Easy examples of Higgs Bundles and Hitchin Equations

1 The motivation

We wish to relate three distinct families of objects over a given manifold: representations $\pi_1(S) \to \mathrm{Sl}(2,\mathbb{C})$; harmonic bundles (E, ∇, h); and Higgs bundles (E, φ).

2 Setting

Let S connected closed oriented surface $g \geq 2$ and X a Riemann surface over S.

Definition 2.1. *A Higgs bundle on X is a pair (E, φ) where $E \to X$ is a rank 2 holomorphic vector bundle with trivial determinant and $\varphi \in H^0(X, K \otimes \mathrm{End}_0(E))$.*

Definition 2.2. *A Higgs bundle (E, φ) is stable if for any $L \subset E$ such that $\varphi(L) \subset L \otimes K$ we have $\deg(L) < 0$. Such L are called φ invariant. If $E = \bigoplus_{i=1}^{2}(E_i, \varphi_i)$ where the components are stable, then say $(E, \varphi)$is polystable.*

Theorem 2.3. *(Hitchin 87) If (E, φ) is polystable then there exists hermitian metric H on E such that the Chern connection A of H satisfies*

$$F_A + [\varphi, \varphi^{*_H}] = 0 \tag{1}$$

(This is the Hitchin equation.)

[[CHERN CONNECTION]] [[H HERMITIAN ON E OR M ?? Whats is *H]]

Remark ... ALGEBRA

Once you have a solution to the Hitchin Equation $\nabla = A + \varphi + \varphi^*$ connection on E .

(A, φ) solution $\Leftrightarrow$ ∇ is a flat connection.

Get a representation $\mathrm{hol}_\nabla : \pi(S) \to \mathrm{SL}(2, \mathbb{C})$.

Moreover the fact that φ is holomorphic is equivalent to H being a harmonic metric.

3 Example 1

(E, φ) is a stable Higgs bundle iff E is a stable vecotr bundles.

Solving hitchins equations for $(E, 0)$ means finding a metric H such that

$$F_A + [0, 0^*] = 0 \tag{2}$$
$$F_A = 0 \tag{3}$$

The Chern connection is already flat.

We recover Narasimhan-seshadin

$$\{E \text{ rank 2, triv det, stable holom vb}\} \leftrightarrow \{\mathrm{Rep}(\pi_1(S), \mathrm{SU}(2))\} \tag{4}$$

Relating the Moduli space $\mathcal{M}_{0,2}(X)$ the Doulbeuat moduli. Complex structure with a complex compact submanifold.

$$M_{0,2}(X) \to T^* M_{0,2}(X) \xrightarrow{\text{open, dense}} \mathcal{M}_{0,2} \tag{5}$$

4 Hyperbolic Geometry and Teichmuller theory

Recall that hyperbolic space $\mathbb{H} = \{(x, y) \in \mathbb{R} | \ y > 0\}$ has hyperbolic metric $g = \frac{dx \wedge dy}{y^2}$. It has constant sectional curvature -1. The orientation preserving automorphism group is $\mathrm{Aut}^+(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$. How does one construct $\mathbb{H}$-structures on S.

[[Let $\Gamma < \mathrm{PSL}(2, \mathbb{R})$]]

We define the following three spaces associated to S. The Fuchisan space $\mathcal{DF}$ is

$$\mathcal{DF}(\pi_1(S), \mathrm{PSL}_2\mathbb{R})) = \tag{6}$$
$$\{\rho : \pi(S) \to \mathrm{PSL}_2(\mathbb{R}) | \text{ injective, discrete}\}/\mathrm{PSL}_2(\mathbb{R}) \tag{7}$$

The Fricke space $\mathcal{F}(S)$ is

$$\mathcal{F}(S) = \{(Y, h, f) | \ h \text{ hyperbolic metric } f : S \to Y\} / \mathrm{Diff}_0(S) \tag{8}$$

The Teichmuller space $\mathcal{T}(S)$ is

$$\mathcal{T}(S) = \{(X, l) | \ l : S \to X\} / \mathrm{Diff}_0(S) \tag{9}$$

There are maps relating these three spaces ad the spaces. cf Killing Hopf, Uniformization (Poincare Koebe)

Theorem 4.1. *As smooth manifolds* $\mathcal{D}F(S) = \mathcal{F}(S) = \mathcal{T}(S) = \mathbb{R}^{6g-6}$

Theorem 4.2. *(Wolf) Fix* $X \in \mathcal{T}(S)$*. Then get homeomorphism*

$$\mathcal{F}(S) \to H^0(X, K^2) \tag{10}$$

Fix $X \in \mathcal{T}(S)$ choose a sqare root of K Take short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathcal{O}^* \to \mathcal{O}^* \tag{11}$$

[WHATS]

Let $E = K^{1/2} \oplus K^{-1/2}$ which is holomorphic of rank 2. Note that this is unstable. $\deg(K^{1/2}) = g - 1 > 0$.

Choose

$$\varphi = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \in H^0(X, K \otimes \mathrm{End}_0(E)) \tag{12}$$

The only φ-invariant subbundle is $0 \oplus K^{-1/2}$ which has degree $1 - g < 0$. Thus (E, φ) is a stable higgs bundle.

Can now apply Hitchins theorem and get H hermitian metric on E, such that the chern connection satisfies $F_A + [\varphi, \varphi^*] = 0$

Remark: E is decomposible.

$$\mathcal{M}_{0,2}(X) \to H^0(X, K^2) \tag{13}$$

$$(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \mapsto \alpha \tag{14}$$