^n)$ by conjugation

$\omega(A, B) = -\mathrm{IM}(\mathrm{tr}(AB^*))$

$\mu(A) = i/2[A, A^*]$

$\Omega^{1,0}(M, \mathrm{End}_0 V)$

$\omega(\Phi, \Phi) = ....$

$\mathcal{G}$ acts on $\Omega^{1,0}(M, \mathrm{End}_0(V))$

$\mu(\Phi) = [\Phi, \Phi^*]$

$\mathcal{G}$ acts on $\mathcal{A} \times \Omega^{1,0}(M, \mathrm{End}_0 V)$

$\mu(A, \Phi) = F(A') + [\Phi, \Phi^*] = 0$

To solve Hitchin equation find $(A, \Phi)$ such that $\mu(A, \Phi) = 0$.

Choose a Riemannain metric on $M$ so that we can consider $\|\mu\|^0$

Then $\|\mu\|_2^2 = \int_M \|F_A + [\Phi, \Phi^*]\|^2$

[[ THIS IS QUITE A NOTATION HEAVY PIECE. A LOT OF ATTENTTION NEEDED ]]