Exercise: Vector-Valued Function (Solution)

1. Let C be a smooth curve whose parametric equation of this form:

$$x = 3t^3, \quad y = te^{-2t}, \quad z = \sin t$$

a) Find the position vector $\vec{r}(t)$ of this curve.

Solution

Since $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}; \quad t_0 \le t \le t_1$

Then, $\vec{r}(t) = 3t^3\vec{i} + te^{-2t}\vec{j} + \sin(t)\vec{k}$

Therfore, the position vector $\vec{r}(t)$ is $3t^3\vec{i} + te^{-2t}\vec{j} + \sin(t)\vec{k}$.

b) Find $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$

Solution

Since $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

Then $\vec{r}\,'(t)$

$$\vec{r}\,'(t) = 9t^2\vec{i} + (-2te^{-2t} + e^{-2t})\vec{j} + \cos(t)\vec{k}.$$

and $\vec{r}\,''(t)$

$$\vec{r}\,''(t) = 18t\vec{i} + (4te^{-2t} - 4e^{-2t})\vec{j} - \sin(t)\vec{k}.$$

2. Find the arc length of curve $\vec{r}(t) = 3\cos(2t)\vec{i} - 3\sin(2t)\vec{j} + 8t\vec{k}, \quad 0 \le t \le 2\pi.$

Solution

Find $\quad \vec{r}\,'(t) = -6\sin(2t)\vec{i} - 6\cos(2t)\vec{j} + 8\vec{k}$

and
$$
\begin{aligned}
\|\vec{r}\,'(t)\| &= \sqrt{\big(-6\sin(2t)\big)^2 + \big(-6\cos(2t)\big)^2 + 8^2} \\
&= \sqrt{36\sin^2(2t) + 36\cos^2(2t) + 64} \\
&= \sqrt{36\big(\sin^2(2t) + \cos^2(2t)\big) + 64} \\
&= \sqrt{36 + 64} \\
&= \sqrt{100} \\
&= 10
\end{aligned}
$$

Find arc length by the formula

$$
\begin{aligned}
s(t) &= \int_0^{2\pi} \left\| \frac{d\vec{r}}{dt} \right\| dt \quad \text{for} \quad a \le t \le b \\
&= \int_0^{2\pi} 10\, dt \\
&= 10t \Big|_0^{2\pi} \\
&= 20\pi \quad \text{unit}
\end{aligned}
$$

Thus, the arc length of the curve $\vec{r}(t) = 3\cos(2t)\vec{i} - 3\sin(2t)\vec{j} + 8t\vec{k}$ is 20π unit.

3. Find the arc length parameterization of the curve $\vec{r}(t) = \cos(3t)\vec{i} + \sin(3t)\vec{j} + 6t^{\frac{3}{2}}\vec{k}$.

Solution

Set $\quad \vec{r}(t) = \cos(3t)\vec{i} + \sin(3t)\vec{j} + 6t^{\frac{3}{2}}\vec{k} = \langle \cos(3t), \sin(3t), 6t^{\frac{3}{2}} \rangle$

Then, $\quad \vec{r}\,'(t) = -3\sin(3t)\vec{i} + 3\cos(3t)\vec{j} + 9\sqrt{t}\vec{k}$

and

$$\begin{aligned}
\|\vec{r}\,'(t)\| &= \sqrt{\big(-3\sin(3t)\big)^2 + \big(3\cos(3t)\big)^2 + \big(9\sqrt{t}\big)^2} \\
&= \sqrt{9\sin^2(3t) + 9\cos^2(3t) + 81t} \\
&= \sqrt{9\big(\sin^2(3t) + \cos^2(3t) + 9t\big)} \\
&= 3\sqrt{9t+1}
\end{aligned}$$

Find arc length $s(t)$

$$\begin{aligned}
s(t) &= \int_0^t \|\vec{r}\,'(\tau)\| d\tau \\
&= \int_0^t 3\sqrt{9\tau + 1} d\tau \\
&= \frac{2}{9}(9\tau+1)^{\frac{3}{2}}\Big|_0^t \\
&= \frac{2}{9}\Big((9t+1)^{\frac{3}{2}} - 1\Big)
\end{aligned}$$

Simplify t in terms of s is $s = \frac{2}{9}\Big((9t+1)^{\frac{3}{2}} - 1\Big)$

$$\begin{aligned}
s &= \frac{2}{9}\Big((9t+1)^{\frac{3}{2}} - 1\Big) \\
\frac{9}{2}s &= (9t+1)^{\frac{3}{2}} - 1 \\
\frac{9}{2}s + 1 &= (9t+1)^{\frac{3}{2}} \\
\left(\frac{9}{2}s + 1\right)^{\frac{2}{3}} &= 9t + 1 \\
\left(\frac{9}{2}s + 1\right)^{\frac{2}{3}} - 1 &= 9t \\
t &= \frac{1}{9}\left[\left(\frac{9}{2}s + 1\right)^{\frac{2}{3}} - 1\right]
\end{aligned}$$

substitute $t = \frac{1}{9}\left[\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right]$ in $\vec{r}(t) = \cos(3t)\vec{i} + \sin(3t)\vec{j} + 6t^{\frac{3}{2}}\vec{k}$ we get,

$$\vec{r}(s) = \cos\left(\frac{1}{3}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)\right)\vec{i} + \sin\left(\frac{1}{3}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)\right)\vec{j}$$
$$+ \frac{2}{9}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)^{\frac{3}{2}}\vec{k}$$

Therefore $\vec{r}(t)$ can be reparametrized in terms of s as

$$\vec{r}(s) = \cos\left(\frac{1}{3}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)\right)\vec{i} + \sin\left(\frac{1}{3}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)\right)\vec{j}$$
$$+ \frac{2}{9}\left(\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - 1\right)^{\frac{3}{2}}\vec{k}.$$

4. Find the equation of the line tangent to $\vec{r}(t)$ at t_0, then sketch the graph of $\vec{r}(t)$ and draw the tangent vector $\vec{r}\,'(t_0)$.

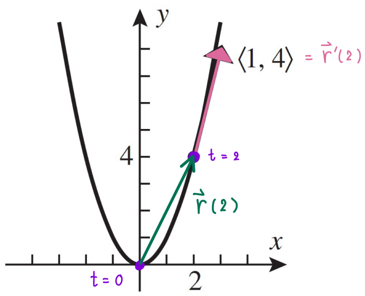
   a) $\vec{r}(t) = \langle t, t^2\rangle; \quad t_0 = 2$

      Solution

      We have $\quad \vec{r}\,'(t) = \langle 1, 2t\rangle$

      and $\quad \vec{r}\,'(t_0) = \vec{r}\,'(2) = \langle 1, 4\rangle$

      Therefore, the graph of $\vec{r}(t)$ and the tangent vector are shown as follows



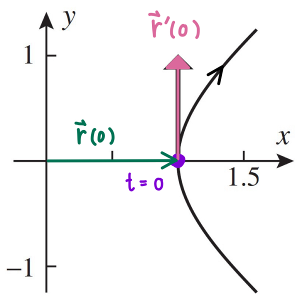
b) $\vec{r}(t) = \sec(t)\vec{i} + \tan(t)\vec{j}; \quad t_0 = 0$

Solution

We have $\quad \vec{r}\,'(t) = \sec(t)\tan(t)\vec{i} + (\sec t)^2\vec{j}$

and
$$\begin{aligned}\vec{r}\,'(t_0) &= \vec{r}\,'(0)\\ &= 1(0)\vec{i} + 1^2\vec{j}\\ &= \vec{j}\end{aligned}$$

Therefore, the graph of $\vec{r}(t)$ and the tangent vector are shown as follows



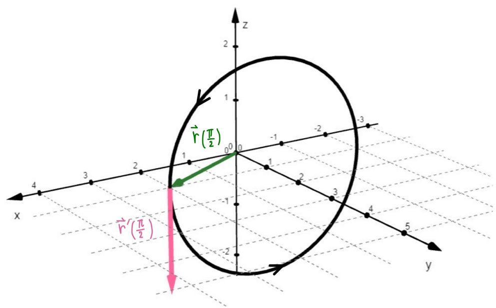
c) $\vec{r}(t) = 2\sin(t)\vec{i} + \vec{j} + 2\cos(t)\vec{k}; \quad t_0 = \dfrac{\pi}{2}$

Solution

We have
$$\begin{aligned}\vec{r}\,'(t) &= 2\cos(t)\vec{i} + 0\vec{j} + 2(-\sin(t))\vec{k}\\ &= 2\cos(t)\vec{i} - 2\sin(t)\vec{k}\end{aligned}$$

and
$$\begin{aligned}\vec{r}\,'(t_0) &= \vec{r}\,'(\frac{\pi}{2})\\ &= 2(0)\vec{i} - 2(1)\vec{k}\\ &= -2\vec{k}\end{aligned}$$

Therefore, the graph of $\vec{r}(t)$ and the tangent vector are shown as follows



5. Find a vector equation of the line tangent to the graph of $\vec{r}(t)$ at the point P_0 on the curve.

a) $\vec{r}(t) = (3t-1)\vec{i} + \sqrt{3t+4}\vec{j}; \quad P_0(-1,2)$

Solution

Consider $\vec{r_0} = \vec{r}(t_0) = (3t_0 - 1)\vec{i} + \sqrt{3t_0+4}\vec{j}$

We get, component of $\vec{r}(t_0)$;

$$x(t_0) = 3t_0 - 1$$
$$y(t_0) = \sqrt{3t_0+4}$$

Find t_0 of $\vec{r}(t)$ at $P_0 = (-1,2)$

From $x(t_0)$;
$$\begin{aligned} 3t_0 - 1 &= -1 \\ 3t_0 &= 0 \\ t_0 &= 0 \end{aligned}$$

$y(t_0)$;
$$\begin{aligned} \sqrt{3t_0+4} &= 2 \\ 3t_0 + 4 &= 4 \\ 3t_0 &= 0 \\ t_0 &= 0 \end{aligned}$$

Then, $t_0 = 0$,

Thus, $\vec{r_0} = \vec{r}(t_0) = \vec{r}(0) = -1\vec{i} + 2\vec{j}$

From
$$\begin{aligned} \vec{v_0} &= \vec{r_0}\,'(t_0) \\ &= 3\vec{i} + \frac{1}{2\sqrt{3t_0+4}}(3)\vec{j} \end{aligned}$$

as $t_0 = 0$; then,
$$\begin{aligned} \vec{v_0} &= \vec{r}\,'(0) \\ &= 3\vec{i} + \frac{3}{2\sqrt{4}}\vec{j} \\ &= 3\vec{i} + \frac{3}{4}\vec{j} \end{aligned}$$

We get,
$$\begin{aligned} \vec{r} &= \vec{r_0} + t\vec{v_0} \\ &= (-\vec{i} + 2\vec{j}) + t(3\vec{i} + \frac{3}{4}\vec{j}) \\ &= (3t-1)\vec{i} + (\frac{3}{4}t + 2)\vec{j} \end{aligned}$$

Thus, the vector equation of the line tangent is $(3t-1)\vec{i} + (\frac{3}{4}t+2)\vec{j}$.

b) $\vec{r}(t) = 4\cos(t)\vec{i} - 3t\vec{j}; \quad P_0(2, -\pi)$

Solution

Consider
$$\vec{r_0} = \vec{r}(t_0)$$
$$= 4\cos(t_0)\vec{i} - 3t_0\vec{j}$$

We get, component of $\vec{r}(t_0)$;

$$x(t_0) = 4\cos(t_0)$$
$$y(t_0) = -3t_0$$

Find t_0 of $\vec{r}(t)$ at $P_0 = (2, -\pi)$

From $x(t_0)$;
$$\begin{aligned} 4\cos(t_0) &= 2 \\ \cos(t_0) &= \frac{2}{4} \\ t_0 &= \frac{\pi}{3} \end{aligned}$$

$y(t_0)$;
$$\begin{aligned} -3t_0 &= -\pi \\ t_0 &= \frac{\pi}{3} \end{aligned}$$

Then,
$$t_0 = \frac{\pi}{3}$$

Thus,
$$\begin{aligned} \vec{r_0} &= \vec{r}(t_0) \\ &= \vec{r}\left(\frac{\pi}{3}\right) \\ &= 4\cos\left(\frac{\pi}{3}\right)\vec{i} - 3\left(\frac{\pi}{3}\right)\vec{j} \\ &= 4\left(\frac{1}{2}\right)\vec{i} - \pi\vec{j} \\ &= 2\vec{i} - \pi\vec{j} \end{aligned}$$

$$
\begin{aligned}
\text{From} \qquad && \vec{v}(t) &= \vec{r}\,'(t) \\
&& &= -4\sin(t)\vec{i} - 3\vec{j} \\
\text{as} \qquad && t_0 &= \frac{\pi}{3}, \\
\text{Then,} \qquad && \vec{v_0} &= -4\sin(\frac{\pi}{3})\vec{i} - 3\vec{j} \\
&& &= -4\Big(\frac{\sqrt{3}}{2}\Big)\vec{i} - 3\vec{j} \\
&& &= -2\sqrt{3}\vec{i} - 3\vec{j} \\
\text{We get} \qquad && \vec{r} &= \vec{r_0} + t\vec{v_0} \\
&& &= (2\vec{i} - \pi\vec{j}) + t(-2\sqrt{3}\vec{i} - 3\vec{j}) \\
&& &= (-2\sqrt{3}t + 2)\vec{i} + (3t - \pi)\vec{j}
\end{aligned}
$$

Thus, the vector equation of the line tangent is $(-2\sqrt{3}t + 2)\vec{i} + (3t - \pi)\vec{j}$.

c) $\vec{r}(t) = t^2\vec{i} - \dfrac{1}{t+3}\vec{j} + (4-t^2)\vec{k}; \quad P_0(4,-1,0)$

Solution

Consider
$$\begin{aligned}\vec{r_0} &= \vec{r}(t_0)\\ &= t_0^2\vec{i} - \frac{1}{t_0+3}\vec{j} + (4-t_0^2)\vec{k}\end{aligned}$$

We get, component of $\vec{r}(t_0)$;

$$\begin{aligned}x(t_0) &= t_0^2\\ y(t_0) &= -\frac{1}{t_0+3}\\ z(t_0) &= 4 - t_0^2\end{aligned}$$

Find t_0 of $\vec{r}(t)$ at $P_0 = (4,-1,0)$

From $x(t_0)$;
$$t_0^2 = 4 = \pm 2$$

$y(t_0)$;
$$\begin{aligned}-\frac{1}{t_0+3} &= -1\\ 1 &= t_0+3\\ t_0 &= -2\end{aligned}$$

$z(t_0)$;
$$\begin{aligned}4 - t_0^2 &= 0\\ t_0^2 &= 4\\ t_0 &= \pm 2\end{aligned}$$

Since $x(t_0)$ and $z(t_0)$ we get, $t_0 = \pm 2$

and $y(t_0)$ we get, $t_0 = -2$

So, $t_0 = -2$

$$
\begin{aligned}
\text{Thus,} \qquad \vec{r_0} &= \vec{r}(t_0) \\
&= \vec{r}(-2) \\
&= (-2)^2\vec{i} - \frac{1}{-2+3}\vec{j} + (4-(-2)^2)\vec{k} \\
&= 4\vec{i} + \vec{j} \\
\text{From} \qquad \vec{v}(t) &= \vec{r}\,'(t) \\
&= 2t\vec{i} - \left(\frac{-1}{(t+3)^2}\right)\vec{j} + (-2t)\vec{k} \\
&= 2t\vec{i} + \left(\frac{1}{(t+3)^2}\right)\vec{j} - 2t\vec{k} \\
\text{as} \qquad t_0 &= -2, \\
\text{Then,} \qquad \vec{v_0} &= 2(-2)\vec{i} + \left(\frac{1}{(-2+3)^2}\right)\vec{j} - 2(-2)\vec{k} \\
&= -4\vec{i} + \vec{j} - 4\vec{k} \\
\text{We get} \qquad \vec{r} &= \vec{r_0} + t\vec{v_0} \\
&= (4\vec{i} + \vec{j}) + t(-4\vec{i} + \vec{j} - 4\vec{k})
\end{aligned}
$$

Thus, the vector equation of the line tangent is $(4\vec{i} + \vec{j}) + t(-4\vec{i} + \vec{j} - 4\vec{k})$.

6. Let $\vec{r}(t)$ be the position vector of particle moving in the plane. Find the velocity, acceleration, and speed at an arbitrary time t.

   a) $\vec{r}(t) = 3\cos(t)\vec{i} + 3\sin(t)\vec{j}; \quad t = \dfrac{\pi}{3}$

   Solution

$$
\begin{aligned}
\vec{v}(t) &= \vec{r}\,'(t) \\
&= -3\sin(t)\vec{i} + 3\cos(t)\vec{j} \\
\vec{v}\Big(\frac{\pi}{3}\Big) &= -3\Big(\frac{\sqrt{3}}{2}\Big)\vec{i} + 3\Big(\frac{1}{2}\Big)\vec{j} \\
&= -\frac{3}{2}\sqrt{3}\vec{i} + \frac{3}{2}\vec{j} \\
\Big\|\vec{v}\Big(\frac{\pi}{3}\Big)\Big\| &= \sqrt{\Big(\frac{-3\sqrt{3}}{2}\Big)^2 + \Big(\frac{3}{2}\Big)^2} \\
&= \sqrt{\frac{27}{4} + \frac{9}{4}} \\
&= \sqrt{9} \\
&= 3 \\
\vec{a}(t) &= \vec{r}\,''(t) \\
&= -3\cos(t)\vec{i} - 3\sin(t)\vec{j} \\
\vec{a}\Big(\frac{\pi}{3}\Big) &= -3\Big(\frac{1}{2}\Big)\vec{i} - 3\Big(\frac{\sqrt{3}}{2}\Big)\vec{j} \\
&= -\frac{3}{2}\vec{i} - \frac{3\sqrt{3}}{2}\vec{j}
\end{aligned}
$$

   Thus, the velocity is $\vec{v}\Big(\dfrac{\pi}{3}\Big) = -\dfrac{3}{2}\sqrt{3}\vec{i} + \dfrac{3}{2}\vec{j}$,
   the acceleration is $\vec{a}\Big(\dfrac{\pi}{3}\Big) = -\dfrac{3}{2}\vec{i} - \dfrac{3\sqrt{3}}{2}\vec{j}$,and speed is $\Big\|\vec{v}\Big(\dfrac{\pi}{3}\Big)\Big\| = 3$.

b) $\vec{r}(t) = e^t\vec{i} + e^{-t}\vec{j}; \quad t = 0$

Solution

$$\begin{aligned}
\vec{v}(t) &= e^t\vec{i} - e^{-t}\vec{j} \\
\vec{v}(0) &= \vec{i} - \vec{j} \\
\|\vec{v}(0)\| &= \sqrt{1+1} \\
&= \sqrt{2} \\
\vec{a}(t) &= e^t\vec{i} + e^{-t}\vec{j} \\
\vec{a}(0) &= \vec{i} + \vec{j}
\end{aligned}$$

Thus, the velocity is $\vec{v}(0) = \vec{i} - \vec{j}$, the acceleration is $\vec{a}(0) = \vec{i} + \vec{j}$, and speed is $\|\vec{v}(0)\| = \sqrt{2}$.

c) $\vec{r}(t) = t\vec{i} + \frac{1}{2}t^2\vec{j} + \frac{1}{3}t^3\vec{k}; \quad t = 2$

Solution

$$\begin{aligned}
\vec{v}(t) &= \vec{i} + \frac{1}{2}(2t)\vec{j} + \frac{1}{3}(3t^2)\vec{k} \\
&= \vec{i} + t\vec{j} + t^2\vec{k} \\
\vec{v}(2) &= \vec{i} + 2\vec{j} + 4\vec{k} \\
\|\vec{v}(2)\| &= \sqrt{1^2 + 2^2 + 4^2} \\
&= \sqrt{21} \\
\vec{a}(t) &= 0\vec{i} + \vec{j} + 2t\vec{k} \\
\vec{a}(2) &= \vec{j} + 4\vec{k}
\end{aligned}$$

Thus, the velocity is $\vec{v}(2) = \vec{i} + 2\vec{j} + 4\vec{k}$, the acceleration is $\vec{a}(2) = \vec{j} + 4\vec{k}$, and speed is $\|\vec{v}(2)\| = \sqrt{21}$.

7. Find the unit vector: $\left(\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right)$

a) $\vec{r}(t) = \ln(2t)\vec{i} + t\vec{j}; \quad t = e$

Solution

$$\begin{aligned}
\vec{r}\,'(t) &= \frac{1}{2t}(2)\vec{i} + \vec{j} \\
\vec{r}\,'(e) &= \frac{1}{e}\vec{i} + \vec{j} \\
\|\vec{r}\,'(e)\| &= \sqrt{\left(\frac{1}{e}\right)^2 + 1^2} \\
&= \sqrt{\frac{1}{e^2} + 1} \\
&= \sqrt{\frac{1+e^2}{e^2}} \\
&= \frac{1}{e}\sqrt{1+e^2} \\
\vec{T}(e) &= \frac{\frac{1}{e}}{\frac{\sqrt{1+e^2}}{e}}\vec{i} + \frac{\frac{1}{\sqrt{1+e^2}}}{e}\vec{j} \\
&= \frac{1}{\sqrt{1+e^2}}\vec{i} + \frac{e}{\sqrt{1+e^2}}\vec{j}
\end{aligned}$$

Therefore, the unit vector is $\vec{T}(e) = \dfrac{1}{\sqrt{1+e^2}}\vec{i} + \dfrac{e}{\sqrt{1+e^2}}\vec{j}$.

b) $\vec{r}(t) = 4\cos(t)\vec{i} + 4\sin(t)\vec{j} + t\vec{k}; \quad t = \dfrac{\pi}{2}$

Solution

$$\begin{aligned}
\vec{r}\,'(t) &= -4\sin(t)\vec{i} + 4\cos(t)\vec{j} + \vec{k} \\
\vec{r}\,'\left(\frac{\pi}{2}\right) &= -4\vec{i} + \vec{j} + \vec{k} \\
\left\|\vec{r}\,'\left(\frac{\pi}{2}\right)\right\| &= \sqrt{(-4)^2 + 0^2 + 1^2} \\
&= \sqrt{17} \\
\vec{T}\left(\frac{\pi}{2}\right) &= -\frac{4}{\sqrt{17}}\vec{i} + \frac{1}{\sqrt{17}}\vec{k}
\end{aligned}$$

Therefore, the unit vector is $\vec{T}\left(\dfrac{\pi}{2}\right) = -\dfrac{4}{\sqrt{17}}\vec{i} + \dfrac{1}{\sqrt{17}}\vec{k}$.

c) $\vec{r}(t) = t\vec{i} + \frac{1}{2}t^2\vec{j} + \frac{1}{3}t^3\vec{k}; \quad t = 0$

Solution

$$\begin{aligned}
\vec{r}\,'(t) &= \vec{i} + \frac{1}{2}(2t)\vec{j} + \frac{1}{3}(3t^2)\vec{k} \\
&= \vec{i} + t\vec{j} + t^2\vec{k} \\
\vec{r}\,'(0) &= \vec{i} + 0\vec{j} + 0^2\vec{k} \\
&= \vec{i} \\
\|\vec{r}\,'(0)\| &= \sqrt{1^2} \\
&= 1 \\
\vec{T}(0) &= \frac{\vec{r}\,'(0)}{\|\vec{r}\,'(0)\|} \\
&= \frac{\vec{i}}{1} \\
&= \vec{i}
\end{aligned}$$

Therefore, the unit vector is $\vec{T}(0) = \vec{i}$.

8. Find the curvature as follows.

   a) Line vector equation in the form $\vec{r}(t) = \vec{r_0} + t\vec{v}$ passing through the terminal point of the position vector $\vec{r_0}$ and parallel to vector $\vec{v}$. Find the line equation $\vec{r}(s)$ with arc length parameter s, and find the curvature at any point.
   $\left(\text{Using } \kappa(s) = \left\|\dfrac{d\vec{T}}{ds}\right\| = \|\vec{T}\,'(s)\| = \|\vec{r}\,''(s)\|\right)$

   Solution

   We compute the ingredients needed for the three unit vector.

$$\vec{r}\,'(t) = \vec{v}$$
$$\|\vec{r}\,'(t)\| = \|\vec{v}\|$$

At $t_0 = 0$:

$$\begin{aligned}
\vec{s}(t) &= \int_{u=0}^{u=t} \|\vec{r}\,'(u)\| du \\
&= \int_{u=0}^{u=t} \|\vec{v}\| du \\
&= \|\vec{v}\| u \Big|_{u=0}^{u=t} \\
&= \|\vec{v}\| t \\
t &= \frac{s}{\|\vec{v}\|}
\end{aligned}$$

So,

$$\vec{r}(s) = \vec{r_0} + \frac{s}{\|\vec{v}\|}\vec{v}$$

Consider curve of linear equation.

$$\begin{aligned}
\vec{r}(s) &= \vec{r_0} + \frac{s}{\|\vec{v}\|}\vec{v} \\
\vec{r}\,'(s) &= \frac{1}{\|\vec{v}\|}\vec{v} \\
\vec{r}\,''(s) &= 0 \\
\kappa(s) &= \|\vec{r}\,''(s)\| = 0
\end{aligned}$$

Therefore, $\kappa(s) = 0$.

b) Find the curvature κ of $\vec{r}(t) = t\vec{i} + \ln(\cos t)\vec{j}$

$\left(\text{Using } \kappa(t) = \dfrac{\|\vec{T}(t)\|}{\|\vec{r}\,'(t)\|}\right)$

Solution

We compute the ingredients needed for the three unit vector.

$$\begin{aligned}
\vec{r}\,'(t) &= \frac{d\vec{r}}{dt} \\
&= \vec{i} - \tan(t)\vec{j} \\
\|\vec{r}\,'(t)\| &= \sqrt{1 + (-\tan(t))^2} \\
&= \sqrt{\sec^2(t)} \\
&= \sec(t)
\end{aligned}$$

Unit tangent vector:

$$\begin{aligned}
\vec{T} &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \\
&= \frac{\vec{i} - \tan(t)\vec{j}}{\sec(t)} \\
&= \frac{\vec{i} - \left(\frac{\sin(t)}{\cos(t)}\right)\vec{j}}{\frac{1}{\cos(t)}} \\
&= \left(\vec{i} - \left(\frac{\sin(t)}{\cos(t)}\right)\vec{j}\right)\cos(t) \\
&= \cos(t)\vec{i} - \sin(t)\vec{j} \\
\frac{d\vec{T}}{dt} &= -\sin(t)\vec{i} - \cos(t)\vec{j}
\end{aligned}$$

and

$$\begin{aligned}
\kappa(t) &= \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} \\
&= \frac{1}{\|\sec(t)\|}\| - \sin(t)\vec{i} - \cos(t)\vec{j}\| \\
&= \frac{1}{\|\frac{1}{\cos(t)}\|}\sqrt{(-\sin(t))^2 + (-\cos(t))^2} \\
&= (\cos(t))(1) \\
&= \cos(t)
\end{aligned}$$

Therefore, $\kappa(t) = \cos(t)$.

c) Find the curvature κ of a circle of radius 2 with the center is (x_0, y_0).
$\left(\text{Using } \kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}\right)$

Solution

Since, $\vec{r}(t) = (x_0 + 2\cos(t))\vec{i} + (y_0 + 2\sin(t))\vec{j}$

We compute the ingredients needed for the three unit vector.

$$\vec{r}\,'(t) = (-2\sin(t))\vec{i} + (2\cos(t))\vec{j}$$

$$\vec{r}\,''(t) = (-2\cos(t))\vec{i} + (-2\sin(t))\vec{j}$$

and
$$\begin{aligned}
\|\vec{r}\,''(t)\| &= \sqrt{(-2\cos(t))^2 + (-2\sin(t))^2} \\
&= \sqrt{2^2(\sin^2(t) + \cos^2(t))} \\
&= \sqrt{2^2} \\
&= 2
\end{aligned}$$

Consider
$$\begin{aligned}
\vec{r}\,'(t) \times \vec{r}\,''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(t) & 2\cos(t) & 0 \\ -2\cos(t) & -2\sin(t) & 0 \end{vmatrix} \\
&= 2^2\sin^2(t)\vec{k} + 2^2\cos^2(t)\vec{k} \\
&= 2^2(\sin^2(t) + \cos^2(t))\vec{k} \\
&= 2^2\vec{k}
\end{aligned}$$

and
$$\begin{aligned}
\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| &= \sqrt{(2^2)^2} \\
&= 2^2
\end{aligned}$$

So,
$$\begin{aligned}
\kappa(t) &= \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \\
&= \frac{2^2}{2^3} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, $\kappa(t) = \dfrac{1}{2}$.

9. Find the unit normal vector as follows.

a) $\vec{r}(t) = 2\sin(2t)\vec{i} + 2\cos(2t)\vec{j} + 4\vec{k}$

$\left(\text{Using } \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right)$

Solution

Find unit tangent vector:

Since
$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \\
&= \frac{(4\cos(2t))\vec{i} + (-4\sin(2t))\vec{j} + 0\vec{k}}{\sqrt{(4\cos(2t))^2 + (-4\sin(2t))^2 + 0^2}} \\
&= \frac{(4\cos(2t))\vec{i} + (-4\sin(2t))\vec{j}}{\sqrt{16\cos^2(2t) + 16\sin^2(2t)}} \\
&= \frac{(4\cos(2t))\vec{i} + (-4\sin(2t))\vec{j}}{\sqrt{16\Big(\cos^2(2t) + \sin^2(2t)\Big)}} \\
&= \frac{(4\cos(2t))\vec{i} + (-4\sin(2t))\vec{j}}{\sqrt{16(1)}} \\
&= \frac{(4\cos(2t))\vec{i} + (-4\sin(2t))\vec{j}}{4} \\
&= \cos(2t)\vec{i} + (-\sin(2t))\vec{j}
\end{aligned}$$

So, $\vec{T}(t) = \cos(2t)\vec{i} + (-\sin(2t))\vec{j}$

Find unit normal vector:

Since
$$\begin{aligned}
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} \\
&= \frac{-2\sin(2t)\vec{i} + (-2\cos(2t))\vec{j}}{\sqrt{(-2\sin(2t))^2 + (-2\cos(2t))^2}} \\
&= \frac{-2\sin(2t)\vec{i} + (-2\cos(2t))\vec{j}}{\sqrt{4\sin^2(2t) + 4\cos^2(2t)}} \\
&= \frac{-2\sin(2t)\vec{i} + (-2\cos(2t))\vec{j}}{\sqrt{4\Big(\sin^2(2t) + \cos^2(2t)\Big)}} \\
&= \frac{-2\sin(2t)\vec{i} + (-2\cos(2t))\vec{j}}{\sqrt{4(1)}} \\
&= -\sin(2t)\vec{i} + (-\cos(2t))\vec{j}
\end{aligned}$$

Therefore, $\vec{N}(t) = -\sin(2t)\vec{i} + (-\cos(2t))\vec{j}$.

b) $\vec{r}(t) = t\vec{i} + 3t\vec{j} + \frac{1}{2}t^2\vec{k}$ at $t = 0$

$\Big($Using $\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}\Big)$

Solution

Find unit tangent vector:

$$\text{Since} \qquad \vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$

$$= \frac{1\vec{i} + 3\vec{j} + \left(\frac{1}{2}\right)(2t)\vec{k}}{\sqrt{1^2 + 3^2 + t^2}}$$

$$= \frac{\vec{i} + 3\vec{j} + t\vec{k}}{\sqrt{10 + t^2}}$$

$$= \left(10 + t^2\right)^{-\frac{1}{2}}\vec{i} + 3\left(10 + t^2\right)^{-\frac{1}{2}}\vec{j} + t\left(10 + t^2\right)^{-\frac{1}{2}}\vec{k}$$

So, $\vec{T}(t) = \left(10 + t^2\right)^{-\frac{1}{2}}\vec{i} + 3\left(10 + t^2\right)^{-\frac{1}{2}}\vec{j} + t\left(10 + t^2\right)^{-\frac{1}{2}}\vec{k}.$

Find unit normal vector at $t = 0$

$$\text{Since} \qquad \vec{N}(0) = \frac{\frac{d\vec{T}}{dt}}{\|\frac{d\vec{T}}{dt}\|}$$

$$\text{Find} \qquad \frac{d\vec{T}(t)}{dt} = -\frac{1}{2}\left(10 + t^2\right)^{-\frac{3}{2}}(2t)\vec{i} + 3\left(-\frac{1}{2}\right)\left(10 + t^2\right)^{-\frac{3}{2}}(2t)\vec{j}$$

$$+ \left[t\left(-\frac{1}{2}\right)\left(10 + t^2\right)^{-\frac{3}{2}}(2t) + \left(10 + t^2\right)^{-\frac{1}{2}}(1)\right]\vec{k}$$

$$= -t\left(10 + t^2\right)^{-\frac{3}{2}}\vec{i} + (-3t)\left(10 + t^2\right)^{-\frac{3}{2}}\vec{j}$$

$$+ \left[(-t^2)\left(10 + t^2\right)^{-\frac{3}{2}} + \left(10 + t^2\right)^{-\frac{1}{2}}\right]\vec{k}$$

$$\text{At} \quad t = 0: \qquad \frac{d\vec{T}(0)}{dt} = 0\left(10 + 0^2\right)^{-\frac{3}{2}}\vec{i} + (0)\left(10 + 0^2\right)^{-\frac{3}{2}}\vec{j}$$

$$+ \left[(0)\left(10 + 0^2\right)^{-\frac{3}{2}} + \left(10 + 0^2\right)^{-\frac{1}{2}}\right]\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + \left(0 + (10)^{-\frac{1}{2}}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + \frac{1}{\sqrt{10}}\vec{k}$$

and
$$\left\|\frac{d\vec{T}(0)}{dt}\right\| = \sqrt{\left(\frac{1}{10}\right)^2}$$
$$= \sqrt{\frac{1}{10}}$$
$$= \frac{1}{\sqrt{10}}$$

We get
$$\vec{N}(0) = \frac{\frac{d\vec{T}(0)}{dt}}{\|\frac{d\vec{T}(0)}{dt}\|}$$
$$= \frac{\frac{1}{\sqrt{10}}\vec{k}}{\frac{1}{\sqrt{10}}}$$
$$= \vec{k}$$

Therefore, unit normal vector at $t = 0$ of $\vec{r}(t) = t\vec{i} + 3t\vec{j} + \dfrac{1}{2}t^2\vec{k}$ is $\vec{k}$.

10. Find the center of circle and equation for the osculating cicle at the origin on the parabola $y = -2x^2$.

Solution

The parametric equation for this curve are $x = t, y = -2t^2$

Thus, $\vec{r}(t) = t\vec{i} + \big(-2t^2\big)\vec{j}$

$$\text{Find} \qquad \vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \vec{i} + (-4t)\vec{j}$$

$$\text{and} \qquad \|\vec{v}(t)\| = \sqrt{1^2 + (-4t)^2} = \sqrt{1 + 16t^2}$$

$$\text{Therefore,} \qquad \vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{1\vec{i} + (-4t)\vec{j}}{\sqrt{1+16t^2}} = \Big(1+16t^2\Big)^{-\frac{1}{2}}\vec{i} + (-4t)\Big(1+16t^2\Big)^{-\frac{1}{2}}\vec{j}$$

$$\begin{aligned} \text{and} \qquad \frac{d\vec{T}(t)}{dt} &= -\frac{1}{2}\Big(1+16t^2\Big)^{-\frac{3}{2}}(32t)\vec{i} \\ &\quad + \left[(-4t)\Big(-\frac{1}{2}\Big)\Big(1+16t^2\Big)^{-\frac{3}{2}}(32t) + \Big(1+16t^2\Big)^{-\frac{1}{2}}(-4)\right]\vec{j} \\ &= -16t\Big(1+16t^2\Big)^{-\frac{3}{2}}\vec{i} + \left[64t^2\Big(1+16t^2\Big)^{-\frac{3}{2}} - 4\Big(1+16t^2\Big)^{-\frac{1}{2}}\right]\vec{j} \end{aligned}$$

So, $\kappa(t)$ at $t = 0$ is

$$\begin{aligned} \kappa(0) &= \frac{1}{\|\vec{v}(0)\|}\left\|\frac{d\vec{T}(0)}{dt}\right\| \\ &= \frac{1}{\sqrt{1+16(0)^2}} \cdot \\ &\quad \left[\sqrt{\Big(-16(0)\big(1+16(0)^2\big)^{-\frac{3}{2}}\Big)^2 + \Big(64(0)^2\Big(1+16(0)^2\Big)^{-\frac{3}{2}} - 4\Big(1+16(0)^2\Big)^{-\frac{1}{2}}\Big)^2}\right] \\ &= \frac{1}{\sqrt{1+0}}\left(\sqrt{0^2 + (0-4)^2}\right) \\ &= \frac{1}{\sqrt{1}}(\sqrt{16}) \\ &= 4 \end{aligned}$$

Radias of curvature: $\rho = \dfrac{1}{\kappa} = \dfrac{1}{4}$

Center of curvature: $\Big(0, -\dfrac{1}{4}\Big)$

Therefore, equation of osculating circle is

$$\begin{aligned}(x-0)^2 + \left(y - \Big(-\frac{1}{4}\Big)\right)^2 &= \Big(\frac{1}{4}\Big)^2 \\ x^2 + \left(y + \frac{1}{4}\right)^2 &= \frac{1}{16}.\end{aligned}$$

11. Find unit binormal vector $\vec{B}$ of the position of a moving particle is given by $\vec{r}(t) = (e^t \cos t)\vec{i} + (e^t \sin t)\vec{j} + 2\vec{k}$.

Solution

Find unit tangent vector:

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}(t)\|} \\
&= \frac{\big(-e^t \sin t + e^t \cos t\big)\vec{i} + \big(e^t \cos t + e^t \sin t\big)\vec{j}}{\sqrt{\big(e^t \cos t - e^t \sin t\big)^2 + \big(e^t \cos t + e^t \sin t\big)^2}} \\
&= \frac{\big(-e^t \sin t + e^t \cos t\big)\vec{i} + \big(e^t \cos t + e^t \sin t\big)\vec{j}}{\sqrt{\big(e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t\big) + \big(e^{2t} \cos^2 t + 2e^2 \sin t \cos t + e^{2t} \sin^2 t\big)}} \\
&= \frac{\big(-e^t \sin t + e^t \cos t\big)\vec{i} + \big(e^t \cos t + e^t \sin t\big)\vec{j}}{\sqrt{2e^{2t} \cos^2 t + 2e^{2t} \sin^2 t}} \\
&= \frac{\big(-e^t \sin t + e^t \cos t\big)\vec{i} + \big(e^t \cos t + e^t \sin t\big)\vec{j}}{\sqrt{2e^{2t}\big(\cos^2 t + \sin^2 t\big)}} \\
&= \frac{\big(-e^t \sin t + e^t \cos t\big)\vec{i} + \big(e^t \cos t + e^t \sin t\big)\vec{j}}{\sqrt{2}e^t}
\end{aligned}$$

So, $\vec{T}(t) = \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)\vec{i} + \Big(\frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t\Big)\vec{j}$

Find unit normal vector:

$$\begin{aligned}
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} \\
&= \frac{\Big(-\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\Big)\vec{i} + \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)\vec{j}}{\sqrt{\Big(-\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\Big)^2 + \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)^2}} \\
&= \frac{\Big(-\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\Big)\vec{i} + \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)\vec{j}}{\sqrt{\Big(\frac{1}{2} \cos^2 t + \sin t \cos t + \frac{1}{2} \sin^2 t\Big) + \Big(\frac{1}{2} \sin^2 t - \sin t \cos t + \frac{1}{2} \cos^2 t\Big)}} \\
&= \frac{\Big(-\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\Big)\vec{i} + \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)\vec{j}}{\sqrt{\sin^2 t + \cos^2 t}}
\end{aligned}$$

So, $\vec{N}(t) = \Big(-\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\Big)\vec{i} + \Big(-\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t\Big)\vec{j}$

Find unit binormal vector:

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{\sqrt{2}}\sin t + \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t & 0 \\ -\frac{1}{\sqrt{2}}\cos t - \frac{1}{\sqrt{2}}\sin t & -\frac{1}{\sqrt{2}}\sin t + \frac{1}{\sqrt{2}}\cos t & 0 \end{vmatrix} \\
&= 0\vec{i} + 0\vec{j} + \left(\sin^2 t + \cos^2 t\right)\vec{k} \\
&= \vec{k}
\end{aligned}$$

Therefore, unit binormal vector is $\vec{k}$.

12. Determine the torsion of $\vec{r}(t) = (6\sin t)\vec{i} + (6\cos t)\vec{j} + 8t\vec{k}$.

Solution

Find unit tangent vector:

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \\
&= \frac{(6\cos t)\vec{i} + (-6\sin t)\vec{j} + 8\vec{k}}{\sqrt{(6\cos t)^2 + (-6\sin t)^2 + 8^2}} \\
&= \frac{(6\cos t)\vec{i} + (-6\sin t)\vec{j} + 8\vec{k}}{\sqrt{36\cos^2 t + 36\sin^2 t + 64}} \\
&= \frac{(6\cos t)\vec{i} + (-6\sin t)\vec{j} + 8\vec{k}}{\sqrt{36 + 64}} \\
&= \frac{(6\cos t)\vec{i} + (-6\sin t)\vec{j} + 8\vec{k}}{\sqrt{100}} \\
&= \frac{(6\cos t)\vec{i} + (-6\sin t)\vec{j} + 8\vec{k}}{10}
\end{aligned}$$

So, $\vec{T}(t) = \Big(\dfrac{3}{5}\cos t\Big)\vec{i} + \Big(-\dfrac{3}{5}\sin t\Big)\vec{j} + \dfrac{4}{5}\vec{k}$

Find unit normal vector:

$$\begin{aligned}
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} \\
&= \frac{\Big(-\frac{3}{5}\sin t\Big)\vec{i} + \Big(-\frac{3}{5}\cos t\Big)\vec{j}}{\sqrt{\Big(-\frac{3}{5}\sin t\Big)^2 + \Big(-\frac{3}{5}\cos t\Big)^2}} \\
&= \frac{\Big(-\frac{3}{5}\sin t\Big)\vec{i} + \Big(-\frac{3}{5}\cos t\Big)\vec{j}}{\sqrt{\Big(\frac{3}{5}\Big)^2}} \\
&= \frac{\Big(-\frac{3}{5}\sin t\Big)\vec{i} + \Big(-\frac{3}{5}\cos t\Big)\vec{j}}{\frac{3}{5}}
\end{aligned}$$

So, $\vec{N}(t) = (-\sin t)\vec{i} + (-\cos t)\vec{j}$

Find unit binormal vector:

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} \\
&= \left(\frac{4}{5}\cos t\right)\vec{i} + \left(-\frac{4}{5}\sin t\right)\vec{j} - \left(\frac{3}{5}\right)\frac{1}{\vec{k}}
\end{aligned}$$

Find torsion:

$$\begin{aligned}
\tau &= -\frac{1}{\|\vec{r}\,'(t)\|}\vec{B}\,'(t) \cdot \vec{N}\,'(t) \\
&= -\frac{1}{10}\langle -\frac{4}{5}\sin t, -\frac{4}{5}\cos t, 0\rangle \cdot \langle -\sin t, -\cos t, 0\rangle \\
&= -\frac{1}{10}\left(\frac{4}{5}\sin^2 t + \frac{4}{5}\cos^2 t\right) \\
&= -\frac{4}{50}(\sin^2 t + \cos^2 t) \\
&= -\frac{2}{25}
\end{aligned}$$

Therefore, torsion is $-\dfrac{2}{25}$.

13. Find $T(t), N(t)$, and $B(t)$ for the given value of t. Then find equations for the osculating, normal, and rectifying planes at the point that corresponds to that value of t when $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + \vec{k}; \quad t = \frac{\pi}{4}.$

Solution

Unit tangent vector: $$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

Consider $$\vec{r}\,'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j} + 0\vec{k}$$

$$\|\vec{r}\,'(t)\| = 1$$

So, $$\vec{T}(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}$$

At $t = \frac{\pi}{4}$:
$$\begin{aligned}
\vec{T}\left(\frac{\pi}{4}\right) &= -\sin\left(\frac{\pi}{4}\right)\vec{i} + \cos\left(\frac{\pi}{4}\right)\vec{j} \\
&= -\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j} \\
&= \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle
\end{aligned}$$

Unit normal vector:
$$\begin{aligned}
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} \\
&= \frac{-\cos(t)\vec{i} - \sin(t)\vec{j}}{\sqrt{(-\cos t)^2 + (-\sin t)^2}}
\end{aligned}$$

$$\vec{N}(t) = -\cos(t)\vec{i} - \sin(t)\vec{j}$$

At $t = \frac{\pi}{4}$:
$$\begin{aligned}
\vec{N}\left(\frac{\pi}{4}\right) &= -\cos\left(\frac{\pi}{4}\right)\vec{i} - \sin\left(\frac{\pi}{4}\right)\vec{j} \\
&= -\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j} \\
&= \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \rangle
\end{aligned}$$

Binormal vector:
$$\begin{aligned}
\vec{B} &= \vec{T} \times \vec{N} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} \\
&= \sin^2 t - (-\cos^2 t)\vec{k}
\end{aligned}$$

$$\vec{B}(t) = \vec{k}$$

At $t = \frac{\pi}{4}$: $$\vec{B}\left(\frac{\pi}{4}\right) = \vec{k}$$

From $\vec{r}(t)$

$$\begin{aligned}\vec{r}\Big(\frac{\pi}{4}\Big) &= \cos\Big(\frac{\pi}{4}\Big)\vec{i} + \sin\Big(\frac{\pi}{4}\Big)\vec{j} + \vec{k} \\ &= \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j} + \vec{k}\end{aligned}$$

and $\vec{N} \cdot \langle P - P_0 \rangle = 0, \quad P_0 = \vec{r}\Big(\frac{\pi}{4}\Big)$

Find rectifying plane from

$$\begin{aligned}\vec{N} \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1\rangle &= 0 \\ \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\rangle \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1\rangle &= 0 \\ \frac{-\sqrt{2}}{2}\Big(x - \frac{\sqrt{2}}{2}\Big) + \Big(-\frac{\sqrt{2}}{2}\Big)\Big(y - \frac{\sqrt{2}}{2}\Big) + 0(z-1) &= 0 \\ -\frac{\sqrt{2}}{2}x + \frac{2}{4} - \frac{\sqrt{2}}{2}y + \frac{2}{4} + 0 &= 0 \\ -\frac{\sqrt{2}}{2}x + \frac{1}{2} - \frac{\sqrt{2}}{2}y + \frac{1}{2} &= 0 \\ -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y + 1 &= 0 \\ -\frac{\sqrt{2}}{2}(x+y) &= -1 \\ x + y &= \frac{2}{\sqrt{2}} \\ x + y &= \sqrt{2}\end{aligned}$$

So, rectifying plane is $x + y = \sqrt{2}$

Since osculating plane is a plan consisting of $\vec{T}$ and $\vec{N}$

Thus, osculating plane:

$$\begin{aligned}\vec{B} \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1\rangle &= 0 \\ \langle 0, 0, 1\rangle \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1\rangle &= 0 \\ 0 + 0 + z - 1 &= 0 \\ z &= 1\end{aligned}$$

So, osculating plane is $z = 1$

Since normal plane is a plan consisting of $\vec{N}$ and $\vec{B}$

Thus, normal plane:

$$\begin{aligned}
\vec{T} \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1 \rangle &= 0 \\
\langle \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle \cdot \langle x - \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z - 1 \rangle &= 0 \\
-\frac{\sqrt{2}}{2}\Big(x - \frac{\sqrt{2}}{2}\Big) + \frac{\sqrt{2}}{2}\Big(y - \frac{\sqrt{2}}{2}\Big) + 0(z-1) &= 0 \\
\frac{-\sqrt{2}}{2}x + \frac{1}{2} + \frac{\sqrt{2}}{2}y - \frac{1}{2} + 0 &= 0 \\
\frac{\sqrt{2}}{2}(-x + y) &= 0 \\
y - x &= 0
\end{aligned}$$

Therefore, normal plane is $y - x = 0$.