Non Abelian hodge theory

Owen

Throughout X is a compact connected Riemann surface.

Theorem 0.1. *(Non Abelain Hodge Theorem)*

$$\mathcal{M}_{\text{flat}}|^{\text{ss}} \to \mathcal{M}_{\text{higgs}} \tag{1}$$

where $\mathcal{M}_{\text{flat}}$ is the moduli space of rank n vector bundles over X with flat connection, and $\mathcal{M}_{\text{Higgs}}$ is the moduli space of rank n, degree 0 vector bundles E over X together with a section $\varphi \in H^0(X, \text{End}(E) \otimes \Omega^1_X)$.

Definition 0.2. (E, ∇) *is simple if it has no proper flat subbundles. It is semisimple if it decomposes as a direct sum of simple bundles.*

By $\mathcal{M}_{\text{flat}}|^{\text{ss}}$ we denote the moduli space restricted to semisimple bundles.
Assume $n = 1$, so and E is a line bundle.
Recall that there is a complex analytic isomorphism.

$$\begin{aligned}
\mathcal{M}_{\text{Betti}} &= \text{Maps}(\pi_1(X), \text{GL}_n\mathbb{C})/\text{GL}_n\mathbb{C} && (2)\\
&= \text{Maps}(\pi_1(X), \mathbb{C}^*) && (3)\\
&= \text{Maps}(\pi_1(X)/[\pi_1(X), \pi_1(X), \mathbb{C}^*) && (4)\\
&= \text{Maps}(H^1(X, \mathbb{Z}, \mathbb{C}^*) && (5)\\
&= H^1(X, \mathbb{C}^*) && (6)
\end{aligned}$$

In the case is a genus g Riemann surface we have

$$H^\bullet(X, \mathbb{Z}) \Big\{ \textit{things} \tag{7}$$

Remark: when $n = 1$ stability and semistability are automatic. Thus we have that NAHT is equivalent in our case to

$$H^1(X, \mathbb{C}^*) \to \mathcal{M}_{\text{Higgs}} \tag{8}$$

Fact: there exists a complex manifold Pic^0 called the Jacobian which parameterizes degree 0 line bundles. In the case that E is of rank 1 so the endomorphism bundle is trivial. Have Higgs field $\varphi \in H^0(X, \Omega^1_X)$

Thus we reformulate NAHT as

$$H^1(X, \mathbb{C}^*) \to \text{Pic}^0(X) \times H^0(\Omega^1_X) \tag{9}$$

The usual hodge decomposition

$$H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(\Omega^1_X) \tag{10}$$

which is almost what we want, but not quite.

The rest of the talk aims to patch over some holes above, and complete the proof of NAHT.

1 Sheaves

Consider the short exact sequence of sheaves of abelian groups

$$0\mathbb{Z} \qquad \mathbb{C}\mathbb{C}^* \qquad 0 \tag{11}$$
$$0\mathbb{Z} \qquad \mathcal{O}\mathcal{O}^* \qquad 0 \tag{12}$$
$$\tag{13}$$

[[MAKE THIS A COMMUTATIVE DIAGRAMME]]

And applying functor $H^\bullet(X, -)$ and consider the long exact sequence. We now play the diagram chasing game to prove the necessary injectivity and surjectivity results.

Claim $H^0(\mathbb{C}^*) \to H^1(\mathbb{Z})$. Same holds for $H^1(\mathbb{Z}) \to H^1(\mathcal{O}_X)$.

By definition $\mathrm{Pic}(X) = H^1(\mathcal{O}_X)$ The $\ker(\deg : H^1(\mathcal{O}_X^*) \to H^2(\mathbb{Z}))$ is precisely the Jacobian $\mathrm{Pic}^0(X)$.

Surjectivity at $H^1(\mathbb{C}^*)$ follows from the universal coefficient theorem since

$$H^2(\mathbb{Z}) \to H^2(\mathbb{C}) \tag{14}$$

is injective.

Example: Suppose X has genus 1. Then $H^1(\mathbb{Z}) = \mathbb{Z}^2$, $H^1(\mathbb{C}) = \mathbb{C}^2$, $H^1(\mathcal{O}_X) = \mathbb{C}$, $H^1(\mathbb{C}^*) = (\mathbb{C}^*)^2$, $H^0(\Omega^1_X)$.

Warning: $H^1(\mathbb{C}^*)$ is not holomorphically equal to $\mathcal{M}_{\text{flat}}$.

Suppose $H^1(\mathbb{C}^*)$ splits as $\mathrm{Pic}^0(X) \times H^0(\Omega^1_X)$. Then there is a section $s : \mathrm{Pic}^0(X) \to H^1(\mathbb{C}^*)$. Pulls back to a map $s : \mathrm{Pic}^0(X) \to H^1(\mathbb{C})$

[[OTHER THINGS I CANNOT READ]]

Proof of $n = 1$ NAHT

[[FILL IN THIS HOLE]]