VECTOR BUNDLES ON CURVES

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1. Vector bundles and locally free sheaves

In this section we introduce vector bundles and locally free sheaves and the correspondence between the two (see also [A1] . The best know case is rank one. Vector bundles of rank one (also called line bundles) correspond to locally free sheaves of rank one (invertible sheaves) and also to divisors modulo linear equivalence.

We start by recalling the definitions and basic properties of locally-free sheaves

1.1. Definition *A locally free sheaf of rank r on a curve C is a sheaf of modules $\mathcal{E}$ such that there is an open covering U_i of C and $\mathcal{E}(U_i) \cong (\mathcal{O}(U_i))^r$.*

When $r = 1$, the sheaf is called invertible.

Consider the intersections $U_i \cap U_j$ of two of the open sets of the covering. There, the sheaf is identified to the trivial rank r sheaf in two different ways coming from the inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Therefore, every such sheaf determines a set of invertible $r \times r$ matrices of functions giving the isomorphisms

$$(\mathcal{O}(U_i \cap U_j))^r \to (\mathcal{O}(U_i \cap U_j))^r$$

Conversely, given $r \times r$ invertible matrices $A_{i,j}$ of functions on $U_i \cap U_j$ satisfying the obvious compatibility conditions $A_{i,i} = Id$, $A_{j,k}A_{i,j} = A_{i,k}$, one can define a locally free sheaf by gluing the trivial sheaves $(\mathcal{O}(U_i))^r$ on $U_i \cap U_j$ using these isomorphisms.

1.2. Lemma *a) A coherent sheaf $\mathcal{E}$ on C is locally free if and only if the fibers $\mathcal{E}_P$ are free at every point $P \in C$.*

b) A subsheaf of a locally free sheaf is locally free.

c) A non-zero map from a rank one locally-free sheaf to a locally-free sheaf is injective.

Proof. (see [H] ex.5.7, p. 124) As a) is a local statement, we can work with an open covering by affine sets $SpecA_i$ on which the sheaf corresponds to an A_i-module M_i. We are assuming $(M_i)_P \cong ((A_i)_P)^r$. Let $m_1, ..., m_r \in M_i$ be such that their localisations provide a basis of

$(M_i)_P$. Consider the map $(A_i)^r \to M_i$ its kernel K and cokernel C are finitely generated and by construction their localisation at P is zero. Hence, there exists f such that $C_f = 0, K_f = 0$. Then, $(M_i)_f$ is free. This proves a). Now b) follows from a) using the fact that over a principal ideal domain, a submodule of a finitely generated free module is free.

Let now $f : \mathcal{E}' \to \mathcal{E}$ be a morphism of locally-free sheaves with $\mathcal{E}'$ of rank one. Let $\mathcal{K}$ be its kernel. From b), $\mathcal{K}$ is locally free. If $\mathcal{K} \neq 0$, then it has rank one. Hence, $\mathcal{E}'/\mathcal{K}$ is a torsion sheaf that inject into $\mathcal{E}$. As $\mathcal{E}$ is locally free, this implies that $\mathcal{E}'/\mathcal{K} = 0$. Hence, $\mathcal{E}' = \mathcal{K}$ and $f = 0$. □

Note that a quotient of a locally free sheaf by a subsheaf is not necessarily locally free. Nevertheless, we can make the subsheaf a little bit bigger so that the quotient is locally free

1.3. Lemma *Let $\mathcal{E}$ be a locally free sheaf of rank r and $\mathcal{E}'$ a subsheaf of rank r'. There exists then a subsheaf $\bar{\mathcal{E}}'$ of rank r' containing $\mathcal{E}'$ such that $\mathcal{E}/\bar{\mathcal{E}}'$ is locally free. In particular, if $\mathcal{E}'$ is maximal the quotient is already locally free.*

Proof. Let T be the torsion submodule of $M = \mathcal{E}/\mathcal{E}'$. The fiber of M at a point of C is a finitely generated module over a principal ideal domains. If $T = 0$, then the fibers of M are torsion-free and therefore free. Then, using 1.2 a), M is free.

Now define $\bar{\mathcal{E}}'$ as the kernel of the natural composition $\mathcal{E} \to M \to M/T$. The new quotient of $\mathcal{E}$ by $\bar{\mathcal{E}}'$ is M/T which has no torsion. Hence it is locally free. By definition of $\bar{\mathcal{E}}'$, it is clear that $\mathcal{E} \subset \bar{\mathcal{E}}'$. Moreover, from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E}' & \to & \mathcal{E} & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{\mathcal{E}}' & \to & \mathcal{E} & \to & M/T & \to & 0 \end{array}$$

and using the snake's lemma, one finds $\bar{\mathcal{E}}'/\mathcal{E}' \cong T$. Hence, $\bar{\mathcal{E}}'$ and $\mathcal{E}'$ have the same rank. □

1.4. Definition *A vector bundle of rank r over C is a variety E together with a morphism $\pi : E \to C$ such that there exists an open affine covering U_i of C and isomorphisms*

$$\varphi_i : (\pi)^{-1}(U_i) \to U_i \times \mathbf{A^r}$$

where $\mathbf{A^r}$ denotes the affine space of dimension r and such that in the intersections $U_i \cap U_j$, the composition

$$\varphi_j(\varphi_i)^{-1}_{|U_i \cap U_j} = (Id, \phi_{i,j}),\ \ \phi_{i,j} \in GL(r)$$

is given by linear maps.

Note that the transition functions $\phi_{i,j}$ satisfy the properties

$$\phi_{j,k}\phi_{i,j} = \phi_{i,k},\ \ \phi_{i,i} = Id.$$

Conversely, given a covering U_i of C and transition functions satisfying the properties above, one can construct a vector bundle by gluing the trivial vector bundles $U_i \times \mathbf{A^r}$ along the intersections $(U_i \cap U_j) \times \mathbf{A^r}$ using these maps.

1.5. Definition *A morphism of vector bundles over C, $E \to E'$ is a morphism of varieties that commutes with the projections to C and restricts to a linear map on each fiber.*

A subbundle F of a vector bundle E is a subvariety of E which is itself a bundle and such that the inclusion is a morphism of bundles.

The usual operations on vector spaces like direct sum, tensor product are defined for vector bundles in the natural way

1.6. Definition *A section of the vector bundle over an open set $U \subset C$ is a map $s: \ U \to E$ such that $\pi s = Id_U$. A global section is a section in which $U = C$*

1.7. Theorem.*There is a natural one to one correspondence between vector bundles and locally-free sheaves on a curve C.*

Proof. Given a vector bundle E, define a sheaf $\mathcal{E}$ as follows: $\mathcal{E}(U)$ is taken to be the set of sections of E over U This has natural module structure obtained by adding two sections on each fiber or multiplying with a function by evaluating and multiplying on each fiber. Moreover, If $U \subset V$ and $s \in \mathcal{E}(U)$, then $s_{|V}$ is a section in V and we get a natural map $\mathcal{E}(U) \to \mathcal{E}(V)$ which satisfies the obvious compatibility conditions if $W \subset V \subset U$.

We need to show that $\mathcal{E}$ is locally free. Take an open covering in which $E \to C$ is trivial $(\pi)^{-1}(U_i) \cong U_i \times \mathbf{A^r}$. Consider the canonical local "coordinate" sections

$$\begin{array}{rccc} x_i: & U_i & \to & U_i \times \mathbf{A^r} \\ & p & \to & (p,(0...1..0)) \end{array}$$

where the 1 appears in the i^{th} place. Then, every section s can be written as $s = f_1x_1 + ... + f_rx_r$ for some functions $f_1, ..., f_r$. Hence

$$\begin{array}{ccc} \mathcal{E}(U_i) & \to & (\mathcal{O}(U_i))^r \\ s & \to & (f_1, ...f_r) \end{array}$$

gives the isomorphism required.

Conversely, given a locally free sheaf $\mathcal{E}$, one can define a vector bundle as follows

$$E = \{(P,t) | P \in C,\ t \in \mathcal{E}_P / \mathcal{M}_P \mathcal{E}_P\}$$

Here $\mathcal{M}_P$ denotes the maximal ideal of the local ring $\mathcal{O}_P$. The map $\pi : E \to C$ is projection on the first component.

We need to show that in this way we obtain a vector bundle. Take an open covering U_i over which $\mathcal{E}$ is free i.e. $\mathcal{E}(U_i) \cong (\mathcal{O}(U_i))^r$. As we are working over an algebraically closed field $\mathbf{C}$, $\mathcal{O}_P/\mathcal{M}_P \cong \mathbf{C}$. Hence, $\mathcal{E}_P/\mathcal{M}_P\mathcal{E}_P \cong \mathbf{A^r}$.

On the intersection $U_i \cap U_j$ of two of these open sets the transition functions on $\mathcal{E}(U_i \cap U_j)$ are given by matrices $A_{i,j} : (\mathcal{O}(U_i \cap U_j))^r \to (\mathcal{O}(U_i \cap U_j))^r$. Then on this intersection of open sets $\varphi_j(\varphi_i)^{-1} = (Id, \phi_{ij})$ with $(\phi_{i,j})_P$ being given by reducing $A_{i,j}$ modulo $\mathcal{M}_P$. As the $A_{i,j}$ are isomorphisms, the reduction modulo $\mathcal{M}_P$ gives an element of the general linear group as required.

Given a map of locally-free sheaves, one defines a map of the associated bundles by taking the reduction modulo P on each fiber.

Conversely, given a map of vector bundles $E \to E'$ and a section $s :\ C \to E$, the composition is a section $C \to E'$. Hence if $\mathcal{E}, \mathcal{E}'$ are the locally-free sheaves associated to E, E', by restricting to open sets, one obtains a morphism $\mathcal{E}(U) \to \mathcal{E}'(U)$ compatible with restrictions. $\square$

It is worth pointing out that subsheaves of a locally-free sheaf do not correspond to subbundles of a bundle: given an injective map of sheaves $\mathcal{E} \to \mathcal{E}'$, the map may fail to be injective when reduced modulo the maximal ideal of some point.

Every vector bundle has the trivial section

$$\begin{array}{ccc} C & \to & E \\ p & \to & (p,0) \end{array}$$

1.8. Lemma *A non-trivial global section of a vector bundle E correspond to a non-zero map of sheaves $\mathcal{O} \to \mathcal{E}$ with $\mathcal{E}$ the sheaf associated to E.*

Proof. Given a section $s : C \to E$, define a map of sheaves by

$$\begin{array}{ccc} \mathcal{O}(U) & \to & \mathcal{E}(U) \\ 1 & \to & s_{|U} \end{array}$$

Conversely, given a morphism of sheaves $f : \mathcal{O} \to \mathcal{E}$ it induces a map on each fiber $f_P :\ \mathcal{O}_P/\mathcal{M}_P \to \mathcal{E}_P/\mathcal{M}_P\mathcal{E}_P$. Define a vector bundle map

$$\begin{array}{ccc} C & \to & E \\ p & \to & (p, f_P(1)) \end{array}$$

$\square$

We want to recall the correspondence between line bundles, Weil divisors and Cartier divisors in the particular case of a non-singular curve. For more details and a much more general setting see for example [H] Ch II section 6.

A (Weil)divisor on a curve is a finite sum $D = \sum n_P P$ where the n_p are integers.The divisor is effective if all the n_P are positive. If f is a meromorphic function on C, the divisor corresponding to f can be defined as $(f) = \sum ord_P(f)P$ where $ord_P(f)$ is the order of f at P which is positive if P is a zero of f, negative if P is a pole and zero otherwise. Two divisors D and D' are linearly equivalent if their difference is the divisor of a function. The degree of a divisor is the integer defined as $\sum n_P$

A Cartier divisor on C is given by a covering U_i of C together with functions f_i one on each U_i so that $\frac{f_i}{f_j}$ is holomorphic (and invertible) on $U_i \cap U_j$. We identify two Cartier divisors (U_i, f_i) and (V_j, g_j) if $f_{i|V_j} = g_{j|U_i}$. Therefore, every Cartier divisor can be identified to a Cartier divisor on a covering which is a refinement of any given covering. In this way we can multiply divisors as follows: assume that they are defined on the same covering and take the product of the corresponding functions.

A principal divisor is given by the cover consisting of C alone and a function f on C. The set of principal divisors is closed under the product defined above and the quotient of the set of divisors by those that are principal has a group structure.

Given a Cartier divisor, one can associate to it a Weil divisor by considering on each open set U_i the zeroes minus the poles of (f_i). This is well defined as f_i/f_j is holomorphic and invertible on $U_i \cap U_j$ and therefore has neither zeroes nor poles on these intersections. Conversely, from a Weil divisor D, one can construct a Cartier divisor by choosing open sets that contain at most one of the point on the support of D and functions that vanish at these points with the assigned multiplicity.

Given a Cartier divisor, one can define a locally-free rank one sheaf by taking the trivial sheaf $\mathcal{O}(U_i)$ and gluing them by the isomorphisms $\frac{f_j}{f_i}$ on $U_i \cap U_j$.

Conversely, given an invertible sheaf and a trivialisation $U_i, f_{i,j}$, one can define a Cartier divisor: Pick one arbitrary open set U_0 and define the Cartier divisor as $(U_i, f_{i,0})$. As $f_{j,0}f_{0,j} = 1,\ f_{0,j} = \frac{1}{f_{j,0}}$. As $f_{0,j}f_{i,0} = f_{i,j}$, we obtain $f_{i,j} = \frac{f_{i,0}}{f_{j,0}}$ which is a unit on $U_i \cap U_j$ (because it gives the isomorphism of $\mathcal{O}(U_i \cap U_j) \to \mathcal{O}(U_i \cap U_j))$. Therefore $\frac{f_{i,0}}{f_{j,0}}$ is holomorphic and invertible on $U_i \cap U_j$ as required.

1.9. Lemma *If $\mathcal{L}$, $\mathcal{L}'$ are invertible sheaves and D, D' are Weil divisors corresponding to them, then $\mathcal{L} \otimes \mathcal{L}'$ corresponds to $D + D'$.*

Proof. : Given two invertible sheaves $\mathcal{L}$, $\mathcal{L}'$ both trivialising over a covering U_i and with transition function $f_{i,j}, f'_{i,j}$, we have isomorphisms

$$\begin{array}{cccc} \mathcal{L} \otimes \mathcal{L}' & \cong & \mathcal{O}(U_i \cap U_j) \otimes \mathcal{O}(U_i \cap U_j) & \to & \mathcal{O}(U_i \cap U_j) \\ & & a \otimes b & \to & ab \\ & & \downarrow f_{i,j} \otimes f'_{i,j} & & \downarrow f_{i,j} f'_{i,j} \\ & & \mathcal{O}(U_i \cap U_j) \otimes \mathcal{O}(U_i \cap U_j) & \to & \mathcal{O}(U_i \cap U_j) \end{array}$$

The divisor of the function $f_{i,j} f'_{i,j}$ is the sum of the divisors corresponding to $f_{i,j}$ and $f'_{i,j}$ as required. □

1.10. Definition. *Let D be an effective divisor, $D = \sum n_P P$, $n_P \geq 0$. Define a sheaf $\mathcal{F}$ on the support of D by $\mathcal{F}_P = \mathcal{C}^{n_P}$. Define the skyscraper sheaf $\mathbf{C}_D$ as the extension of $\mathcal{F}$ by zero outside of D.*

1.11. Lemma *A line bundle $\mathcal{L}$ corresponds to an effective divisor D if and only if $\mathcal{L}$ has a section. In this case one has an exact sequence*

$$0 \to \mathcal{O} \to \mathcal{L} \to \mathbf{C}_D \to 0$$

where $\mathbf{C}_D$ denotes the skyscraper sheaf with support on D. One then writes $\mathcal{L} = \mathcal{O}(D)$

Proof. : Assume that $\mathcal{L}$ corresponds to an effective divisor D. Take a representation as a Cartier divisor (U_i, f_i) where f_i has zeros on $U_i \cap D$ and no poles on U_i. By definition of the associated invertible sheaf $\mathcal{L}$, $\mathcal{L}(U_i)$ is isomorphic to $\mathcal{O}(U_i)$ and the gluing on the intersections $U_i \cap U_j$ is given by $\frac{f_j}{f_i}$. Define now the morphism of sheaves

$$\begin{array}{ccccc} \mathcal{O}(U_i) & \to & \mathcal{L}(U_i) & \cong & \mathcal{O}(U_i) \\ 1 & \to & & & f_i \end{array}$$

This definition is compatible with the gluing and therefore gives rise to a global map

$$\mathcal{O} \to \mathcal{L}.$$

One checks locally that the map is injective and an isomorphism except where the f_i have zeroes, namely on D. By construction, the cokernel at a point P in D has dimension n_P. Hence, the sequence above is exact as claimed.

Conversely, assume that $\mathcal{L}$ has a section. Take an open covering where the line bundle is trivial. Over an open set of this trivialisation, the map must be given by

$$\begin{array}{ccccc} \mathcal{O}(U_i) & \to & \mathcal{L}(U_i) & \cong & \mathcal{O}(U_i) \\ 1 & \to & & & f_i \end{array}$$

where the f_i are regular functions (and therefore do not have poles) on U_i. As by assumption, these are local representations of a global morphism, the transition functions for $\mathcal{L}$ are given by $\frac{f_j}{f_i}$. Hence, $\mathcal{L}$ corresponds to an effective divisor. □

1.12. Riemman-Roch Theorem *If $\mathcal{L}$ is an invertible sheaf, then the Euler-Poincare characteristic can be computed as*

$$\chi(\mathcal{L}) = deg(\mathcal{L}) + \chi(\mathcal{O}) = deg(\mathcal{L}) + 1 - g$$

where $g = h^1(\mathcal{O})$ is by definition the genus of the curve.

Proof. Assume first that D is effective. Then we have the exact sequence above

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathbf{C}_D \to 0$$

This gives rise to a long exact sequence of homology from which one computes that $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \chi(\mathbf{C}_D)$. As $\mathbf{C}_D$ is a skyscraper sheaf, it has support on a zero-dimensional set. Hence, $h^1(\mathbf{C}_D = 0, \chi(\mathbf{C}_D) = h^0(\mathbf{C}_D)$ and this last number coincides with $deg(D)$ by definition of $\mathcal{C}_D$.

In the general case, assume that $\mathcal{L}$ corresponds to the divisor $D - D'$ with both D, D' effective. From 1.9, $\mathcal{L} = \mathcal{O}(D) \otimes (\mathcal{O}(D'))^*$. Therefore, tensoring the exact sequence above with $(\mathcal{O}(D'))^* \cong \mathcal{O}(-D')$, we obtain

$$0 \to \mathcal{O}(-D') \to \mathcal{L} \to \mathbf{C}_D(-D') \to 0$$

Hence, $\chi(\mathcal{L}) = \chi(\mathcal{O}(-D')) + degD$. On the other hand, replacing D by D' in the sequence in 1.11 and tensoring with $\mathcal{O}(-D') = (\mathcal{O}(D'))^*$, we obtain

$$0 \to \mathcal{O}(-D') \to \mathcal{O} \to \mathbf{C}'_D(-D') \to 0$$

Hence, $\chi(\mathcal{O}(-D')) = \chi(\mathcal{O}) - degD'$. Putting the two equations together, the result follows. □

1.13. Definition *The determinant bundle of a vector bundle E of rank r (or a locally-free sheaf of rank r), is defined as the r^{th}- wedge power of the bundle (or sheaf).*

1.14. Definition *The degree of a vector bundle E of rank r is defined as the degree of the associated locally free sheaf which is in turn defined as*

$$deg(\mathcal{E}) = \chi(\mathcal{E}) - r\chi(\mathcal{O}) = \chi(E) - r(1-g)$$

The degree of a locally-free sheaf or of the corresponding vector bundle could also be defined as the degree of the determinant. Our initial definition is easier to generalize to the case of torsion-free sheaves over a singular curve. We will soon check that both definitions agree.

1.15. Lemma. *Given a locally-free sheaf $\mathcal{E}$ of rank r there exists an exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}' \to 0$$

where $\mathcal{L}$ is an invertible sheaf and $\mathcal{E}'$ a locally-free sheaf of rank $r-1$

Proof. We want to show first that tensoring $\mathcal{E}$ with a line bundle of positive degree $\mathcal{O}(D)$, the resulting sheaf $\mathcal{E}(D) = \mathcal{E} \otimes \mathcal{O}(D)$ has a section (this is in fact true for all coherent sheaves). If $\mathcal{E}$ has a section, we are done. Otherwise, choose an effective divisor D such that $rdegD > h^1(\mathcal{E})$. From the exact sequence in 1.11 tensored with $\mathcal{E}$ and taking homology, we obtain

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{E}(D)) \to H^0(\mathcal{E}_D) \to H^1(\mathcal{E})$$

where we write $(\mathcal{E})_D$ for $\mathcal{E}$ tensored with the skyscraper sheaf $\mathbf{C}_D$.

As $\mathcal{E}_D$ is a skyscraper sheaf with support on D and each fiber of dimension r, its space of sections has dimension $rdegD$. Then by our choice of $deg(D)$, the last map in the exact sequence cannot be injective and the result follows. We obtain in this way a non-zero map $\mathcal{O} \to \mathcal{E}(D)$ and therefore a non-zero map $\mathcal{O}(-D) \to \mathcal{E}$ whose image is a rank one locally-free sheaf. If the cokernel of this map is not locally free, then using 1.3, we can obtain a larger rank one subsheaf of $\mathcal{E}$ such that the quotient is locally free. □

1.16. Lemma *If $\mathcal{E}_1, \mathcal{E}_2$ are locally-free sheaves of ranks r_1, r_2, then $deg(\mathcal{E}_1 \otimes \mathcal{E}_2) = r_1 deg(\mathcal{E}_2) + r_2 deg(\mathcal{E}_1)$*

Proof. We use induction on r_2. Assume first $r_2 = 1$. If $\mathcal{E}_2 = \mathcal{O}(D_2)$, then using the exact sequence in 1.11 and tensoring with $\mathcal{E}_1$, one obtains

$$0 \to \mathcal{E}_1 \to \mathcal{E}_1(D_2) \to \mathcal{E}_{1D_2} \to 0.$$

This shows that $\chi(\mathcal{E}_1(D_2)) = \chi(\mathcal{E}_1) + r_1 deg(D_2)$. Using the definition of degree, the result follows. For a general line bundle $\mathcal{E}_2$, $\mathcal{E}_2 = \mathcal{O}(D_2 - D_2')$. We have the exact sequence

$$0 \to \mathcal{E}_1(D_2 - D_2') \to \mathcal{E}_1(D_2) \to \mathcal{E}_{1D_2'} \to 0$$

Hence $\chi(\mathcal{E}_1(D_2 - D_2')) = \chi(\mathcal{E}_1(D_2)) - r_1 deg(D_2')$. We already know the result for $\mathcal{E}_1(D_2)$. Hence, the result follows from the definition of degree.

We now assume that the result has been proved for all sheaves of rank $r_2 - 1$. Using 1.15, there is an exact sequence

$$0 \to \mathcal{E}_2' \to \mathcal{E}_2 \to \mathcal{E}_2'' \to 0$$

with $\mathcal{E}_2', \mathcal{E}_2''$ locally-free of ranks one and $r_2 - 1$. Tensoring this sequence with $\mathcal{E}_1$, computing Euler-Poincare characteristic and using the induction assumption, the result follows. $\square$

1.17. Definition *A sheaf $\mathcal{E}$ (or the corresponding vector bundle if $\mathcal{E}$ is locally-free) is said to be generated by global sections if the natural map*

$$H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$$

is onto.

For example, a line bundle is generated by global sections if and only if the complete linear series has no fixed points.

A quotient of a sheaf that is generated by global sections is again generated by global sections.

1.18. Lemma. *If $\mathcal{E}$ is a locally free sheaf on C, there exists a positive divisor D on C such that $\mathcal{E}(D)$ is generated by global sections.*

Proof. The result is in fact true for all coherent sheaves and this can be taken as definition of ampleness (see [H] p.153).

An alternative ad hoc proof can be obtained by induction: for line bundles the result is true. Using Riemman-Roch and Serre duality, one can check that any line bundle of degree at least $2g$ has no fixed points.

We can now use 1.15. Tensoring with a line bundle of sufficiently high degree, we can assume that we have an exact sequence

$$0 \to \mathcal{E}'(D) \to \mathcal{E}(D) \to \mathcal{E}''(D) \to 0$$

where both $\mathcal{E}'(D)$ and $\mathcal{E}''(D)$ are generated by global sections. Moreover, as for line bundles of sufficiently high degree the first cohomology group vanishes, we can assume that $h^1(\mathcal{E}'(D)) = 0$. Consider now the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{E}'(D)) \otimes \mathcal{O} & \to & H^0(\mathcal{E}(D)) \otimes \mathcal{O} & \to & H^0(\mathcal{E}''(D)) \otimes \mathcal{O} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}' & \to & \mathcal{E} & \to & \mathcal{E}'' & \to & 0 \end{array}$$

Then, the result follows from the snake's Lemma. $\square$

1.19. Theorem *(*[A3] *p.426, Thm 2) Given a locally-free sheaf E of rank r generated by global sections, there exists an exact sequence*

$$0 \to \mathcal{O}^{r-1} \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is invertible.

Proof. By assumption, we have a surjective map

$$H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E} \to 0$$

We denote by $\mathcal{E}'$ the subsheaf of $H^0(\mathcal{E}) \otimes \mathcal{O}$ that is the kernel of this map. From 1.2, $\mathcal{E}'$ is locally free.

As we are working over an algebraically closed field, for each point $P \in C$, $H^0(\mathcal{E}) \otimes \mathcal{O}_P/\mathcal{M}_P \cong H^0(\mathcal{E})$. Define $V_P = Im(\mathcal{E}'_P/\mathcal{M}_P\mathcal{E}'_P \to H^0(E))$. As $H^0(\mathcal{E}) \to \mathcal{E}_P/\mathcal{M}_P$ is onto, $dimV_P = h^0(\mathcal{E}) - r$. Define now

$$V = \cup_P(V_P) \subset H^0(\mathcal{E}).$$

then, $dim(V) \leq h^0(\mathcal{E}) - r + 1$. Therefore, there exists a linear space W of dimension $r-1$ in $H^0(\mathcal{E})$ not intersecting V. By definition the composition of the inclusion of $W \otimes \mathcal{O}$ in $H^0(\mathcal{E}) \otimes \mathcal{O}$ with the natural map to $\mathcal{E}$ is injective. It remains to show that the cokernel $\mathcal{L}$ has no torsion (and therefore is a torsion-free sheaf of rank one). This can be proved as follows: by definition of W for every point P, the map $W \otimes \mathcal{O}_P/\mathcal{M}_P \to \mathcal{E}_P/\mathcal{M}_P\mathcal{E}_P$ is an injective map of vector spaces. Therefore, the cokernel is a one-dimensional vector space. Hence $\mathcal{L}_P$ cannot have torsion. ☐

1.20. Lemma *Let*

$$0 \to E' \xrightarrow{i} E \xrightarrow{\pi} E'' \to 0$$

be an exact sequence of vector bundles of ranks r', r, r'' respectively. Then,

a)

$$\wedge^r E \cong \wedge^{r'} E' \otimes \wedge^{r''} E''$$

b) If $r' = 1$, for all $k \leq r$, one has exact sequences

$$0 \to E' \otimes \wedge^{k-1} E'' \to \wedge^k E \to \wedge^k E'' \to 0$$

Proof. We prove b), the proof of a) is similar.

The map $E \to E''$ can be described locally over open sets U by

$$\begin{array}{ccc} U \times \mathbf{C}^r & \to & U \times \mathbf{C}^{r''} \\ (p, e) & \to & (p, \pi(e)) \end{array}$$

We write here $\mathbf{C^r}$ rather than $\mathbf{A^r}$ to emphasize the vector bundle structure.

The map

$$\wedge^k E \to \wedge^k E''$$

is defined as $\wedge^k(\pi)$.

The map

$$E' \otimes \wedge^{k-1} E'' \to \wedge^k E$$

is defined as follows: Let $(p, e"_1), ..., (p, e"_{k-1}) \in U \times \mathbf{C^{r"}}$ and $(p, e') \in U \times \mathbf{C}$. Take $(p, e_1), ..., (p, e_{k-1}) \in U \times \mathbf{C^r}$, $\pi((p, e_i)) = ((p, e_i"))$. Map $(p, e"_1 \wedge ... \wedge e"_{k-1} \otimes e') \to (p, e_1 \wedge ... \wedge e_{k-1} \wedge e')$. This does not depend on the choice of the e_i: If $\pi(\bar{e}_i) = \pi(e_i)$, then $(p, t_i) = (p, \bar{e}_i - e_i)$ is in E'. By assumption, E' has rank one, therefore $\wedge^2 E' = 0$. Then $e_1 \wedge ... \wedge t_i \wedge ... \wedge e_{k-1} \wedge e' = 0$ □

1.21. Theorem *The degree of a locally-free sheaf as defined in 1.14 equals the degree of its determinant*

Proof. Let $\mathcal{E}$ be a torsion-free sheaf of rank r. From 1.18 and 1.19 there exists an effective divisor D and an exact sequence

$$0 \to \mathcal{O}^{r-1} \to \mathcal{E}(D) \to L \to 0$$

From the additivity of the Euler-Poincare characteristic on exact sequences and the definition of degree, $deg(\mathcal{E}(D)) = deg\mathcal{L}$. From 1.20 the determinant of $\mathcal{E}(D)$ is $\wedge^r(\mathcal{E}(D)) = \mathcal{L}$. From 1.16

$$deg(\mathcal{E} \otimes \mathcal{O}(D)) = deg(\mathcal{E}) + r deg(D)$$

One can then complete the proof by using

$$\wedge^r(\mathcal{E} \otimes \mathcal{O}(D)) = (\wedge^r \mathcal{E}) \otimes \mathcal{O}(rD)$$

This is clear if $r = 1$. In the general case, it can be checked by induction using 1.15 and 1.20. □

2. Extensions

One of the most useful ways to give vector bundles is by presenting them as extensions of other bundles. We start by recalling some results from homological algebra (see[R]).

2.1. Definition *A module P is projective if for every surjective map $\pi : M \to M"$ and map $\varphi : P \to M"$ there exists a "lifting" $\bar{\varphi} : P \to M$ such that $\pi\bar{\varphi} = \varphi$. Equivalently, a module is projective if and only if the functor $Hom(P, -)$ is exact.*

A module I is injective if for every injective map $i : M' \to M$ and map $\varphi : M' \to I$ there exists an "extension" $\bar{\varphi} : M \to I$ such that $\bar{\varphi}i = \varphi$. Equivalently, a module is injective if and only if the functor $Hom(-, I)$ is exact.

For example, free modules are projective. Therefore, every module M has a projective resolution, namely an exact sequence of the form

$$(*) ... \to P_n \overset{d_n}{\to} P_{n-1} \to ... \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective

Similarly it can be proved that every module N has an injective resolution

$$(**) 0 \to N \to I_0 \to I_1 \to ... \to I_n \overset{d_n}{\to} I_{n+1} \to ...$$

with all the I_j injective.

2.2. Definition *The module $Ext^n(M, N)$ is defined as $Ker d^*_n / Im d^*_{n+1}$ where $d^*_n : Hom(P_{n-1}, N) \to Hom(P_n, N)$ is the n^{th} map in the sequence obtained from (*) by applying the functor $Hom(-, N)$.*

Alternatively, the module $Ext^n(M, N)$ is defined as $Ker d^{**}_{n-1} / Im d^{**}_n$ where $d^{**}_n : Hom(M, I_n) \to Hom(M, I_{n+1})$ is the n^{th} map in the sequence obtained from (**) by applying the functor $Hom(M, -)$.

It can be checked that these definitions do not depend on the resolutions chosen and that both definitions give the same module.

By the general properties of derived functors, given a short exact sequence of modules

$$0 \to T' \to T \to T" \to 0,$$

one obtains two long exact sequences

$$0 \to Hom(M, T') \to Hom(M, T) \to Hom(M, T") \to$$
$$\to Ext^1(M, T') \to Ext^1(M, T) \to Ext^1(M, T") \to Ext^2(M, T')...$$

and

$$0 \to Hom(T", N) \to Hom(T, N) \to Hom(T', N) \to$$

$$\to Ext^1(T", N) \to Ext^1(T, N) \to Ext^1(T', N) \to Ext^2(T", N)...$$

It is worth pointing out that if N is free, then $Ext^i(M, N) = 0$. Moreover, Ext is additive in both M, N, namely

$$Ext^i(M_1 \oplus M_2, N) \cong Ext^i(M_1, N) \oplus Ext^i(M_2, N)$$

$$Ext^i(M, N_1 \oplus N_2) \cong Ext^i(M, N_1) \oplus Ext^i(M, N_2)$$

We will be interested mainly on Ext^1 because of its relationship with extensions in the following sense

2.3. Definition. *An extension of a module $M"$ by a module M' is an exact sequence*

$$0 \to M' \to M \to M" \to 0.$$

Two such sequences are said to be equivalent if there exists an isomorphism $\varphi : M \to M$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M" & \to & 0 \\ & & \downarrow Id & & \downarrow \varphi & & \downarrow Id & & \\ 0 & \to & M' & \to & M & \to & M" & \to & 0. \end{array}$$

We can denote by $e(M", M')$ the set of extensions of $M"$ by M' modulo this equivalence.

This can be turned into a covariant functor $e^{M"} = e(M", -)$: given a map $\varphi : M_1' \to M_2'$ define $e^{M"} : e(M", M_1') \to e(M", M_2')$ as follows. Given an extension $0 \to M_1' \to M_1 \to M" \to 0$, define an extension of $M"$ by M_2' by taking M_2 as a push out diagram in the left corner square below

$$\begin{array}{ccccccccc} 0 & \to & M_1' & \xrightarrow{i} & M_1 & \to & M" & \to & 0 \\ & & \downarrow \varphi & & \downarrow & & \downarrow & & . \\ 0 & \to & M_2' & \to & M_2 & \to & M" & \to & 0 \end{array}$$

$M_2 = M_1 \times M_2' / \{(\varphi(m_1'), i(m_1')) | m_1' \in M_1'\}$.

Similarly, define a contravariant functor $e(-, M')$: given a map $\varphi : M"_1 \to M"_2$ define $e_{M'} : e(M_2", M') \to e(M_1", M')$ as follows. Given an extension $0 \to M' \to M_2 \to M"_2 \to 0$, define an extension of $M"_1$ by M' by taking M_1 as a pull-back diagram in the right corner square below

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M_1 & \to & M_1" & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & . \\ 0 & \to & M' & \to & M_2 & \xrightarrow{\pi} & M_2" & \to & 0 \end{array}$$

$M_1 = \{(m_2, m"_1) \in M_2 \times M_1" | (\varphi(m_2") = \pi(m_2)\}$.

2.4. Theorem. *The elements of* $Ext^1(M", M')$ *correspond bijectively to extensions of* $M"$ *by* M' *modulo the equivalence defined above.*

The functors $e(-, M')$ *and* $Ext^1(-, M')$*(resp the functors* $e(M", -)$ *and* $Ext^1(M", -)$*) are naturally equivalent.*

Proof. (sketch, see[R] p.202-206 for details) Consider an extension of $M"$ by M'.

$$0 \to M' \to M \to M" \to 0$$

Take an injective resolution of M'

$$0 \to M' \to I'_0 \to I'_1 \to ...$$

By the properties of injective modules, and the definition of kernels, one can complete a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M" & \to & 0 \\ & & \downarrow Id & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \to & M' & \to & I'_0 & \to & I'_1 & \to & I'_2 \end{array}$$

Then, φ determines an element of $Ext^1(M", M')$

Conversely, given $e \in Ext^1(M'', M')$ one has a corresponding map $M" \to I'_1$ such that the composition $M" \to I'_1 \to I'_2$ is zero. Construct then a push out diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M" & \to & 0 \\ & & \downarrow Id & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \to & M' & \to & I'_0 & \stackrel{d_0}{\to} & I'_1 & \to & I'_2 \end{array}$$

where $M = \{(i_0, m")|d_0(i_0) = \varphi(m")\}$. Then the first row is exact.

A dual construction could be carried out with projective resolutions of $M"$. $\square$

We will later need the following basic properties of Ext

2.5. Lemma *a) The zero element* $0 \in Ext^1(M", M')$ *corresponds to the direct sum extension (also called trivial extension)*

$$0 \to M' \to M' \oplus M" \to M" \to 0.$$

b') If $M' = M'_1 \oplus M'_2$ *and* $e = (e_1, e_2) \in Ext^1(M", M') = Ext^1(M", M'_1) \oplus Ext^1(M", M'_2)$ *is such that* $e_2 = 0$*, then* $M = M_1 \oplus M'_2$ *where* M_1 *is the extension corresponding to* e_1*.*

b") If $M" = M_1" \oplus M_2"$ *and* $e = (e_1, e_2) \in Ext^1(M", M') = Ext^1(M_1", M') \oplus Ext^1(M_2", M')$ *is such that* $e_2 = 0$*, then* $M = M_1 \oplus M_2"$ *where* M_1 *is the extension corresponding to* e_1*.*

c) Let

$$0 \to M' \to M \to M" \to 0.$$

be an exact sequence. Consider the functorial map

$$e^{M"} : \ Hom(T, M") \to e(T, M").$$

Using the identification of $e(T, M")$ *with* $Ext^1(T, M")$*,* $e^{M"}$ *is identified to the connecting homomorphism in the long exact sequence*

$$0 \to Hom(T, M') \to Hom(T, M) \to Hom(T, M") \to Ext^1(T, M') \to ...$$

Proof. a) We use the notations in 2.4. The zero element in $Ext^1(M", M')$ can be represented by the zero map $M' \to I'_1$. In the construction of the corresponding extension, one must then take

$$M = \{(i_0, m")|d_1(i_0) = 0\} = \{(i_0, m")|i_0 \in M'\} = M' \oplus M"$$

as stated.

In order to prove b'), assume that $M' = M'_1 \oplus M'_2$. An injective resolution of M' can be obtained by taking the direct sum of an injective resolution of M'_1 and an injective resolution of M'_2. Then the proof follows the lines of part a). The proof of b" works better using the dual construction with projective resolutions of $M"$.

c) Consider injective resolutions of $M', M, M"$ fitting in an exact diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \xrightarrow{j} & M & \xrightarrow{\pi} & M" & \to & 0 \\ & & \downarrow d' & & \downarrow d & & \downarrow d" & & \\ 0 & \to & I'_0 & \xrightarrow{j_0} & I_0 & \xrightarrow{\pi_0} & I"_0 & \to & 0 \\ & & \downarrow d'_0 & & \downarrow d_0 & & \downarrow d"_0 & & \\ 0 & \to & I'_1 & \xrightarrow{j_1} & I_1 & \xrightarrow{\pi_1} & I"_1 & \to & 0 \end{array}$$

The connecting homomorphism in the long exact sequence is computed as follows: let $\varphi : T \to M"$ be a morphism. There exists then

$$\varphi_0 : T \to I_0, \ d"\varphi = \pi_0\varphi_0$$

$$\varphi_1 : T \to I'_1, \ j_1\varphi_1 = d_0\varphi_0$$

Then, the image of φ in the connecting homomorphism is the class of φ_1 in $Ext^1(T, M')$. The extension of T by M' corresponding to this class has central term (see 2.4)

$$X = \{(t, i'_0) \in T \times I'_0|\varphi_1(t) = d'_0(i'_0)\}$$

On the other hand $e^{M"}(\varphi)$ is an extension of T by M' with central term (see 2.4)

$$\bar{X} = \{(t, m) \in T \times M|\varphi(t) = \pi(m)\}$$

One can define a morphism $\psi : \ X \to \bar{X}$ as follows: if $\varphi_1(t) = d'_0(i'_0)$ then $d_0(\varphi_0(t)) = j_1(\varphi_1(t)) = j_1(d'_0(i'_0)) = d_0(j_0(i'_0))$. Hence $\varphi_0(t) - j_0(i'_0) = d(m), \ m \in M$. Then, $\psi((t, i'_0)) = (t, m)$.

Define similarly a morphism $\bar{\psi}: \bar{X} \to X$ as follows: if $\varphi(t) = \pi(m)$ then $\pi_0(\varphi_0(t)) = d"(\varphi(t)) = d"(\pi(m)) = \pi_1(d(m))$. Hence $\varphi_0(t) - d(m) = j_0(i'_0),\ i'_0 \in I'_0$. Then, $\bar{\psi}((t,m)) = (t, i'_0)$.

The two morphisms defined above are inverse of each other.

□

The definitions above can be sheafified . If $\mathcal{F}, \mathcal{G}$ are sheaves, a morphism $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for every open set $U \subset X,\ \mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}(U)$ modules. Denote by $Hom(\mathcal{F}, \mathcal{G})$ the group of homomorphisms of $\mathcal{F}$ to $\mathcal{G}$.

Then $Ext^i(\ ,\ )$ are defined as the derived functors of $Hom(\ ,\ )$ (see [Lan] for some of the properties of the constuction).

2.6. Proposition *a) If $E"$ is locally-free, $Ext^1(E", E') \cong H^1((E")^* \otimes E')$.*

b) If $E' = \mathcal{O}^r$, the class of non-split E fitting in an extension

$$0 \to \mathcal{O}^r \to E \to E" \to 0$$

with non-split E corresponds with injective maps $\mathbf{C}^r \to H^1((E")^*)$ modulo the action of the linear group of order r.

c) Given an exact sequence as in b), the coboundary map in homology $H^0(E") \to H^1(E')$ is obtained by taking cup-product with the class of the extension $e \in H^1(E"^* \otimes E')$.

Proof. a) For any sheaves $E', E"$, one can define a natural map

$$(E")^* \otimes E' = Hom(E", \mathcal{O}) \otimes E' \to Hom(E", E')$$

If $E"$ is locally-free, this gives rise to an isomorphism. Then, the derived functor of the sheaf of global sections $H^0((E")^* \otimes -)$ coincides with the derived functor of $Hom(E", -)$.

c) From 2.5 c), the class of the extension in $Ext^1(E", E')$ is the image of the identity map in $Hom(E", E")$ by the connecting homomorphism in the long exact sequence

$$...Hom(E", E) \to Hom(E", E") \to Ext^1(E", E') \to Ext^1(E", E)...$$

With the identification of the functors $Hom(E", -)$ and $H^0((E")^* \otimes -)$, this becomes the long exact sequence

$$...H^0((E")^* \otimes E) \to H^0((E")^* \otimes E") \to H^1((E")^* \otimes E') \to H^1((E")^* \otimes E)...$$

obtained from the long exact sequence $H^0(E) \to H^0(E") \to H^1(E') \to H^1(E)...$ by tensoring with $Id_{E"}$ Then the statement follows from basic linear algebra.

b) (See [NR] Lemma 3.3 for a statement of a similar sort). If $E' = \mathcal{O}^r$, then $Ext^1(E', \mathcal{O}^r) = H^1((E")^*)^r$. An element in this vector space corresponds to a map $\mathbf{C}^r \to H^1((E")^*)$. If the map is not injective, let $\mathbf{C}^{r-j}$ be its kernel and write $\mathcal{O}^r = \mathcal{O}^j \oplus \mathcal{O}^{r-j}$. Then, the extension class is given by $(e_1, 0) \in Ext^1(E", \mathcal{O}^j) \oplus Ext^1(E", \mathcal{O}^{r-j})$ and therefore (see 2.5) $E = E_1 \oplus \mathcal{O}^{r-j}$ against the assumption that E is indecomposable.

Given a non-split extension

$$0 \to \mathcal{O}^r \xrightarrow{i} E \to E" \to 0$$

and an automorphism $g' : \mathcal{O}^r \to \mathcal{O}^r$, one obtains a new extension

$$0 \to \mathcal{O}^r \xrightarrow{ig'} E \to E" \to 0$$

with the same E.

It remains to check that the extensions obtained by modifying with an automorphism g of $\mathcal{O}^r$ are not equivalent as extensions. If this were the case, there would be an isomorphism $g : E \to E$ giving rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}^r & \xrightarrow{i} & E & \to & E" & \to & 0 \\ & & \downarrow Id & & \downarrow g & & \downarrow Id & & \\ 0 & \to & \mathcal{O}^r & \xrightarrow{ig'} & E & \to & E" & \to & 0 \end{array}$$

Take a covering of the curve by suitable affine open sets U_i. As $H^1(U_i, \mathcal{F}) = 0$ for any coherent sheaf (see [H] ThIII3.5) $\mathcal{F}$, the extension is trivial over U_i. We can write $E_{U_i} = \mathcal{O}^r_{U_i} \oplus E"_{U_i}$. Considering this decomposition, the gluing on $U_i \cap U_j$ is given by a 2×2-matrix. As the gluing must respect $\mathcal{O}^r$ as a subsheaf, the matrix takes the form

$$\begin{pmatrix} Id & \varphi_{i,j} \\ 0 & Id \end{pmatrix}$$

with $\varphi_{i,j} \in H^0(U_i \cap U_j, Hom(E", \mathcal{O}^r))$.

Use b) and computing cohomology with a Cech resolution with the representation of E as above. One sees then that $\varphi_{i,j}$ gives the class of the extension $0 \to \mathcal{O}^r \to E \to E" \to 0$.

Using the same open cover, the map g is given by a matrix

$$\begin{pmatrix} g' & g_i \\ 0 & Id \end{pmatrix}$$

The compatibility with the gluing gives then

$$\begin{pmatrix} g' & g_j \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & \varphi_{i,j} \\ 0 & Id \end{pmatrix} = \begin{pmatrix} Id & \varphi_{i,j} \\ 0 & Id \end{pmatrix} \begin{pmatrix} g' & g_i \\ 0 & Id \end{pmatrix}$$

In particular $g'\varphi_{i,j} + g_j = g_i + \varphi_{ij}$. Hence, $(g' - Id)\varphi_{i,j} = g_i - g_j$. It follows that $(g' - Id)\varphi_{i,j}$ is a coboundary. Therefore, as an element of $Ext^1(E, \mathcal{O}^r) = Hom(H^0(\mathcal{O}^r), H^1(E^*))$, $(g' - Id)\varphi_{i,j} = 0$. As by assumption the class $\varphi_{i,j}$ corresponds to an injective map $H^0(\mathcal{O}^r) \to H^1(E^*)$, this implies $g - Id = 0$. This shows that $g = Id$ and therefore the two extensions are in fact the same. $\square$

3. Stable and semistable vector bundles

3.1. Definition *The slope of a vector bundle or a locally-free sheaf E on a curve C is defined as $\mu(E) = \frac{degE}{rankE}$.*

A sheaf or bundle E is said to be (semi)stable if for every proper subsheaf or subbundle E' $\mu(E')(\leq) < \mu(E)$

Note that there is no ambiguity in the definition above. Using 1.3 one can see that a locally free sheaf is stable or semistable if and only if the corresponding bundle is stable or semistable.

3.2. Lemma *A vector bundle E is (semi)stable if and only if for every quotient E", $\mu(E)(\leq) < \mu(E")$*

Proof. Given $E, E"$, one has an exact sequence

$$0 \to E' \to E \to E" \to 0$$

with E' a subbundle. As

$$deg(E') + deg(E") = deg(E),\ rk(E') + rk(E") = rk(E),$$

one checks that the conditions in terms of subbundles and in terms of quotients are the same. □

3.3. Lemma *If E is (semi)stable and L is a line bundle, then $E \otimes L$ is (semi)stable.*

Proof. Note that $\mu(E \otimes L) = \mu(E) + deg(L)$. If E' is a subbundle of $E \otimes L$ that contradicts stability, then $E' \otimes (L)^{-1}$ is a subbundle of E that contradicts stability. □

3.4. Lemma *a) If E is (semi)stable of (negative) non strictly positive degree then E has no sections.*

b)If E is (semi)stable $degE(>) \geq 2r(g-1)$. Then $h^1(E) = 0$.

c) If E is (semi)stable $degE(>) \geq r(2g-1)$. Then E is generated by global sections

Proof. a) Assume that E has a section. Then $\mathcal{O}$ is a subsheaf of E. Hence, by the (semi)stability condition, $deg(E)(\geq) > 0$.

b) Assume $h^1(E) \neq 0$. By Serre duality this is equivalent to $h^0(K \otimes E^*) \neq 0$. Notice that if E is (semi)stable, then so is E^*: a subsheaf $E' \to E$ gives rise to a quotient $E^* \to (E'^*)$ and (semi)stability can be checked using 3.2. Then, from 3.3, $K \otimes E^*$ is also (semi)stable. As $deg(K \otimes E^*) = 2r(g-1-degE)$, then b) follows from a).

c) If $degE(>) \geq r(2g-1)$, then $degE(-P)(>) \geq 2r(g-1)$. From b), this implies $h^1(E(-P)) = 0$. Therefore, the map $H^0(E) \otimes \mathcal{O} \to E_P$ is onto as required. □

3.5. Lemma *If E is stable, every non-zero morphism of E to itself is an isomorphism.*

Proof. Let $\varphi : E \to E$. Then we have $E \to ImE \to E$ where the first map is onto and the second injective. By assumption, $ImE \neq 0$. From 3.1 and 3.2, if $ImE \neq E$, then $\mu(E) < \mu(ImE) < \mu(E)$ which is a contradiction. Hence $ImE = E$. This implies that the kernel is or rank zero. Therefore the kernel is zero because E has no torsion subsheaves. $\square$

3.6. Proposition *If E is stable over an algebraically closed field, every morphism $E \to E$ is an homotethy.*

Proof. Let $\varphi : E \to E$ be a morphism, $P \in C$ and λ an eigenvalue of $\varphi_P : E_P/\mathcal{M}_P E_P \to E_P/\mathcal{M}_P E_P$. Then, $\varphi - \lambda Id$ is non-injective. Using the Lemma above, this means that it is zero. Therefore, $\varphi = \lambda Id$. $\square$

3.7. Proposition *Let $0 \to E' \to E \to E" \to 0$ be an exact sequence. Then, $h^0(E"^* \otimes E') = 0$.*

The dimension of the set of stable extensions E that can be obtained in this way with a fixed pair $E', E"$ depends only on the ranks $r', r"$ and degrees $d', d"$ of $E'E"$ and is given as $r'd" - r"d' + r'r"(g-1)$.

Proof. (see [RT] 1.1) If $h^0(E"^* \otimes E') \neq 0$, we have a non-zero morphism $E" \to E'$ Let F be its image. Then, F is a quotient of $E"$ and therefore a quotient of E. It is a submodule of E' and therefore a submodule of E. By the stability of E (using 3.1 and 3.2) it follows that $F = 0$ and therefore $\varphi = 0$.

The second statement can be deduced directly from the first. $\square$

The interest of the statement above is that no conditions are imposed on $E', E"$, only E is required to be stable.

3.8. Lemma *Let E be any vector bundle, $E' \subset E$ a subbundle of E of maximum slope. Then E' is semistable. If E' is maximal with this condition, then E' is stable*

The proof of this result follows immediately from the definition of stability.

3.9. Proposition *Let E be a semistable vector bundle. There exists a filtration*

$$0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E$$

with E_i/E_{i-1} stable and $\mu(E_i/E_{i-1}) = \mu(E)$.

Moreover, $Gr(E) = \oplus(E_i/E_{i-1})$ is uniquely determined by E.

Proof. If E is stable the result is correct as one must take the trivial filtration. In particular, the result holds for all vector bundles of rank one.

Let us now assume that the result is correct for all semistable vector bundles of rank less that $r = rk(E)$. If E is semistable but not stable, choose E_{n-1} maximal among those vector subbundles contradicting stability, so that $\mu(E_{n-1}) = \mu(E)$. By the maximality, E/E_{n-1} is stable. The result is then correct for E_{n-1} as it has rank less than r. Then, the filtration of E_{n-1} gives rise to a filtration of E with the properties required.

Uniqueness of the graduate follows from Jordan-Holder's Theorem using the fact that stable vector bundles are simple. $\square$

4. Vector bundles on rational and elliptic curves

In this section we study vector bundles on rational and elliptic curves. A vector bundle is said to be indecomposable if it is not the direct sum of two vector bundles both of smaller rank . By definition of decomposable, every vector bundle is the direct sum of indecomposable ones . Therefore, it suffices to know the indecomposable vector bundles on a curve in order to know them all. In fact, it can be proved that the decomposition of a vector bundle into direct sum of indecomposables is unique up to the order(see [A2].

It turns out that vector bundles on rational curves are trivial in the sense that they are direct sum of line bundles (see **??**). This result is usually referred to as Grothendieck's Theorem, although some people think it was known classically.

The case of an elliptic curve was studied by Atiyah ([A3]). The description is still extremely simple : for a fixed rank and degree the set of indecomposable vector bundles of this rank and degree is isomorphic to the jacobian of the curve (which in turn is isomorphic to the curve itself) (see 4.11, 4.12). The isomorphisms are canonical for vector bundles of degree zero. For higher degree, they depend on the choice of a line bundle of degree one (or equivalently a point on the curve). Moreover, the multiplicative structure (tensor product of two of these indecomposable bundles) can also be explicitly described (see 4.17).

One can then try to look at vector bundles on rational and elliptic curve in the way in which we look at vector bundles on curves of higher genus. In particular, it makes sense to discuss stability and semistability and to see if there is a moduli space of the right dimension for vector bundles which are stable or semistable.

For rational curves, there are no stable vector bundles of rank greater than one and semistable ones exist only if the degree is a multiple of the rank. We shall see that the "least unstable" vector bundles of given rank and degree on a rational curve are all isomorphic and have an r^2-dimensional family of automorphisms. For curves of genus g stable vector bundles move in an $r^2(g-1)+1$ dimensional family and have a one-dimensional family of endomorphisms, namely the homotheties. The difference of the two numbers $r^2(g-1)$ then agrees with the number in this case $0-r^2$. If we consider more unstable vector bundles on the rational curve, this number becomes more negative.

In the case of the elliptic curve when $(r,d)=h>1$, there are no stable vector bundles of rank r and degree d. Semistable ones exist and if one considers them up to isomorphism, the most "general" element is a direct sum of h vector bundles of rank $\frac{r}{h}$ and degree $\frac{d}{h}$. Again subtracting from the dimension of the family of elements of this form the number of automorphisms of the generic member one obtains zero which agrees with the expected number of $r^2(g-1)$. For other types of decomposition, the number is smaller (see [Tu], [Te1] p.347).

We start with rational curves.

4.1. Grothendick's Theorem. *Every vector bundle on* $\mathbf{P^1}$ *is of the form* $\mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_r)$, $a_1 \geq ... \geq a_r$. *The decomposition above is unique. Here* $\mathcal{O}(a)$ *denotes the* a*-multiple of a point in* $\mathbf{P^1}$.

Proof. (see for example [OSS] p.22) The proof is by induction on r, the case $r = 1$ being well known (see [H] Cor.II6.17). Assume the result known for every vector bundle of rank $r-1$, we want to prove it for a vector bundle E of rank r.

From 1.15, there exist bundles E', E" of ranks $1, r-1$ such that there is an exact sequence

$$0 \to E' \to E \to E" \to 0$$

. In fact, we can choose E' to be a sublinebundle of maximum degree in E. From the induction assumption, this sequence takes the form

$$0 \to \mathcal{O}(a_1) \to E \to \mathcal{O}(a_2) \oplus ... \oplus \mathcal{O}(a_r) \to 0$$

Therefore, E corresponds to an element in $Ext^1(E"^* \otimes E') = \oplus_{i=2}^r H^1(\mathcal{O}(a_i - a_1))$. We shall prove that this vector space is zero and therefore the extension is split.

From Serre duality, $h^1(\mathcal{O}(a)) = h^0(\mathcal{O}(-2-a))$ and the latter is zero if and only if $-2-a < 0$.

From the assumption that $\mathcal{O}(a_1)$ is the sublinebundle of maximal degree of E, it follows that $E(-a_1-d) = 0$ for all $d \geq 1$. Then, from the exact sequence

$$0 \to \mathcal{O}(-1) \to E(-a_1-1) \to \mathcal{O}(a_2-a_1-1) \oplus ... \oplus \mathcal{O}(a_r-a_1-1) \to 0$$

and the fact that $h^1(\mathcal{O}(-1)) = 0$, we obtain $h^0(\mathcal{O}(a_2-a_1-1) \oplus ... \oplus \mathcal{O}(a_r-a_1-1)) = 0$. Hence $a_i - a_1 - 1 < 0$ and a fortiori $a_i - a_1 - 2 < 0$. Therefore the extension is split as claimed.

Uniqueness of the decomposition follows from [A2]. It can also be checked directly as follows. Assume

$$(*)\mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_r) \cong \mathcal{O}(b_1) \oplus ... \oplus \mathcal{O}(b_r),\ a_1 \geq ... \geq a_r,\ b_1 \geq ... \geq b_r$$

. Assume $a_1 = b_1, a_{k-1} = b_{k-1}, a_k > b_k$. Compute then $h^0(E(-a_k))$. Using the right hand side in (*), one gets $b_1 - a_k - 1 + ... + b_{k-1} - a_k - 1 = a_1 - a_k - 1 + ... + a_{k-1} - a_k - 1$. Using the left hand side, one obtains $a_1 - a_k - 1 + ... + a_{k-1} - a_k - 1 + t$ where t is the number of times that the term a_k appears in the left hand side of (*). By assumption, $t \geq 1$ giving a contradiction. $\square$

4.2. Proposition. *Let $d = rd_1 + d_2, d_2 < r$. The "most general" vector bundle on $\mathbf{P^1}$ of degree d and rank r is*

$$\mathcal{O}(d_1+1) \oplus ... \oplus \mathcal{O}(d_1+1) \oplus \mathcal{O}(d_1) \oplus ... \oplus \mathcal{O}(d_1)$$

where the term $d_1 + 1$ appears d_2 times.

Proof. We want to check that the vector bundle described above is the one with the smallest number (r^2) of automorphisms. Write a vector bundle E on $\mathbf{P^1}$ as

$$(\mathcal{O}(a_1))^{r_1} \oplus ... \oplus (\mathcal{O}(a_j))^{r_j}, \ a_1 > ... > a_j, \ r_1 + ... + r_j = r$$

Then

$h^0(E^* \otimes E) = r_1^2 + ... + r_j^2 + r_1r_2(a_1 - a_2 + 1) + ... + r_1r_j(a_1 - a_j + 1) + r_2r_3(a_2 - a_3 + 1) + ... + r_{j-1}r_j(a_{j-1} - a$

$\geq r_1^2 + ... + r_j^2 + 2r_1r_2 + ... + 2r_1r_j + 2r_2r_3 + ... + 2r_{j-1}r_j = (r_1 + ... + r_j)^2 = r^2$
Moreover, equality occurs only if $a_i - a_k = 1$ for all $i > k$. This implies $j = 2, a_2 = a_1 - 1$ as in our description. $\square$

4.3. Lemma. *A vector bundle on any variety is decomposable if and only if it has two endomorphisms $\varphi_1, \ \varphi_2$ such that $\varphi_1 + \varphi_2 = Id$ and $(\varphi_i)^2 = \varphi_i$.*

Proof. If $E = E_1 \oplus E_2$ is decomposable, denote by $\varphi_i : \ E_1 \oplus E_2 \to E_i \to E_1 \oplus E_2$ the composition of the projection with the inclusion. These maps satisfy the properties stated in a). Conversely, if $\varphi_1, \ \varphi_2$ exist satisfying the conditions, define $E_i = Im\varphi_i$. One can check that $E = E_1 \oplus E_2$. $\square$

4.4. Lemma *Over a field of characteristic zero, given a vector bundle E on an arbitrary variety, $End(E) \equiv \mathcal{O} \oplus E'$ where E' is the set of traceless endomorphisms.*

Proof. Every morphism φ can be written in the form

$$\varphi = (\varphi - \frac{Tr(\varphi)}{r}Id) + \frac{Tr(\varphi)}{r}Id$$

$\square$

In the rest of this section C will denote an elliptic curve and all vector bundles will be vector bundles on C.

4.5. Lemma *a) An indecomposable vector bundle of negative degree has no sections.*

b)For an indecomposable vector bundle E of strictly positive degree $d > 0$, $h^0(E) = d$ (and in particular $H^0(E) \neq 0$).

Proof. We prove the first statement by induction on the rank r of the vector bundle E. If $r = 1$, the result is well known. Assume now that the degree is negative and the vector bundle has sections. Let D be the divisor corresponding to the zero locus of such a section ($D \geq 0$). Then, we have an exact sequence

$$0 \to \mathcal{O}(D) \to E \to E" \to 0.$$

with $E"$ a vector bundle (reffeixquocient)¿ These extensions are parameterized by $H^1((E")^* \otimes \mathcal{O}(D)) \cong H^0(E" \otimes \mathcal{O}(-D))$ (as the canonical on an elliptic curve is trivial). Write $E"$ as a direct sum of several indecomposable summands , $E" = \oplus E"_i$. Then, the class of the extension gives rise to a non-zero element of $H^0(E"_i \otimes \mathcal{O}(-D))$ for each i, otherwise the corresponding $E"_i$ is a direct summand of E (2.5). By the induction assumption, $deg(E"_i(-D)) \geq 0$ for all i. A fortiori, $deg(E"_i) \geq 0$. Hence, $deg(E) = \sum deg(E"_i) \geq 0$ contradicting the assumption.

The second statement follows from the first using Riemman-Roch and the fact that the dual of an indecomposable vector bundle of positive degree is an indecomposable vector bundle of negative degree. □

4.6. Proposition *For every positive r there exists a unique indecomposable vector bundle $\bar{E}_{r,0}$ of rank r and degree zero with sections. Moreover*

a) $\bar{E}_{r,0}$ has only one section.

b) $\bar{E}_{r,0} \cong \bar{E}^*_{r,0}$

c) There is an exact sequence

$$0 \to \mathcal{O} \to \bar{E}_{r,0} \to \bar{E}_{r-1,0} \to 0.$$

d) If L is a line bundle and $L \to \bar{E}_{r,0}$ is a non-zero map, then $degL < 0$ or $degL = 0$ and $L \cong \mathcal{O}$.

Proof. We proceed by induction on r, the case $r = 1$ being clear.

We now assume that the result is correct for $r - 1$ and prove it for r. Let E be an indecomposable vector bundle of rank r and degree zero which has at least one section. This section cannot have zeroes, otherwise $E(-D)$ would have a section for some positive divisor D contradicting 4.5. Hence, there is an exact sequence

$$(*)0 \to \mathcal{O} \to E \to E" \to 0.$$

Such an extension with an indecomposable E corresponds to a non-zero element of $H^1((E")^*) \cong H^0(E")^*$. Therefore $h^0(E") \neq 0$. Assume that $E" = \oplus E"_i$. From 2.5 b"), $H^0(E"_i)$ must be different from zero, otherwise the corresponding $E"_i$ is a direct summand of E. As the sum of the degrees of the $E"_i$ is zero, using again 2.5 and 4.5, each $E"_i$ must have degree zero and have sections. By induction assumption, $E"_i = \bar{E}_{r_i,0}$ where $\bar{E}_{r_i,0}$ is the unique vector bundle of rank r_i possessing sections. As the $r_i \leq r-1 < r$, b) and c) hold for them. Order the ranks $r_1 \leq r_2 \leq ... \leq r_k$. Dualising the sequence in c), there is an inclusion $\bar{E}_{r_1,0} \to \bar{E}_{r_i,0}$ that sends the section of the first vector bundle to the section of the second. There is then an isomorphism $\oplus \bar{E}_{r_i,0} \to \oplus \bar{E}_{r_i,0}$ that composed with the map $H^0(\mathcal{O}) \to H^1(\oplus \bar{E}_{r_i,0})$ is zero in all components except the first. If $k > 1$, $\bar{E}_{r_2,0}, ... \bar{E}_{r_k,0}$ are direct summands of E contradicting the assumption that E is indecomposable. Hence, $k = 1$ and $E" = \bar{E}_{r-1,0}$. From the induction assumption, $E" \cong (E")^*$ and $h^0(E") = 1$. Then by Riemman-Roch, $h^1(E") = h^0((E")^*) = h^0(E") = 1$. Hence, by the one to one correspondence between extensions and elements in $H^1((E")^*)$, E is unique with the condition of having sections.

The exact sequence (*) gives rise to a long exact sequence of cohomology

$$0 \to H^0(\mathcal{O}) \to H^0(E) \to H^0(E") \to H^1(\mathcal{O}) \to ...$$

Notice that, by Riemman-Roch and the induction assumption, $h^1(\mathcal{O}) = 1 = h^0((E")^*)$. The connecting morphism in the long exact sequence is cup-product with the extension class (2.6 c), therefore it is non-zero. As both spaces are of dimension one, it is an isomorphism. Hence $h^0(E) = 1$.

Dualising the exact sequence above, we obtain

$$0 \to (E")^* \to E^* \to \mathcal{O} \to 0.$$

Note that E^* is again an indecomposable vector bundle of rank r and degree zero. Then $h^0(E^*) \geq h^0((E")^*) = 1$. Hence, by the uniqueness of the indecomposable vector bundle with sections $E \cong E^*$.

It remains to prove d). Assume that $\varphi : \ L \to \bar{E}_{r,0}$ is a non-zero map. Using c), this implies that there is a non-zero map from L either to $\mathcal{O}$ or to $\bar{E}_{r-1,0}$. Then the result follows by induction. □

4.7. Lemma. *Let E be an indecomposable vector bundle of rank r and degree d generated by global sections. Then, either $E = \mathcal{O}$ or $d > r$ and E has a sublinebundle of degree at least one.*

Proof. From 4.5, $d \geq 0$. From 4.6 a), if $d = 0$, then $E = \mathcal{O}$. Assume now $d > 0$. From the surjectivity of the evaluation map of sections $H^0(E) \otimes \mathcal{O} \to E$ and 4.5 b), $d \geq r$. The kernel of the evaluation map is a locally-free sheaf by 1.2 b). If this inequality were an equality, this kernel would need to be zero and the map an isomorphism contradicting the indecomposability of E. Now, if $d > r$ and L is any line bundle of degree one, $E \otimes (L)^{-1}$ has positive degree and therefore (4.5 b) it has a section. Hence, E has a line subbundle of degree at least one. □

4.8. Proposition. *Let E be an indecomposable vector bundle of rank r and degree zero. Then E is isomorphic to $\bar{E}_{r,0} \otimes L$ for a unique line bundle L of degree zero.*

Proof. Let M be a line bundle of degree one. We claim that $E \otimes M$ has a line subbundle of degree at least one. Note that $E \otimes M$ has degree r and therefore has r independent sections. By definition of independence of sections, the kernel of the evaluation map of sections cannot be of the form $\mathcal{O}^j$. If this map is not injective, then by 4.7 the image will contain a subline bundle of degree one and the claim is proved. If the map is injective, it cannot be an isomorphism because E is indecomposable. As both $H^0(E\otimes M)$ and $E\otimes M$ have the same rank r, the cokernel is torsion. Then if P is on the support of the torsion part, $\mathcal{O}(P)$ is a subsheaf of $E \otimes M$ proving the claim.

Let M' be asublinebundle of $E \otimes M$ of degree at least one. Then $E \otimes M \otimes M'^*$ has a section. From 4.5, this implies that M' has degree one and then from the uniqueness statement in 4.6, $E\otimes M\otimes M'^* \cong \bar{E}_{r,0}$. Hence $L = M \otimes M'^{-1}$ satisfies the condition of the statement.

It remains to prove the uniquenes of L. Assume now that L, L' are line bundles of degree zero and $\bar{E}_{r,0} \otimes L = \bar{E}_{r,0} \otimes L'$. Then, $\bar{E}_{r,0} \otimes L \otimes L'^* = \bar{E}_{r,0}$. Hence, $L \otimes L'^{-1}$ is a sublinebundle of $\bar{E}_{r,0}$. From 4.6 d), $L \otimes L'^* = \mathcal{O}$. □

4.9. Proposition *Denote by $U(r,d)$ the set of isomorphism classes of indecomposable vector bundles of rank r and degree d. Let L be a fixed line bundle of degree one. There is an isomorphism*

$$\begin{array}{ccc} U(r,d) & \to & U(r,d+kr) \\ E & \to & E\otimes L^k \end{array}$$

Proof. The inverse of the map above is given by tensoring with L^*. □

4.10. Proposition *Denote by $U(r,d)$ the set of isomorphism classes of indecomposable vector bundles of rank r and degree $d > 0$. There is an isomorphism*

$$\begin{array}{ccc} U(r,d) & \to & U(r+d,d) \\ E" & \to & E \end{array}$$

where E is given by the extension

$$0 \to \mathcal{O}^d \to E \to E" \to 0$$

Proof. We are going to prove first that the map above is onto: Let E be an indecomposable vector bundle of degree $d > 0$ and rank $r_1, d < r_1$. Write $r_1 = d + r, r \geq 1$. Then, $h^0(E) = d$. Therefore, there is a natural morphism $\mathcal{O}^d \to E$. If the morphism is not injective, using 4.7, the image has a line subbundle L of positive degree. This would imply that $h^0(E \otimes L^{-1}) \geq 1$. As $degE \otimes L^{-1} = d - (degL)(d + r) < 0$, this contradicts 4.5. Therefore, there is an exact sequence

$$(*)0 \to \mathcal{O}^d \to E \to E" \to 0$$

Again, here $E"$ is a vector bundle otherwise E would contain a sub-linebundle of degree at least one. Write $E" = \oplus E"_i$ and each $E"_i$ is an indecomposable vector bundle of rank r_i and degree d_i. If $d_i < 0$ for some i, by 4.5, the corresponding vector bundle has no sections. This would imply (by 2.5 b")) that $E"_i$ is a direct summand of E. Similarly, if $d_i = 0$ for some i, the corresponding vector bundle E_i must have sections and therefore $E"_i = \bar{E}_{r_i,0}$. Dualising the surjective map $E" \to \bar{E}"_{r_i,0}$ and using that $\bar{E}^*_{r_i,0} = \bar{E}_{r_i,0}$ has a section, we obtain that E^* has a section contradicting 4.5. Therefore, $d_i > 0$ for all i and $\sum_{i=1}^k h^0(E"^*_i) = \sum d_i = d$. It follows that the map $H^0(\mathcal{O})^d \to H^1(\oplus E"^*_i)$ is an isomorphism. With a change of basis in $\mathcal{O}^d$, the map can be written as the direct sum of k different maps $H^0(\mathcal{O}^{d_i}) \to H^1(E"^*_i)$, $\sum d_i = d$. As E is indecomposable, this implies that $k = 1$ and therefore $E"$ is in $U(r, d)$.

From 2.6 b), there is a one to one correspondence between the quotient of the set of injective maps $\mathbf{C}^d \to H^1((E")^*)$ by the action of the linear group. The dimension of $h^1((E")^*) = h^0(E") = d$. On the other hand, the linear group is d^2-dimensional. Hence, there is a unique such extension.

□

4.11. Theorem. *Let r, d be integers $r > 0$ and h the greatest common divisor of r, d. Then $U(r, d)$ is isomorphic to $U(h, 0)$ which in turn is isomorphic to the Jacobian of C.*

The isomorphism is completely determined up to the choice of a line bundle L of degree one. The element in $U(r, d)$ corresponding to $\bar{E}_{h,0}$ will be denoted by $E^L_{r,d}$.

Proof. Using 4.9, we can assume that $d \geq 0$. From 4.9, $U(r, d) \cong U(r, d - r)$ if $d \geq r$. From 4.10, $U(r, d) \cong U(r - d, d)$ if $d < r$. Hence, we can construct a sequence of isomorphisms $U(r_i, d_i) \cong U(r_{i+1}, d_{i+1})$ with $0 < r_i, 0 \leq d_i$ for all i and same greatest common divisor $(r_i, d_i) = (r_{i+1}, d_{i+1})$, $r_i + d_i > r_{i+1} + d_{i+1}$. As the numbers involved are positive, the process terminates. Then, $d_i = 0$ and therefore $r_i = h$ as required. □

4.12. Theorem. *a) Every vector bundle E of rank r and degree d can be written as $E^L_{r,d} \otimes L'$ for some line bundle L' of degree zero.*

b) If $h = (r, d)$, $E \otimes L' \equiv E$ if and only if $(L')^{\frac{r}{h}} = \mathcal{O}$.

c) Using the isomorphism between the moduli space and the Jacobian, the determinant map is equivalent to taking the $h-^{th}$ power.

Proof. From 4.11, $U(r, d)$ is isomorphic to $U(h, 0)$ which in turn is isomorphic to C. Then, the map

$$\begin{array}{ccc} Pic^0(C) & \to & U(r,d) \\ L' & \to & E^L_{r,d} \otimes L' \end{array}$$

is either onto or constant. As $det(E_{r,d} \otimes L') = det(E_{r,d}) \otimes (L')^r \neq det(E_{r,d})$ for generic L', the map is not constant. This proves a).

From 4.8, every element in $U(h, 0)$ can be written in a unique way in the form $\bar{E}_{h,0} \otimes L'$. Therefore b) holds in this case. Moreover, the determinant map

$$\begin{array}{cccc} det: & U(h,0) & \to & U(r,d) \\ & E & \to & detE \end{array}$$

satisfies $det(E \otimes L') = det(E) \otimes (L')^h$. Therefore c) holds in this case too.

We now want to see that each step of the isomorphism $U(r, d) \cong U(h, 0)$ preserves condition c). In the case $U(r, d) \cong U(r, d + kr)$ the isomorphism is obtained by multiplying with a line bundle and therefore corresponds to a translation in the Jacobian. So, the result is clear. The isomorphism between $U(r, d)$ and $U(r + d, r)$ is obtained by writing $E \in U(r + d, r)$ as an extension of an $E' \in U(r, d)$ of the form $0 \to \mathcal{O}^d \to E \to E' \to 0$ and the determinant is preserved. This completes the proof of c)

From c), for $h = 1$ the determinant map is an isomorphism. Hence, $E \cong E \otimes L'$ if and only if $det(E) = det(E \otimes L') = det(E) \otimes L'^r$. This proves b) in this case. The proof of b) when $h > 1$ is postponed to 4.17 f) . □

4.13. Lemma. *If $(r, d) = 1$, then $E^L_{r,d} \otimes (E^L_{r,d})^* = \oplus_{i=1}^{r^2} L_i$ where the L_i run over the set of line bundles of order r.*

Proof. From 4.12 b) (which we proved already for $(r, d) = 1$), for any line bundle L_i of order r $E \otimes L_i \cong E$ and therefore, $E \otimes E^* \cong E \otimes E^* \otimes L_i$. As $E \otimes E^*$ contains $\mathcal{O}$ as a direct summand (**??**), it follows that it contains L_i as a direct summand. As there are r^2 of them, the result follows. □

The decomposition of a vector bundle into direct sum of indecomposables is unique up to the order and up to isomorphism [A2]. For the sake of completeness, we provide here an ad hoc proof for the case that we need.

4.14. Proposition. *Assume that we have an isomorphism*

$$(*)\bar{E}_{r_1,0} \oplus ... \oplus \bar{E}_{r_k,0} \equiv \bar{E}_{s_1,0} \oplus ... \oplus \bar{E}_{s_j,0},\ r_1 \leq .. \leq r_k,\ s_1 \leq ... \leq s_j.$$

Then, $k = j, r_i = s_i$.

Proof. The result is true if the total rank is two as $\mathcal{O} \oplus \mathcal{O}$ is not isomorphic to $\bar{E}_{2,0}$ (because the first has two sections and the second only one).

We now proceed by induction on the total rank. Using 4.6 a), one obtains that $k = j$ and that there is a unique subbundle $\mathcal{O}^k$ of both the left hand side and the right hand side of (*). The quotient by this well defined subbundle is

$$\bar{E}_{r_1-1,0} \oplus ... \oplus \bar{E}_{r_k-1,0} \equiv \bar{E}_{s_1-1,0} \oplus ... \oplus \bar{E}_{s_j-1,0}.$$

We can now conclude using induction on the total rank. □

4.15. Remark. *The result above also holds if we take a direct sum of arbitrary vector bundles of degree zero: each summand is determined up to isomorphism.*

The proof would be as above replacing $\mathcal{O}^j$ by the direct sum of the uniquely determined line subbundles of each of the bundles.

4.16. Proposition *An indecomposable vector bundle of degree zero is semistable but not stable.*

Proof. From 4.8, it suffices to prove the result for $\bar{E}_{r,0}$. Using, 4.6 c), it is clear that the vector bundle is not stable. Assume now that $E_{s,d} \to \bar{E}_{r,0}$ is a non-zero map. We want to show that then $d \leq 0$. This will suffice to prove the result. We use induction on r. If $r = 1$, the map above gives rise to a non-zero map $\mathcal{O} \to \bar{E}^*_{s,d}$. The latter is an indecomposable vector bundle of degree $-d$ which has a non-zero section. From 4.5, $-d \leq 0$ as needed.

Assume now the result for $r - 1$ and prove it for r. From 4.6 c), a non-zero map $E_{s,d} \to \bar{E}_{r,0}$ gives rise either to a non-zero map $E_{s,d} \to \bar{E}_{r-1,0}$ or a non-zero map $E_{s,d} \to \mathcal{O}$. This contradicts the induction assumption. □

We now look at the multiplicative structure on the set of vector bundles

4.17. Proposition (Atiyah). *In characteristic zero, one has the following relations*

a) If $r \geq s$, *then* $\bar{E}_{r,0} \otimes \bar{E}_{s,o} = \bar{E}_{r-s+1} \oplus \bar{E}_{r-s+3} \oplus \ldots \oplus \bar{E}_{r+s-1},$.

b) If $(r,d) = 1$, *then* $E^L_{r,d} \otimes \bar{E}_{h,0} = E^L_{rh,dh}$.

c) If $(r,r') = (r,d) = (r',d') = 1$, *then* $E^L_{r,d} \otimes E^L_{r',d'} = E^L_{rr',rd'+r'd}$

d) If $a_1 > a_2, (d_1,p) = (d_2,p) = 1$ *then,* $E^L_{p^{a_1},d_1} \otimes E^L_{p^{a_2},d_2} = \mathcal{O}^{p^{a_2}} \otimes E^L_{p^{a_1},d_1+p^{a_2-a_1}d_2}$.

e) If $(d_1,p) = (d_2,p) = 1$, *then* $E^L_{p^a,d_1} \otimes E^L_{p^a,d_2} = \mathcal{O}^{p^b} \otimes (\oplus L_i) \otimes E^L_{p^b,d_3},\ p^{a-b} = (p^a, d_1+d_2)$. *Here the* L_i *are line bundles of order* p^a *and a representative is taken modulo the class of elements of order* p^b.

f) If L' *is a line bundle of degree zero and* $h = (r,d)$ $E^L_{r,d} \otimes L' = E^L_{r,d}$ *if and only if* L' *is a line bundle of order* r/h.

Note that these rules allow us to compute the tensor product of any two vector bundles E_{r_1,d_1}, E_{r_2,d_2}. First, using b) we can write $E_{r_i,d_i} = E_{h_i,0} \otimes E_{r'_i,d'_i}, h_i = (r_i,d_i)$. Then, using c), we can write $E_{r'_i,d'_i}$ as product of vector bundles of ranks the power of a prime. These can be multiplied together using d)e) if the primes are the same and c) if they are not. One can then multiply the $\bar{E}_{h_i,0}$ using a). Finally, using b) and c) one can go back to direct sum of indecomposable bundles.

Before giving a proof of this result, we will give several useful consequences. The first of them follows from a computation using the description above. The details are left to the reader:

4.18. Corollary. *The tensor product of two indecomposable vector bundles is a direct sum of indecomposable vector bundles all of the same slope.*

4.19. Corollary. *Indecomposable vector bundles are semistable, they are stable if and only if* $(r,d) = 1$ *(see also* [Tu]*).*

Proof. We first show that if $(r,d) = h > 1$, an indecomposable vector bundle of rank r and degree d is not stable.Write $r = hr_1,\ d = hd_1$. We use the exact sequence

$$0 \to \mathcal{O} \to \bar{E}_{h,0} \to \bar{E}_{h-1,0} \to 0.$$

Tensoring with E_{r_1,d_1} and using 4.17 b), we obtain

$$0 \to E_{r_1,d_1} \to E_{r,d} \to E_{(h-1)r_1,(h-1)d_1} \to 0.$$

This exhibits a subsheaf of $E_{r,d}$ with the same slope.

Assume now that $E_{s,f}$ is a subbundle of $E_{r,d}$ contradicting stability (i.e. $\frac{s}{f} \geq \frac{r}{d}$). Then, there is a nonzero map $E_{s,f} \to E_{r,d}$. Therefore, $(E_{s,f})^* \otimes E_{r,d}$ has a section. Using 4.5, this implies that one of the direct summands in the decomposition has slope greater than or equal to zero. Then, from 4.18, all summands in the decomposition have positive slope. This contradicts the initial assumption on the slopes $\square$

Remark Note that the proof above only requires to know the ranks and degrees of the indecomposable summand appearing in the description in 4.17 d), e). It would suffice to know that the right hand side holds without the superscript L.

In the proof of 4.17, we shall use the following results:

4.20. Lemma $h^0(\bar{E}_{r,0} \otimes \bar{E}_{s,0}) = min(r, s)$

Proof. Assume $r \leq s$ and use induction on r. The case $r = 1$ is trivial. Assume that the result has been proved for $r - 1$ and prove it for r. Tensoring the exact sequence from 4.6 c) with $\bar{E}_{s,0}$ and taking homology one obtains

$$0 \to H^0(\bar{E}_{r-1,0} \otimes \bar{E}_{s,0}) \to H^0(\bar{E}_{r,0} \otimes \bar{E}_{s,0}) \to H^0(\bar{E}_{s,0})$$

The left term and right term have dimensions $r - 1$ and one respectively. We need to show that the last arrow is onto or equivalently non-zero. This follows considering the composition of the unique section of $\bar{E}_{r,0}$ with the inclusion of $\bar{E}_{r,o}$ into $\bar{E}_{s,0}$. $\square$

Proof. (of 4.17 a) Notice that $H^0(L \otimes \bar{E}_{r,0} \otimes \bar{E}_{s,0}) \neq 0$ if and only if $degL > 0$ or $L = \mathcal{O}$ as can be proved by induction on r using the exact sequence in 4.6 c).

Therefore, from 4.6d), 4.8 and 4.5, if $r \leq s$, $\bar{E}_{r,0} \otimes \bar{E}_{s,0} = \oplus_{i=1}^{r} \bar{E}_{r_i,0}$. Then, from 4.4, $\bar{E}_{2,0} \otimes \bar{E}_{2,0} = \mathcal{O} \oplus \bar{E}_{3,0}$

We prove by induction on s that $\bar{E}_{2,0} \otimes \bar{E}_{s,0} = \bar{E}_{s-1,0} \otimes \bar{E}_{s+1,0}$. We assume the result true for $s - 1$. Compute then

$$(\bar{E}_{2,0} \otimes \bar{E}_{s-1,0}) \otimes \bar{E}_{s,0} = \bar{E}_{2,0} \otimes (\bar{E}_{s-1,0} \otimes \bar{E}_{s,0})$$

The left hand side is $((\bar{E}_{s-2,0} \oplus \bar{E}_{s,0}) \otimes \bar{E}_{s,0}$ which has a decomposition with $2s - 2$ terms. The right hand side is $\bar{E}_{s-1,0} \otimes (\bar{E}_{t,0} \oplus \bar{E}_{2s-t,0})$ for some t, $, t \leq s$. If $t \leq s-1$ this expression has $t+s-1$ terms. Equating this to $2s - 2$, this gives us $t = s - 1$ as needed. It only remains to exclude the case $t = s$. If

$$(\bar{E}_{2,0} \otimes \bar{E}_{s,0}) = \bar{E}_{s,0} \oplus \bar{E}_{s,0},$$

then

$$(\bar{E}_{2,0} \otimes \bar{E}_{s,0}) \otimes \bar{E}_{s,0} = (\bar{E}_{s,0} \oplus \bar{E}_{s,0}) \otimes \bar{E}_{s,0}.$$

The right hand side has $2s$ terms . On the other hand from 4.4 $\bar{E}_{s,0} \otimes \bar{E}_{s,0} = \mathcal{O} \oplus \sum \bar{E}_{r_i,0}$ and $\bar{E}_{2,0} \otimes (\bar{E}_{s,0} \otimes \bar{E}_{s,0})$ has at most $1 + 2(s - 1)$

terms, which is a contradiction. Therefore the formula in 4.17 holds for $r = 2$.

We prove it now for arbitrary r, s by induction on s. We can assume $s \geq 3$ and the formula holds for $s-1$ and $s-2$. Then,

$$\bar{E}_{2,0} \otimes (\bar{E}_{s-1,0} \otimes \bar{E}_{r,0}) = (\bar{E}_{2,0} \otimes \bar{E}_{s-1,0}) \otimes \bar{E}_{r,0})$$

By induction assumption, the left hand side is

$$\bar{E}_{2,0} \otimes (\bar{E}_{r-s+2,0} \oplus .. \oplus \bar{E}_{r+s-2,0}) =$$

$$\bar{E}_{r-s+1,0} \oplus 2\bar{E}_{r-s+3,0} \oplus ... \oplus 2\bar{E}_{r+s-3,0} \oplus \bar{E}_{r+s-1,0}$$

The right hand side is

$$(\bar{E}_{s-2,0} \oplus \bar{E}_{s,0}) \otimes \bar{E}_{r,0} = \bar{E}_{r-s+3,0} \oplus \bar{E}_{r-s+5,0} \oplus ... \oplus \bar{E}_{r+s-3,0} \oplus \bar{E}_{s,0} \otimes \bar{E}_{r,0}$$

Then by 4.14,the result follows. □

Proof. of 4.17 b) We start by proving that $F = E^L_{r,d} \otimes \bar{E}_{h,0}$ is indecomposable. Recall (cf 4.3) that the algebra structure on the set of global sections of the automorphisms of the vector bundle determines whether it is decomposable or not. Now

$$E_{r,d} \otimes \bar{E}_{h,0} \otimes E^*_{r,d} \otimes \bar{E}^*_{h,0} = (E_{r,d} \otimes E^*_{r,d}) \otimes (\bar{E}_{h,0} \otimes \bar{E}^*_{h,0})$$

the latter parenthesis is a direct sum of indecomposable vector bundles $\bar{E}_{r_i,0}$. It follows then from 4.6 d) and 4.13 that the space of sections of $F \otimes F^*$ is the same as that of $\bar{E}_{h,0} \otimes \bar{E}^*_{h,0}$ and therefore, $E_{L,d} \otimes \bar{E}_{h,0}$ is indecomposable.

We prove now that $E^L_{r,d} \otimes \bar{E}_{h,0} = E^L_{rh,dh}$ by induction on h and for each value of h by induction on r. The case $h = 1$ is trivial as $\bar{E}_{1,0} = \mathcal{O}$. Similarly for $r = 1$, $E^L_{1,d} = L^d$ and $\bar{E}_{h,0} \otimes L^k = E^L_{h,hk}$ by definition. Moreover, using that $E^L_{r,d} \otimes L^k = E^L_{r,d+kr}$, one can reduce to the case $d < r$. From the exact sequence, 4.6 c), tensored with $E^L_{r,d}$, we obtain

$$0 \to E^L_{r,d} \to \bar{E}_{h,0} \otimes E^L_{r,d} \to \bar{E}_{h-1,0} \otimes E^L_{r,d} \to 0$$

By induction assumption, the last term in the sequence is $E^L_{(r-d)(h-1),d(h-1)}$. Then we have an exact diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}^d & \to & \mathcal{O}^{dh} & \to & \mathcal{O}^{d(h-1)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^L_{r,d} & \to & \bar{E}_{h,0} \otimes E^L_{r,d} & \to & \bar{E}_{h-1,0} \otimes E^L_{r,d} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^L_{r-d,d} & \to & E & \to & E^L_{(r-d)(h-1),d(h-1)} & \to & 0
\end{array}$$

where the lower right term comes from induction assumption. Note that E corresponds to an element in $Ext^1(E^L_{(r-d)(h-1),d(h-1)}, E^L_{(r-d),d})$ and the dimension of this space is

$$h^1((E^L_{(r-d)(h-1),d(h-1)})^* \otimes E^L_{(r-d),d}) = h^0(E^L_{(r-d)(h-1),d(h-1)} \otimes (E^L_{(r-d),d})^*) =$$
$$= h^0(\bar{E}_{h-1,0} \otimes E^L_{r-d,d} \otimes (E^L_{(r-d),d})^*) = 1$$

the last identity coming from 4.13 and 4.6 c). Hence E corresponds to the unique extension of this form. From the exact sequence in 4.6 c) tensored with $E^L_{r-d,d}$, $E^L_{r-d,d} \otimes \bar{E}_{h,0}$ corresponds to such an extension. But by induction assumption, $E^L_{r-d,d} \otimes \bar{E}_{h,0} = E^L_{(r-d)h,dh}$. Then the result follows by the definition of $E^L_{rh,dh}$ from $E^L_{(r-d)h,dh}$. □

Proof. of c) Write $F = E^L_{r_1,d_1} \otimes E^L_{r_2,d_2}$. Then from part d), $F \otimes F^* = \oplus L^1_i \otimes L^2_j = \oplus L^1_i L^2_j$ where the L^1_i are the line bundles of order r_1 and the L^2_j are the line bundles of order r_2. As $(r_1, r_2) = 1$, $L^1_i L^2_j$ are the line bundles of order $r_1 r_2$. Hence, there is only one global automorphism of F and F is indecomposable. As $degF = r_2 d_1 + r_1 d_2,\ rkF = r_1 r_2$, $(r_1 r_2, r_2 d_1 + r_1 d_2) = 1$, the determinant of F completely determines F (see 4.12 c)). Now $det(F) = (L^{d_1})^{r_2} \otimes (L^{d_2})^{r_1} = detE^L_{r_1 r_2, r_2 d_1 + r_1 d_2}$ and the result is proved. □

Proof. of f) The result has been proved already for $(r, d) = 1$ in 4.12. Write now $h = (d, r), r = hr_1, d = hd_1$. Then $E^L_{r,d} = \bar{E}_{h,0} \otimes E^L_{r_1,d_1}$. If $(L')^{r_1} = \mathcal{O}$, then $E^L_{r_1,d_1} \otimes L' = E^L_{r_1,d_1}$ and therefore the same is true for $E^L_{r,d}$. Conversely, suppose that $E^L_{r,d} \otimes L' = E^L_{r,d}$. Then,

$$(E^L_{r_1,d_1})^* \otimes E^L_{r,d} \otimes L' = (E^L_{r_1,d_1})^* \otimes E^L_{r,d}.$$

From b) and 4.13, the left hand side is $\oplus(L_i L') \otimes \bar{E}_{h,0}$ where the L_i are the line bundles of order r_1. The right hand side is $\oplus(L_i) \otimes \bar{E}_{h,0}$ and therefore has exactly one section. Hence the same is true of the left hand side. This implies from 4.6 d) that there is one i such that $L_i L' = \mathcal{O}$. As the inverse of a line bundle of a given order has the same order, the result is proved. □

Proof. of d),e). Let p be prime, $a_1 \geq a_2 \geq 1$ and d_1, d_2 such that $(p, d_1) = 1, (p, d_2) = 1$. Write

$$F = E^L_{p^{a_1},d_1} \otimes E^L_{p^{a_2},d_2} = \oplus E_{r_i,d_i}$$

where the E_{r_i,d_i} are indecomposable. Then, from 4.13, $F \otimes F^* = (\oplus^{p^{a_1}}_{i=1} L_i)^{p^a_2}$ where the L_i are the line bundles of order p^{a_1}. In particular, $F \otimes F^*$ is semistable and the decomposition in sum of indecomposables is unique (4.14). As every $E_{r_i,d_i} \otimes E^*_{r_j,d_j}$ is a direct summand of $F \otimes F^*$, it follows that each of them is also a direct sum of line bundles. Then, from 4.17 b) and a), $(r_i, d_i) = 1$. Moreover, as the line bundles appearing in the decomposition are those of order p^{a_1} , from 4.13 one can write $r_i = p_i^b,\ b_i \leq a_1$. Now, the semistability of $F \otimes F^*$ and the fact that the total degree is zero, implies that the degree of any summand is zero. In particular, $deg(E_{r_i,d_i} \otimes E^*_{r_j,d_j}) = 0$. This degree can be computed as (assuming $b_i \geq b_j$) $p^{b_j}(d_i - p^{(b_i-b_j)}d_j) = 0$. As $(d_i, p) = 1$, this implies that $b_i = b_j = c,\ d_i = d_j = d$. Then,

$$(*)F = E^L_{p^c,d} \otimes (\oplus M_i)$$

where the M_i are line bundles of degree zero.

Computing the rank and degree of both sides we obtain

$$p^{a_1}p^{a_2} = p^c t,\ p^{a_2}(p^{(a_1-a_2)}d_2 + d_1) = td$$

Hence, $t = p^{a_1+a_2-c}$. As $(d, p) = 1$,if $a_1 > a_2$ or $a_1 = a_2$ and $(p, d_1 + d_2) = 1$, we get $a_1 + a_2 - b \leq a_2$. Therefore, $c = a_1$.

Compute then $F \otimes F^*$ using both the left and right hand side of (*). We obtain,

$$(\oplus L_i)^{p^{a_2}} = (\oplus L_i) \otimes (\oplus_{k,l} M_k \otimes M_l^*)$$

Hence, $M_k \otimes M_l^*$ has order p_1^a. Then, using g), $E^L_{p^{a_1},d} \otimes M_k = E^L_{p^{a_1},d} \otimes M_l$. therefore, we can write

$$(**)E^L_{p^{a_1},d_1} \otimes E^L_{p^{a_2},d_2} = F = E^L_{a_1,d} \otimes (\oplus M)^{p^{a_2}}.$$

Consider now the automorphism φ of order two of C that assigns to P the element of C equivalent to $2L - P$. It leaves L invariant and therefore preserves $E^L_{r,d}$. Then, applying φ^* to equation (**), we obtain

$$E^L_{p^{a_1},d} \otimes (\oplus M)^{p^{a_2}} = E^L_{p^{a_1},d_1} \otimes E^L_{p^{a_2},d_2} = E^L_{a_1,d} \otimes (\oplus \varphi^*(M))^{p^{a_2}}.$$

We know that $E^L_{p^{a_1},d}$ is stable and therefore $\oplus E^L_{p^{a_1},d} \otimes M$ is semistable and equal to its graduate. Therefore, the decomposition is unique. Then, $E^L_{p^{a_1},d} \otimes M \equiv E^L_{p^{a_1},d} \otimes \varphi^*(M)$. Hence, $\varphi^*(M) \otimes M$ has order p^{a_1}. Hence, $(\varphi^*(M))^{p^{a_1}} = M^{p_1^a}$. As the only line bundles invariant by φ^* are those of order two, M has order dividing $2p^{a_1}$. Moreover,

$$det((E^L_{p^{a_1},d} \otimes M)^{p^{a_2}}) = (L^d M^{p^{a_1}})^{p^{a_2}} = L^{p^{a_2}d_1 + p^{a_1}d_2} M^{p^{a_1+a_2}}$$

while

$$det(E^L_{p^{a_1},d_1} \otimes E^L_{p^{a_2},d_2}) = L^{p^{a_2}d_1 + p^{a_1}d_2}$$

Hence, M has order dividing $p^{a_1+a_2}$. It follows then that M has order dividing p^{a_1} and then $E^L_{p^{a_1},d} \otimes M = E^L_{p^{a_1},d}$.

It remains to treat the case in which $a_1 = a_2$. Write $d_1 + d_2 = p^{a-b}d_3$ with $(d_3, b) = 1$. Then, reasoning as before, $d = d_3,\ t = p^{2a-b}$ so $c = b$ and $F = E_{p^b,d_3} \otimes (\oplus M_i)$. If $b < a$, tensor this equation with $(E^L_{p^{a_1},d_1})^*$

and use 4.13 to conclude. If $b < a$, consider the set of automorphisms of F and conclude as above.

$\square$

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