Part2 | Vector-Valued Functions

We are familiar with **real valued functions** whose values is a real number. This module introduces **vector-valued functions** whose values are vectors. We will use the calculus of vector-valued functions to describe the motion of objects in a plane or in space.

1 Vector-valued functions

A vector-valued function (or simply, **vector function**) consists of two parts: a **domain** I, which is a subset of $\mathbb{R}$, and a **rule**, which assigns to each number t in I and a unique vector in $\mathbb{R}^n$. The range of a vector-valued function is a set of vectors.

The rule is usually given as formula of the function which can be expressed as

$$\vec{\mathbf{r}}(t)=\langle x_1(t), x_2(t),\ldots, x_n(t)\rangle$$

where $x_1(t), x_2(t),\ldots, x_n(t)$ are real-valued function called the **component functions** of $\vec{\mathbf{r}}(t)$.

For $n=2$ (2-space) : A vector-valued function

$\vec{\mathbf{r}}(t)=\langle x(t), y(t)\rangle$ is sometimes written as $\vec{\mathbf{r}}(t)=x(t)\vec{\mathbf{i}}+y(t)\vec{\mathbf{j}}$

For $n=3$ (3-space) : A vector-valued function

$\vec{\mathbf{r}}(t)=\langle x(t), y(t), z(t)\rangle$ is sometimes written as $\vec{\mathbf{r}}(t)=x(t)\vec{\mathbf{i}}+y(t)\vec{\mathbf{j}}+z(t)\vec{\mathbf{k}}$

Example 1.1

a) The component functions of $\vec{\mathbf{r}}(t)=\sqrt{t}\,\vec{\mathbf{i}}+t\,\vec{\mathbf{j}}$ are

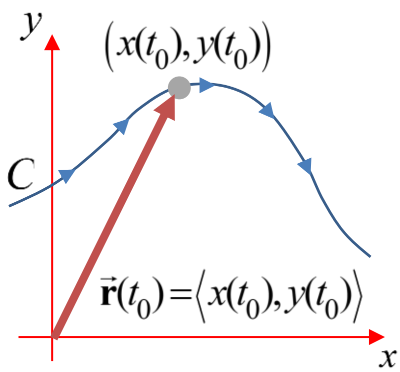
$$x(t)=\sqrt{t}\,,\quad y(t)=t$$

b) The component functions of $\vec{\mathbf{r}}(t)=(1+2t)\vec{\mathbf{i}}+3t\,\vec{\mathbf{j}}+\frac{1}{t}\vec{\mathbf{k}}$ are

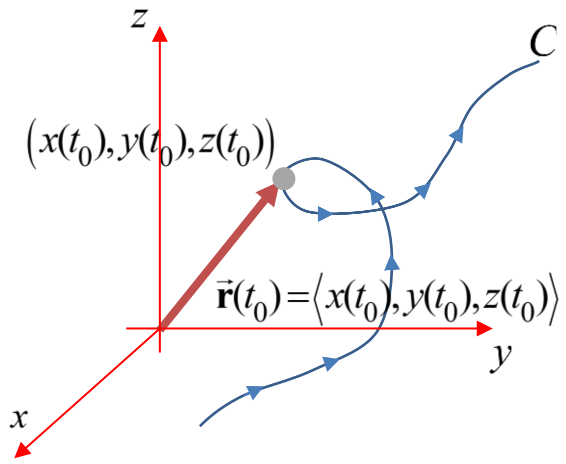
$$x(t)=1+2t\,,\quad y(t)=3t\,,\quad z(t)=\frac{1}{t}$$

Graphs of Vector-valued functions

- For a given $t_0 \in I$, the value of function $\vec{\mathbf{r}}(t_0)$ is called the **position vector** at t_0. As t varies, a **curve** C being traced out by the arrowhead of the moving vector $\vec{\mathbf{r}}(t)$.



2-space



3-space

The arrowhead on the curve C indicates the curve's **orientation** by pointing in the direction of increasing values of t.

- The graph of a vector-valued function $\vec{\mathbf{r}}(t) = x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}}$ consists of the set of all points $(x(t), y(t))$ and curve C is called a **plane curve**.
- The graph of a vector-valued function $\vec{\mathbf{r}}(t) = x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}} + z(t)\vec{\mathbf{k}}$ consists of the set of all points $(x(t), y(t), z(t))$ and curve C is called a **space curve**.

Note Vector-valued functions are closely related to parametric equations of graphs. We plot points $(x(t), y(t))$ or $(x(t), y(t), z(t))$ to produce a graph, which each point represents a vector in the context of vector functions.

Parametric equations for a curve are equations of the form

$$x = x(t), \ \ y = y(t), \text{ and } \ z = z(t)$$

that describe the (x, y, z) coordinates of a point on a curve in $\mathbb{R}^3$

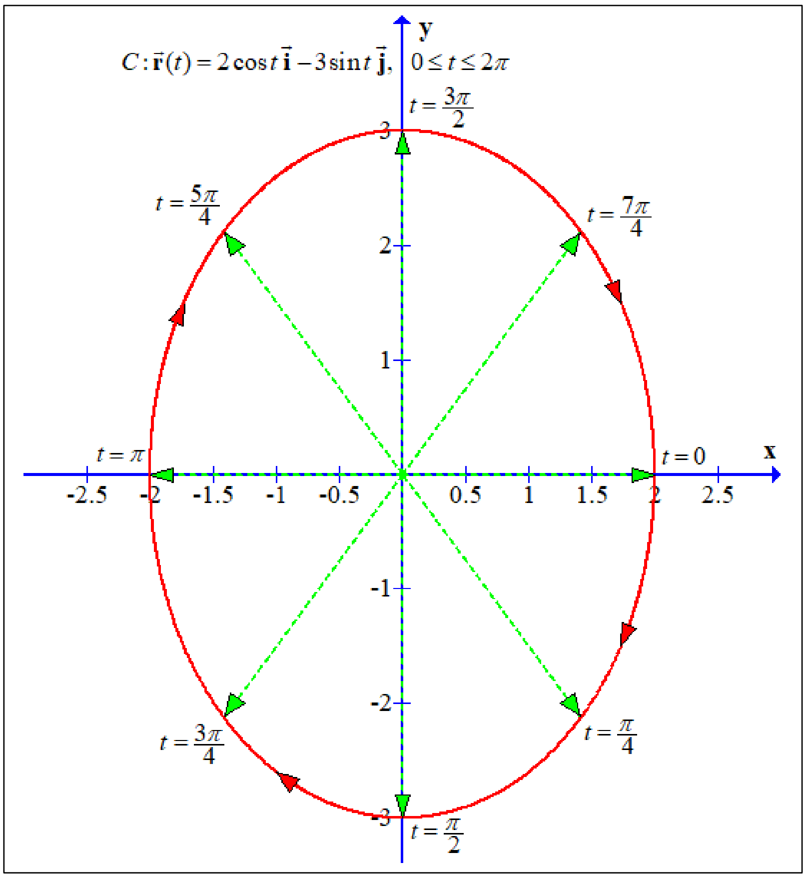
Example 1.2 (video) Sketch the curve traced by the following vector-valued function

a) $\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} - 3\sin t\,\vec{\mathbf{j}},\ 0 \le t \le 2\pi$

<u>Solution</u>

Method I : We start with a table of values t and $\vec{\mathbf{r}}(t)$, then graph each of vectors $\vec{\mathbf{r}}(t)$ (green dotted arrow) and connect the terminal points of each vector to form a curve (red curve).

t	$\vec{\mathbf{r}}(t)$
0	$2\vec{\mathbf{i}}$
$\frac{\pi}{4}$	$\sqrt{2}\,\vec{\mathbf{i}} - \frac{3\sqrt{2}}{2}\vec{\mathbf{j}}$
$\frac{\pi}{2}$	$-3\vec{\mathbf{j}}$
$\frac{3\pi}{4}$	$-\sqrt{2}\,\vec{\mathbf{i}} - \frac{3\sqrt{2}}{2}\vec{\mathbf{j}}$
π	$-2\vec{\mathbf{i}}$
$\frac{5\pi}{4}$	$-\sqrt{2}\,\vec{\mathbf{i}} + \frac{3\sqrt{2}}{2}\vec{\mathbf{j}}$
$\frac{3\pi}{2}$	$3\vec{\mathbf{j}}$
$\frac{7\pi}{4}$	$\sqrt{2}\,\vec{\mathbf{i}} + \frac{3\sqrt{2}}{2}\vec{\mathbf{j}}$
2π	$2\vec{\mathbf{i}}$



The graph of $\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} - 3\sin t\,\vec{\mathbf{j}}$, $0 \le t \le 2\pi$ is the ellipse. The curve has a clockwise orientation, i.e., as t increases from 0 to 2π, the position vector $\vec{\mathbf{r}}(t)$ moves clockwise and its terminal point traces the ellipse.

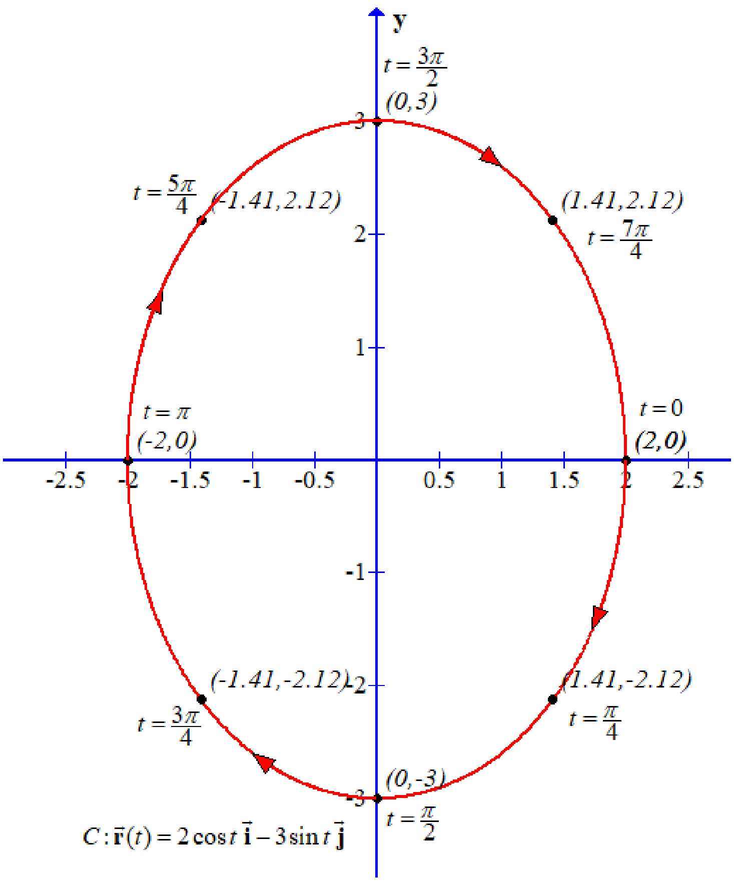
Method II : From the position vector $\vec{\mathbf{r}}(t)$, the corresponding parametric equations are

$$x(t) = 2\cos t,\quad y(t) = -3\sin t \qquad \text{(component functions of } \vec{\mathbf{r}}(t)\text{)}$$

Solving for $\cos t$ and $\sin t$ using the identity $\cos^2 t + \sin^2 t = 1$ produces the ellipse equation $\dfrac{x^2}{2^2} + \dfrac{y^2}{3^2} = 1$.

We make a table of x and y values with enough t values. Plotting these points and sketch the full graph.

t	$x(t)$	$y(t)$
0	2	0
$\frac{\pi}{4}$	$\sqrt{2}$	$\frac{-3\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	-3
$\frac{3\pi}{4}$	$-\sqrt{2}$	$\frac{-3\sqrt{2}}{2}$
π	-2	0
$\frac{5\pi}{4}$	$-\sqrt{2}$	$\frac{3\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	3
$\frac{7\pi}{4}$	$\sqrt{2}$	$\frac{3\sqrt{2}}{2}$
2π	2	0



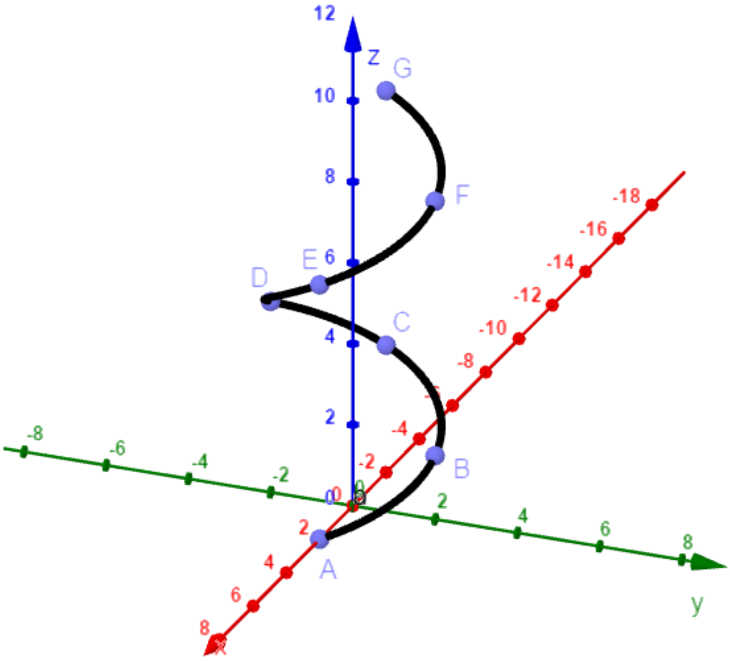
b) $\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}} + t\vec{\mathbf{k}}$, $t \geq 0$.

<u>Solution</u> The parametric equations for this curve are

$$x = 2\cos t, \quad y = 2\sin t, \quad z = t$$

We make a table of x, y and z values with enough t values. Plotting these points and sketch the full graph.

t	$x(t)$	$y(t)$	$z(t)$	point
0	2	0	0	A
$\frac{\pi}{2}$	0	2	$\frac{\pi}{2}$	B
π	-2	0	π	C
$\frac{3\pi}{2}$	0	-2	$\frac{3\pi}{2}$	D
2π	2	0	2π	E
$\frac{5\pi}{2}$	0	2	$\frac{5\pi}{2}$	F
3π	-2	0	3π	G
$\frac{7\pi}{2}$	0	-2	$\frac{7\pi}{2}$	
4π	0	2	4π	



As the parameter t increases, the point (x, y, z) moves upward in a spiral shape wraps around the circular cylinder $x^2 + y^2 = 4$. This curve is called a **circular helix**.

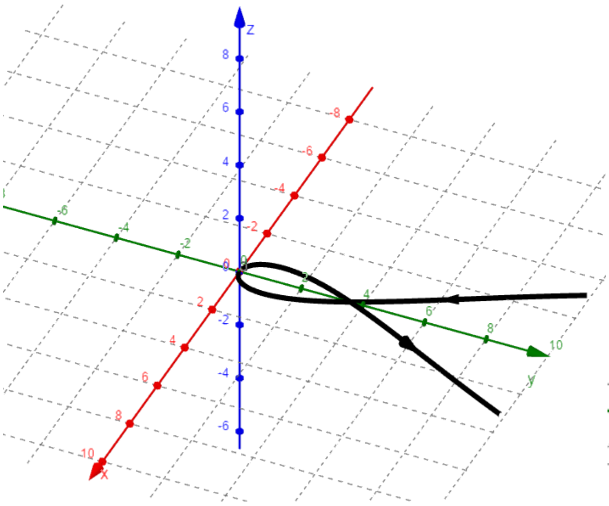
Exercise 1.1 Match the vector functions with the graphs (labeled I-IV).

a) $\vec{\mathbf{r}}(t) = (t^2 - 1)\vec{\mathbf{i}} + (2t - 3)\vec{\mathbf{j}}$, $-\pi \leq t \leq \pi$

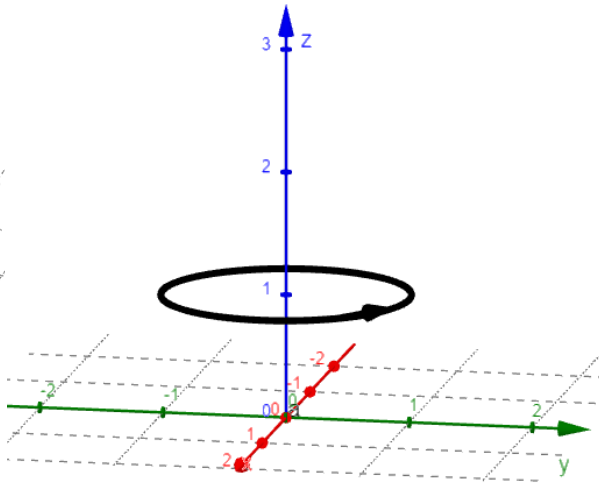
b) $\vec{\mathbf{r}}(t) = (t - 2\sin t)\vec{\mathbf{i}} + t^2\vec{\mathbf{j}}$, $-\pi \leq t \leq \pi$

c) $\vec{\mathbf{r}}(t) = \cos 2t\,\vec{\mathbf{i}} + \sin 2t\,\vec{\mathbf{j}} + \vec{\mathbf{k}}$, $0 \leq t \leq 2\pi$

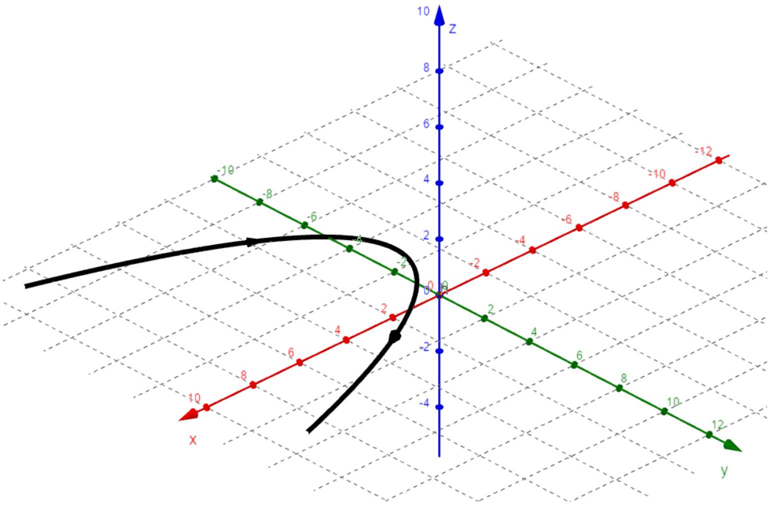
d) $\vec{\mathbf{r}}(t) = \cos t\,\vec{\mathbf{i}} - \cos t\,\vec{\mathbf{j}} + \sin t\,\vec{\mathbf{k}}$, $0 \leq t \leq 2\pi$



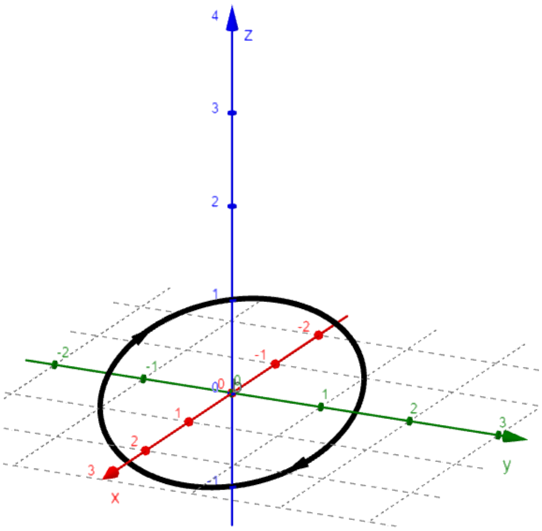
I



II



III



IV

Answer a) III b) I c) II d) IV

2 Derivative of vector-valued functions

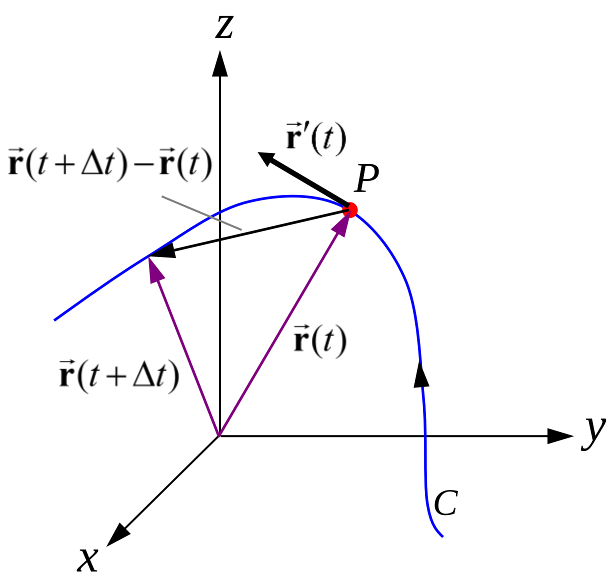
The derivative of a vector-valued function is defined by a limit similar to that for the derivative of a real-valued function.

Definition Let $\vec{\mathbf{r}}(t)$ be a vector-valued function. The derivative of $\vec{\mathbf{r}}$ is the vector-valued function $\vec{\mathbf{r}}'(t)$ defined by

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

for all t such that the limit exists.

When the limit exists for $t = a$, we say that $\vec{\mathbf{r}}(t)$ is **differentiable** at $t = a$.



Other notations for the derivative of $\vec{\mathbf{r}}(t)$ are

$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t)\right], \quad \frac{d\vec{\mathbf{r}}}{dt}, \quad \vec{\mathbf{r}}'(t) \quad \text{or} \quad \vec{\mathbf{r}}'$$

When the component functions are differentiable, $\vec{\mathbf{r}}'(t)$ is obtained by simply differentiating its component functions.

Theorem If $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ where f, g and h are differentiable, then

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Higher order derivatives of a vector-valued function are also obtained by successive differentiating its components.

Example 2.1 (video) Find $\vec{\mathbf{r}}'(t)$ of $\vec{\mathbf{r}}(t)=\left\langle t^6, \sin 2t, \ln(t+1)\right\rangle$.

Solution According to theorem, we differentiate each component of $\vec{\mathbf{r}}$:

$$\vec{\mathbf{r}}'(t)=\left\langle 6t^5, 2\cos 2t, \frac{1}{t+1}\right\rangle$$

Exercise 2.1 Find the derivative of each vector-valued function.

a) $\vec{\mathbf{r}}'(t)$ where $\vec{\mathbf{r}}(t)=\sin(t^2)\,\vec{\mathbf{i}}+e^{\cos t}\,\vec{\mathbf{j}}+t\ln t\,\vec{\mathbf{k}}$

Solution $\vec{\mathbf{r}}'(t)=2t\cos(t^2)\vec{\mathbf{i}}-(\sin t)e^{\cos t}\vec{\mathbf{j}}+(1+\ln t)\vec{\mathbf{k}}$

b) $\vec{\mathbf{r}}''(t)$ where $\vec{\mathbf{r}}(t)=(t^3-2t^2)\,\vec{\mathbf{i}}+4t\,\vec{\mathbf{j}}+te^{-t}\vec{\mathbf{k}}$

Solution $\vec{\mathbf{r}}'(t)=(3t^2-4t)\vec{\mathbf{i}}+4\vec{\mathbf{j}}+(-te^{-t}+e^{-t})\vec{\mathbf{k}}$

$\vec{\mathbf{r}}''(t)=(6t-4)\vec{\mathbf{i}}+(te^{-t}-2e^{-t})\vec{\mathbf{k}}$

c) $\vec{\mathbf{r}}''(t)$ where $\vec{\mathbf{r}}(t)=(\cos t+t\sin t)\,\vec{\mathbf{i}}+(\sin t-t\cos t)\,\vec{\mathbf{j}}+\frac{1}{t}\vec{\mathbf{k}}$

Solution $\vec{\mathbf{r}}'(t)=(t\cos t)\vec{\mathbf{i}}+(t\sin t)\vec{\mathbf{j}}-\frac{1}{t^2}\vec{\mathbf{k}}$

$\vec{\mathbf{r}}''(t)=(\cos t-t\sin t)\vec{\mathbf{i}}+(\sin t+t\cos t)\vec{\mathbf{j}}+\frac{2}{t^3}\vec{\mathbf{k}}$

> **Theorem** (Chain rule)
>
> If $\vec{\mathbf{r}}(s)$ is a differentiable vector function with respect to s and $s=u(t)$ is a differentiable scalar function, then the derivative of $\vec{\mathbf{r}}(s)$ with respect to t is
>
> $$\frac{d\vec{\mathbf{r}}}{dt}=\frac{d\vec{\mathbf{r}}}{ds}\frac{ds}{dt}=\vec{\mathbf{r}}'(s)u'(t)$$

Example 2.2 (video) If $\vec{\mathbf{r}}(s)=(\cos 2s)\,\vec{\mathbf{i}}+(\sin 2s)\,\vec{\mathbf{j}}+e^{-3s}\vec{\mathbf{k}}$, where $s=t^4$ then

$$\begin{aligned}\frac{d\vec{\mathbf{r}}}{dt}=\frac{d\vec{\mathbf{r}}}{ds}\frac{ds}{dt}&=\left[(-2\sin 2s)\vec{\mathbf{i}}+(2\cos 2s)\vec{\mathbf{j}}-3e^{-3s}\vec{\mathbf{k}}\right](4t^3)\\&=-8t^3(\sin 2t^4)\vec{\mathbf{i}}+8t^3(\cos 2t^4)\vec{\mathbf{j}}-12t^3e^{-3t^4}\vec{\mathbf{k}}\end{aligned}$$

Differentiation Rules

The differentiation formulas for real-valued functions have analogs in the context of differentiating vector-valued functions.

Theorem Let $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$ be differentiable vector-valued function, $u(t)$ be a differentiable real-valued function and c be a scalar. Then

(i) $\frac{d}{dt}\left[\vec{\mathbf{r}}_1(t)+\vec{\mathbf{r}}_2(t)\right]=\vec{\mathbf{r}}_1'(t)+\vec{\mathbf{r}}_2'(t)$

(ii) $\frac{d}{dt}\left[c\vec{\mathbf{r}}_1(t)\right]=c\vec{\mathbf{r}}_1'(t)$

(iii) $\frac{d}{dt}\left[u(t)\vec{\mathbf{r}}_1(t)\right]=u(t)\vec{\mathbf{r}}_1'(t)+u'(t)\vec{\mathbf{r}}_1(t)$

(iv) $\frac{d}{dt}\left[\vec{\mathbf{r}}_1(t)\cdot\vec{\mathbf{r}}_2(t)\right]=\vec{\mathbf{r}}_1'(t)\cdot\vec{\mathbf{r}}_2(t)+\vec{\mathbf{r}}_1(t)\cdot\vec{\mathbf{r}}_2'(t)$

(v) $\frac{d}{dt}\left[\vec{\mathbf{r}}_1(t)\times\vec{\mathbf{r}}_2(t)\right]=\vec{\mathbf{r}}_1'(t)\times\vec{\mathbf{r}}_2(t)+\vec{\mathbf{r}}_1(t)\times\vec{\mathbf{r}}_2'(t)$

Example 2.3 Suppose $\vec{\mathbf{r}}(t)$ is differentiable. Find the derivative of $\|\vec{\mathbf{r}}(t)\|$.

Solution Since $\|\vec{\mathbf{r}}(t)\|$ is a real-valued function of t, then by the chain rule for real-valued function, we know that

$$\frac{d}{dt}\|\vec{\mathbf{r}}(t)\|^2 = 2\|\vec{\mathbf{r}}(t)\|\frac{d}{dt}\|\vec{\mathbf{r}}(t)\|.$$

But $\|\vec{\mathbf{r}}(t)\|^2 = \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)$, therefore

$$\frac{d}{dt}\|\vec{\mathbf{r}}(t)\|^2 = \frac{d}{dt}\left[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)\right] = \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) = 2\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t).$$

Hence, we have $$2\|\vec{\mathbf{r}}(t)\|\frac{d}{dt}\|\vec{\mathbf{r}}(t)\| = 2\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t).$$

If $\|\vec{\mathbf{r}}(t)\|\neq 0$ then $\frac{d}{dt}\|\vec{\mathbf{r}}(t)\| = \frac{\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)}{\|\vec{\mathbf{r}}(t)\|}$.

Theorem if $\|\vec{\mathbf{r}}(t)\|\neq 0$, then $\|\vec{\mathbf{r}}(t)\|$ is constant if and only if $\vec{\mathbf{r}}'(t)\perp\vec{\mathbf{r}}(t)$ for all t.

Proof: $\|\vec{\mathbf{r}}(t)\|$ is constant if and only if $\frac{d}{dt}\|\vec{\mathbf{r}}(t)\| = 0$. From example 2.3, we must have $\frac{\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)}{\|\vec{\mathbf{r}}(t)\|}=0$. Thus $\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)=0$, which means that $\vec{\mathbf{r}}'(t)\perp\vec{\mathbf{r}}(t)$.

Smooth curves

Let $\vec{\mathbf{r}}(t)$ be a vector function. We say that $\vec{\mathbf{r}}(t)$ is a **smooth function** if $\vec{\mathbf{r}}'(t)$ is continuous and $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t on an open interval (a,b). A curve C is said to be **smooth curve** if it is trace out by a smooth function $\vec{\mathbf{r}}(t)$. Geometrically, a smooth curve has no corners or cusps.

Example 2.4 Find the intervals on which the curve C given by

$$\vec{\mathbf{r}}(t) = (5\cos t - \cos 5t)\vec{\mathbf{i}} + (5\sin t - \sin 5t)\vec{\mathbf{j}}, \quad 0 \le t \le 2\pi$$

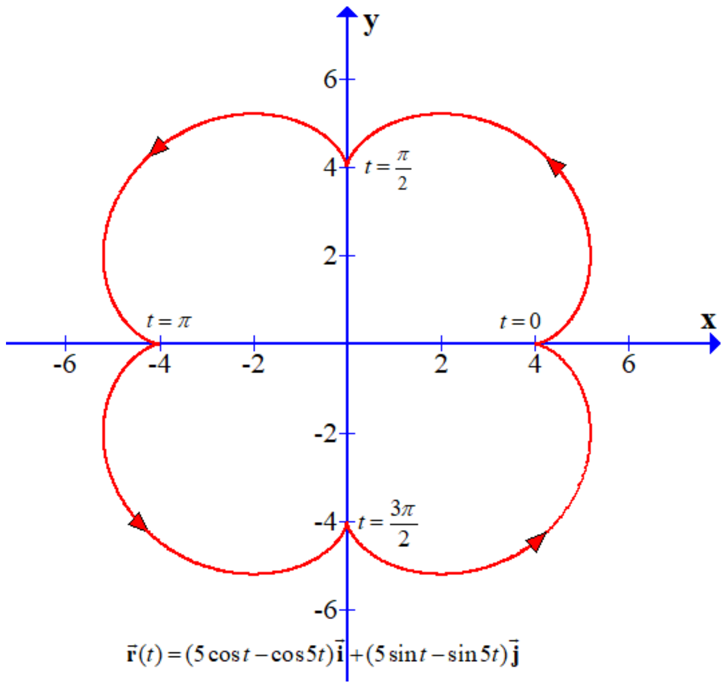
is smooth.

Solution The derivative of $\vec{\mathbf{r}}$ is

$$\vec{\mathbf{r}}'(t) = (-5\sin t + 5\sin 5t)\vec{\mathbf{i}} + (5\cos t - 5\cos 5t)\vec{\mathbf{j}}.$$

In the interval $[0, 2\pi]$, the only values of t for which $\vec{\mathbf{r}}'(t) = 0\vec{\mathbf{i}} + 0\vec{\mathbf{j}}$ are $t = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π.

Therefore, C is smooth in the intervals $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.



Note: The curve is not smooth at points at which the curve makes abrupt changes in direction. The point where a curve has a sharp point or corner is called a cusp or node.

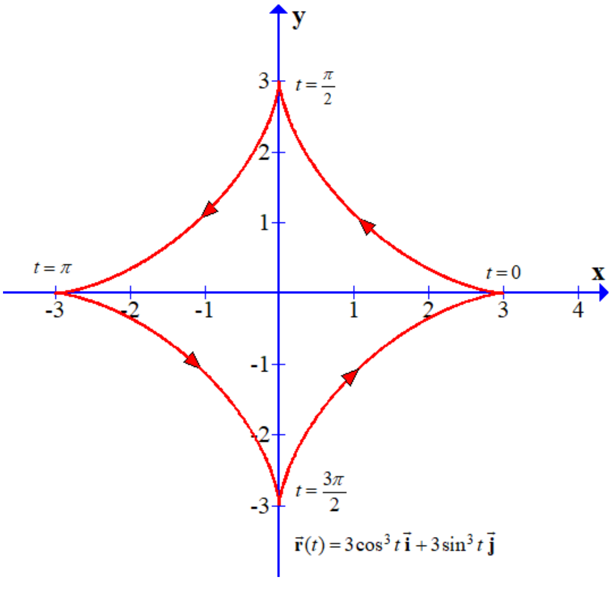
Exercise 2.2

a) Determine the open intervals for $\vec{\mathbf{r}}(t)=\left\langle 3\cos^3 t,\ 3\sin^3 t\right\rangle$ is smooth on $\left[0,2\pi\right]$.

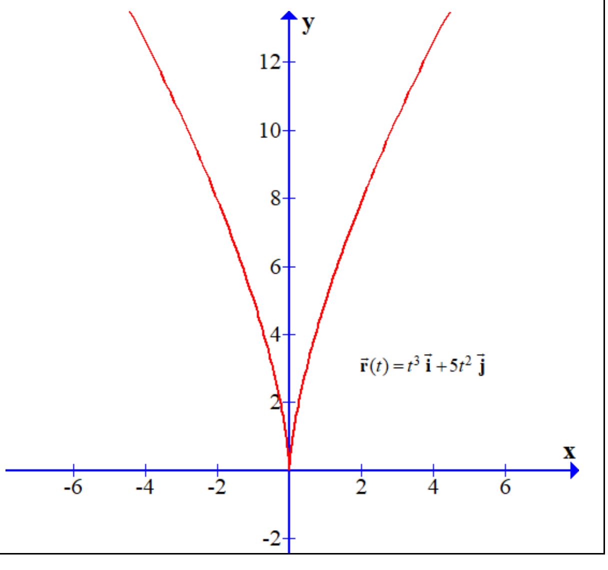
b) Determine whether the curved represented by $\vec{\mathbf{r}}(t)=\left\langle t^3,\ 5t^2\right\rangle$ is smooth on $(-\infty,\infty)$?

Answer

a)



b)



Geometric Interpretation of $\vec{\mathbf{r}}'(t)$

Tangent vector and tangent line

Recall that the derivative of a real-valued function at a point can be interpreted as the slope of the *tangent line* to the graph at that point. Similarly, the derivative of a vector-valued function is a *tangent vector* to the curve represented by the function.

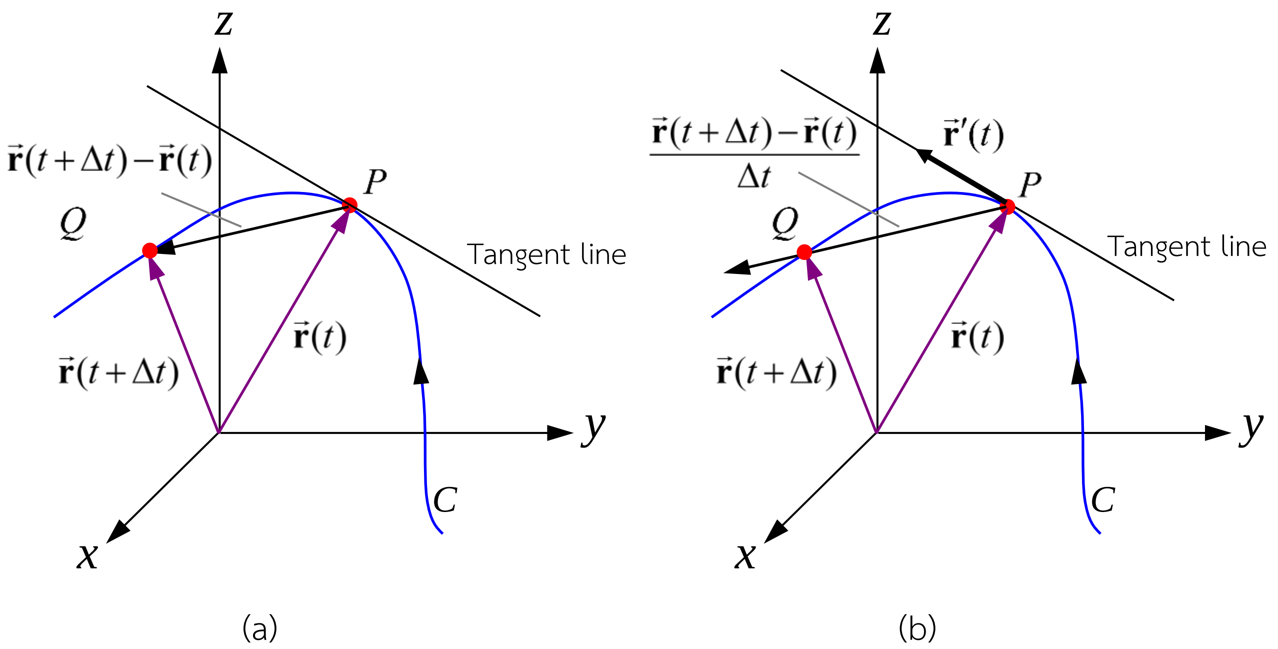


Figure 2.1

Let the points P and Q have position vector $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}(t+\Delta t)$, then $\overrightarrow{PQ}$ represents the vector $\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)$ (Figure 2.1a). If $\Delta t>0$, the scalar multiple $\frac{1}{\Delta t}(\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t))$ has the same direction as $\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)$. As $\Delta t\to 0$, this vector approaches a vector $\vec{\mathbf{r}}'(t)$ that lies on the tangent line (Figure 2.1b). If $\vec{\mathbf{r}}'(t)$ exits and $\vec{\mathbf{r}}'(t)\neq 0$, then we call $\vec{\mathbf{r}}'(t)$ the **tangent vector** to the curve C at P, and we call the line through P parallel to $\vec{\mathbf{r}}'(t)$ the **tangent line** to C at P.

Equation of the tangent line

An equation of the tangent line to the graph of $\vec{\mathbf{r}}(t)$ at $t=t_0$ is the line through $\vec{\mathbf{r}}(t_0)$ with direction parallel to $\vec{\mathbf{r}}'(t_0)$ is

$$\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(t_0)+t\,\vec{\mathbf{r}}'(t_0)$$

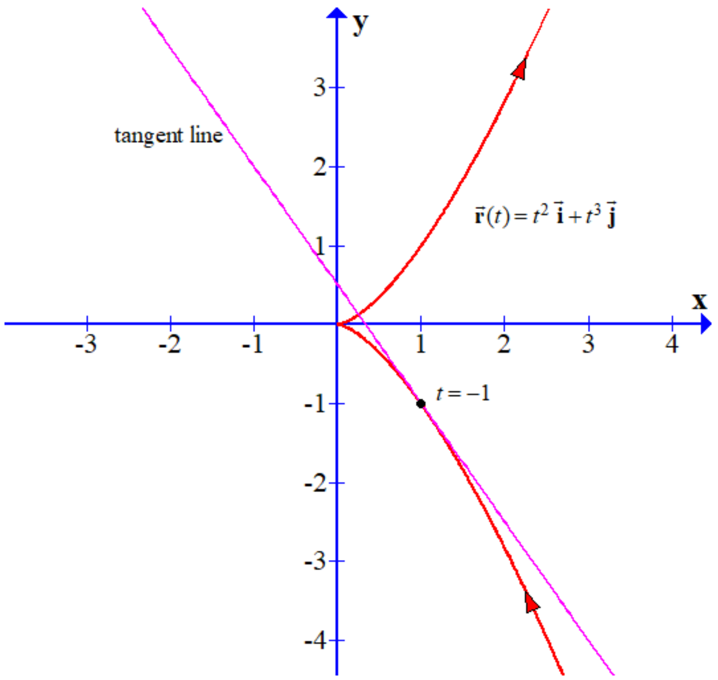
Example 2.5 Find the equation of the lines tangent to $\vec{\mathbf{r}}(t)=\langle t^2, t^3\rangle$ at $t=-1$ and $t=0$.

Solution We differentiate each component of $\vec{\mathbf{r}}$: $\vec{\mathbf{r}}'(t)=\langle 2t, 3t^2\rangle$.

At $t=-1$, we have $\vec{\mathbf{r}}(-1)=\langle 1,-1\rangle$ and $\vec{\mathbf{r}}'(-1)=\langle -2,3\rangle$,

so the equation of the line tangent to the graph of $\vec{\mathbf{r}}(t)$ at $t=-1$ is

$$\vec{\mathbf{r}}(t)=\langle 1,-1\rangle+t\langle -2,3\rangle$$

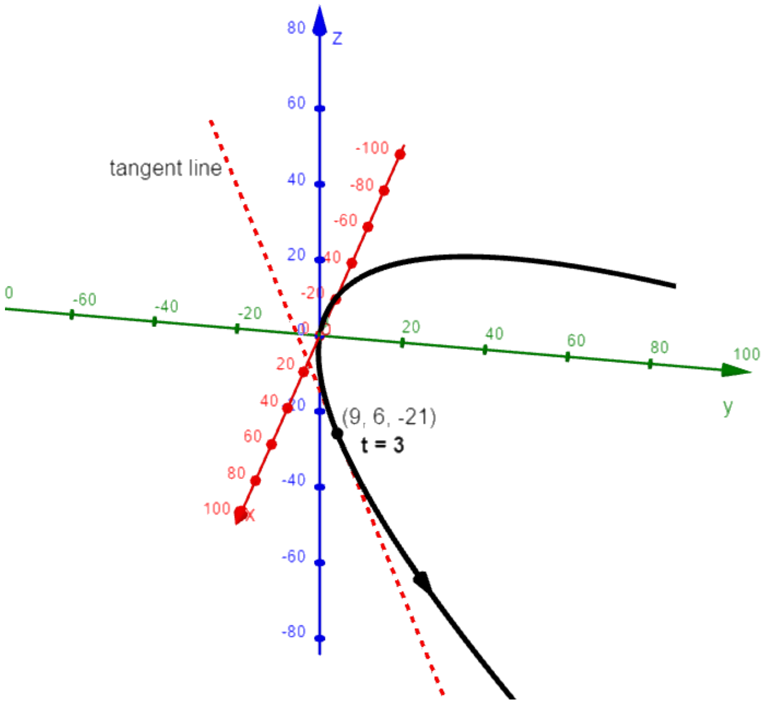


At $t=0$, we have $\vec{\mathbf{r}}(0)=\langle 0,0\rangle$ and $\vec{\mathbf{r}}'(0)=\langle 0,0\rangle$.

This implies that the tangent line has no direction, hence cannot find the equation of the tangent line.

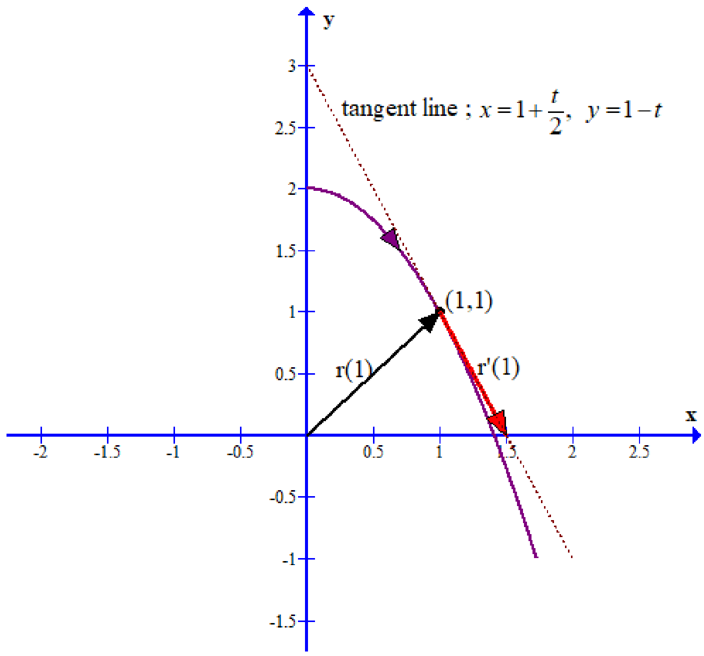
Example 2.6 Find parametric equations of the tangent line to the graph of the curve C whose parametric equations are $x=t^2$, $y=t^2-t$, $z=-7t$ at $t=3$.

Solution .



Exercise 2.3 Find equation for the tangent line to the curve $\vec{\mathbf{r}}(t) = \sqrt{t}\,\vec{\mathbf{i}} + (2-t)\,\vec{\mathbf{j}}$, and sketch the position vector $\vec{\mathbf{r}}(1)$ and the tangent vector $\vec{\mathbf{r}}'(1)$.

Solution



Unit tangent vector

Let C be a smooth curve represented by $\vec{\mathbf{r}}$ on an open interval (a,b). The **unit tangent vector** $\vec{\mathbf{T}}(t)$ at t is defined as

$$\vec{\mathbf{T}}(t)=\frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|},\quad \vec{\mathbf{r}}'(t)\neq\vec{\mathbf{0}}$$

Example 2.7 Find the unit tangent vector of $\vec{\mathbf{r}}(t)=(1+t^3)\vec{\mathbf{i}}+te^{-t}\vec{\mathbf{j}}+\sin 2t\vec{\mathbf{k}}$ at the point where $t=0$.

Solution We differentiate each component of $\vec{\mathbf{r}}$:

$$\vec{\mathbf{r}}'(t)=3t^2\,\vec{\mathbf{i}}+(1-t)e^{-t}\vec{\mathbf{j}}+2\cos 2t\vec{\mathbf{k}}$$

At $t=0$, we have $\vec{\mathbf{r}}(0)=\vec{\mathbf{i}}$ and $\vec{\mathbf{r}}'(0)=\vec{\mathbf{j}}+2\vec{\mathbf{k}}$, the unit tangent vector at the point $(1,0,0)$ is

$$\vec{\mathbf{T}}(0)=\frac{\vec{\mathbf{r}}'(0)}{\|\vec{\mathbf{r}}'(0)\|}=\frac{\vec{\mathbf{j}}+2\vec{\mathbf{k}}}{\sqrt{1+4}}=\frac{1}{\sqrt{5}}\vec{\mathbf{j}}+\frac{2}{\sqrt{5}}\vec{\mathbf{k}}$$

3 Length of space curve

If C is a smooth curve given by $\vec{\mathbf{r}}(t)=x(t)\vec{\mathbf{i}}+y(t)\vec{\mathbf{j}}+z(t)\vec{\mathbf{k}}$ on an interval $[a,b]$, then the length of C on the interval is

$$L=\int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt=\int_a^b \|\vec{\mathbf{r}}'(t)\|\,dt \qquad (3.1)$$

Example 3.1 Find the length of the curve given by

a) (video) $\vec{\mathbf{r}}(t)=\langle 2t, 3\sin 2t, 3\cos 2t\rangle$ for $0\le t\le 2\pi$.

Solution

b)** $\vec{\mathbf{r}}(t)=\left\langle t, \frac{4}{3}t^{\frac{3}{2}}, \frac{1}{2}t^2\right\rangle$ for $0\le t\le 2$.

Solution Since $\vec{\mathbf{r}}'(t)=\left\langle 1, 2t^{\frac{1}{2}}, t\right\rangle$, we have $\|\vec{\mathbf{r}}'(t)\|=\sqrt{1^2+(2t^{\frac{1}{2}})^2+t^2}=\sqrt{1+4t+t^2}$

The length of the curve is

$$\int_0^2 \|\vec{\mathbf{r}}'(t)\|\,dt=\int_0^2\sqrt{1+4t+t^2}\,dt=\int_0^2\sqrt{(t+2)^2-3}\,dt$$

$$=\int_0^2\sqrt{(t+2)^2-3}\,dt$$

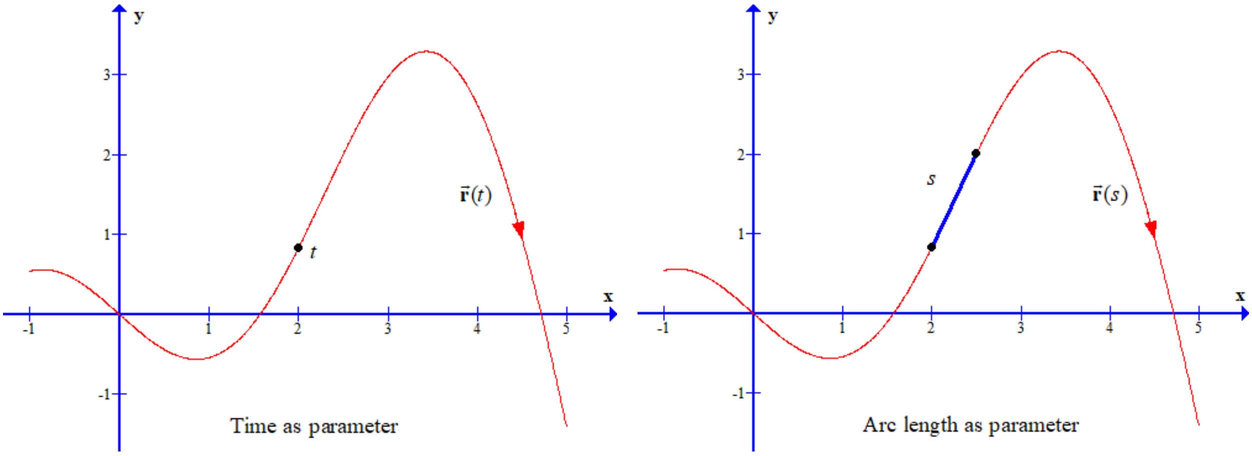
$$=\left[\frac{t+2}{2}\sqrt{(t+2)^2-3}-\frac{3}{2}\ln\left|(t+2)+\sqrt{(t+2)^2-3}\right|\right]_0^2$$

$$=2\sqrt{13}-\frac{3}{2}\ln(4+\sqrt{13})-1+\frac{3}{2}\ln 3 \quad \approx 4.816 \text{ units}$$

Formula : $\int\sqrt{u^2-a^2}\,du=\frac{u}{2}\sqrt{u^2-a^2}-\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+C$

Arc length as a parameter

The curves can be represented by vector-valued functions in different ways, depending on the choice of parameter, for motion along a curve, the convenient parameter is time t, for studying the geometric properties of a curve, the convenient parameter is often arc length s.



If C is a smooth curve given by $\vec{\mathbf{r}}(t) = x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}} + z(t)\vec{\mathbf{k}}$ defined on the closed interval $[a,b]$, For $a \le t \le b$, the **arc length function** is given by

$$s(t) = \int_a^t \|\vec{\mathbf{r}}'(\tau)\|\, d\tau = \int_a^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau \qquad (3.2)$$

The arc length s is called the **arc length parameter**.

Note (1) Using the definition of the arc length function and the second Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\| \qquad (3.3)$$

or $$ds = \|\vec{\mathbf{r}}'(t)\|\, dt$$

(2) If C is a smooth curve given by $\vec{\mathbf{r}}(s) = x(s)\vec{\mathbf{i}} + y(s)\vec{\mathbf{j}} + z(s)\vec{\mathbf{k}}$ where s is the arc length parameter, then $\|\vec{\mathbf{r}}'(s)\| = 1$ (Let $t = s$ in formula (3.3)).

Example 3.2 (video) Find the arc length parameterization of the helix

$$\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}} + t\vec{\mathbf{k}}, \quad t \ge 0$$

Solution Since $\vec{\mathbf{r}}'(t) = -2\sin t\,\vec{\mathbf{i}} + 2\cos t\,\vec{\mathbf{j}} + \vec{\mathbf{k}}$ and $\|\vec{\mathbf{r}}'(t)\| = \sqrt{5}$

The length of the curve from $\vec{\mathbf{r}}(0)$ to an arbitrary $\vec{\mathbf{r}}(t)$ is

$$s(t)=\int_0^t \|\vec{\mathbf{r}}'(\tau)\|\,d\tau=\int_0^t \sqrt{5}\,d\tau=\sqrt{5}t \qquad \text{(arc length function)}$$

Therefore, s and t are related by $s=\sqrt{5}t$.

Using $t=\dfrac{s}{\sqrt{5}}$, a vector equation of the helix can be parameterized in terms of s as

$$\vec{\mathbf{r}}(s)=2\cos\frac{s}{\sqrt{5}}\,\vec{\mathbf{i}}+2\sin\frac{s}{\sqrt{5}}\,\vec{\mathbf{j}}+\frac{s}{\sqrt{5}}\vec{\mathbf{k}}.$$

Parametric equation of the helix are then

$$x(s)=2\cos\frac{s}{\sqrt{5}},\quad y(s)=2\sin\frac{s}{\sqrt{5}},\quad z(s)=\frac{s}{\sqrt{5}}.$$

4 Velocity, Acceleration and Speed

If x, y and z are twice-differentiable functions of t, and $\vec{\mathbf{r}}$ is a position function of a particle moving along a curve given by $\vec{\mathbf{r}}(t)=x(t)\vec{\mathbf{i}}+y(t)\vec{\mathbf{j}}+z(t)\vec{\mathbf{k}}$, then the **velocity** vector, **acceleration** vector and **speed** of the particle at time t are defined by

Velocity : $$\vec{\mathbf{v}}(t)=\vec{\mathbf{r}}'(t)=x'(t)\vec{\mathbf{i}}+y'(t)\vec{\mathbf{j}}+z'(t)\vec{\mathbf{k}} \qquad (4.1)$$

Acceleration : $$\vec{\mathbf{a}}(t)=\vec{\mathbf{r}}''(t)=x''(t)\vec{\mathbf{i}}+y''(t)\vec{\mathbf{j}}+z''(t)\vec{\mathbf{k}} \qquad (4.2)$$

Speed : $$\|\vec{\mathbf{v}}(t)\|=\|\vec{\mathbf{r}}'(t)\|=\sqrt{\left[x'(t)\right]^2+\left[y'(t)\right]^2+\left[z'(t)\right]^2} \qquad (4.3)$$

Example 4.1 The position of a moving particle is given by $\vec{\mathbf{r}}(t)=t\,\vec{\mathbf{i}}+t^3\vec{\mathbf{j}}+3t\,\vec{\mathbf{k}}$, $t\geq 0$. Graph the curve defined by $\vec{\mathbf{r}}(t)$ and the vectors $\vec{\mathbf{v}}(1)$ and $\vec{\mathbf{a}}(1)$.

Solution

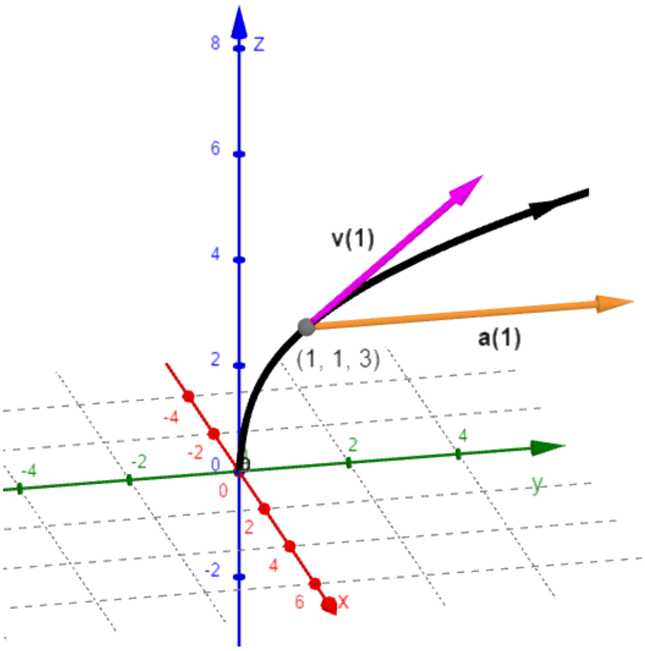


Figure 4.1

5 Curvature and radius of curvature

Curvature is the measure of how sharply a curve bends. We can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector $\vec{\mathbf{T}}$ with respect to the arc length s (Keep in mind that has constant length, only its direction changes).

Suppose C is the graph of a smooth vector-valued function that parameterized in terms of arc length as shown in Figure 5.1. If C is a straight line (no bend), then the direction of $\vec{\mathbf{T}}$ remains constant (Figure 5.1a). If C bends slightly, then $\vec{\mathbf{T}}$ undergoes a gradual change of direction (Figure 5.1b). If C bends sharply, then $\vec{\mathbf{T}}$ undergoes a rapid change of direction (Figure 5.1c)

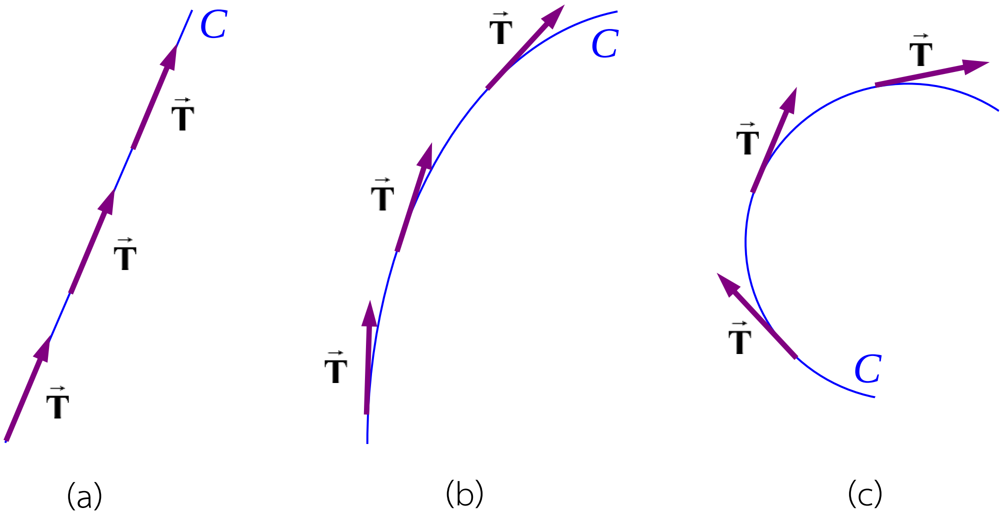


Figure 5.1

Let C be a smooth curve given by $\vec{\mathbf{r}}(s)$, where s is the arc length parameter. The curvature κ ("kappa") at s is given by

$$\kappa(s) = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \vec{\mathbf{T}}'(s) \right\| = \left\| \vec{\mathbf{r}}''(s) \right\| \tag{5.1}$$

Since the curve are generally not parameterized by arc length, it is convenient to express Formula (5.1) in terms of a general parameter t. Using the chain rule, we can write

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt} \quad \text{and consequently} \quad \frac{d\vec{\mathbf{T}}}{ds} = \frac{\frac{d\vec{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = \frac{1}{\left\|\vec{\mathbf{r}}'(t)\right\|}\frac{d\vec{\mathbf{T}}}{dt}$$

Let C be a smooth curve given by $\vec{\mathbf{r}}(t)$, then curvature κ of C at t is given by

$$\kappa(t) = \frac{\left\|\vec{\mathbf{T}}'(t)\right\|}{\left\|\vec{\mathbf{r}}'(t)\right\|} \tag{5.2}$$

$$\kappa(t) = \frac{\left\|\vec{\mathbf{r}}'(t)\times\vec{\mathbf{r}}''(t)\right\|}{\left\|\vec{\mathbf{r}}'(t)\right\|^3} \tag{5.3}$$

Example 5.1 Find the curvature of a circle of radius a.

Solution

Example 5.2* Find $\kappa(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct \quad \text{where } a, c > 0$$

Solution The vector function for the helix is

$$\vec{\mathbf{r}}(t) = a\cos t\,\vec{\mathbf{i}} + a\sin t\,\vec{\mathbf{j}} + ct\,\vec{\mathbf{k}}$$

Thus, $$\vec{\mathbf{r}}'(t) = -a\sin t\,\vec{\mathbf{i}} + a\cos t\,\vec{\mathbf{j}} + c\,\vec{\mathbf{k}}$$

$$\vec{\mathbf{r}}''(t) = -a\cos t\,\vec{\mathbf{i}} - a\sin t\,\vec{\mathbf{j}}$$

$$\vec{\mathbf{r}}'(t)\times\vec{\mathbf{r}}''(t) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ac\sin t\,\vec{\mathbf{i}} - ac\cos t\,\vec{\mathbf{j}} + a^2\vec{\mathbf{k}}$$

Therefore, $\quad \left\|\vec{\mathbf{r}}'(t)\right\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2+c^2}$

and $\quad \left\|\vec{\mathbf{r}}'(t)\times\vec{\mathbf{r}}''(t)\right\| = \sqrt{(ac\sin t)^2 + (ac\cos t)^2 + a^4}$

$$= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2+c^2}$$

So $\quad \kappa(t) = \dfrac{\left\|\vec{\mathbf{r}}'(t)\times\vec{\mathbf{r}}''(t)\right\|}{\left\|\vec{\mathbf{r}}'(t)\right\|^3} = \dfrac{a\sqrt{a^2+c^2}}{\left(\sqrt{a^2+c^2}\right)^3} = \dfrac{a}{a^2+c^2}$

Note that κ does not depend on t, which tell us that the helix has constant curvature.

Exercise 5.1

a) Find the curvature of the curve given by $\vec{\mathbf{r}}(t) = 2t\vec{\mathbf{i}} + t^2\vec{\mathbf{j}} - \frac{1}{3}t^3\vec{\mathbf{k}}$

b) Find the curvature of the curve given by $\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} + 3\sin t\,\vec{\mathbf{j}}$, $\quad 0 \le t \le 2\pi$ at $t=0$ and $t=\frac{\pi}{2}$.

Answer a) $\dfrac{2}{(t^2+2)^2}$ $\qquad$ b) $\kappa(0)=\dfrac{2}{9}$, $\kappa(\frac{\pi}{2})=\dfrac{3}{4}$

Radius of curvature

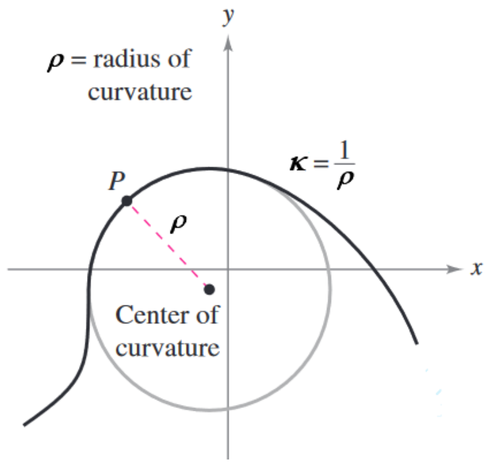
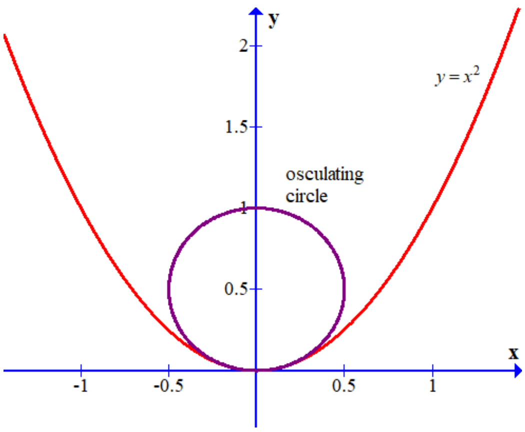


Figure 5.2

Let C be a curve with curvature κ at point P. The circle passing through point P with radius $\rho = \dfrac{1}{\kappa}$ sharing a common tangent with C at P, and centered on the concave side of the curve at P, is called the **circle of curvature** or **osculating circle** at P. The radius ρ of the osculating circle at P is called the **radius of curvature** at P, and the center of the circle is called the **center of curvature**.

Example 5.3 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

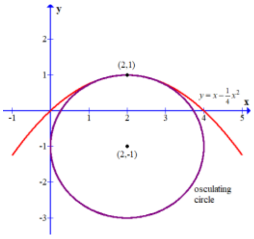
Solution



Exercise 5.2 Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$.

Sketch the circle of curvature at $(2,1)$.

Answer $\kappa = \frac{1}{2}$



6 Unit normal vector and binormal vector

Unit normal vector

Let $\vec{\mathbf{r}}(t)$ be a smooth space curve with unit tangent vector $\vec{\mathbf{T}}(t)$. Recall that if $\|\vec{\mathbf{r}}(t)\|$ is constant then $\vec{\mathbf{r}}'(t)\perp\vec{\mathbf{r}}(t)$, and because $\left\|\vec{\mathbf{T}}(t)\right\|=1$, therefore $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{T}}'(t)$ are orthogonal vectors. This implies that $\vec{\mathbf{T}}'(t)$ is perpendicular to the tangent line to C at t, so we say that $\vec{\mathbf{T}}'(t)$ **normal** to C at t. If $\vec{\mathbf{T}}'(t)\neq\vec{\mathbf{0}}$, then we can define the principal unit normal vector $\vec{\mathbf{N}}(t)$ (or simply **unit normal**) as

$$\vec{\mathbf{N}}(t)=\frac{\vec{\mathbf{T}}'(t)}{\left\|\vec{\mathbf{T}}'(t)\right\|} \tag{6.1}$$

Note The unit normal vector is defined only at points where $\vec{\mathbf{T}}'(t)\neq\vec{\mathbf{0}}$.

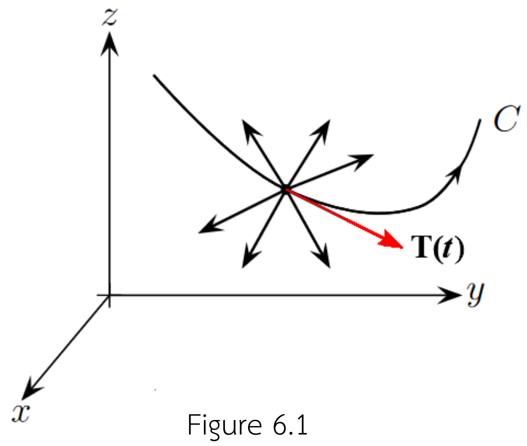


Figure 6.1

There are many vectors at a given point on a smooth curve C that are orthogonal to the unit tangent vector $\vec{\mathbf{T}}(t)$ (Figure 6.1). The unit normal vector $\vec{\mathbf{N}}(t)$ is point in the direction of $\vec{\mathbf{T}}'(t)$ and this vector tells us the direction the curve is turning at each point (i.e. points inward toward the concave side of the curve) as shown in Figure 6.2

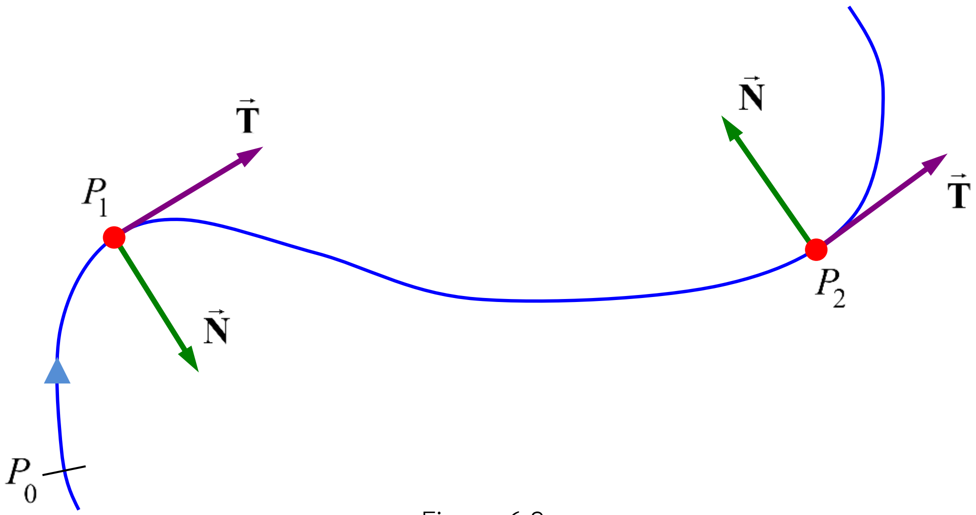


Figure 6.2

Assume that s is an arc length parameter for a smooth vector-valued function $\vec{\mathbf{r}}(s)$, it follows from (6.1) that $\vec{\mathbf{N}}(s)=\dfrac{\vec{\mathbf{T}}'(s)}{\left\|\vec{\mathbf{T}}'(s)\right\|}$. Since the curvature $\kappa(s)=\left\|\dfrac{d\vec{\mathbf{T}}}{ds}\right\|$, so we have another formula for unit normal vector as

$$\vec{\mathbf{N}}(s)=\frac{\vec{\mathbf{T}}'(s)}{\left\|\vec{\mathbf{T}}'(s)\right\|}=\frac{1}{\kappa}\vec{\mathbf{T}}'(s) \qquad (6.2)$$

Binormal vector

If C is the graph of a vector-valued function $\vec{\mathbf{r}}(t)$, the vector defined by

$$\vec{\mathbf{B}}(t)=\vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t) \qquad (6.3)$$

Is called the **binormal vector** to C at t. It is perpendicular to both $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ and is also a unit vector (Figure 6.3a).

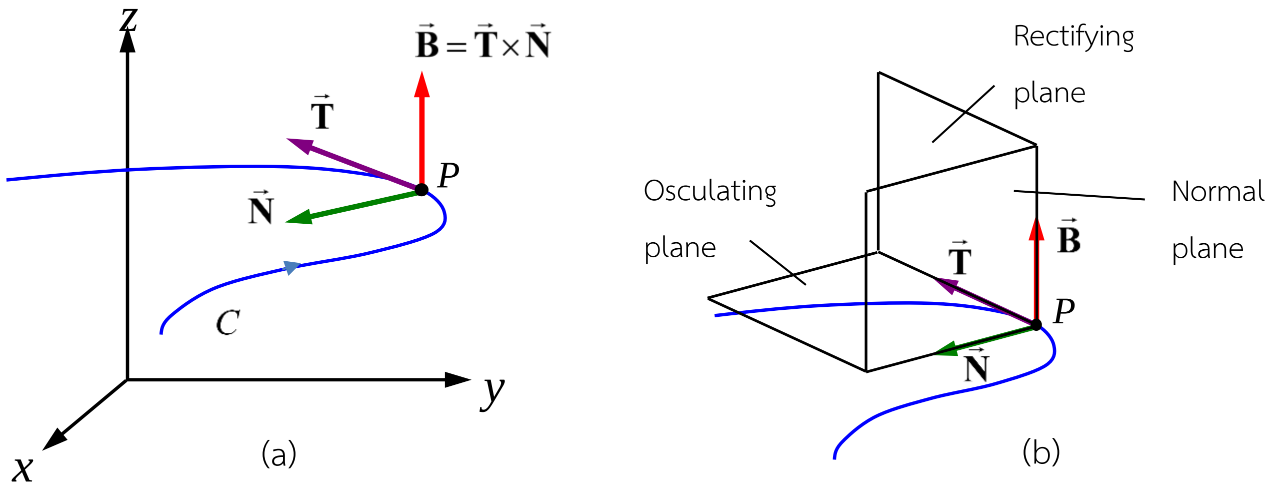


Figure 6.3

The three unit vectors $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$ form a right-handed set of mutually orthogonal vectors called $\vec{\mathbf{T}}\vec{\mathbf{N}}\vec{\mathbf{B}}$-frame (or **Frenet frame**) that moves along the curve as t varies (Figure 6.3b).

The plane of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ (or $\vec{\mathbf{T}}\vec{\mathbf{N}}$-plane) is called the **osculating plane** (It is the plane that contains the curve).

The plane of $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$ (or $\vec{\mathbf{N}}\vec{\mathbf{B}}$-plane) is called the **normal plane**.

The plane of $\vec{\mathbf{T}}$ and $\vec{\mathbf{B}}$ (or $\vec{\mathbf{N}}\vec{\mathbf{B}}$-plane) is called the **rectifying plane**.

Example 6.1 The position of a moving particle is given by

$$\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}} + 3t\vec{\mathbf{k}}\,.$$

Find the vectors $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$, the curvature and an equation of the osculating plane at $t = \dfrac{2\pi}{3}$.

Solution We compute the ingredients needed for the three unit vectors.

$$\vec{\mathbf{r}}'(t) = -2\sin t\,\vec{\mathbf{i}} + 2\cos t\,\vec{\mathbf{j}} + 3\vec{\mathbf{k}}$$

$$\left\|\vec{\mathbf{r}}'(t)\right\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2} = \sqrt{13}$$

Unit tangent vector : $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{\left\|\vec{\mathbf{r}}'(t)\right\|} = \dfrac{1}{\sqrt{13}}(-2\sin t\,\vec{\mathbf{i}} + 2\cos t\,\vec{\mathbf{j}} + 3\vec{\mathbf{k}})$

$$\vec{\mathbf{T}}'(t) = \frac{1}{\sqrt{13}}(-2\cos t\,\vec{\mathbf{i}} - 2\sin t\,\vec{\mathbf{j}})$$

$$\left\|\vec{\mathbf{T}}'(t)\right\| = \sqrt{\left(\frac{-2\cos t}{\sqrt{13}}\right)^2 + \left(\frac{-2\sin t}{\sqrt{13}}\right)^2} = \frac{2}{\sqrt{13}}$$

Unit normal vector : $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}'(t)}{\left\|\vec{\mathbf{T}}'(t)\right\|} = -\cos t\,\vec{\mathbf{i}} - \sin t\,\vec{\mathbf{j}}$

Binormal vector : $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\dfrac{2}{\sqrt{13}}\sin t & \dfrac{2}{\sqrt{13}}\cos t & \dfrac{3}{\sqrt{13}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$$= \frac{1}{\sqrt{13}}\left(3\sin t\,\vec{\mathbf{i}} - 3\cos t\,\vec{\mathbf{j}} + 2\vec{\mathbf{k}}\right)$$

At $t = \dfrac{2\pi}{3}$: $\vec{\mathbf{T}}\left(\dfrac{2\pi}{3}\right) = \dfrac{1}{\sqrt{13}}(-2\sin\dfrac{2\pi}{3}\vec{\mathbf{i}} + 2\cos\dfrac{2\pi}{3}\vec{\mathbf{j}} + 3\vec{\mathbf{k}}) = \dfrac{1}{\sqrt{13}}(-\sqrt{3}\,\vec{\mathbf{i}} - \vec{\mathbf{j}} + 3\vec{\mathbf{k}})$

$$\vec{\mathbf{N}}\left(\frac{2\pi}{3}\right) = -\cos\frac{2\pi}{3}\vec{\mathbf{i}} - \sin\frac{2\pi}{3}\vec{\mathbf{j}} = \frac{1}{2}\vec{\mathbf{i}} - \frac{\sqrt{3}}{2}\vec{\mathbf{j}}$$

$$\vec{\mathbf{B}}\left(\frac{2\pi}{3}\right) = \frac{1}{\sqrt{13}}\left(3\sin\frac{2\pi}{3}\vec{\mathbf{i}} - 3\cos\frac{2\pi}{3}\vec{\mathbf{j}} + 2\vec{\mathbf{k}}\right) = \frac{1}{\sqrt{13}}\left(\frac{3\sqrt{3}}{2}\vec{\mathbf{i}} + \frac{3}{2}\vec{\mathbf{j}} + 2\vec{\mathbf{k}}\right)$$

$$\kappa\left(\frac{2\pi}{3}\right) = \frac{\left\|\vec{\mathbf{T}}'\left(\frac{2\pi}{3}\right)\right\|}{\left\|\vec{\mathbf{r}}'\left(\frac{2\pi}{3}\right)\right\|} = 2$$

An equation of the plane through the point (x_0, y_0, z_0) with normal vector $\langle a, b, c\rangle$ is

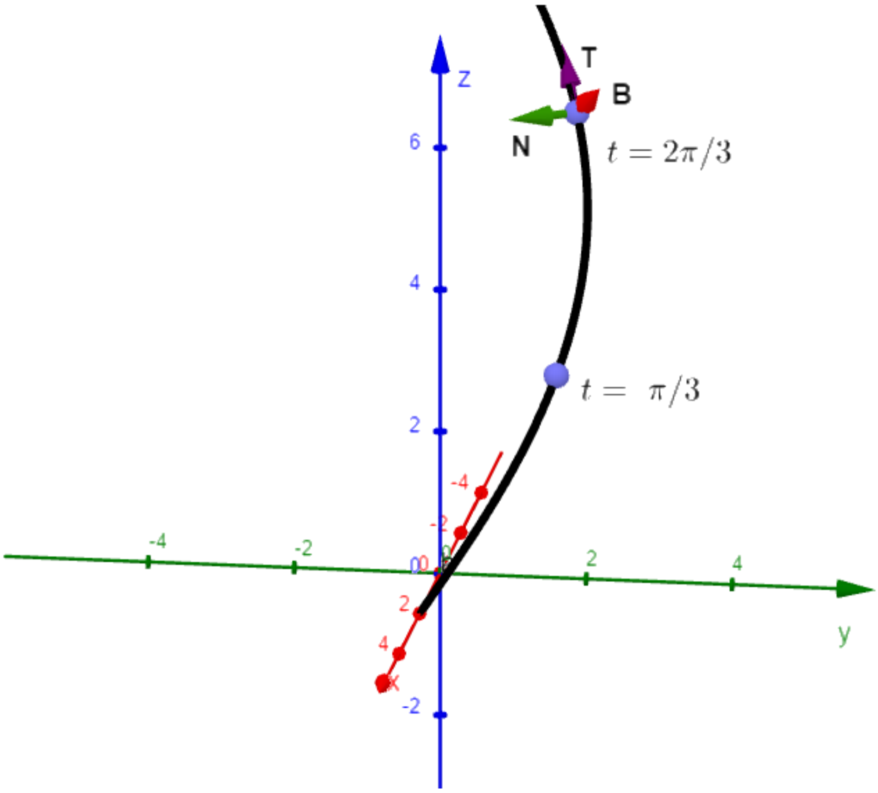
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

The osculating plane at $t=\frac{2\pi}{3}$ contains the point $\left(-1,\sqrt{3},2\pi\right)$ and the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, so its normal vector is $\vec{\mathbf{B}}\left(\frac{2\pi}{3}\right)=\frac{1}{\sqrt{13}}\left(\frac{3\sqrt{3}}{2}\vec{\mathbf{i}}+\frac{3}{2}\vec{\mathbf{j}}+2\vec{\mathbf{k}}\right)$.

Therefore, an equation of the osculating plane is

$$\frac{3\sqrt{3}}{2\sqrt{13}}(x+1)+\frac{3}{2\sqrt{13}}(y-\sqrt{3})+\frac{2}{\sqrt{13}}(z-2\pi)=0$$

or $$3\sqrt{3}x+3y+4z=8\pi \qquad \#$$



Exercise 6.1 The position of a moving particle is given by the circular helix

$$\vec{\mathbf{r}}(t)=\cos t\,\vec{\mathbf{i}}+\sin t\,\vec{\mathbf{j}}+t\vec{\mathbf{k}}$$

Find the binormal vectors, the equations of normal plane and osculating plant at the point $\left(0,1,\frac{\pi}{2}\right)$.

Solution

7 Torsion

Assume that s is an arc length parameter for a smooth vector-valued function $\vec{\mathbf{r}}(s)$ and $\dfrac{d\vec{\mathbf{T}}}{ds}$ and $\dfrac{d\vec{\mathbf{N}}}{ds}$ exist at each point on the curve. The rate of change of binormal vector is expressed by the vector

$$\begin{aligned}\frac{d\vec{\mathbf{B}}}{ds} &= \frac{d}{ds}(\vec{\mathbf{T}}\times\vec{\mathbf{N}})\\ &= \vec{\mathbf{T}}\times\frac{d\vec{\mathbf{N}}}{ds}+\frac{d\vec{\mathbf{T}}}{ds}\times\vec{\mathbf{N}}\\ &= \vec{\mathbf{T}}\times\frac{d\vec{\mathbf{N}}}{ds}+\kappa\vec{\mathbf{N}}\times\vec{\mathbf{N}} && \left(\text{formula 6.2 : } \vec{\mathbf{N}}(s)=\frac{1}{\kappa}\frac{d\vec{\mathbf{T}}}{ds}\right)\\ &= \vec{\mathbf{T}}\times\frac{d\vec{\mathbf{N}}}{ds}\end{aligned}$$

Therefore, the vector $\dfrac{d\vec{\mathbf{B}}}{ds}$ is perpendicular to vectors $\vec{\mathbf{T}}$ and $\dfrac{d\vec{\mathbf{N}}}{ds}$.

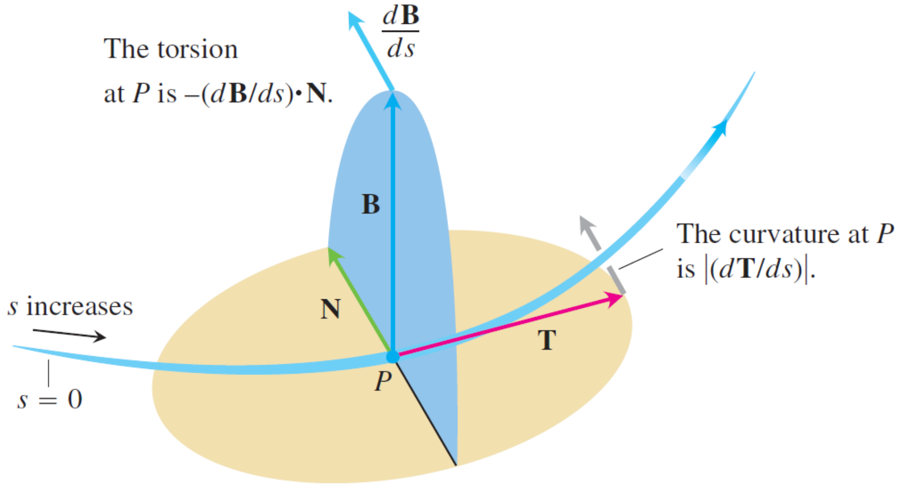
Since $\vec{\mathbf{B}}$ is a unit vector, $\|\vec{\mathbf{B}}\|=1$, then $\dfrac{d\vec{\mathbf{B}}}{ds}$ is perpendicular to vectors $\vec{\mathbf{B}}$.

It follows that $\dfrac{d\vec{\mathbf{B}}}{ds}$ is perpendicular to vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{B}}$, that is $\dfrac{d\vec{\mathbf{B}}}{ds}//\vec{\mathbf{N}}$.

Thus $\dfrac{d\vec{\mathbf{B}}}{ds}$ is a scalar multiple of $\vec{\mathbf{N}}$, the negative of this scalar is called the **torsion** of $\vec{\mathbf{r}}(s)$ and is denoted by $\tau(s)$. Thus,

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau\vec{\mathbf{N}} \tag{6.4}$$

The torsion τ is related to the twisting properties of the curve, and $\tau(s)$ is regarded as a numerical measure of the tendency for the curve to twist out of the osculating plane. The minus sign is chosen to make the torsion of a right-handed helix positive.



Note: $\dfrac{d\vec{\mathbf{B}}}{ds}\cdot\vec{\mathbf{N}} = -\tau\vec{\mathbf{N}}\cdot\vec{\mathbf{N}} = -\tau$ or $\tau = -\dfrac{d\vec{\mathbf{B}}}{ds}\cdot\vec{\mathbf{N}}$

Since $\dfrac{d\vec{\mathbf{B}}}{dt} = \dfrac{d\vec{\mathbf{B}}}{ds}\dfrac{ds}{dt}$,

we have that $\dfrac{d\vec{\mathbf{B}}}{ds} = \dfrac{\dfrac{d\vec{\mathbf{B}}}{dt}}{\dfrac{ds}{dt}} = \dfrac{\vec{\mathbf{B}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$

Therefore, $\tau(t) = -\dfrac{1}{\|\vec{\mathbf{r}}'(t)\|}\vec{\mathbf{B}}'(t)\cdot\vec{\mathbf{N}}(t)$

Example 6.2 Determine the torsion of $\vec{\mathbf{r}}(t) = 3\cos t\,\vec{\mathbf{i}} + 3\sin t\,\vec{\mathbf{j}} + 4t\vec{\mathbf{k}}$

Solution