Integration Techniques

1. Substitution

2. By Parts

3. Partial Fractions

Partial Fractions

denominator

<u>case 1</u> all roots are linear,
all roots are different

$$\frac{?}{(x-5)(x+3)}$$

$$\frac{?}{(x+1)(x-2)(x+3)}$$

(1.) $\int \frac{5x-4}{(x+1)(2x-1)}\,dx$ $x \neq -1, \frac{1}{2}$

Solution: $\frac{5x-4}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$

$$= \frac{A}{x+1} \cdot \frac{(2x-1)}{(2x-1)} + \frac{B}{2x-1} \cdot \frac{(x+1)}{(x+1)}$$

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{A(2x-1) + B(x+1)}{(x+1)(2x-1)}$$

$$\boxed{5x-4 = A(2x-1)+B(x+1)} \quad — \; ★$$

Since $\frac{5x-4}{(x+1)(2x-1)}$ undefines if $x = -1, \frac{1}{2}$.

plug in $x=-1$ to ★ ; $5(-1)-4 = A(2(-1)-1)$

$$-5-4 = A(-2-1)$$

$$-9 = A(-3)$$

$$-9 = -3A$$

$A = 3$

plug in $x = \frac{1}{2}$ to ⊛ ;

$$5\left(\frac{1}{2}\right) - 4 = B\left(\frac{1}{2} + 1\right)$$

$$\frac{5}{2} - 4 = B\left(\frac{3}{2}\right)$$

$$-\frac{3}{2} = B\left(\frac{3}{2}\right)$$

$$B = -1$$

Hence,

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{3}{x+1} + \frac{(-1)}{2x-1}$$

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{3}{x+1} - \frac{1}{2x-1}$$

$$\int \frac{5x-4}{(x+1)(2x-1)}\,dx = 3\int \frac{1}{x+1}\,dx - \int \frac{1}{2x-1}\,dx$$

$$= 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C$$

$$\int \frac{1}{2x-1}\,dx$$

$u = 2x-1$

$du = 2\,dx \Rightarrow dx = \frac{1}{2}du$

$$\int \frac{1}{2x-1}\,dx = \int \frac{1}{u} \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|2x-1| + C$$

② $\int \frac{x^2+18x+5}{(x+1)(x-2)(x+3)}\,dx$

Solution:

$$\frac{x^2+18x+5}{(x+1)(x-2)(x+3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A}{x+1}\cdot\frac{(x-2)(x+3)}{(x-2)(x+3)} + \frac{B}{x-2}\cdot\frac{(x+1)(x+3)}{(x+1)(x+3)} + \frac{C}{x+3}\cdot\frac{(x+1)(x-2)}{(x+1)(x-2)}$$

$$\frac{x^2+18x+5}{(x+1)(x-2)(x+3)} = \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)}{(x+1)(x-2)(x+3)}$$

$$\boxed{x^2+18x+5 = A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)} \quad - \star$$

Since $\frac{x^2+18x+5}{(x+1)(x-2)(x+3)}$ undefines if $x = -1, 2, -3$.

Plug in $x=-1$ to $\star$;

$$(-1)^2+18(-1)+5 = A(-1-2)(-1+3)$$
$$1-18+5 = A(-3)(2)$$
$$-12 = -6A$$
$$A = 2$$

Plug in $x=2$ to $\star$;

$$2^2+18(2)+5 = B(2+1)(2+3)$$
$$4+36+5 = B(3)(5)$$
$$45 = B(15)$$

$B = 3$

Plug in $x=-3$ to ②; $C = -4$

Hence,

$$\frac{x^2+18x+5}{(x+1)(x-2)(x+3)} = \frac{2}{x+1} + \frac{3}{x-2} + \frac{-4}{x+3}$$

$$\frac{x^2+18x+5}{(x+1)(x-2)(x+3)} = \frac{2}{x+1} + \frac{3}{x-2} - \frac{4}{x+3}$$

$$\int \frac{x^2+18x+5}{(x+1)(x-2)(x+3)}\,dx = 2\int \frac{1}{x+1}\,dx + 3\int \frac{1}{x-2}\,dx - 4\int \frac{1}{x+3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-2| - 4\ln|x+3| + C$$

Answer

① $\displaystyle\int \frac{11-15x}{(3x-1)(3x+2)}\,dx$ ② $\displaystyle\int \frac{4x^2-23x+19}{(x-6)(x-1)(x+3)}\,dx$

Substitution

3. $\int 2x\sqrt{1+x^2}\,dx$

Solution: $u = 1+x^2$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int 2x\sqrt{1+x^2}\,dx = \int 2\sqrt{1+x^2}\,x\,dx$$

$$= \int \cancel{2}\sqrt{u}\cdot\frac{1}{\cancel{2}}du$$

$$= \int \sqrt{u}\,du$$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{1/2+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{3}(1+x^2)^{3/2} + C$$

$$\int \frac{4x}{\sqrt{2x^2+1}}\,dx$$

$$\int 2xe^{x^2-5}\,dx$$

$$\int -2x\sin(1-x^2)\,dx$$

④ $\displaystyle\int \frac{(\ln(x))^2}{x}\,dx = \int (\ln(x))^2 \cdot \frac{1}{x}\,dx$

Solution: Let $u = \ln(x)$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}\,dx$$

Then, $\displaystyle\int \frac{(\ln(x))^2}{x}\,dx$

$$= \int (\ln(x))^2 \cdot \frac{1}{x}\,dx$$

$$= \int u^2 \cdot du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{(\ln(x))^3}{3} + C$$

Answer

(5.) $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx = \int \arctan(x)\cdot\frac{1}{1+x^2}\,dx$

Solution: $u = \arctan(x)$

$$\frac{du}{dx} = \frac{1}{1+x^2} \Rightarrow du = \frac{1}{1+x^2}\,dx$$

Then, $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx$

$$= \int \arctan(x)\cdot\frac{1}{1+x^2}\,dx$$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\arctan(x))^2}{2} + C$$

By Parts

6. $\int x^4 \ln x \, dx$

Choose $u = \ln x$

$dv = x^4 \, dx$

	u	dv
$+$	$\ln x$	x^4
$-\int$	$\frac{1}{x}$	$\frac{x^5}{5}$

★ 3rd stop: When you can integrate of the product of that row.

$$\int x^4 \ln x \, dx = +\ln x\left(\frac{x^5}{5}\right) - \int \frac{1}{x} \cdot \frac{x^5}{5} \, dx$$

$$= \frac{1}{5}x^5 \ln x - \int \frac{x^4}{5} \, dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{x^5}{25} + C$$

Try $\int \ln\sqrt{x} \, dx$

7. $\int \ln\sqrt{x}\, dx = \int \ln\sqrt{x} \cdot 1\, dx$

Solution: $u = \ln\sqrt{x}$

$dv = 1\, dx$

	u	dv
$+$	$\ln\sqrt{x}$	1
$-$	$\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$	x

Then, $\int \ln\sqrt{x}\, dx = +\ln\sqrt{x}\,(x) - \int \frac{1}{2\cancel{x}} \cdot \cancel{x}\, dx$

$$= x\ln\sqrt{x} - \int \frac{1}{2}\, dx$$

$$= x\ln\sqrt{x} - \frac{1}{2}x + C$$

Integration Techniques
(Substitution, By Parts, and Partial Fraction)

1) $\int \cos(3-7x)\mathrm{d}x$ 8.

2) $\int e^{3x-2}\mathrm{d}x$ 11.

3) $\int -2x\sin\left(1-x^2\right)\mathrm{d}x$ 5.

4) $\int \frac{\sin(3x)}{9-\cos(3x)}\mathrm{d}x$ 13.

5) $\int 2x^2\sqrt{x^3+1}\mathrm{d}x$ 3.

6) $\int \sin^4(x)\cos(x)\mathrm{d}x$ 15.

7) $\int \frac{x-4}{x^2-8x+3}\mathrm{d}x$ 1.

8) $\int \left(e^{3x}+4\right)^5 e^{3x}\mathrm{d}x$ 2.

9) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\mathrm{d}x$ 16.

10) $\int \frac{x}{e^{2x}}\mathrm{d}x$ 4.

11) $\int x^3\ln\left(x^2\right)\mathrm{d}x$ 14.

12) $\int \frac{\ln(x)}{\sqrt{x}}\mathrm{d}x$ 7.

13) $\int \frac{\ln(x+1)}{x^2}\mathrm{d}x$ 12.

14) $\int e^{-2x}\sin(3x)\mathrm{d}x$ 10.