Hodge theory

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1 Four worlds of Riemann surfaces

Definition 1.1. *A Riemann surface X is a complex manifold of dimension* 1.

examples: $\mathbb{P}$....

Denote two sheaves of particular interest. For open subset $U \subset X$,

$$\mathcal{O}_X(U) = \text{ holomorpic functions on } U \tag{1}$$

$$\mathcal{M}_X(U) = \text{ meromorphic functions on } U \tag{2}$$

Let T be a topological closed oriented 2-manifold with genus g, M a differential manifold homeomorphic to T, and Σ be a closed Riemann surface of genus g.

	Topological	Differential	Dolbeaut	algebraic
Classification results	T is homeomorphic to a sphere with g handles.	M is unique upto diffeomorphism.		GAGA
Fundamental group	$\pi_1(\Sigma_g) = \langle a_i, b_i \vert\ \Pi_{i=1}^{g} a_i b_i a_i^{-1} b^{-1} \rangle$			
Cohomology	Hurwicz	d'Rham	Dolbeaut	

Riemann surfaces are in 1-1 correspondence to projective curves, and holomorphic maps are in 1-1 correspondence to regular morphisms.

Example: Complex tori relate elliptic curves. $\Gamma(1, \omega)$ and j-invariants

2 Line Bundles

Definition 2.1. *A holomorphic line bundle $L \to X$ is a complex vector bundle. (Axioms stated here)*

The set of (holomorphic) line bundles (upto isomorphism) forms a group with tensor power acting as the binary operator. The trivial bundle is then the identity element, and inverse is given by duality. Denote this group $\mathrm{Pic}(X)$.

Definition 2.2. *Let L be a line bundle with trivialisation $\{U_i\}$ A holomorphic section s of line bundle ... (Axioms stated) A meromorphic section s of line bundle ... (Axioms stated) $\mathcal{O}_L$ denotes the sheaf of holomorphic sections.*

[[DEFINITION OF A SHEAF]]

Definition 2.3. *A sheaf $\mathcal{F}$ on X is an invertible if for all U_i in the cover $\{U_i\}$ such that $\mathcal{F}_i$ is a free $\mathcal{O}_{U_i}$ module.*

Theorem 2.4. *There exists a canonical isomorphism*

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \tag{3}$$

And isomorphic to the group of invertible sheaves on X under $\otimes$.

Definition 2.5. *The canonical bundle K_X is the determinant of the cotangent bundle.*

Theorem 2.6. *(Riemann - Roch)*

$$h^0(X, \mathcal{O}_L) - h^0(X, \mathcal{O}_{L^* \otimes K_X}) = \deg(L) + 1 - g \tag{4}$$

where $h := \dim H$

Theorem 2.7. *(Serre duality)*

$$H^k(X, \mathcal{O}_L) \cong H^{n-k}(X, \mathcal{O}_{L^* \otimes K})^* \tag{5}$$

3 Divisors

Definition 3.1. *A divisor* $D : X \to \mathbb{Z}$ *which is 0 for all but finitely many* $x \in X$*. Represent* D *as the formal sum* $\sum D(x)x$*. The set of all divisors is denoted* $\mathrm{Div}(X)$ *The degree of a divisor* D *is* $\deg(D) = \sum D(x)$*. For any function* f *have the associated divisor* $\mathrm{Div}(f) = \sum \mathrm{ord}_x(f)x$

Divisors form a group under formal addition in a natural way. We induce on the set of divisors a partial ordering induced by $\mathbb{Z}$. There are notable subgroups.

Say two divisors are equivalent if they differ by a principle divisors The class group is defined as

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)} \tag{6}$$

And the analogue for degree 0 .

[[STATE 0 ANALOGUE]]

For a line bundle we can define the associated divisor, which is defined upto principle divisor. That is ...

Theorem 3.2. *There exists*

$$\frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)} \cong \mathrm{Cl}(X) \cong \mathrm{Pic}(X) \tag{7}$$

And the 0 *analogue.*

Weiestrass problem. Given divisor D

[[STATE]]

4 Jacobian

Denote by $H^0(X, \Omega^1_X)$ the space of holomorphic 1-forms. Let

$$\lambda_C : H^0(X, \Omega^1_X) \to \mathbb{C} \tag{8}$$

$$\omega \mapsto \int_C \omega \tag{9}$$

By Stokes λ only depends on the class $[C] \in H_1(X, \mathbb{Z})$ of curve C.

Letting Λ be the image of λ.

Definition 4.1. *The jacobian of* X *is given by*

$$\mathrm{Jac}(X) = H^0(\Omega^1_X)/\Lambda \tag{10}$$

Thus we identify $H^0(X, \Omega^1_X)^*$ *with* $\mathbb{C}^g$

Definition 4.2. *Fix a base point* $p_0 \in X$ *define*

$$A : X \to \mathrm{Jac}(X) \tag{11}$$

$$p \mapsto \int \gamma_p \omega \tag{12}$$

where γ_p *is a path* p_0 *to* p*. Abel-Jacobi map.*

[[WHAT IS]]

Extend linearly to $A : \mathrm{Div}(X) \to \mathrm{Jac}(X)$*.*

Theorem 4.3. *(Abel) If* $D \in \mathrm{Div}_0$ *then* $A_0(D) = 0$ *iff* $D \in \mathrm{PDiv}(X)$*.*

Theorem 4.4. *(Jacobi) The map* A_0 *is surjective and* $\mathrm{Pic}_0 \cong \mathrm{Jac}(X)$*.*

5 Hodge theory

Let X now be compact complex manifold. The induced almost complex structure (here defined) We have the decomposition of the exterior algebra into (p, q) forms. Denote the sheaf section of $\Lambda^{p,q}X$ by $A^{p,q}(X)$

We can then decompose d into ∂ and $\bar{\partial}$

[[STATE HOW THIS IS DONE]]

Theorem 5.1. *(Dolbeault)*

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p) \tag{13}$$

Really long list defining the operators

[[ADD THIS LIST]]

Theorem 5.2. *(Hodge theorem)*

[[STATE]]

Theorem 5.3. $\mathcal{H}_{\bar{\partial}}^{p,q}(X) \cong H^{p,q}(X)$

[[WHAT IS THE CONTENT HERE]]

Theorem 5.4. $H_\Delta^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ *In the case where* $k = 1$ $H_\Delta^1(X, \mathbb{C}^*) \cong \mathrm{Jac}(X) \oplus H^0(X, \Omega_X^1)$

Hodge theory gives us the set of 1 dim representations $\pi_1(X)$ is isomorphic to the set of holomorphic line bundles with degree 0

[[MORE EXPLICITLY DISTINGUISH BETWEEN WHICH COHOMOLOGY IS BEING USED THORUGHOUT]]