Thermal fin

Finite Element Approximation

(a) We consider the five PDEs :

$$-k^i \Delta \mathbf{u}^i = 0, i = 0, ..., 4$$

we wish to write the variational formulation. We apply rightly the inner product by any standard test function $\mathbf{v} \in X^e$ then integrate on Ω^i for each $i = 0, ..., 4$ before taking the sum of these five terms :

$$-\sum_{i=0}^{4} k^i \int_{\Omega^i} \Delta \mathbf{u}^i \cdot \mathbf{v} = 0$$

Considering $k^0 = 1$ and integrating by parts, we get :

$$\sum_{i=0}^{4} k^i \left[\int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \int_{\Gamma^i} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} \right] = 0$$

But since $\Gamma^0 = \Gamma_{root} \cup \Gamma^0_{int} \cup \Gamma^0_{ext}$ and $\Gamma^i = \Gamma^i_{int} \cup \Gamma^i_{ext}$ for $i = 0, ..., 4$, we may write :

$$\begin{aligned}
0 = \quad & \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \sum_{i=1}^{4} k^i \left[\int_{\Gamma^i_{int}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} + \int_{\Gamma^i_{ext}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} \right] \\
& -k^0 \left[\int_{\Gamma_{root}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} + \int_{\Gamma^0_{ext}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \right] - k^0 \int_{\Gamma^0_{int}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \\
= \quad & \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \left[\sum_{i=1}^{4} k^i \int_{\Gamma^i_{int}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} + \int_{\Gamma^0_{int}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \right] \\
& - \sum_{i=0}^{4} k^i \int_{\Gamma^i_{ext}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} - \int_{\Gamma_{root}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v}
\end{aligned}$$

We now use the facts :

$$\begin{array}{ll}
\bigcup_{i=1}^{4} \Gamma^i_{int} = \Gamma^0_{int} & \\
\mathbf{n}^i = -\mathbf{n}^0 & \\
-\nabla \mathbf{u}^0 \cdot \mathbf{n}^i = -k^i (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) & \text{on } \Gamma^i_{int}, i = 1, ..., 4 \\
-k^i (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) = \mathrm{Bi}\ \mathbf{u}^i & \text{on } \Gamma_{ext}
\end{array}$$

yielding :

$$0 = \quad \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - [0] + \mathrm{Bi} \sum_{i=0}^{4} \int_{\Gamma^i_{ext}} \mathbf{u}^i \cdot \mathbf{v} - \int_{\Gamma_{root}} \mathbf{v}$$