Vector bundle over Riemann surface

Definition 0.1. *Let M be a Riemannian manifold then a smooth vector bundle (E, M, V, π) such that the projection $\pi : E \to M$ is smooth and there exists a cover of M such that all $p \in M$, $\pi^{-1}(U_i) \cong U_i \times V$, for vector space V.*

We get transition maps $\{\varphi_{ij} : U_{ij} \to \mathrm{GL}(V)$ satisfying the cocycle condition.

From a cover $\{U_i\}$ and transition maps φ_{ij} we can recover the vector bundle.

Examples: Trivial. Cotangent space.

In the case E is of rank 1, then call E a line bundle. For $\xi \in H^1(X\mathcal{O}^*)$ holomorphic line bundles.

Vector bundles from surface group representation X a Riemann surface.

$\rho : \pi_1(X) \to \mathrm{GL}(V)$, $\hat{X} \times_\rho V := \hat{X} \times V/\sim$ where $\hat{X}$ is the universal cover of X . And the equivalence is as it should ....

Definition 0.2. *As smooth section $s : X \to E$ is a smooth map such that $\pi \circ s = \mathrm{id}|_X$*

$\Gamma(TX)$ denotes the space of vector fields.

We can construct new bundles from existing of ones with $\oplus, \otimes$ and $\wedge$. Direct some, tensor product, exterior products

Definition 0.3. *Let $E \to X$ and $E' \to X'$ be smooth vector bundles over X. Then a bundle map is a pair $(f, u) : (E, X) \to (E', X')$ such that the relevant maps commute. The set of all such is $\mathrm{Hom}(E, E')$ and pro... ()*

If we have a map $f : Y \to X$ then we can consider the pullback of $e \to X$.

Proposition 0.4. *Let $E \to X$ be a smooth bundle, let $f : Y \to X$ and $g : Y \to X$ be homotopic then $f^*E \cong g^*E$, topological classification of rank n vector bundles over X*

Definition 0.5. *The Grassmanian $G(n, n + k)$ is the space n-planes in $\mathbb{R}^{n+k}$*

Definition 0.6. *The Tautological bundle $\gamma^n(\mathbb{R}^{n+k})$ over $G(n, n + k)$ to be the bundle with elements are of the form (V, v) where $v \in V$.*

Theorem 0.7. *Any rank n vector bundle over a paracompact space admits a bundle map to γ^n. Any two bundle maps are isomorphic.*

Have a correspondence smooth rank n vector bundles over X and the homotopy classes of smooth maps from X to $G(n)$.

Degree for line bundles look at the degree of the associated divisor. For higher rank, do as above for $\det(E)$.

Over X a Riemann surface. The trivial complex line bundle with 2 open sets. Pick $p_0 \in X$, U_1 is a contractible neighbourhood of p. $U_2 = U_1 \setminus \{p\}$. $f : S^1 \to S^1$

Have short exact sequence

$$0 \to \mathbb{Z} \to C^\infty \xrightarrow{\exp} C^{\infty*} \tag{1}$$

Get the long exact sequence which results in

$$H^1(X, C^\infty) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z} \tag{2}$$

which is otherwise known as the degree map

Definition 0.8. *A connection $D : \Omega^0(X, E) \to \Omega^1(X, E)$ is a linear map satisfying a Leibniz rule: for $\sigma \in \Omega^0(X, E)$ and $f \in C^\infty$, then $D(f\sigma) = df \otimes \sigma + fD\sigma$.*

Pick a local frame. Then there exists some ω such that $D = d + \omega$. ω is known as the connection 1-form. ω satisfies the computational condition that changing frames by g

$$\omega_{e'} = dg \cdot g^{-1} + g\omega_e g^{-1} \tag{3}$$

Definition 0.9. *A hermitian structure H on a complex vector bundle E is a smooth family of Hermitiain scalar products on E such that for any sections s, s', $H(s, s')$ is smooth.*

Definition 0.10. *A hermitian structure H is a unitary with respect to a connection D if for $s_1, s_2 \in \Omega^0(E)$ we have $dH(s_1, s_2) = H(Ds_1, s_2) + H(s_1, Ds_2)$.*

Definition 0.11. *$E \to X$ is a holomorphic vector bundle if $\pi : E \to X$ is holomorphic.*

Proposition 0.12. *The following are equivalent. 1. $\pi : E \to X$ is holomorphic. 2. $\varphi_{ij} : U_{ij} \to \mathrm{GL}(U)$ holomorphic 3. $\exists \bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the Leibniz rule, and $\bar{\partial}^2 = 0$.*

Theorem 0.13. *Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle. Let H be a holomorphic sturcture on E. Then there exists a connection that is unitary with respect to H, and $D^{0,1} = \bar{\partial}_E$.*

Where $D^{0,1} = \pi^{0,1} \circ D$.
Have $J : TX \to TX$. Proof: Calculation using connection 1-form $\omega\partial H \cdot H^{-1}$ and parallel transport.

Definition 0.14. *A section is parallel with respect to a connection D if $Ds = 0$*

Parallel transport is, in general, sensitive to the underlying curve. It depends on the underlying curvature of the connection.

Definition 0.15. *The curvature of D is given by D^2*

This is sensible by extending the initial definition of D to the exterior algebra. Say D is flat, if $D^2 = 0$.

Theorem 0.16. *D is flat then the holonomy only depends on the homotopy of the curve.*