Exercise: Mathematical Induction, Sequence, and Series (Solution)

1. Use mathematical induction to prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true for all n is positive integers.

Solution Let $P(n)$: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ Where n is a positive integers.

(1) Since

$$\frac{(1)(1+1)(2(1)+1)}{6} = \frac{(1)(2)(3)}{6} = 1^2$$

Therefore $P(1)$ is $1^2 = \dfrac{(1)(1+1)(2(1)+1)}{6}$ is ture

(2) Let k is positive integer, If $P(k)$ is ture, then

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \tag{1}$$

We show that $P(k+1)$ is ture, then

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

By addition $(k+1)^2$ substitute equation (1)

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{2k^3 + 3k^2 + k + 6(k^2 + 2k + 1)}{6} \\
&= \frac{2k^3 + 3k^2 + k + 6k^2 + 12k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Therfore $P(k+1)$ is ture

By Mathematical Induction, $P(n)$ is ture for all n is a positive integers.

2. Determine if the following sequences converge or diverge. If it converges, find its limit.

2.1 $\left\{\left(\dfrac{2n+3}{2n-5}\right)^n\right\}$

Solution Let $K = \left(\dfrac{2x+3}{2x-5}\right)^x$ we get $\ln K = x\ln\left(\dfrac{2x+3}{2x-5}\right)$

Consider

$$\lim_{x\to\infty}\ln K = \lim_{x\to\infty} x\ln\left(\frac{2x+3}{2x-5}\right)$$
$$\ln\lim_{x\to\infty} K = \lim_{x\to\infty}\frac{\ln\left(\frac{2x+3}{2x-5}\right)}{\frac{1}{x}}$$

Since $\lim_{x\to\infty}\dfrac{\ln\left(\frac{2x+3}{2x-5}\right)}{\frac{1}{x}}$ in the form $\dfrac{0}{0}$ using L' Hopital's rule

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\left(\frac{2x+3}{2x-5}\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(\frac{2x+3}{2x-5}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{\frac{2x-5}{2x+3}\left(\frac{(2x-5)(2)-(2x+3)(2)}{(2x+5)^2}\right)}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty}\frac{2x-5}{2x+3}\left(\frac{4x-10-4x-6}{(2x-5)^2}\right)(-x^2)\\
&= \lim_{x\to\infty}\frac{16x^2}{(2x+3)(2x-5)}\\
&= \lim_{x\to\infty}\frac{16x^2}{4x^2-4x-15}\\
&= 4
\end{aligned}$$

So $\lim_{x\to\infty} K = e^4$

We get $\lim_{x\to\infty}\left(\dfrac{2x+3}{2x-5}\right)^x = e^4$

Therefore, $\left\{\left(\dfrac{2n+3}{2n-5}\right)^n\right\}$ converges to e^4

2.2 $\left\{ \ln(n) - \ln(n+1) \right\}$

Solution

Since $a_n = \ln(n) - \ln(n+1)$

Let $f(x) = \ln\left(\frac{x}{x+1}\right)$

Then
$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \ln\left(\frac{x}{x+1}\right)\\ &= \ln \lim_{x\to\infty}\left(\frac{x}{x+1}\right)\\ &= \ln 1\\ &= 0\end{aligned}$$

Therefore, $\left\{ \ln(n) - \ln(n+1) \right\}$ converges to 0

2.3 $\left\{ \frac{n^2}{2n-1} \sin\left(\frac{1}{n}\right) \right\}$

Solution

Since $a_n = \frac{n^2}{2n-1}\sin\left(\frac{1}{n}\right)$

Let $f(x) = \frac{x^2}{2x-1}\sin\left(\frac{1}{x}\right)$

Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2x-1}\sin\left(\frac{1}{x}\right)$

We get
$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{2x-1}\sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{x}{2x-1}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)\\ &= \lim_{x\to\infty} \frac{x}{2x-1} \cdot \lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)\end{aligned}$$

Consider $\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = 1$

Thus
$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{2x-1}\sin\left(\frac{1}{x}\right) &= \left(\frac{1}{2}\right)(1)\\ &= \frac{1}{2}\end{aligned}$$

Therefore, $\left\{ \frac{n^2}{2n-1} \sin\left(\frac{1}{n}\right) \right\}$ Converges to $\frac{1}{2}$

3. Consider the following sequences. Are they monotone? bounded? if the following sequences are monotone, check that it increasing,or decreasing ?

3.1 $\left\{\dfrac{1}{2^n}\right\}$

Solution

$$\begin{aligned}
\text{Consider} \qquad a_{n+1} - a_n &= \frac{1}{2^{n+1}} - \frac{1}{2^n} \\
&= \frac{1}{(2 \cdot 2^n)} - \frac{1}{2^n} \\
&= \frac{1-2}{2 \cdot 2^n} \\
&= -\frac{1}{2^{n+1}} < 0 \\
\text{Then} \qquad a_{n+1} - a_n &< 0 \\
a_{n+1} &< a_n
\end{aligned}$$

So, $\left\{\dfrac{1}{2^n}\right\}$ is monotonic, and decreasing.

The squence $\left\{\dfrac{1}{2^n}\right\} = \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \cdots, \dfrac{1}{2^n}, \cdots$ is bounded above by every real number greater then or equal to $\dfrac{1}{2}$. The squence is also bounded below by every number less then or equal to 0, which is its greatest lower bound.

Therfore, $\left\{\dfrac{1}{2^n}\right\}$ is bounded.

3.2 $\left\{\dfrac{2^{n+1}}{n+2}\right\}$

Solution

$$
\begin{aligned}
\text{Consider} \qquad \frac{a_{n+1}}{a_n} &= \frac{\frac{2^{n+2}}{n+3}}{\frac{2^{n+1}}{n+2}} \\
&= \frac{2^{n+2}}{n+3} \cdot \frac{n+2}{2^{n+1}} \\
&= \frac{2^n \cdot 2^2 \cdot (n+2)}{(n+3) \cdot 2^n \cdot 2} \\
&= \frac{2(n+2)}{n+3} \\
&= \frac{2n+4}{n+3} > 1 \\
\text{Then} \qquad \frac{a_{n+1}}{a_n} &> 0 \\
a_{n+1} &> a_n
\end{aligned}
$$

So, $\left\{\dfrac{2^{n+1}}{n+2}\right\}$ is monotonic, and increasing.

The squence $\left\{\dfrac{2^{n+1}}{n+2}\right\} = \dfrac{4}{3}, 2, \dfrac{16}{5}, \cdots$ is bounded above by every real number greater then or equal to $\dfrac{4}{3}$. The squence is also unbounded below.

Therfore, $\left\{\dfrac{2^{n+1}}{n+2}\right\}$ is unbounded.

3.3 $\left\{2ne^{-2n}\right\}$

Solution

Since $\quad a_n = 2ne^{-2n} \quad$ for all n is positive integers.

Let $\quad f(x) = 2xe^{-2x} \quad$ for all $x \in [1, \infty)$

We get
$$f'(x) = -4xe^{-2x} + 2e^{-2x}$$
$$= (-4x+1)e^{-2x}$$
$$= \frac{-4x+1}{e^{2x}} < 0 \qquad \text{for all } x \in [1, \infty)$$

So, $\left\{2ne^{-2n}\right\}$ is monotonic, and decreasing.

The squence $\left\{2ne^{-2n}\right\} = \frac{2}{e^2}, \frac{4}{e^4}, \frac{6}{e^6} \cdots$ is bounded above by every real number greater then or equal to $\frac{2}{e^2}$. The squence is also bounded below by every number less then or equal to 0, which is its greatest lower bound.

Therfore, $\left\{2ne^{-2n}\right\}$ is bounded.

4. Determine if the following Infinite series converge or diverge.

4.1 Telescoping Series : $\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$

(Hint: Use partial fractions)

Solution

$$
\begin{aligned}
\text{Let} \qquad S_n &= \left(\frac{1}{\ln 3}-\frac{1}{\ln 2}\right)+\left(\frac{1}{\ln 4}-\frac{1}{\ln 3}\right)+\left(\frac{1}{\ln 5}-\frac{1}{\ln 4}\right)+\cdots \\
&\quad +\left(\frac{1}{\ln(n-1+2)}-\frac{1}{\ln(n-1+1)}\right) \\
&\quad +\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right) \\
&= -\frac{1}{\ln 2}-\frac{1}{\ln(n+2)}
\end{aligned}
$$

$$
\text{Let} \qquad f(x) = -\frac{1}{\ln 2}-\frac{1}{\ln(x+2)}
$$

$$
\begin{aligned}
\text{Consider } \lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\left(-\frac{1}{\ln 2}-\frac{1}{\ln(x+2)}\right) \\
&= -\frac{1}{\ln 2}
\end{aligned}
$$

Therefore, The series is Telescoping Series, and convergent to $-\frac{1}{\ln 2}$

4.2 Geometric Series: $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

(Hint: Write out the first few terms of the series to find a and r)

Solution

Consider

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{3}{6}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

We get,

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ is geometric series with $a = 1$ and $r = \frac{1}{2} < 1$

convergent to $\frac{1}{1-\frac{1}{2}} = 2$

$\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$ is geometric series with $a = 1$ and $r = \frac{1}{6} < 1$

convergent to $\frac{1}{1-\frac{1}{6}} = \frac{6}{5}$

Then

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \frac{4}{5}$$

4.3 Geometric Series: $\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$

(Hint: Write out the first few terms of the series to find a and r)

Solution

Consider $$\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n = 1+\frac{e}{\pi}+\left(\frac{e}{\pi}\right)^2+\cdots$$

We get $a = 1$ and $r = \frac{e}{\pi} \approx \frac{2.18}{3.14} < 1$

Thus, $\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$ is geometric series with $a = 1$ and $r = \frac{e}{\pi} < 1$

and convergent to $\frac{a}{1-r} = \frac{1}{1-\frac{e}{\pi}} = \frac{\pi}{\pi - e}$

4.4 p-Series: $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{4}}}$

Solution

Consider $$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{4}}} = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{5}{4}}$$

This is p-series with $p = \frac{5}{4} > 1$

Thus, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{4}}}$ is converges.

4.5 p-Series: $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

Solution

Consider
$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{1}{n\sqrt{n}} + \frac{1}{n^2\sqrt{n}} \right)$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{3}{2}}} + \frac{1}{n^{\frac{5}{2}}} \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} + \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$$

We get,

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is p-series with $p = \frac{3}{2} > 1$, then converges.

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ is p-series with $p = \frac{5}{2} > 1$, then converges.

Let
$$a_n = \frac{n+1}{n^2\sqrt{n}}$$
$$= \frac{n+1}{n^{(2+\frac{1}{2})}}$$
$$= \frac{n+2}{n^{\frac{5}{2}}}$$

Thus
$$\lim_{x\to\infty} \frac{x+2}{x^{\frac{5}{2}}} = \lim_{x\to\infty} \left(\frac{1}{x^{\frac{3}{2}}} + \frac{2}{x^{\frac{5}{2}}} \right)$$
$$= 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges.

4.6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (Using Integral Test)

Solution

Let $\quad a_n = \dfrac{1}{n^2+1}, \qquad f(x) = \dfrac{1}{x^2+1}$

Then
$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{1}{x^2+1}dx \\
&= \tan^{-1} x \Big|_1^{\infty} \\
&= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1 \right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent to $\dfrac{\pi}{4}$.

4.7 $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ (Using Integral Test)

Solution

Let $$a_n = \frac{2}{1+e^n}, \qquad f(x) = \frac{2}{1+e^x}$$

Then
$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{2}{1+e^x}dx \\
&= \int_e^{\infty} \frac{2}{1+u} \cdot \frac{1}{u}du \\
&= \int_e^{\infty} \Big(\frac{2}{u} - \frac{2}{1+u}\Big)du \\
&= \lim_{b\to\infty} 2\Big(\ln u - \ln|1+u|\Big)\Big|_e^b \\
&= \lim_{b\to\infty} 2\ln\left(\frac{u}{1+u}\right)\Big|_e^b \\
&= \lim_{b\to\infty} \left[2\ln\left(\frac{b}{b+1}\right) - 2\ln\left(\frac{e}{e+1}\right)\right] \\
&= 2\ln 1 - 2\ln\left(\frac{e}{e+1}\right) \\
&= -2\ln\left(\frac{e}{e+1}\right)
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ is convergent to $-2\ln\left(\frac{e}{e+1}\right)$.

4.8 $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (Using Comparison Test)

Solution

$$\text{Since} \qquad \ln n < n, \qquad n \geq 1$$
$$\frac{\ln n}{n^3} < \frac{n}{n^3}$$
$$= \frac{1}{n^2}$$
$$\text{And} \qquad a_n = \frac{\ln n}{n^3}, \qquad b_n = \frac{1}{n^2}$$

We get, b_n is p-series with $p = 2 > 1$ then, b_n converges

Since $a_n = \frac{\ln n}{n^3} < \frac{1}{n^2} = b_n$

By comparison then, $\sum_{n=}^{\infty} \frac{\ln n}{n^3}$ converges.

4.9 $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ (Using Ratio Test)

Solution

$$
\begin{aligned}
&\text{Let} && a_n = \frac{2^n+5}{3^n} \\
&\text{Then} && a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}} \\
&\text{Consider} && \frac{a_{n+1}}{a_n} = \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5} \\
& && = \frac{1}{3}\left(\frac{2^{n+1}+5}{2^n+5}\right) \\
& && = \frac{1}{3}\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \\
&\text{So,} && \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \\
& && = \frac{1}{3}\left(\frac{2+5(0)}{1+5(0)}\right) \\
& && = \frac{1}{3}\left(\frac{2}{1}\right) \\
& && = \frac{2}{3} < 1
\end{aligned}
$$

The series converges because $L = \frac{2}{3} < 1$

This does not mean that $\frac{2}{3}$ is the sum of series. In fact,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} &= \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\frac{5}{3^n} \\
&= \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}} \\
&= \frac{21}{2}
\end{aligned}
$$

Therefore, $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ is convergent to $\frac{21}{2}$.

5. Determine if the following alternating series converge or diverge?

5.1 $\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{3n}{2n+1}$

Solution

Let $\quad a_n = \frac{3n}{2n+1}$

Consider $\quad \lim_{n\to\infty} a_n = 0 \quad$ or not

Since
$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3n}{2n+1} \\ &= \lim_{n\to\infty} \frac{\frac{3n}{n}}{\frac{2n}{n}+\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{3}{2+\frac{1}{n}} \\ &= \frac{3}{2}\end{aligned}$$

We get, $\lim_{n\to\infty} a_n \neq 0$

Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{3n}{2n+1}$ diverge.

5.2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}$

Solution $\quad$ Let $a_n = \frac{1}{n5^n}$ $\quad$ and $a_{n+1} = \frac{1}{(n+1)5^{(n+1)}}$

Consider $\quad a_{n+1} < a_n$

Then $\quad \frac{1}{(n+1)5^{(n+1)}} < \frac{1}{n5^n} \quad$ decreasing

And consider $\quad \lim_{n\to\infty} a_n = 0 \quad$ or not

Since
$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{n5^n} \\ &= 0\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}$ converges.

6. Determine if $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!7^{(-n)}}$ absolutely converge or conditionally converge or diverge. (Using Ratio Test)

Solution

$$\text{Since} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!7^{(-n)}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 7^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-7)^n}{n!}$$

$$\text{Let} \qquad a_n = \frac{(-7)^n}{n!}$$

$$\text{Then} \qquad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{7 \cdot 7^n}{(n+1)n!} \cdot \frac{n!}{7^n}\right)$$

$$= \lim_{n\to\infty} \frac{7}{n+1}$$

$$= 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!7^{(-n)}}$ absolutely converge.

7. Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(3n+5)^2}$ absolutely converge or conditionally converge or diverge. (Using Comparison Test)

Solution Let $a_n = \frac{(-1)^{n+1}}{(3n+5)^2}$ and $b_n = \frac{1}{n^2}$

Consider
$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{(3n+5)^2} \right|$$
$$= \sum_{n=1}^{\infty} \frac{1}{(3n+5)^2}$$
$$\leq \frac{1}{n^2}$$

Since $a_n \leq b_n$

And $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p -$ series as $p = 2 < 1$

So, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{(3n+5)^2} \right|$ converge

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(3n+5)^2}$ absolutely converge.

8. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{3^{2n}}$.

Solution

Consider
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{2(n+1)}}\cdot\frac{3^{2n}}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{(2n+2)}}\cdot\frac{3^{2n}}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{3^2}\right| \\ &= \frac{|x|}{9}\end{aligned}$$

We get, converge to
$$\frac{|x|}{9} < 1$$
$$|x| < 9$$

So,
$$-9 < x < 9$$

Consider at

$x = 9;$ $\quad \sum_{n=0}^{\infty} \frac{9^n}{3^{2n}} = \sum_{n=0}^{\infty} 1$ $\quad$ diverges

$x = -9;$ $\quad \sum_{n=0}^{\infty} \frac{(-9)^n}{3^{2n}} = \sum_{n=0}^{\infty} (-1)^n$ $\quad$ diverges

Therefore, $\sum_{n=0}^{\infty} \frac{x^n}{3^{2n}}$ the convergence interval of x where $-9 < x < 9$.

9. Find the Taylor polynomial of $f(x) = \dfrac{1}{x}$ about the given point $x = 1$.

Solution

$$\begin{aligned}
&\text{Let} && f(x) = \frac{1}{x}, && f(1) = 1 \\
&\text{Then} && f'(x) = -\frac{1}{x^2}, && f'(1) = -1 \\
& && f''(x) = \frac{2}{x^3}, && f''(1) = 2 \\
& && f^{(3)}(x) = -\frac{6}{x^4}, && f^{(3)}(1) = -6 \\
& && f^{(4)}(x) = \frac{24}{x^5}, && f^{(4)}(1) = 24 \\
& && f^{(5)}(x) = -\frac{120}{x^6}, && f^{(5)}(1) = -120 \\
& && \vdots && \vdots \\
& && f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}, && f^{(n)}(1) = \frac{(-1)^n n!}{1^{n+1}} \\
& && && =(-1)^n n!
\end{aligned}$$

Therefore, the Taylor polynomial of f about $x = 1$ is

$$\begin{aligned}
&f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f^{(3)}(1)(x-1)^3 + \frac{1}{4!}f^{(4)}(1)(x-1)^4 \\
&\qquad + \frac{1}{5!}f^{(5)}(1)(x-1)^5 + \cdots + \frac{1}{n!}f^{(n)}(1)(x-1)^n \\
&= 1 + (-1)(x-1) + \frac{1}{2!}(2)(x-1)^2 + \frac{1}{3!}(-6)(x-1)^3 + \frac{1}{4!}(24)(x-1)^4 \\
&\qquad + \frac{1}{5!}(-120)(x-1)^5 + \cdots + \frac{1}{n!}(-1)^n(n!)(x-1)^n \\
&= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1^5) + \cdots + (-1)^n(x-1)^n
\end{aligned}$$

10. Find Maclaurin series of $f(x) = \ln(1-x)$.

Solution

$$
\begin{aligned}
&\text{Let} \quad & f(x) &= \ln(1-x), & f(0) &= \ln(1-0) = 0 \\
&\text{Then} \quad & f'(x) &= -\frac{1}{1-x} & & \\
& & &= (-1)(1-x)^{-1}, & f'(0) &= -1 \\
& & f''(x) &= (-1)(-1)(1-x)^{-2}(-1) & & \\
& & &= (-1)(1-x)^{-2}, & f''(0) &= -1 \\
& & f^{(3)}(x) &= (-1)(-2)(1-x)^{-3}(-1), & f^{(3)}(0) &= -2 \\
& & f^{(4)}(x) &= (-1)(-2)(-3)(1-x)^{-4}(-1)(-1), & f^{(4)}(0) &= -3! \\
& & &\vdots & &\vdots \\
& & f^{(n)}(x) &= (-1)^{2n-1}(n-1)!(1-x)^{-n} & & \\
& & &= -(n-1)!(1-x)^{-n}, & f^{(n)}(0) &= -(n-1)!
\end{aligned}
$$

$$
\text{Therefore} \quad \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!}x^n = -\sum_{n=1}^{\infty} \frac{x^n}{n}
$$

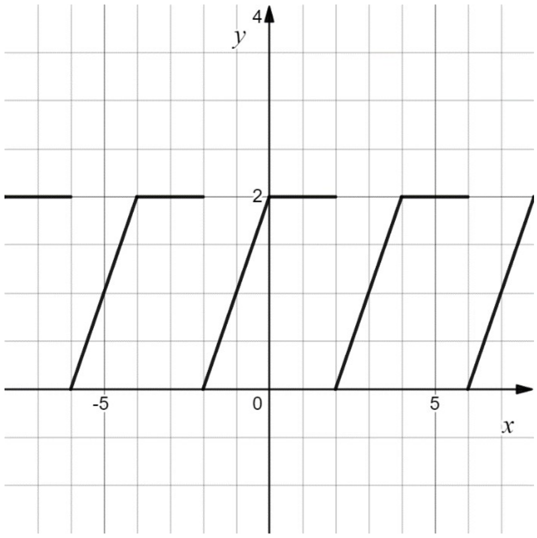
11. Find the power series of $f(x) = \dfrac{1}{\sqrt{4-x}}$.

Solution

$$
\begin{aligned}
\text{Since} \qquad f(x) &= \frac{1}{\sqrt{4-x}} \\
&= \frac{1}{2(1-\frac{x}{4})^{\frac{1}{2}}} \\
&= \frac{1}{2}\Big(1-\frac{x}{4}\Big)^{-\frac{1}{2}} \\
\text{And} \qquad (1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\
\text{Then} \qquad k &= -\frac{1}{2}, \qquad x = -\frac{x}{4} \\
\binom{-\frac{1}{2}}{n} &= \frac{\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{5}{2}\Big)\cdots\Big(-\frac{1}{2}-n+1\Big)}{n!} \\
&= \frac{(-1)^n\Big((1)(3)(5)\cdots(2n-1)\Big)}{2^n \cdot n!} \\
\text{Therefore,} \qquad \frac{1}{2}\Big(1-\frac{x}{4}\Big)^{-\frac{1}{2}} &= \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}\Big(-\frac{x}{4}\Big)^n \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \Big[\frac{(-1)^n\Big((1)(3)(5)\cdots(2n-1)\Big)}{2^n \cdot n!} \cdot \frac{(-1)^n x^n}{4^n}\Big] \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \Big[\frac{(1)(3)(5)\cdots(2n-1)x^n}{8^n \cdot n!}\Big] \\
&= \frac{1}{2}\Bigg[1+\frac{x}{8}+\frac{3x^2}{8^2\cdot 2!}+\frac{15x^3}{8^3\cdot 3!}+\cdots\Bigg]
\end{aligned}
$$

12. Draw graphs and Write Fourier series of $f(x) = \begin{cases} x+2, & -2 \le x < 0 \\ 2, & 0 < x \le 2 \end{cases}$

Solution Since $f(x) = \begin{cases} x+2, & -2 \le x < 0 \\ 2, & 0 < x \le 2 \end{cases}$ graph of $f(x)$ is



From

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

$f(x)$ has period $2L$ is 4 then $L = 2$

Find a_0

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) dx \\ &= \frac{1}{2} \int_{-2}^{2} f(x) dx \\ &= \frac{1}{2} \left(\int_{0}^{2} 2dx + \int_{-2}^{0} (x+2) dx \right) \\ &= \frac{1}{2} \left(2x \Big|_0^2 + \left(\frac{x^2}{2} + 2x \right) \Big|_{-2}^{0} \right) \\ &= \frac{1}{2} \left((4-0) + \Big(0 - (2-4) \Big) \right) \\ &= \frac{1}{2}(4+2) \\ a_0 &= 3 \end{aligned}$$

Find a_n

$$
\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\Big(\frac{n\pi x}{2}\Big)dx \\
&= \frac{1}{2}\Bigg(\int_{0}^{2} 2\cos\Big(\frac{n\pi x}{2}\Big)dx + \int_{-2}^{0}(x+2)\cos\Big(\frac{n\pi x}{2}\Big)dx\Bigg) \\
&= \frac{1}{2}\Bigg(\frac{4}{n\pi}\sin\big(\frac{n\pi x}{2}\big)\Big|_{0}^{2} + \Bigg(\frac{2}{n\pi}\Big((x+2)\sin\big(\frac{n\pi x}{2}\big)\Big)\Big|_{-2}^{0} - \frac{2}{n\pi}\int_{-2}^{0}\sin\big(\frac{n\pi x}{2}\big)dx\Bigg)\Bigg) \\
&= \frac{1}{2}\Bigg(\frac{4}{(n\pi)^2}\cos\Big(\frac{n\pi x}{2}\Big)\Big|_{-2}^{0}\Bigg) \\
&= \frac{1}{2}\Bigg(\frac{4}{(n\pi)^2}\Big(1-\cos(n\pi)\Big)\Bigg) \\
a_n &= -\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big)
\end{aligned}
$$

Find b_n

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\Big(\frac{n\pi x}{2}\Big)dx \\
&= \frac{1}{2}\Bigg(\int_{0}^{2} 2\sin\Big(\frac{n\pi x}{2}\Big)dx + \int_{-2}^{0}(x+2)\sin\Big(\frac{n\pi x}{2}\Big)dx\Bigg) \\
&= \frac{1}{2}\Bigg(-\frac{4}{n\pi}\cos\big(\frac{n\pi x}{2}\big)\Big|_{0}^{2} + \Bigg(-\frac{2}{n\pi}\Big((x+2)\cos\big(\frac{n\pi x}{2}\big)\Big)\Big|_{-2}^{0} + \frac{2}{n\pi}\int_{-2}^{0}\cos\big(\frac{n\pi x}{2}\big)dx\Bigg)\Bigg) \\
&= \frac{1}{2}\Bigg(-\frac{4}{n\pi}\big(\cos(n\pi)-1\big) + \Bigg(-\frac{4}{n\pi} + \frac{4}{(n\pi)^2}\sin\Big(\frac{n\pi x}{2}\Big)\Big|_{-2}^{0}\Bigg)\Bigg) \\
&= \frac{1}{2}\Bigg(-\frac{4}{n\pi}(-1)^n + \frac{4}{n\pi} - \frac{4}{n\pi}\Bigg) \\
b_n &= -\frac{2}{n\pi}(-1)^n
\end{aligned}
$$

Thus, Fourier series of $f(x)$ is

$$
f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\Bigg(-\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\Big(\frac{n\pi x}{2}\Big) - \frac{2}{n\pi}(-1)^n\sin\Big(\frac{n\pi x}{2}\Big)\Bigg)
$$

13. Write Fourier series of $f(x) = |x| - 1$ with period $-2 \leq x \leq 2.$

Solution

From

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$f(x)$ has period $2L$ is 4 then $L = 2$

Consider $f(x)$ is odd or even

$$f(-x) = |-x| - 1$$
$$= |x| - 1$$

So, $$f(-x) = f(x)$$

Thus, $f(x)$ is even then fourier series of $y = f(x)$ has $b_n = 0$

Find a_0

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2}\int_{-2}^{2} f(x)dx \\
&= \int_{0}^{2} (x-1)dx \\
&= \left(\frac{x^2}{2} - x\right)\Big|_0^2 \\
&= (2-2) - 0 \\
a_0 &= 0
\end{aligned}$$

Find a_n

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \int_{0}^{2} (x-1)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{2}{n\pi}\left((x-1)\sin\left(\frac{n\pi x}{2}\right)\right)\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 \\
a_n &= \frac{4}{(n\pi)^2}\Big((-1)^n - 1\Big)
\end{aligned}$$

Thus, Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\left(\frac{n\pi x}{2}\right)$$

14. Write Fourier series of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ -2, & 0 < x \leq 2 \end{cases}$

Solution

From

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$f(x)$ has period $2L$ is 4 then $L = 2$

Consider $f(x)$ is odd or even

$$f(-1) = 2$$

$$f(1) = -2$$

So, $$f(-1) = -f(1)$$

Thus, $f(x)$ is odd,then fourier series of $y = f(x)$ has $a_0 = a_n = 0$

Find b_n

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\ &= -2\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{4}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 \\ b_n &= \frac{4}{(n\pi)^2}\Big((-1)^n - 1\Big) \end{aligned}$$

Thus, Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\Big((-1)^n - 1\Big)\sin\left(\frac{n\pi x}{2}\right)$$

15. Determine Fourier series of $f(x) = \begin{cases} x+\pi, & -\pi \le x < 0 \\ x-\pi, & 0 < x \le \pi \end{cases}$

is $f(x) = -\sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$

and show the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$.

Solution

From

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$f(x)$ has period $2L$ is 2π then $L = \pi$

Consider $f(x)$ is odd or even

$$f(-\frac{\pi}{2}) = \frac{\pi}{2}$$
$$f(\frac{\pi}{2}) = -\frac{\pi}{2}$$

So, $$f(-x) = -f(x)$$

Thus, $f(x)$ is odd, then fourier series of $y = f(x)$ has $a_0 = a_n = 0$

Find b_n

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{\pi}\int_0^{\pi}(x-\pi)\sin(nx)dx \\ &= -\frac{2}{n\pi}(x-\pi)\cos(nx)\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos(nx)dx \\ &= 0 - \frac{2}{n} + \frac{2}{n^2\pi}\sin(n\pi)\Big|_0^{\pi} \\ b_n &= -\frac{2}{n} \end{aligned}$$

Thus, Fourier series of $f(x)$ is

$$f(x) = -\sum_{n=1}^{\infty}\frac{2}{n}\sin(nx)$$

Consider at $x = \dfrac{\pi}{2}$, then

$$\begin{aligned}
f\left(\frac{\pi}{2}\right) &= -\sum_{n=1}^{\infty} \frac{2}{n} \sin\left(\frac{n\pi}{2}\right) \\
\frac{\pi}{2} - \pi &= -2\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \\
-\frac{\pi}{2} &= -2\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \\
\frac{\pi}{4} &= \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \\
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + \cdots
\end{aligned}$$

Thus, $$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

16. Determine Fourier series of $f(x) = \begin{cases} x+2, & -2 < x \leq 0 \\ 0, & 0 < x \leq 2 \end{cases}$

and find convergent values when $x = 0$.

Solution

From

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

$f(x)$ has period $2L$ is 4 then $L = 2$

Find a_0

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{2} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{0} (x+2)dx \\ &= \frac{1}{2}\left(\frac{x^2}{2} + 2x\right)\Big|_{-2}^{0} \\ &= \frac{1}{2}\Big(0 - (2-4)\Big) \\ &= \frac{1}{2}(2) \\ a_0 &= 1 \end{aligned}$$

Find a_n

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{1}{2}\int_{-2}^{0} (x+2)\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{1}{2}\left(\frac{2}{n\pi}\left((x+2)\sin\left(\frac{n\pi x}{2}\right)\right)\Big|_{-2}^{0} - \frac{2}{n\pi}\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right)dx\right) \\ &= \frac{1}{2}\left(\frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0}\right) \\ &= \frac{1}{2}\left(\frac{4}{(n\pi)^2}\Big(1 - \cos(n\pi)\Big)\right) \\ a_n &= -\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big) \end{aligned}$$

Find b_n

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} (x+2)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\Bigg(-\frac{2}{n\pi}\left((x+2)\cos\left(\frac{n\pi x}{2}\right)\right)\Bigg|_{-2}^{0} + \frac{2}{n\pi}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx\Bigg) \\
&= \frac{1}{2}\Bigg(-\frac{4}{n\pi} + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0}\Bigg) \\
&= \frac{1}{2}\Bigg(-\frac{4}{n\pi}\Bigg) \\
b_n &= -\frac{2}{n\pi}
\end{aligned}
$$

Thus, Fourier series of $f(x)$ is

$$
f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\Bigg(-\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg)
$$

Consider at $x = 0$ is discontinuous point.

Therefore, convergent values is $\dfrac{f(0^+) + f(0^-)}{2} = \dfrac{0+2}{2} = 1$