Metric structure on $\mathcal{M}$

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1 Overview of notation and conventions

In some of the previous talks we were introduced to the affine space of connections, $\mathcal{A}$, and the space of Higgs fields, $\Omega^{1,0}(M, \mathrm{ad}P \otimes \mathbb{C})$. In Talk 5 we saw these spaces constructed via dimensional reduction from the affine space of connections over $\mathbb{R}^4$. Just to restate it here $\mathcal{A}$ is an affine space modeled on $\Omega^{0,1}(M, \mathrm{ad}P \otimes \mathbb{C})$. By this we mean that for $A, B \in \mathcal{A}$ and $a \in \Omega^{0,1}(M, \mathrm{ad}P \otimes \mathbb{C})$ we have that

$$\bar{\partial}_A - \bar{\partial}_B \in \Omega^{0,1}(M, \mathrm{ad}P \otimes \mathbb{C}), \tag{1}$$

$$\bar{\partial}_{A+\alpha} = \bar{\partial} + A^{(0,1)} + \alpha \in \mathcal{A}, \tag{2}$$

where we are using the usual abuse of notation and denoting a connection by the connection one-form piece and since we are working on a holomorphic vector bundle where the connection is compatible with a fixed hermitian metric the connection is specified by its $(0, 1)$ piece[1]. We will use the notation of the previous talks where possible and are working on a vector bundle $\mathrm{ad}P$ over the Riemann surface M where P was the principal G bundle over $\mathbb{R}^4$.

DISCLAIMER: In [1] Hitchin proves that $\mathcal{M}$ is a smooth manifold of dimension $12(g - 1)$, where g is the genus of the underlying Riemann surface M. However, I will not prove this here and refer those interested to §5 in [1]. Also I will be assuming that we are working with the $SO(3)$ self-duality equations, as was done in Talk 8.

On a notational note, as this was a point of confusion in the talk, I will use $N = \mathcal{A} \times \Omega^{0,1}(M, \mathrm{ad}P \otimes \mathbb{C})$ for the space of connections and Higgs fields, (A, φ) for a connection and Higgs field pair in N and $(\dot{A}, \dot{\varphi})$ for a vector in the tangent space $T_{(A,\varphi)}N \simeq \Omega^{(0,1)}(M; \mathrm{ad}P \otimes \mathbb{C}) \oplus \Omega^{(1,0)}(M; \mathrm{ad}P \otimes \mathbb{C})$. The only difference between my notation and that of [1] is that I am using $\dot{A}$ rather than $\dot{A}^{(0,1)}$ for the tangent vector to the connection and not explicitly showing that $\dot{A}$ only stands for the $(0, 1)$ piece.

2 The Moduli space of solutions to the self-duality equations

Let $(A, \varphi) \in N$ be a connection, Higgs field pair.

Definition 2.1. *The Moduli space of solutions to the Hitchin equations is*

$$\mathcal{M} = \{(A, \varphi) \in N : \bar{\partial}_A \varphi = 0, F_A + [\varphi, \varphi^*] = 0\}/\mathcal{G}, \tag{3}$$

where $\mathcal{G}$ is the group of gauge transformations.

Note that as $g = SO(3)$ we have that $\mathcal{G} = \Omega^0(M; End_0(\mathrm{ad}P))$.
We also have that $\mathcal{G}$ acts on N as

$$g \in \mathfrak{g} \cong \Omega^0(M, \mathrm{ad}P \otimes K), \quad X = (\bar{\partial}_A g, [\varphi, g]) \in T_{A,\varphi}N. \tag{4}$$

The Hermitian metric on $T_{A,\varphi}N$ given in [1] is

$$g((\dot{A}_1, \dot{\varphi}_1), (\dot{A}_2, \dot{\varphi}_2)) = 2i \int_M \mathrm{tr}(\dot{A}_1^* \dot{A}_2 + \dot{\varphi}_1 \dot{\varphi}_2^*) \tag{5}$$

[1] I think that this was discussed in Talk 3 and is due to the $(1, 0)$ part being related to the $(0, 1)$ part via conjugation with respect to the hermitian metric.

Remark 2.2. *$T_{(A,\varphi)}N \simeq \Omega^{0,1} \oplus \Omega^{1,0}$ has a hyperkahler structure. Explicitly*

$$J(\dot{A}, \dot{\varphi}) = (i\dot{\varphi}^*, -i\dot{A}^*), \tag{6}$$

$$K(\dot{A}, \dot{\varphi}) = (-\dot{\varphi}^*, \dot{A}^*), \tag{7}$$

$$I(\dot{A}, \dot{\varphi}) = (i\dot{A}, i\dot{\varphi}^*), \tag{8}$$

for which we can check that I, J, K obey the quaternion algebra, $IJ = K$ etc.

To each of these complex structures we have an associated kahler form $\omega_J, \omega_K, \omega_I$.
ω_I is the kahler form associated to I and is invariant under the action of $\mathcal{G}$ and so has a moment map associated to it. We saw in Talk 8 that this moment map is

$$\mu_I(A, \varphi) = F_A + [\varphi, \varphi^*] \tag{9}$$

We can also combing the other kahler forms to get the holomorphic symplectic structure $\Omega_I = \omega_J + i\omega_K$[2].

$$\Omega_I((\dot{A}_1, \dot{\varphi}_1), (\dot{A}_2, \dot{\varphi}_2)) = \int_M \mathrm{tr}(\dot{\varphi}_2 \dot{A}_1 - \dot{\varphi}_1 \dot{A}_2). \tag{10}$$

In this case the action of $\mathcal{G}$ induces a moment map through

$$i_{X_g}\Omega_I = df_{X_g} = d\,\langle \mu, g\rangle\,. \tag{11}$$

Example 2.3. *To explicitly construct μ consider*

$$\begin{aligned}(i_X\Omega_I)(\dot{A}, \dot{\varphi}) &= \omega((\bar{\partial}_A g, [\varphi, g]), (\dot{A}, \dot{\varphi})), && (12)\\ &= \int_M \mathrm{tr}\left(\dot{\varphi}\bar{\partial}_A g - [\varphi, g]\dot{A}\right), && (13)\\ &= \int_M \mathrm{tr}\left(-g\bar{\partial}_A\dot{\varphi} - g[\dot{A}, \varphi]\right), && (14)\\ &= d_N\Big(-\int_M \mathrm{tr}\left(\bar{\partial}_A \varphi g\right)\Big)(\dot{A}, \dot{\varphi}), && (15)\end{aligned}$$

so we have that

$$\mu(A, \varphi) = \bar{\partial}_A \varphi. \tag{16}$$

Split this moment map into real and imaginary parts $\mu = \mu_J + i\mu_K$ to get the moment maps associated with J and K.
Now for $j = I, J, K$ we have that

$$\mu_i(A', \varphi') = 0 \tag{17}$$

is equivalent to the self duality equations. Going back our definition of $\mathcal{M}$ we have that

$$\mathcal{M} = \bigcap_{i=I,J,K} \mu_i^{-1}(0)/\mathcal{G} \tag{18}$$

which is the form of a hyperkahler quotient. If we had that N was a finite dimensional manifold and $\mathcal{G}$ a finite dimensional group then by a variant of the Kempf-Ness theorem, which we encountered in Talk 4, we would have that $\mathcal{M}$ inherited a hyperkahler structure. However, in the infinite dimensional case we don't have it so easy and need to proceed on a case by case basis.

Want to construct a hyperkahler structure over $\mathcal{M}$. Let $P : \bigcap_i \mu_i^{-1}(0) \to \mathcal{M}$, suppose that $\bar{\omega}_i$ is pullbacked by $P^*\bar{\omega}_i = \omega_i|_{\bigcap \mu^{-1}(0)}$ then $d\omega_i = 0$ implies that $d\bar{\omega}_i = 0$. This is a hypersymplectic structure. To make it hyperkahler we need to see that the complex structure, and the metric descends[3] to $\mathcal{M}$. It is somewhat surpising that this works but it does, for the details see the proof of Theorem 6.7 in [1].
Remember that here we are taking for granted that the Moduli space $\mathcal{M}$ is smooth and that the dimension is $12(g-1)$. We can now see that $\mathcal{M}$ is a smooth hypekahler manifold.

[2]It is called a holomorphic symplectic form as it is a closed non-degenerate $(2,0)$ form. More details were given in Talk 12

[3]To see this consider that $df_X = 0$ on $\bigcap_i \mu_i^{-1}(0)$ so the metric is degenerate there and in particular it is zero for vectors tangent to the gauge orbits and we can also see that the horizontal subspace is preserved by I, J, K. Now the $\mathcal{G}$ action also preserves I, J, K so the three complex structures descend to $\mathcal{M}$.

Remark 2.4. *Take $x \in S^2$ then*

$$(x_1 I + x_2 J + x_3 K)^2 = -1 \tag{19}$$

so we have a sphere of complex structures on $\mathcal{M}$ which we call a twistor sphere of complex structure.

Now consider $\mathcal{M} \times S^2$. Recall that we have a $U(1)$ action on the solution to Hitchins equations; If (A, φ) a solution then so to is $(A, e^{i\vartheta}\varphi)$. The $U(1)$ action induces an action on $T_{(A,\varphi)}\mathcal{M}$ and $\mathcal{M} \times S^2$ under which

$$\omega_I \to \omega_I, \quad \Omega_I \to e^{i\vartheta}\Omega_I. \tag{20}$$

This preserves one kahler form, so on the S^2 of $\mathbb{C}$ structures there are two fixed points, the two poles $\pm I$. I is then the 'preferred' $\mathbb{C}$ structure.

Aside: consider stable Higgs bundles then if (V, φ) is stable, so too is $(V, e^{i\vartheta}\varphi)$. Using this aside and the equivalence of stable Higgs bundles to solutions of the self-duality equations seen in Talk 8 we can say that $(\mathcal{M}, I)$ is isomorphic to the space of stable Higgs bundles with complex structure I.
In [1] it was proven that all the complex structures other than $\pm I$ are equivalent, to see this consider that on the stable Higgs bundles side the $U(1)$ action can be extended to a $\mathbb{C}^*$ action. Picking one of these other complex structures, J we can show that $(\mathcal{M}, J)$ is isomorphic to the moduli space of $PSL(2,\mathbb{C})$ flat connections $A + \varphi + \varphi^* + A^*$. Note that more usually the flat connection would just be denoted by $A + \varphi + \varphi^*$ where the difference is down to my using A to denote $A^{(0,1)}$ so that $A + A^*$ here would be the full connection in the notation of [1] and I think some of the earlier talks.

3 Extra topics

There were a couple of topics that I did not have time to mention in my talk which I think are interesting enough to get a mention here.

1. As ω_I is invariant under the $U(1)$ action we can find the moment map associated to this action which will be $\mu_{U(1)}(A, \varphi) = -\frac{1}{2}||\varphi||^2_{L^2}$. This can be used as a Morse function to explore the topology of $\mathcal{M}$.
2. A related feature is that the fixed points of this action include the flat holomorphic vector bundles, when $\varphi = 0$, and the case when the Higgs bundle splits into two parts $(E, \varphi) = \oplus_{i=1}^2 (E_i, \varphi_i)$ with φ nilpotent. In the case of this splitting the self-duality equations seem to reduce to the vortex equations on M which are another set of interesting equations.
3. $\mathcal{M}$ can also be seen to be related to $\mathcal{M}_{Betti}$ when complex structure K is picked.
4. We can interpret where the various complex structures come from and find that I is inherited from the Riemann surface M and K comes from the representations of the fundamental group in $Gl(n, \mathbb{C})$.

References

[1] N. J. Hitchin. The Self-Duality Equations on a Riemann Surface. Proc. London Math. Soc., (3) 55 (1987) 59-126.