

Easy examples of Higgs Bundles and Hitchin Equations

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1 The motivation

We wish to relate three distinct families of objects over a given manifold: representations $\pi_1(S) \rightarrow \mathrm{Sl}(2, \mathbb{C})$; harmonic bundles (E, ∇, h) ; and Higgs bundles (E, φ) . Denote these moduli spaces by $\mathcal{M}_{\mathrm{Betti}}$, $\mathcal{M}_{\mathrm{Har}}$, and $\mathcal{M}_{\mathrm{Higgs}}$.

2 Setting

Let S connected closed oriented surface $g \geq 2$ and X a Riemann surface over S .

Definition 2.1. A Higgs bundle on X is a pair (E, φ) where $E \rightarrow X$ is a rank 2 holomorphic vector bundle with trivial determinant and $\varphi \in H^0(X, K \otimes \mathrm{End}_0(E))$.

Definition 2.2. A subset $L \subset E$ of Higgs bundle (E, φ) is φ invariant if L is a subbundle such that $\varphi(L) \subset L \otimes K$. Say that (E, φ) is stable if for any φ invariant subbundle $L \subset E$, we have $\deg(L) < 0$. If $E = \bigoplus_{i=1}^2 (E_i, \varphi_i)$ where the components are stable, then say (E, φ) is polystable.

Theorem 2.3. (Hitchin 87) If (E, φ) is polystable then there exists hermitian metric H on E such that the Chern connection A of H satisfies

$$F_A + [\varphi, \varphi^{*H}] = 0 \quad (1)$$

(This is the Hitchin equation.)

[[CHERN CONNECTION ??]]

Remark ... ALGEBRA

[[HOLE HERE]]

Once you have a solution to the Hitchin Equation $\nabla = A + \varphi + \varphi^*$ connection on E .

[[WHAT DOES THIS MEAN]]

(A, φ) solution $\Leftrightarrow \nabla$ is a flat connection. Get a representation $\mathrm{hol}_{\nabla} : \pi_1(S) \rightarrow \mathrm{SL}(2, \mathbb{C})$.

Moreover the fact that φ is holomorphic is equivalent to H being a harmonic metric.

[[WRITE SYMBOLICALLY AS WELL]]

3 Example 1

(E, φ) is a stable Higgs bundle iff E is a stable vector bundles.

Solving Hitchin's equations for $(E, 0)$ means finding a metric H such that

$$F_A + [0, 0^*] = 0 \quad (2)$$

$$F_A = 0 \quad (3)$$

The Chern connection is already flat.

We recover Narasimhan-Seshadri

$$\{E \text{ rank 2, triv det, stable holom vb}\} \leftrightarrow \{\mathrm{Rep}(\pi_1(S), \mathrm{SU}(2))\} \quad (4)$$

Relating the Moduli space $\mathcal{M}_{0,2}(X)$ the Doublbeuat moduli. Complex structure with a complex compact submanifold.

$$M_{0,2}(X) \rightarrow T^*M_{0,2}(X) \xrightarrow{\text{open, dense}} \mathcal{M}_{0,2} \quad (5)$$

4 Hyperbolic Geometry and Teichmuller theory

Wish to endow in S hyperbolic structure. Recall that hyperbolic space $\mathbb{H} = \{(x, y) \in \mathbb{R} \mid y > 0\}$ has hyperbolic metric $g = \frac{dx^2 + dy^2}{y^2}$. It has constant sectional curvature -1 . The orientation preserving automorphism group is $\text{Aut}^+(\mathbb{H}) = \text{PSL}(2, \mathbb{R})$.

[[Let $\Gamma < \text{PSL}(2, \mathbb{R})$]]

We define the following three spaces associated to S . The Fuchisan space \mathcal{DF} is

$$\mathcal{DF}(\pi_1(S), \text{PSL}_2(\mathbb{R})) = \quad (6)$$

$$\{\rho : \pi(S) \rightarrow \text{PSL}_2(\mathbb{R}) \mid \text{injective, discrete}\} / \text{PSL}_2(\mathbb{R}) \quad (7)$$

The Fricke space $\mathcal{F}(S)$ is

$$\mathcal{F}(S) = \{(Y, h, f) \mid h \text{ hyperbolic metric } f : S \rightarrow Y\} / \text{Diff}_0(S) \quad (8)$$

The Teichmuller space $\mathcal{T}(S)$ is

$$\mathcal{T}(S) = \{(X, l) \mid l : S \rightarrow X\} / \text{Diff}_0(S) \quad (9)$$

[[DEFINE THE QUOTIENT SPACES HERE]]

[[COMMUTATIVE DIAGRAMME]] There are maps relating these three spaces. cf Killing Hopf, Uniformization (Poincare Koebe)

Theorem 4.1. *As smooth manifolds $\mathcal{DF}(S) = \mathcal{F}(S) = \mathcal{T}(S) = \mathbb{R}^{6g-6}$*

Theorem 4.2. *(Wolf) Fix $X \in \mathcal{T}(S)$. Then there is a homeomorphism*

$$\mathcal{F}(S) \rightarrow H^0(X, K^2) \quad (10)$$

Fix $X \in \mathcal{T}(S)$ choose a square root of K Take short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{O}^* \rightarrow \mathcal{O}^* \quad (11)$$

Let $E = K^{1/2} \oplus K^{-1/2}$ which is holomorphic of rank 2. Note that this is unstable. $\deg(K^{1/2}) = g - 1 > 0$. Choose

$$\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H^0(X, K \otimes \text{End}_0(E)) \quad (12)$$

The only φ -invariant subbundle is $0 \oplus K^{-1/2}$ which has degree $1 - g < 0$. Thus (E, φ) is a stable higgs bundle.

Can now apply Hitchins theorem and get H hermitian metric on E , such that the chern connection satisfies $F_A + [\varphi, \varphi^*] = 0$

Remark: E is decomposable.

$$\mathcal{M}_{0,2}(X) \rightarrow H^0(X, K^2) \quad (13)$$

$$\left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \right) \mapsto \alpha \quad (14)$$