Chapter 3 Power Series

Definition 3.1: A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_k(x-a)^k + \ldots \quad (1)$$

where $a, c_0, c_1, c_2, \ldots, c_k, \ldots$ are constant and x is a variable.

The constant "a" is called the *center* of the power series, while the constants "$c_0, c_1, c_2, \ldots, c_k, \ldots$" are called *coefficients* of the power series.

If $a = 0$, we call (1) a *power series in x*.

Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n! x^n$ and $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ are all power series in x

If $a \neq 0$, we call (1) a *power series in* $x-a$ or a *power series centered at a*,

Example: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n+1}$ is a power series in $x-1$**,** and $\sum_{n=0}^{\infty} (-1)^n \frac{(x+3)^n}{n!}$ is a power series centered at -3 .

Since x can be any number, the power series (1) may be either convergent or divergent depending on the value of x.

For example, consider the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$.

If $x = 6$, the above series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p-series with $p = 2$. It then **converges**.

If $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2}$ which is a **divergent** alternating series.

Question? For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, what are the values of x that make this series converges?

Observe that any power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_k(x-a)^k + ...$$

always converges at its center $x = a$. (Why?)

Then where else does it converge? The following theorem states about the convergence of a power series.

Theorem 3.2: (Radius of convergence)

Let $F(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series in *x*-*a*.

Then there are only three possibilities:

(i) $F(x)$ converges only at $x=a$, or

(ii) $F(x)$ converges for all x, or

(iii) there is a real number $R>0$ such that $F(x)$ converges absolutely if $|x-a|<R$ ($x\in(a-R,a+R)$) and diverges if $|x-a|>R$ ($x\in(-\infty,a-R)\cup(a+R,\infty)$). It may or may not converge at the end points $|x-a|=R$ ($x=a-R$ or $x=a+R$).

In case *(i)*, set $R=0$, and in case *(ii)*, set $R=\infty$. We call R the *radius of convergence* of the power series $F(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$.

By the above theorem, we note that the set of all *x* that make the series converge in each case can be expressed as an interval. Namely, $\{a\}=[a,a]$ for case *(i)*, $(-\infty,\infty)$ for case *(ii)*, and one of the following intervals $(a-R,a+R)$, $(a-R,a+R]$, $[a-R,a+R)$, or

$[a-R, a+R]$ for case *(iii)*. This leads to the following definition.

Definition 3.3: The set of all values of x that make a power series converge is called the *interval of convergence* of the power series.

The procedure used to find the interval of convergence of a power series.

This procedure contains 2 steps.

<u>Step 1</u>: Finding the radius of convergence of the power series by applying the ratio test or the n^{th} root test so that we get the convergence interval of x where $|x-a|<R$ or $a-R<x<a+R$. To do this, we set

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1, \text{ or } \lim_{n\to\infty}\sqrt[n]{|a_n|}<1,$$

where a_n is the n-th term of a given power series. (Here, $a_n = c_n(x-a)^n$).

<u>Step 2</u>: Test the convergence of this power series at the end points $x = a + R$ and $x = a - R$ by the methods we know such as comparison test, integral test and alternating series test.

Example: Find x which makes the series converge.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

Problem: Find the intervals of convergence of the following series.

a. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n(n+1)}$

b. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$

Exercise 3.1

Find the radius of convergence and the interval of convergence of the following power series.

1. $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$

2. $\sum_{n=0}^{\infty} nx^n$

3. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$

5. $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$

6. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

7. $\sum_{n=0}^{\infty} \frac{4}{4^n}(2x-1)^n$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{\sqrt{n}}$

9. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

10. $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$

11. $\sum_{n=0}^{\infty} \frac{n}{(n^2+1)4^n}(x-10)^n$

12. $\sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^n (x+6)^n$

13. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^3}$

14. $\sum_{n=0}^{\infty} \frac{n}{\sqrt{n+1}}(x-e)^n$

15. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} \cdot x^n$

16. $\sum_{n=1}^{\infty} \frac{nx^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

17. $\sum_{n=2}^{\infty} \frac{(-1)^n (2x+3)^n}{n \ln n}$

18. $\sum_{n=0}^{\infty} \frac{x^n}{(\ln n)^n}$

19. $\sum_{n=0}^{\infty} \frac{n!}{(10)^n}(x-\pi)^n$

20. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

Solution of Exercise 3.1

1. $1, [-1,1)$	2. $1, (-1,1)$	3. $\infty, (-\infty, \infty)$	4. $2, (-2,2]$
5. $\frac{1}{3}, \left[-\frac{1}{3}, \frac{1}{3}\right]$	6. $1, [-1,1]$	7. $2, \left(-\frac{3}{2}, \frac{5}{2}\right)$	8. $1, (0,2]$
9. $\infty, (-\infty, \infty)$	10. $\frac{1}{2}, \left[\frac{5}{2}, \frac{7}{2}\right)$	11. $1, [-1,1)$	12. $0, \{-6\}$
13. $\frac{1}{2}, [0,1]$	14. $1, (e-1, e+1)$	15. $1, (-1,1)$	16. $\infty, (-\infty, \infty)$
17. $\frac{1}{2}, (-2,-1]$	18. $\infty, (-\infty, \infty)$	19. $0, \{\pi\}$	20. $\infty, (-\infty, \infty)$

Taylor and Maclaurin Series

Both series are power series used to approximate other functions.

Definition 3.4 (Taylor's Series)

Let f be a function that has derivatives of all orders at a point a. The *Taylor Series* of f about $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \ldots \qquad (1)$$

Definition 3.5 (Taylor's Polynomial)

Let f be a function that has derivatives up to order n at a point a, the n-th *Taylor Polynomial* of f about $x = a$ is

$$f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ . \ . \ . \ + f^{(n)}(a)\frac{(x-a)^n}{n!}. \qquad (2)$$

We can define the polynomial (of degree n) $P_n(x)$ as follows

$$P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ . \ . \ . \ + f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

If a = 0 in either (1) or (2), we call it *Maclaurin Series* or *Maclaurin Polynomial*, respectively.

Example: Given $f(x) = e^x$, write down the Maclaurin polynomial of f.

Solution Let $f(x) = e^x$.

Then $f'(x) = f''(x) = f'''(x) = ... = f^{(n)}(x) = e^x$.

Hence, $f'(0) = f''(0) = f'''(0) = ... = f^{(n)}(0) = e^0 = 1$.

$$\therefore \quad P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1 + x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

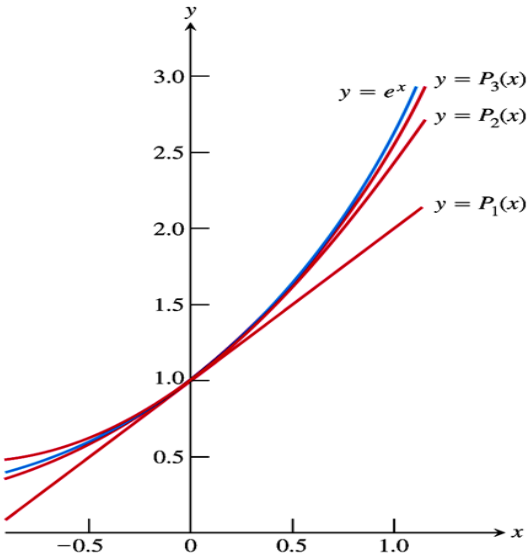

Figure 1

Figure 1 shows that the higher order of n, the closer polynomial $P_n(x)$ gets to the function $f(x) = e^x$.

The Maclaurin Polynomial is just an approximation of a function since we cut off the tail of an infinite series. Thus, there exists a *truncation error*.

We have also found that the Maclaurin series is more accurate when *x* gets closer to zero. That's why we have better use a Taylor series when *x* is far from zero.

Exercise: Find the 3rd Taylor polynomial of $f(x)=\sin x$ about $x=\frac{\pi}{3}$.

Solution: Let $f(x)=\sin x$.

Since
$$f(x)=\sin x\ ,\quad f\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$
$$f'(x)=\cos x\ ,\quad f'\left(\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2}$$
$$f''(x)=-\sin x\ ,\quad f''\left(\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$
$$f'''(x)=-\cos x\ ,\quad f'''\left(\frac{\pi}{3}\right)=-\cos\frac{\pi}{3}=-\frac{1}{2}$$

Therefore,

$$P_3(x)=f\left(\tfrac{\pi}{3}\right)+f'\left(\tfrac{\pi}{3}\right)\left(x-\tfrac{\pi}{3}\right)+\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\tfrac{\pi}{3}\right)^2+\frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\tfrac{\pi}{3}\right)^3$$
$$=\frac{\sqrt{3}}{2}+\frac{1}{2}\left(\mathrm{x}-\tfrac{\pi}{3}\right)-\frac{\sqrt{3}}{2.2!}\left(x-\tfrac{\pi}{3}\right)^2-\frac{1}{2.3!}\left(x-\tfrac{\pi}{3}\right)^3.$$

3.6 Differentiation and Integration of Power Series

We can do it term by term:

Let $f(x)=\sum_{n=1}^{\infty}c_n(x-a)^n$ be a power series with radius of convergence $R>0$. Then we have

1. $f(x)$ is continuous for all $|x-a|<R$
2. $\int f(x)dx=\sum_{n=0}^{\infty}\dfrac{c_n(x-a)^{n+1}}{n+1}+C;\quad |x-a|<R$
3. $f'(x)=\sum_{n=0}^{\infty}nc_n(x-a)^{n-1};\quad |x-a|<R$

Well known Maclaurin Series

1. $\dfrac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots=\sum_{n=0}^{\infty}x^n\ ,\ -1<x<1$

2. $e^x=1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+\ldots=\sum_{n=0}^{\infty}\left(\dfrac{x^n}{n!}\right)\ ,\quad -\infty<x<\infty$

3. $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots=\sum_{n=0}^{\infty}\left(\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\ ,-\infty<x<\infty$

4. $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\ldots=\sum_{n=0}^{\infty}\left(\dfrac{(-1)^n x^{2n}}{(2n)!}\right)\ ,\quad -\infty<x<\infty$

5. $\ln|1+x|=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots+\dfrac{(-1)^n x^{n+1}}{n+1}+\ldots=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{n+1}}{n+1}\ ,\quad -1<x\le 1$

Example Find Maclaurin series of $f(x) = \tan^{-1} x$.

Solution Since we know

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

$$= \int (1 - x^2 + x^4 - x^6 + x^8 - \ldots)dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Evaluate C by plugging in $x = 0$

Thus $C = \tan^{-1}(0) = 0$

Then $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

The radius of convergence of $f(x) = \frac{1}{1+x^2}$ is 1.

Hence, the radius of convergence of $f(x) = \tan^{-1} x$ is also 1, and its interval of convergence is $(-1,1)$.

3.7 The Binomial Series

Let k be any real number and $|x|<1$.

$$(1+x)^k = 1+kx+k(k-1)\frac{x^2}{2!}+k(k-1)(k-2)\frac{x^3}{3!}+\ldots$$

$$=\sum_{n=0}^{\infty}\binom{k}{n}x^n$$

where $\binom{k}{n}=\dfrac{k(k-1)\ldots(k-n+1)}{n!}, n\geq 1$ and $\binom{k}{0}=1$

calculated from Maclaurin series of $f(x)=(1+x)^k$

as follows:

$$f(x)=(1+x)^k \qquad \text{and } f(0)=1$$

$$f'(x)=k(1+x)^{k-1}$$

$$f''(x)=k(k-1)(1+x)^{k-2}$$

$$f'''(x)=k(k-1)(k-2)(1+x)^{k-3}$$

$$f^{(n)}(x)=k(k-1)\ldots(k-n+1)(1+x)^{k-n}$$

and $f^{(n)}(0)=k(k-1)\ldots(k-n+1)$.

Hence, Maclaurin series of $f(x)=(1+x)^k$ is

$$\sum_{n=0}^{\infty}f^{(n)}(0)\frac{x^n}{n!}=\sum_{n=0}^{\infty}k(k-1)\ldots(k-n+1)\frac{x^n}{n!}.$$

Find the interval of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{k(k-1)...(k-n+1)(k-n)x^{n+1}}{(n+1)!}}{\dfrac{k(k-1)...(k-n+1)x^n}{n!}}\right|$$

$$= |x|\lim_{n\to\infty}\frac{|k-n|}{n+1} = |x| < 1$$

Example Find the power series of $f(x) = 1/(1+x)^2$ and its interval of convergence.

<u>**Solution**</u> Apply binomial series when $k = -2$:

$$\binom{-2}{n} = \frac{-2(-3)...(-2-n+1)}{n!}, n \geq 1$$

$$= \frac{(-1)^n 2(3)...n(n+1)}{n!} = (-1)^n (n+1)$$

Thus $f(x) = 1/(1+x)^2$

$$= \sum_{n=0}^{\infty}\binom{-2}{n}x^n = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n, |x| < 1$$

Example Find the power series of $f(x) = 1/\sqrt{4-x}$ and its radius of convergence.

Solution Consider

$$f(x)=1/\sqrt{4-x}=\frac{1}{\sqrt{4(1-\frac{x}{4})}}=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}.$$

Apply binomial series at $k=-1/2$ and replace x by $-x/4$. We obtain

$$f(x)=\frac{1}{\sqrt{4-x}}=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-\frac{x}{4}\right)^{n}$$

$$=\frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^{2}+\right.$$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^{3}+\ldots$$

$$\left.\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^{n}+\ldots\right]$$

$$=\frac{1}{2}\left[1+\frac{x}{8}+\frac{1\cdot 3x^{2}}{2!8^{2}}+\ldots+\frac{1\cdot 3\cdot 5\cdots(2n-1)x^{n}}{n!8^{n}}+\ldots\right]$$

where $|-x/4|<1$, i.e., $|x|<4$.

Thus its radius of convergence is 4.

Exercise 3.2

For each of the problem 1-8, find the Taylor's series of the function f about the given point $x = a$ and specify its radius of convergence.

1. $f(x) = e^{2x}$, $a = 0$
2. $f(x) = \sin x$, $a = \frac{\pi}{4}$
3. $f(x) = \frac{1}{x+1}$, $a = 0$
4. $f(x) = \frac{1}{(x+1)^2}$, $a = -2$
5. $f(x) = \frac{1}{x}$, $a = 1$
6. $f(x) = e^{x}$, $a = 3$
7. $f(x) = \sqrt{x}$, $a = 4$
8. $f(x) = \tan^{-1} 2x$, $a = 0$

For each of the problem 9-14, find the Macluarin's series of the function f and its radius of convergence by using the well-known Macluarin's series including differentiation and integration of the series.

9. $f(x) = \frac{1}{x+1}$
10. $f(x) = \frac{1}{(x+1)^2}$
11. $f(x) = \frac{1}{1+4x^2}$
12. $f(x) = \frac{1}{4+x^2}$
13. $f(x) = \frac{1}{1-x^2}$
14. $f(x) = \ln \left| \frac{1+x}{1-x} \right|$

For each of the problem 15-22, apply any techniques to find the Macluarin's series of the function *f* and its radius of convergence.

15. $f(x) = e^{3x}$

16. $f(x) = x^2 \cos x$

17. $f(x) = x \sin \dfrac{x}{2}$

18. $f(x) = \sin^2 x$ (Hint: $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$

19. $f(x) = \sqrt{1+x}$

20. $f(x) = \dfrac{1}{\sqrt[3]{1-x}}$

21. $f(x) = (1+x)^{-3}$

22. $f(x) = \ln|5+x|$

Answer to exercise 3.2

1. $\sum_{n=0}^{\infty} \left(\dfrac{2^n x^n}{n!} \right)$, $R = \infty$

2. $\sum_{n=0}^{\infty} \left(\dfrac{(-1)^{\frac{n}{2}(n-1)}}{\sqrt{2}} \cdot \dfrac{\left(x - \frac{\pi}{4} \right)}{n!} \right)$, $R = \infty$

3. $\sum_{n=0}^{\infty} \left((-1)^n x^n \right)$, $R = 1$

4. $\sum_{n=0}^{\infty} \left((n+1)(x+2)^n \right)$, $R = 1$

5. $\sum_{n=0}^{\infty} \left((-1)^n (x-1)^n \right)$, $R = 1$

6. $\sum_{n=0}^{\infty} \left(\dfrac{e^3}{n!} (x-3)^n \right)$, $R = \infty$

7. $2+\frac{1}{4}(x-4\sum_{n=2}^{\infty}\left((-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}{n!2^{3n-1}}(x-4)^n)\right)+$, $R=\frac{1}{2}$

8. $\sum_{n=0}^{\infty}\left((-1)^n\frac{(2x)^{2n+1}}{(2n+1)}\right)$, $R=\frac{1}{4}$

9. $\sum_{n=0}^{\infty}\left((-1)^n x^n\right)$, $R=1$

10. $\sum_{n=0}^{\infty}\left((-1)^n(x-1)x^n\right)$, $R=1$

11. $\sum_{n=0}^{\infty}\left((-1)^n 4^n x^{2n}\right)$, $R=\frac{1}{2}$

12. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{4^{n+1}}x^{2n}\right)$, $R=2$

13. $\sum_{n=0}^{\infty}x^{2n}$, $R=1$

14. $\sum_{n=0}^{\infty}\left(\frac{2x^{2n+1}}{2n+1}\right)$, $R=1$

15. $\sum_{n=0}^{\infty}\left(\frac{3^n x^n}{n!}\right)$, $R=\infty$

16. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+2}}{(2n)!}\right)$, $R=\infty$

17. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+2}}{2^{2n+1}(2n+1)!}\right)$, $R=\infty$

18. $\sum_{n=1}^{\infty}\left(\frac{(-1)^n\cdot 2^{2n-1}\cdot x^{2n}}{(2n)!}\right)$, $R=\infty$

19. $2+\frac{1}{4}(x-4)+\sum_{n=2}^{\infty}\left((-1)^n\frac{1\cdot 3\cdot 5\cdot \ \ldots\ \cdot(2n-3)}{2^n n!}\cdot x^n\right)$, $R=1$

20. $1+\sum_{n=1}^{\infty}\left(\frac{1\cdot 4\cdot 7\cdot \ \ldots\ \cdot(3n-2)}{3^n n!}\cdot x^n\right)$, $R=1$

21. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2}(n+1)(n+2)x^n\right)$, $R=1$

22. $\ln 5+\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{5^n\cdot n}\right)$, $R=5$