Title

Ben

1 BPS states

Definition 1.1. *BPS states or counts are invariants of 3CY categories.*

A CY category is a category with a duality condition on associated algebras.

Examples of 3 CY categories:

1. Coherent sheaves on X where X is a smooth projective 3-fold over $\mathbb{C}$ with trivial canonical bundle. Serre duality $E, F \in \mathrm{Coh}(X)$ then $\mathrm{Ext}^i(E,F) \cong \mathrm{Ext}(F,E)^*$.
2. Local systems on a real 3 dimensional closed orientable manifold. IN this case $H^i * (M, E^* \otimes F) \cong H^{3-i}(M, F^* \otimes E^*)^*$.
3. M as above. Then Representations of $\mathbb{C}[\pi_1(M)]$.

Any one of these examples gives us the input of DT theory.

Let C be 3CY category. Introduce the notion of charge lattice. Target for some [[SOMETHING]] invariant of algebras of $\mathbb{C}$

Let $N = \mathbb{Z}$ - dimension of $\mathbb{C}[\pi(M)]$ - module

$\mathcal{M}_\gamma$ for $\gamma \in N$ stack of objects in C of class γ. This is manageably small.

Vanishing cycles: Say $\mathrm{Spec}(\mathbb{C}[x]/x^d)$ is a moduli space of objects in some category B. This naively has a only a single object of a single point. However a deformation of x^d will see the point split into d objects. Thus we would like $\#\mathrm{Spec}(\mathbb{C}[x]/x^d)$ to be d and not 1.

Consider $H^*(X, \mathbb{Q})$.... [[CANT SEE]]

DT theory is the study of the cohomology $H(M(C), \mathcal{Q})$ of the stack $M(C)$ over the category into sheaf $\mathcal{Q}$ We use the charge lattice and mixed hodge structures to decompose this space $D(MHS_N)$.

More precisely $[H(M(C), \mathcal{Q})] \in K_0(MHS_N)$ where $K_0(MHS_N) = \mathbb{Z}[[L]]/([L'] + [L''] = [L])$ if $L' \to L \to L'' \to \Delta \in D(MHS_N)$. This is huge and unwieldly.

How to even write this down? Answer: via Plethystic exponential EXP, $f = \sum_{d \in \mathbb{Z}^m} a_d x^d$, $\mathrm{EXP}(f) = \prod_{d \in \mathbb{Z}^m} (1 - x^d)^{-a_d}$.

Where does this come from? Origin: consider

$$K(\mathrm{Vect}_{\mathbb{Z}^m}) \xrightarrow{\chi} \mathbb{Z}[[\mathbb{Z}^m]] \tag{1}$$

$$[v] \mapsto \sum_d \mathrm{dim} V_d x^d \tag{2}$$

but this ill defined...

cf Grothendeick something. Commutative diagramme between characters and symmetric algebra maps.

[[ERRRRRRR]]

Question: How does on understand $[H(M(C), \mathcal{Q}]$? Its just too big.

Integrality conjecture (historically misnamed): This says that $[H(M(C), \mathcal{Q})] = \mathrm{EXP}(\oplus_\gamma BPS_\gamma \otimes H_{\mathbb{C}^*}()$ where BPS_γ is a finite dimensional mixed hodge structures.

Many stupid things have appeared in this talk. We should try to address these...

Let M be an n dimensional orientable closed real manifold. $\mathbb{C}[\pi_1(M)]$ -modules. nCY categories, with $\mathrm{Ext}^i(E,F) \cong \mathrm{Ext}^{n-i}(F,E)^*$ we want $n = 2$ not 3.

Theorem 1.2. *(BEN) Let B be a 2CY category. Let C be a category of pairs (E, f) for $E \in B$, and $f : E \to E^*$. Then C is 3CY and $H(M(C), \mathcal{Q}) \cong H(M(B), \mathbb{Q})$*

Conclusion: $[M(\mathbb{C}[\pi_1(\Sigma_g)])] = \mathrm{EXP}([\oplus_n BPS_{n,g} \otimes H_{\mathbb{C}^*}(pts)])$
Hausel Villegas : $[BPS_{n,g}] = [H(\mathrm{Rep}_n^1(\Sigma_g)]$
$\mathrm{Rep}_n^1(\Sigma_g) = [A, B...]/\mathrm{PGL}_n$
and by NAHT $\cong \mathrm{Higgs}_{n,1}(C)$
Categorified Integrability theorem. this is much much stronger.
What does this have to do with the $P = W$? Only conjectures to follow?
Conjecture: $BPS_{g,n} \cong H(\mathrm{Rep}_n^1(\Sigma_g)$
Theres analogue of the conjecture on the Higgs side. There's a network of conjectures.
Conjecture: $H(\mathrm{Rep}(\mathbb{C}[\pi_1(\Sigma_g)])) \cong \mathrm{Sym}(\bigoplus H(\mathrm{Rep}_n^1(\Sigma_g) \otimes H_{\mathbb{C}}(pt)))$
$H(\mathrm{Higgs}_0^{ss}) \cong \mathrm{Sym}(\bigoplus H(\mathrm{Higgs}(C) \oplus H_{\mathbb{C}^*}(pt)))$
If we believe these conjectures, then $P = W$ becomes equivalent to the an isomorphism between the two LHS above (call it α). α is an isomorphism between 2 qunantum groups. They contain lie algebra structures on respective BPSg and BPS-Higgs
Claim: P, W filtrations come from induced $\mathfrak{sl}_2$ on these Lie algebra.