Vectors

In physics and engineering applications we use two kinds of quantities, scalars and vectors .

A scalar is a quantity having magnitude but no direction, for example mass, length, temperature and any real number.

A vector is a quantity having both ***magnitude and direction***, for instance force, velocity and acceleration etc.

A vector denotes the line having the arrow as shown in Figure 1.

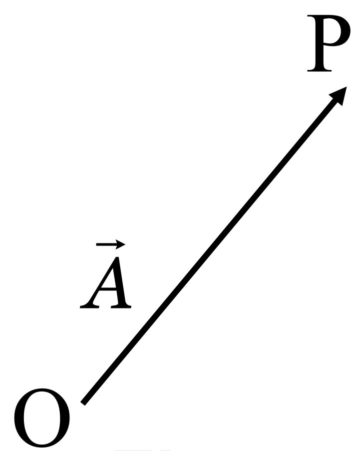


Figure 1

In Figure 1. The tail end *O* of the arrow is called the *origin* or *initial point* and the head *P* is called the *terminal point or terminus*.

Notation

A vector is represented by a letter with an arrow over it, as $\vec{A}, \vec{B}, \vec{a}, \vec{b}, \vec{u}, \vec{v}, etc,$

- A vector whose initial point is A and whose terminal point is B is written as $\overrightarrow{AB}$
- The magnitude of a vector is written as $\| \vec{A} \|$ or $\| \overrightarrow{AB} \|$

Vector Algebra

1. Two vectors $\vec{A}$ and $\vec{B}$ are equal if they have the same magnitude and direction. Thus in Figure 2,we have

$$\vec{A} = \vec{B}.$$

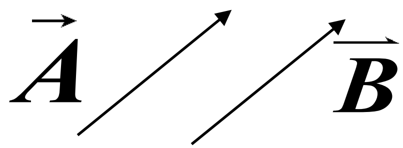


Figure 2

2. Zero vector, denoted by $\vec{0}$, has zero magnitude and no specific direction.

3. The negative of a vector $\vec{A}$ written $-\vec{A}$ is a vector that has the same magnitude as $\vec{A}$ but is opposite in direction as show in Figure 3.

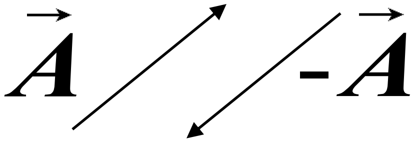


Figure 3

Vector Algebra

4. **Vectors are said to be** *free*, **which means that a vector can be moved from one position to another provided its** ***magnitude and direction are not changed*** **as in Figure 4.**

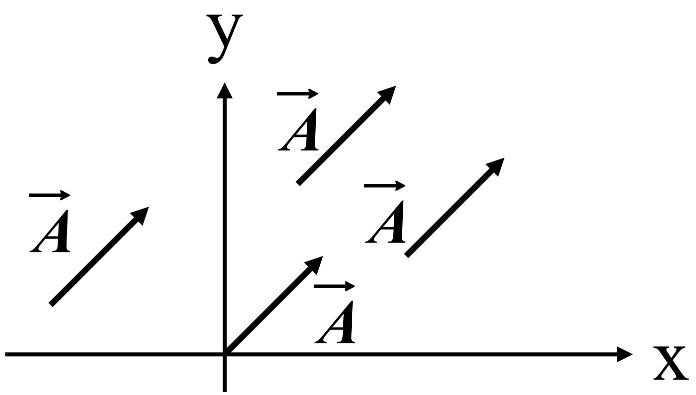


Figure 4

Vector Algebra

5. Two vectors $\vec{A}$ and $\vec{B}$ are *parallel* if and only if they are nonzero scalar, *k*, multiples of each other, that is

$$\boxed{\vec{A} = k\vec{B}}$$

5.1 If $k > 0$, then $k\vec{A}$ has the same direction as $\vec{A}$, see Figure 5(a).

Vector Algebra

5.2 **If $k < 0$, then $k\vec{A}$ has the direction opposite that of $\vec{A}$, see Figure 5(b).**

5.3 If $k = 0$, then $k\vec{A} = \vec{0}$ is a zero vector, see Figure 5(c).

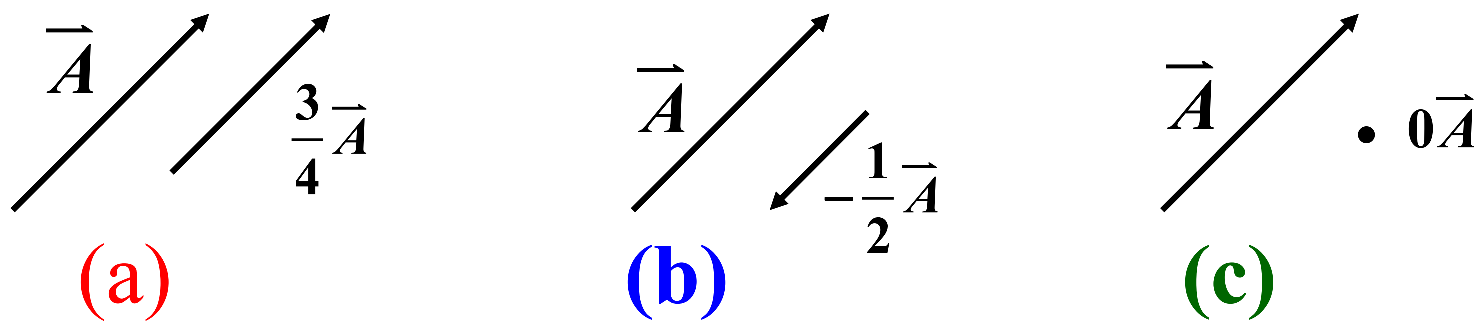
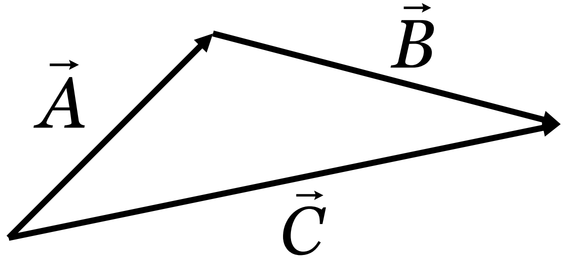


Figure 5

Vector Algebra

6. The sum of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ formed by placing the initial point of $\vec{B}$ on the terminal point of $\vec{A}$ and then joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$.



$$\boxed{\vec{C} = \vec{A} + \vec{B}}$$

Figure 6

Note that vecter $\vec{C}$ has the initial point at the initial point of $\vec{A}$ and the terminal point at the terminal point of $\vec{B}$

7. The difference of two vectors $\vec{A}$ and $\vec{B}$ is denoted by

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

See Figure 7.

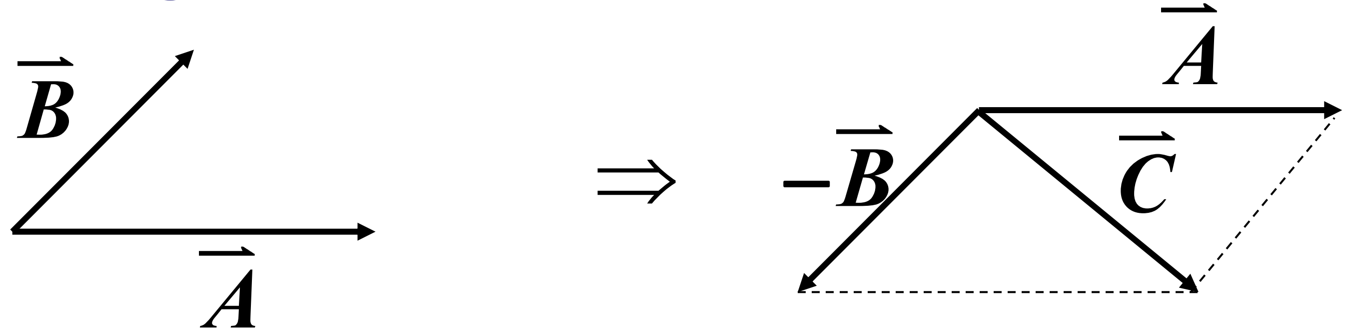


Figure 7

Vector in a Coordinate Plane

In two dimension or 2 – space

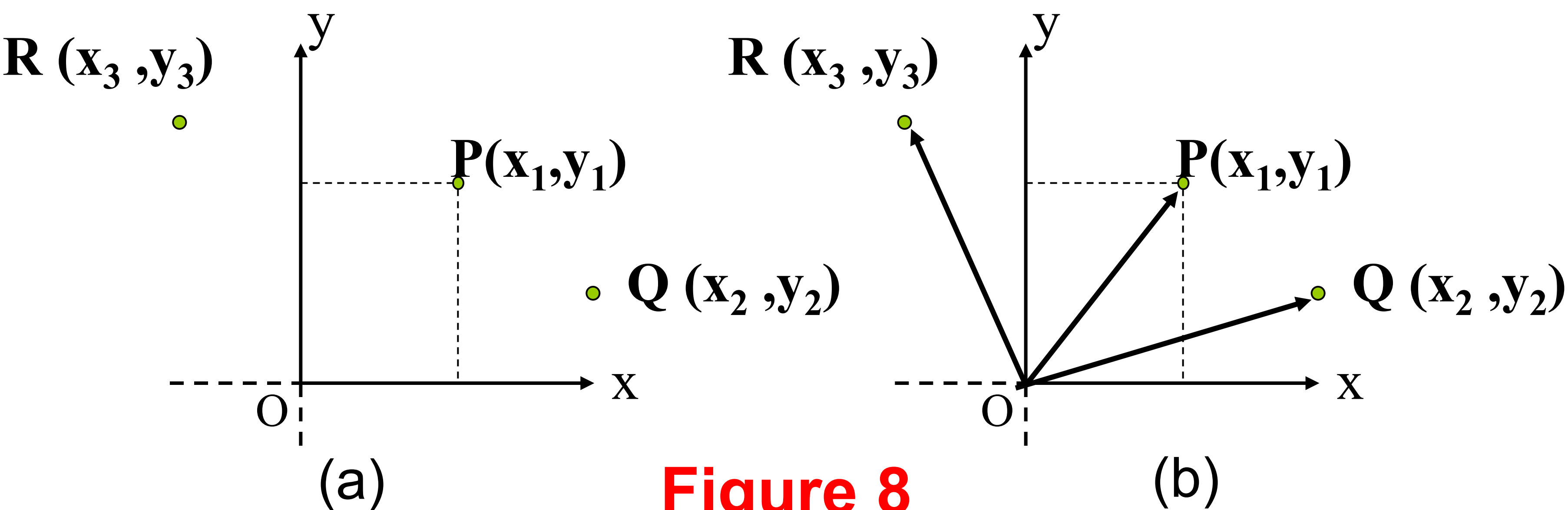


Figure 8

- **The set of all vectors in the plane is denoted by R^2, see Figure 8(b).**

- Every point in the plane (see Figure 8(a)) can be represented by the vectors and it is called the position vector of the point P,Q and R.

- The ***position vector*** of the point P is written by

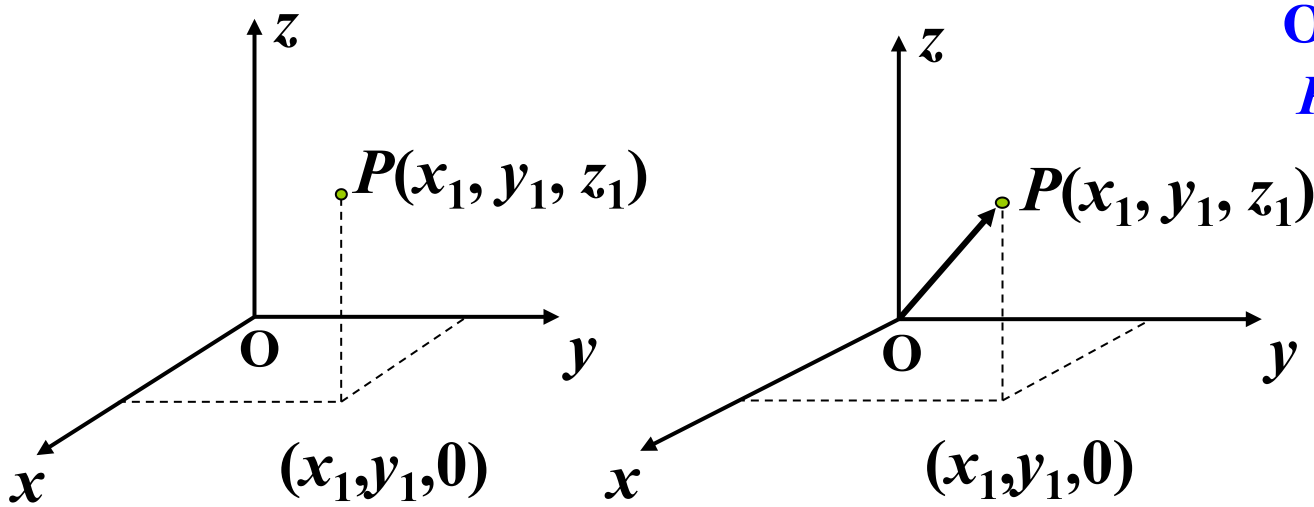
$$\overrightarrow{OP} = \langle x_1, y_1 \rangle$$

The numbers x_1, y_1 are called the ***components*** of the vector $\overrightarrow{OP} = \vec{P}$.

Example **Sketch the position vectors for**

$$\vec{a} = \langle -3, 2 \rangle, \quad \vec{\mathrm{b}} = \langle 0, -2 \rangle$$

In three dimension or 3-space



O is initial point
P is terminal point

Figure 9

- **The set of all vectors in 3-space is denoted by R^3.**
- **The *position vector* of the point P is written by** $\overrightarrow{OP} = \langle x_1, y_1, z_1 \rangle$

The numbers x_1, y_1, z_1 are called the ***components*** of the vector $\overrightarrow{OP} = \vec{P}$.

Example **Sketch the position vectors for**

$$\vec{A} = \langle -3, 2, 2 \rangle, \quad \vec{B} = \langle 0, -2, 3 \rangle$$

Vectors with initial point not at the origin

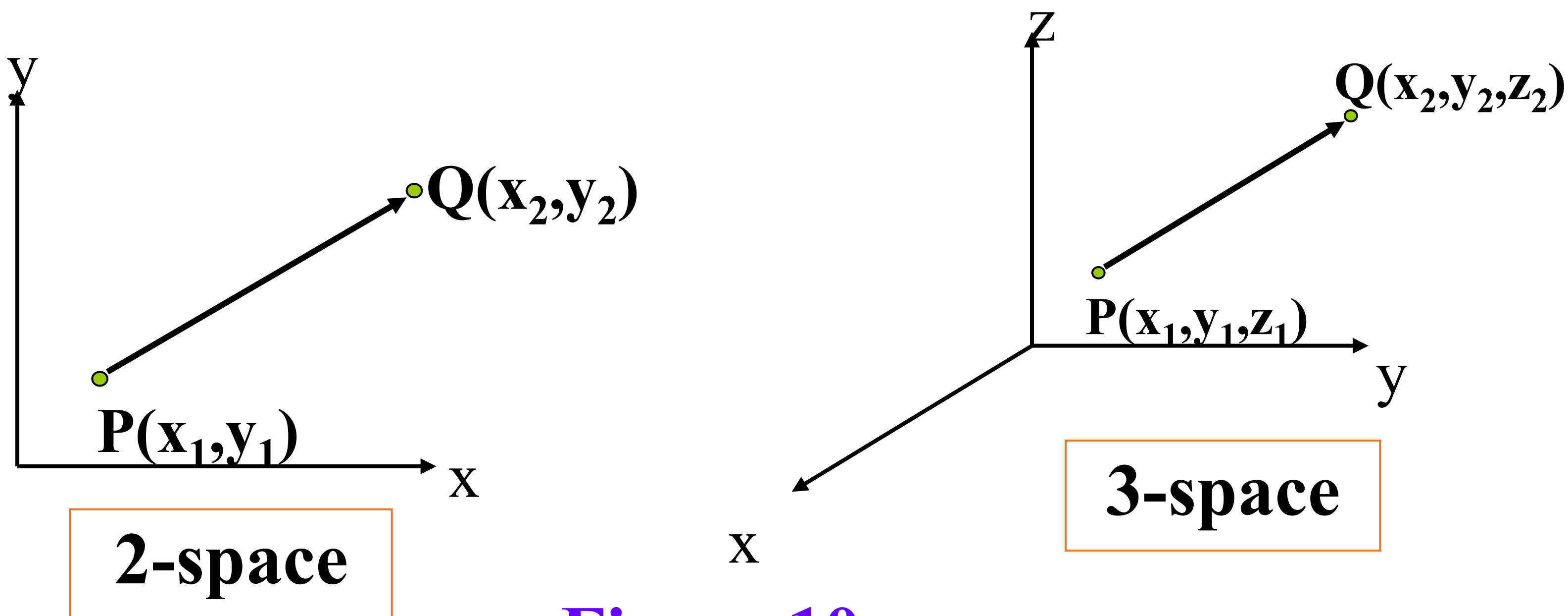


Figure 10

The vector from *P* to *Q* is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle \qquad \text{(2-space)}$$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \qquad \text{(3-space)}$$

Example Given the points P(-2, 3) and Q(4, 5), find vectors $\vec{a}$ and $\vec{b}$ in R^2 that correspond to $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

Example Given the points P(-2, 3,4) and Q(4, 5,-2), find vectors $\vec{a}$ and $\vec{b}$ in R^3 that correspond to $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

Definition

Let $\vec{A} = \langle a_1, a_2 \rangle$ and $\vec{B} = \langle b_1, b_2 \rangle$ be vector in R^2.

1) Addition: $\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$

2) Scalar multiplication: $k\vec{A} = \langle ka_1, ka_2 \rangle$

3) Equality: $\vec{A} = \vec{B} \Leftrightarrow a_1 = b_1 \text{ and } a_2 = b_2$

4) Zero vector: $\vec{0} = \langle 0, 0 \rangle$

Example If $\vec{a} = \langle 1, 4 \rangle$ and $\vec{b} = \langle -6, 3 \rangle$, find

(a) $\vec{a} + \vec{b}$ (b) $\vec{a} - \vec{b}$ and (c) $2\vec{a} + 3\vec{b}$

Solution

Definition

Let $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$ be vectors in $\mathbb{R}^3$.

(i) Addition: $\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(ii) Scalar multiplication: $k\vec{A} = \langle ka_1, ka_2, ka_3 \rangle$

(iii) Equality: $\vec{A} = \vec{B} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$

(iv) Zero vector: $\vec{0} = \langle 0, 0, 0 \rangle$

Magnitude of a Vector

Definition The *magnitude, length* or *norm* of a vector $\vec{A}$ is the distance between its initial and terminal points and is denote by $\|\vec{A}\|$.

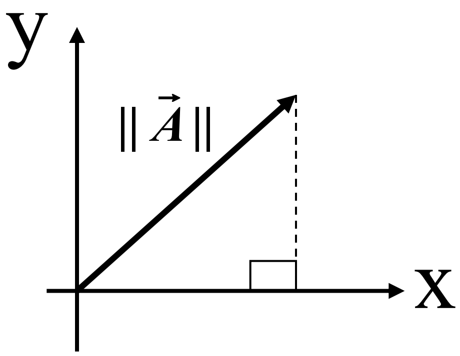


Figure 10(a)

If $\vec{A} = \langle a_1, a_2 \rangle$,then the length of $\vec{A}$ is given by

$$\boxed{\|\vec{A}\| = \sqrt{a_1^2 + a_2^2}}$$

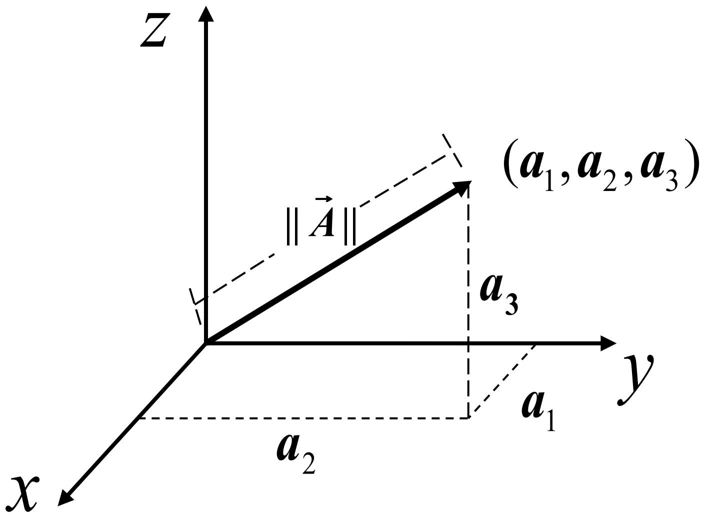


Figure 10(b)

The length of a vector

$\vec{A} = \langle a_1, a_2, a_3 \rangle$ **is given by**

$$\boxed{\|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Note that

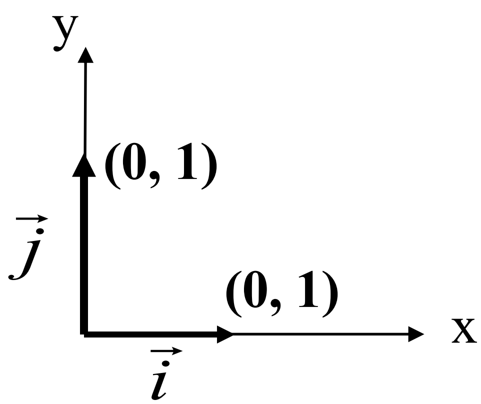
1. If $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, then

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. $\|k\vec{v}\| \equiv |k| \ \|\vec{v}\|$

The $\vec{i}, \vec{j}, \vec{k}$ vectors

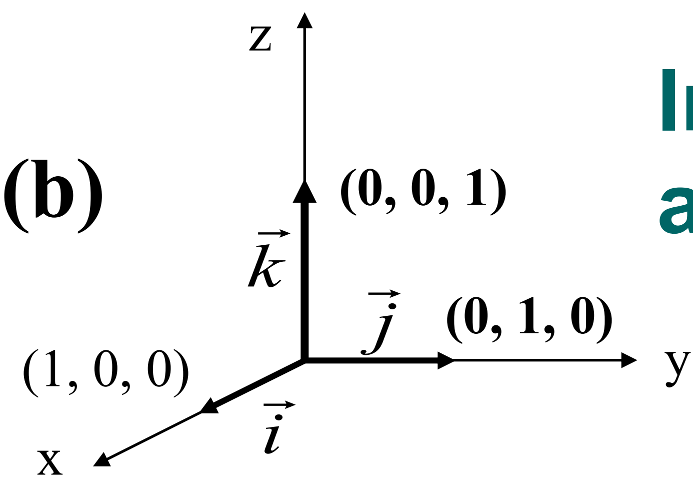
(a)



In 2-space, the unit vectors along the x and y- axis are

$$\vec{i} = \langle 1,0 \rangle, \ \vec{j} = \langle 0,1 \rangle$$

(b)



In 3-space, the unit vectors along the x, y and z- axis are

$$\vec{i} = \langle 1,0,0 \rangle, \ \vec{j} = \langle 0,1,0 \rangle$$

$$\vec{k} = \langle 0,0,1 \rangle$$

Remark The vector $\vec{i},\vec{j},\vec{k}$ are called the standard unit vector since $\|\vec{i}\|=\|\vec{j}\|=\|\vec{k}\|=1$

(2) The unit vector $\vec{i}=\langle 1,0\rangle$ and $\vec{j}=\langle 0,1\rangle$ are a basis for the system of vectors in 2-space.

Similarly the unite vectors $\vec{i}=\langle 1,0,0\rangle$, $\vec{j}=\langle 0,1,0\rangle$ and $\vec{k}=\langle 0,0,1\rangle$ are a basis for the system of vector in 3-space.

(3) Every vectors in 2-space or 3-space can be written as a linear combination of the standard unit vectors. For example,

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$= v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle$$

$$= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

Example

$$\langle 2,-3,4 \rangle = 2\vec{i} - 3\vec{j} + 4\vec{k}$$

$$\langle 0,3,0 \rangle = 3\vec{j}$$

Unit vector

A unit vector is a vector having *unit magnitude*. If $\vec{\mathbf{A}}$ is a vector with magnitude $\|\vec{\mathbf{A}}\|$, then $\vec{\mathbf{A}}/\|\vec{\mathbf{A}}\|$ is a unit vector having the same direction as $\vec{\mathbf{A}}$.

Example **Find the unit vector in the same direction of $3\vec{w}-\vec{v}$ if**

$$\vec{w}=2\vec{i}-\vec{j}+2\vec{k} \quad \text{and} \quad \vec{v}=3\vec{i}+4\vec{j}-5\vec{k}$$

Solution $3\vec{w}-\vec{v}=3\vec{i}-7\vec{j}+11\vec{k}$

$$\|3\vec{w}-\vec{v}\|=\sqrt{(3)^2-(7)^2+(11)^2}=\sqrt{179}$$

Hence, the unit vector in the same direction of $3\vec{w}-\vec{v}$ is

$$\vec{U}=\frac{1}{\sqrt{179}}\left(3\vec{i}-7\vec{j}+11\vec{k}\right)$$

Basic Properties

For any vectors $\vec{u},\vec{v}$ and $\vec{w}$ and scalar k the following relationships hold:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
3. $\vec{u}+\vec{0}=\vec{0}+\vec{u}$
4. $\vec{u}+(-\vec{u})=\vec{0}$
5. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
6. $1\vec{u}=\vec{u}$

The Dot Product or Inner Product

The dot product, inner product, or scalar product, yields *a* scalar.

Definition (Dot Product of Two Vectors)

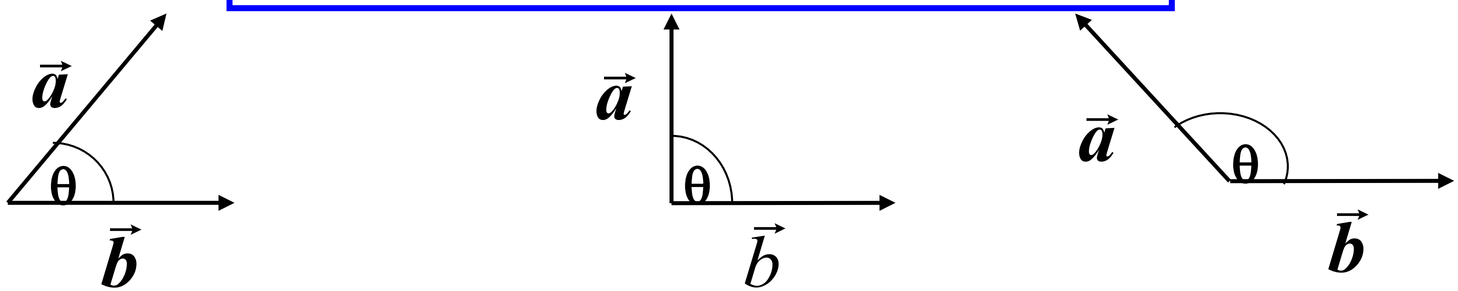
The dot product of two vectors $\vec{a}$ and $\vec{b}$ is a scalar

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\| \ \|\vec{b}\| \cos\theta} \qquad \ldots (1)$$

where $0 \le \theta \le \pi$ is the angle between $\vec{a}$ and $\vec{b}$

From (1) we obtain the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\ \ \|\vec{b}\|}\right)} \quad \ldots\ (2)$$



$\vec{a}\cdot\vec{b} > 0$	$\vec{a}\cdot\vec{b} = 0$	$\vec{a}\cdot\vec{b} < 0$
θ is acute	θ is π/2	θ is obtuse

Figure11

Theorem If $\vec{a}$ and $\vec{b}$ are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then

$$\theta \text{ is acute} \Leftrightarrow \vec{a}\cdot\vec{b} > 0$$

$$\theta \text{ is obtuse} \Leftrightarrow \vec{a}\cdot\vec{b} < 0$$

$$\theta = \frac{\pi}{2} \Leftrightarrow \vec{a}\cdot\vec{b} = 0$$

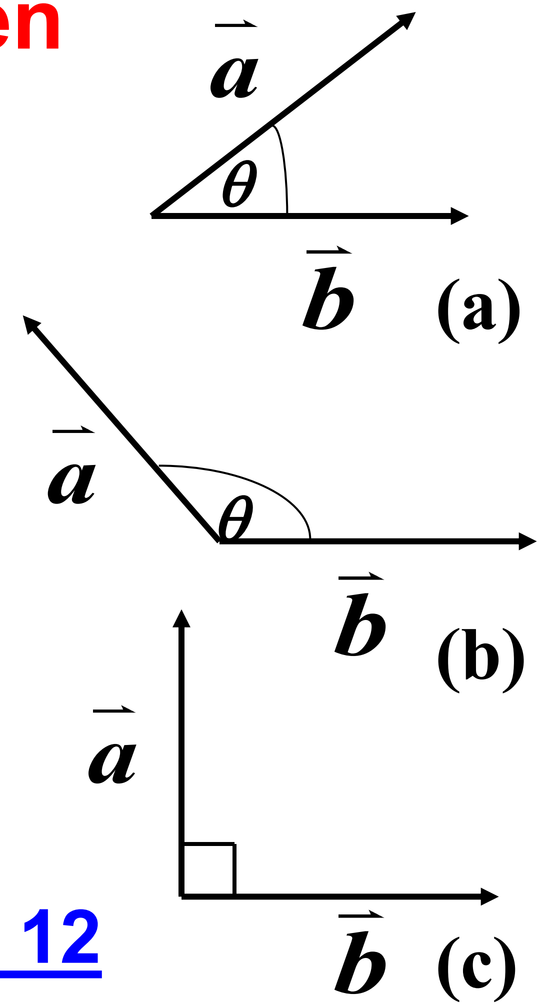


Figure 12

Orthogonal Vectors

Nonzero vectors $\vec{a}$ and $\vec{b}$ are

orthogonal if and only if

$$\boxed{\vec{a} \cdot \vec{b} = 0} \qquad \dots (3)$$

Note that

- The zero vector is orthogonal to every vectors.

- Length in terms of inner product

$$\boxed{\vec{a} \cdot \vec{a} = \|\vec{a}\|^2 \text{ or } \|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}} \qquad \dots (4)$$

Component Form of the Dot Product

If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then component form of the dot product is given by

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3} \quad \ldots (5)$$

Note that **Equation(5) shows that the dot product of two vector is the sum of the products of their corresponding components.**

Example If $\vec{a} = \langle 1, -2, 3 \rangle$, $\vec{b} = \langle 1, -2, 3 \rangle$ and $\vec{c} = \langle 1, -2, 3 \rangle$ then

$$\vec{a} \cdot \vec{b} = 1(-3) + (-2)(4) + 3(2) = -5$$

$$\vec{b} \cdot \vec{c} = (-3)(3) + (4)(6) + 2(3) = 21$$

$$\vec{a} \cdot \vec{c} = 1(3) + (-2)(6) + 3(3) = 0$$

It is found that

$$\vec{a}\cdot\vec{b} = -5 < 0$$

$\vec{a}$ and $\vec{b}$ make an obtuse angle,

$$\vec{a}\cdot\vec{c} = 21 > 0$$

$\vec{b}$ and $\vec{c}$ make an acute angle,

$$\vec{a}\cdot\vec{c} = 0$$

$\vec{a}$ and $\vec{c}$ are perpendicular.

Example Find the inner product and the length of $\vec{a} = \langle 1,2,0 \rangle$ and $\vec{b} = \langle 3,-2,1 \rangle$ as well as the angle between these vectors.

Solution $\vec{a} \cdot \vec{b} = -1$

$$\|\vec{a}\| = \sqrt{(1)^2 + (2)^2 + (0)^2} = \sqrt{5}$$

$$\|\vec{b}\| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$$

Hence the angle between $\vec{a}$ and $\vec{b}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \; \|\vec{b}\|}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{70}}\right)$$

Properties of Dot Product

For any vectors $\vec{a}, \vec{b}, \vec{c}$ and k is a scalar.

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (Commutative law)

2. $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ (Distributive law)

3. $\vec{a} \cdot \vec{a} \geq 0$ and $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

4. $k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$

Properties of Dot Product

5. $\left|\vec{a}\cdot\vec{b}\right| \leq \|\vec{a}\|\ \|\vec{b}\|$ (Schwarz inequality)

6. For $\vec{i},\vec{j}$ and $\vec{k}$ are unit vectors, we have

$$\|\vec{i}\|^2 = \vec{i}\cdot\vec{i} = 1,$$

$$\|\vec{j}\|^2 = \vec{j}\cdot\vec{j} = 1,$$

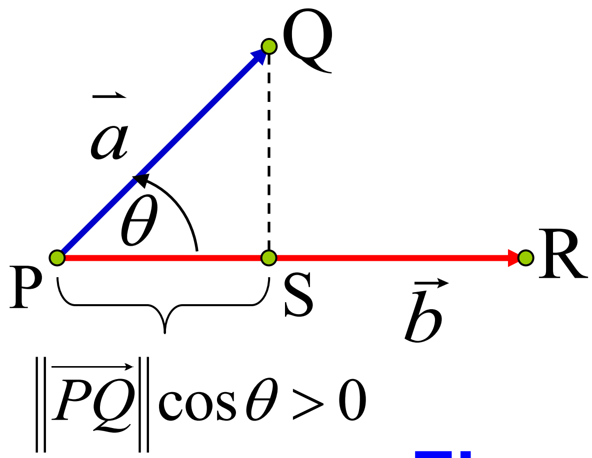
$$\|\vec{k}\|^2 = \vec{k}\cdot\vec{k} = 1,$$

and $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

Definition The scalar component of $\vec{a}$ along $\vec{b}$ equals the dot product of $\vec{a}$ with the unit vector that has the same direction as $\vec{b}$, denoted by

$$\boxed{\operatorname{comp}_{\vec{b}}\vec{a} = \vec{a} \cdot \frac{\vec{b}}{\|\vec{b}\|}}$$

Remark the component of $\vec{a}$ along $\vec{b}$ means that finding the component of $\vec{a}$ in direction of $\vec{b}$



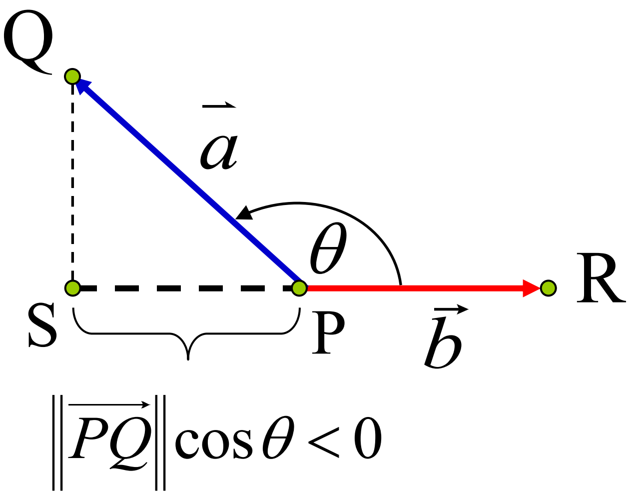
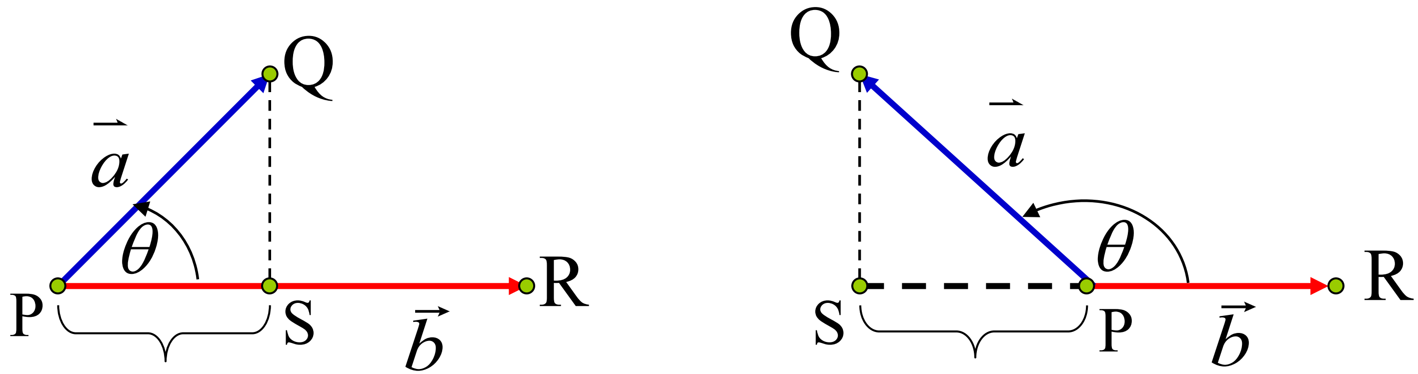


Figure 14

Let $\vec{a} = \overrightarrow{PQ}$ **and** $\vec{b} = \overrightarrow{PR}$**, then**

$$\text{comp}_{\vec{b}}\, \vec{a} = \vec{a} \cdot \frac{\vec{b}}{\left\|\vec{b}\right\|} = \frac{1}{\left\|\vec{b}\right\|}(\vec{a} \cdot \vec{b})$$

$$= \frac{\left\|\vec{a}\right\| \left\|\vec{b}\right\|}{\left\|\vec{b}\right\|}\cos\theta = \left\|\vec{a}\right\|\cos\theta$$



$\left\|\overrightarrow{PQ}\right\|\cos\theta > 0$ $\qquad$ $\left\|\overrightarrow{PQ}\right\|\cos\theta < 0$

Figure 14

Note that

- **The scalar** $\left\|\overrightarrow{PQ}\right\|\cos\theta$ **is positive if** $0 \le \theta < \pi/2$,
- **The scalar** $\left\|\overrightarrow{PQ}\right\|\cos\theta$ **is negative if** $\pi/2 < \theta \le \pi$,
- **The scalar** $\left\|\overrightarrow{PQ}\right\|\cos\theta$ **is zero if** $\theta = \pi/2$.

Example **Find the scalar component of the vector $\vec{F} = \langle 2, -1, 3 \rangle$ in the direction of the following vectors:**

a) $\vec{i}$,

b) $\vec{a} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$, **and**

c) $\vec{b} = \langle 4, 2, -1 \rangle$.

Solution

a) Since $\vec{i} = \langle 1,0,0 \rangle$ is unit vector so the scalar component of $\vec{F}$ in the direction of $\vec{i}$ is

$$\mathrm{comp}_{\vec{i}}\vec{F} = \vec{F} \cdot \vec{i} = \left\langle 2,-1,3 \right\rangle \cdot \left\langle 1,0,0 \right\rangle = 2$$

Solution

b) Since $\|\vec{a}\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1$, then the vector $\vec{a}$ is a unit vector. Thus the scalar component of $\vec{F}$ in the direction of $\vec{a}$ is

$$\mathrm{comp}_{\vec{a}}\vec{F} = \vec{F}\cdot\vec{a}$$

$$= \langle 2,-1,3\rangle . \left\langle \frac{1}{3},\frac{2}{3},\frac{2}{3} \right\rangle = 2$$

Solution

c) Since $\left\|\vec{b}\right\| = \sqrt{21}$ indicate that $\vec{b}$ is not unit vector then the component of $\vec{F}$ in the direction of $\vec{b} = \langle 4,2,\text{-}1 \rangle$ is

$$\mathrm{comp}_{\vec{b}}\vec{F} = \vec{F} \cdot \frac{\vec{b}}{\|\vec{b}\|}$$

$$= \langle 2,-1,3 \rangle \cdot \frac{\langle 4,2,-1 \rangle}{\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}} \langle 8-2-3 \rangle = \frac{3}{\sqrt{21}}$$

Example If $\vec{a} = 4\vec{i} - \vec{j} + 5\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} - 2\vec{k}$, find $\text{comp}_{\vec{a}}\, \vec{b}$ and $\text{comp}_{\vec{a}+2\vec{b}}\, \vec{a}$.

solution $\left\|\vec{a}\right\| = \sqrt{16+1+25} = \sqrt{42}$

Thus, $\text{comp}_{\vec{a}}\vec{b} = \vec{b} \cdot \dfrac{\vec{a}}{\left\|\vec{a}\right\|}$

$$= \frac{1}{\sqrt{42}}(24-3-10)$$

$$= \frac{11}{\sqrt{42}}$$

$$\vec{a} + 2\vec{b} = 16\vec{i} + 5\vec{j} + \vec{k}$$

$$\left\|\vec{a} + 2\vec{b}\right\| = \sqrt{256 + 25 + 1} = \sqrt{282}$$

Thus $\quad \text{comp}_{\vec{a}+2\vec{b}}\, \vec{a} = \vec{a} \cdot \dfrac{(\vec{a} + 2\vec{b})}{\left\|\vec{a} + 2\vec{b}\right\|}$

$$= \frac{1}{\sqrt{282}}(64 - 5 + 5)$$

$$= \frac{64}{\sqrt{282}} \approx 3.81$$

Projection Vectors

The projection vector of $\vec{a} = \overrightarrow{PQ}$ along a nonzero vector $\vec{b} = \overrightarrow{PR}$ is the vector determine by dropping a perpendicular from Q to the line PS. The notation for this vector is $\mathrm{proj}_{\vec{b}}\vec{a}$

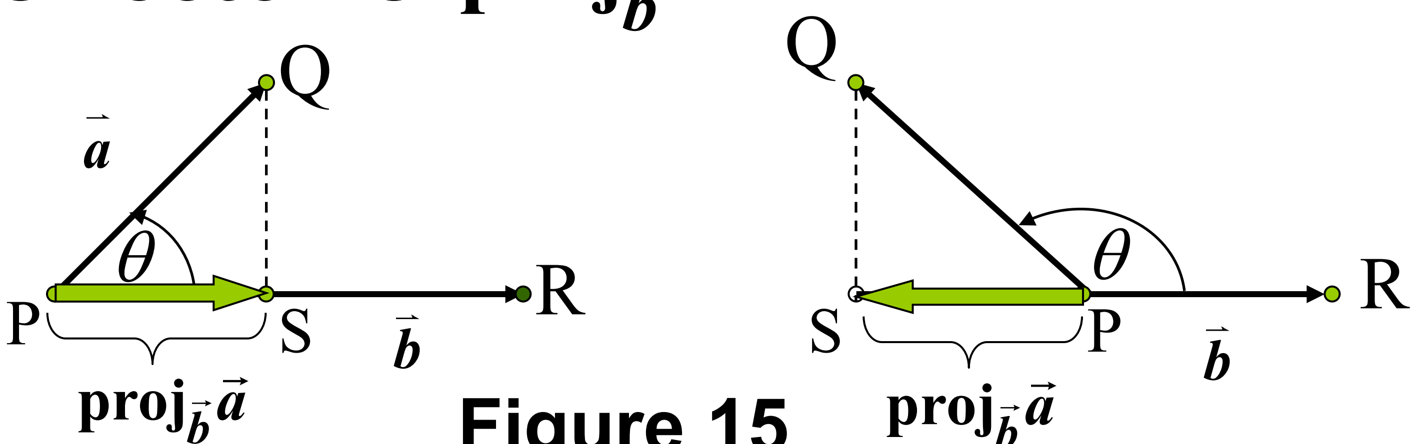


Figure 15

Theorem If $\vec{a}$ and $\vec{b}$ are vector in 2-space or 3-space and if $\vec{a} \neq \vec{0}$ then the ***projection vector*** *of* $\vec{a}$ *along* $\vec{b}$ is a vector

$$\mathrm{proj}_{\vec{b}}\vec{a} = \left(\mathrm{comp}_{\vec{b}}\vec{a}\right)\frac{\vec{b}}{\left\|\vec{b}\right\|}$$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{\left\|\vec{b}\right\|^2}\right)\vec{b}$$

Example Let $\vec{u} = \langle 2, -1, 3 \rangle$ and $\vec{a} = \langle 4, -1, 2 \rangle$

Find the projection vector of $\vec{u}$ along $\vec{a}$.

Solution

$$\vec{u} \cdot \vec{a} = 2(4) + (-1)(-1) + 3(2) = 15$$

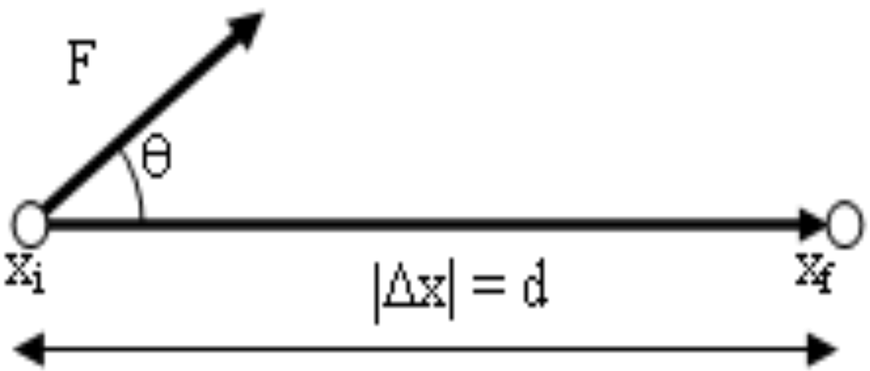
and $$\|\vec{a}\| = \sqrt{4^2 + (-1)^2 + 2^2} = \sqrt{21}$$

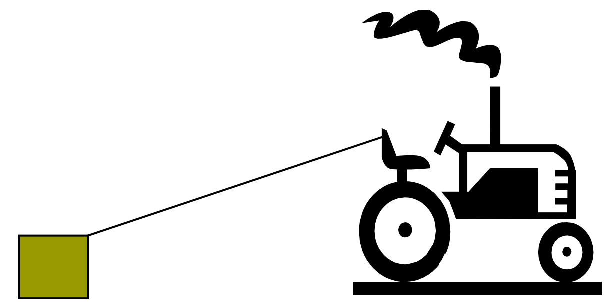
The projection vector of $\vec{u}$ along $\vec{a}$ is

$$\mathrm{proj}_{\vec{a}}\vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{15}{21}\langle 4, -1, 2 \rangle = \left\langle \frac{20}{7}, \frac{-5}{7}, \frac{10}{7} \right\rangle$$

Application: Work done by a Force

Consider an object moving while constant force is applied. The object moves along some axis through a displacement of magnitude d





Work done by a force is $W = \text{comp}\vec{F}\ d$

where $comp\vec{F}$ is the component of $\vec{F}$ in the direction of displacement.

If the force is measured in Newtons, distance is measured in metres, then work is measured in Newton-metres = Joules

Example **Calculate the work done by a 10 N force $\vec{F}$ in moving a particle from *A*(2,1) to *B*(8,5), when the force acts at an angle of 45 degrees to $\overrightarrow{AB}$ and the distance is measured in meters.**

Vector Product or Cross Product

Definition (Cross Product)

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = (\|\vec{a}\| \ \|\vec{b}\| \sin\theta)\vec{n}$$

where θ is the angle between the vectors such that $0 \le \theta \le \pi$ and $\vec{n}$ is *a unit vector* perpendicular to the plane of $\vec{a}$ and $\vec{b}$ with direction given by the right-hand rule.

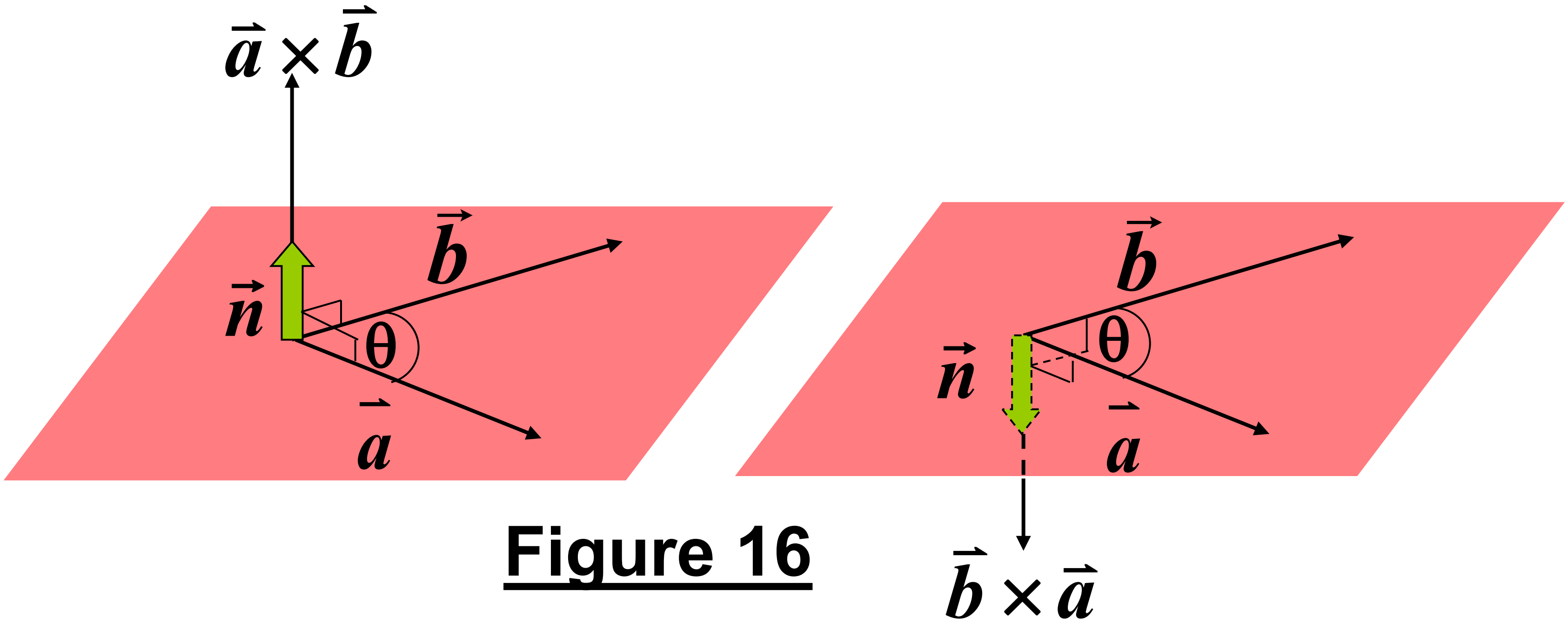


Figure 16

Note that

- $\vec{a} \times \vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.
- $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and
- $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$.

Alternative Definition of the Cross Product

Definition

If $\vec{a} = \langle a_1, a_2, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are vector in 3-space, then the cross Product $\vec{a} \times \vec{b}$ is the vector defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example Let $\vec{a} = \langle 4, 0, -1 \rangle$ and $\vec{b} = \langle -2, 1, 3 \rangle$

Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix} = \vec{i}(0+1) - \vec{j}(12-2) + \vec{k}(4-0)$$

$$= \vec{i} - (10)\vec{j} + 4\vec{k}$$

and $\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$

$$= -\vec{i} + 10\vec{j} + 4\vec{k}$$

Remarks

- **The cross product is defined *only for vector in 3-space*, whereas the dot product is defined for vector in 2-space or 3-space.**
- **The cross product of two vector is *a vector*, whereas the dot product of two vector is *a scalar*.**

Algebraic Properties of Cross Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in 3-space, then

1. $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

2. $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

3. $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

4. $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$

   for every scalar k.

5. $\vec{a} \times \vec{0} = \vec{0} \times \vec{a} = \vec{0}$

6. $\vec{a} \times \vec{a} = \vec{0}$

7. $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Example From the property (6), we have $\vec{i}\times\vec{i}=\vec{0}$, $\vec{j}\times\vec{j}=\vec{0}$, $\vec{k}\times\vec{k}=\vec{0}$

Theorem 3 Parallel Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a}\times\vec{b}=\vec{0}$

Example If $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} - 3\vec{k}$ then we obtain $\vec{a} \times \vec{b} = \vec{0}$. Hence, $\underline{\vec{a}}$ and $\vec{b}$ are parallel vectors.

Example The cross products of any pair of vectors in the set $\vec{i}, \vec{j}, \vec{k}$ can be obtained by the diagram in Figure 17.

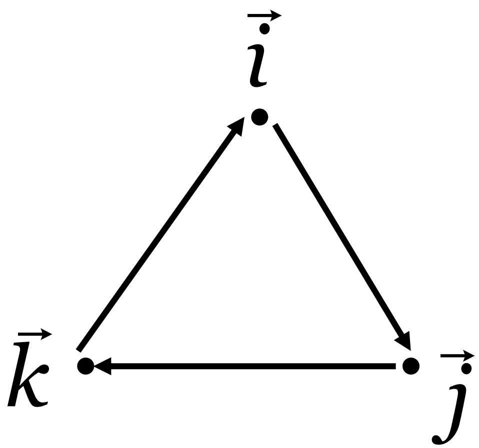


Figure 17

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i}$$

$$\vec{j} \times \vec{k} = \vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

$$\vec{k} \times \vec{i} = \vec{j} \qquad \vec{j} \times \vec{i} = -\vec{k}$$

Theorem If $\vec{a}$ and $\vec{b}$ are vectors in 3-space, then :

a) $\vec{a}\cdot(\vec{a}\times\vec{b})=0$ ($\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$)

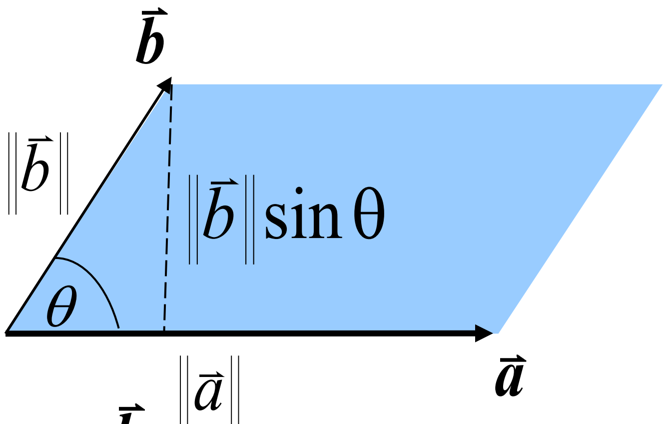
b) $\vec{b}\cdot(\vec{a}\times\vec{b})=0$ ($\vec{a}\times\vec{b}$ is orthogonal to $\vec{b}$)

c) $\left\|\vec{a}\times\vec{b}\right\|^2=\left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2-\left(\vec{a}\cdot\vec{b}\right)^2$

(Lagrange's identity)

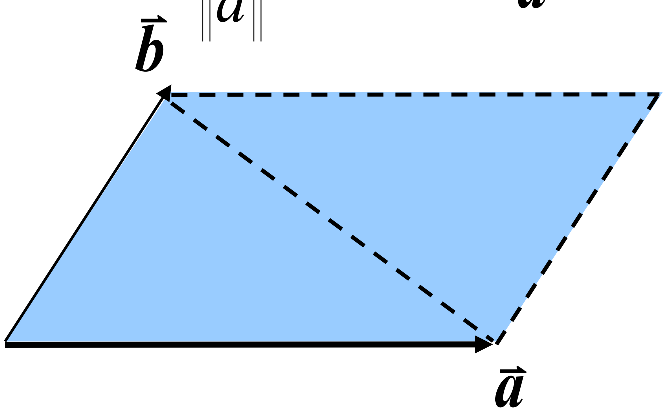
Areas and Volume

Two nonzero and nonparallel vectors $\vec{a}$ and $\vec{b}$ can be considered to be the sides of a parallelogram.



The area A of a parallelogram is given by

$$A = \|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$$



The area of a triangle with sides $\vec{a}$ and $\vec{b}$ is

$$A = \frac{1}{2}\|\vec{a} \times \vec{b}\| = \frac{1}{2}\|\vec{a}\|\|\vec{b}\|\sin\theta$$

Figure 18

Example Find the area of the parallelogram with two adjacent sides determined by

$\vec{a} = \langle -6, 4 \rangle$ and $\vec{b} = \langle 1, 3 \rangle$.

Solution Let A be the area of a parallelogram.

since

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 4 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(-18 - 4)$$

$$= -22\vec{k}$$

then $$A = \left\|\vec{a} \times \vec{b}\right\| = 22 \quad \text{unit}^2$$

Example Find the area of the parallelogram with two adjacent sides determined by

$\vec{a} = \langle 3, 2, -1 \rangle$ and $\vec{b} = \langle 1, 4, 2 \rangle.$

Solution Since

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 1 & 4 & 2 \end{vmatrix} = \vec{i}(4+4) - \vec{j}(6+1) + \vec{k}(12-2)$$

$$= 8\vec{i} - 7\vec{j} + 10\vec{k}$$

then $A = \left\| \vec{a} \times \vec{b} \right\| = \sqrt{213} \approx 14.59 \text{ unit}^2$

Example Find the area of the triangle having vertices at points $P_1(1, 1, 1)$, $P_2(2, 3, 4)$, $P_3(3, 0, -1)$

Solution since $\overrightarrow{P_1P_2} = \langle 1,2,3 \rangle$ and $\overrightarrow{P_2P_3} = \langle 1,-3,-5 \rangle$

We have $$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -3 & -5 \end{vmatrix} = -\vec{i} + 8\vec{j} - 5\underline{\vec{k}}$$

Hence, the area is

$$A = \frac{1}{2}\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} \right\| = \frac{1}{2}(1 + 64 + 25) = 45 \ \text{unit}^2$$

The Scalar Triple Product or Box Product

Definition If $\vec{a},\vec{b}$ and $\vec{c}$ are vectors in 3-space, then $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the ***scalar triple product*** of $\vec{a},\vec{b}$ and $\vec{c}$.

Let $\vec{a}=\langle \vec{a}_1,\vec{a}_2,\vec{a}_3 \rangle,\vec{b}=\langle \vec{b}_1,\vec{b}_2,\vec{b}_3 \rangle$ and $\vec{c}=\langle \vec{c}_1,\vec{c}_2,\vec{c}_3 \rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example **Calculate the scalar triple product of the vectors**

$$\vec{a} = 3\vec{i} - 2\vec{j} - 5\vec{k},\ \vec{b} = \vec{i} + 4\vec{j} - 4\vec{k}\ ,\vec{c} = 3\vec{j} + 2\vec{k}$$

Solution

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(20) + 2(2) - 5(3) = 49$$

Geometric Properties of the Scalar Triple Product

Volume of the parallelepiped

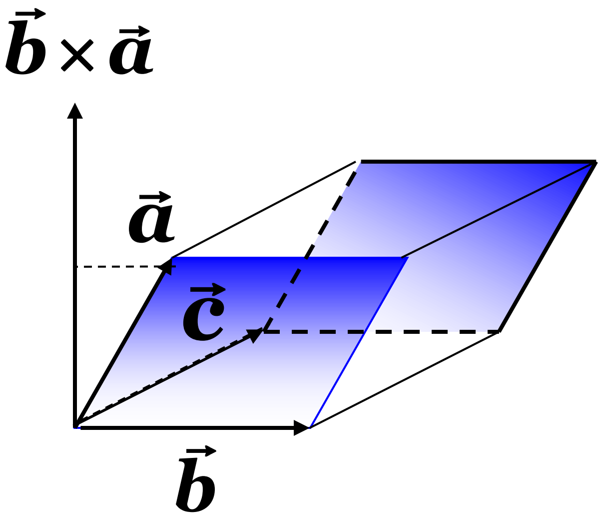


Figure 19

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ do not lie in the same plane, then the volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ shown in Figure 19 is

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Example Find the volume of the parallelepiped with sides $\vec{a}, \vec{b}$ and $\vec{c}$ where

$$\vec{a} = \langle 3,1,2 \rangle,\ \vec{b} = \langle 4,5,1 \rangle,\ \vec{c} = \langle 1,2,4 \rangle$$

Solution Let V be the volume of the parallelepiped.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 18\vec{i} - 15\vec{j} + 3\vec{k}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 3,1,2 \rangle \cdot \langle 18,-15,3 \rangle$$

Therefore the volume is

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \quad 45 \ \text{unit}^3$$

Coplanar Vectors

Vectors that lie in the same plane are said to be *coplanar*.

Theorem **If the vector** $\vec{a} = \langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle$, $\vec{b} = \langle \vec{b}_1, \vec{b}_2, \vec{b}_3 \rangle$ **and** $\vec{c} = \langle \vec{c}_1, \vec{c}_2, \vec{c}_3 \rangle$ **have the same initial point, then they lie in the same plane if and only if** $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0.$

Example If $\vec{a}$, $\vec{b}$ and $\vec{c}$ have the same initial points, determine whether $\vec{a}=\langle 4,-8,1\rangle$, $\vec{b}=\langle 2,1,-2\rangle$ and $\vec{c}=\langle 3,-4,12\rangle$ lie in the same plane.

Solution

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}4 & -8 & 1\\ 2 & 1 & -2\\ 3 & -4 & 1\end{vmatrix}=4\begin{vmatrix}1 & -2\\ -4 & 12\end{vmatrix}-2\begin{vmatrix}-8 & 1\\ -4 & 12\end{vmatrix}+3\begin{vmatrix}-8 & 1\\ 1 & -2\end{vmatrix}$$

$$=4(4)-2(-92)+3(15)=245$$

Since $\vec{a}\cdot(\vec{b}\times\vec{c})\neq 0$ then

$\vec{a}$, $\vec{b}$ and $\vec{c}$ do not lie in the same plane.

Example If $\vec{a}$, $\vec{b}$ and $\vec{c}$ have the same initial points, determine whether $\vec{a} = \langle 1, -2, 3 \rangle$, $\vec{b} = \langle 2, -4, 6 \rangle$ and $\vec{c} = \langle 5, -8, 1 \rangle$ lie in the same plane.

The triple Vector Product

The triple vector product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c})}$$

Remarks

1) $\vec{a}\times(\vec{b}\times\vec{c})=\left(\vec{a}\cdot\vec{b}\right)\vec{c}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

2) $(\vec{a}\times\vec{b})\times\vec{c}=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$

Lines and Planes

A line in 2-space and 3-space can be determined by specifying a point on the line and a nonzero vector parallel to the line see Figure 20.

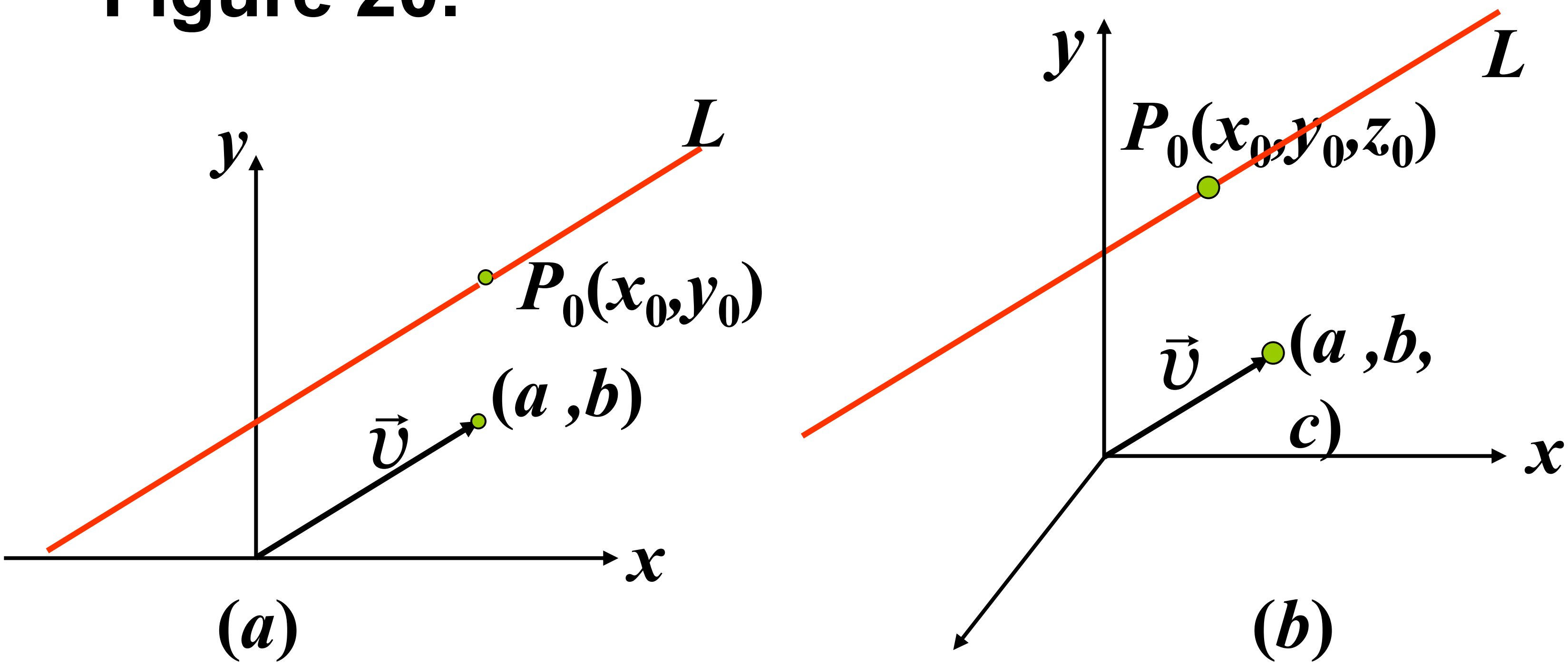


Figure20

Figure 20 shows a unique line L passes through P_0 and the vector $\vec{v}$ is parallel to L.

Vector equation for the line

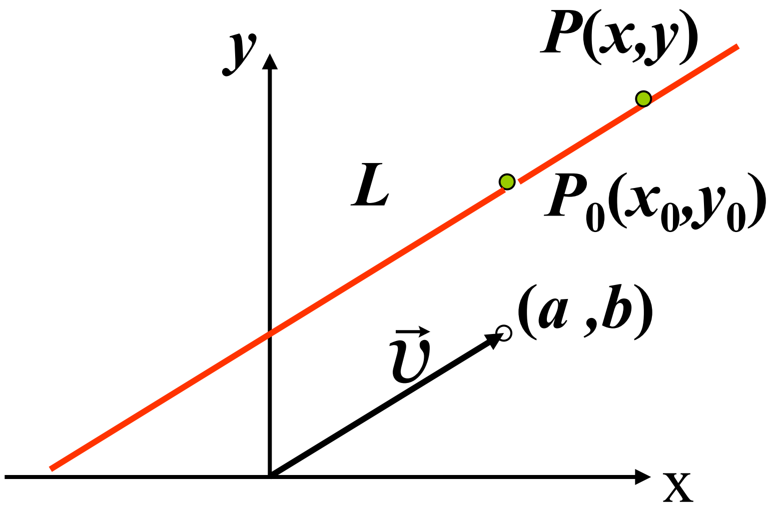


Figure 21

Let $P(x, y)$ is any point on the line L, we have

$$\overrightarrow{P_0P} = \langle x - x_0, y - y_0 \rangle$$
$$= \langle x, y \rangle - \langle x_0, y_0 \rangle$$
$$= \vec{P} - \overrightarrow{P_0}$$

and $\vec{P} - \overrightarrow{P_0} \,//\, \vec{v}$ so that $\vec{P} - \overrightarrow{P_0} = t\vec{v}$

Then, the ***vector equation*** for the line L is

$$\boxed{\vec{P} = \overrightarrow{P_0} + t\vec{v}}$$

Example The vector equation

$$\langle x, y\rangle = \langle -3, 2\rangle + t\langle 1, 1\rangle$$

represents a segment of the line in 2–space that passes through the point $(-3,2)$ and is parallel to the vector $(1,1)$.

Similarly, the line in 3-space

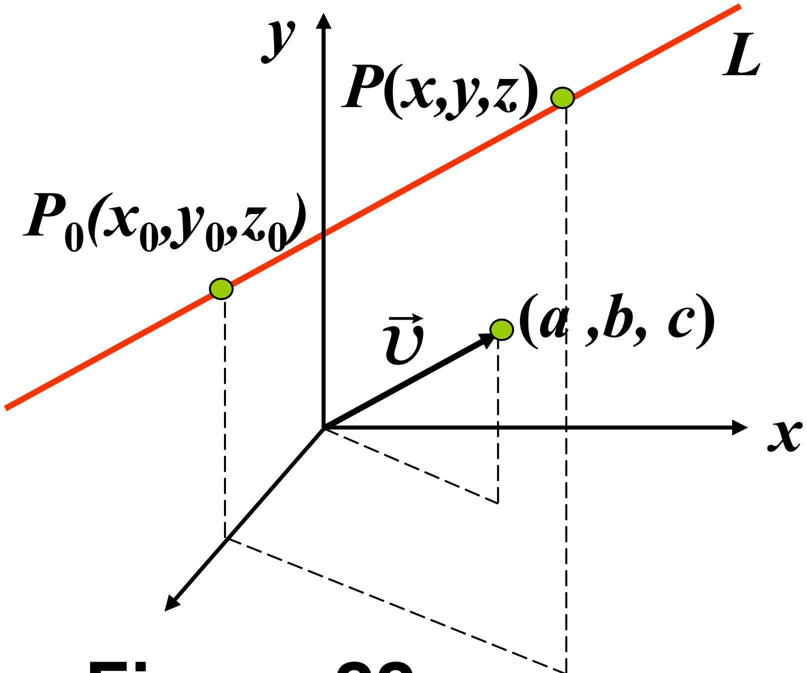


Figure 22

Let $P(x, y, z)$ is any point on the line L, we have

$$\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$$
$$= \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle$$
$$= \vec{P} - \overrightarrow{P_0}$$

and $\vec{P} - \overrightarrow{P_0} \,//\, \vec{v}$ so that $\vec{P} - \overrightarrow{P_0} = t\vec{v}$

Then, the ***vector equation*** for the line L is

$$\boxed{\vec{P} = \overrightarrow{P_0} + t\vec{v}}$$

Example Find a vector equation of the line in 3–space that passes through the point $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution The vector $\overrightarrow{P_1P_2} = \langle 3, -4, 8 \rangle$ is parallel to the given line. Thus, a vector equation of the line through the points P_1 and P_2 is

$$\langle x, y, z \rangle = \langle 2, 4, -1 \rangle + t \langle 3, -4, 8 \rangle.$$

Parametric Equations of The Line

In 2-space

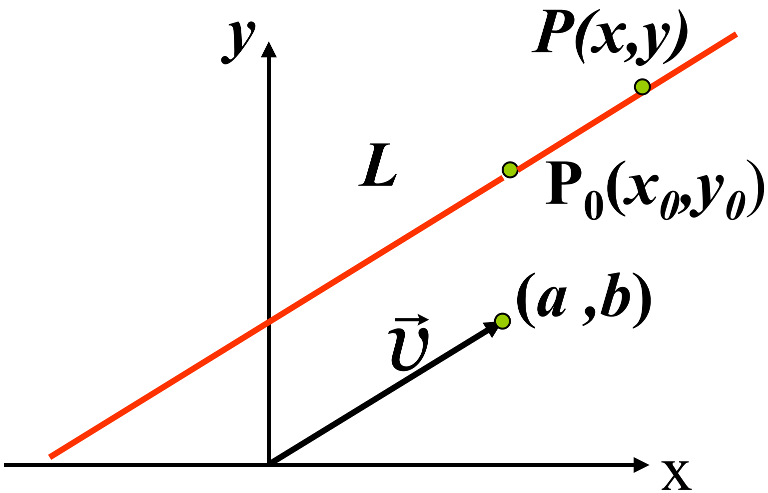


Figure 23

The vector equation for the line L is

$$\overrightarrow{P} = \overrightarrow{P_0} + t\vec{v}$$

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle a, b \rangle$$

$$= \langle x_0 + ta, y_0 + tb \rangle$$

Equating components, we obtain the *parametric equations* to the line through P_0 as

$$\boxed{x = x_0 + ta, y = y_0 + tb}$$

Similarly, in 3-space

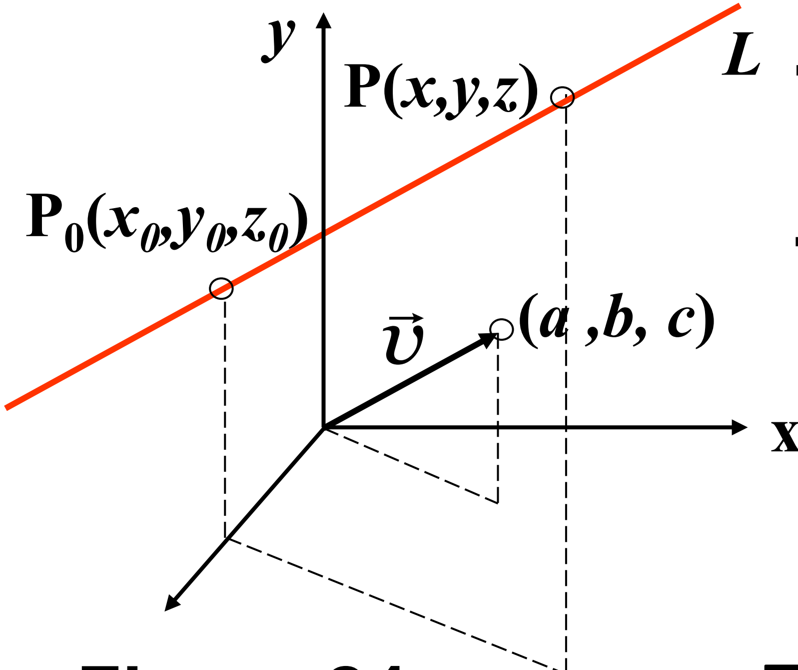


Figure 24

The *vector equation* for the line L is $\overrightarrow{P} = \overrightarrow{P_0} + t\vec{v}$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Equating components, we obtain the ***parametric equations*** to the line through P_0 as

$$\boxed{x = x_0 + ta, y = y_0 + tb, z = z_0 + tc}$$

symmetric equation for the line

From the parametric equations for the line in 2-space or 3-space we can clear the parameter by writing

In 2-space, $x = x_0 + ta, y = y_0 + tb$

$$\boxed{t = \frac{x - x_0}{a} = \frac{y - y_0}{b}} \qquad \ldots (1)$$

In 3-space, $x = x_0 + ta, y = y_0 + tb, z = z_0 + tc$

$$\boxed{t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \qquad \ldots (2)$$

Equations (1) and (2) are called *symmetric equation* for the line.

Example Write vector, parametric and symmetric equations for the line through (4,6,-3) and parallel to $\vec{v} = 5\vec{i} - 10\vec{j} + 2\vec{k}$.

Solution

Vector equation: $\langle x, y, z \rangle = \langle 4, 6, -3 \rangle + t\langle 5, -10, 2 \rangle$

parametric equation: x= 4+5t,y=6-10t,z=-3+2t

symmetric equation: $\dfrac{x-4}{5} = \dfrac{y-6}{-10} = \dfrac{z+3}{2}$

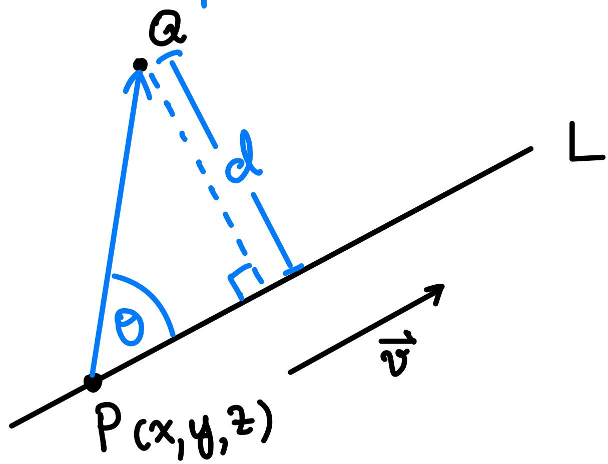
Example Find the parametric equations of the line

a) passing through $(4,2)$ and parallel to $\vec{v} = \langle 1,-5 \rangle$,

b) passing through $(1,2,-3)$ and parallel to $\vec{v} = \langle 4,5,7 \rangle$,

c) passing through the origin in 3-space and parallel to $\vec{v} = \langle 1,1,1 \rangle$.

Distance between point and line



Assume that L passes through a point P(x, y, z) and is parallel with a vector $\vec{v}$.

The distance between point Q and line L

$$d = \|\overrightarrow{PQ}\| \sin\theta = \frac{\|\overrightarrow{PQ} \times \vec{v}\|}{\|\vec{v}\|}$$

Example: A sphere has a tangent line:

$$x = 1+3t, \quad y = 6-2t, \quad z = 4t.$$

If $Q(2, 3, -4)$ is the center of this sphere, then find the radius of this sphere.

Plane in 3–space

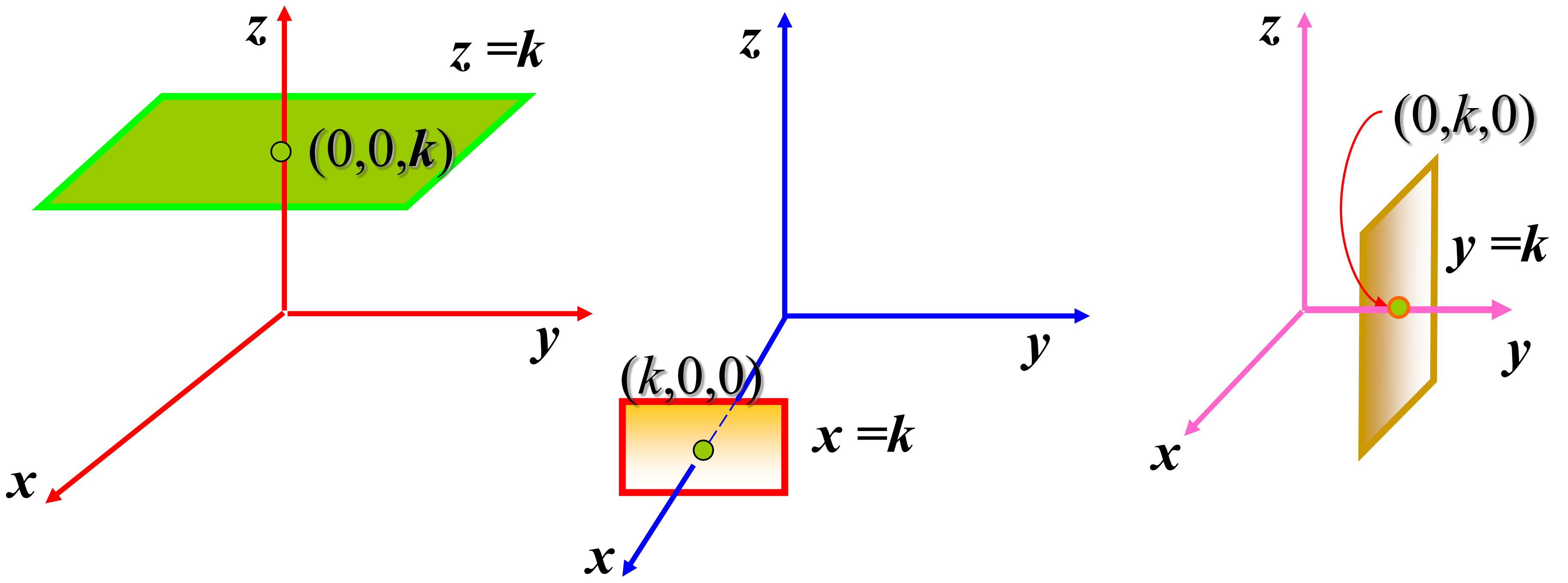


Figure 25

Vector Equation of the Plane

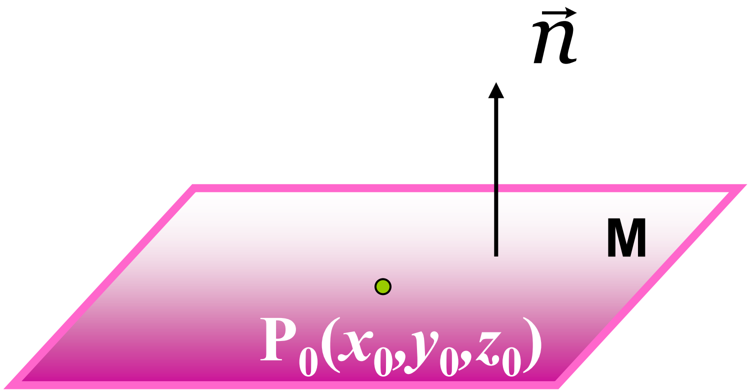


Figure 26

A plane in 3-space is uniquely determined by ***specifying a point*** P_0 in the plane and a ***nonzero vector*** $\vec{n}$ ***perpendicular*** to the plane

(see Figure 26).

This vector $\vec{n}$ is called a **normal vector**.

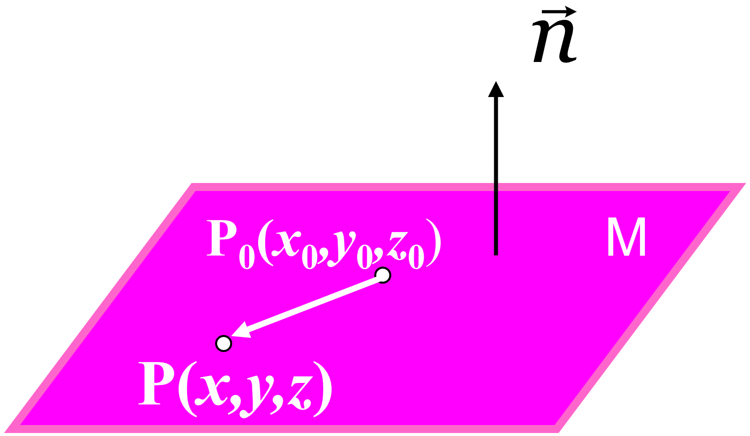


Figure 27

Let $P(x,y,z)$ is any point on the plane M. The vector equation of the plane is

$$\boxed{\vec{n}\cdot\overrightarrow{P_0P}=0} \quad \ldots (1)$$

Since $\overrightarrow{P_0P}=\langle x-x_0, y-y_0, z-z_0\rangle$ and let $\vec{n}=\langle a,b,c\rangle$, then (1) yields a Cartesian equation of the plane containing $P_0(x_0,y_0,z_0)$:

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0} \quad \ldots (2)$$

Example Find the equation of the plane passing through point $(-3,-1,7)$ and perpendicular to the vector $\vec{n} = \langle 4, 2, -5 \rangle$

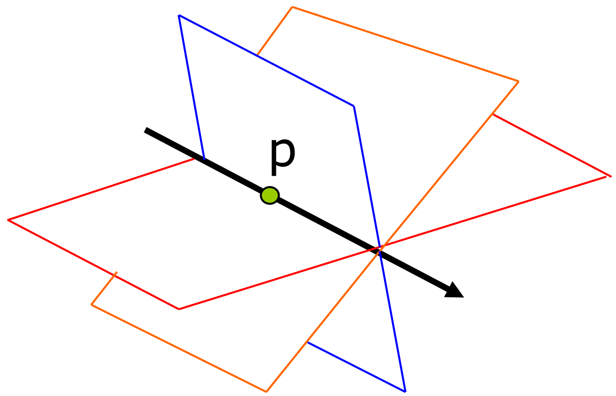
Solution From (2), we have

$$4(x-3)+2(y+1)-5(z-7)=0$$

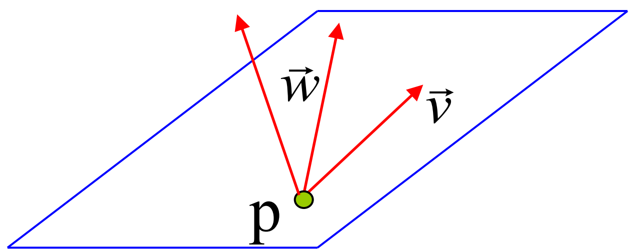
Thus, equation of the plane is

$$4x+2y-5z+25=0$$

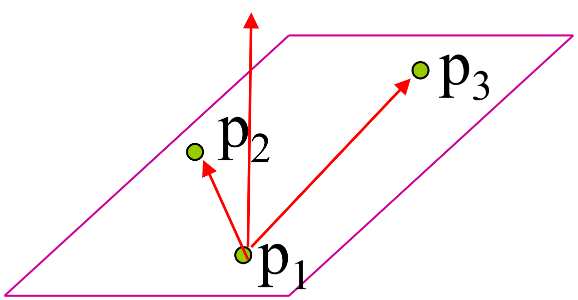
Theorem **The graph of any linear equation** $\mathrm{a}x + \mathrm{b}y + \mathrm{c}z + d = 0$, a, b, c **not all zero, is a plane having the vector** $\vec{n} = \langle a, b, c \rangle$ **as a** ***normal vector*****.**



There are infinitely many planes containing P and parallel to $\vec{v}$



There is a unique plane through P that is parallel to both $\vec{v}$ and $\vec{w}$



There is a unique plane through three noncollinear points.

Figure 28

Example Find an equation of the plane through the points $P_1(1,2,-1)$, $P_2(2,3,1)$ and $P_3(3,-1,2)$.

Solution

Definition Orthogonal and parallel planes

Let $\vec{n}_1$ be a normal vector to plane M_1 and $\vec{n}_2$ be a normal vector to plane M_2. Then

(i) M_1 and M_2 are orthogonal if

$$\vec{n}_1 \cdot \vec{n}_2 = 0.$$

(ii) M_1 and M_2 are parallel if $\vec{n}_1 = k\vec{n}_2$ for some non scalar k.

Example Determine whether the planes

$3x-4y+5z=0$ and $-6x+8y-10z-4=0$

are parallel.

Solution Let $M1 : 3x-4y+5z=0$

$M2 : -6x+8y-10z-4=0$

So, we have $\vec{n}_1 = 3\vec{i} - 4\vec{j} + 5\vec{k} \quad \perp \ M_1$

and $\vec{n}_2 = -6\vec{i} + 8\vec{j} - 10\vec{k} \quad \perp \ M_2$

since $\vec{n}_1 = -2\vec{n}_2$, then the plane M_1 is parallel to the plane M_2.

Example Determine whether the line $x = 3+8t$, $y = 4+5t$, $z = -3-t$ is parallel to the plane $x-3y+5z = 12$.

Solution Let L be the line $x = 3+8t$,

$y = 4+5t$, $z = -3-t$

M be the plane $x-3y+5z = 12$.

It is seen that

$\vec{v} = \langle 8,5,-1 \rangle$ is parallel to L and

$\vec{n} = \langle 1,-3,5 \rangle$ is normal to M.

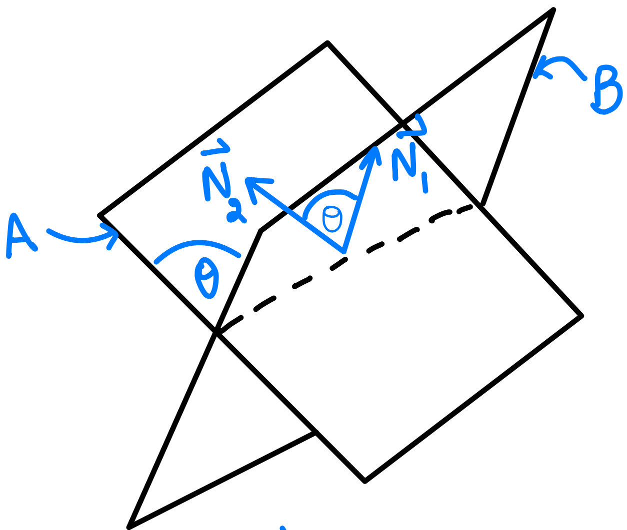
In order for the line and plane to be parallel, the vector $\vec{v}$ and $\vec{n}$ must be perpendicular.

$$\vec{v} \cdot \vec{n} = 8(1) + 5(-3) + (-1)(5) = -12 \neq 0$$

Thus, the line and plane are not parallel.

Angle and Trace between Planes

- Angle between Planes
  is the angle between normal vectors of the two planes.



Let $\vec{N}_1$ and $\vec{N}_2$ be normal vectors of planes A and B, respectively.
Let θ be the angle between these two planes.
Then

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{\|\vec{N}_1\| \, \|\vec{N}_2\|}$$

Trace of two planes

If two planes A and B are non-parallel, then A and B will intersect.

The trace between A and B is a line in 3D-space. The vector $\vec{N}_1 \times \vec{N}_2$ is parallel to the trace.

<u>Example</u>: Consider the planes

$$T_1 : 2x - y + z - 10 = 0$$

$$T_2 : x + y + z - 5 = 0$$

1) Find the angle between T_1 and T_2.

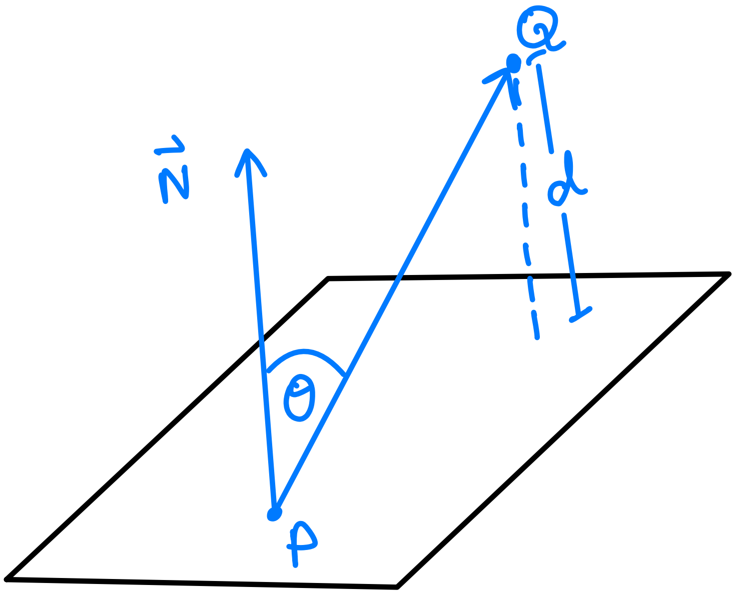
2) Find the parametric equation of the line passing through $(1,2,3)$ and parallel to the trace of T_1 and T_2.

<u>Example</u>: Find the parametric equation of the trace between planes

$$3x - 6y - 2z = 15 \quad \text{and}$$

$$2x + y - 2z = 5$$

Distance between point and plane



$$d = \|\overrightarrow{PQ}\| \cos\theta$$

$$= \left| \frac{\overrightarrow{PQ} \cdot \vec{N}}{\|\vec{N}\|} \right|$$

<u>Example:</u> Find the distance between $Q(2,-3,4)$ and the plane $x+2y+2z=13$

Example: Find the distance between planes

$$2x - 3y + z = 7 \quad \text{and}$$

$$6x - 9y + 3z = 9$$