Exercise: Mathematical Induction, Sequence, and Series

1. Use mathematical induction to prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true for all n is positive integers.

2. Determine if the following sequences converge or diverge. If it converges, find its limit.

   2.1 $\left\{ \left(\dfrac{2n+3}{2n-5} \right)^n \right\}$

   2.2 $\left\{ \ln(n) - \ln(n+1) \right\}$

   2.3 $\left\{ \dfrac{n^2}{2n-1} \sin\left(\dfrac{1}{n}\right) \right\}$

3. Consider the following sequences. Are they monotone? bounded? if the following sequences are monotone, check that it increasing,or decreasing ?.

   3.1 $\left\{ \dfrac{1}{2^n} \right\}$

   3.2 $\left\{ \dfrac{2^{n+1}}{n+2} \right\}$

   3.3 $\left\{ 2ne^{-2n} \right\}$

4. Determine if the following Infinite series converge or diverge.

   4.1 Telescoping Series : $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right)$

   (Hint: Use partial fractions)

   4.2 Geometric Series: $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

   (Hint: Write out the first few terms of the series to find a and r)

   4.3 Geometric Series: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{e}{\pi} \right)^n$

   (Hint: Write out the first few terms of the series to find a and r)

   4.4 p-Series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}}$

   4.5 p-Series: $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

   4.6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (Using Integral Test)

4.7 $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ (Using Integral Test)

4.8 $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (Using Comparison Test)

4.9 $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ (Using Ratio Test)

5. Determine if the following alternating series converge or diverge?

5.1 $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{3n}{2n+1}$

5.2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}$

6. Determine if $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!7^{(-n)}}$ absolutely converge or conditionally converge or diverge. (Using Ratio Test)

7. Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(3n+5)^2}$ absolutely converge or conditionally converge or diverge. (Using Comparison Test)

8. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{3^{2n}}$.

9. Find the Taylor polynomial of $f(x) = \frac{1}{x}$ about the given point $x = 1$.

10. Find Maclaurin series of $f(x) = \ln(1-x)$.

11. Find the power series of $f(x) = \frac{1}{\sqrt{4-x}}$.

12. Draw graphs and Write Fourier series of $f(x) = \begin{cases} x+2, & -2 \le x < 0 \\ 2, & 0 < x \le 2 \end{cases}$

13. Write Fourier series of $f(x) = |x| - 1$ with period $-2 \le x \le 2$.

14. Write Fourier series of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ -2, & 0 < x \le 2 \end{cases}$

15. Determine Fourier series of $f(x) = \begin{cases} x+\pi, & -\pi \le x < 0 \\ x-\pi, & 0 < x \le \pi \end{cases}$

is $f(x) = -\sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$

and show the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$.

16. Determine Fourier series of $f(x) = \begin{cases} x+2, & -2 < x \le 0 \\ 0, & 0 < x \le 2 \end{cases}$

and find convergent values when $x = 0$.

Answers

1. -

2. 2.1 Converges to e^4

   2.2 Converges to 0

   2.3 Diverges

3. 3.1 $\left\{ \dfrac{1}{2^n} \right\}$ is monotonic, decreasing, and bounded by every real number greater then or equal to $\dfrac{1}{2}$. The squence is also bounded below by every number less then or equal to 0, which is its greatest lower bound.

   3.2 $\left\{ \dfrac{2^{n+1}}{n+2} \right\}$ is monotonic, increasing, and unbouded by every real number greater then or equal to $\dfrac{4}{3}$. The squence is also unbounded below.

   3.3 $\left\{ 2ne^{-2n} \right\}$ is monotonic, decreasing, and bounded by every real number greater then or equal to $\dfrac{2}{e^2}$. The squence is also bounded below by every number less then or equal to 0, which is its greatest lower bound.

4. 4.1 Converges

   4.2 Converges

   4.3 Converges

   4.4 Converges

   4.5 Converges

   4.6 Converges

   4.7 Converges

   4.8 Converges

   4.9 Converges

5. 5.1 Diverges

   5.2 Converges

6. The series is absolutely converge.

7. The series is absolutely converge.

8. Convergence interval of x where $-9 < x < 9$

9. The Taylor polynomial of $f(x)$ about $x = 1$ is

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x - 1^5) + \cdots + (-1)^n (x-1)^n$$

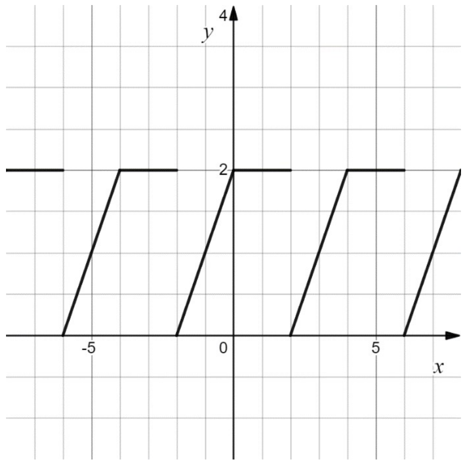
10. Maclaurin series of $f(x)$ is $-\sum_{n=1}^{\infty} \frac{x^n}{n}$

11. Power series of $f(x)$ is $\frac{1}{2}\left[1 + \frac{x}{8} + \frac{3x^2}{8^2 \cdot 2!} + \frac{15x^3}{8^3 \cdot 3!} + \cdots\right]$

12. Fourier series of $f(x)$ is

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{2}\right)\right)$$

and graph of $f(x)$ is



13. Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\left(\frac{n\pi x}{2}\right)$

14. Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\Big((-1)^n - 1\Big)\sin\left(\frac{n\pi x}{2}\right)$

15. -

16. Fourier series of $f(x)$ is $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{(n\pi)^2}\Big((-1)^n - 1\Big)\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right)$
And convergent values when $x = 0$ is 1