Spectral curves and integrability of Dolbeaut moduli

Philip

Rationale setup: Moduli space $\mathcal{M}_{Dol}$ of stable Higgs Bundles.

[[WHY IS THIS NOT Mhiggs??]]

Remark: $\mathcal{M} \subset T^*\mathcal{N}$ open dense for $\mathcal{N}$ is the moduli space of stable vector bundles Can generalise this: Let $P \to C$ be a principal G - bundle, then let V be an associated vector bundle to the fundamental representation. Implicitly assuming that G is $\mathrm{GL}_n\mathbb{C}$??

Claim: $\mathcal{M}_{Dol}$ is an algebraically completely integrable system.

Want: proof by construction Hitchin fibration $h : \mathcal{M} \to \mathcal{B}$ There exists curve $\Sigma \to C$ such that the fibres of the Hitchin fibration are $\mathrm{Jac}(\Sigma)$ spectral curves.

1 Integral Systems

Let (M, ω) be a (holomorphic) symplectic manifold.

Example: Cotangent spaces, Kahler manifolds or hyperkahler manifolds. The latter is holomorphic symplectic $\Omega = \omega_J + i\Omega_K$.

Definition 1.1. *For $f \in C^\infty(M)$, the Hamiltonian of f is the vector field X_f, such that*

$$df = \omega(X_f, \cdot) \tag{1}$$

Definition 1.2. *For $f, g \in C^\infty(M)$, the Poisson Bracket $\{f, g\} = \omega(X_f, X_g)$. Say f, g Poisson commute if their Poisson bracket is 0.*

Let f,g be G-invariant functions on M, then can define $\tilde{f}, \tilde{g}$ on quotient $\mu^{-1}(0)/G$. They Poisson commute if and only if $\tilde{f}, \tilde{g}$ Poisson commute.

Definition 1.3. *Let (M, ω) of dim $2n$ is called (holomorphically) completely integrable system if: there exists n functions $\{f_1, \dots, f_n\}$ which pairwise Poisson commute, and are functionally independent ie $\Lambda_i df_i \neq 0$ on an open dense set of M, denoted M_0.*

Remark: The level sets of $\{f_i\}$ give a foliation on M_0

Theorem 1.4. *(Arnold - Louiville) Let $(M, \omega, \{f_i\})$ be a (holomorphic) complete integrable system. Let N be a connected complement of the level set of f. Then N is a diffeo (biholomorphic) to $\mathbb{R}^k \times T^{n-k}$ ($\mathbb{C}^k \times T^{n-k}$)*

In particular complete connected components are diffeomorphic (biholomorphic) to a torus.

Definition 1.5. *Algebraically complete integrable system is a holomorphic completely integrable system if the generic fibres of $f : M \to \mathbb{C}^n$ are abelian varieties*

2 Hitchin Fibration

Consider $\mathcal{M}_{Dol}$.

Want map φ to a basis of $n = \dim(\mathcal{M})/2$ functions given by polynomials

$$h : \mathcal{M} \to \mathcal{B} := \bigoplus_{i=1}^{r} H^0(C, K)(V, \varphi) \qquad \mapsto (a_i(\varphi), \dots, a_r(\varphi)) \tag{2}$$

$\det(\lambda - \varphi) = \lambda^r + a_1\lambda^{r-1} + \dots + a_{r-1}\lambda + a_r$

Actually take $\pi : K \to C$ for $\{a_i\} \in \mathcal{B}$, let φ be the tautological section of π^*K. This map is proper and surjective and $\dim(\bigoplus_i^r H^0(C, K^i)) = n$.

3 Spectral Curve

Consider $\det(\lambda - \varphi) = \mathrm{Char}_\varphi(\lambda)$, gives an algebraic curve Σ in $K = T^*C$ and is called a spectral curve. A generic curve (generic $\{a_i\} \in \mathcal{B}$) is smooth. We have map $\pi : T^*C \to C$, restricting to $\pi : \Sigma \to C$.

Thus we view Σ as a ramified r cover of C. At generic points on C, the $\lambda_1, \ldots, \lambda_r$ distinct . Eigenspaces give a line bundle $L \to \Sigma$ implies a point in $\mathrm{Jac}(\Sigma)$

Claim: The fibre of $h : \mathcal{M} \to \mathcal{B}$ are given by $\mathrm{Jac}(\Sigma)$ for $\Sigma = \Sigma_{\{a_i\}}$

Some technicalities about how all this work is omitted here. Use the direct image construction to get line bundle over Σ.

$u \subset C$ $H^0(U, \pi_* L) = H^0(\pi^{-1}(U), L)$

To retrieve φ via the functorial structure. [[COMMUTATIVE DIAGRAM HERE]]

Stability: Σ is irreducible. If there exists $M \subset V = \pi_* L$, φ invariant then $\mathrm{char}_\varphi|_M$ divides the character variety.