Duality based verification techniques for Pose Graph Optimization

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***Abstract*—This paper reviews duality-based verification techniques for Pose Graph Optimization (PGO), focusing on the SE-Sync algorithm with Semidefinite Programming (SDP) relaxations. We propose to develop a custom SE-Sync implementation and detail the transformation of PGO into a relaxed SDP problem. We aim to enhance comprehensibility by reformulating key steps from the SE-Sync paper. Additionally, we incorporate techniques from Cartan-Sync to solve the SDP problem and investigate the impact of redundant constraints on certificate tightness under increasing noise levels.**

I. Introduction

We consider the problem of Pose Graph Optimization (PGO), which consists of estimating a set of poses, rotations, and translations from pairwise relative pose measurements. This problem is typically formulated as a maximum a posteriori (MAP) or maximum-likelihood estimation (MLE) under an assumed probability distribution for the measurement noise, which results in a nonlinear least squares problem. Standard approaches to solving PGO employ iterative methods such as Gauss-Newton or Levenberg-Marquardt, which achieve locally optimal solutions. However, these methods do not guarantee the quality of the solution, as convergence to different local optima can vary depending on the initial conditions. Ideally, we seek to understand how close we are to the global optimum and whether further solution refinement is necessary, especially in safety-critical applications.

In the past decade, duality-based verification techniques for Simultaneous Localization and Mapping (SLAM) have gained significant attention. A notable advancement in this field is SE-Sync [1]: a certifiably correct algorithm for synchronization over the Special Euclidean Group. SE-Sync leverages Lagrangian duality [2] and Semidefinite Programming (SDP) [3] relaxations to provide optimality bounds. This paper reviews the prior work that led to the development of SE-Sync, analyzes the SE-Sync algorithm itself, and discusses some of the remaining challenges in 2024 related to duality-based verification techniques for PGO.

II. Literature Review

In [4], Carlone and Dellaert present a technique to verify if a given 2D SLAM solution is globally optimal. They establish lower and upper bounds for the optimal solution, enabling the assessment of the quality of any solution returned by an iterative solver. By re-parameterizing the rotation matrices $R_i \in \mathrm{SO}(2)$, they reformulate the initial nonlinear least-squares minimization problem into a Quadratically Constrained Quadratic Programming (QCQP) minimization problem. They reduce the dimensionality of the problem by eliminating the position vectors using the Schur complement. Subsequently, they compute the dual problem of the QCQP, providing a method to verify global optimality.

Lagrangian duality theory provides two key insights: the dual problem is always convex, regardless of the primal problem's nature, and the optimal value of the dual problem offers a lower bound on the primal's optimal value. In cases where strong duality holds, this bound is tight. The dual problem is an SDP (Semidefinite Programming) problem, solvable by off-the-shell convex solvers. Any feasible solution to the primal problem provides an upper bound for the minimization problem, ensuring that the solution quality can be effectively assessed. Experimental results indicate that, in all tested cases, the bound provided by the dual problem was tight, suggesting that strong duality may hold, even if not formally proven.

However, a significant challenge remains. Pose Graph Optimization (PGO) is a large-scale problem requiring the estimation of thousands of poses, making the dual SDP problem difficult for current off-the-shelf solvers to handle efficiently. With most real datasets, solving the SDP problem required approximately one hour. Despite the appealing quality of the certificate gap, the framework is impractical for real-world application due to the scalability issues of off-the-shelf SDP solvers.

In a follow-up paper [5], the same authors generalize their approach to 3D SLAM. In this work, while re-parameterizing the matrices in $\mathrm{SO}(3)$, they drop the constraint $\det(R) = 1$, leading to a QCQP minimization problem where the equality constraints correspond to matrices being part of $\mathrm{O}(3)$ instead of $\mathrm{SO}(3)$. Consequently, the resulting bounds are looser. However, given a candidate solution we can still check if a solution is optimal and still provides a certificate gap, albeit a looser one. Additionally, they introduce a technique to verify whether a candidate solution is optimal without solving a large-scale SDP problem. This method relies on solving a linear system and checking the positive semi-definite nature of a matrix. This test is fast, but the result is binary, qualifying a solution as optimal or not, without providing bounds. Experimental results demonstrate that the bounds are tight when the noise level, modeled as a Gaussian isotropic noise $\mathcal{N}\left(0,\ \sigma_R^2 I\right)$, is below a certain threshold $\sigma_R = 0.1$. However, the bounding gap increases as σ_R increases. The certificate quality decreases with increasing noise levels.

In [6], Briales and Gonzalez-Jimenez improve upon the results from [5] by providing a novel formulation of the QCQP problem that results in smaller matrices. The previous work systematically uses the Kronecker product to transform matrix products into vector products, resulting in each rotation matrix producing a 9×9 block. In contrast, this paper employs a trace-based reformulation, keeping the rotation matrices as 3×3 blocks. Additionally, some diagonal blocks corresponding to constant terms are dropped, leading to sparser matrices. This reformulation preserves the quality of the results while reducing computation time by a factor of 50. However, solving the SDP problem with more than 1000 poses remains prohibitively slow, requiring more than 15 minutes.

In [7], Briales and Gonzalez-Jimenez, and in [8], Brynte et al. consider the applicability of these techniques to different problems, including the SLAM front-end. They use these techniques for registration problems (e.g., point cloud registration) with point-to-point, point-to-line, and point-to-plane correspondences. They also apply them to hand-eye calibration and rotation averaging, to name a few. The fewer poses to estimate, the smaller the SDP problem, and the more applicable these techniques are.

In [7], the authors maintain the constraint $\det(R)=1$ when deriving the QCQP formulation. They replace the default cubic determinant constraint with quadratic constraints in the form of three cross products $R^{(i)}\times R^{(j)}=R^{(k)}$ for $i,j,k=\text{cyclic}\,(1,2,3)$. This ensures that each column of a rotation matrix adheres to the right-hand rule, resulting in a rotation matrix instead of a reflection matrix. Consequently, we have 9 additional constraints per rotation matrix. Adding these constraints remarkably improves the quality of the duality gap. In [7], the authors achieve tight experimental results regarding the duality gap. This approach particularly applies to smaller-scale SDP problems, such as those related to SLAM front-end processing.

In [9], Rosen and Carlone design a custom SDP solver to enable fast, real-time computations on large-scale PGO problem sets. Solving the SDP problem requires finding an SDP matrix $Z=VV^T$ of dimension $dn\times dn$, where $d=3$ and n corresponds to a few thousand poses. However, in practice, the solution Z is of rank r, not much greater than d. Thus, the main idea is to search for $Z=V_{nd\times r}V^T_{r\times nd}$. This approach dramatically reduces the search space size and renders the positive semidefiniteness constraint redundant since $VV^T\succeq0$ for any choice of V. Consequently, the rank-restricted form of the problem becomes a low-dimensional nonlinear program instead of a semidefinite program. In [10], Burer and Monteiro originally proposed a method to solve this problem based on an augmented Lagrangian procedure. However, Rosen, Carlone et al. in [1], [9] adopt Riemannian optimization. The problem we have to solve is of the form

$$\begin{aligned}\min_{V\in\mathbb{R}^{dn\times r}} \quad & \operatorname{Tr}\left(CVV^T\right)\\ s.t. \quad & \operatorname{Tr}\left(A_iVV^T\right)=b_i \quad i=1,\ldots,m\end{aligned}$$

The set

$$\mathcal{M}=\left\{V\in\mathbb{R}^{dn\times r}\mid \operatorname{Tr}\left(A_iVV^T\right)=b_i, i=1,\ldots,m\right\}$$

is a smooth Riemannian manifold under certain conditions on A_i, as explained by Majumdar et al. in [11]. The objective function $V\mapsto\operatorname{Tr}\left(CVV^T\right)$ is smooth, making this problem a Riemannian optimization problem. If $m<\frac{r(r+1)}{2}$, any second-order critical point is globally optimal and Riemannian trust-region methods can return such a point. These principles underpin the real-time SDP solver presented in [9]. In their experiments, the SE-Sync solution [1], regularly enhanced by their research findings, outperformed GTSAM regarding speed while providing an optimality certificate. This research culminated in the seminal paper on SE-Sync [1], which has significantly influenced the field. We have reviewed the origin and evolution of the key concepts described in [1].

In [12], Holmes, Dümbgen and Barfoot emphasize that certifiable methods like SE-Sync rely on simplifying assumptions to facilitate problem formulation, notably assuming an isotropic measurement noise distribution. They address a localization problem, specifically estimating a sequence of poses based on measurements from known landmarks using stereo-camera data. The conversion of stereo pixels to Euclidean coordinates results in a noise distribution that should not be modeled isotropically. Consequently, the resulting maximum-likelihood optimization incorporates matrix rather than scalar weighting factors. Their experiments reveal that semidefinite relaxations were tight only at lower noise levels when matrix weighting factors were used instead of scalar ones.

III. Proposed Work

We will develop a custom SE-Sync implementation in Python, created from scratch, detailing the steps involved in transforming the original Pose Graph Optimization (PGO) problem into the final relaxed SDP problem. Our primary objective is to revisit Appendix B of the SE-Sync paper [1], reformulating the estimation problem from problem 1 to problem 7 in a didactic manner. This reformulation aims to enhance comprehensibility for a broader audience. To solve the SDP problem, we plan to use pymanopt, a Python toolbox for optimization on Riemannian manifolds that supports the truncated-Newton Riemannian trust-region (RTR) algorithm. Additionally, we may adopt strategies from Cartan-Sync [13], where Briales and Gonzalez-Jimenez demonstrate comparable performance speeds to SE-Sync but relies less on problem-specific customizations. The customization in Cartan-Sync is limited to a custom preconditioner. For comparative benchmark results with various SDP solvers (MOSEK, SDPLR, SDPNAL+, STRIDE, ManiSDP), we refer the reader to to Wang et al. [14].

Additionally, understanding the limitations of SDP relaxations under varying noise levels is crucial. It is known from [12] that the tightness of SDP relaxations may be lost when increasing the level of noise. However, as highlighted in [15] and [12] by the same authors, in range-only localization and stereo camera localization problems, the use of redundant constraints enabled to regain tightness. Though redundant in the original problem formulation, these constraints may become essential in the semidefinite relaxation formulation. We will investigate the impact of redundant constraints in PGO, their effectiveness in the presence of noise, and the implications on computation time.

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