P=W conjecture

[[MISSED OPENING REMARKS]]
Σ is a Riemann surface with genus $g \geq 2$ Let $n > 0$ and $d \in \mathbb{Z}$ and gcd = 1.

$$\mathcal{M}_{\text{betti}} \to \mathcal{M}_{\text{Dol}} \tag{1}$$

$$\mathcal{M}_{\text{Betti}}\{A_i, B_i \in GLnC, i = 1, \ldots, g | \pi[A_i, B_i] = \xi_n^d I_n\} // GLn\mathbb{C} \tag{2}$$

$$\mathcal{M}_{\text{Dol}} = \{\text{stable higgs bundles of rank } n \text{ and degree } d \ \} \tag{3}$$

NAHT implies $H^*(\mathcal{M}_{\text{Betti}}) = H^*\mathcal{M}_{\text{Dol}}$ Want to see he difference on the level of coholomogy groups How? Mixed hodge structures.

1 Sheaf cohomologies

Definition 1.1. *Let X be a complex algebraic variety. $\mathcal{D}_C^b(X)$ is the derived category of constructible sheaves.*

The objects are complexes of sheaves of abelian groups such that their cohomologies are constructable. That is for a given chain there exists a stratification such that ...

The morphisms are chain maps, modulo null-homotopic ones and invert quasi isomorphism.

$E_p \in dcat$ gives rise to $H^*(X, E_\bullet)$ Choose $E_\bullet = I_\bullet$ and I such that I_i are injective objects. and apply $\Gamma(X, I_\bullet)$ $H^i(X, E_\bullet) := H^i(\Gamma(X, I_\bullet))$

For instance if we have a finite filtration

$$E_\bullet^1 \to E_\bullet^2 \to \cdots \bigcup_i E_\bullet^i = E_\bullet \tag{4}$$

It induces a filtration at the level of homology groups $H^*(X, E_\bullet)$.

2 W

Recall from usual Hodge theory we know that $\mathbb{C} \cong \Omega_X^*$ (analytic). Thus $H^*(X, \mathbb{C}) \cong H^*(X, \Omega_X^*)$

Ω_X^* admits a stupid filtration

$$(F^i \Omega_X^*) = \begin{cases} 0 & n < p \\ \Omega_X^n & n \geq p \end{cases} \tag{5}$$

Hodge: If X is a Kahler and compact, then:

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} = H^{p,q}(X, \mathbb{C}) \tag{6}$$

and $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$

Let us suppose that X is not compact, but smooth. and we have a compactification $X \to \bar{X}$ and $Y \subset \bar{X}$ is the complement set. It is a collection of divisors on $\bar{X}$

On $\bar{X}$ we can define meromorphic 1-forms, locally modelled on dz_i/z_i where $Y = \bigcap Y_i$. Want divisor to have at worst normal crossings

Definition 2.1.

$$\Omega^1_X \langle Y \rangle = \{\textit{sheaf of 1-forms generated by } \Omega^1_X \textit{ and } dz_i/z_i\}, \quad \Omega^p_X \langle Y \rangle = \Lambda^p(\Omega^1_X \langle Y \rangle) \tag{7}$$

Fact

$$H^*(X, \mathbb{C}) = H^*(...) \tag{8}$$

Definition 2.2. *$W_n(\Omega^p_{\bar{X}} \langle Y \rangle) := \langle \alpha \wedge dz_{i_1}/z_{i_1} ... \rangle$ weighted filtration.*

Now we have two filtration on $H^*(X, \mathbb{C})$: $W_\bullet$ is increasing, $F^\bullet$ is decreasing.

Theorem 2.3. *(Deligne) Let X be an algebraic variety over $\mathbb{C}$. In every cohomological degree n, the graded piece $\mathrm{Gr}^W_L := W_L/W_{L-1}$ admits a canonical decomposition*

$$\mathrm{Gr}^W_L = \bigoplus \mathrm{Gr}^F_P(\mathrm{Gr}^W_L), \quad \mathrm{Gr}^P_F \cong \overline{\mathrm{Gr}^{L-P}_F} \tag{9}$$

$H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ grading gives map $H^2 \to H^2_{(2)} \oplus H^2_{(2)} \oplus H^2_{(2)}$ No longer have 'pure hodge' decomposition. have a filtrated.

Properties: $W_\bullet$ and $F^\bullet$ commute with pullbacks. Compatible with Kunneth cup product. On H^j, W has non-zero degrees. $[j, z_j]$ is X is smooth non-compact. $[0, j]$ if X is singular compact. $[j]$ if X is smooth compact. $[0, 2j]$ in general.

Non abelian hodge theory in rank 1.

$\mathcal{M}_{\mathrm{Betti}} = (\mathbb{C}^*)^{2g} \to \mathrm{Jac}_\Sigma \times \mathbb{C}^g = \mathcal{M}_{\mathrm{Dol}}$

$H^*(\mathcal{M}_{\mathrm{Betti}}) = H^*(\mathbb{C}^*)^{\otimes 2g}$

$H^*(\mathbb{C}^*) = \mathbb{C} \oplus \mathbb{C}$ cohomology degree (0,1) Weight $(0, 2)$.

On the side of $\mathcal{M}_{\mathrm{Dol}}$ the weight filtration is trivial.

So question is what happens if the filtration is carried across. The answer comes in the form of perverse sheaves....

Hitchin-Base $\mathcal{B}$

$$\mathcal{M}_{\mathrm{Dol}} \to \mathcal{B} \tag{10}$$

Have

$$H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{C}) \cong H^*(\mathcal{B}, \pi_* \underline{\mathbb{C}}) \tag{11}$$

taking the stupid filtration. Gives Leray filtration, and this is not the right one!!

Instead take the category inside perverse sheaves...

The conjecture: Under natural diffeomorphism $\mathcal{M}_{\mathrm{Dol}} \cong \mathcal{M}_{\mathrm{Betti}}$ then $P = W$

Low rank cases dealt with.

[[A LOT MISSING HERE]]