MTH 102: Mathematics II

Chapter 1. Sequence of Numbers

1.1 What's a sequence?

A sequence of numbers is a function defined on the set of positive integers. The numbers in the sequence are called *terms*. Another way, the sequence is a set of quantities u_1, u_2, u_3... stated in a definite order and each term formed according to a fixed pattern.

For example: 1, 3, 5, 7, 9, 11, . . .

3, 6, 12, 24, . . .

Let f be a function defined on a set of natural numbers N. Then the sequence f is a set of ordered pairs $(n, f(n))$. In general, we represent $f(n)$ by a_n where n is called the index. The explicit formula for a_n is called a *bracket notation*.

For example, the bracket notation of the above sequences are $\{2n-1\}_{n=1}^{\infty}$ and $\{3\cdot 2^{n-1}\}_{n=1}^{\infty}$, respectively.

- **Infinite sequence**

This kind of sequence is unending sequence like all natural numbers: 1, 2, 3,...

- **Finite sequence**

This kind of sequence contains only a finite number of terms.

- A good example of a finite sequence is the page number of a book.

Exercises

1. Write the first 4 terms of the following infinite sequences:

a) $\left\{ \frac{1}{3n} \right\}_{n=1}^{\infty}$

b) $\left\{ \sin \frac{n\pi}{2} - \cos n\pi \right\}_{n=0}^{\infty}$

2. Write the first ten terms of Fibonacci Sequence $\{F_n\}$ which is recursively defined as

$$F_{n+1} = F_n + F_{n-1} \qquad \text{for } n \geq 2$$

where $F_1 = 1$, $F_2 = 1$.

3. Find the n^{th} - term (bracket notation) of the following sequences:

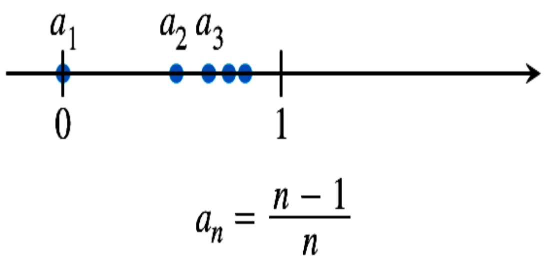
   a) 2, -2, 2, -2, . . .

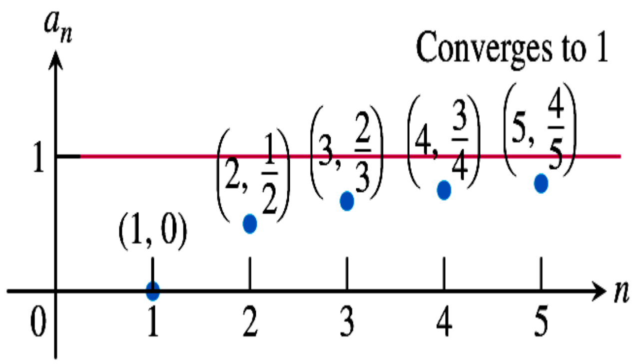
   b) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$

1.2 Graph of a sequence

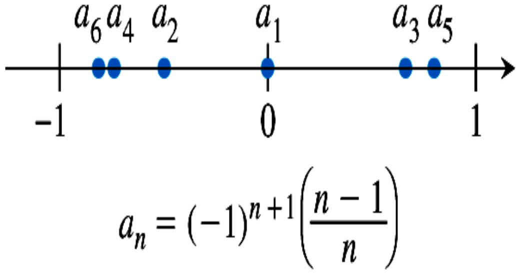
Since a sequence is considered as a function, we can draw a graph for each sequence.

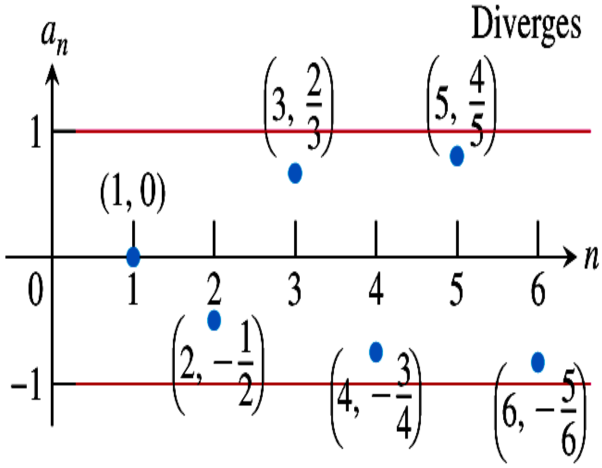
Example1: Here is the graph of $\left\{\dfrac{n-1}{n}\right\}$.



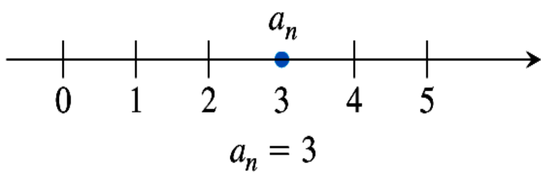


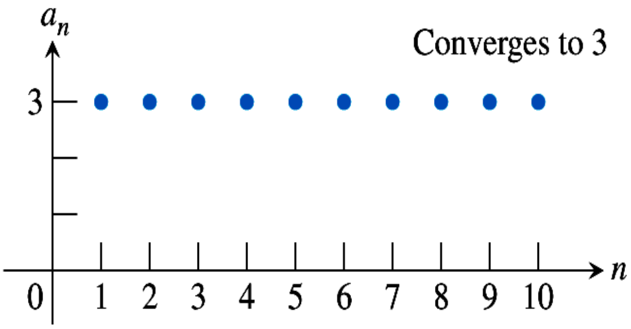
Example2: $\left\{\dfrac{(-1)^{n+1}(n-1)}{n}\right\}$





Example3: {3,3,3,. . . .}





All three graphs are discontinuous. The first and the third graphs converge to 1 and 3, respectively, while the second one diverges.

1.3 Limit of a sequence

Suppose a_1, a_2, ... is a sequence of real numbers. We say that a unique real number L is the *limit* of this sequence and we write $\lim_{n\to\infty} a_n = L$ (where $L \neq \pm\infty$) if and only if for every real number $\epsilon > 0$ there exists a natural number n_0 (which may depend on ε) such that for all $n > n_0$, we have $|\, a_n - L \,| < \varepsilon$.

Intuitively, this means that eventually most elements (infinite numbers) of the sequence get as close as we want to the limit, since the absolute value $|\, a_n - L \,|$ can be interpreted as the "distance" between a_n and L. Note that not every sequence has a limit; if it does exist, we call it a <u>*convergent sequence*</u>. Otherwise, we say that it is <u>*divergent*</u>.

Exercise2

Test if the following sequences converge or diverge:

a) $\left\{(-1)^n \dfrac{3n^2}{n^2-4}\right\}$

b) $\left\{(-1)^n \dfrac{2}{n^2+1}\right\}$

Calculating limit of Sequence

Theorem 1.1.1 Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers and A, B, k be some real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

1. $\lim_{n\to\infty}(a_n + b_n) = A + B$ (Sum Rule)
2. $\lim_{n\to\infty}(a_n - b_n) = A - B$ (Difference Rule)
3. $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$ (Product Rule)
4. $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, $B \neq 0$ (Quotient Rule)
5. $\lim_{n\to\infty} kb_n = kB$ (Constant Multiple Rule)

Theorem 1.2 Let $f(x)$ be a function defined on $[n_0, \infty)$ and $\{a_n\}$ be a sequence of real number such that $a_n = f(n)$ for all $n > n_0$. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

If $\lim_{x\to\infty} f(x)$ is one of the indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty$, 0^0, ∞^0, 1^∞, $\infty - \infty$, the L' Hopital's rule for a sequence will be needed.

Exercise: Check if each of the following sequence converges or diverges

1. $\left\{ \dfrac{1-6n^4}{n^4+8n^3} \right\}$

<u>Solution</u>

$$\lim_{x\to\infty} \frac{1-6x^4}{x^4+8x^3} = \lim_{x\to\infty} \frac{\frac{1}{x^4} - \frac{6x^4}{x^4}}{\frac{x^4}{x^4} + \frac{8x^3}{x^4}} = \lim_{x\to\infty} \frac{\frac{1}{x^4} - 6}{1 + \frac{8}{x}} = -6$$

Thus $\left\{ \dfrac{1-6n^4}{n^4+8n^3} \right\}$ converges to -6.

2. $\left\{ \dfrac{n^2-2n+1}{n-1} \right\}$

<u>Solution</u>

$$\lim_{x\to\infty} \frac{x^2-2x+1}{x-1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x}{x^2} - \frac{1}{x^2}} = \infty$$

Thus $\left\{ \dfrac{n^2-2n+1}{n-1} \right\}$ diverges.

3. $\left\{ \dfrac{2^{1000}+2^{n-1}+3^{n-2}}{2^n+3^n+5} \right\}$

4. $\left\{ n - \sqrt{n^2 - n} \right\}$

5. $\left\{ \ln n - \ln(2n^3 + 1) \right\}$

Well-known limits of sequences:

1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

2) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3) $\lim\limits_{n\to\infty} \dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

4) $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1 \quad , x > 0$

5) $\lim\limits_{n\to\infty} x^n = 0 \quad , |x| < 1$

6) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$

7) $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

cases 4-7, x is a constant

Example: Find the limit of the infinite sequence $\left\{\left(\dfrac{1+n}{n-1}\right)^n\right\}$

<u>Solution</u> Let $K=\left(\dfrac{1+x}{x-1}\right)^x$, then $\ln K = x\ln\left(\dfrac{1+x}{x-1}\right)$.

$$\lim_{x\to\infty}\ln K = \lim_{x\to\infty} x\ln\left(\frac{1+x}{x-1}\right)$$

$$\ln\lim_{x\to\infty} K = \lim_{x\to\infty}\frac{\ln\left(\dfrac{1+x}{x-1}\right)}{\dfrac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\dfrac{x-1}{1+x}\left(\dfrac{1\cdot(x-1)-1\cdot(1+x)}{(x-1)^2}\right)}{-\dfrac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{2x^2}{(1+x)(x-1)}=2$$

So, $\ln\lim\limits_{x\to\infty} K = 2$ implies $\lim\limits_{x\to\infty} K = e^2$.

Therefore, $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x-1}\right)^x = e^2$.

Thus $\left\{\left(\dfrac{1+n}{n-1}\right)^n\right\}$ converges to e^2.

The Sandwich Theorem of Sequence

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $a_n \le b_n \le c_n$ for all n. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example: Show that $\left\{\dfrac{\sin n}{n}\right\}$ converges

<u>Solution</u> Note that $-1 \le \sin n \le 1$

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$0 = \lim_{n\to\infty} -\frac{1}{n} \le \lim_{n\to\infty} \frac{\sin n}{n} \le \lim_{n\to\infty} \frac{1}{n} = 0$$

By Sandwich Theorem: $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0$, thus $\left\{\dfrac{\sin n}{n}\right\}$ converges.

The Continuous Function Theorem for Sequence

Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = L$ and f be a continuous function defined at all a_n's and L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Note: We may write: if $a_n \to L$, then $f(a_n) \to f(L)$

Example: $\left\{\sin\left(\frac{n\pi+2}{2n}\right)\right\}$ converges or diverges?

1.4 Monotone Sequences

Definition: A sequence $\{a_n\}$ is called

increasing	if	$a_1 < a_2 < a_3 < \ldots < a_n < \ldots$
non-decreasing	if	$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le \ldots$
decreasing	if	$a_1 > a_2 > a_3 > \ldots > a_n > \ldots$
non-increasing	if	$a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge \ldots$

All of these sequences are considered **monotone**. If a sequence is either increasing or decreasing, it is called **strictly monotone**.

To check if a given sequence is monotone or not, we can use one of the following methods:

1) difference: $a_{n+1} - a_n$

2) ratio: $\frac{a_{n+1}}{a_n}$

3) derivative: let $f(n) = a_n$, then check the sign of $f'(x)$, when $x \geq 1$.

<u>Example</u>: Check if the following sequences are monotone:

(a) $1, \frac{2}{2!}, \frac{3}{3!}, \frac{4}{4!}, \ldots$

(b) $\left\{ n - \left(\frac{1}{2} \right)^n \right\}$

Theorem 1.4.1: For a non-decreasing sequence $\{a_n\}$, there are two possibilities:

1) There exists a constant *M* such that $a_n \le M$ for all *n*. Then $\lim_{n\to\infty} a_n = L \le M$. We say that the sequence a_n is *bounded above* by *M*.

2) There is no such *M*. In this case, $\lim_{n\to\infty} a_n = \infty$.

Theorem 1.4.2: For a non-increasing sequence $\{a_n\}$, there are two possibilities:

1) There exists a constant *M* such that $a_n \ge M$ for all *n*. Then $\lim_{n\to\infty} a_n = L \ge M$. Also, we say that the sequence a_n is *bounded below* by *M*.

2) There is no such *M*. In this case, $\lim_{n\to\infty} a_n = -\infty$.

Definition 1.4.2: The sequence $\{a_n\}$ is called a ***bounded sequence*** if there exists a real number *M* and *N* such that $N \le a_n \le M$ for all *n*. Here, we called *M* as an ***upper bound*** and *N* as a ***lower bound*** of the sequence $\{a_n\}$. The smallest upper bound is called the *least upper bound*, while the *biggest lower bound* is called the greatest lower bound.

Note:

1) Every bounded monotone sequence is convergent, but not all bounded sequences are convergent. Example?
2) Every unbounded sequence is divergent.

Examples: Consider the following sequences. Are they monotone? bounded? convergent or divergent?

1) $\{\cos(n\pi)\}$

2) $\left\{(-1)^{n+1}\dfrac{n}{n^2+1}\right\}$

Exercise 1

1. Determine if the following sequences converge or diverge. If it converges, find its limit.

1.1 $\left\{\dfrac{4n-1}{8n+3}\right\}$ 1.2 $\left\{5(-1)^{n+1}\right\}$ 1.3 $\left\{\dfrac{n^3-1}{2n}\right\}$

1.4 $\left\{\dfrac{\sqrt{2n+1}}{n}\right\}$ 1.5 $\left\{\dfrac{\ln n}{n}\right\}$ 1.6 $\left\{\dfrac{6-2^{-n}}{3+4^{-n}}\right\}$

1.7 $\left\{n^{\frac{2}{n+1}}\right\}$ 1.8 $\left\{\ln\left(\dfrac{4n+1}{5n-1}\right)\right\}$ 1.9 $\left\{\dfrac{\sin^2 n}{4^n}\right\}$

1.10 $\left\{(-1)^n \dfrac{5n^3}{n^3+1}\right\}$ 1.11 $\left\{n\sin\dfrac{\pi}{n}\right\}$ 1.12 $\left\{\dfrac{(-1)^{n+1}}{n^2}\right\}$

1.13 $\left\{\dfrac{e^n}{4^n}\right\}$ 1.14 $\left\{\sqrt{n^2+3n}-n\right\}$ 1.15 $\left\{\left(\dfrac{n+3}{n+1}\right)^n\right\}$

2. Determine if the following sequences are monotone, increasing or decreasing.

2.1 $\left\{\dfrac{n}{2n+1}\right\}$ 2.2 $\left\{n-2^n\right\}$ 2.3 $\left\{\dfrac{n!}{3^n}\right\}$

Answer to exercise 1

1. 1.1 Con. to $\frac{1}{2}$ 1.2 Div. 1.3 Div.

   1.4 Con. to 0 1.5 Con. to 0 1.6 Con. to 2

   1.7 Con. to 1 1.8 Con. to $\ln\dfrac{4}{5}$ 1.9 Con. to 0

   1.10 Div. 1.11 Con. to π 1.12 Con. to 0

   1.13 Con. to 0 1.14 Con. to $\dfrac{3}{2}$ 1.15 Con. to e^2

2. 2.1 monotone and increasing

   2.2 monotone and decreasing

   2.3 not monotone

Chapter 2 Series

2.1 A series is formed by the sum of the terms of a sequence. For example:

$$1, 3, 5, 7, \ldots$$

is a sequence, but

$$1 + 3 + 5 + 7 + \ldots$$

is a series.

We shall indicate the terms of a series as follows: a_1 will represent the first term, a_2 the second term, a_3 the third term, etc., so that a_n will be represent the n-th term. Also, the partial sum of the first 5 terms will be indicated as S_5 and the partial sum of the first n-terms will be stated as S_n.

Infinite Series

Let $\{a_n\}$ be an infinite sequence. Then we define:

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

Each S_n is called the ***n*-th *partial sum*** of the infinite series $\sum_{n=1}^{\infty} a_n$.

Definition 2.1: An infinite series $\sum_{n=1}^{\infty} a_n$ is said to converge to a real number S if $\lim_{n\to\infty} S_n = S$, where $S \neq \pm\infty$. Otherwise, the series is divergent.

Theorem 2.2: If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.3: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 2.4: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series differing only the first m terms, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

- **Infinite Geometric Series:**

The series $\sum_{n=1}^{\infty} a_n$ is said to be a *geometric series* if a_n can be written of the form $a_n = a \cdot r^{n-1}$ where a, r are real numbers such that $r \neq 0$. The real number r is also called the *common ratio*. The partial sum of any geometric series can be expressed as $S_n = \dfrac{a(r^n - 1)}{r-1} = \dfrac{ra_n - a}{r-1}$. In general, there are 3 cases of geometric series

1) If $0 < |r| < 1$, the series converges to $\lim\limits_{n\to\infty} S_n = \dfrac{a}{1-r}$.

2) If $|r| \geq 1$ and $a \neq 0$, then the series diverges.

3) If $a = 0$, then the series converges to 0.

- **Harmonic Series**: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$

- ***P*-Series**: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \cdots$, where $p \in \mathbb{R}^+ \setminus \{1\}$

Theorem 2.5: A harmonic series always diverges. A *p*-series is convergent if $p > 1$ and divergent if $0 < p < 1$.

Exercise 2.1

Determine if the following series converge or diverge. If it converges, find its value.

1. $\displaystyle\sum_{n=1}^{\infty} \left((-1)^{n-1} \frac{7}{6^{n-1}} \right)$

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{(4n-3)(4n+1)} \right)$

3. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{9n^2+3n-2}\right)$

4. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$

5. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sqrt{n^4+1}}{3n^2+1}\right)$

7. $\displaystyle\sum_{n=2}^{\infty}\left[\frac{1}{n^2-1}+\left(-\frac{3}{2}\right)^{n+1}\right]$

8. $\displaystyle\sum_{n=0}^{\infty}\left[\ln\left(\frac{n}{n+1}\right)\right]$

9. $\displaystyle\sum_{n=1}^{\infty}\left[\ln\left(\frac{n}{2n+1}\right)\right]$

10. $\displaystyle\sum_{n=1}^{\infty}(\cos n\pi)$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n-1}{3^n}$

12. $\displaystyle\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

14. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n}$

Answer 2.1

1. Conv. to 6
2. Conv. to 1
3. Conv. to $\frac{1}{6}$
4. Conv. to $\frac{17}{6}$
5. Conv. to $\frac{23}{2}$
6. Div.
7. Div.
8. Div.
9. Div.
10. Div.
11. Conv. to $\frac{3}{2}$
12. Conv. to $\ln\frac{1}{2}$
13. Conv. to 1
14. Div.
15. Conv. to 1

2.2 Tests for Convergence of Series with Positive Terms

If we know the partial sum S_n , we can determine if $\sum_{n=1}^{\infty} a_n$ converges or diverges directly. However, sometimes it's not easy to find S_n. Here we will learn five techniques to check if a given sequence is convergent or divergent.

2.2.1 Integral Test

Let $\sum a_n$ be a series such that $a_n > 0$ and let $f(x)$ be a *positive decreasing continuous* function for all $x \in [1, \infty)$ such that $f(n) = a_n$ for all integer *n*. Then $\sum a_n$ and $\int_1^{\infty} f(x)dx$ are both convergent or both divergent.

Example1: Check if the following series converge.

a. $\sum_{n=1}^{\infty} ne^{-n}$ b. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{\ln(n+1)}}$

Solution

a. Let $f(x) = xe^{-x}$

Evaluate $\int_1^{\infty} xe^{-x}dx$ (use integration by part), then

$$\lim_{b\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big|_1^b = \lim_{b\to\infty}\left(-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right) = 2e^{-1}$$

Thus $\sum_{n=1}^{\infty} ne^{-n}$ converges.

b.

Exercise 2.2.1

Questions 1 - 10 Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)$ 2. $\sum_{n=1}^{\infty}\left(\frac{1}{1+4n^2}\right)$ 3. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+5}}\right)$

4. $\sum_{n=1}^{\infty}\left(\frac{n}{\ln(n+1)}\right)$ 5. $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^2}\right)$ 6. $\sum_{n=1}^{\infty}\left(\frac{e^n}{1+e^{2n}}\right)$

7. $\sum_{n=1}^{\infty}\left(\frac{8\tan^{-1} n}{n^2+1}\right)$ 8. $\sum_{n=2}^{\infty}\left(\frac{1}{n\ln n}\right)$

9. $\sum_{n=3}^{\infty}\left(\frac{1}{n\ln n\sqrt{\ln^2 n-1}}\right)$ 10. $\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)\ln^2(n+1)}\right)$

11. Show that $\sum_{n=2}^{\infty}\left(\frac{1}{n\ \ln^p n}\right)$ converges if $p>1$ and diverges if $p\le 1$.

12. Show that $\sum_{n=3}^{\infty}\left(\frac{1}{n\ln n\ (\ln^p(\ln n))}\right)$ converges if $p>1$ and diverges if $p\le 1$

Answer to ex. 2.2.1

1. Div.	2. Conv.	3. Div.	4. Div.
5. Conv.	6. Conv.	7. Conv.	8. Div.
9. Conv.	10. Conv.		

2.2.2 Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive series $(a_n \geq 0, b_n \geq 0$ for all $n \geq 1)$.

1. If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

2. If $a_n \geq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Example2:

Check if the following series converge?

a. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}+1}$ b. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ c. $\sum_{n=1}^{\infty} \frac{1}{3^n - \cos n}$

Exercise 2.2.2

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{\sin^2 n}{2^n}\right)$
2. $\sum_{n=1}^{\infty}\left(\frac{1-n}{n2^n}\right)$
3. $\sum_{n=1}^{\infty}\left(\frac{1}{3^{n-1}+1}\right)$
4. $\sum_{n=1}^{\infty}\left(\sin\frac{1}{n}\right)$
5. $\sum_{n=1}^{\infty}\left(\frac{1}{1+2+3+...+n}\right)$
6. $\sum_{n=1}^{\infty}\left(\frac{n-1}{n^3+1}\right)$
7. $\sum_{n=2}^{\infty}\left(\frac{\sqrt{n}}{n+1}\right)$
8. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+1}}\right)$
9. $\sum_{n=1}^{\infty}\left(\frac{3+\cos n}{3^n}\right)$
10. $\sum_{n=1}^{\infty}\left(\frac{\sin^2 n}{n\sqrt{n}}\right)$
11. $\sum_{n=1}^{\infty}\left(\frac{2^n}{1+3^n}\right)$
12. $\sum_{n=1}^{\infty}\left(\frac{\arctan n}{n^3}\right)$
13. $\sum_{n=1}^{\infty}\left(\frac{1+2^n}{1+3^n}\right)$
14. $\sum_{n=1}^{\infty}\left(\frac{n}{(n+1)2^n}\right)$
15. $\sum_{n=1}^{\infty}\left(\frac{n+1}{n2^n}\right)$
16. $\sum_{n=1}^{\infty}\left(\frac{n!}{n^2}\right)$
17. $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)$
18. $\sum_{n=1}^{\infty}\left(\frac{1}{n^n}\right)$
19. $\sum_{n=1}^{\infty}\left(\frac{2^n-1}{3^n+5^n}\right)$
20. $\sum_{n=1}^{\infty}\left(\frac{3\sin^2 n}{n!}\right)$

Answers to ex. 2.2.2

1. conv.	2. conv.	3. conv.
4. div.	5. conv.	6. conv.
7. div.	8. conv.	9. conv.
10. conv.	11. conv.	12. conv.
13. conv.	14. conv.	15. conv.
16. div.	17. div.	18. conv.
19. conv.	20. conv.	

2.2.3 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive series ($a_n \geq 0, b_n \geq 0$ for all $n \geq 1$)

1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

3. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = +\infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Example3: Check if the following series converge.

a. $\sum_{n=1}^{\infty} \frac{n}{4n^3 - 2}$ b. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n+1}}$ c. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Exercise 2.2.3

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{2\sqrt{n}+\sqrt[3]{n}}\right)$ 2. $\sum_{n=3}^{\infty}\left(\frac{1}{\ln(\ln n)}\right)$ 3. $\sum_{n=1}^{\infty}\left(\frac{(\ln n)^2}{n^3}\right)$

4. $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n}\ln n}\right)$ 5. $\sum_{n=1}^{\infty}\left(\frac{1}{1+\ln n}\right)$ 6. $\sum_{n=2}^{\infty}\left(\frac{\ln(n+1)}{(n+1)}\right)$

7. $\sum_{n=1}^{\infty}\left(\frac{1}{n\sqrt{n^2-1}}\right)$ 8. $\sum_{n=1}^{\infty}\left(\frac{10n+1}{n(n+1)(n+2)}\right)$ 9. $\sum_{n=1}^{\infty}\left(\frac{\tan^{-1} n}{n^{1.1}}\right)$

10. $\sum_{n=1}^{\infty}\left(\frac{\coth n}{n^2}\right)$ 11. $\sum_{n=1}^{\infty}\left(\frac{1}{n\sqrt[n]{n}}\right)$ 12. $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^3}\right)$

13. $\sum_{n=1}^{\infty}\left(\frac{n!\ln n}{n(n+2)!}\right)$ 14. $\sum_{n=1}^{\infty}\left(\frac{\arctan n}{n}\right)$ 15. $\sum_{n=1}^{\infty}\left(\frac{5n^2+2n}{\sqrt{1+n^5}}\right)$

16. $\sum_{n=1}^{\infty}\left(\frac{2}{3^n-1}\right)$ 17. $\sum_{n=1}^{\infty}\left(\frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}\right)$ 18. $\sum_{n=1}^{\infty}\left(\frac{1}{1+\sqrt{n}}\right)$

19. $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}\ln n}{n^3+1}\right)$ 20. $\sum_{n=1}^{\infty}\left(\frac{n(n+3)}{(n+1)\left(\sqrt{n}+2\right)(3n+7)}\right)$

Answers to ex. 2.2.3

1. div. 2. div. 3. conv. 4. div. 5. div.

6. div. 7. conv. 8. conv. 9. conv. 10. conv.

11. div. 12. conv. 13. conv. 14. div. 15. div.

16. conv. 17. conv. 18. div. 19. conv. 20. div.

2.2.4 Ratio Test

Let $\sum a_n$ be a given series. Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

(1) If $L < 1$, then the series $\sum a_n$ converges

(2) If $L > 1$, then the series $\sum a_n$ diverges

(3) If $L = 1$, then the test fails. We are not able to conclude anything from the test.

Example 4: Check if the following series converge

a. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Exercise 2.2.4

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{n^{\sqrt{2}}}{2^n}\right)$ 2. $\sum_{n=1}^{\infty}\left(n!e^{-n}\right)$ 3. $\sum_{n=1}^{\infty}\left(\frac{n^{10}}{10^n}\right)$

4. $\sum_{n=1}^{\infty}\left(\frac{\ln n}{e^n}\right)$ 5. $\sum_{n=1}^{\infty}\left(\frac{(n+1)(n+2)}{n!}\right)$ 6. $\sum_{n=1}^{\infty}\left(\frac{n!}{(2n+1)!}\right)$

7. $\sum_{n=1}^{\infty}\left(\frac{2^n n! n!}{(2n)!}\right)$ 8. $\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot\ \ldots\ \cdot(2n-1)}{4^n 2^n n!}\right)$ 9. $\sum_{n=1}^{\infty}\left(\frac{5^n+n}{n!+3}\right)$

10. $\sum_{n=1}^{\infty}\left(\frac{3}{2+n\cdot 5^n}\right)$ 11. $\sum_{n=1}^{\infty}\left(\frac{(2n)!}{n!(2n)^n}\right)$ 12. $\sum_{n=1}^{\infty}\left(\frac{(2n)!}{3^n}\right)$

Answers to ex. 2.2.4

1. conv. 2. div. 3. conv. 4. conv.

5. conv. 6. conv. 7. conv. 8. conv.

9. conv. 10. conv. 11. conv. 12. div.

2.2.5 The n^{th}-Root Test

Let $\sum a_n$ be a given series. Consider $\lim_{n\to\infty} \sqrt[n]{|a_n|} = R$.

(1) If $R < 1$, then the series $\sum a_n$ converges

(2) If $R > 1$, then the series $\sum a_n$ diverges

(3) If $R = 1$, then the test fails. We are not able to conclude anything from the test.

Example 5: $\sum_{n=1}^{\infty} \frac{1}{[\ln(n+1)]^n}$ converges or diverges?

Exercise 2.2.5

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$
2. $\sum_{n=1}^{\infty}\left(\frac{2+(-1)^{n}}{(1.25)^{n}}\right)$
3. $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^{n}$
4. $\sum_{n=1}^{\infty}\left(\frac{n}{(\ln n)^{n}}\right)$
5. $\sum_{n=1}^{\infty}\left(\frac{(n!)^{n}}{(n^{n})^{2}}\right)$
6. $\sum_{n=1}^{\infty}\left(\frac{n^{n}}{2^{n^{2}}}\right)$
7. $\sum_{n=1}^{\infty}\left(1-e^{-n}\right)^{n}$
8. $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^{n^{2}}$
9. $\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}\right)^{n}$
10. $\sum_{n=1}^{\infty}\left(\frac{5n+2}{3n-1}\right)^{n}$

Answers to exercise 2.2.5

1. conv.	2. conv.	3. div.	4. conv.
5. div.	6. conv.	7. div.	8. conv.
9. conv.	10. div.		

2.3 Alternating Series

The alternating series are the series of the following form:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1}a_n + \ldots \quad \text{, or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \ldots + (-1)^{n}a_n + \ldots$$

where a_n is a positive sequence ($a_n > 0$ for all n).

The following theorem states about the convergence of these alternating series.

Theorem 2.3.1 (alternating series test)

If an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ (or $\sum_{n=1}^{\infty}(-1)^{n}a_n$) satisfies the following two properties:

(1) $0 < a_{n+1} < a_n$ for all $n > 1$ (positive decreasing seq.)

(2) $\lim_{n\to\infty} a_n = 0$,

then the alternating series converges.

<u>**Remark**</u>: If $\lim_{n\to\infty} a_n \neq 0$, then the above alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $\sum_{n=1}^{\infty}(-1)^{n}a_n$ diverge.

Example 6: Determine if the following alternating series converge or diverge?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n^2}$

Exercise 2.3

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{3n+1}$
3. $\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n}$
4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$
5. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$
6. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$
7. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{\ln n^2}$
8. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$
9. $\sum_{n=2}^{\infty} (-1)^{n} \frac{\ln n}{n}$
10. $\sum_{n=2}^{\infty} (-1)^{n+1} e^{-n}$

Answers to exercise 2.3

1. conv.
2. div.
3. conv.
4. conv.
5. div.
6. conv.
7. div.
8. conv.
9. conv.
10. conv.

2.4 Absolute and Conditional Convergence

Definition 2.4.1 $\sum_{n=1}^{\infty} a_n$ is an *absolutely convergent* series if $\sum_{n=1}^{\infty} |a_n|$ converges.

Definition 2.4.2 $\sum_{n=1}^{\infty} a_n$ is a *conditionally convergent* series if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 2.4.3 If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges and $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$.

Example7: Check if $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}$ converges.

<u>Solution</u>: $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2} = \frac{1}{2} - \frac{1}{8} - \frac{1}{9} - \frac{1}{32} + \frac{1}{50} + \frac{1}{36} + \frac{1}{98} - \ldots$

Consider $\sum_{n=1}^{\infty} \left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}\right|$. Since $\left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}\right| \le \frac{1}{n^2}$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series as $p = 2 > 1$.

By comparison test, $\sum_{n=1}^{\infty}\left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}\right|$ also converges.

By Theorem 1.8.3 $\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}$ converges.

Theorem 2.4.4: Ratio Test

Let $\sum a_n$ be an infinite series. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then

(1) If $L<1$, then the series **converges absolutely**.

(2) If $L>1$, then the series diverges.

(3) If $L=1$, the test fails.

Exercises:

a. $\sum_{n=1}^{\infty}\frac{(-5)^n}{n!}$ b. $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$ c. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$

Solution

a. $a_n = \frac{(-5)^n}{n!}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n}$

Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5}{n+1} = 0$

Therefore, $\sum_{n=1}^{\infty}\frac{(-5)^n}{n!}$ converges absolutely.

Exercise 2.4

Determine if the following series absolutely converge or conditionally converge or diverge.

1. $\sum_{n=1}^{\infty}\left(-\frac{3}{5}\right)^n$
2. $\sum_{n=1}^{\infty}(-1)^n \frac{3}{n^2}$
3. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n^3}{e^n}\right)$
4. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$
5. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+2}{3n-1}\right)^n$
6. $\sum_{n=1}^{\infty}(-1)^n \frac{n+2}{n(n+3)}$
7. $\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}$
8. $\sum_{n=2}^{\infty}\left(-\frac{3}{\ln n}\right)^n$
9. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+1}{n^3+2}$
10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\tan^{-1} n}{n^2+2}$
11. $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$
12. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$
13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)^n}{(2n)^n}$
14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!}{2^n n! n}$
15. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$
16. $\sum_{n=1}^{\infty}\frac{k\cos k\pi}{k^2+1}$
17. $\sum_{n=3}^{\infty}\frac{(-1)^n \ln n}{n}$
18. $\sum_{n=2}^{\infty}\left(-\frac{1}{\ln n}\right)^n$
19. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$
20. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

Answers to exercise 2.4

1. abs. conv.	2. abs. conv.	3. abs. conv.	4. cond. conv.
5. abs. conv.	6. cond. conv.	7. div.	8. abs. conv.
9. cond. conv.	10. abs. conv.	11. abs. conv.	12. abs. conv.
13. abs. conv.	14. div.	15. abs. conv.	16. div.
17. cond. conv.	18. abs. conv.	19. cond. conv.	20. div.
