MTH 102: Mathematics II

Chapter 0: Mathematical Induction

This chapter explains a powerful proof technique called **mathematical induction**. To motivate the discussion, let's first examine the kinds of statements that induction is used to prove.

Consider the following statement

"The sum of the first n odd natural numbers equals n^2."

The following table illustrates what this conjecture says. Each row is headed by a natural number n, followed by the sum of the first n odd natural numbers, and followed by n^2.

n	**Sum of the first n odd natural numbers**	n^2
1	1	1
2	1+3	4
3	1+3+5	9
4	1+3+5+7	16
5	1+3+5+7+9	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+\ldots+(2n-1)$	n^2
$\vdots$	$\vdots$	$\vdots$

Note that in the first five lines of the table, the sum of the first n odd numbers really does add up to n^2. Notice also that these first five lines indicate that the n^{th} odd natural number (the last number in each sum) is $2n$-1. For instance, when $n = 2$, the second odd natural number is $2 \cdot 2 - 1 = 3$; when $n = 3$, the third odd natural number is $2 \cdot 3 - 1 = 5$, etc.

The table raises a very simple question.

"Does the sum $1 + 3 + 5 + 7 + \ldots + (2n-1)$ really equal n^2 for all natural number n?"

Let's rephrase this as follows. For each natural number n, we define a statement S_n as follows:

$$S_1\text{:}\quad 1 = 1^2$$

$$S_2\text{:}\quad 1+3 = 2^2$$

$$S_3\text{:}\quad 1+3+5 = 3^2$$

$$\vdots$$

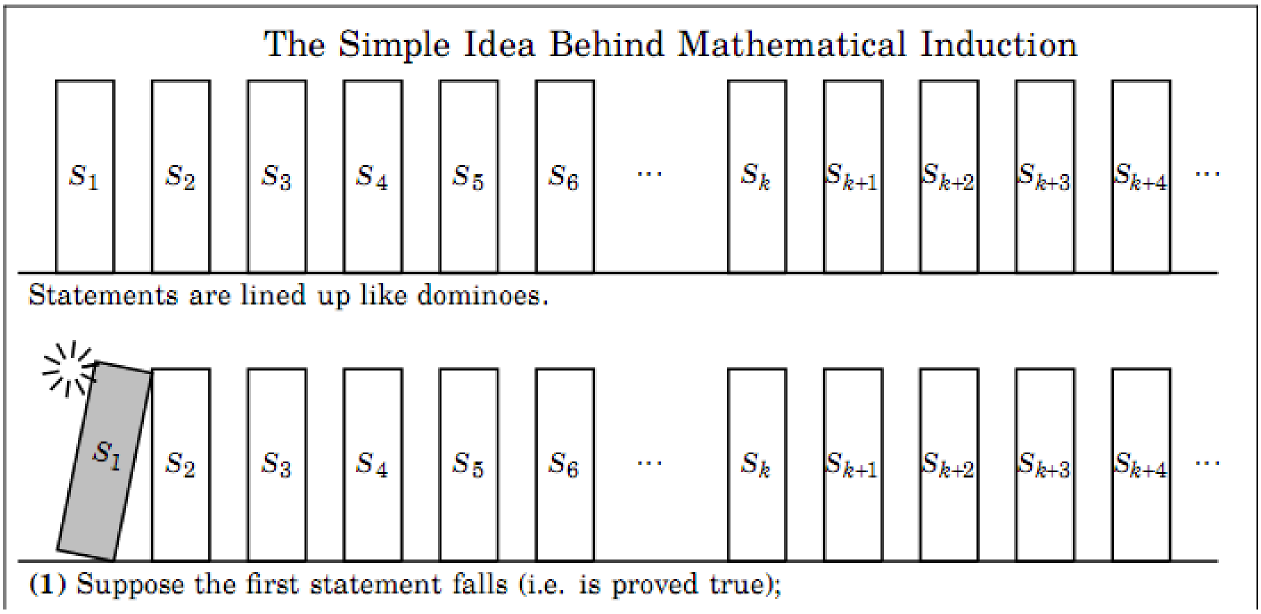
$$S_n\text{:}\quad 1+3+5+\ldots+(2n-1) = n^2$$

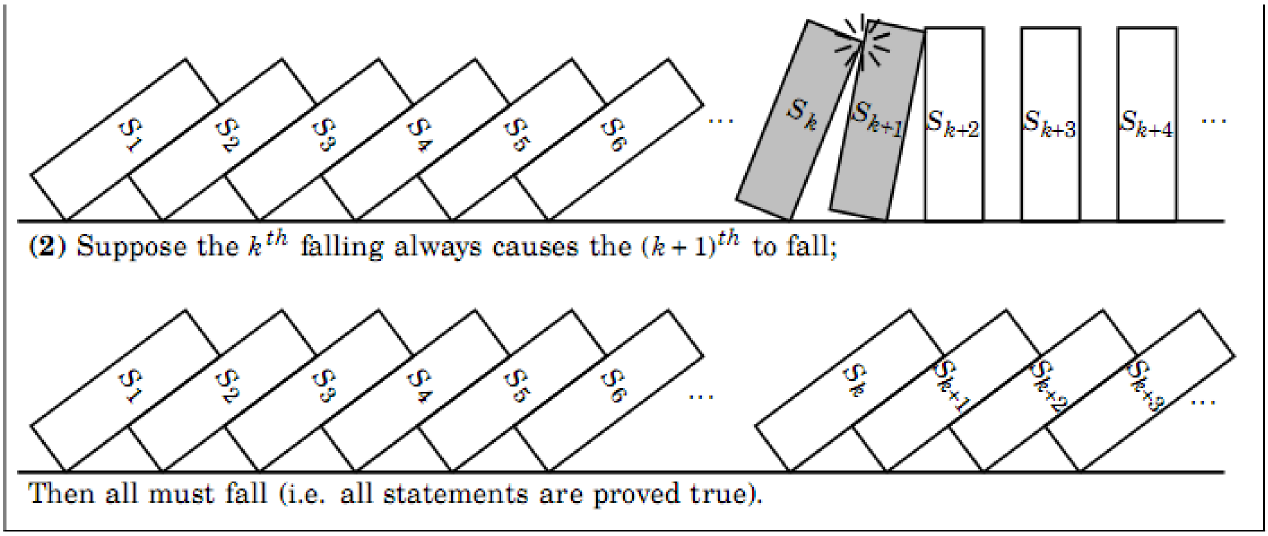
$$\vdots$$

Our question is: Are all of these statements true?

Mathematical induction is a proof technique designed to answer this kind of question. It is used when we have a set of statements S_1, S_2, S_3, . . . , S_n, . . ., and we need to prove that they are all true. The method is really quite simple.

To visualize it, think of these statements as dominoes lined up in a row. Imagine you can prove the first statement S_1, and symbolize this as domino S_1 being knocked down. Additionally, imagine that you can prove that any statement S_k being true (falling) forces the next statement S_{k+1} to be true (to fall). Then S_1 falls, and knocks down S_2. Next S_2 falls and knocks down S_3, then S_3 knocks down S_4, and so on. The inescapable conclusion is that all the statements are knocked down (proved to be true).





In this setup, the first step (1) is called the **basis step**.

The second step (2) is called the **inductive step**.

Because S_1 is usually a very simple statement, the basis step is often quite easy to do. In the inductive step, the direct proof is most often used to prove S_k implies S_{k+1}. So, this inductive step is usually carried out by assuming S_k is true and showing that it forces S_{k+1} to be true.

The assumption that S_k is true is called the **inductive hypothesis.**

Now let's apply this technique to our original conjecture that the sum of the first n odd natural numbers equals n^2.

Our goal is to show that for each $n \in N$, the statement

$$S_n : 1+3+5+7+\ldots+(2n-1) = n^2 \text{ is true.}$$

Before getting started, observe that S_k is obtained from S_n by plugging k in for n. Thus S_k is the statement

$$S_k : 1+3+5+7+\ldots+(2k-1)= k^2.$$

Also, we get S_{k+1} by plugging in k+1 for n, so that

$$S_{k+1} : 1+3+5+7+\ldots+(2(k+1)-1)= (k+1)^2.$$

Example If $n \in N$, then $1+3+5+7+\ldots+(2n-1)= n^2$.

Proof: We will prove this with mathematical induction.

(1) Basis step: Observe that if $n = 1$, this statement is $1 = 1^2$, which is obviously true.

(2) Inductive step: We must now prove that S_k implies S_{k+1} for any $k \geq 1$. That is, we must show that $1 + 3 + 5 + 7 + \ldots + (2k-1) = k^2$ implies $1+3+5+7+\ldots+ (2(k+1)-1) =(k+1)^2$. We use direct proof. Suppose that $k \geq 1$ and $1+3+5+7+\ldots+ (2k-1) = k^2$. Next, we consider

$$1 + 3 + 5 + 7 + \ldots + (2(k+1)-1)$$
$$= 1 + 3 + 5 + 7 + \ldots + (2k-1) + (2(k+1)-1)$$
$$= [1 + 3 + 5 + 7 + \ldots + (2k-1)] + (2(k+1)-1)$$

$= k^2 + (2(k+1)-1)$ (by the inductive hypothesis)

$= k^2 + 2k + 1$

$= (k+1)^2$

Thus $1+3+5+7+\ldots+(2(k+1)-1) = (k+1)^2$.

This proves that S_k implies S_{k+1} for all $k \geq 1$. It follows by mathematical induction that $1+3+5+7+\ldots+(2n-1) = n^2$ for every $n \in N$. □

Example Prove that $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$ for all $n \in N$.

Example Show that, for every $n \in N$,

$$1+2+3+4+\cdots+n=\frac{n^2+n}{2}.$$

In induction proofs, it is usually the case that the first statement S_1 is indexed by the natural number 1, but this need not always be so. Depending on the problem, the first statement could be S_0, or S_m for any other integer m. In the next example the statements are $S_0, S_1, S_2, S_3, \ldots$ The same outline is used, except that the basis step verifies S_0, not S_1.

Example Show that $5 \mid (n^5 - n)$ for all non-negative integers n.

Proof. We will prove this with mathematical induction. Observe that the first non-negative integer is 0, so the basis step involves $n = 0$.

(1) When $n = 0$, the statement S_0 is $5 \mid (0^5 - 0)$ or $5 \mid 0$, which is obviously true.

(2) Let $k \geq 0$. We need to prove that if $5 \mid (k^5 - k)$, then $5 \mid ((k+1)^5 - (k+1))$.

We use direct proof. Suppose $5 \mid (k^5 - k)$. Thus $k^5 - k = 5a$ for some $a \in Z$. Then observe that

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k\end{aligned}$$

$$= 5a+5k^4+10k^3+10k^2+5k$$

$$= 5(a+k^4+2k^3+2k^2+k).$$

This shows $(k+1)^5-(k+1)$ is an integer multiple of 5, so $5 \mid ((k+1)^5 - (k+1))$. We have now shown $5 \mid (k^5 - k)$ implies $5 \mid ((k+1)^5 - (k+1))$. It follows by induction that $5 \mid (n^5 - n)$ for all non-negative integers n. □

Example For all integers $n \geq 4$ we have $n! > 2^n$

Exercise

Use mathematical induction to prove that the following assertions are true for $n \geq 1$

1. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4} n^2 (n+1)^2$

2. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

3. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}$

4. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

5. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$