Hitchin's equations: four dimensionsal motivation from Yang Mills theory, and reduction in two dimensions.

Raul Sanchez Galan

1 Principal bundles

Let $\pi : P \to M$ be a principal G-bundle over M with trivialisation $\{U_\alpha\}$. Suppose we have a representation $\rho : G \to \mathrm{GL}(V)$, then the associated vector bundle is defined to be $P \times V/\sim$ where the equivalence classes are given by $(p, v) \sim (pg, \rho(g^{-1})v)$.

Example representation: The adjoint representation. For each $g \in G$, set $\mathrm{Ad}_g : G \to G$, where $\mathrm{Ad}_g : h \mapsto ghg^{-1}$. Consider the map $\mathrm{Ad} : G \to \mathrm{Aut}(G)$, defined $g \mapsto \mathrm{Ad}_g$. We can identify the tangent space T_eG at the identity with the Lie algebra $\mathfrak{g}$, and hence get the derivative $d_e(\mathrm{Ad}_g) : \mathfrak{g} \to \mathfrak{g}$. The adjoint representation $\mathrm{ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is defined by $g \mapsto \mathrm{ad}_g := d_e(\mathrm{Ad}_g)$. Note that $\mathrm{Aut}(\mathfrak{g})$ denotes the group of automorphisms respecting the Lie Bracket.

Definition 1.1. *A connection on P is a 1-form ω satisfying the following conditions:*

(i) ω is $\mathfrak{g}$ valued, under the decomposition $T^*P = T^*M \oplus \mathfrak{g}$.

(ii) for $g \in G$, $R_g^*\omega_{pg} = \mathrm{ad}_{g^{-1}} \circ \omega_p$, where R_g denotes the action of the Lie group on P.

(iii) restricted to fibres, ω agrees with the Maurer-Cartan right invariant form, with values in the Lie algebra $\mathfrak{g}$, $\omega(dRX) = X$ for $X \in \mathfrak{g}$

Relating to the covariant derivative given by Charles talk (and so need to have (locally) a one form on M with values in $\mathfrak{g}$).

Take a local trivialisation $\{U_\alpha\}$, and set $A_\alpha = s^*(\omega)$ where $s : U_\alpha \to U_\alpha \times G$ is a section. The covariant derivative is locally defined by $\nabla = d + A$.

The curvature of $\nabla : \Omega^0(E) \to \Omega^1(E)$ But induces map d_A extending to the exterior algebra. So define curvature $F_A = d_A^2 : \Omega^0(E) \to \Omega^2(0)$

In local coordinates we can write $\nabla_i = \frac{\partial}{\partial x_i} + A_i$ where $A_\alpha = \sum A_i dx_i$

2 Yang-Mills equations

On a manifold M with metric and orientation, we can define the Yang mills functional is defined on the space of connections of a vector bundles

$$YM(A) = \int |F_A|^2 d\mu \tag{1}$$

The critical points are the solutions of the associated Euler Lagrange

[[EXPAND ON THIS]]

$$d_A F_A = 0, \quad d_A * F_A = 0 \tag{2}$$

The first of these is the Bianchi identity and is automatically satisfied. The latter is 2nd order PDE on A. Minimum of the Yang-Mills equations with finite action are called instantons. The Yang-Mills functional for a vector bundle E is bounded from below by $YM \geq 8\pi^2 c_2(E)$ where $c_2(E) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F_A^2)$. Thus if the integral is defined for some connection, there exists a minimum.

Note that the Hodge star $*$, implicitly appearing in the functional, requires a metric and orientation to be defined. $* : \Omega^2(\mathrm{ad}P) \to \Omega^2(\mathrm{ad}P)$

In the case that $c_2(E) = 0$, then the minimum of the Yang-Mills functional is attained by self dual, and anti-self dual curvatures. This motivates wishing to solve in A, $F_A = *F_A$.

3 Restricting to Euclidean space

For now we restrict to $M = \mathbb{R}^4$ with the standard metric. Then the hodge squared $*^2 = (-1)^{n(n-k)}$ since $*^2 = 1$. Thus we have eigenspace decomposition by $*$ of Ω^2 as $\Omega^2(\mathrm{ad}P) = \Omega^+ \oplus \Omega^-$. Let the decomposition $F_A = F_A^+ + F_A^-$. Expanding F_A^2 we can see that case on minimum critical points we have that $F_A = *F_A$ which is only first order in A. This is if and only if A dual or anti-self dual. (Switching orientation of $\mathbb{R}^4$ takes dual to anti-self-dual).

[[Note: Some issue computing Chern classes over the real bundles?]]

Let G be a compact group acting $\mathbb{R}^4$. $F = F_{ij} dx_i \wedge dx_j$. Then the self dual equation become:

$$F_{12} = F_{34} \qquad F_{13} = F_{42} \qquad F_{14} = F_{23} \tag{3}$$

Suppose A is translation invariant in x_3 and x_4. Relabel $A_3 = \varphi_1$ and $A_4 = \varphi_2$. Then on $\mathbb{R}^2$ consider the connection $A = A_1 dx_1 + A_2 dx_2$.

[[ALGEBRA]]

Let $\varphi = \varphi_1 + i\varphi_2$. Then $F_{12} = i/2[\varphi, \varphi^*]$ and $[\nabla_1 + i\nabla_2, \varphi] = 0$. Setting $z = x_1 + ix_2$ and introducing $\Phi = \frac{1}{2}\varphi dz \in \Omega^{1,0}(\mathbb{R}^2, \mathrm{ad}P \otimes \mathbb{C})$ then $\Phi^* = \frac{1}{2}\varphi * d\bar{z}$.

The equations become

$$F = -[\Phi, \Phi^*] \tag{4}$$

$$\bar{\partial}_A \Phi = 0 \tag{5}$$

These are invariant under changes of coordinates and trivialisations of $\Omega^1(\mathrm{ad}P)$ Observe that the second equation says that Φ is a holomorphic section of $\Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$.

[[MISSING CRUCIAL PART OF ARGUEMENT RELATING TRANSLATIONAL INVARIANCE AND VECTOR BUNDLES OVER RIEMANN SURFACES.]]

4 Hitchin equations

Examples: Let M a compact Riemann surface and G a compact Lie croup. Then $\Phi = 0 \to F_A = 0$ unitary flat connections are in one to one correspondence with stable holomorphic bundles.

$g = h dz \wedge d\bar{z}$ compatible with the ex structure. $G = \mathrm{U}(1)$. Take the Levi- Civita connection. Induces a connection on K by $d + A$, on $K^{1/2}$ by $d + 1/2A$.

[[CLARIFY WHAT THIS MEANS]] On $V = K^{1/2} \oplus K^{-1/2}$ have direct sum of two connections.

Define $\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \Omega^{1,0}(\mathrm{End}(V))$

Have $\mathrm{Hom}(K^{1/2}, K^{-1/2}) \cong K^{-1}$ so 1 denotes the canonical section of $K^{-1} \otimes K$

$F_A = -2h dz d\bar{z}$ and $\mathrm{Ricci}_g = -2g$. We are restricted to negative sectional curvature.

[[THEN WHAT]]