Thermal fin

I - Finite Element Approximation

(a) We consider the five PDEs :

$$-k^i \Delta \mathbf{u}^i = 0, i = 0, ..., 4$$

we wish to write the variational formulation. We apply rightly the inner product by any standard test function $\mathbf{v} \in X^e$ then integrate on Ω^i for each $i = 0, ..., 4$ before taking the sum of these five terms :

$$-\sum_{i=0}^{4} k^i \int_{\Omega^i} \Delta \mathbf{u}^i \cdot \mathbf{v} = 0$$

Considering $k^0 = 1$ and integrating by parts, we get :

$$\sum_{i=0}^{4} k^i \left[\int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \int_{\Gamma^i} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} \right] = 0$$

But since $\Gamma^0 = \Gamma_{root} \cup \Gamma^0_{int} \cup \Gamma^0_{ext}$ and $\Gamma^i = \Gamma^i_{int} \cup \Gamma^i_{ext}$ for $i = 0, ..., 4$, we may write :

$$\begin{aligned}
0 = \ & \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \sum_{i=1}^{4} k^i \left[\int_{\Gamma^i_{int}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} + \int_{\Gamma^i_{ext}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} \right] \\
& -k^0 \left[\int_{\Gamma_{root}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} + \int_{\Gamma^0_{ext}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \right] - k^0 \int_{\Gamma^0_{int}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \\
= \ & \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - \left[\sum_{i=1}^{4} k^i \int_{\Gamma^i_{int}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} + \int_{\Gamma^0_{int}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v} \right] \\
& -\sum_{i=0}^{4} k^i \int_{\Gamma^i_{ext}} (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) \cdot \mathbf{v} - \int_{\Gamma_{root}} (\nabla \mathbf{u}^0 \cdot \mathbf{n}^0) \cdot \mathbf{v}
\end{aligned}$$

We now use the facts :

$$\begin{aligned}
& \bigcup_{i=1}^{4} \Gamma^i_{int} = \Gamma^0_{int} & \\
& \mathbf{n}^i = -\mathbf{n}^0 & \\
& -\nabla \mathbf{u}^0 \cdot \mathbf{n}^i = -k^i (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) & \text{on } \Gamma^i_{int}, i = 1, ..., 4 \\
& -k^i (\nabla \mathbf{u}^i \cdot \mathbf{n}^i) = \text{Bi } \mathbf{u}^i & \text{on } \Gamma_{ext}
\end{aligned}$$

yielding :

$$0 = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \nabla \mathbf{v} - [0] + \text{Bi} \sum_{i=0}^{4} \int_{\Gamma^i_{ext}} \mathbf{u}^i \cdot \mathbf{v} - \int_{\Gamma_{root}} \mathbf{v}$$

or equivalently :

$$\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{u}^i \cdot \mathbf{v} + \text{Bi} \int_{\partial\Omega \setminus \Gamma_{root}} \mathbf{u}^i \cdot \mathbf{v} = \int_{\Gamma_{root}} \mathbf{v}$$

It is easy to see that it is written in the form $a(\mathbf{u}, \mathbf{v}; \mu) = l(\mathbf{v}; \mu)$ where :

$$a(\mathbf{w}, \mathbf{v}; \mu) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{w}^i \cdot \mathbf{v} + \text{Bi} \int_{\partial\Omega \setminus \Gamma_{root}} \mathbf{w}^i \cdot \mathbf{v} \ \text{ and } \ l(\mathbf{v}; \mu) = \int_{\Gamma_{root}} \mathbf{v}$$

are respectively a bilinear symmetric form and a linear form. We conclude by saying that $\mathbf{u}^e(\mu)$ verifies :

$$a(\mathbf{u}^e(\mu), v; \mu) = l(v; \mu) \ , \ \forall \mathbf{v} \in X^e$$

(b) Consider :

$$J(\mathbf{w}) = \frac{1}{2}a(\mathbf{w}, \mathbf{w}; \mu) - l(\mathbf{w}; \mu)$$

an let us look for its differential. It is the linear part of the following map :

$$\begin{array}{rcl} X^e & \longrightarrow & \mathbb{R} \\ h & \longmapsto & J(\mathbf{w}+h) \end{array}$$

We compute :

$$\begin{array}{rcl} J(\mathbf{w}+h) = & & \frac{1}{2}a(\mathbf{w}+h, \mathbf{w}+h; \mu) - l(\mathbf{w}+h; \mu) \\ & = & \frac{1}{2}\left(a(\mathbf{w}, \mathbf{w}; \mu) + a(h, h; \mu) + a(\mathbf{w}, h; \mu) + a(h, \mathbf{w}; \mu)\right) - l(\mathbf{w}; \mu) - l(h; \mu) \\ \text{since } a \text{ is bilinear} & = & \left[\frac{1}{2}(a(\mathbf{w}, \mathbf{w}; \mu) + a(h, h; \mu)) - l(\mathbf{w}; \mu)\right] + \left[a(\mathbf{w}, h; \mu) - l(h; \mu)\right] \end{array}$$

thus the differential of J at $\mathbf{w}$ is $D_{\mathbf{w}}J(h) = a(\mathbf{w}, h; \mu) - l(h; \mu)$. But according to **(a)**, for all $bv \in X^e$, we have : $a(\mathbf{u}^e(\mu), \mathbf{v}; \mu) = l(\mathbf{v}; \mu)$, or equivalently $J(\mathbf{u}^e(\mu)) \equiv 0$. Since $a(h, h; \mu) = |||h|||_\mu^2 \geqslant 0$ for all $h \in X^e$, $\mathbf{u}^e(\mu)$ is a local minimum for J. Since $\mathbf{u}^e(\mu)$ is the only argument vanishing $D.J$, it is a global minimum for J.

II - Reduced-basis approximation

(a) By definition of $\mathbf{u}_N \in W_N$, we have for all $\mathbf{w}_N \in W_N$:

$$J(\mathbf{u}_N) = \frac{1}{2}a(\mathbf{u}_N, \mathbf{u}_N; \mu) - l(\mathbf{u}_N; \mu) \leqslant \frac{1}{2}a(\mathbf{w}_N, \mathbf{w}_N; \mu) - l(\mathbf{w}_N; \mu) = J(\mathbf{w}_N)$$

But :
— $l(\mathbf{u}_N; \mu) = a(\mathbf{u}(\mu), \mathbf{u}_N; \mu)$
— $l(\mathbf{w}_N; \mu) = a(\mathbf{u}(\mu), \mathbf{w}_N; \mu)$
thus :

$$\begin{array}{rcl} \frac{1}{2}a(\mathbf{u}_N, \mathbf{u}_N; \mu) - a(\mathbf{u}(\mu), \mathbf{u}_N; \mu) & \leqslant & \frac{1}{2}a(\mathbf{w}_N, \mathbf{w}_N; \mu) - a(\mathbf{u}(\mu), \mathbf{w}_N; \mu) \\ a(\mathbf{u}(\mu) - \mathbf{u}_N, \mathbf{u}(\mu) - \mathbf{u}_N; \mu) & \leqslant & a(\mathbf{u}(\mu) - \mathbf{w}_N, \mathbf{u}(\mu) - \mathbf{w}_N; \mu) \\ |||\mathbf{u}(\mu) - \mathbf{u}_N|||_\mu & \leqslant & |||\mathbf{u}(\mu) - \mathbf{w}_N|||_\mu \end{array}$$

(b) We have respectively :

$$\begin{cases} T_{rootN}(\mu) = l(\mathbf{u}_N(\mu); \mu) \\ T_{root}(\mu) = l(\mathbf{u}(\mu); \mu) \end{cases}$$

thus :

$$\begin{array}{rcl} T_{root}(\mu) - T_{rootN}(\mu) = & & l(\mathbf{u}(\mu); \mu) - l(\mathbf{u}_N(\mu); \mu) \\ & = & l(\mathbf{u}\mu; \mu) - l(\mathbf{u}_N(\mu); \mu) + l(\mathbf{u}_N(\mu); \mu) - l(\mathbf{u}_N(\mu); \mu) \\ & = & a(\mathbf{u}(\mu), \mathbf{u}(\mu); \mu) - a(\mathbf{u}(\mu), \mathbf{u}_N(\mu); \mu) + a(\mathbf{u}_N(\mu), \mathbf{u}_N(\mu); \mu) - a(\mathbf{u}_N(\mu), \mathbf{u}(\mu); \mu) \\ & = & a(\mathbf{u}(\mu), \mathbf{u}(\mu); \mu) - 2a(\mathbf{u}(\mu), \mathbf{u}_N(\mu); \mu) + a(\mathbf{u}_N(\mu), \mathbf{u}_N(\mu); \mu) \\ & = & a(\mathbf{u}(\mu) - \mathbf{u}_N(\mu), \mathbf{u}(\mu) - \mathbf{u}_N(\mu); \mu) \\ & = & |||\mathbf{u}(\mu) - \mathbf{u}_N(\mu)|||_\mu^2 \end{array}$$

(c) We know that for all $\mathbf{w}_N \in W_N$, the following holds :

$$a(\mathbf{u}_N(\mu), \mathbf{w}_N; \mu) = l(\mathbf{w}_N; \mu)$$

Moreover, $\mathbf{u}_N(\mu) = \sum_{i=1}^{N} \underline{\mathbf{u}}_N^i a \xi_i$ so we can write for all $j = 1, ..., N$:

$$a(\mathbf{u}_N(\mu), \xi_j; \mu) = \sum_{i=1}^{N} \underline{\mathbf{u}}_N^i a(\xi_i, \xi_j; \mu) = l(\xi_j; \mu)$$

We build the matrices $\underline{A}_N(\mu), \underline{F}_N(\mu)$ and $\underline{L}_N$ whose coefficients are defined by :

$$\underline{A}_N(\mu)_{ij} = a(\xi_i, \xi_j; \mu) \text{ and } \underline{F}_N(\mu)_j = l(\xi_j; \mu) \text{ and } \underline{L}_N^i = \int_{\Gamma_{root}} \xi_i$$

such that :

$$\underline{A}_N(\mu) \cdot \underline{\mathbf{u}}_N(\mu) = \underline{F}_N(\mu)$$

$$T_{rootN}(\mu) = \int_{\Gamma_{root}} \mathbf{u}_N(\mu) = \sum_{i=1}^{N} \underline{\mathbf{u}}_N(\mu) \int_{\Gamma_{root}} \xi_i = \underline{L}_N^T \cdot \underline{\mathbf{u}}_N$$

Packing the shapshots' components into an $\mathcal{N} \times N$ matrix Z yields :

$$\underline{A}_N(\mu) = Z^T \cdot \underline{A}_{\mathcal{N}}(\mu) \cdot Z \text{ and } \underline{F}_N(\mu) = Z^T \cdot \underline{F}_{\mathcal{N}}(\mu) \text{ and } \underline{L}_N = Z^T \cdot \underline{L}_{\mathcal{N}}$$

(d) We know that :

$$a(\mathbf{w}, \mathbf{v}; \mu) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla \mathbf{w} \cdot \nabla \mathbf{v} + \text{Bi} \int_{\partial\Omega \backslash \Gamma_{root}} \mathbf{w}\mathbf{v} = \sum_{i=1}^{5} \mu_i \int_{\Omega^{i-1}} \nabla \mathbf{w} \cdot \nabla \mathbf{v} + \mu_6 \int_{\partial\Omega \backslash \Gamma_{root}} \mathbf{w}\mathbf{v}$$

Let :

$$\begin{array}{ll} \theta^q(\mu) = \mu_q, & a^q(\mathbf{w}, \mathbf{v}) = \int_{\Omega^{q-1}} \nabla \mathbf{w} \cdot \nabla \mathbf{v}, \quad q = 1, ..., 5 \\ \theta^6(\mu) = \mu_6, & a^6(\mathbf{w}, \mathbf{v}) = \int_{\partial\Omega \backslash \Gamma_{root}} \mathbf{w}\mathbf{v} \end{array}$$

in such a way we can rewrite :

$$a(\mathbf{w}, \mathbf{v}; \mu) = \sum_{q=1}^{6} \theta^q(\mu) a^q(\mathbf{w}, \mathbf{v}) \ , \ \forall \mathbf{w}, \mathbf{v} \in X, \ \forall \mu \in \mathcal{D}.$$

We now build the matrices $\underline{A}^{\mathcal{N}q} \in \mathcal{M}_{\mathcal{N} \times \mathcal{N}}(\mathbb{R})$ and $\underline{A}_N^q \in \mathcal{M}_{N \times N}(\mathbb{R})$ whose coefficients are :

$$\underline{A}_{ij}^{\mathcal{N}q} = a^q(\varphi_i, \varphi_j) \text{ and } \underline{A}_{Nij}^q = a^q(\xi_i, \xi_j) = Z^T \cdot \underline{A}^{\mathcal{N}} \cdot Z = \sum_{q=1}^{6} \theta^q(\mu) Z^T \cdot \underline{A}^{\mathcal{N}q} \cdot Z$$

where φ_i denotes the ith FEM basis function. This yields the following two decompositions :

$$\underline{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{6} \theta^q(\mu) \underline{A}^{\mathcal{N}q} \text{ and } \underline{A}_N(\mu) = \sum_{q=1}^{6} \theta^q(\mu) \underline{A}_N^q.$$

(e) Let $\bar{\mu} \in \mathcal{D}$ be a fixed parameter. We endow X^e with the subsequent inner product : $(\mathbf{u}, \mathbf{v})_{X^e} = a(\mathbf{u}, \mathbf{v}; \bar{\mu})$ and the associated norm $\| \cdot \|_{X^e} : \mathbf{v} \in X^e \mapsto \sqrt{(\mathbf{v}, \mathbf{v})_{X^e}}$. We now suppose the basis $(\xi_i)_{i=1,...,N}$ is orthonormalized with respect to this inner product. Note that it is not necessarily orthonormal with respect to another inner product $a(\cdot, \cdot; \mu), \ \mu \neq \bar{\mu}$.

The condition number of a matrix is the quotient of its largest eigenvalue by its least one. Assume we have $\underline{A}_N(\mu)$'s eigenvalues $\lambda_1 \geqslant ... \geqslant \lambda_N$ sorted in decreasing order. Thus we have :

$$\text{cond}(\underline{A}_N(\mu)) = \frac{\lambda_1}{\lambda_N}$$

Since $W_N \simeq \mathbb{R}^N$, the least and the largest eigenvalues verify :

$$\lambda_N(\mu) = \inf_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\underline{\mathbf{v}}_N^T \cdot \underline{\mathbf{v}}_N} \quad \text{and} \quad \lambda_1(\mu) = \sup_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\underline{\mathbf{v}}_N^T \cdot \underline{\mathbf{v}}_N}$$

On the other hand, the continuity and the coercivity constants are defined by similar quotients, namely :

$$\gamma(\mu) = \sup_{\mathbf{v},\mathbf{w} \in X^e} \frac{a(\mathbf{v},\mathbf{w};\mu)}{\|\mathbf{v}\|_{X^e}\|\mathbf{w}\|_{X^e}} \quad \text{and} \quad \alpha(\mu) = \inf_{\mathbf{v} \in X^e} \frac{a(\mathbf{v},\mathbf{v};\mu)}{\|\mathbf{v}\|_{X^e}^2}$$

Let us compute :

$$\begin{aligned}
\alpha(\mu) = & \inf_{\mathbf{v} \in X^e} \frac{a(\mathbf{v},\mathbf{v};\mu)}{\|\mathbf{v}\|_{X^e}} \\
\leqslant & \inf_{\mathbf{v}_N \in W_N} \frac{a(\mathbf{v}_N,\mathbf{v}_N;\mu)}{\|\mathbf{v}_N\|_{X^e}^2} && \text{since } W_N \subset X^e \\
= & \inf_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\sum_{i,j=1}^N \underline{\mathbf{v}}_N^i \underline{\mathbf{v}}_N^j (\xi^i, \xi_j)} \\
= & \inf_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\underline{\mathbf{v}}_N^T \cdot \underline{\mathbf{v}}_N} && \text{since } (\xi_i)_{i=1,\dots,N} \text{ is orthonormal} \\
= & \lambda_N(mu) \\
\gamma(\mu) = & \sup_{\mathbf{v},\mathbf{w} \in X^e} \frac{a(\mathbf{v},\mathbf{w};\mu)}{\|\mathbf{v}\|_{X^e}\|\mathbf{w}\|_{X^e}} \\
\geqslant & \sup_{\mathbf{v}_N,\mathbf{w}_N \in W_N} \frac{a(\mathbf{v}_N,\mathbf{w}_N;\mu)}{\|\mathbf{v}_N\|_{X^e}\|\mathbf{w}_N\|_{X^e}} && \text{since } W_N \subset X^e \\
= & \sup_{\mathbf{v}_N,\mathbf{w}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{w}}_N}{\sqrt{\sum_{i,j=1}^N \underline{\mathbf{v}}_N^i \underline{\mathbf{v}}_N^j (\xi^i, \xi_j)}\sqrt{\sum_{i,j=1}^N \underline{\mathbf{w}}_N^i \underline{\mathbf{w}}_N^j (\xi^i, \xi_j)}} \\
\geqslant & \sup_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\sum_{i,j=1}^N \underline{\mathbf{v}}_N^i \underline{\mathbf{v}}_N^j (\xi^i, \xi_j)} \\
= & \sup_{\mathbf{v}_N \in W_N} \frac{\underline{\mathbf{v}}_N^T \cdot \underline{A}_N(\mu) \cdot \underline{\mathbf{v}}_N}{\underline{\mathbf{v}}_N^T \cdot \underline{\mathbf{v}}_N} && \text{since } (\xi_i)_{i=1,\dots,N} \text{ is orthonormal} \\
= & \lambda_1(\mu)
\end{aligned}$$

Since $\lambda_1(\mu) \leqslant \gamma(\mu)$ and $\alpha(\mu) \leqslant \lambda_N(\mu)$, the following result is proved :

$$\operatorname{cond}(\underline{A}_N(\mu)) = \frac{\lambda_1(\mu)}{\lambda_N(\mu)} \leqslant \frac{\gamma(\mu)}{\alpha(\mu)}.$$