GIT Versus symplectic reduction

1 The problem

Suppose we have group G acting on space X. We would like to describe X/G of G-orbits and inherit properties of X, however there exists bad points.

Example: Consider $G = \mathbb{C}^*$, $X = \mathbb{C}^2$ with action

$$s(x, y) \mapsto (sx, sy) \tag{1}$$

The quotient $\mathbb{C}^2/\mathbb{C}^*$ is non-hausdorff as some orbits are not closed.

Alternatively suppose

$$s(x, y) \mapsto (sx, s^{-1}y) \tag{2}$$

Now both axis are in the limit of $xy - \alpha$ as $\alpha \to 0$.

Both GIT and symplectic reduction choose some 'unstable' orbits and deal with them.

Let G act on space X where $\mathrm{SL}(n, \mathbb{C})$ and $X \subset \mathbb{P}^n$. The topological characterisation of semi-stability. Let $X \subset \mathbb{P}^n$ has associated affine $\tilde{X} \subset \mathbb{C}^{n+1}$. G acts in $\tilde{X}$. so for any $x \in X$, pick $\tilde{x} \in \tilde{X}$ in lift. Say that x is stable if $0 \notin \overline{G.\hat{x}}$ Say that x is polystable if $G.\hat{x}$ is closed Say that x is stable if it is polystable and has a finite stabliser.

Theorem 1.1. *There exists a projective variety $(X//G)$ such that there exists a surjective morphism $\varphi : X^{ss} \to X//G$ which is a good git. That is*

Consider our earlier example $\mathbb{C}^*$ acts on $\mathbb{C}^2$ now sat in $\mathbb{P}^2$ then $s(x : y : z) \mapsto (sx : sy : z)$. We have three types of orbit, $(x, y, z) \mapsto (x, y, z/s)$ then $0 \in \overline{G.(0, 0, 1)}$ so $X//G = (\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^* \cong \mathbb{P}^1$ $(x, y, z) \mapsto (sx, sy, z)$ then $X//G$ is just one point. $(x, y, z) \mapsto (s^2x, s^2y, sz)$ then $X//G$ is empty.

Here there is choice of embedding, which results in different $X//G$.

An alternative definition runs along the following lines.

Let $X \subset \mathbb{P}^n$ and $G \subset \mathrm{PSL}(n)$ such that G acts on X. Let $K = \Gamma(\bigoplus_{k \in \mathbb{N}} \mathcal{O}(k)|_X)$. Set K^G be the set of elements invariant under G. Then the git reduction of X by G is simply $\mathrm{Proj}(K^G)$

Symplectic reduction.

Let $K = G \cap \mathrm{SU}(n+1)$ The action of K on X is smooth, but also symplectic and Hamiltonian.

Have

$$\mathfrak{k} \to C^{\infty}(\mathbb{P}, \mathbb{R}) \tag{3}$$

$$v \mapsto [m_v] \tag{4}$$

Put together all of the Hamiltonians m_v to give a moment map $m : X \to Lie(K)^*$ such that $\langle m(x), v \rangle = m_v(x)$, for all $vinLie(K)$.

A moment map is unique up to addition of a central element in $Lie(L)^*$

Theorem 1.2. *(Marsden - Weinstenn Meyer) If the action of K on $m^{-1}(0)$ is free and proper, then the symplectic reduction $X^{red} = m^{-0}/K$ is a symplectic manifold with dimension* $\dim(X) - 2 * \dim(K)$.

Consider one of the examples above. $K = \mathrm{U}(1)$ ...

Theorem 1.3. *(Kempf- Ness) A G - orbit contains a zero of the moment map if and only if it's polystable.*

If $x \in X$ is polystable, then if the orbit $G.x$ meets $m^{-1}(e)$ in a single K-orbit.

$x \in X$ is semistable if and if its orbit closure meets $m^{-1}(0)$.

$m^{-1}(0) \subset X^{ss}$ which gives a homeomorphism $m^{-1}(0)/K \to X//G$

Moduli of Vector bundles over $(X, \mathcal{L})$ compact ?? moduli space we have to consider coherent sheaves E of the same hilbert polynomial

For any coherent sheaf E $E(r) \otimes L^{otimesr}$, then for $r >> 0$, $E(r)$ is generated by global sections and has no higher cohomology.

$$0 \to \varphi \to \mathcal{O}_X^{\bigoplus h^0}(E(r)) \xrightarrow{\varphi} E(r) \to 0 \tag{5}$$

$$\chi(E) = \dim H^0(X, E(r)) \tag{6}$$

we fix an isomorphism $H^0(E(r)) \cong \mathbb{C}^N$ where $N = \chi(E(r))$ then all $E's$ are a quotient of $\mathcal{O}(-r) \oplus N$ which are parameterised by a subset of the Grasmannian

Subset $H^0(\mathrm{Ker}(\varphi(s)) \subset H^0(\mathcal{O}(s)^{\otimes N})$ So we divide by the choice of isomorphism. That is $\mathrm{SL}(N, \mathbb{C}^N)$ to get a moduli space of semistable bundles.

$\chi(E(r)) = \sum_i a_i r^{n-i}$ $P_E(r) = \chi(E(r))/a_0$ and $\mu(E) = a_1/a_0$. Depending on the line bundle, then are 2 different notions of stability,

E is (semi)-stable if and only if, fo rnay coherent subsheaf $F \to E$, we have Geiseker-stability if $P_F(r) \leq P_E(r)$ for all r sufficiently large, and slope stable if $\Gamma(F) \leq \Gamma(E)$.

In the case where X is a compact Riemann surface

Theorem 1.4. *(Narasimhan- Seshadri) An indecomposible holomorphic bundle E is slope stable if and only if there is a unitary connection on E having constant central curvature $F = -2\pi i\mu(E)$ such a connection unique up to isomorphism.*