Session 1: Games, Automata and Synthesis

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Introduction

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Approach (How to prove)

- Establish invariant
- Use some form of induction
- May rely on external solvers like SAT or SMT

Needs

A clear first input then output relation

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Model checking

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int next_move(state* cstate)
  // ...
  return nm;
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void next_state(state* cstate,
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Works on *infinite* behavior, reasons about continuous systems

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- Some logical specification The elevator will at some point open its door at every floor to which it is called.

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A continuously executing system

Model checking: A closer look

Correctness

The model may not exhibit a single trace violating the specification

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The elevator model

- *Specification*: How the elevator should behave governed by the
- *Controller*: Choosing the next action as a function of
- *State*: the current position of the elevator and the
- *Environment*: behavior, that is which floors are demanded

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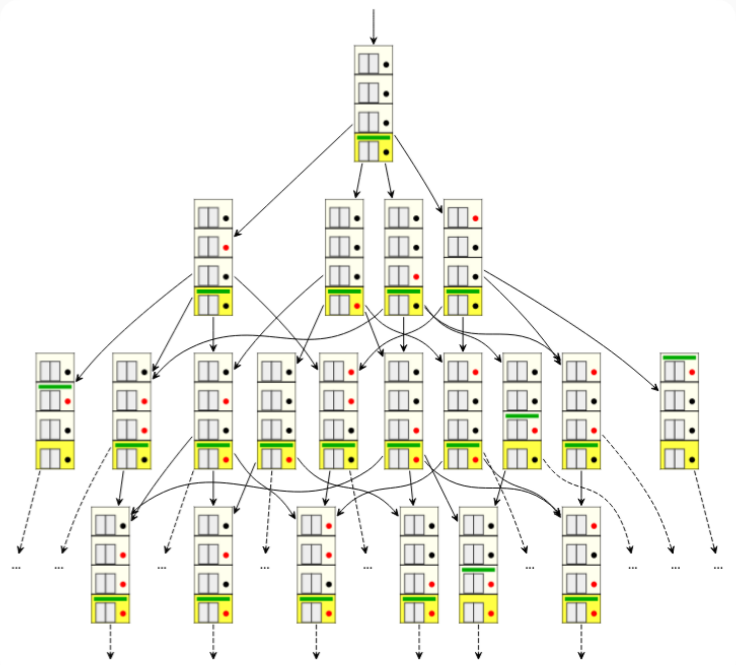
The model is **correct** if the controller can guarantee the specification for **all** valid moves of the environment.

Model checking: A closer look



Introducing: Synthesis

Synthesis is so to speak the flip-side of verification:

Instead of verifying that a model/controller verifies a specification, why not directly generate it such a manner?

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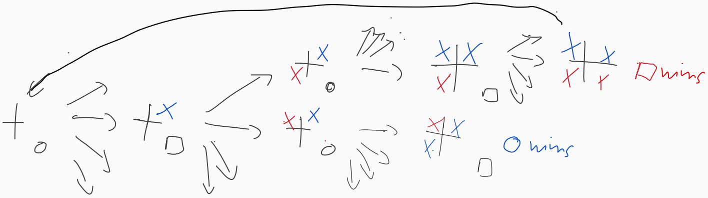
Intuitively, the idea is to make the *choices* explicit and create a two-player game between the environment (*env*, player 0) and the controller (*player*, player 1).

If the controller can always *win* the game, he has an answer or strategy for all possible environment behaviors. This strategy then by construction verifies the specification.

Simplified 2 wins: Env wins if it can occupy a diagonal at any point



Definition 1

We are only concerned with 2 player games with perfect information.

The game is played between player 0 (the *Env*) and player 1 (the *Player*). To ease notations, we also define player i ($i \in [0, 1]$) and his opponent player $i - 1$.

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Arena

An arena $\mathcal{A} = (V, V_0, V_1, E)$ - a finite set of vertices V - the set of vertices V_i owned by player i partitioning V - $E \subseteq V \times V$ the set of directed edges - for every vertex v the set $\{v' | (v, v') \in E\}$ is non-empty

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We say arena is **alternating** if for every edge $(v, v') \in E$ we have $v \in V_i$ and $v' \in V_{i-1}$

Definition 2

Sub-Arena

Let $\mathcal{A}$ be an arena and $V' \subseteq V$ a subset of vertices.
The sub-arena of $\mathcal{A}$ induced by V' called $\mathcal{A}_{V'}$ is defined as
$\mathcal{A}_{V'} = (V', V_0 \cap V', V_1 \cap V', E \cap (V' \times V'))$

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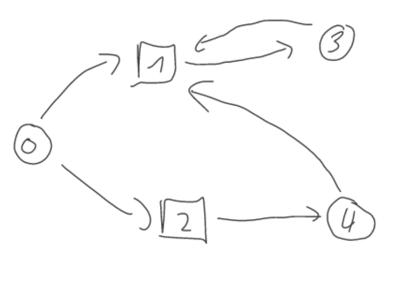

- $V' = \{1, 3\}$ induces a sub-arena
- $V' = \{2, 4\}$ does not

Definition 3

Intuitively, during the play the players push a token along the edges of the arena, whoever own the vertex decides on the next edge to use. During this process, we record the vertices (and sometimes) edges seen.

A Play

A play in $\mathcal{A}$ is an infinite sequence $\rho = \rho_0\rho_1\rho_2\rho_3\cdots \in V^\omega$ such that $\rho_n\rho_{n+1} \in E$ holds for all $n \in \mathbb{N}$.

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A Strategy

A strategy for player i ($i \in \{0, 1\}$) in an arena $\mathcal{A}$ is a function $\delta_i : V^*V_i \to V$ s.t. $\delta_i(wv) = v'$ implies $(v, v') \in E$ for every w and v.

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In other words, a strategy decides what to do next given the history of the game and its choice is possible in $\mathcal{A}$.

Definition 4

A consistent play

A play in $\mathcal{A}$ is **consistent** with a strategy δ_i for player i if $\rho_{n+i} = \delta_i(\rho_0, \ldots, \rho_n)$ for every $n \in \mathbb{N}$.

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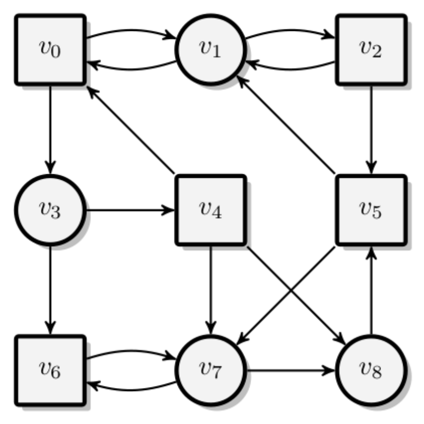
And

Positional strategy

A strategy δ_i is called positional (or memoryless) if $\delta_i(wv) = \delta_i(v)$ for all $w \in V^*$ and $v \in V$.

Since positional strategies are function from V to V (instead of $(V^* \times V)$ to V) we also denote them as such.

Definition 3&4 illustrations



player 0: circles
player 1: squares

- $v_0 v_3 (v_6 v_7)^\omega$ is a play
- *go right* is a positional strategy for player 0
- $v_0 (v_1 v_2 v_1 v_2 v_5 v_1)^\omega$ is a play consistent with the positional strategy *go right* for player 0

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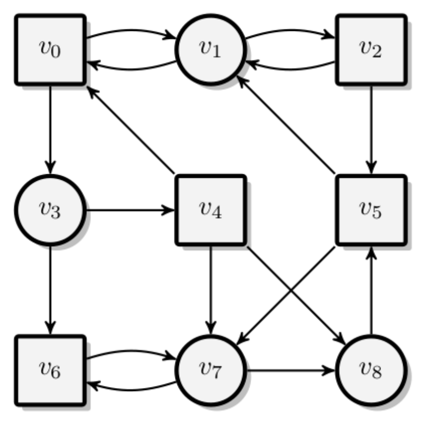

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- $v_0 v_3 (v_6 v_7)^\omega$ is a play
- *go right* is a positional strategy for player 0
- $v_0 (v_1 v_2 v_1 v_2 v_5 v_1)^\omega$ is a play consistent with the positional strategy *go right* for player 0
- Give the positional strategy δ_1 consistent with $v_0 (v_1 v_2 v_1 v_2 v_5 v_1)^\omega$
- Given we start in v_0, is there a strategy for player 1 to reach v_6?

Definition 5 (and final) [at least for now]

We can finally properly define a game

Game

A game $\mathcal{G}$ is defined by the tuple $(\mathcal{A}, Win)$ with $\mathcal{A}$ being the arena and $Win \subseteq V^\omega$ being the winning sequences.
We say the game is won by player 1 if and only if $\rho \in Win$. Otherwise it is won by player 0.

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A strategy δ_i for player i is called a **winning strategy** for vertex v if all plays starting v and consistent with δ_i are won by player i.

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A Winning Region

The winning region $W_i(\mathcal{G})$ of player i is the set of vertices from which player i has a winning strategy.

On winning regions

Lemma: Winning regions do not intersect

or $W_0(\mathcal{G}) \cap W_1(G) = \emptyset$

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Proof. Let $\mathcal{G} = (\mathcal{A}, Win)$. Towards a contradiction, assume there exists a vertex $v \in W_0(\mathcal{G}) \cap W_1(\mathcal{G})$.

Then, both players have a winning strategy from v, call them δ_0 and δ_1. Let $\rho = \rho(v, \delta_0, \delta_1)$, i.e., we let the players both use their winning strategy against each other, the play is consistent with both strategies.

Then, $\rho \in Win$, as δ_1 is a winning strategy for player 1 and $\rho \notin Win$, as δ_0 is a winning strategy for player 0.

Hence, we have derived the desired contradiction.

On winning regions and determinacy

Last slide we have shown that winning regions are always disjoint.

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[Positional] Determinancy

Let $\mathcal{G}$be a game with vertex set V.
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Note the subtility "one of the playes has **a** positional winning stragtegy [for each vertex v]".

This indicates that the strategy may depend on the initial vertex.

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