Module 1 Part2

Functions of several variables

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4. Partial Derivatives
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Functions of several variables

Definition: A **Real-Valued Function of Two Variables** denoted $z = f(x, y)$ is a rule that assigns to each point $(x, y) \in D(f)$ a unique real number $f(x, y) \in R(f)$. The **Domain of f** denoted $D(f)$ is the set of points (x, y) for which f is defined. The **Range of f** denoted $R(f)$ is the set defined as $R(f) = \{f(x, y) : (x, y) \in D(f)\}$.

Definition: A **Real-Valued Function of Three Variables** denoted $w = f(x, y, z)$ is a rule that assigns to each point $(x, y, z) \in D(f)$ a unique real number $f(x, y, z) \in R(f)$. The **Domain of f** denoted $D(f)$ is the set of points (x, y, z) for which f is defined. The **Range of f** denoted $R(f)$ is the set defined as $R(f) = \{f(x, y, z) : (x, y, z) \in D(f)\}$.

Example

1. Let $f(x,y) = x^2 + 3y$. Evaluate the following:

   a. $f(1,2)$ b. $f(-1,3)$

2. Find the domain of

   a. $f(x,y) = x^2 + 3y$ b. $f(x,y) = \dfrac{y}{\sqrt{x - y^2}}$

Exercise Find the domain of the following functions

1. $f(x,y) = \sqrt{36 - 9x^2 - 9y^2}$
2. $f(x,y) = \ln(x + y)$
3. $f(x,y,z) = (\sin xy)\left(\sqrt{y - z}\right)$
4. $f(x,y,z) = \dfrac{\sqrt{x + y}}{x^2 - z}$

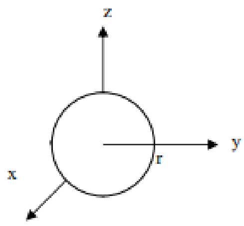
Graphs of Quadratic Surfaces

The important quadratic surfaces you should know are the following:

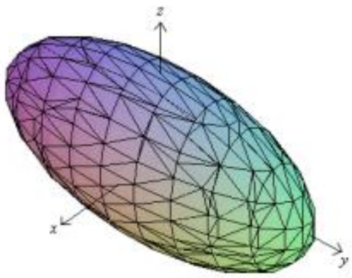
1. Sphere
2. Ellipsoid
3. Hyperboloid of one sheet
4. Hyperboloid of two sheets
5. Elliptic cone
6. Elliptic paraboloid
7. Hyperbolic paraboloid

Sphere

The equation of a sphere centered at (0,0,0) with radius r has the form: $x^2 + y^2 + z^2 = r^2$.



Ellipsoid



The equation of an ellipsoid centered at $(0,0,0)$ has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b, c are some constants.

Hyperboloid of one sheet

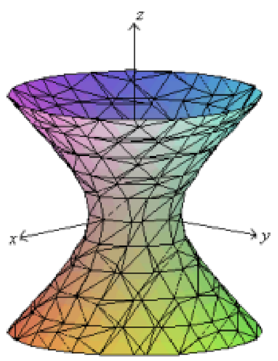
The equation of a hyperboloid of one sheet centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



Hyperboloid of two sheets

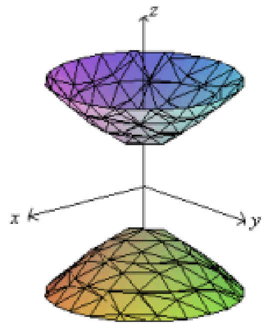
The equation of a hyperboloid of two sheets centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along y-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

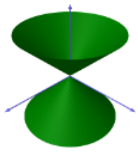
Lie along x-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



Elliptic cone

The equation of an elliptic cone centered at $(0,0,0)$ has the following forms:



Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

where a, b, c are some constants.

Elliptic paraboloid

The equation of a paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis and open on positive z: $$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

open on negative z: $$z=-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

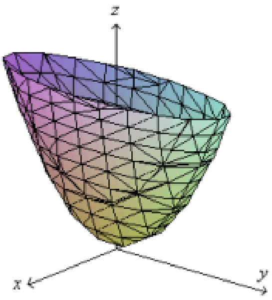
Lie along y-axis and open on positive y: $$y=\frac{x^2}{a^2}+\frac{z^2}{c^2}$$

open on negative y: $$y=-\frac{x^2}{a^2}-\frac{z^2}{c^2}$$

Lie along x-axis and open on positive x: $$x=\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

open on negative x: $$x=-\frac{y^2}{b^2}-\frac{z^2}{c^2}$$

where a, b, c are some constants.



Hyperbolic paraboloid

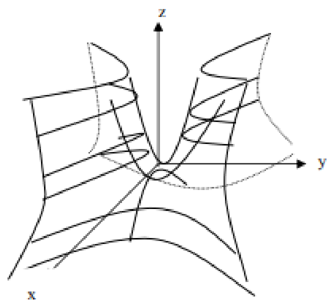
The equation of a hyperbolic paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis : $z=-\frac{x^2}{a^2}+\frac{y^2}{b^2}$, $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

Lie along y-axis : $y=-\frac{x^2}{a^2}+\frac{z^2}{c^2}$, $y=\frac{x^2}{a^2}-\frac{z^2}{c^2}$

Lie along x-axis : $x=-\frac{y^2}{b^2}+\frac{z^2}{c^2}$, $x=\frac{y^2}{b^2}-\frac{z^2}{c^2}$

where a,b,c are some constants.



Example Specify the quadratic surfaces for the following equation.

quadratic surfaces	Name of graph
$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{2}=1$	
$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{36}=1$	
$x^2+\frac{y^2}{4}=\frac{z}{2}$	
$16x^2+4y^2-z^2=0$	
$\frac{(x-1)^2}{4}-\frac{(y-2)^2}{9}-\frac{(z-1)^2}{16}=1$	
$x^2-2y^2+2x+4y-z+2=0$	

<u>Example</u> Specify the type and draw the surface of $-x^2 - 4y^2 = -z$ and find the traces after cutting the surface by each plane.

Limits and Continuity of Functions of Several Variables

Limits

We say that the limit as (x, y) approaches (a, b) of f is L and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if the value of $f(x, y)$ tends to L along all possible trajectories that approach (a, b).

Consequently, if we can find two trajectories approaching (a, b) for which the limits are different, we conclude that the limit $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

The properties of the limit

Suppose that $\lim f = L$ and $\lim g = M$, then

1. $\lim(f \pm g) = L \pm M$.
2. $\lim f \cdot g = L \cdot M$.
3. $\lim \frac{f}{g} = \frac{L}{M}$, $M \neq 0$.
4. $\lim(f)^a = L^a$.

Example Consider the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2 - y^2}{x^2 + 2y^2}$ along y=x and y=x^2.

Exercise If $f(x, y) = \dfrac{x^3 y}{x^6 + y^2}$, dose $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist?

Continuity

A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example

1. Determine the set of points at which the function is continuous, where

$$f(x,y)=\frac{x^3-y^3}{x^2+y^2}$$

2. Let $f(x,y)=\begin{cases}\dfrac{x+3y+1}{x-1}, & (x,y)\neq(0,1)\\ -4, & (x,y)=(0,1)\end{cases}$, Is the function continuous at (0,1) ?

Exercise

Let $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2-y^2}, & (x,y)\neq(0,0)\\ 1 & ,(x,y)=(0,0)\end{cases}$, Is the function continuous at (0,0) ?

Partial Derivatives

- **the partial derivative of f(x,y) with respect to x and y are**

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = (\frac{\partial f}{\partial x})_y = f_x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = (\frac{\partial f}{\partial y})_x = f_y$$

- **for general n-variable**

$$\frac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots x_i + \Delta x_i, \ldots x_n) - f(x_1, x_2, \ldots x_i, \ldots x_n)}{\Delta x_i}$$

<u>Example</u> Find the partial derivative of $f(x, y) = 2y^2 - 3xy$ by definition.

The easy way how to calculate the partial derivative

To find a partial derivative of with respect to one variable, we consider another input variable as a constant. For example, to find the partial derivative of with respect to x, we consider y as a constant.

<u>Example</u> Find the partial derivative of $f(x, y) = 2y^2 - 3xy$.

Find the partial derivative of $f(x, y) = e^{x+2y} \log(3y - x)$

<u>Exercise</u> Find the partial derivative of the following functions

1. $f(x,y)=\dfrac{2x+3y}{5x-4y}$
2. $f(x,y)=\dfrac{x^2}{y}+\sqrt{x-2y}$
3. $f(x,y)=y\sin(xy)$
4. $f(x_1,x_2,x_3,x_4)=x_1\cos(x_2+2x_3)+e^{3x_2}x_4\ln x_3$

Partial Derivatives of Higher Order

Second Order

Taking a second-order partial derivative means taking a partial derivative of the first partial derivative. If $z = f(x, y)$, then

$$f_{xx} = f_{xx}(x, y) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{xy} = f_{xy}(x, y) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yx} = f_{yx}(x, y) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$f_{yy} = f_{yy}(x, y) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Example

Find all the second order partial derivatives of $f(x,y) = x^5 - x^2 y + y^3 + 2x$

Exercise

1. Find all the second order partial derivatives of $f(x,y) = xye^{-y^2}$
2. Find $f_{xy}(x,y)$ where $f(x,y) = \sin(xy) + xe^y$
3. Find $f_{xy}(x,y)$ where $f(x,y) = \sin(xy) + xe^y$ at (0,1)

Higher Order

Partial Derivatives of Higher Order

Partial derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ may be applied several times such as

$$\frac{\partial^3 f}{\partial y \partial x^2} \quad \text{or} \quad f_{xxy}, \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} \quad \text{or} \quad f_{xxyy}.$$

Example

Let $f(x,y,z) = z^3 y^2 \ln(x)$. Find f_{xxyzz}

Exercise

Find $f_{xxy} = \dfrac{\partial^3 f}{\partial y \partial x^2}$ for $f(x,y) = e^{xy}$

Chain Rules for Partial Derivatives

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

1. Let $z = x^2 - y^2$ $, x = \cos(t)$ $, y = \sin(t)$, find $\dfrac{dz}{dt}$

2. Let $w = f(x, y, z), x = g(t), y = h(t), z = j(t)$, find $\dfrac{dw}{dt}$

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example

1. Let $z = x^2 + y^2$ $, x = s - t$ $, y = s + t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

Exercise

1. Let $z(x,y)=x^2y \quad ,x=t^2 \quad ,y=t^3$. Find $\frac{dz}{dt}$
2. Let $f(x,y)=xy+\ln(x^2+y^2) \quad ,x=\cos\theta \quad ,y=\sin\theta$. Find $\frac{df}{d\theta}$
3. Let $w=e^u\sin v \quad ,u=st^2 \quad ,v=s^2t$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$
4. Let $z=e^{xy} \quad ,x=2u+v \quad ,y=\frac{u}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$
5. Let $f=ue^{u+v} \quad ,u=\ln x^2y \quad ,v=2x$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Implicit Differentiation

Implicit Differentiation

Let y be a function defined implicitly in term of x.

We can find $\dfrac{dy}{dx}$ by following this procedure.

1. Set up $F(x, y) = 0.$
2. Differentiate $F(x, y) = 0$ with respect to x on both sides:

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0.$$

Then, we get

$$0 = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

Therefore, $$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

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<u>Example</u>

1. Suppose that $y = f(x)$ and $x^3 + y^3 = 6xy$, find $\dfrac{dy}{dx}$

2. Suppose that $y = f(x)$ and $2x\ln y - y^2 \sin x = 0$, find $\dfrac{dy}{dx}$

If we suppose that $z = f(x,y)$ be an implicitly function where $F(x,y,z)=0$. By Chain Rule, we have that:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Example

1. Suppose that $z = f(x,y)$ and $2xz^2 + y\sin z = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. Suppose that $z = f(x,y)$ and $xy^2 + z^3 + \sin(xyz) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Exercise

1. Let $z = f(x,y)$ and $\sin(x+y)+\sin(y+z)+\sin(x+z)=0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
2. Let $w = f(x,y,z)$ and $x^2y - xe^w + 2wz = 0$, find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ (Hint consider Chain rule of F(x,y,z)=0)

Implicit differentiations of a system of equations

We need to use Jacobian to solve it

<u>Theorem</u> Let u and v be functions x and y.

Now, we have $F(u,v,x,y)=0$ and $G(u,v,x,y)=0$.

By Cramer's rule: If $F_uG_v - F_vG_u \neq 0$, then

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}, \qquad u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

<u>Example</u>

Let u and v be functions x and y such that $u+2uv=x^2+xy$ and $v^2-u=2x-y^2$.

Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

<u>Exercise</u>

1. Let u and v be functions x and y such that $xu^2+v=y^3$ and $2yu-xv^3=4x$.

   Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Total Differential

Definition: The total differential of $f(x, y)$, denoted by df , is defined to be

$$df = f_x(x, y)dx + f_y(x, y)dy$$

Example

1. Find df where $f(x, y) = \sin y + x\ln y$

2. Let $f(x, y) = x^2 + xy + y^2$, find df and Δf where $x = 2, y = 3, \Delta x = 0.2\ and\ \Delta x = -0.1$

Application of total difference/differential

We know that $\Delta f \approx df$ when $\Delta x, \Delta y \to 0$.

Thus, $f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x(x, y)dx + f_y(x, y)dy$.

i.e. $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)dx + f_y(x, y)dy$.

Example

1. Estimate $\sqrt{2.01^2 + 3.99^2}$

2. Estimate $(6.04)(3.1)(2.96)$

Exercise

Estimate the volume of the cone if the radius is reduced from 3.15 to 3 cm and the height is reduced from 9 to 8.89 cm.

Max and Min of multivariable functions

Definition: A point (a,b) in domain of f is called a critical point of f if either both $f_x(a,b)=0$ and $f_y(a,b)=0$ or at least one partial derivative of f does not exist at (a,b).

The test of local max and local min

Theorem: Let $z=f(x,y)$. Suppose f has continuous second partial derivatives in an open disk around (x_0,y_0) such that both $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$. There are three possibilities:

1. If $f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2>0$,
   - and $f_{xx}(x_0,y_0)>0$, then f has a local min at (x_0,y_0).
   - and $f_{xx}(x_0,y_0)<0$, then f has a local max at (x_0,y_0).
2. If $f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2<0$, then the point (x_0,y_0) is a saddle point of f.
3. If $f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2=0$, then the test fails. We have no conclusion about the point (x_0,y_0). It may be the local max, the local min, the saddle point or none of these.

Example

Find the width, length, and height of the rectangular box with the maximum volume when the sum of all 12 edges is 60 cm.