Metric structure on M

Let $(A, \varphi) \in \mathcal{A} \times \Omega^{1,0}(M, \mathrm{ad}P \otimes \mathbb{C})$. $\mathcal{A}$ is the space of connections compatible with a fixed hermitian metric.

$$\mathcal{M} = \{(A, \varphi) : \bar{\partial}_A \varphi = 0, F_A + [\varphi, \varphi^*] = 0\}/\mathcal{G} \tag{1}$$

$\mathcal{G}$ acts on $\mathcal{A} \times \Omega^{1,0}$

$$g \in \mathfrak{g} \cong \Omega^0(M, \mathrm{ad}P \otimes K), \quad (\bar{\partial}_A g, [g, \varphi]) \in T_{A,\varphi}\mathcal{A} \times \Omega^{1,0} \tag{2}$$

$$g((A_1, \varphi_1), (A_2, \varphi_2)) = 2i \int_M \mathrm{tr}(A_1^* A_2 + \varphi_1 \varphi_2^*) \tag{3}$$

Claim that $\Omega^{0,1} \oplus \Omega^{1,0}$ has a hyperkahler structure. Explicitly

$$J(A, \varphi) = (i\varphi, -i\varphi^*) \tag{4}$$
$$K(A, \varphi) = (-\varphi, -i\varphi^*) \tag{5}$$
$$I(A, \varphi) = (i\varphi, i\varphi^*) \tag{6}$$

To each of these we have the associated symplectic $\omega_J, \omega_K, \omega_I$.

ω_I is a symplectic form and so has an associated group action and moment map.

$$\mu(A, \varphi) = F_A + [\varphi, \varphi^*] \tag{7}$$

Have holomorphic symplectic structure Ω_I

$$\Omega_I^{((A_1,\varphi_1),(A_2,\varphi_2))} = \int_M \mathrm{tr}(\varphi_2 A_1 - \varphi_1 A_2) \tag{8}$$

We have the action $g \in \mathfrak{g}$ from the lie algebra associated to the Gauge group.

$$g \in \mathfrak{g} \cong \Omega^0(M, \mathrm{ad}P \otimes K), \quad (\bar{\partial}_B g, [g, \psi]) \in T_{B,\psi}\mathcal{A} \times \Omega^{1,0} \tag{9}$$

induces moment map

$$i_X \Omega_I = df_{X_g} = d\,\langle \mu, g \rangle \tag{10}$$

Consider the symplectic contraction on two forms

$$\Omega_I((\bar{\partial}_A g, [\varphi', g])(A^{0,1}, \Phi)) = \int \tag{11}$$
$$= -\int_M \mathrm{tr}(-g\bar{\partial}_A \Phi - g[A^{0,1}, \varphi']) \tag{12}$$
$$= d\int_M \left(\int_M \mathrm{tr}(\bar{\partial}_A \Phi g)\right)(A^{0,1}, \Phi) \tag{13}$$

Have $\mu(A', \varphi') = \bar{\partial}_{A'} \varphi'$

Split moment maps into real and imaginary parts $\mu = \mu_J + i\mu_K$. For $i = I, J, K$, have $\mu_i(A', \varphi') = 0$. This is equivalent to the self duality equations.

$$\mathcal{M} = \bigcap_i \mu_i^{-1}(0)/\mathcal{G} \tag{14}$$

Want to construct a hyperkahler structure over $\mathcal{M}$. Let $P : \bigcap_i \mu_i^{-1}(0) \to \mathcal{M}$, suppose that $\bar{\omega}_i$ is pullbacked by $P^*\bar{\omega}_i = \omega_i|_{\bigcap \mu}$ This is hypersymplectic structure. Complex structure (requires a descension of metric).

We are taking for granted that the Moduli space $\mathcal{M}$ is smooth. And that the dimension is $12(g-1)$. Claim we have shown it is hyperkahler. So we have a sphere of complex structures on $\mathcal{M}$.

Recall we have a $U(1)$ action on the solution to Hitchins equations. If (A, φ) a solution then so to is $(A, e^{i\vartheta})$. The $U(1)$ action induces action on ...

$$\omega_I \to \omega_I, \quad \Omega_I \to e^{i\vartheta}\Omega_I \tag{15}$$

preserves one kahler form, so on S^2 of $\mathbb{C}$ structures there are two fixed points, the two poles $\pm I$.

I is then the 'preferred' $\mathbb{C}$ structure.

Aside: consider stable Higgs bundles then if (V, φ) is stable, so too is $(V, e^{i\vartheta}\varphi)$ is also stable. $(\mathcal{M}, I)$ is isomorphic to the space of stable Higgs bundles.

$(\mathcal{M}, J)$ is isomorphic to the moduli space of flat connections $A + \varphi + \varphi^* + A^*$.