Extending to non compact Riemann surfaces

Non-abelian hodge theory for non compact curves (tame or wild) .

Have a punctured Riemann surface $X \setminus \{p_1, \ldots, p_k\}$

Recap the compact case :

1. Harmonic Bundle,
2. flat (stable) complex bundle,
3. Irreducible representations $\pi_1(X) \to GLn$
4. Higgs bundles

Maps:

1. 1) —¿ 2)
2. 2) —¿ 1) Couvette donladson
3. 2) —¿ 3) RH
4. 1) —¿ 4) ¡¡ Hitchn Kobeshi

[[MAKE THIS INTO COMMUTATIVE DIAGRAMME]]

[[ADD NON-COMPACT ANALOGUES THAT WE'LL SEE.]]

Includes parabolic bundles, parabolic Higgs bundle

Definition 0.1. *A parabolic bundle $E \to \bar{X}$ over compact Riemann surface $\bar{X}$ is a bundle together with a subset $D \subset \bar{X}$, such that for each $p \in D$, we have a full flag*

$$E|_p > E|_{1,p} > \cdots > E|_{r,p} > \{0\} \tag{1}$$

$$0 \leq \alpha_1 < \cdots < \alpha_r < 1 \tag{2}$$

This models holomorphic bundles on X.

Parabolic Higgs field (E, E_α, ϑ) such that $E_{\alpha,p} \to \Omega_X \otimes E_{\alpha,p}$.

Parabolic flat bundle $(E, \{E_\alpha\}, \nabla)$ around the punctures we model these in local coordinates $\frac{dz}{z}$ weighted by α.

Both ϑ and ∇ allow poles of order at most one.

Stability for parabolic Higgs-bundle.

Definition 0.2. *Let $(E, \{E_\alpha\}$ parabolic bundle. For subbundle $F \to E$ we inherit the flag structure by restricting the flags $\{E_\alpha\}$.*

$\mathrm{pardeg}(E) = \deg(E) + \sum_p \sum_i \alpha_i(p)$.

Then E is stable if it admits the stability where we consider the pardeg *instead of* deg

Definition 0.3. *(E, D, D'', h) is tame if for each puncture the standard metric on D^2 1) Higgs bundles side $|\lambda_i|_{D_z} < c/|z|$ 2) Flat bundle ∇ flat connection have at most polynomial growth at p v my flat section then $|v|^2 < r^2$*

As expressed in local coordinates

We have a Riemann Hilbert singularities [[WHICH MEANS ??]] for connections with regular singularities

$$\mathrm{Rep}\{\pi_1(\bar{X} \setminus D) \to GLnC\} \to \{\text{ flat connections with regular singularities}\} \tag{3}$$

		(E, ϑ)	(E, ∇)	(L, μ)
Tabulate	Weights			
	Eigenvalues			

Moduli spaces

Once we fix the weight α recover all the results we want.

$m\alpha$ hyperkaheler

$\mathcal{M}_{\text{Betti}}$ is the usual $\mathcal{M}_{\text{dR}}$ has log connections $\mathcal{M}_{\text{Dol}}$ If we consider the special case of nilpotent Higgs fields $\vartheta = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ then $E_{i,p} \to E_{i+1,p} \otimes_p \Omega^1_X$

Then consider the strong parabolic endomorphism.

What about the wild case??

$\nabla = \sum A_i z^{-i} dz$

(Boalch - Biquard)

Still get a correspondence

But we dont get $\mathcal{M}_{\text{Betti}}$