Exercise: Vectors, Lines and Planes (Solution)

1. Let

$$\vec{A} = \langle 3, 4\rangle \qquad \vec{B} = \langle -3, 0\rangle \qquad \vec{C} = \langle -3, 2, -4\rangle$$
$$\vec{D} = \langle 6, -4, 8\rangle \qquad \vec{E} = \langle 1, 3, 2\rangle \qquad \vec{F} = \langle 2, -3, 1\rangle$$

1.1 Find $\vec{B} - 3\vec{A}$

Solution

Find $\vec{B} - 3\vec{A}$, then

$$\begin{aligned}\vec{B} - 3\vec{A} &= -3\vec{i} - 3(3\vec{i} + 4\vec{j}) \\ &= -3\vec{i} - 9\vec{i} - 12\vec{j} \\ &= -12\vec{i} - 12\vec{j}\end{aligned}$$

Therefore, $\vec{B} - 3\vec{A} = -12\vec{i} - 12\vec{j}$.

1.2 Find $\| - 4\vec{B}\|\vec{C} + 5\vec{F}$

Solution

Find $\| - 4\vec{B}\|$, then

$$\begin{aligned}\| - 4\vec{B}\| &= \sqrt{((-4)(-3))^2 + 0} \\ &= 12\end{aligned}$$

Find $\| - 4\vec{B}\|\vec{C} + 5\vec{F}$, then

$$\begin{aligned}\| - 4\vec{B}\|\vec{C} + 5\vec{F} &= 12(-3\vec{i} + 2\vec{j} - 4\vec{k}) + 5(2\vec{i} - 3\vec{j} + \vec{k}) \\ &= -36\vec{i} + 24\vec{j} - 48\vec{k} + 10\vec{i} - 15\vec{j} + 5\vec{k} \\ &= -26\vec{i} + 9\vec{j} - 43\vec{k}\end{aligned}$$

Therefore, $\| - 4\vec{B}\|\vec{C} + 5\vec{F} = -26\vec{i} + 9\vec{j} - 43\vec{k}$.

1.3 Find the vector of length 2 that has the same direction as $\vec{D}$

Solution

Find $\|\vec{D}\|$, then

$$\begin{aligned}\|\vec{D}\| &= \sqrt{(6)^2+(-4)^2+(8)^2}\\ &= \sqrt{36+16+64}\\ &= \sqrt{116}\end{aligned}$$

Find the vector of length 2 that has same direction as $\vec{D}$

$$\begin{aligned}2\frac{\vec{D}}{\|\vec{D}\|} &= \frac{2}{\sqrt{116}}(6\vec{i}-4\vec{j}+8\vec{k})\\ &= \frac{1}{\sqrt{29}}(6\vec{i}-4\vec{j}+8\vec{k})\end{aligned}$$

Therefore, the vector of length 2 that has same direction as $\vec{D}$ is $\dfrac{1}{\sqrt{29}}(6\vec{i}-4\vec{j}+8\vec{k})$.

1.4 Show that the vector $\vec{C}$ and $\vec{D}$ are parallel.

Solution

$$\begin{aligned}\vec{C}\times\vec{D} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & 2 & -4\\ 6 & -4 & 8\end{vmatrix}\\ &= (16-16)\vec{i}-(-24+24)\vec{j}+(12-12)\vec{k}\\ &= \vec{0}\end{aligned}$$

Since $\quad \vec{C}\times\vec{D}=\vec{0}$

Therefore, vector $\vec{C}$ and $\vec{D}$ are parallel.

1.5 Find the initial point of $\vec{E}$, if the terminal points is $(4, -3, 1)$.

Solution Since the vector $\vec{E}$ can be obtained by subtracting the coordinates of its initial point (a, b, c) from the coordinates of its terminal point. Then, we have

$$\vec{E} = (4-a)\vec{i} + (-3-b)\vec{j} + (1-c)\vec{k}$$

$$\vec{i} + 3\vec{j} + 2\vec{k} = (4-a)\vec{i} + (-3-b)\vec{j} + (1-c)\vec{k}$$

Consider $$1 = 4 - a$$

$$a = 3$$

and $$3 = -3 - b$$

$$b = -6$$

and $$2 = 1 - c$$

$$c = -1$$

Thus, the initial point of $\vec{E}$ is $(3, -6, -1)$.

2. Let $\vec{v} = 10\vec{i} + 11\vec{j} - 2\vec{k}$ and $\vec{u} = 3\vec{j} + 4\vec{k}$. Find

2.1 $\vec{v} \cdot \vec{u}$, $\|\vec{v}\|$, $\|\vec{u}\|$.

Solution

$$\begin{aligned}
\vec{v} \cdot \vec{u} &= (10\vec{i} + 11\vec{j} - 2\vec{k}) \cdot (3\vec{j} + 4\vec{k}) \\
&= \langle 10, 11, -2 \rangle \cdot \langle 0, 3, 4 \rangle \\
&= 10(0) + 11(3) + (-2)(4) \\
&= 25 \\
\|\vec{v}\| &= \sqrt{(10)^2 + (11)^2 + (-2)^2} \\
&= \sqrt{100 + 121 + 4} \\
&= \sqrt{225} \\
&= 15 \\
\|\vec{u}\| &= \sqrt{(0)^2 + (3)^2 + (4)^2} \\
&= \sqrt{0 + 9 + 16} \\
&= \sqrt{25} \\
&= 5
\end{aligned}$$

Therefore, $\vec{v} \cdot \vec{u} = 25$, $\|\vec{v}\| = 15$, $\|\vec{u}\| = 5$.

2.2 Cosine of the angle between $\vec{u}$ and $\vec{v}$.

Solution

Since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

We get

$$\begin{aligned} \cos\theta &= \frac{25}{(5)(15)} \\ &= \frac{25}{75} \\ &= \frac{1}{3} \end{aligned}$$

Therefore, $\cos\theta = \dfrac{1}{3}$.

2.3 The scalar component of $\vec{u}$ in the direction of $\vec{v}$ (Comp$_{\vec{v}}\vec{u}$).

Solution

From the scalar component of $\vec{u}$ in the direction of $\vec{v}$ is

$$\text{Comp}_{\vec{v}}\vec{u} = \|\vec{u}\| \cos\theta = \vec{u} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

We get

$$\begin{aligned} \text{Comp}_{\vec{v}}\vec{u} &= \langle 0, 3, 4\rangle \cdot \frac{\langle 10, 11, -2\rangle}{15} \\ &= (0)\left(\frac{10}{15}\right) + (3)\left(\frac{11}{15}\right) + (4)\left(\frac{-2}{15}\right) \\ &= 0 + \frac{33}{15} - \frac{8}{15} \\ &= \frac{5}{3} \end{aligned}$$

Therefore, the scalar component of $\vec{u}$ in the direction of $\vec{v}$ is $\dfrac{5}{3}$.

2.4. The projection vector of $\vec{u}$ along $\vec{v}$ (proj$_{\vec{v}}\vec{u}$).

Solution

$$
\begin{aligned}
\text{Since}\quad \text{proj}_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\
&= \left(\frac{25}{15^2}\right)\langle 10, 11, -2\rangle \\
&= \frac{25}{225}\langle 10, 11, -2\rangle \\
&= \frac{1}{9}\langle 10, 11, -2\rangle \\
&= \frac{1}{9}(10\vec{i} + 11\vec{j} - 2\vec{k})
\end{aligned}
$$

Therefore, the projection vector of $\vec{u}$ along $\vec{v}$ is $\frac{1}{9}(10\vec{i} + 11\vec{j} - 2\vec{k})$.

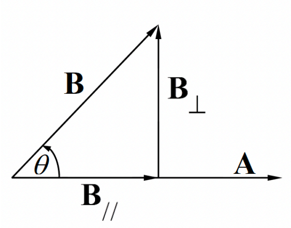
3. Find the angle between $\vec{u} = \sqrt{3}\vec{i} - 7\vec{j}$, $\quad \vec{v} = \sqrt{3}\vec{i} + \vec{j} - 2\vec{k}$

Solution

$$
\begin{aligned}
\text{Since}\quad \theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) \\
&= \cos^{-1}\left(\frac{(\sqrt{3})(\sqrt{3}) + (-7)(1) + (0)(-2)}{\left(\sqrt{(\sqrt{3})^2 + (-7)^2 + 0^2}\right)\left(\sqrt{(\sqrt{3})^2 + 1^2 + (-2)^2}\right)}\right) \\
&= \cos^{-1}\left(\frac{3 - 7 + 0}{(\sqrt{3 + 49})(\sqrt{3 + 1 + 4})}\right) \\
&= \cos^{-1}\left(\frac{-4}{(\sqrt{52})(\sqrt{8})}\right) \\
&= \cos^{-1}\left(\frac{-4}{\sqrt{416}}\right) \\
&= \cos^{-1}\left(\frac{-4}{4\sqrt{26}}\right) \\
&= \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \\
&\approx 101.3^\circ
\end{aligned}
$$

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 101.3^\circ$

4. For general vectors $\vec{A}$ and $\vec{B}$, the vectors $\vec{B}_{\perp}$ and $\vec{B}_{/\!/}$ are defined as in the below diagram:



If $\vec{A} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{B} = 5\vec{j} - 3\vec{k}$

Find $\vec{B}_{/\!/}$ and $\vec{B}_{\perp}$.

Solution

Since

$$
\begin{aligned}
\vec{B}_{/\!/} &= \text{proj}_{\vec{A}}\vec{B} \\
&= \left(\frac{\vec{B}\cdot\vec{A}}{\|\vec{A}\|^2}\right)\vec{A} \\
&= \left(\frac{(0)(1)+(5)(1)+(-3)(1)}{1^2+1^2+1^2}\right)\langle 1,1,1\rangle \\
&= \left(\frac{0+5-3}{3}\right)\langle 1,1,1\rangle \\
&= \frac{2}{3}\langle 1,1,1\rangle \\
&= \langle \frac{2}{3},\frac{2}{3},\frac{2}{3}\rangle \\
&= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{B}_{\perp} &= \vec{B} - \vec{B}_{/\!/} \\
&= \langle 0,5,-3\rangle - \langle \frac{2}{3},\frac{2}{3},\frac{2}{3}\rangle \\
&= \langle 0-\frac{2}{3}, 5-\frac{2}{3}, -3-\frac{2}{3}\rangle \\
&= \langle -\frac{2}{3}, \frac{13}{3}, -\frac{11}{3}\rangle \\
&= -\frac{2}{3}\vec{i} + \frac{13}{3}\vec{j} - \frac{11}{3}\vec{k}
\end{aligned}
$$

Therefore, $\vec{B}_{/\!/} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$ and $\vec{B}_{\perp} = -\frac{2}{3}\vec{i} + \frac{13}{3}\vec{j} - \frac{11}{3}\vec{k}$.

5. Let $\vec{A} = \langle 1, 2, 0 \rangle, \vec{B} = \langle 2, -1, 5 \rangle$ and $\vec{C} = \langle 7, 3, 1 \rangle$. Find

5.1 $\vec{A} \times (\vec{B} \times \vec{C})$

Solution

Find
$$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 5 \\ 7 & 3 & 1 \end{vmatrix}$$
$$= (-1\vec{i} + 35\vec{j} + 6\vec{k}) - (-7\vec{k} + 15\vec{i} + 2\vec{j})$$
$$= (-1 - 15)\vec{i} + (35 - 2)\vec{j} + (6 - (-7))\vec{k}$$
$$= -16\vec{i} + 33\vec{j} + 13\vec{k}$$

Find
$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ -16 & 33 & 13 \end{vmatrix}$$
$$= (26\vec{i} + 0\vec{j} + 33\vec{k}) - (-32\vec{k} + 0\vec{i} + 13\vec{j})$$
$$= (26 - 0)\vec{i} + (0 - 13)\vec{j} + (33 - (-32))\vec{k}$$
$$= 26\vec{i} - 13\vec{j} + 65\vec{k}$$
$$= \langle 26, -13, 65 \rangle$$

Therefore, $\vec{A} \times (\vec{B} \times \vec{C}) = \langle 26, -13, 65 \rangle$.

5.2 $\vec{A} \times (\vec{C} - 2\vec{B})$

Solution

Find
$$\begin{aligned}\vec{C} - 2\vec{B} &= (7\vec{i} + 3\vec{j} + 1\vec{k}) - 2(2\vec{i} - 1\vec{j} + 5\vec{k}) \\ &= (7\vec{i} + 3\vec{j} + 1\vec{k}) - (4\vec{i} - 2\vec{j} + 10\vec{k}) \\ &= (7-4)\vec{i} + (3-(-2))\vec{j} + (1-10)\vec{k} \\ &= 3\vec{i} + 5\vec{j} - 9\vec{k}\end{aligned}$$

Find
$$\begin{aligned}\vec{A} \times (\vec{C} - 2\vec{B}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & 5 & -9 \end{vmatrix} \\ &= (-18\vec{i} + 0\vec{j} + 5\vec{k}) - (6\vec{k} + 0\vec{i} - 9\vec{j}) \\ &= (-18-0)\vec{i} + (0-(-9))\vec{j} + (5-6)\vec{k} \\ &= \langle -18, 9, -1 \rangle\end{aligned}$$

Therefore, $\vec{A} \times (\vec{C} - 2\vec{B}) = \langle -18, 9, -1 \rangle$.

5.3 $4\vec{C} \times (3\vec{A} \times 2\vec{A})$

Solution

Find
$$\begin{aligned}3\vec{A} \times 2\vec{A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3(1) & 3(2) & 3(0) \\ 2(1) & 2(2) & 2(0) \end{vmatrix} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & 0 \\ 2 & 4 & 0 \end{vmatrix} \\ &= (0\vec{i} + 0\vec{j} + 12\vec{k}) - (12\vec{k} + 0\vec{i} + 0\vec{j}) \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k}\end{aligned}$$

Find
$$\begin{aligned}4\vec{C} \times (3\vec{A} \times 2\vec{A}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4(7) & 4(3) & 4(1) \\ 0 & 0 & 0 \end{vmatrix} \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} \\ &= \langle 0, 0, 0 \rangle \quad \text{or} \quad \vec{0}\end{aligned}$$

Therefore, $4\vec{C} \times (3\vec{A} \times 2\vec{A}) = \langle 0, 0, 0 \rangle$ or $\vec{0}$

5.4 $\dfrac{1}{10}\left[\left[\dfrac{1}{5}(\vec{A}\times\vec{B})\right]\times(\vec{A}\times\vec{C})\right]$

Solution

Find
$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & -1 & 5 \end{vmatrix}$$
$$= (10\vec{i}+0\vec{j}-1\vec{k})-(4\vec{k}+0\vec{i}+5\vec{j})$$
$$= (10-0)\vec{i}+(0-5)\vec{j}+(-1-4)\vec{k}$$
$$= 10\vec{i}-5\vec{j}-5\vec{k}$$

Find
$$\frac{1}{5}(\vec{A}\times\vec{B}) = \frac{1}{5}(10\vec{i}-5\vec{j}-5\vec{k})$$
$$= 2\vec{i}-1\vec{j}-1\vec{k}$$

Find
$$\vec{A}\times\vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 7 & 3 & 1 \end{vmatrix}$$
$$= (2\vec{i}+0\vec{j}+3\vec{k})-(14\vec{k}+0\vec{i}+1\vec{j})$$
$$= (2-0)\vec{i}+(0-1)\vec{j}+(3-14)\vec{k}$$
$$= 2\vec{i}-1\vec{j}-11\vec{k}$$

Find
$$\frac{1}{5}\left[(\vec{A}\times\vec{B})\right]\times(\vec{A}\times\vec{C}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 2 & -1 & -11 \end{vmatrix}$$
$$= (11\vec{i}-2\vec{j}-2\vec{k})-(-2\vec{k}+1\vec{i}-22\vec{j})$$
$$= (11-1)\vec{i}+(-2-(-22))\vec{j}+(-2-(-2))\vec{k}$$
$$= 10\vec{i}+20\vec{j}+0\vec{k}$$

Find
$$\frac{1}{10}\left[\left[\frac{1}{5}(\vec{A}\times\vec{B})\right]\times(\vec{A}\times\vec{C})\right] = \frac{1}{10}(10\vec{i}+20\vec{j}+0\vec{k})$$
$$= 1\vec{i}+2\vec{j}+0\vec{k}$$
$$= \langle 1,2,0\rangle$$

Therefore, $\dfrac{1}{10}\left[\left[\dfrac{1}{5}(\vec{A}\times\vec{B})\right]\times(\vec{A}\times\vec{C})\right] = \langle 1,2,0\rangle$

6. Let $\vec{A} = \langle 3, 1, 1\rangle, \vec{B} = \langle -1, 2, 1\rangle$ and $\vec{C} = \langle 4, -8, -4\rangle$. Which vectors are perpendicular to each other? And which vectors are parallel?

Solution

Consider
$$\begin{aligned}\vec{A} \cdot \vec{B} &= \langle 3, 1, 1\rangle \cdot \langle -1, 2, 1\rangle \\ &= (3)(-1) + (1)(2) + (1)(1) \\ &= -3 + 2 + 1 \\ &= 0\end{aligned}$$

Therefore, $\vec{A}$ and $\vec{B}$ are perpendicular implies that $\vec{A}$ and $\vec{B}$ are not parallel.

Consider
$$\begin{aligned}\vec{A} \cdot \vec{C} &= \langle 3, 1, 1\rangle \cdot \langle 4, -8, -4\rangle \\ &= (3)(4) + (1)(-8) + (1)(-4) \\ &= 12 - 8 - 4 \\ &= 0\end{aligned}$$

Therefore, $\vec{A}$ and $\vec{C}$ are perpendicular implies that $\vec{A}$ and $\vec{C}$ are not parallel.

Consider
$$\begin{aligned}\vec{B} \cdot \vec{C} &= \langle -1, 2, 1\rangle \cdot \langle 4, -8, -4\rangle \\ &= (-1)(4) + (2)(-8) + (1)(-4) \\ &= -4 + (-16) + (-4) \\ &= -24 \neq 0\end{aligned}$$

Therefore, $\vec{B}$ and $\vec{C}$ are not perpendicular because $\vec{B} \cdot \vec{C} \neq 0$.

or
$$\begin{aligned}\vec{B} \times \vec{C} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 4 & -8 & -4 \end{vmatrix} \\ &= (-8\vec{i} + 4\vec{j} + 8\vec{k}) - (8\vec{k} - 8\vec{i} + 4\vec{j}) \\ &= (-8 - (-8))\vec{i} + (4 - 4)\vec{j} + (8 - 8)\vec{k} \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} \\ &= \langle 0, 0, 0\rangle\end{aligned}$$

Therefore, $\vec{B}$ and $\vec{C}$ are parallel.

7. Determine whether the given vectors $\vec{A}, \vec{B}$ and $\vec{C}$ lie in the same plane or not?

7.1 $\vec{A} = \langle -2, 5, 2\rangle, \vec{B} = \langle 3, 0, -2\rangle, \vec{C} = \langle 1, 5, 0\rangle$

Solution

Consider
$$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -2 \\ 1 & 5 & 0 \end{vmatrix}$$
$$= (0\vec{i} - 2\vec{j} + 15\vec{k}) - (0\vec{k} - 10\vec{i} + 0\vec{j})$$
$$= (0 - (-10))\vec{i} + (-2 - 0)\vec{j} + (15 - 0)\vec{k}$$
$$= 10\vec{i} - 2\vec{j} + 15\vec{k}$$
$$= \langle 10, -2, 15\rangle$$

Consider
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \langle -2, 5, 2\rangle \cdot \langle 10, -2, 15\rangle$$
$$= (-2)(10) + (5)(-2) + (2)(15)$$
$$= -20 - 10 + 30$$
$$= 0$$

or
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} -2 & 5 & 2 \\ 3 & 0 & -2 \\ 1 & 5 & 0 \end{vmatrix}$$
$$= (0 + (-10) + 30) - (0 + 20 + 0)$$
$$= 20 - 20$$
$$= 0$$

Therefore, vector $\vec{A}, \vec{B}, \vec{C}$ are in the same plane.

7.2 $\vec{A} = \langle -3, 1, -2 \rangle, \vec{B} = \langle 2, 2, 7 \rangle, \vec{C} = \langle 1, -1, -1 \rangle$

Solution

Consider $$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 7 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= (-2\vec{i} + 7\vec{j} - 2\vec{k}) - (2\vec{k} - 7\vec{i} - 2\vec{j})$$

$$= (-2 - (-7))\vec{i} + (7 - (-2))\vec{j} + (-2 - 2)\vec{k}$$

$$= 5\vec{i} + 9\vec{j} - 4\vec{k}$$

$$= \langle 5, 9, -4 \rangle$$

Consider $$\vec{A} \cdot (\vec{B} \times \vec{C}) = \langle -3, 1, -2 \rangle \cdot \langle 5, 9, -4 \rangle$$

$$= (-3)(5) + (1)(9) + (-2)(-4)$$

$$= -15 + 9 + 8$$

$$= 2$$

or $$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} -3 & 1 & -2 \\ 2 & 2 & 7 \\ 1 & -1 & -1 \end{vmatrix}$$

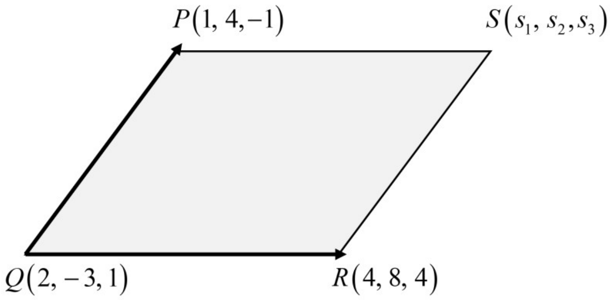
$$= (6 + 7 + 4) - (-4 + 21 + (-2))$$

$$= 17 - 15$$

$$= 2$$

Therefore, vector $\vec{A}, \vec{B}, \vec{C}$ are not in the same plane.

8. Let $P(1,4,-1), Q(2,-3,1), R(4,8,4)$ be the vertices of the parallelogram $PQRS$. Find the coordinates of the point S.



(NOT TO SCALE)

Solution

Find $\overrightarrow{QP}$ and $\overrightarrow{QR}$

$$\begin{aligned}\overrightarrow{QP} &= (1-2)\vec{i} + (4-(-3))\vec{j} + (-1-1)\vec{k} \\ &= -\vec{i} + 7\vec{j} - 2\vec{k} \\ \overrightarrow{QR} &= (4-2)\vec{i} + (8-(-3))\vec{j} + (4-1)\vec{k} \\ &= 2\vec{i} + 11\vec{j} + 3\vec{k}\end{aligned}$$

Find $\overrightarrow{QS}$ by the vector $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are the adjacent edges of the parallelogram $PQRS$. So, the vector $\overrightarrow{QS}$ it from $\overrightarrow{QP}$ and $\overrightarrow{QR}$, then

$$\begin{aligned}\overrightarrow{QS} &= \overrightarrow{QP} + \overrightarrow{QR} \\ &= -\vec{i} + 7\vec{j} - 2\vec{k} + 2\vec{i} + 11\vec{j} + 3\vec{k} \\ &= \vec{i} + 18\vec{j} + \vec{k}\end{aligned}$$

If point $S(s_1, s_2, s_3)$, then

$$\overrightarrow{QS} = (s_1 - 2)\vec{i} + (s_2 - (-3))\vec{j} + (s_3 - 1)\vec{k}$$

$$\vec{i} + 18\vec{j} + \vec{k} = (s_1 - 2)\vec{i} + (s_2 + 3)\vec{j} + (s_3 - 1)\vec{k}$$

Consider

$$1 = s_1 - 2$$

$$s_1 = 3$$

And

$$18 = s_2 + 3$$

$$s_2 = 15$$

And

$$1 = s_3 - 1$$

$$s_3 = 2$$

Therefore, the coordinate of point $S(s_1, s_2, s_3)$ of the parallelogram $PQRS$ is $(3, 15, 2)$.

9. Let

$$\vec{A} = \vec{i} + 5\vec{j} - 3\vec{k} \qquad \vec{B} = -4\vec{i} + \vec{j} - 2\vec{k}$$
$$\vec{C} = 7\vec{i} - \vec{k} \qquad \vec{D} = 4\vec{j} + 3\vec{k}$$

Find

9.1 $\| - 2\vec{D}\|(\vec{A} + 3\vec{B}) \cdot \vec{D}$

Solution

Find $\| - 2\vec{D}\|$, then

$$\begin{aligned} -2\vec{D} &= -2(4\vec{j} + 3\vec{k}) \\ &= -8\vec{j} - 6\vec{k} \\ \| - 2\vec{D}\| &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64 + 36} \\ &= 10 \end{aligned}$$

Find $\vec{A} + 3\vec{B}$, then

$$\begin{aligned} \vec{A} + 3\vec{B} &= \vec{i} + 5\vec{j} - 3\vec{k} + 3(-4\vec{i} + \vec{j} - 2\vec{k}) \\ &= \vec{i} + 5\vec{j} - 3\vec{k} - 12\vec{i} + 3\vec{j} - 6\vec{k} \\ &= -11\vec{i} + 8\vec{j} - 9\vec{k} \end{aligned}$$

Find $\| - 2\vec{D}\|(\vec{A} + 3\vec{B}) \cdot \vec{D}$, then

$$\begin{aligned} \| - 2\vec{D}\|(\vec{A} + 3\vec{B}) \cdot \vec{D} &= 10(-11\vec{i} + 8\vec{j} - 9\vec{k}) \cdot (4\vec{j} + 3\vec{k}) \\ &= 10(0 + 32 - 27) \\ &= 50 \end{aligned}$$

Therefore, $\| - 2\vec{D}\|(\vec{A} + 3\vec{B}) \cdot \vec{D} = 50$.

9.2 The angle between vector $\vec{B}$ and $\vec{C}$

Solution

Since $\vec{B} = -4\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{C} = 7\vec{i} - \vec{k}$

Find $\|\vec{B}\|$ and $\|\vec{C}\|$, then

$$\begin{aligned}
\|\vec{B}\| &= \sqrt{(-4)^2 + 1^2 + (-2)^2} \\
&= \sqrt{16 + 1 + 4} \\
&= \sqrt{21} \\
\|\vec{C}\| &= \sqrt{(7)^2 + (-1)^2} \\
&= \sqrt{50}
\end{aligned}$$

Find the angle between vector $\vec{B}$ and $\vec{C}$, by the formula

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\vec{B} \cdot \vec{C}}{\|\vec{B}\|\|\vec{C}\|}\right) \\
&= \cos^{-1}\left(\frac{-28 + 0 + 2}{\sqrt{21} \times \sqrt{50}}\right) \\
&= \cos^{-1}\left(\frac{-26}{\sqrt{21 \times 50}}\right) \\
&= \cos^{-1}\left(\frac{-26}{5\sqrt{42}}\right) \\
&\approx 143.36^\circ
\end{aligned}$$

Therefore, the angle between vector $\vec{B}$ and $\vec{C}$ is $\theta = \cos^{-1}\left(\frac{-26}{5\sqrt{42}}\right) \approx 143.36^\circ$

9.3 The volume of the rectangular cuboid with $\vec{A}, \vec{B}$ and $\vec{C}$ are adjacent sides.

Solution

Since $\vec{A} = \vec{i} + 5\vec{j} - 3\vec{k}$ and $\vec{B} = -4\vec{i} + \vec{j} - 2\vec{k}$

Find $\vec{A} \times \vec{B}$, then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -3 \\ -4 & 1 & -2 \end{vmatrix}$$

$$= -7\vec{i} + 14\vec{j} + 21\vec{k}$$

Find the volume of the rectangular cuboid with adjacent edges given by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$, then

$$\text{Rectangular cuboid} = \left|\vec{C} \cdot (\vec{A} \times \vec{B})\right|$$

$$= \left|(7\vec{i} - \vec{k}) \cdot (-7\vec{i} + 14\vec{j} + 21\vec{k})\right|$$

$$= \left| - 49 + 0 - 21\right|$$

$$= 70 \quad unit^3$$

Therefore, the volume of the rectangular cuboid with adjacent edges given by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ is $70\ unit^3$.

9.4 The vector of length $\sqrt{5}$ and orthogonal to $\vec{C}$ and $\vec{D}$.

Solution

Since $\vec{C} = 7\vec{i} - \vec{k}$ and $\vec{D} = 4\vec{j} + 3\vec{k}$

Find $\vec{C} \times \vec{D}$, then

$$\begin{aligned} \vec{C} \times \vec{D} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -1 \\ 0 & 4 & 3 \end{vmatrix} \\ &= 4\vec{i} - 21\vec{j} + 28\vec{k} \end{aligned}$$

Find $\|\vec{C} \times \vec{D}\|$, then

$$\begin{aligned} \|\vec{C} \times \vec{D}\| &= \sqrt{(4)^2 + (-21)^2 + 28^2} \\ &= \sqrt{16 + 441 + 784} \\ &= \sqrt{1241} \end{aligned}$$

Find the vector of length $\sqrt{5}$ and orthogonal to $\vec{C}$ and $\vec{D}$

$$\sqrt{5}\frac{\vec{C} \times \vec{D}}{\|\vec{C} \times \vec{D}\|} = \frac{\sqrt{5}}{\sqrt{1241}}(4\vec{i} - 21\vec{j} + 28\vec{k})$$

Therefore, the vector of length $\sqrt{5}$ and orthogonal to $\vec{C}$ and $\vec{D}$ is $\frac{\sqrt{5}}{\sqrt{1241}}(4\vec{i} - 21\vec{j} + 28\vec{k})$.

10. Let

$$A(0,2,2), \quad B(8,8,-2), \quad C(9,12,6), \quad D(2,0,4), \quad E(5,-1,3).$$

Answer the following questions:

10.1 The vector $\overrightarrow{AB}$ is orthogonal to vector $\overrightarrow{CD}$ or not?

Solution

$$\begin{aligned}
\text{Find} \qquad \overrightarrow{AB} &= \langle 8-0, 8-2, -2-2\rangle \\
&= \langle 8, 6, -4\rangle \\
\text{Find} \qquad \overrightarrow{CD} &= \langle 2-9, 0-12, 4-6\rangle \\
&= \langle -7, -12, -2\rangle \\
\text{Consider} \qquad \overrightarrow{AB}\cdot\overrightarrow{CD} &= \langle 8, 6, -4\rangle \cdot \langle -7, -12, -2\rangle \\
&= -56 - 72 + 8 \\
&= -120 \\
\text{We get} \qquad \overrightarrow{AB}\cdot\overrightarrow{CD} &\neq 0
\end{aligned}$$

Therefore, vector $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not orthogonal.

10.2 If $\|k\overrightarrow{CB}\| = 9$. Find the value of k.

Solution

$$\begin{aligned}
\text{Find} \qquad \overrightarrow{CB} &= \langle 8-9, 8-12, -2-6\rangle \\
&= \langle -1, -4, -8\rangle \\
\text{We get} \qquad k\overrightarrow{CB} &= \langle -k, -4k, -8k\rangle \\
\text{Since} \qquad \|k\overrightarrow{CB}\| &= 9 \\
\text{Then,} \qquad \sqrt{(-k)^2 + (-4k)^2 + (-8k)^2} &= 9 \\
\sqrt{k^2 + 16k^2 + 64k^2} &= 9 \\
\sqrt{k^2(1+16+64)} &= 9 \\
|k|\sqrt{81} &= 9 \\
|k| &= 1
\end{aligned}$$

Therefore, $k = \pm 1$

10.3 Find the area of the triangle A, B, C.

Solution

SInce $A(0,2,2), \quad B(8,8,-2), \quad C(9,12,6)$

The formula of the area of the triangle with vertices at A, B, C is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

From 10.1 we have, $\overrightarrow{AB} = \langle 8, 6, -4 \rangle$

Find $\overrightarrow{AC} = \langle 9-0, 12-2, 6-2 \rangle$

$$= \langle 9, 10, 4 \rangle$$

Consider
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 6 & -4 \\ 9 & 10 & 4 \end{vmatrix}$$

$$= 64\vec{i} - 68\vec{j} + 26\vec{k}$$

We get,
$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{64^2 + (-68)^2 + 26^2}$$
$$= \sqrt{9396}$$
$$= 18\sqrt{29}$$

Therefore, the area of the triangle having vertices at points A, B, C is $\frac{18\sqrt{29}}{2} = 9\sqrt{29}\ unit^2$.

10.4 Find the equation of the plane passing through points A, D, E.

Solution

Since $A(0,2,2), \quad D(2,0,4), \quad E(5,-1,3)$

Let $\vec{N} = \langle a, b, c \rangle$

be the normal vector of the plane $a(x-0)+b(y-y_0)+c(z-z_0)=0$

that, is $\vec{N} = \overrightarrow{AD} \times \overrightarrow{AE}$

Find the vector $\overrightarrow{AD} = \langle 2-0, 0-2, 4-2 \rangle$

$= \langle 2, -2, 2 \rangle$

And $\overrightarrow{AE} = \langle 5-0, -1-2, 3-2 \rangle$

$= \langle 5, -3, 1 \rangle$

Then, $\vec{N} = \overrightarrow{AD} \times \overrightarrow{AE}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 2 \\ 5 & -3 & 1 \end{vmatrix}$$

$= 4\vec{i} + 8\vec{j} + 4\vec{k}$

$= \langle 4, 8, 4 \rangle$

Therefore, the equation of the plane passing through point A and orthogonal with $\overrightarrow{AD}, \overrightarrow{AE}$ is

$$4(x-0)+8(y-2)+4(z-2)=0$$

$$4x+8y-16+4z-8=0$$

$$4x+8y+4z-24=0$$

10.5 Find the symmetric equations of the line passing through the points E and parallel to $\overrightarrow{DC}$.

Since $C(9, 12, 6), \quad D(2, 0, 4), \quad E(5, -1, 3)$

$$\text{Find the vector} \quad \overrightarrow{DC} = \langle 9-2, 12-0, 6-4 \rangle$$
$$= \langle 7, 12, 2 \rangle$$

From, the symmetric equation of the line passing through the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c \rangle$ is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Therefore, the symmetric equation of the line passing through the point E and parallel to the vector $\overrightarrow{DC}$ is

$$\frac{x-5}{7} = \frac{y+1}{12} = \frac{z-3}{2}$$

11. Find symmetric equations of the line passing through the point $(3, 0, 1)$ and parallel to the line.

$$x = y + 2, \quad z = 4.$$

Solution

Let L_1 has the symmetric equation as

$$x = y + 2, \quad z = 4$$

When compare with symmetric equation in case $c = 0$, in the form as

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad z = z_0$$

We get,

$$\frac{x-0}{1} = \frac{y+2}{1} \quad \text{and} \quad z = 4$$

So, L_1 have the direction vector $V = \langle a, b, c \rangle = \langle 1, 1, 0 \rangle$

Let L_2 be the line passing through $(3, 0, 1)$ and parallel to L_1

Since L_1 parallel to L_2

Thus, V also parallel to the vector direction of L_2

Therefore, L_2 will has symmetric equation,

$$\frac{x-3}{1} = \frac{y-0}{1} \qquad \text{and} \qquad z = 1$$

Hence, $x - 3 = y$ and $z = 1$

12. Find an equation of the plane that passes through the point $(1, -3, 6)$ and perpendicular to the planes $3x + y - z = 3$ and $x - y + 3z = 6$.

Solution

Since the equation of the plane, $3x + y - z = 3$ has a normal vector $\vec{N_1} = \langle 3, 1, -1 \rangle$

and the equation of the plane, $x - y + 3z = 6$ has a normal vector $\vec{N_2} = \langle 1, -1, 3 \rangle$

Find the orthogonal vector of $\vec{N_1}$ and $\vec{N_2}$ by $\vec{N} = \vec{N_1} \times \vec{N_2} = \langle a, b, c \rangle$

$$\begin{aligned}
\text{Then,} \quad \vec{N} &= \vec{N_1} \times \vec{N_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ 1 & -1 & 3 \end{vmatrix} \\
&= 2\vec{i} - 10\vec{j} - 4\vec{k} \\
&= \langle 2, -10, -4 \rangle
\end{aligned}$$

Thus, the equation of the plane that passing through point $(1, -3, 6)$ is

$$\begin{aligned}
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\
2(x - 1) - 10(y + 3) - 4(z - 6) &= 0 \\
2x - 2 - 10y - 30 - 4z + 24 &= 0 \\
2x - 10y - 4z - 8 &= 0
\end{aligned}$$

13. Determine whether the line and plane are parallel or perpendicular?

13.1 $x = 4 + 2t, \quad y = -t, \quad z = 3 - 4t, \quad 3x + 2y + z - 5 = 0.$

Solution

Since the parametric equation of the line L; is

$$x = 4 + 2t, \quad y = -t, \quad z = 3 - 4t$$

It have parallel vector is

$$\vec{v} = \langle 2, -1, -4 \rangle$$

Since an equation of the plane; $3x + 2y + z - 5 = 0$

It have a normal vector is

$$\vec{N} = \langle 3, 2, 1 \rangle$$

Consider $\quad \vec{v} \cdot \vec{N} = 2(3) + (-1)2 + (-4)1$

$$= 6 - 2 - 4$$

$$= 0$$

As $\quad \vec{v} \cdot \vec{N} = 0$

then, $\quad \vec{v} \perp \vec{N}$

This mean that, this line is parallel to the planes.

13.2 $x = -1 + 2t, \quad y = 4 + t, \quad z = 1 - t, \quad -2x - y + z = 12.$

Solution

Since the parametric equation of the line L; is

$$x = -1 + 2t, \quad y = 4 + t, \quad z = 1 - t$$

It have parallel vector is

$$\vec{v} = \langle 2, 1, -1 \rangle$$

Since an equation of the plane; $-2x - y + z = 12$

It have a normal vector is

$$\vec{N} = \langle -2, -1, 1 \rangle$$

Consider $$\begin{aligned} \vec{v} \cdot \vec{N} &= 2(-2) + 1(-1) + (-1)1 \\ &= -4 - 1 - 1 \\ &= -6 \end{aligned}$$

Then $$\vec{v} \cdot \vec{N} \neq 0$$

That is the vector $\vec{v}$ not parallel to vector $\vec{N}$

Thus, consider $\vec{v} \times \vec{N}$

$$\begin{aligned} \vec{v} \times \vec{N} &= (1-1)\vec{i} + (2-2)\vec{j} + (-2-(-2))\vec{k} \\ &= \langle 0, 0, 0 \rangle \\ &= \vec{0} \end{aligned}$$

That is $\vec{v} /\!/ \vec{N}$ ($\vec{v}$ parallel to $\vec{N}$)

This mean that, this line is perpendicular to the planes.

14. Find the acute angle of intersection of the planes $x + y - 2z = 5$ and $3y - 4z = 6$.

Solution

From an equation of the plane $x+y-2z = 5$, it has a normal vector $\vec{N_1} = \langle 1, 1, -2 \rangle$ and equation of the plane $3y - 4z = 6$, it has a normal vector $\vec{N_2} = \langle 0, 3, -4 \rangle$

Find $$\|\vec{N_1}\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

Find $$\|\vec{N_2}\| = \sqrt{0^2 + 3^2 + (-4)^2} = \sqrt{25} = 5$$

By the formula, $$\cos\theta = \frac{\vec{N_1} \cdot \vec{N_2}}{\|\vec{N_1}\| \cdot \|\vec{N_2}\|} = \frac{\langle 1, 1, -2 \rangle \cdot \langle 0, 3, -4 \rangle}{5\sqrt{6}} = \frac{1(0) + 1(3) + (-2)(-4)}{5\sqrt{6}}$$

that is $$\cos\theta = \frac{11}{5\sqrt{6}}$$

Then, $$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{6}}\right) \approx 26^\circ$$

Therefore, the angle of intersection of the planes $x + y - 2z = 5$ and $3y - 4z = 6$ is $\cos^{-1}\left(\frac{11}{5\sqrt{6}}\right) \approx 26^\circ$

15. Find the distance between the point and the plane.

15.1 $(1, -2, 3)$; $2x - 2y + z = 5$

Solution

Let $P = (x_0, y_0, z_0)$ be the point on the plane $2x - 2y + z = 5$

Let $Q = (1, -2, 3)$,

Then $\overrightarrow{PQ} = \langle 1 - x_0, -2 - y_0, 3 - z_0 \rangle$

From an equation of the plane, $2x - 2y + z = 5$, it has a normal vector is

$$\vec{N} = \langle 2, -2, 1 \rangle$$

And it magnitude is
$$\begin{aligned} \|\vec{N}\| &= \sqrt{4 + 4 + 1} \\ &= 3 \end{aligned}$$

By the formula,
$$\begin{aligned} d = \|\overrightarrow{PQ}\| \cos\theta &= \frac{\left|\overrightarrow{PQ} \cdot \vec{N}\right|}{\|\vec{N}\|} \\ &= \frac{\left|2(1 - x_0) - 2(-2 - y_0) + 3 - z_0\right|}{3} \\ &= \frac{1}{3}\left|2 - 2x_0 + 4 + 2y_0 + 3 - z_0\right| \\ &= \frac{1}{3}\left|9 - (2x_0 - 2y_0 + z_0)\right| \\ &= \frac{1}{3}\left|9 - 5\right| \\ &= \frac{4}{3} \end{aligned}$$

Thus, the distence is $\frac{4}{3}$ unit.

15.2 $(0,1,5)$; $\quad 3x+7y-2z-5=0$

Solution

Let $\quad P=(x_0,y_0,z_0)$ be the point on the plane $\quad 3x+7y-2z=5$

Let $\quad Q=(0,1,5)$

Then $\quad \overrightarrow{PQ}=\langle 0-x_0, 1-y_0, 5-z_0\rangle$

From an equation of the plane, $3x+7y-2z=5$, it has a normal vector is

$$\vec{N}=\langle 3,7,-2\rangle$$

And it magnitude is $\quad \|\vec{N}\|=\sqrt{3^2+7^2+(-2)^2}$

$$=\sqrt{62}$$

By the formula, $\quad d=\dfrac{\left|\overrightarrow{PQ}\cdot\vec{N}\right|}{\|\vec{N}\|}$

$$\begin{aligned}
&=\frac{1}{\sqrt{62}}\Big|3(-x_0)+7(1-y_0)-2(5-z_0)\Big| \\
&=\frac{1}{\sqrt{62}}\Big|-3x_0-7y_0+2z_0+7-10\Big| \\
&=\frac{1}{\sqrt{62}}\Big|-3-(3x_0+7y_0-2z_0)\Big| \\
&=\frac{1}{\sqrt{62}}|-3-5| \\
&=\frac{1}{\sqrt{62}}|-8| \\
&=\frac{8}{\sqrt{62}}
\end{aligned}$$

Thus, the distence is $\dfrac{8}{\sqrt{62}}$ unit.