

MTH10102

Mathematics I

Module 2 Integrals

## INTEGRATION

### 1. Antiderivative

This lesson concerns the reverse process of taking derivative of a function as we learned last section. In particular, if  $y' = f'(x)$ , we want to find  $y = f(x)$ . Consider the following:

If  $f(x) = x^2$ ,                      then  $f'(x) = 2x$

If  $f(x) = x^2 + 1$ ,                  then  $f'(x) = 2x$

If  $f(x) = x^2 + 2$ ,                  then  $f'(x) = 2x$

.....

If  $f(x) = x^2 + C$ ,                  then  $f'(x) = 2x$

Thus  $f'(x) = 2x$  may have  $f(x) = x^2$  or in general,  
 $f(x) = x^2 + C$ , where  $C$  is some constant.

We call  $x^2 + C$  an **antiderivative** of  $2x$ .

**Definition 1.1** Function  $F(x)$  such that  $F'(x) = f(x)$  is called an “an antiderivative of  $f(x)$ ”

For examples, for any constant  $C$

$$1. F(x) = x^2 + \frac{1}{x} + C \text{ is an antiderivative of } f(x) = 2x - \frac{1}{x^2}$$

$$\text{since } F'(x) = 2x - \frac{1}{x^2}.$$

$$2. F(x) = \sin x + C \text{ is an antiderivative of } f(x) = \cos x \text{ since } F'(x) = \cos x.$$

$$3. F(x) = e^x + \tan^{-1} x + C \text{ is an antiderivative of}$$

$$f(x) = e^x + \frac{1}{1+x^2} \text{ since } F'(x) = e^x + \frac{1}{1+x^2}.$$

### Properties of an antiderivative of $f(x)$

1. Every continuous function  $f(x)$  has infinitely many antiderivatives of  $f(x)$ .
2. If  $F_1(x), F_2(x)$  are both antiderivatives of  $f(x)$ , then the difference  $F_1(x) - F_2(x) = \text{constant}$ .
3. If  $F(x)$  is an antiderivative  $f(x)$ , then  $F(x) + C$  where  $C$  is some constant is also the antiderivative of  $f(x)$ . Thus we say that all antiderivatives of  $f(x)$  are in the form of  $F(x) + C$ .

**Definition 1.2** The process of finding an antiderivative of  $f(x)$  is called an integration

$$f(x) \rightarrow F(x) \text{ if } F'(x) = f(x)$$

**Notion:**  $\int f(x)dx$  is called an “integral of  $f(x)$  with respect to  $x$ ”

$\int$  is an integration notation,  $dx$  refers to the independent variable  $x$  and  $f(x)$  is called an integrand.

There are two types of integrations: indefinite and definite Integrals.

## 2 Indefinite Integral

Since  $\frac{d}{dx} F(x) = f(x)$  or  $dF(x) = f(x)dx$ , then

$$\int dF(x) = \int f(x)dx = F(x) + C \text{ where } C \text{ is some constant.}$$

**Note** that the notation  $\int$  is the reverse operation of the derivative notation and we call this process an indefinite integral. Examples:

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(ax) = a$$

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\int 1dx = \int dx = x + C$$

$$\int adx = ax + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\int \cos x dx = \sin x + C$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\int \sec^2 x dx = \tan x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\int \csc x \cot x dx = -\csc x + C$$

### Rule of Algebra for Antiderivative

#### 1. Constant multiplication

$$\int af(x)dx = a \int f(x)dx, \text{ } a \text{ is some constant}$$

#### 2. Addition and subtraction

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

**Example 1** Evaluate  $\int (5x - x^2 + 2)dx$

Solution 
$$\begin{aligned}\int (5x - x^2 + 2)dx &= \int 5x dx - \int x^2 dx + \int 2 dx \\ &= 5 \int x dx - \int x^2 dx + 2 \int dx \\ &= 5 \left( \frac{x^2}{2} + c_1 \right) - \left( \frac{x^3}{3} + c_2 \right) + 2(x + c_3) \\ &= \frac{5x^2}{2} + 5c_1 - \frac{x^3}{3} - c_2 + 2x + 2c_3 \\ &= \frac{5x^2}{2} - \frac{x^3}{3} + 2x + C\end{aligned}$$

where  $C = 5c_1 - c_2 + 2c_3$

**Example 2** Evaluate  $\int (8x^3 + 4x - 6\sqrt{x} - \frac{2}{\sqrt[3]{x}} + \frac{5}{x^2})dx$

Solution

**Example 3** Evaluate  $\int (3e^x - 7 \sin x + \frac{5}{x}) dx$

Solution

**Example 4** Evaluate  $\int \frac{\cos x}{\sin^2 x} dx$

Solution

### 3. Definite Integral

A definite integral of  $f(x)$  from  $a$  to  $b$  is written as

$$\int_a^b f(x)dx$$

$a$  and  $b$  are called limits of integration, where  $a$  is the lower limit and  $b$  is the upper limit.

**Definition 3.1** A definite integral of  $f(x)$  is a continuous function on  $a \leq x \leq b$  such that

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

**Definition 3.2** If  $a < b$  and  $f(x)$  is integrable on  $a \leq x \leq b$

$$1. \int_a^a f(x)dx = 0$$

$$2. \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$3. \int_a^b f(x)dx > 0 \text{ where } f(x) > 0, \text{ and}$$

$$\int_a^b f(x)dx < 0 \text{ where } f(x) < 0$$

## Evaluation process of an definite integral

Step 1 Find antiderivative of  $F(x)$

Step 2 Calculate  $F(b) - F(a)$  by plugging  $x = b$  and  $x = a$  into  $F(x)$  we found in step 1

## Properties of a definite integral

Let  $f(x)$  and  $g(x)$  be integrable functions on  $a \leq x \leq b$  and  $C$  be some constant.

$$1. \int_a^b C dx = C(b - a)$$

$$2. \int_a^b Cf(x) dx = C \int_a^b f(x) dx$$

$$3. \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$



**Example 5** Evaluate  $\int_1^2 \left[ 5x^2 + 3x - 1 - \frac{6}{x} \right] dx$

Solution

$$\begin{aligned}
 \int_1^2 \left[ 5x^2 + 3x - 1 - \frac{6}{x} \right] dx &= \int_1^2 5x^2 dx + \int_1^2 3x dx - \int_1^2 dx - \int_1^2 \frac{6}{x} dx \\
 &= 5 \int_1^2 x^2 dx + 3 \int_1^2 x dx - \int_1^2 dx - 6 \int_1^2 \frac{1}{x} dx \\
 &= 5 \left. \frac{x^3}{3} \right|_1^2 + 3 \left. \frac{x^2}{2} \right|_1^2 - x \Big|_1^2 - 6 \ln |x| \Big|_1^2 \\
 &= 5 \left( \frac{8-1}{3} \right) + 3 \left( \frac{4-1}{2} \right) - (2-1) - 6(\ln 2 - \ln 1) \\
 &= 5 \left( \frac{7}{3} \right) + 3 \left( \frac{3}{2} \right) - (1) - 6(\ln 2 - 0) \\
 &= \frac{35}{3} + \frac{9}{2} - 1 - 6 \ln 2 \\
 &= \frac{70 + 27 - 6}{6} - 6 \ln 2 \\
 &= \frac{91}{6} - 6 \ln 2
 \end{aligned}$$

Thus  $\int_1^2 \left[ 5x^2 + 3x - 1 - \frac{6}{x} \right] dx = \frac{91}{6} - 6 \ln 2$

**Example 6** Evaluate  $\int_{\pi}^{\pi} [e^x + 4 \sin x] dx$

Solution

**Example 7** Evaluate  $\int_0^3 |x-2| dx$

Solution From  $f(x) = |x-2|$

We can write  $f(x) = \begin{cases} x-2; & x \geq 2 \\ -(x-2); & x < 2 \end{cases}$

$$\begin{aligned} \text{Thus } \int_0^3 |x-2| dx &= \int_0^2 |x-2| dx + \int_2^3 |x-2| dx \\ &= \int_0^2 (-x+2) dx + \int_2^3 (x-2) dx \\ &= -\int_0^2 x dx + \int_0^2 2 dx + \int_2^3 x dx - \int_2^3 2 dx \\ &= -\frac{x^2}{2} \Big|_0^2 + 2x \Big|_0^2 + \frac{x^2}{2} \Big|_2^3 - 2x \Big|_2^3 \\ &= -\frac{1}{2}[4-0] + 2[2-0] \\ &\quad + \frac{1}{2}[9-4] - 2[3-2] \\ &= -2 + 4 + \frac{5}{2} - 2 \\ &= \frac{5}{2} \end{aligned}$$

$$\text{Hence } \int_0^3 |x-2| dx = \frac{5}{2}$$

**Example 8** Evaluate

$$\int_{-2}^1 f(x)dx \text{ where } f(x) = \begin{cases} 2 - x^2; & x \geq 0 \\ x + 2; & x < 0 \end{cases}$$

Solution

## 4. Techniques of Integration

### 4.1 Integration by Substitution

We change the integrand by substitution.

**Example 9** Evaluate  $\int (3x - 5)^{20} dx$

Solution Let  $u = 3x - 5$ . Then  $du = 3dx$  or  $dx = \frac{du}{3}$

$$\begin{aligned} \text{Thus} \quad \int (3x - 5)^{20} dx &= \int u^{20} \frac{du}{3} \\ &= \frac{1}{3} \int u^{20} du \\ &= \frac{1}{3} \cdot \frac{u^{21}}{21} + C \\ &= \frac{(3x - 5)^{21}}{63} + C \end{aligned}$$

$$\text{Hence} \quad \int (3x - 5)^{20} dx = \frac{(3x - 5)^{21}}{63} + C$$

**Example 10** Evaluate  $\int \frac{(\ln x)^2}{x \ln 9} dx$

Solution

**Example 11** Evaluate  $\int (x+3)\sqrt{x+1}dx$

Solution

### The procedure of integration by substitution

1. Define  $u = g(x)$  and find  $du = g'(x)dx$
2. Rewrite  $\int f(x)dx$  in terms of new variable  $u$  to get  $\int h(u)du$
3. Find the integral  $\int h(u)du = H(u) + C$
4. Plug  $u = g(x)$  back into the resulting function in step 3.

$$\int f(x)dx = H(u) + C = H(g(x)) + C = F(x) + C$$

**Example 12** Evaluate  $\int x^2(1-x)^{100} dx$

Solution Let  $u = 1 - x$  or  $x = 1 - u$

Then  $x^2 = (1-u)^2$  and  $du = -dx$

$$\begin{aligned} \text{Thus } \int x^2(1-x)^{100} dx &= \int (1-u)^2 u^{100} (-du) \\ &= \int (1-2u+u^2)(-u^{100})du \\ &= \int -u^{100} du + \int 2u^{101} du - \int u^{102} du \\ &= -\frac{u^{101}}{101} + 2\frac{u^{102}}{102} - \frac{u^{103}}{103} + C \\ &= \frac{2(1-x)^{102}}{102} - \frac{(1-x)^{101}}{101} - \frac{(1-x)^{103}}{103} + C \\ \text{Hence } \int x^2(1-x)^{100} dx &= \frac{2(1-x)^{102}}{102} - \frac{(1-x)^{101}}{101} \\ &\quad - \frac{(1-x)^{103}}{103} + C \end{aligned}$$



**Example 13** Evaluate  $\int \frac{\sec^2 2x dx}{1 + \tan 2x}$

Solution

**Example 14** Evaluate  $\int \frac{(x^2 + 1)dx}{2x - 3}$

Solution Let  $u = 2x - 3$ . Then  $du = 2dx$  or  $\frac{du}{2} = dx$

$$\text{and } x = \frac{u + 3}{2}, \quad x^2 = \left(\frac{u + 3}{2}\right)^2, \quad x^2 = \frac{1}{4}(u^2 + 6u + 9)$$

$$\text{then } x^2 + 1 = \frac{1}{4}(u^2 + 6u + 9 + 4)$$

Substitution:

$$\begin{aligned} \int \frac{(x^2 + 1)dx}{2x - 3} &= \int \frac{1}{4} \cdot \frac{(u^2 + 6u + 9 + 4)}{u} \cdot \frac{du}{2} \\ &= \frac{1}{8} \int \frac{(u^2 + 6u + 13)}{u} du \\ &= \frac{1}{8} \int \left(u + 6 + \frac{13}{u}\right) du \\ &= \frac{1}{8} \left[ \frac{u^2}{2} + 6u + 13 \ln|u| \right] + C \\ &= \frac{1}{8} \left\{ \frac{(2x - 3)^2}{2} + (2x - 3) \right. \\ &\quad \left. + 13 \ln|2x - 3| \right\} + C \end{aligned}$$

Thus

$$\int \frac{(x^2 + 1)dx}{2x - 3} = \frac{1}{8} \left[ \frac{(2x - 3)^2}{2} + (2x - 3) + 13 \ln|2x - 3| \right] + C$$

**Remark**

Once we change the variable in the definite integral by substitution technique, we also need to change the limits of integration.

**Example 15** Evaluate  $\int_0^1 xe^{4x^2+1} dx$

Solution Let  $u = 4x^2 + 1$ . Then  $du = 8xdx$  or  $xdx = \frac{du}{8}$

When  $x = 0$ , then  $u = 1$ . And when  $x = 1$ , then  $u = 5$

$$\begin{aligned} \text{Substitution: } \int_0^1 xe^{4x^2+1} dx &= \int_0^1 e^{4x^2+1} xdx \\ &= \int_1^5 \frac{e^u du}{8} \\ &= \frac{1}{8} \int_1^5 e^u du \\ &= \frac{1}{8} e^u \Big|_1^5 \\ &= \frac{1}{8} [e^5 - e^1] \end{aligned}$$

$$\text{Thus } \int_0^1 xe^{4x^2+1} dx = \frac{1}{8} (e^5 - e)$$

**Example 16** Evaluate  $\int_0^3 x(1+x)^{\frac{1}{2}} dx$

Solution

## 4.2 Integration by Parts

We use this technique when integration by substitution doesn't work. We consider the integral as  $\int u dv$  where  $dv$  is a part of the function consisting of  $dx$  and  $f(x)$  or  $g(x)$ .

Formula used to find the integration by parts:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

or 
$$d(uv) = u dv + v du$$

$$u dv = d(uv) - v du$$

and 
$$\int u dv = \int d(uv) - \int v du$$

$$\int u dv = uv - \int v du$$

### Remark

This technique is to express  $\int u dv$  in terms of  $uv$  and  $\int v du$  which is easier to be integrated. Thus choosing appropriate  $u$  and  $v$  is a crucial step for doing integration by parts.

### Summary

$$\text{Let } \int f(x)g(x)dx = \int h(x)dx = \int u dv = uv - \int v du$$

To pick  $u$  and  $v$ , we consider

1.  $dv$  is easy to get integrated so that we have  $v$
2.  $\int v du$  exists

In case of, definite integral:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

**Example 17** Evaluate  $\int x \ln x dx$

Solution Let  $u = \ln x$  and  $dv = x dx$

$$du = \frac{dx}{x} \text{ and } \int dv = \int x dx \quad \text{or} \quad v = \frac{x^2}{2}$$

From  $\int u dv = uv - \int v du$

Then  $\int x \ln x dx = \int \ln x (x dx)$

$$= \ln(x) \left( \frac{x^2}{2} \right) - \int \left( \frac{x^2}{2} \right) \left( \frac{dx}{x} \right)$$

$$= \ln(x) \left( \frac{x^2}{2} \right) - \frac{1}{2} \int x dx$$

$$= \ln(x) \left( \frac{x^2}{2} \right) - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Thus  $\int x \ln x dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

**Example 18** Evaluate  $\int_1^2 \ln x dx$

Solution

**Example 19** Evaluate  $\int \tan^{-1} x \, dx$

Solution



Note: Some integrals may need several integrations by parts.

**Example 20** Evaluate  $\int e^{2x} \sin x \, dx$

Solution Let  $u = e^{2x}$  and  $dv = \sin x \, dx$

$$du = 2e^{2x} \, dx \quad \text{and} \quad v = -\cos x$$

Then

$$\begin{aligned} \int e^{2x} \sin x \, dx &= e^{2x}(-\cos x) - \int -\cos x(2e^{2x} \, dx) \\ &= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \end{aligned}$$

Next, consider  $2 \int e^{2x} \cos x \, dx$

Let  $u = e^{2x}$  and  $dv = \cos x \, dx$

$$du = 2e^{2x} \, dx \quad \text{and} \quad v = \sin x$$

$$\begin{aligned} 2 \int e^{2x} \cos x \, dx &= 2 \left[ e^{2x} \sin x - \int \sin x(2e^{2x} \, dx) \right] \\ &= 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx \end{aligned}$$

Then

$$\begin{aligned} \int e^{2x} \sin x \, dx &= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx + C \end{aligned}$$

$$5 \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x + C$$

$$\text{Hence,} \quad \int e^{2x} \sin x \, dx = \frac{1}{5} \left[ -e^{2x} \cos x + 2e^{2x} \sin x \right] + C$$

### Rules to pick $u$ and $dv$

1.  $u$  should have a simple derivative.
2.  $dv$  may be complicated but easy to get integrated.
3.  $\int v du$  is easier to evaluate than  $\int u dv$

### Examples of $u$ and $dv$

1.  $\int x^n e^{ax} dx$ ,  $\int x^n \cos ax dx$ ,  $\int x^n \sin ax dx$

Then  $u = x^n$  and  $dv$  is the rest of the integrand

2.  $\int x^n \sin^{-1} x dx$ ,  $\int x^n \cos^{-1} x dx$ ,  $\int x^n \tan^{-1} x dx$

Then  $u = \sin^{-1} x$  or  $u = \cos^{-1} x$  or  $u = \tan^{-1} x$ , respectively

and  $dv$  is the rest

3.  $\int x^m [\ln x]^n dx$  where  $m \neq -1$

Then  $u = [\ln x]^n$  and  $dv$  is the rest

### 4.3 Integration of Rational Function by Partial Fraction

It is used when the integrand is in a form of rational function  $\frac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_{m-1}x^{m-1} + b_mx^m}; n < m$$

Express  $\frac{f(x)}{g(x)}$  as a partial fraction: ex  $\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$

which is found by

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

and  $5x-3 = A(x-3) + B(x+1)$

$$5x-3 = (A+B)x + (-3A+B)$$

Calculating  $A$  and  $B$  by comparing coefficients of  $x$

$$A+B=5 \quad \text{and} \quad -3A+B=-3$$

Solve to get  $A=2$  and  $B=3$ . We call  $A$  and  $B$  constants calculated by undetermined coefficients.

### Conditions on partial fractions:

$\frac{f(x)}{g(x)}$  can be expressed as a partial fraction if

1. Power of  $f(x)$  is higher than or equal to power of  $g(x)$  ( $n \geq m$ ),

we first have to divide  $g(x)$  by  $f(x)$  to get

$$\frac{f(x)}{g(x)} = \phi(x) + \frac{h(x)}{g(x)}$$

where  $h(x)$ ,  $g(x)$  are both polynomials and power of  $h(x)$  is less than power of  $g(x)$ .

2.  $g(x)$  can be factor out as linear or quadratic factors

#### 2.1 Types of factors

- a. Linear factor is in a form of  $(ax + b)$  where  $a, b$  are real.
- b. Irreducible quadratic factor is in a form of  $(ax^2 + bx + c)$   
where  $a, b, c$  are real

*Procedure of Integration by Partial Fraction*

**Consider a rational function**  $\frac{f(x)}{g(x)}$

**Case 1**  $g(x)$  has only non-repeated linear factors

$$g(x) = (a_1x + b_1)(a_2x + b_2)\dots\dots(a_nx + b_n)$$

where  $\frac{b_1}{a_1} \neq \frac{b_2}{a_2} \neq \dots\dots \neq \frac{b_n}{a_n}$  and  $a_1, a_2, \dots\dots, a_n \neq 0$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots\dots + \frac{A_n}{a_nx + b_n}$$

where  $A_1, A_2, \dots\dots, A_n$  are all constants we need to find.

**Example 37** Evaluate  $\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx$

Solution Consider  $x^3 + x^2 - 2x$

$$x^3 + x^2 - 2x = x(x-1)(x+2)$$

Thus 
$$\frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+2}$$

$$\begin{aligned} 2x^2 + 5x - 1 &= A_1(x-1)(x+2) + A_2(x)(x+2) + A_3x(x-1) \\ &= A_1(x^2 + x - 2) + A_2(x^2 + 2x) + A_3(x^2 - x) \end{aligned}$$

Compare coefficients:

$$A_1 + A_2 + A_3 = 2$$

$$A_1 + 2A_2 - A_3 = 5$$

$$-2A_1 = -1$$

Solve to get  $A_1 = \frac{1}{2}, A_2 = 2, A_3 = -\frac{1}{2}$

Thus 
$$\frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} = \frac{1}{2x} + \frac{2}{x-1} - \frac{1}{2(x+2)}$$

Plug it back into the integral:

$$\begin{aligned} \int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx &= \int \frac{1}{2x} dx + \int \frac{2}{x-1} dx - \int \frac{1}{2(x+2)} dx \\ &= \frac{1}{2} \ln|x| + 2 \ln|x-1| - \frac{1}{2} \ln|x+2| + C \end{aligned}$$

Hence

$$\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx = \frac{1}{2} \ln|x| + 2 \ln|x-1| - \frac{1}{2} \ln|x+2| + C$$

**Case 2**  $g(x)$  has only repeated linear factors.

$$g(x) = (ax + b)^n$$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$$

where  $A_1, A_2, \dots, A_n$  are all constant we need to find.

**Example 38** Evaluate  $\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx$

Solution Consider

$$\frac{x^2 + 2x + 3}{(x-1)(x+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}$$

$$\begin{aligned} x^2 + 2x + 3 &= A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1) \\ &= A_1(x^2 + 2x + 1) + A_2(x^2 - 1) + A_3(x-1) \end{aligned}$$

Compare the coefficients:

$$A_1 + A_2 = 1$$

$$2A_1 + A_3 = 2$$

$$A_1 - A_2 - A_3 = 3$$

Solve to get  $A_1 = \frac{3}{2}, A_2 = -\frac{1}{2}, A_3 = -1$

Thus 
$$\frac{x^2 + 2x + 3}{(x-1)(x+1)^2} = \frac{3}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{(x+1)^2}$$

Plug it back to the integral:

$$\begin{aligned} \int \frac{(x^2 + 2x + 3)dx}{(x-1)(x+1)^2} &= \frac{3}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} \\ &= \frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + \frac{1}{(x+1)} + C \end{aligned}$$

Hence,

$$\int \frac{(x^2 + 2x + 3)dx}{(x-1)(x+1)^2} = \frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + \frac{1}{(x+1)} + C$$

**Case 3**  $g(x)$  has only non repeated irreducible quadratic factors  $ax^2 + bx + c$ :

$$\frac{f(x)}{g(x)} = \frac{Ax + B}{ax^2 + bx + c}$$

where  $A, B$  are constants we need to find.

**Example 39** Evaluate  $\int \frac{5x^2 + 3x - 2}{x^3 - 1} dx$

Solution Consider

$$\frac{5x^2 + 3x - 2}{x^3 - 1} = \frac{5x^2 + 3x - 2}{(x-1)(x^2 + x + 1)}$$

$$\frac{5x^2 + 3x - 2}{x^3 - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}$$

$$5x^2 + 3x - 2 = A(x^2 + x + 1) + (Bx + C)(x-1)$$

$$= A(x^2 + x + 1) + Bx^2 - Bx + Cx - C$$

Compare the coefficients:

$$A + B = 5$$

$$A - B + C = 3$$

$$A - C = -2$$

Solve to get  $A = 2, B = 3, C = 4$

Thus

$$\frac{5x^2 + 3x - 2}{x^3 - 1} = \frac{2}{x-1} + \frac{3x + 4}{x^2 + x + 1}$$



Plug it back into the integral:

$$\begin{aligned}\int \frac{(5x^2 + 3x - 2)dx}{x^3 - 1} &= \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2 + x + 1} \\ &= \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2 + x + 1} \\ &= 2\ln|x-1| + \int \frac{(3x+4)dx}{x^2 + x + 1}\end{aligned}$$

Next consider  $\int \frac{(3x+4)dx}{x^2 + x + 1} = \int \frac{(3x+4)dx}{\left[x + \frac{1}{2}\right]^2 + \frac{3}{4}}$

Let  $u = x + \frac{1}{2}$  and  $du = dx$

Thus

$$\begin{aligned}\int \frac{(3x+4)dx}{\left[x + \frac{1}{2}\right]^2 + \frac{3}{4}} &= \int \frac{3\left[u - \frac{1}{2}\right] + 4}{u^2 + \frac{3}{4}} du \\ &= \int \frac{3u + \frac{5}{2}}{u^2 + \frac{3}{4}} du \\ &= 3 \int \frac{udu}{u^2 + \frac{3}{4}} + \frac{5}{2} \int \frac{du}{u^2 + \frac{3}{4}} \\ &= \frac{3}{2} \ln(u^2 + \frac{3}{4}) + \frac{5(2)}{2(\sqrt{3})} \tan^{-1} \frac{2}{\sqrt{3}} u + C\end{aligned}$$

$$= \frac{3}{2} \ln(x^2 + x + 1) + \frac{5}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$$

Hence

$$\int \frac{(5x^2 + 3x - 2)dx}{x^3 - 1} = 2 \ln|x-1| + \frac{3}{2} \ln(x^2 + x + 1) + \frac{5}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$$

**Case 4**  $g(x)$  has only repeated irreducible quadratic factors:

$(ax^2 + bx + c)^n, n \geq 2$ :

$$\frac{f(x)}{g(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where  $A_1, \dots, A_n, B_1, \dots, B_n$  are all constants we need to find.

**Example 40** Evaluate  $\int \frac{(x^3 + 1)dx}{(x^2 + 4)^2}$

Solution Consider

$$\begin{aligned} \frac{x^3 + 1}{(x^2 + 4)^2} &= \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} \\ x^3 + 1 &= (Ax + B)(x^2 + 4) + (Cx + D) \\ &= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D \end{aligned}$$

Compare coefficients:

$$A = 1$$

$$B = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

Solve to get  $A = 1$ ,  $B = 0$ ,  $C = -4$ ,  $D = 1$

Thus 
$$\frac{x^3 + 1}{(x^2 + 4)^2} = \frac{1x + 0}{x^2 + 4} + \frac{-4x + 1}{(x^2 + 4)^2}$$

Plug it back into the integral:

$$\begin{aligned} \int \frac{(x^3 + 1)dx}{(x^2 + 4)^2} &= \int \frac{xdx}{x^2 + 4} - 4 \int \frac{xdx}{(x^2 + 4)^2} + \int \frac{dx}{(x^2 + 4)^2} \\ &= \frac{1}{2} \ln(x^2 + 4) - 4 \int \frac{xdx}{(x^2 + 4)^2} + \int \frac{dx}{(x^2 + 4)^2} \quad \text{Next} \end{aligned}$$

consider 
$$-4 \int \frac{xdx}{(x^2 + 4)^2}$$

Let  $u = x^2 + 4$  and  $du = 2xdx$

So we have 
$$\begin{aligned} -4 \int \frac{xdx}{(x^2 + 4)^2} &= -2 \int \frac{du}{u^2} \\ &= 2u^{-1} + C \\ &= \frac{2}{x^2 + 4} + C \end{aligned}$$

And for 
$$\int \frac{dx}{(x^2 + 4)^2}$$

We let  $x = 2 \tan \theta$  and  $dx = 2 \sec^2 \theta d\theta$

We then have 
$$\int \frac{dx}{(x^2 + 4)^2} = \int \frac{2 \sec^2 \theta d\theta}{(4 \tan^2 \theta + 4)^2}$$

$$\begin{aligned}
&= \frac{1}{8} \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^2} \\
&= \frac{1}{8} \int \frac{d\theta}{\sec^2 \theta} \\
&= \frac{1}{8} \int \cos^2 \theta d\theta \\
&= \frac{1}{16} \int (1 + \cos 2\theta) d\theta \\
&= \frac{1}{16} \left[ \theta + \frac{\sin 2\theta}{2} \right] + C \\
&= \frac{1}{16} \left[ \tan^{-1} \frac{x}{2} + \frac{2x}{(x^2 + 4)} \right] + C
\end{aligned}$$

Hence

$$\begin{aligned}
\int \frac{(x^3 + 1)dx}{(x^2 + 4)^2} &= \frac{1}{2} \ln(x^2 + 4) + \frac{2}{x^2 + 4} + \frac{1}{16} \tan^{-1} \frac{x}{2} \\
&\quad + \frac{x}{8(x^2 + 4)} + C
\end{aligned}$$

**Example 41** Evaluate  $\int \frac{(x^5 - x^4 - 3x + 5)}{x^4 - 2x^3 + 2x^2 - 2x + 1} dx$

Solution

**Exercise 1**

Evaluate the following integrals

1.  $\int 3x^2(x^3 + 2)^2 dx$
2.  $\int x^2 \sqrt{x^3 + 2} dx$
3.  $\int \frac{8x^2}{(x^3 + 2)} dx$
4.  $\int \frac{x^2}{\sqrt{x^3 + 2}} dx$
5.  $\int 3x\sqrt{1 - 2x^2} dx$
6.  $\int \frac{x + 3}{\sqrt[3]{x^2 + 6x}} dx$
7.  $\int (3x^2 - 2)(x^3 - 2x) dx$
8.  $\int \frac{x + 1}{x^2 + 2x + 5} dx$
9.  $\int x^2 \sqrt{1 + x} dx$
10.  $\int \frac{x^2}{1 - 2x^3} dx$
11.  $\int (e^x + 1)^3 dx$
12.  $\int \cos^3 2x \sin 2x dx$
13.  $\int e^{\cos x} \sin x dx$
14.  $\int \frac{\cos x dx}{\sqrt{4 - \sin^2 x}}$
15.  $\int \frac{e^{\sqrt{1+x}}}{\sqrt{1+x}} dx$
16.  $\int \cos 2x \sqrt{1 - \sin 2x} dx$
17.  $\int 3^{2x+1} dx$
18.  $\int \frac{e^{\tan^{-1} 2x}}{1 + 4x^2} dx$
19.  $\int \left[ \frac{\ln x}{x} \right]^3 dx$
20.  $\int \frac{1}{x \ln x} dx$

Evaluate the following definite integrals

$$21. \int_1^5 \frac{x+3}{\sqrt{2x-1}} dx$$

$$22. \int_0^1 \frac{x}{x^2+4} dx$$

$$23. \int_1^8 \sqrt{1+3x} dx$$

$$24. \int_4^8 \frac{xdx}{\sqrt{x^2-15}}$$

$$25. \int_0^{2\pi} \sin \frac{x}{2} dx$$

### Answers to exercise 1

$$1. \left[ \frac{x^3+2}{3} \right]^3 + C$$

$$2. \frac{2}{9}(x^3+2)^{\frac{3}{2}} + C$$

$$3. \frac{-4}{3(x^3+2)^2} + C$$

$$4. \frac{2}{3}\sqrt{x^3+2} + C$$

$$5. -\frac{1}{2}(1-2x^2)^{\frac{3}{2}} + C$$

$$6. \frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$$

$$7. \frac{1}{6}(x^3-2x)^6 + C$$

$$8. \frac{1}{2}\ln|x^2+2x+5| + C$$

$$9. \frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C$$

$$10. -\frac{1}{6}\ln|1-2x^3| + C$$

$$11. \frac{1}{4}(e^x+1) + C$$

$$12. -\frac{\cos^4 2x}{8} + C$$

$$13. -e^{\cos x} + C$$

14.  $\sin^{-1}\left(\frac{\sin x}{2}\right) + C$

15.  $2e^{\sqrt{1+x}} + C$

16.  $-\frac{1}{3}(1 - \sin 2x)^{\frac{3}{2}} + C$

17.  $\frac{3^{2x+1}}{2\ln 3} + C$

18.  $\frac{1}{2}e^{\tan^{-1} 2x} + C$

19.  $\frac{1}{4}[\ln x]^4 + C$

20.  $\ln|\ln x| + C$

21. 20

22.  $\frac{1}{2}\ln \frac{5}{4}$

23. 26

24. 6

25. 4

**Exercise 2**

Evaluate the following integrals

1.  $\int x \sin x dx$

2.  $\int x e^x dx$

3.  $\int x^2 \ln x dx$

4.  $\int x \sqrt{1+x} dx$

5.  $\int \sec^3 x dx$

6.  $\int x^2 \sin x dx$

7.  $\int x^2 e^{2x} dx$

8.  $\int x \cos x dx$

9.  $\int x \sec^2 3x dx$

10.  $\int \cos^{-1} 2x dx$

11.  $\int \tan^{-1} x dx$

12.  $\int \frac{x e^x}{(1+x)^2} dx$

13.  $\int x \tan^{-1} x dx$

14.  $\int x^2 e^{-3x} dx$

15.  $\int x^3 \sin x dx$

16.  $\int x \sin^{-1} x^2 dx$

17.  $\int \sin x \sin 3x dx$

18.  $\int \sin(\ln x) dx$

19.  $\int e^{ax} \cos b x dx$

20.  $\int e^{ax} \sin b x dx$



Show how to use reduction formula to the following integrals.

$$21. \int u^n e^{au} du$$

$$22. \int u^n \cos bu du$$

Evaluate the following definite integrals

$$23. \int_1^e \ln x dx$$

$$24. \int_0^{\frac{\pi}{3}} x^2 \sin 3x dx$$

$$25. \int_0^{\sqrt{2}} x^3 e^{x^2} dx$$

### Answers to exercise 2

$$1. -x \sin x + \sin x + C$$

$$2. xe^x - e^x + C$$

$$3. \frac{x^3 \ln x}{3} - \frac{1}{9} x^3 + C$$

$$4. \frac{2}{3} x(1+x)^{\frac{3}{2}} - \frac{4}{15} (1+x)^{\frac{5}{2}} + C$$

$$5. \frac{1}{2} (\sec x \tan x + \ln |\sec x \tan x|) + C$$

$$6. -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

$$7. \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + C$$

$$8. x \sin x + \cos x + C$$

$$9. \frac{1}{3} x \tan 3x - \frac{1}{9} \ln |\sec 3x| + C$$

$$10. x \cos^{-1} 2x - \frac{1}{2} \sqrt{1-4x^2} + C$$

$$11. x \tan^{-1} x - \ln \sqrt{1+x^2} + C$$

$$12. \frac{e^x}{1+x} + C$$

$$13. \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C$$

$$14. -\frac{1}{3} e^{-3x} \left( x^2 + \frac{2}{9} x + \frac{2}{9} \right) + C$$

$$15. -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

$$16. \frac{1}{2} x^2 \sin^{-1} x^2 + \frac{1}{2} \sqrt{1-x^4} + C$$

$$17. \frac{1}{8} \sin 3x \cos x - \frac{3}{8} \cos 3x \sin x + C$$

$$18. \frac{1}{2} [x \sin(\ln x) - x \cos(\ln x)] + C$$

$$19. \frac{e^{ax} (b \sin bx + a \cos bx)}{a^2 + b^2} + C$$

$$20. \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} + C$$

$$21. \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} du$$

$$22. \frac{1}{b} u^n \sin bu - \frac{n}{b} \int u^{n-1} \sin bu du$$

$$23. 1$$

$$24. \frac{1}{27}(\pi^2 - 4)$$

$$25. \frac{1}{2}(e^2 + 1)$$

### Exercise 3

Evaluate the following integrals

$$1. \int \frac{1}{x^2 - 4} dx$$

$$2. \int \frac{x+1}{x^3 + x^2 - 6x} dx$$

$$3. \int \frac{1}{x^2 + 7x + 6} dx$$

$$4. \int \frac{x}{x^2 - 3x - 4} dx$$

$$5. \int \frac{x^2 - 3x - 1}{x^3 + x^2 - 2x} dx$$

$$6. \int \frac{x^2 + 3x - 4}{x^2 - 2x - 8} dx$$

$$7. \int \frac{x}{(x-2)^2} dx$$

$$8. \int \frac{3x+5}{x^3 - x^2 - x + 1} dx$$

$$9. \int \frac{x^4 - x^3 - x - 1}{x^3 - x^2} dx$$

$$10. \int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} dx$$

$$11. \int \frac{x^2}{a^4 - x^4} dx$$

$$12. \int \frac{2x^2 + 3}{(x^2 + 1)^2} dx$$

$$13. \int \frac{1}{x^3 + x} dx$$

$$14. \int \frac{x^3 + x^2 + x + 3}{(x^2 + 1)(x^2 + 3)} dx$$

$$15. \int \frac{2x^3}{(x^2 + 1)^2} dx$$

$$16. \int \frac{2x^3 + x^2 + 4}{(x^2 + 4)^2} dx$$

$$17. \int \frac{x^3 + x - 1}{(x^2 + 1)^2} dx$$

$$18. \int \frac{x^4}{(1-x)^3} dx$$

$$19. \int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2 + 2)^3} dx$$

$$20. \int \frac{1}{e^{2x} - 3e^x} dx$$

$$21. \int \frac{\sin x}{\cos x(1 + \cos^2 x)} dx$$

$$22. \int \frac{(2 + \tan^2 \theta) \sec^2 \theta}{1 + \tan^3 \theta} d\theta$$

Evaluate the following definite integrals

$$23. \int_{-1}^2 \frac{1}{x^2 - 9} dx$$

$$24. \int_{-8}^{-3} \frac{x + 2}{x(x - 2)^2} dx$$

$$25. \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x dx}{\cos^2 x - 5 \cos x + 4}$$

### Answers to exercise 3

$$1. \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$2. \frac{1}{30} \ln \left| \frac{(x-2)^9}{(x)^5 (x+3)^4} \right| + C$$

$$3. \frac{1}{5} \ln \left| \frac{x+1}{x+6} \right| + C$$

$$4. \frac{1}{5} \ln |(x-4)(x+1)^4| + C$$

$$5. \frac{1}{2} \ln \left| \frac{x(x+2)^3}{(x-1)^2} \right| + C$$

$$6. x + \ln |(x-4)^4 (x+2)| + C$$

$$7. \ln |x-2| - \frac{2}{x-2} + C$$

$$8. -\frac{4}{x-1} + \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + C$$

9.  $\frac{x^2}{2} - \frac{1}{x} - 2 \ln \left| \frac{x-1}{x} \right| + C$
10.  $\tan^{-1} x + \frac{1}{2} \ln |x^2 + 2| + C$
11.  $\frac{1}{4a} \ln \left| \frac{a+x}{a-x} \right| - \frac{1}{2a} \tan^{-1} \frac{x}{a} + C$
12.  $\frac{5}{2} \tan^{-1} x + \frac{x}{2(x^2 + 1)} + C$
13.  $\ln \left| \frac{x}{\sqrt{x^2 + 1}} \right| + C$
14.  $\ln |\sqrt{x^2 + 3}| + \tan^{-1} x + C$
15.  $\ln |x^2 + 1| + \frac{1}{x^2 + 1} + C$
16.  $\ln |x^2 + 4| + \frac{1}{2} \tan^{-1} \frac{x}{2} + \frac{4}{x^2 + 4} + C$
17.  $\frac{1}{2} \ln |x^2 + 1| - \frac{1}{2} \tan^{-1} x - \frac{x}{2(x^2 + 1)} + C$
18.  $-\frac{x^2}{2} - 3x - 6 \ln |1 - x| - \frac{4}{1 - x} + \frac{1}{2(1 - x)^2} + C$
19.  $\frac{1}{2} \ln |x^2 + 2| - \frac{\sqrt{2}}{2} \tan^{-1} \frac{x}{\sqrt{2}} - \frac{x}{(x^2 + 2)^2} + C$
20.  $\frac{1}{3e^x} + \frac{1}{9} \ln \left| \frac{e^x - 3}{e^x} \right| + C$
21.  $\ln \left| \frac{\sqrt{1 + \cos^2 x}}{\cos x} \right| + C$

$$22. \ln|1 + \tan \theta| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2 \tan \theta - 1}{\sqrt{3}} + C$$

$$23. -\frac{1}{6} \ln 10$$

$$24. \frac{1}{2} \ln \frac{3}{4} + \frac{1}{5}$$

$$25. \frac{1}{3} \ln \left| \frac{-\frac{\sqrt{2}}{2} - 1}{-\frac{\sqrt{2}}{2} - 4} \right| - \frac{1}{3} \ln \left| \frac{-\frac{\sqrt{2}}{2} - 1}{\frac{\sqrt{2}}{2} - 4} \right|$$

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## Applications of the Definite Integral

If  $y = f(x)$  is a continuous function on  $a \leq x \leq b$  and  $F(x)$  is an antiderivative of  $f(x)$  and may be denoted by

$$\int f(x) dx = F(x) + C, \text{ where } C \text{ is some constant.} \quad (1)$$

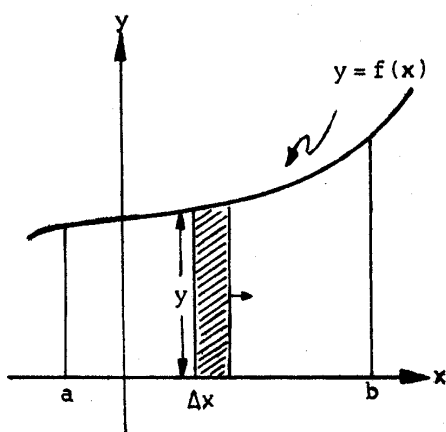
The definite integral of  $f(x)$  on the interval  $(a, b)$  is

$$\int_a^b f(x) dx = F(b) - F(a) \quad (2)$$

The definite integrals have a lot of applications in geometry and physics such as area under a curve, area between curves, volume, arc length, surface area, moment and work.

### 1. Area Under a Curve

If  $y = f(x)$  is a non-negative and continuous function on  $a \leq x \leq b$  as shown in figure 1



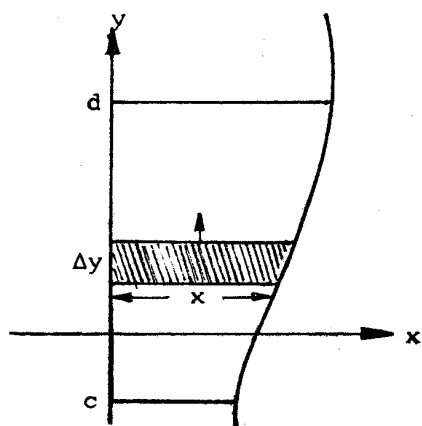
Area under the curve of  $y = f(x)$  from  $x = a$  to  $x = b$  as shown here is

$$A = \int_a^b y dx .$$

**Figure 1**



If we want to find the area covered by the curves of  $x = g(y)$  where  $g(y) \geq 0$ ,  $y$ -axis,  $y = c$  and  $y = d$  as shown in Figure 2, we partition the area into  $n$  small parts, all parts' widths are denoted by  $\Delta y_1, \Delta y_2, \dots, \Delta y_n$  and each length is  $x = g(y)$ .



**Consider the  $i^{\text{th}}$  partition.**

Area  $\Delta A_i \approx x \cdot \Delta y_i = \text{width} \times \text{length}$ .

Then,  $A \approx \sum_{i=1}^n \Delta A_i = \sum_{i=1}^n (x \cdot \Delta y_i)$ .

If  $n \rightarrow \infty$  ( or  $\Delta y_i \rightarrow 0$  ), we have

$$A = \lim_{\Delta y_i \rightarrow 0} \sum_{i=1}^n (x \cdot \Delta y_i) = \int_c^d x dy.$$

**Figure 2**

### Summary Area under a curve

1. Area covered by the curves of  $y = f(x)$  where

$f(x) \geq 0$ ,  $x$ -axis,  $x = a$ , and  $x = b$  is

$$A = \int_a^b y dx. \quad (3)$$

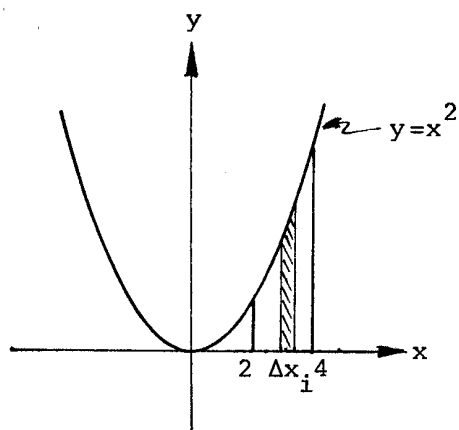
2. Area covered by the curves of  $x = g(y)$  where

$g(y) \geq 0$ ,  $y$ -axis,  $y = c$  and  $y = d$  is

$$A = \int_c^d x dy. \quad (4)$$

**Example 1** Compute the area covered by  $y = x^2$ , the  $x$ -axis,  $x = 2$  and  $x = 4$ .

**Solution**



Partition along the  $x$ -axis

$$\Delta A_i = y \cdot \Delta x_i$$

$$A = \int_2^4 y \, dx$$

$$= \int_2^4 x^2 \, dx = \left[ \frac{x^3}{3} \right]_2^4$$

$$= \frac{4^3}{3} - \frac{2^3}{3} = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

$$= 18\frac{2}{3} \text{ unit}^3.$$

**Example 2** Find the area covered by  $y = x^3$ ,  $x = -1$ ,  $x = 2$  and the  $x$ -axis.

**Solution**

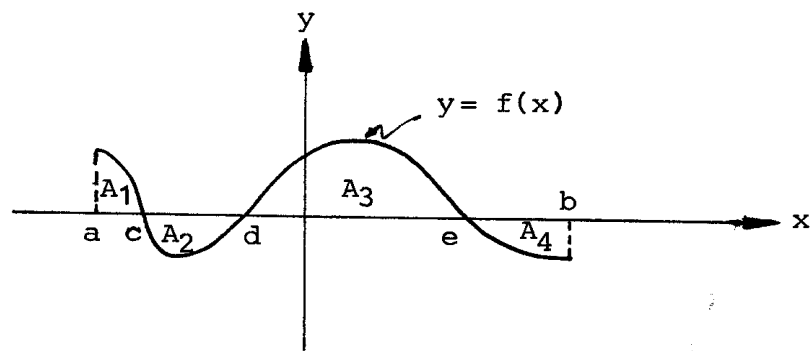
**Remark** The area is a non-negative value, but the definite integral may be negative. So, we may write the area as  $A = \left| \int_a^b f(x) dx \right|$ .

If we integrate along the  $x$ -axis, the definite integral is positive when the graph is above  $x$ -axis and negative when the graph is below  $x$ -axis. For example, as in example 2,

$$\int_{-1}^2 y dx = \int_{-1}^2 x^3 dx = \left[ \frac{x^4}{4} \right]_{x=-1}^{x=2} = \left[ \frac{2^4}{4} - \frac{(-1)^4}{4} \right] = 4 - \frac{1}{4} = 3\frac{3}{4}.$$

It is the area under the curve above  $x$ -axis from 0 to 2 minus the area below the  $x$ -axis from -1 to 0.

To find the total area of under the curve of  $y = f(x)$ ,  $a \leq x \leq b$  as shown here



**Figure 5**

### Total area

$$\begin{aligned}
 A &= |A_1| + |A_2| + |A_3| + |A_4| \\
 &= \left| \int_a^c f(x) dx \right| + \left| \int_c^d f(x) dx \right| + \left| \int_d^e f(x) dx \right| + \left| \int_e^b f(x) dx \right|.
 \end{aligned}$$

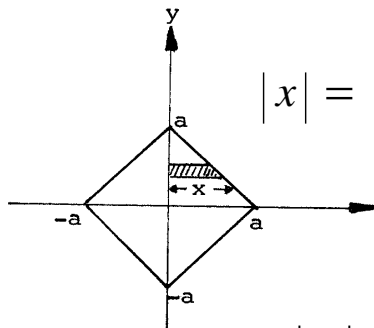
1. Analogously, if we integrate along the  $y$ -axis, the definite integral is positive when the graph is on the right and negative when the graph is on the left of the  $y$ -axis.

2. If a graph is symmetric, we can integrate just one part and multiply by number of symmetries as shown in example 3.

**Example 3** Compute the area covered by  $|x| + |y| = a$ .

### Solution

By definition of absolute value



**Figure 6**

$$\begin{aligned}
 |x| &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}, \text{ we have} \\
 |x| + |y| = a &\rightarrow \begin{cases} x + y = a & \text{when } x \geq 0 \text{ and } y \geq 0 \\ x - y = a & \text{when } x \geq 0 \text{ and } y < 0 \\ -x + y = a & \text{when } x < 0 \text{ and } y \geq 0 \\ -x - y = a & \text{when } x < 0 \text{ and } y < 0 \end{cases}
 \end{aligned}$$

As we can see, this graph is symmetric about the origin. So we can just find the area in the first Quadrant, called it  $A_1$ . The total area is then four times  $A_1$ .

**Consider**  $A_1$  If partition along the  $y$ -axis,

$$\Delta A_1 = x \cdot \Delta y \quad \text{where} \quad x = a - y.$$

$$\begin{aligned} A_1 &= \int_0^a (a - y) dy = \left[ ay - \frac{y^2}{2} \right]_{y=0}^{y=a} \\ &= a^2 - \frac{a^2}{2} = \frac{a^2}{2}. \end{aligned}$$

Finally, we obtain

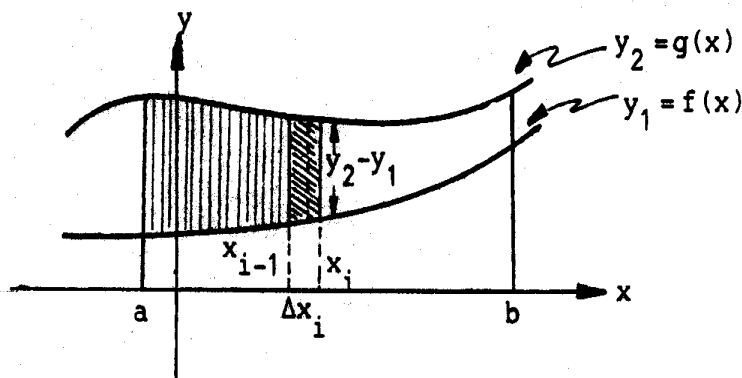
$$A = 4A_1 = \frac{4a^2}{2} = 2a^2 \text{ unit}^3.$$


---

## 2. Area Between Curves

### 2.1 Rectangular Form

If  $y_1 = f(x)$  and  $y_2 = g(x)$  are continuous functions such that  $y_2 \geq y_1$  for  $a \leq x \leq b$ , we may compute the areas between these two curves  $y_1, y_2$  from  $x = a$  and  $x = b$  as shown below.



**Figure 7**

Partition the area into small  $n$  parts with widths  $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ .

Let  $\Delta A_i$  = the area of the  $i^{th}$  partition.

Then  $\Delta A_i \approx (y_2 - y_1) \cdot \Delta x_i = \text{width} \times \text{length}$

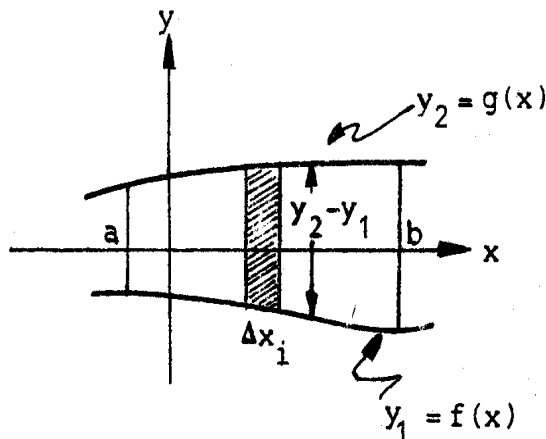
Thus, the total area

$$A \approx \sum_{i=1}^n \Delta A_i \approx \sum_{i=1}^n (y_2 - y_1) \cdot \Delta x_i.$$

If  $\Delta x_i \rightarrow 0$ , the length  $(y_2 - y_1)$  of the interval  $(x_{i-1}, x_i)$  will approach  $(y_2 - y_1)$  at  $x_{i-1}$  and  $x_i$ . Thus, the approximation is closer and closer to the exact area. Therefore,

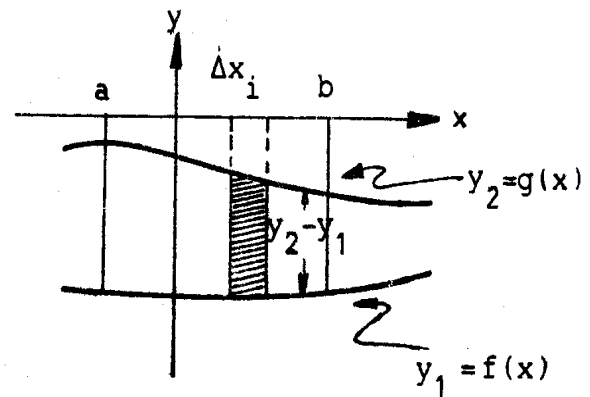
$$A = \lim_{\substack{\Delta x_i \rightarrow 0 \\ n \rightarrow \infty}} \sum_{i=1}^n \Delta A_i = \lim_{\substack{\Delta x_i \rightarrow 0 \\ n \rightarrow \infty}} \sum_{i=1}^n (y_2 - y_1) \Delta x_i = \int_a^b (y_2 - y_1) dx.$$

This formula is always valid if  $y_2 > y_1$ . The above or below  $x$ -axis locations do not matter. Here are some examples.



$$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$$

**Figure 8**



$$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$$

**Figure 9**

**Note** If  $y_2 \geq y_1$ ,  $y_2$  is always above  $y_1$ .

**Summary** If  $y_1 = f(x)$  and  $y_2 = g(x)$  are continuous functions such that  $y_2 \geq y_1$  for  $a \leq x \leq b$ , then the area covered by the curves  $y_1$  and  $y_2$  from  $x = a$  to  $x = b$  is

$$A = \int_a^b (y_2 - y_1) dx = \int_a^b (g(x) - f(x)) dx . \quad (5)$$

Analogously if  $g_1(y)$  and  $g_2(y)$  are continuous function such that  $g_2(y) \geq g_1(y)$  for  $c \leq y \leq d$ , we may compute the area covered by  $x_1 = g_1(y)$ ,  $x_2 = g_2(y)$  from  $y = c$  to  $y = d$ . For  $x_2 > x_1$  as shown in three figures below, we partition along the  $y$ -axis to  $n$  parts with widths  $\Delta y_1, \Delta y_2, \dots, \Delta y_n$ .

Area of the  $i^{th}$  partition is

$$\Delta A_i \approx (x_2 - x_1) \cdot \Delta y_i = \text{length} \times \text{width} .$$

Thus, the total area

$$A = \lim_{\substack{\Delta y_i \rightarrow 0 \\ (n \rightarrow \infty)}} \sum_{i=1}^n \Delta A_i = \lim_{\substack{\Delta y_i \rightarrow 0 \\ (n \rightarrow \infty)}} \sum_{i=1}^n (x_2 - x_1) \Delta y_i = \int_c^d (x_2 - x_1) dy .$$

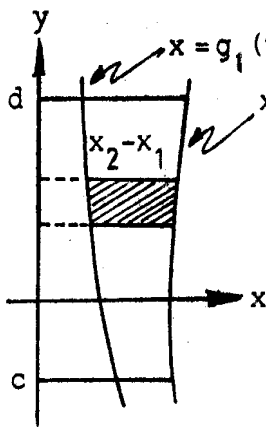


Figure 10

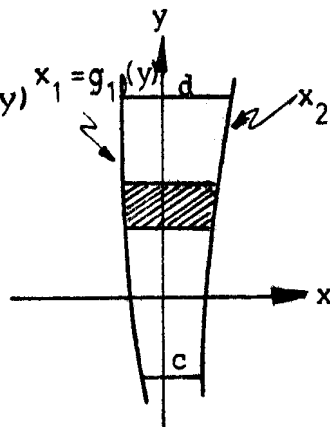


Figure 11

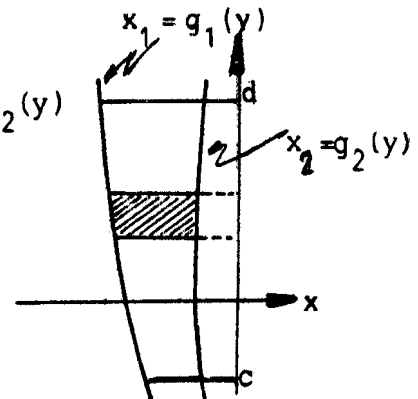


Figure 12

**Summary** If  $x_1 = g_1(y)$  and  $x_2 = g_2(y)$  are continuous functions such that  $x_2 \geq x_1$  for  $c \leq y \leq d$ , then the area enclosed by  $x_1$ ,  $x_2$ ,  $y = c$  and  $y = d$  is

$$A = \int_c^d (x_2 - x_1) dy = \int_c^d (g_2(y) - g_1(y)) dy . \quad (6)$$



**Example 4** Compute the area covered by  $x^2 = y$ ,  $x^2 = 4y$  and the line  $x = 2$ .

**Approach 1** Partition along the  $x$ -axis.

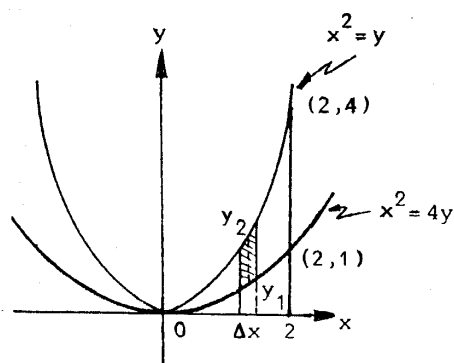


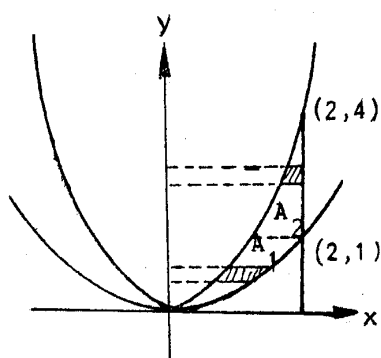
Figure 13

$$\text{Then } \Delta A = (y_2 - y_1) \cdot \Delta x$$

$$\text{where } y_2 = x^2 \text{ and } y_1 = \frac{x^2}{4}.$$

$$\begin{aligned} A &= \int_0^2 (y_2 - y_1) dx = \int_0^2 \left( x^2 - \frac{x^2}{4} \right) dx \\ &= \frac{3}{4} \int_0^2 x^2 dx = \frac{3}{4} \left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = 2. \end{aligned}$$

**Approach 2** Partition along the  $y$ -axis: there are 2 parts.



$A_1$ :  $y$  from  $0 \rightarrow 1$ , we have

$$\Delta A_1 = (x_2 - x_1) \Delta y$$

$$\text{where } x_2 = \sqrt{4y} \text{ and } x_1 = \sqrt{y}.$$

(Do not forget that  $x_2$  is on the right of  $x_1$ )

$$\begin{aligned} A_1 &= \int_0^1 (x_2 - x_1) dy = \int_0^1 (\sqrt{4y} - \sqrt{y}) dy \\ &= \int_0^1 \sqrt{y} dy = \left[ \frac{2}{3} y^{3/2} \right]_{y=0}^{y=1} = \frac{2}{3}. \end{aligned}$$

$A_2$ :  $y$  from  $1 \rightarrow 4$ , we have

$$\Delta A_1 = (x_2 - x_1) \Delta y \quad \text{where } x_2 = 2 \text{ and } x_1 = \sqrt{y}.$$

$$\begin{aligned} A_2 &= \int_1^4 (x_2 - x_1) dy = \int_1^4 (2 - \sqrt{y}) dy \\ &= \left[ 2y - \frac{2}{3} y^{3/2} \right]_{y=1}^{y=4} = \frac{4}{3}. \end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = 2.$$

No matter which approach you choose, the correct answer is always the same.

**Example5** Compute the area covered by  $y^2 = 2x$  and  $x - y = 4$ .

**Solution**

**Example 6** Compute the area covered by  $y = -x^2 - 2x + 3$ , its tangent line at  $(2, -5)$  and the  $y$ -axis.

**Solution** Consider  $y = -x^2 - 2x + 3 = -(x^2 + 2x + 1) + 4$ .

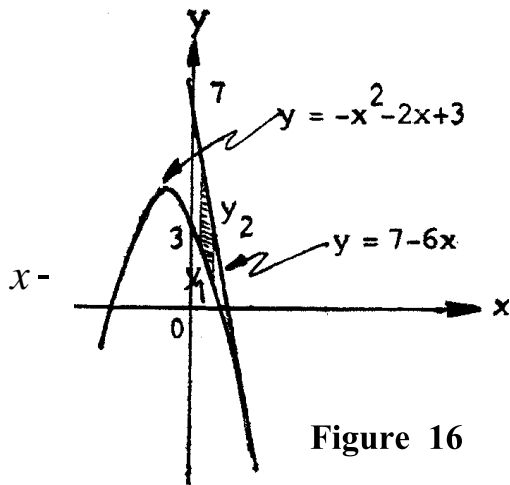


Figure 16

$y - 4 = -(x + 1)^2$  is a parabolic curve having a vertex at  $(-1, 4)$ . This curve has the  $y$ -intercept as  $(0, 3)$  and has the intercept at  $x = -3$ , and  $x = 1$ .

Consider  $\frac{dy}{dx} = -2x - 2$ .

The slope of the tangent line at  $(2, -5)$  is  $-2(2) - 2 = -6$ .

The equation of this tangent line can be found by  $y - y_1 = m(x - x_1)$ .

Here, we have  $y_1 = -5$ ,  $x_1 = 2$ ,  $m = -6$ .

So,  $y - (-5) = -6(x - 2)$ . That is, we have the equation of the tangent line of this parabola at  $(2, -5)$  is  $y = 7 - 6x$ .

If partition on  $x$ -axis,

$$\Delta A = (y_2 - y_1) \Delta x$$

where  $y_2 = 7 - 6x$  and  $y_1 = -x^2 - 2x + 3$ .

$$\begin{aligned} \text{Then, } A &= \int_0^2 \left[ (7 - 6x) - (-x^2 - 2x + 3) \right] dx \\ &= \int_0^2 \left[ 4 - 4x + x^2 \right] dx = \frac{8}{3}. \end{aligned}$$

**Exercise 1**

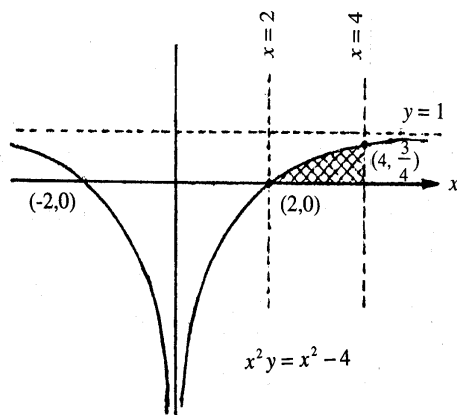
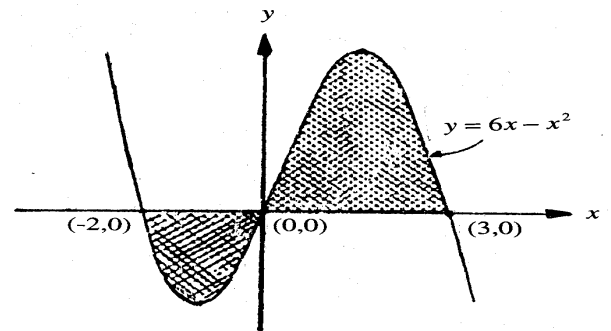
Compute each area covered by the following graphs

1.  $x$  - axis,  $y = 2x - x^2$
2.  $y$  - axis,  $x = y^2 - y^3$
3.  $y^2 = x$ ,  $x = 4$
4.  $y = 2x - x^2$ ,  $y = -3$
5.  $y = x^2$ ,  $y = x$
6.  $x = 3y - y^2$ ,  $x + y = 3$
7.  $y = x^4 - 2x^2$ ,  $y = 2x^2$
8. First part of  $y = \sin x$
9.  $y$  - axis,  $y^2 - 4x - 4 = 0$
10. Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
11.  $x = y^2$ ,  $x = y$
12.  $y^2 = 8x$ ,  $x^2 = 4y$
13.  $x^2 - 5x + y = 0$ ,  $y = x$
14.  $y^2 = 9x$ ,  $y^2 = x^3$
15.  $y = x^2$ ,  $y = x$ ,  $y = 2x$
16.  $y^2 = 4x$ ,  $2x - y - 4 = 0$
17.  $y = x^3 - 4x$ ,  $x$  - axis
18.  $x + 2y = 2$ ,  $y - x = 1$ ,  $2x + y = 7$
19.  $x^2y = x^2 - 4$ ,  $x$  - axis,  $x = 2$  and  $x = 4$

20.  $y = 6x + x^2 - x^3$ ,  $x$ -axis
21.  $f(x) = \begin{cases} x^2 & , \ x \leq 2 \\ -x + 6, & x > 2 \end{cases}$  from  $x = 0$  and  $x = 3$
22.  $y = x(x-3)(x+3)$ ,  $y = -5x$
23.  $y = x^2$ ,  $y = 8 - x^2$  and  $y = 4x + 12$
24.  $x = 0$ ,  $x = 2$ ,  $y = 2^x$  and  $y = 2x - x^2$
25.  $x = -2y^2$ ,  $x = 1 - 3y^2$
26.  $y = x + 1$ ,  $y = \cos x$  and the  $x$ -axis (largest region)
27. One loop of  $y^2 = (x-1)(x-2)^2$
28.  $y = x^2 - 2x + 2$ , its tangent line at the point  $M(3, 5)$ , the  $y$ -axis
29.  $\sqrt{x} + \sqrt{y} = 1$  and  $x + y = 1$
30.  $y = x^2$ ,  $y = 4$  This area is divided into 2 equal parts by the line  $y = c$ . Evaluate the value of  $c$ .
31.  $x^2 = 4y$ ,  $y = \frac{8}{x^2 + 4}$
32. One loop of  $y^2 = (x-1)^2$
33.  $y^2 = 4x$ ,  $x^2 = 4y$  and  $x^2 + y^2 = 5$  where  $x \geq 0$ ,  $y \geq 0$
34. Hypocycloid:  $x^{2/3} + y^{2/3} = a^{2/3}$
35.  $y^3 = x^2$  the cord connecting  $(-1, 1)$  and  $(8, 4)$
36.  $y^2 = x^2(1 - x^2)$
37.  $xy = 4$ ,  $y = x$ ,  $x = 5$  and  $x = \sqrt{-y}$

**Answer 1**

- |                     |                                  |                    |
|---------------------|----------------------------------|--------------------|
| 1. $\frac{4}{3}$    | 2. $\frac{1}{12}$                | 3. $\frac{32}{3}$  |
| 4. $\frac{32}{3}$   | 5. $\frac{1}{6}$                 | 6. $\frac{4}{3}$   |
| 7. $\frac{128}{15}$ | 8. 2                             | 9. $\frac{8}{3}$   |
| 10. $\pi ab$        | 11. $\frac{1}{6}$                | 12. $\frac{16}{3}$ |
| 13. $\frac{32}{3}$  | 14. $\frac{24\sqrt{3}}{5}$       | 15. $\frac{7}{6}$  |
| 16. 9               | 17. 8                            | 18. 6              |
| 19. 1 (Figure 17)   | 20. $\frac{253}{12}$ (Figure 18) |                    |

**Figure 17****Figure 18**

- |                                     |                   |                   |
|-------------------------------------|-------------------|-------------------|
| 21. $\frac{37}{6}$                  | 22. 8             | 23. 64            |
| 24. $\frac{3}{\ln 2} - \frac{4}{3}$ | 25. $\frac{4}{3}$ | 26. $\frac{3}{2}$ |
| 27. $\frac{8}{15}$                  | 28. 9             | 29. $\frac{1}{3}$ |

30.  $\frac{32}{3}, c = \sqrt[3]{16}$

31.  $2\pi - \frac{4}{3}$  (Figure 19)

32.  $\frac{8}{15}$  (Figure 20)

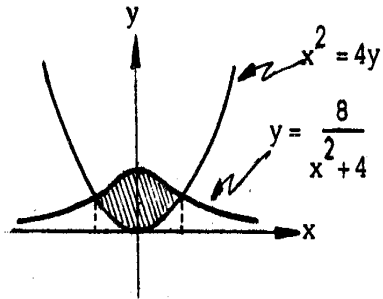


Figure 19

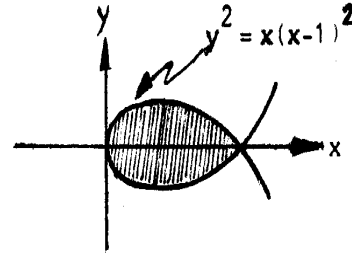


Figure 20

33.  $\frac{2}{3} + \frac{5}{2} \sin^{-1} \frac{3}{5}$

34.  $\frac{3}{8} \pi a^2$

35. 2.7

36.  $\frac{4}{3}$

37.  $4(\ln 5 - \ln 2) + \frac{131}{3}$



## IMPROPER INTEGRALS

### Introduction

Previously we learned about the definite integral  $\int_a^b f(x)dx$  where  $a$  and  $b$  are constants. The integrand  $f(x)$  is continuous and bounded on the interval  $[a, b]$ .

In this chapter we are interested in  $\int_a^b f(x)dx$  where limits of integration  $a$  and  $b$  may be infinity or the integrand function is unbounded at some points in the interval  $[a, b]$ . This type of integral called “*Improper Integral*.” There are three cases:

**Case 1** Limit of integration is infinity (Infinite interval)

$$[a, +\infty) \quad (-\infty, b] \quad (-\infty, +\infty)$$

For examples,

$$\int_1^{+\infty} \frac{dx}{x^2} \quad \int_{-\infty}^0 e^x dx \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

**Case 2** Integrand  $f(x)$  is unbounded at some point  $x=c$  in  $[a, b]$

$$\lim_{x \rightarrow c} f(x) = \pm\infty$$

For examples,

$$\int_{-3}^3 \frac{dx}{x^2} \quad \int_1^2 \frac{dx}{x-1} \quad \int_0^{\pi} \tan x dx$$

**Case 3** Combination of both case 1 and case 2

For examples,

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}} \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9} \quad \int_1^{+\infty} \sec x dx$$

### 1. Evaluation of improper integral case 1

**Definition** Let  $a$  be a real number and  $f$  be bounded and integrable on  $[a, t]$  for all  $t$  such that  $t > a$ . Thus the improper integral of  $f(x)$  on  $[a, +\infty)$  denoted by  $\int_a^{+\infty} f(x) dx$  is defined by

$$\int_a^{+\infty} f(x) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x) dx$$

**Remark**

- If  $\lim_{t \rightarrow +\infty} \int_a^t f(x) dx$  exists, then  $\int_a^{+\infty} f(x) dx$  converges.
- If  $\lim_{t \rightarrow +\infty} \int_a^t f(x) dx$  does not exist, then  $\int_a^{+\infty} f(x) dx$  diverges.

**Definition** Let  $b$  be a real number and  $f$  be bounded and integrable on  $[t, b]$  for all  $t$  such that  $t < b$ . Thus the improper integral of  $f(x)$  on  $(-\infty, b]$  denoted by  $\int_{-\infty}^b f(x) dx$  is defined by

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

**Remark**

- If  $\lim_{t \rightarrow -\infty} \int_t^b f(x) dx$  exists, then  $\int_{-\infty}^b f(x) dx$  converges.
- If  $\lim_{t \rightarrow -\infty} \int_t^b f(x) dx$  does not exist, then  $\int_{-\infty}^b f(x) dx$  diverges.

**Definition** Let  $f$  be a bounded and integrable function on  $[a, b]$  for constants  $a$  and  $b$ , where  $a < b$ . Thus the improper integral of  $f(x)$  on  $(-\infty, +\infty)$  denoted by  $\int_{-\infty}^{+\infty} f(x) dx$  is defined by

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx \quad \text{where } c \in \mathbb{R}$$

**Remark**

- $\int_{-\infty}^{+\infty} f(x) dx$  converges if both integral on the right converge.
- $\int_{-\infty}^{+\infty} f(x) dx$  diverges if at least one integral on the right diverges.

**Example** Evaluate  $\int_1^{+\infty} \frac{1}{x^3} dx$

**Solution**

$$\begin{aligned} \int_1^{+\infty} \frac{1}{x^3} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^3} dx \\ &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2x^2} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2t^2} \right) = 1/2 \end{aligned}$$

Thus we conclude that the given integral converges to  $1/2$  . ■

**Example** Evaluate  $\int_1^{+\infty} \frac{1}{x} dx$

**Solution**

**Example** Identify  $p$  such that  $\int_1^{+\infty} \frac{1}{x^p} dx$  converges or diverges

**Solution**

$$\begin{aligned} \int_1^{+\infty} \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx \\ &= \lim_{t \rightarrow \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) \end{aligned}$$

If  $p > 1$ , then  $1-p < 0$  and  $t^{1-p} \rightarrow 0$  as  $t \rightarrow +\infty$ .

Thus

$$\lim_{t \rightarrow \infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \frac{1}{p-1}$$

If  $p < 1$ , then  $1-p > 0$  and  $t^{1-p} \rightarrow +\infty$  as  $t \rightarrow +\infty$

Thus

$$\lim_{t \rightarrow \infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = +\infty$$

If  $p = 1$ , then  $\int_1^{+\infty} \frac{1}{x} dx$  diverges. From all three cases, we have

$$\int_1^{+\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & , p > 1 \\ \text{diverge} & , p \leq 1 \end{cases}$$

**Example** Determine if the following improper integrals converges or diverges.

1)  $\int_1^{+\infty} \frac{1}{x^{2/3}} dx$

2)  $\int_1^{+\infty} \frac{1}{x^5} dx$

3)  $\int_1^{+\infty} \frac{1}{x^{3/2}} dx$

**Answer** 1)  $\int_1^{+\infty} \frac{1}{x^{2/3}} dx$  diverges

2)  $\int_1^{+\infty} \frac{1}{x^5} dx$  converges to  $\frac{1}{5-1} = \frac{1}{4}$

3)  $\int_1^{+\infty} \frac{1}{x^{3/2}} dx$  converges to  $\frac{1}{(3/2)-1} = 2$

**Example** Evaluate  $\int_0^{+\infty} (1-x)e^{-x} dx$

**Solution**

**Example** Determine if  $\int_{-\infty}^1 xe^{-x^2} dx$  converges or diverges to which value.

**Solution**

$$\begin{aligned}
 \int_{-\infty}^1 xe^{-x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^1 xe^{-x^2} dx \\
 &= \lim_{t \rightarrow -\infty} \left[ \frac{-e^{-x^2}}{2} \right]_t^1 \\
 &= \lim_{t \rightarrow -\infty} \left[ -\frac{1}{2e} + \frac{e^{-t^2}}{2} \right] \\
 &= -\frac{1}{2e} + 0 \\
 &= -\frac{1}{2e}
 \end{aligned}$$

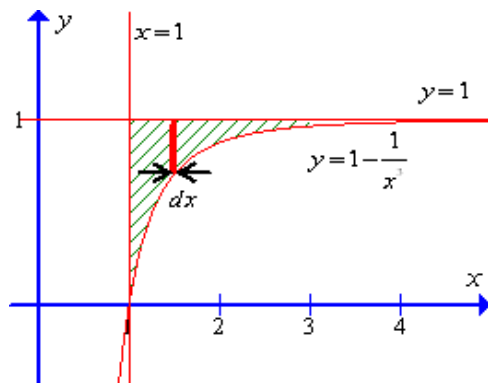
Hence,  $\int_{-\infty}^1 xe^{-x^2} dx$  converges to  $-1/2e$  ■

**Example** Evaluate  $\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx$

**Solution**

**Example** Find the area bounded by curve  $y = 1 - \frac{1}{x^3}$ , line  $x = 1$  and line  $y = 1$

**Solution** Area  $A$  as shown below can be computed by



$$\begin{aligned}
 A &= \int_1^{\infty} (y_2 - y_1) dx \\
 &= \int_1^{\infty} \left[ 1 - \left( 1 - \frac{1}{x^3} \right) \right] dx = \int_1^{\infty} \frac{1}{x^3} dx \\
 &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^3} dx \\
 &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2x^2} \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2t^2} + \frac{1}{2} \right] \\
 A &= 1/2 \text{ unit}^2
 \end{aligned}$$

## 2. Evaluation of improper integral case 2

**Definition** Let  $a$  and  $b$  be real numbers such that  $a < b$ . Suppose  $f$  is a bounded and integrable function on  $[t, b]$  for all  $t$  such that  $a < t < b$ , but  $f$  goes to infinity at  $x = a$ , i.e.

$$\lim_{x \rightarrow a^+} f(x) = \pm \infty$$

Thus the improper integral of  $f(x)$  on  $[a, b]$  is defined by

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

**Remark**

- If the limit exists, then the integral  $\int_a^b f(x) dx$  converges
- Otherwise  $\int_a^b f(x) dx$  diverges.

**Definition** Let  $a$  and  $b$  be real numbers such that  $a < b$ . Suppose  $f$  is a bounded and integrable function on  $[a, t]$  for all  $t$  such that  $a < t < b$ , but  $f$  goes to infinity at  $x = b$ , i.e.

$$\lim_{x \rightarrow b^-} f(x) = \pm \infty$$

Thus the improper integral of  $f(x)$  on  $[a, b]$  is defined by

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

**Remark**

- If the limit exists, then  $\int_a^b f(x) dx$  converges
- Otherwise  $\int_a^b f(x) dx$  diverges

**Definition** Let  $a$  and  $b$  be real numbers such that  $a < b$ . Suppose  $f$  is a bounded and integrable function on  $[a, b]$ , but  $f$  goes to infinity at  $x = c$  in  $(a, b)$ . Thus the improper integral of  $f$  on  $[a, b]$  is defined by

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

**Remark**

- $\int_a^b f(x) dx$  converges if both integral on the right converge.
- $\int_a^b f(x) dx$  diverges if at least one integral on the right diverge.

**Example** Evaluate  $\int_0^1 \frac{1}{\sqrt{x}} dx$

**Solution**

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{t \rightarrow 0^+} \left[ 2\sqrt{x} \right]_t^1 \\ &= \lim_{t \rightarrow 0^+} (2 - 2\sqrt{t}) = 2 \end{aligned}$$

Hence,  $\int_0^1 \frac{1}{\sqrt{x}} dx$  converges to 2. ■



**Example** Evaluate  $\int_1^2 \frac{dx}{1-x}$

**Solution**

$$\begin{aligned} \int_1^2 \frac{dx}{1-x} &= \lim_{t \rightarrow 1^+} \int_t^2 \frac{dx}{1-x} \\ &= \lim_{t \rightarrow 1^+} \left[ -\ln |1-x| \right]_t^2 \\ &= \lim_{t \rightarrow 1^+} (-\ln |-1| + \ln |1-t|) \\ &= 0 + \lim_{t \rightarrow 1^+} \ln |1-t| = -\infty \end{aligned}$$

Hence, we have  $\int_1^2 \frac{dx}{1-x}$  diverges. ■

**Example** Evaluate  $\int_0^1 \frac{dx}{\sqrt{1-x}}$

**Solution**

**Example** Determine if  $\int_1^4 \frac{dx}{(x-2)^{2/3}}$  converges or diverges

**Solution**

### 3. Evaluation of improper integral case 3

**Example** Evaluate  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

**Solution**

$$\begin{aligned} \int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)} \\ &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{\sqrt{x}(x+1)} \\ &= \lim_{t \rightarrow 0^+} \left[ 2 \tan^{-1} \sqrt{x} \right]_t^1 + \lim_{t \rightarrow \infty} \left[ 2 \tan^{-1} \sqrt{x} \right]_1^t \\ &= 2 \left[ \frac{\pi}{4} - 0 \right] + 2 \left[ \frac{\pi}{2} - \frac{\pi}{4} \right] \\ &= \pi \end{aligned}$$

Hence,  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$  diverges to  $\pi$ . ■

**Exercise** Separate the following integrals into several parts according to their impropriety.

1)  $\int_{-3}^{\infty} \frac{dx}{x+2}$

**Solution**

2)  $\int_{-\infty}^0 \frac{dx}{(x+3)^2}$

**Solution**

3)  $\int_{-\infty}^{+\infty} \frac{dx}{x^3}$

**Solution**

**Exercise 7.1**

1 Determine if the following improper integral converges or diverges and find its value.

1.1  $\int_2^{\infty} \frac{1}{(x+1)^2} dx$

1.2  $\int_0^{\infty} \cos x dx$

1.3  $\int_1^{\infty} \frac{\ln x}{x} dx$

1.4  $\int_e^{\infty} \frac{1}{x \ln^3 x} dx$

1.5  $\int_0^{\infty} \frac{1}{1+2^x} dx$

1.6  $\int_{-1}^{\infty} \frac{x}{1+x^2} dx$

1.7  $\int_2^{\infty} \frac{1}{x^2+4} dx$

1.8  $\int_0^{\infty} \frac{1}{\sqrt{e^x}} dx$

1.9  $\int_0^{\infty} x e^{-x} dx$

1.10  $\int_0^{\infty} e^{-x} \cos x dx$

1.11  $\int_0^{\infty} \frac{1}{e^{2x} + e^x} dx$

1.12  $\int_1^{\infty} \frac{1}{\sqrt{x}(1+e^{\sqrt{x}})^2} dx$

1.13  $\int_{-\infty}^1 \frac{1}{3-2x} dx$

1.14  $\int_{-\infty}^0 e^{3x} dx$

1.15  $\int_{-\infty}^{-1} \frac{x}{\sqrt{1+x^2}} dx$

1.16  $\int_{-\infty}^0 \frac{1}{(1-x)^{5/2}} dx$

1.17  $\int_{-\infty}^0 \frac{e^x}{3-2e^x} dx$

1.18  $\int_{-\infty}^0 \frac{1}{(x-8)^{2/3}} dx$

1.19  $\int_{-\infty}^0 \frac{1}{2x^2+2x+1} dx$

1.20  $\int_{-\infty}^{\infty} \frac{|x+1|}{x^2+1} dx$

1.21  $\int_{-\infty}^{\infty} \frac{x^2}{x^2+1} dx$

1.22  $\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2} dx$

1.23  $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$

1.24  $\int_{-\infty}^{\infty} x e^{-x^2} dx$

2 Find value of  $a$  such that  $\int_0^{\infty} e^{-ax} dx = 5$

3 Show that  $\int_1^{\infty} \frac{1}{x^p} dx$  converges if  $p > 1$  and diverges if  $p \leq 1$

4 Find the area between the curve  $y = \frac{8}{x^2-4}$  and  $x$ -axis where  $x \geq 3$

5 Find the area between the curves  $y = \frac{1}{x}$  and  $y = \frac{1}{x^2}$  where  $x \in [1, \infty)$

6 Let  $R = \{(x, y) \mid x \geq 4 \text{ and } 0 \leq y \leq x^{-3/2}\}$ . Find

**6.1** area of region  $R$

**6.2** volume of a solid generated when the region  $R$  is revolved about the  $x$ -axis

7 Let  $R$  be the region between the curve  $y = \frac{4}{x^2 + 1}$  and  $x$ -axis where  $x \geq 0$ . Find

**7.1** area of region  $R$

**7.2** volume of a solid generated when the region  $R$  is revolved about the  $x$ -axis

### Answer 7.1

**1**

**1.1**  $1/3$

**1.2** diverges

**1.3** diverges

**1.4**  $1/2$

**1.5** 1

**1.6** diverges

**1.7**  $\pi/8$

**1.8** 2

**1.9** 1

**1.10**  $1/2$

**1.11**  $1 - \ln 2$

**1.12**  $2 \left( \ln(1+e) - 1 - \frac{1}{1+e} \right)$

**1.13** diverges

**1.14**  $1/3$

**1.15** diverges

**1.16**  $2/3$

**1.17**  $\frac{1}{2} \ln 3$

**1.18** diverges

**1.19**  $3\pi/4$

**1.20** diverges

**1.21** diverges

**1.22** 0

**1.23**  $\pi/2$

**1.24** 0

**2**  $1/5$

**4**  $2 \ln 5$

**5** undefined

**6**

**6.1** 1

**6.2**  $\pi/32$

**7**

**7.1**  $2\pi$

**7.2**  $4\pi^2$

**Exercise 7.2**

1 Determine if the following improper integral converges or diverges and find its value.

$$1.1 \quad \int_0^9 \frac{1}{\sqrt{x}} dx$$

$$1.2 \quad \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

$$1.3 \quad \int_3^4 \frac{1}{(x-3)^2} dx$$

$$1.4 \quad \int_0^4 \frac{1}{(4-x)^{3/2}} dx$$

$$1.5 \quad \int_1^2 \frac{x}{\sqrt{x-1}} dx$$

$$1.6 \quad \int_0^1 x \ln x dx$$

$$1.7 \quad \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}} dx$$

$$1.8 \quad \int_0^{\pi/2} \sec^2 x dx$$

$$1.9 \quad \int_0^2 \frac{2x+1}{x^2+x-6} dx$$

$$1.10 \quad \int_0^1 \ln x dx$$

$$1.11 \quad \int_0^4 \frac{\ln \sqrt{x}}{\sqrt{x}} dx$$

$$1.12 \quad \int_0^1 \frac{1}{\sqrt{1-\sqrt{x}}} dx$$

$$1.13 \quad \int_2^4 \frac{x}{\sqrt[3]{x-2}} dx$$

$$1.14 \quad \int_0^2 \frac{x}{(x^2-1)^2} dx$$

$$1.15 \quad \int_{-1}^8 \frac{1}{\sqrt[3]{x}} dx$$

$$1.16 \quad \int_{-2}^7 \frac{1}{(x+1)^{2/3}} dx$$

$$1.17 \quad \int_{-1}^1 \frac{1}{\sqrt{|x|}} dx$$

$$1.18 \quad \int_2^4 \frac{1}{(x-3)^7} dx$$

$$1.19 \quad \int_0^3 \frac{1}{x^2+2x-3} dx$$

$$1.20 \quad \int_1^3 \frac{x}{(x^2-4)^3} dx$$

$$1.21 \quad \int_{-1}^2 \frac{1}{x^2} \cos \frac{1}{x} dx$$

$$1.22 \quad \int_0^2 \frac{1}{\sqrt{2x-x^2}} dx$$

$$1.23 \quad \int_{-1}^2 \frac{1}{x^2-x-2} dx$$

$$1.24 \quad \int_0^1 \frac{1}{x(\ln x)^{1/5}} dx$$

2 Show that  $\int_0^1 \frac{1}{x^p} dx$  converges if  $p < 1$  and diverges if  $p \geq 1$

3 Find the area between the curve  $y = \frac{1}{(1-x)^2}$  and  $x$ -axis where  $x \in [0, 4]$

4 Find (a) Area of region  $R$ , and

(b) Volume of a solid generated by revolving region  $R$  about the  $x$ -axis, when  $R$  is given as follows.

$$4.1 \quad R = \{(x, y) \mid -4 \leq x \leq 4 \text{ and } 0 \leq y \leq 1/(x+4)\}$$

$$4.2 \quad R = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1/\sqrt{x}\}$$

5 Find the area between the curves  $y = \frac{1}{x}$  and  $y = \frac{1}{x(x^2+1)}$  where  $x \in [0, 1]$

**Answer 7.2****1**

**1.1** 6

**1.2**  $\pi/2$

**1.3** diverges**1.4** diverges

**1.5**  $8/3$

**1.6**  $-1/4$

**1.7** 1

**1.8** diverges**1.9** diverges

**1.10**  $-1$

**1.11**  $4(\ln 2 - 1)$

**1.12**  $8/3$

**1.13**  $21\sqrt[3]{4}/5$

**1.14** diverges

**1.15**  $9/2$

**1.16** 3

**1.17** 4

**1.18** diverges**1.19** diverges**1.20** diverges**1.21** diverges

**1.22**  $\pi$

**1.23** diverges**1.24** diverges**3** Undefined**4****4.1** (a) DNE

(b) DNE

**4.2** (a) 2

(b) DNE

**5**  $\frac{1}{2}\ln 2$

**Exercise 7.3****1.** Determine if the following improper integral converges or diverges and find its value.

**1.1**  $\int_0^{\infty} \frac{1}{(x-1)^{2/3}} dx$

**1.2**  $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$

**1.3**  $\int_{-1}^{\infty} \frac{1}{x^2-1} dx$

**1.4**  $\int_1^{\infty} \frac{1}{x \ln x} dx$

**1.5**  $\int_0^{\infty} x^{-0.1} dx$

**1.6**  $\int_0^{\infty} \frac{1}{\sqrt{x}(x+4)} dx$

$$1.7 \quad \int_1^{\infty} \frac{1}{x^2 - 6x + 8} dx$$

$$1.8 \quad \int_0^{\infty} \frac{\sqrt{x}}{1 - \sqrt{x}} dx$$

$$1.9 \quad \int_2^{\infty} \frac{1}{(x+7)\sqrt{x-2}} dx$$

$$1.10 \quad \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 1} dx$$

$$1.11 \quad \int_{-\infty}^{\infty} \frac{1}{x^2 - 3x + 2} dx$$

$$1.12 \quad \int_{-\infty}^{\infty} \frac{e^x}{e^x - 1} dx$$

### Answer 7.3

1.1 diverges

1.2  $\pi / 2$

1.3 diverges

1.4 diverges

1.5 diverges

1.6  $\pi / 2$

1.7 diverges

1.8 diverges

1.9  $\pi / 3$

1.10 diverges

1.11 diverges

1.12 diverges

## Numerical Integration

We use the material on the following website. It has an interactive program.

Visit website:

<http://www.zweigmedia.com/RealWorld/integral/numint.html>