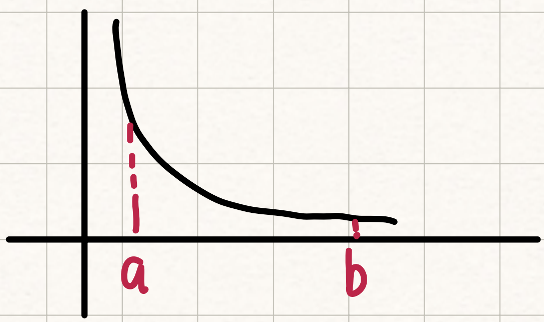
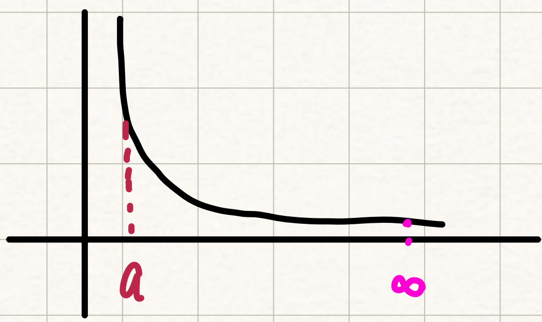
Improper Integral





Area $A = \int_a^b f(x)\,dx$

Area $A = \int_a^\infty f(x)\,dx$

<u>Definition</u> $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

- lim exists ⇒ converge to...
- lim D.N.E. ⇒ diverge

① $\int_1^\infty \frac{1}{x^2}\,dx$

Solution: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

$\int x^{-2}\,dx = \frac{x^{-1}}{-1} + C$

$= -\frac{1}{x} + C$

$$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} - \left(-\frac{1}{1}\right)\right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} + 1\right]$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}\right) + 1$$

$$= 0+1$$

$$= 1$$

Thus, $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges to 1.

② $\int_1^{\infty} \frac{1}{x}\,dx$

Solution: $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \Big[\ln|x|\Big]\Big|_1^t$$

$$= \lim_{t\to\infty} \Big[\ln|t| - \underbrace{\ln|1|}_{0}\Big]$$

$$= \infty$$

Thus, $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

Definition $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx$

- lim exists $\Rightarrow$ converge to...
- lim D.N.E. $\Rightarrow$ diverge

③ $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx$

Solution: $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{x}\,dx$

	u	dv
⊕	x	e^x
⊖	1	e^x
⊕	0	e^x

$$= \lim_{t\to-\infty}\left[xe^{x} - e^{x}\right]\Big|_{t}^{0}$$

$$= \lim_{t\to-\infty}\left[(0e^{0} - e^{0}) - (te^{t} - e^{t})\right]$$

$0 - 1 - (te^{t} - e^{t})$

$= -1 - te^{t} + e^{t}$

$$= \lim_{t\to-\infty}\left[-1 - te^{t} + e^{t}\right]$$

$\displaystyle\lim_{t\to\infty} te^{t}$ $\left(-\infty \cdot \frac{1}{-\infty}\ \text{I.F}\right)$

$\displaystyle= \lim_{t\to\infty}\frac{t}{e^{-t}}$ (Rewrite)

$\displaystyle= \lim_{t\to-\infty}\frac{1}{-e^{-t}}$

$\displaystyle= \lim_{t\to-\infty} -e^{t}$

$= 0$

$$= \lim_{t\to-\infty}(-1) - \lim_{t\to-\infty}(te^{t}) + \lim_{t\to-\infty}(e^{t})$$

$$= -1 - 0 + 0$$

$$= -1$$

Thus, $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx$ converges to -1.

Definition $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

where $a \in \mathbb{R}$

④ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

Solution: Choose $a = 0$

Then, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx}_{①} + \underbrace{\int_{0}^{\infty} \frac{1}{1+x^2}\,dx}_{②}$

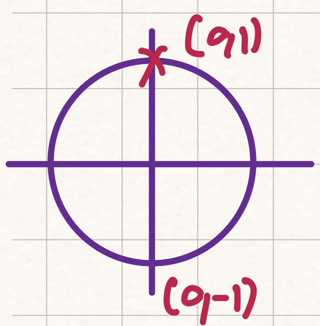
① $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \Big[\arctan(x)\Big]\Big|_{t}^{0}$

$$= \lim_{t\to-\infty} \left[\arctan(0) - \arctan(t)\right]$$

$$= \lim_{t\to-\infty} \left[0 - \arctan(t)\right]$$

$$= -\left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$



$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

② $$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \Big[\arctan(x)\Big]\Big|_0^t$$

$$= \lim_{t\to\infty} \Big[\arctan(x) - \arctan(0)\Big]$$

$$= \lim_{t\to\infty} \Big[\arctan(x)\Big]$$

$$= \frac{\pi}{2}$$

Hence, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Thus, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges to π.