Character Varieties

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Let X be a fixed Riemann surface.

We have three associated spaces that correspond to topological, smooth, and holomorphic perspectives.

$$\begin{aligned} R_{\text{Betti}} &= \text{Hom}_{\text{grp}}(\pi_1(X), \text{GL}_n\mathbb{C}) && (1)\\ R_{\text{dR}} &= \text{rank } n \text{ flat vector bundles} && (2)\\ R_{\text{Higgs}} &= \text{rank } n \text{ Higgs bundles} && (3)\\ & && (4) \end{aligned}$$

Quotienting by relevant subgroups gives

$$\begin{aligned} \mathcal{M}_{\text{Betti}} &= R_{\text{Betti}}/\text{ conj} && (5)\\ \mathcal{M}_{\text{dR}} &= R_{\text{dR}}/\text{ gauge} && (6)\\ \mathcal{M}_{\text{Higgs}} &= R_{\text{Higgs}}/\text{ equiv} && (7)\\ & && (8) \end{aligned}$$

These are all equivalent as topological spaces.

Let γ be a group, and G a Lie group. $R(\Gamma, G) = \text{Hom}(\Gamma, G)$.

Example: $\Gamma = \mathbb{Z}$ and $G = \text{SL}_2(\mathbb{R})$ then $R(\Gamma, G) = \text{SL}_2(\mathbb{R})$. The closure of an orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ contains the identity. Something bad happens here. We lack the ability to find an invariant decomposition.

Definition 0.1. *A representation ρ is reductive or semisimple it is a sum of irreducible representations.*

$$\begin{aligned} \text{Hom}(\Gamma, G) &\to G^d && (9)\\ \rho &\mapsto \begin{bmatrix} \rho(\gamma_1) \\ \dots \\ \rho(\gamma_1) \end{bmatrix} && (10)\\ & && (11) \end{aligned}$$

where $\Gamma = \langle \gamma_i \rangle$. In the case that Γ is a fundamental group...

Suppose $n = 1$. Then

$$\begin{aligned} \mathcal{M}_{\text{Betti}} &= \text{hom}(\pi_1(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g} && (12)\\ \mathcal{M}_{\text{Higgs}} &= T^*\text{Jac}(X) && (13)\\ & && (14) \end{aligned}$$

Which can (sort of) easily be seen to be diffeomorphic, but not necessarily analytic.

Consider $\frac{\partial f}{\partial z} = \frac{1}{2z} f$ on $\mathbb{C}^*$. Locally a solution will have the form $f = cz^{1/2}$ for $c \in \mathbb{C}$. Considering this solution as it varies along a curve circumventing the origin we return to $z^{-1/2}$.

[[REPHRASE]]

Repesentations of $\pi_1(X, x)$. Local systems $\mathcal{F}$ on X, Locally constant sheaf $\mathcal{F}_x = \text{colim}_U \mathcal{F} = \mathcal{F}(U_x)$, that is their exists a neighbourhood, which agrees with the stalk??. Vector bundles with flat connection on X

First we wish to construct a map from local systems $\mathcal{F}$ on X to reps of $\pi_1(X)$.

[[WHAT IS A LOCAL SYSTEM]]

Fix base point $x \in X$, and a trivialisation of the sheaf $\mathcal{F}$. For loop $\gamma : [0, 1] \to X$ anchored at x, Let $\gamma : \mathcal{F}_x \to \mathcal{F}_x$ by concanotaion of inclusion and restriction maps from the sheaf. It transpires that in the case of locally constant sheaf, we can show that this depends only on the homotopic class of the curve. Thus we get a map from locally constant sheaves and representations of $\pi_1(X)$ to $\mathrm{Aut}(\mathcal{F})$

Secondly, we consider vector bundles with flat connections on X to locally constant sheafs. $\mathrm{Ker}(\nabla)$.

We have previously seen the monodromy representation of a flat connection (E, ∇) by considering parallel transport about loops.

The Riemann Hilbert correspondence gives us

$$\mathcal{M}_{\mathrm{Betti}} \to \mathcal{M}_{\mathrm{dR}} \tag{15}$$

equivalence as real analytic spaces, but the algebraic structures may differ.