Character Varieties through examples

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In what follows we aim to cover the following: definition of character variety for any finitely generated group; crash course in algebraic geometry; state some useful results to do computations; symplectic structure when $\Gamma = \pi_1(\Sigma_g)$; affine cubic curves.

1 Character varieties

Definition 1.1. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be complex reductive linear group, and Γ a finitely generated group. The character variety of Γ in relation to G is*

$$\chi(\Gamma, G) = \mathrm{Hom}(\Gamma, G)//G \tag{1}$$

2 Algebraic geometry

Definition 2.1. *Affine variety $X \subset \mathbb{A}^n_{\mathbb{C}}$.*

Definition 2.2. *Regular functions.*

Examples: Suppose X is the zero locus of the polynomials $p_1, \ldots, p_k \in \mathbb{C}[x_1, \ldots, x_n]$. Then the regular functions are elements of ring

$$\mathbb{C}[X] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(p_1, \ldots, p_k)} \tag{2}$$

The functor Spec takes quotient rings of the form above to their associated varieties.

Let G act on the algebra A. Denote the elements of A fixed by G by A^G. We define

$$X//G = \mathrm{Spec}(\mathbb{C}[X]^G) \tag{3}$$

known as the categorical quotient.

From now on $G = \mathrm{SL}_2\mathbb{C}$.

$$A(\Gamma) = \frac{\mathbb{C}[M^{\gamma}_{ij} \mid i,j = 1,2 \quad \gamma \in \Gamma]}{(\det M = 1,\ M^{\gamma\delta} = M^{\gamma} M^{\delta})} \tag{4}$$

$$\mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C}) = \mathrm{Spec} A(\Gamma) \tag{5}$$

$$A(\Gamma)^{\mathrm{SL}_2\mathbb{C}} \to \chi(\Gamma) = \mathrm{Spec}(A(\Gamma)^{\mathrm{SL}_2\mathbb{C}}) \tag{6}$$

Example: $\Gamma = \mathbb{Z}$. Then $\rho(gen) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A(\mathbb{Z}) = \frac{\mathbb{C}[a,b,c,d]}{(ab - cd) = 1)} \tag{7}$$

$$A(\mathbb{Z})^{\mathrm{SL}_2\mathbb{C}} = \mathbb{C}[a+d] \cong \mathbb{C}[z] \tag{8}$$

so character variety $\chi(\mathbb{Z}) = \mathbb{C}$

Skein algebra

$$B(\Gamma) = \frac{\mathbb{C}[M^\gamma \mid \gamma \in \Gamma]}{(M^e = 2, M^\gamma M^\delta = M^{\gamma\delta} + M_{\gamma\delta^{-1}})} \tag{9}$$

[[RED ?? DONT GET THIS]]

$$\chi_{\mathrm{Skein}}(\Gamma) = \mathrm{Spec} B(\Gamma) \tag{10}$$

Theorem 2.3.

$$\chi^{\mathrm{red}}_{\mathrm{Skein}}(\Gamma) = \chi^{\mathrm{red}}(\Gamma) \tag{11}$$

Corollary 2.4. *If $A(\Gamma)$ and $B(\Gamma)$ are nice. $\chi(\Gamma) \cong \chi_{\mathrm{Skein}}(\Gamma)$.*

Theorem 2.5. *$\Gamma = \pi_1(M)$ for M manifold (with boundary). Then*

$$B(\Gamma) = \bigoplus_\gamma \mathbb{C}[\gamma] \tag{12}$$

where γ runs over isotopy classes of multicurves (immersed 1-submanifolds not bounding a disc)

[[SOMETHING FUNKY HERE]]

Example: $M = S^2 \setminus \{p_1, p_2, p_3\}$. Then $\pi_1 = \mathbb{F}_2$ the free group on two generators. Claim then that the multicurves are given by $a, b, a * b$ so

$$B(\Gamma) = \mathbb{C}[x_1, x_2, x_3], \quad \chi(\mathbb{F}_2) = \mathbb{C}^3 \tag{13}$$

$$\mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}_2\mathbb{C}) = (\mathrm{SL}_2\mathbb{C}) \tag{14}$$

Have that $\mathbb{C}[\mathrm{SL}_2\mathbb{C}]^{\mathrm{SL}_2\mathbb{C}}$ is generated by some trace functions

$$(A, B) \mapsto \begin{cases} \mathrm{Tr}(A) \\ \mathrm{Tr}(B) \\ \mathrm{Tr}(AB) \end{cases} \tag{15}$$

3 Affine cubic curves

Let $M = S^2 \setminus \{p_1, \dots, p_4\}$ then $\pi_1(M) = \mathbb{F}_3$, $\chi(\pi_1(M)) = (\mathrm{SL}_2\mathbb{C})^3 // \mathrm{SL}_2\mathbb{C}$

$A(\pi_1(M))^{\mathrm{SL}_2\mathbb{C}} = \langle x, y, z, m_1, m_2, m_3, m_4 | \sim \rangle$

where $m_1 = \mathrm{Tr}(M_1)$, $m_2 = \mathrm{Tr}(M_2)$, $m_3 = \mathrm{Tr}(M_3)$, $x = \mathrm{Tr}(M_2M_3)$, $y = \mathrm{Tr}(M_1M_3)$, $z = \mathrm{Tr}(M_1M_2)$, $m_4 = \mathrm{Tr}(M_1M_2M_3)$,

It turns out that the $\mathrm{Spec}(A(\pi_1(M))^{\mathrm{SL}_2\mathbb{C}})$ is a cubic hypersurface in $\mathbb{C}^7$ in the so called Frieke family Family of cubic surfaces

$$xyz + x^2 + y^2 + z^2 + b_1 x + b^2 y + b_3 z + c = 0 \tag{16}$$

Theorem 3.1. *(Goldman - Toledo) Every cubic surface in the Frieke family arises as a relative character variety.*