Chapter4: Fourier Series

1. Periodic Functions

Definition: A periodic function $f(x)$ has period p if for every real number x, we have $f(x+p) = f(x)$.

Example1: Draw graphs of the following functions

1. $f(x) = \sin x$
2. $g(x) = \cos(nx)$

<u>Solution</u>

1. Let $\mathrm{f}(\mathrm{x}) = \sin \mathrm{x}$. Since

$$\mathrm{f}(\mathrm{x}+2\pi) = \sin(\mathrm{x}+2\pi) = \sin(\mathrm{x}+4\pi) = \cdots = \sin \mathrm{x} = \mathrm{f}(\mathrm{x}),$$

then $\mathrm{f}(\mathrm{x}) = \sin \mathrm{x}$ is a periodic function with periods $2\pi, 4\pi, 6\pi, \ldots$, where 2π is the smallest one.

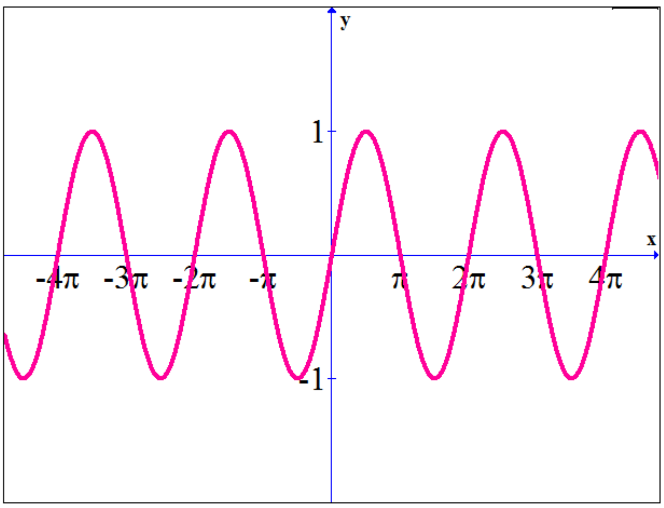


Figure 1

2. Let $g(x) = \cos nx$, Since

$$f(x+\frac{2\pi}{n}) = \cos\left(n\left(x+\frac{2\pi}{n}\right)\right) = \cos\left(n\left(x+\frac{4\pi}{n}\right)\right) = \cdots = \cos nx,$$

then $\cos nx$ is a periodic function with periods $\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots$, where $n = 1,2,3,\ldots$ and $\frac{2\pi}{n}$ is the smallest one.

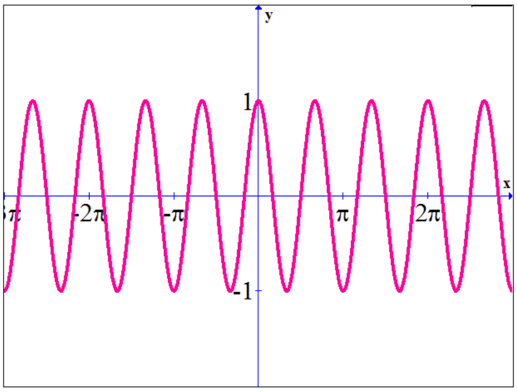


Figure 2: $y = \cos 3x$

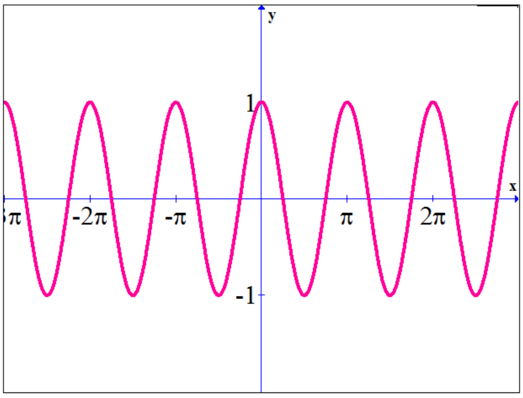


Figure 3: $y = \cos 2x$

Example2: Which function is periodic?

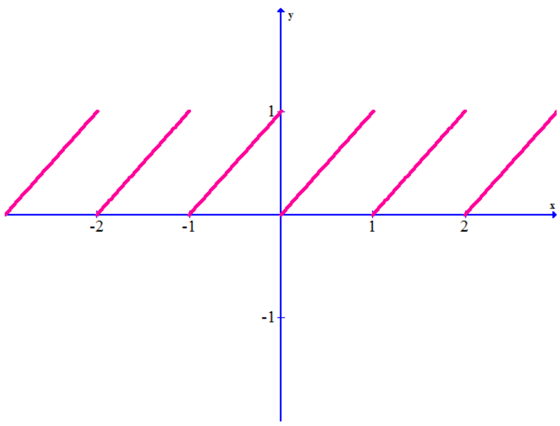


Figure 4.1

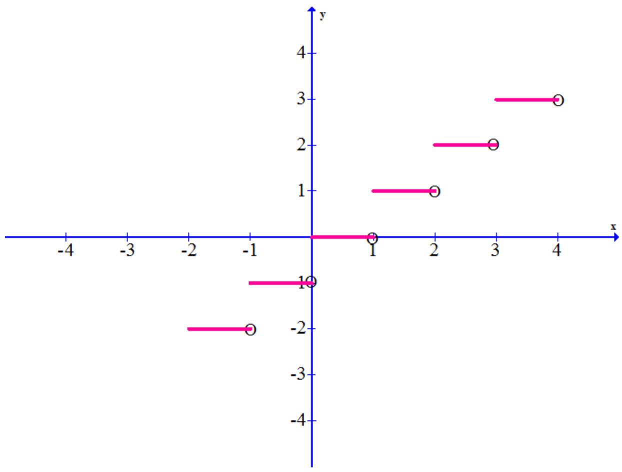


Figure 4.2

Properties of periodic functions

1. If f and g are periodic functions with period p, *and* a, b are constants, then

$$h(x) = af(x) + bg(x)$$

is also periodic.

2. If $f(x)$ has period p, then np is also a period of $f(x)$ for any positive integer n and the smallest p is called the **fundamental period** or **primitive period**.

In general, the word "period" refers to fundamental period such as $f(x) = \sin x$ with period 2π.

3. If $f(x)$ has period p, then

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx$$

for any real numbers a and b

2. Fourier Series

Fourier series is an infinite series of a periodic function. The series consists of sine and cosine functions.

Suppose $f(x)$ is a periodic function with period $2L$ in $(-L, L)$ or $[-L, L]$. The **Fourier series** of $f(x)$ is

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

That is $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad ; n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad ; n = 1, 2, 3, \ldots$$

where a_n and b_n are *Fourier Coefficients*:

Any 2π period function f(x) can be written as Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

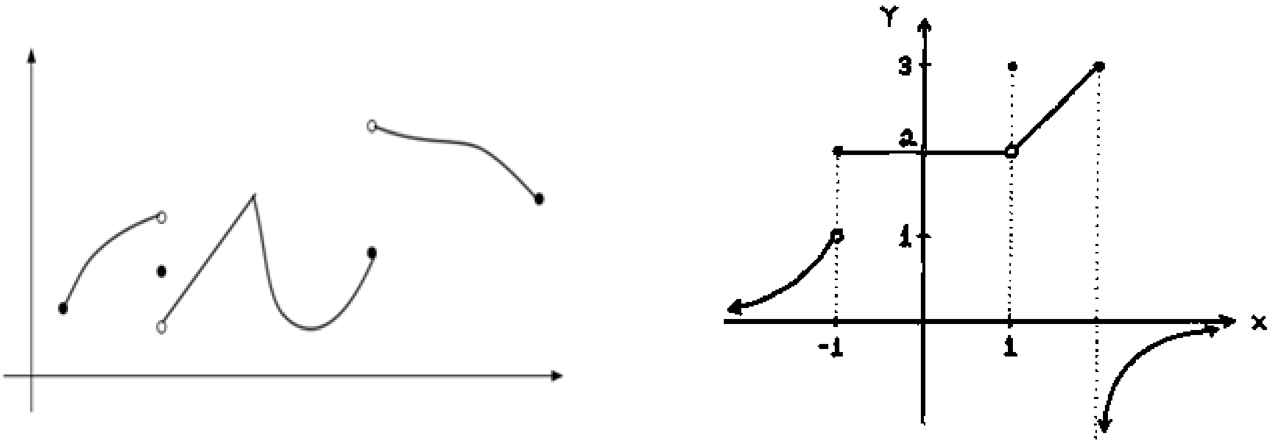
where a_n and b_n are Fourier coefficients calculated by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ; n = 0,1,2,...$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad ; n = 1,2,3,...$$

Remark:

1. Piecewise continuous functions

Function $f(x)$ is called a **piecewise continuous function** if function f(x) is continuous in each subdomain of f(x) and each interval has finite left and right limit values.



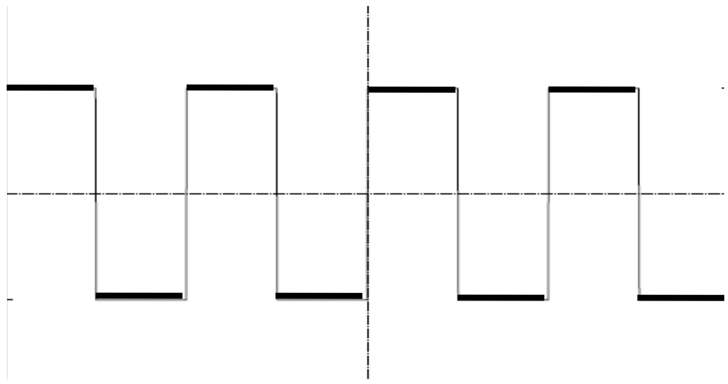
Piecewise continuous   Not Piecewise continuous

Figure 5

2. For Fourier series, a function *f*(*x*) can be either piecewise continuous or continuous. But for Taylor and Maclaurin series, the function has to be continuous and differentiable to any order *n*.

Example 3: Write Fourier series of

$$f(x) = \begin{cases} -k, -\pi < x < 0 \\ k\ ,\ 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$



Solution: We calculate the Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\left(\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx\right) = 0$$

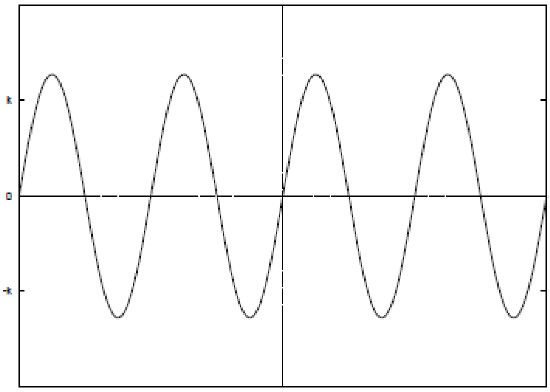
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\, dx + \int_{0}^{\pi} k\cos nx\, dx\right] \\ &= \frac{1}{\pi}\left\{\left[-\frac{k}{n}\sin nx\right]_{-\pi}^{0} + \left[\frac{k}{n}\sin nx\right]_{0}^{\pi}\right\} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{k}{n}\cos nx\right]_{-\pi}^{0} - \left[\frac{k}{n}\cos nx\right]_{0}^{\pi}\right\} \\
&= \frac{k}{n\pi}\{\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\} \\
&= \frac{2k}{n\pi}\{1 - \cos n\pi\} \\
&= \frac{2k}{n\pi}[1 - (-1)^n] \\
&= \begin{cases} 0, & n \text{even} \\ \frac{4k}{n\pi}, & n \text{odd} \end{cases}
\end{aligned}
$$

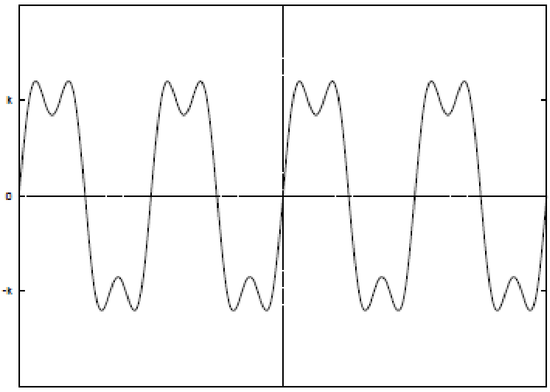
Thus $b_1 = \dfrac{4k}{\pi}$, $b_2 = 0$, $b_3 = \dfrac{4k}{3\pi}$, $b_4 = 0$, ...

The Fourier series is

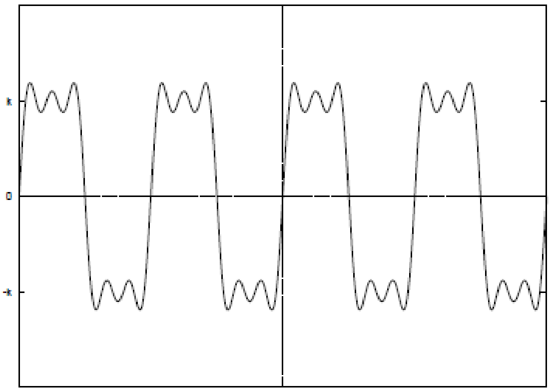
$$
\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right).
$$



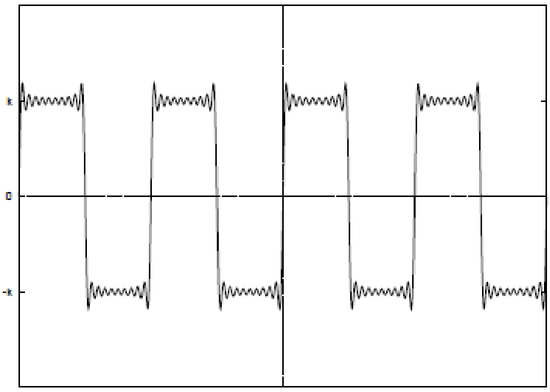
1 term of series



2 terms of series



3 terms of series



10 terms of series

Example 4: Write Fourier series of $f(x) = x \; ; x \in (-\pi, \pi)$

<u>Solution</u> Draw a graph of $f(x) = x \; ; -\pi < x < \pi$ with period 2π

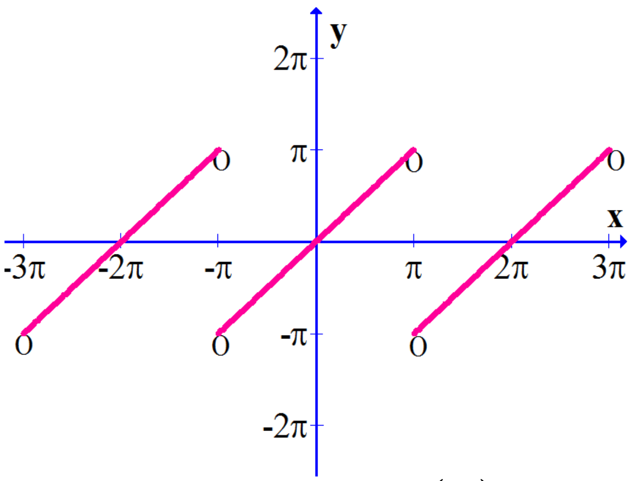


Figure 6: Graph of $f(x) = x \; ; -\pi < x < \pi$

From $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ ; n = 0,1,2,\ldots$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \ ; n = 1,2,3,\ldots$$

So, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \ = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx \ = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx$$

$$= \frac{1}{n\pi}\left[x\sin nx\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx$$

$$= \frac{1}{n\pi}\left[-x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx dx$$

$$= -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n} \ ; n = 1,2,3,\ldots$$

Thus, Fourier series of $f(x) = x \ ; x \in (-\pi, \pi)$ is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \ \square$$

From figure 7, when $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$, we got the sum of Fourier series is 0. We say the Fourier series converges to 0 even though the function is undefined at these points.

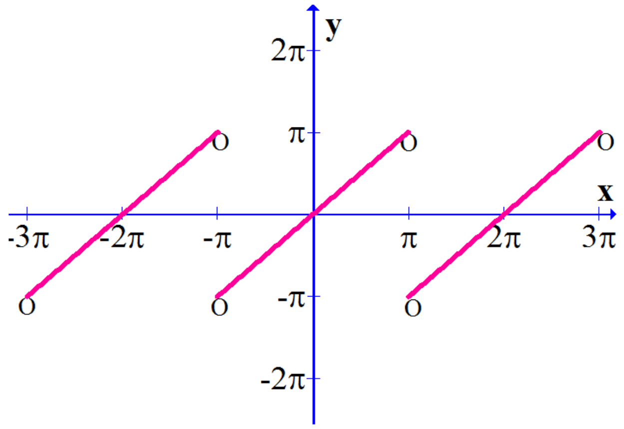


Figure 7

Therefore, if we want to find the convergence of Fourier series, we may use the following theorem.

Theorem 1: Dirichlet's Theorem

Let $f(x)$ be a periodic function with period $2L$ and be piecewise continuous on $(-L, L)$. Its Fourier series converges to $f(x)$ at every continuous point and coverges to the average value of left and right limits of each discontinuous point.

Dirichlet's Theorem

(1) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)=f(x)$$ for all continuous points x

(2) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x_0}{L}+b_n\sin\frac{n\pi x_0}{L}\right)=\frac{1}{2}\left(\lim_{x\to x_0^-}f(x)+\lim_{x\to x_0^+}f(x)\right)$$ for any discontinuous point x_0.

Conditions of Dirichlet's Theorem:

1. $f(x)$ must be a periodic function.
2. $f(x)$ must be continuous or piecewise continuous.
3. $f(x)$ must have left and right limits everywhere.

From example 4, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right)=f(x)\,;\quad x\neq\pm\pi,\pm3\pi,\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right)$$
$$=\frac{1}{2}\left(\lim_{x\to x_0^-}f(x)+\lim_{x\to x_0^+}f(x)\right);\quad x_0=\pm\pi,\ \pm3\pi$$

Thus, if $x=\frac{\pi}{2}$, then $2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right)=x$

becomes $$2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)=\frac{\pi}{2}$$

That is $$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

Example 5: Write Fourier series of periodic function

$$f(x)=\begin{cases} 0\ ;-\pi \le x \le 0 \\ 1\ ;0 < x < \dfrac{\pi}{2} \\ 0\ ;\dfrac{\pi}{2} \le x \le \pi \end{cases} \quad \text{and} \quad f(x+2\pi)=f(x)$$

and compute the convergent values when $x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi$.

Solution: $f(x)$ is periodic on $x\in[-\pi,\pi]$ with period 2π

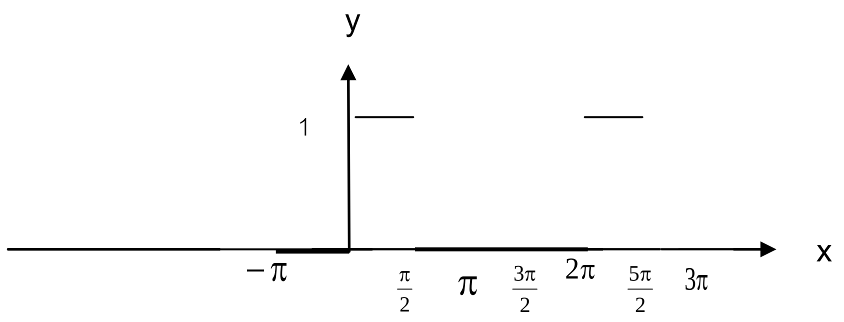


Figure 9 graph of $f(x)$

From $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)dx+\int_{0}^{\frac{\pi}{2}}f(x)dx+\int_{\frac{\pi}{2}}^{\pi}f(x)dx\right)=\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}1dx=\frac{1}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}\cos nxdx=\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

Thus, $a_1,a_2,a_3,\ldots=\frac{1}{\pi},0,-\frac{1}{3\pi},\ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \sin nx dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

Thus, $b_1, b_2, b_3, ... = \frac{1}{\pi}, \frac{1}{\pi}, \frac{1}{3\pi}, ...$

Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{4} + \frac{1}{\pi}\cos x - \frac{1}{3\pi}\cos 3x + \cdots + \frac{1}{\pi}\sin x + \frac{1}{\pi}\sin 2x + \frac{1}{3\pi}\sin 3x + \cdots$$

$$= \frac{1}{4} + \frac{1}{\pi}\left(\cos x + \sin x + \sin 2x - \frac{\cos 3x}{3} + \frac{\sin 3x}{3} + \cdots\right)$$

$$= f(x) \quad ; x \in [-\pi, \pi], x \neq 0, \frac{\pi}{2}$$

For $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, consider Dirichlet's Theorem:

$x = 0$; series converges to $\frac{1}{2}\left(\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right) = \frac{1}{2}(0+1) = 0.5$

$x = \frac{\pi}{4}$; series converges to $f(x) = f\left(\frac{\pi}{4}\right) = 1$

$x = \frac{\pi}{2}$; series converges to $\frac{1}{2}\left(\lim_{x\to \frac{\pi}{2}^-} f(x) + \lim_{x\to \frac{\pi}{2}^+} f(x)\right) = \frac{1}{2}(1+0) = 0.5$

$x = \frac{3\pi}{4}$; series converges to $f(x) = f\left(\frac{3\pi}{4}\right) = 0$

$x = \pi$; series converges to $f(x) = f(\pi) = 0$

Example 6: Write Fourier series of $f(x)=\begin{cases} 0 & ;-2<x<-1 \\ k & ;-1<x<1 \\ 0 & ;1<x<2 \end{cases}$

$f(x+4)=f(x)$, and compute the convergent values when $x=-1,\ 0,\ 1.5.$

<u>Solution</u> f(x) has period $2L=4$

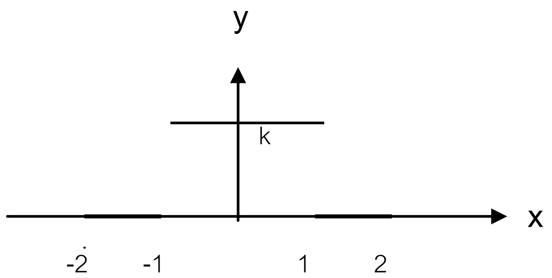


Figure 10 Graph of period function $f(x),\ x\in(-2,2)$

From $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$ and

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx \quad ;n=0,1,2,...$$
$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx \quad ;n=1,2,3,...$$

Consider convergent values when $x = -1, 0, 1.5.$

$x = -1$, series converges to $\frac{1}{2}\left(\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right) = \frac{1}{2}(0+k) = \frac{k}{2}$

$x = 0$, series converges to $f(x) = f(0) = k$

$x = 1.5$, series converges to $f(x) = f(1.5) = 0$

Example 7: Suppose $f(x)=\begin{cases}\pi, & 0<x<\dfrac{\pi}{2}\\ 0, & \text{other } x\in(-\pi,\pi)\end{cases}$

and $f(x+2\pi)=f(x)$. Find

1. Fourier series of $f(x)$ and its sum when x = $-\pi,-\dfrac{\pi}{2},0,\dfrac{\pi}{2}$

2. The sum of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$

Solution $f(x)$ has period 2π as follows:

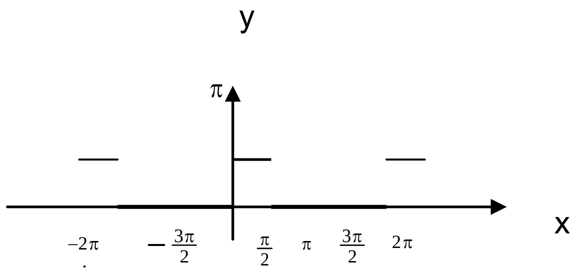


Figure 11 Graph of a periodic function $f(x)$

From $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx \quad ;n=0,1,2,\ldots$$
$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx \quad ;n=1,2,3,\ldots$$

The Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\left[a_1\cos x+a_2\cos 2x+a_3\cos 3x+\cdots\right]+\left[b_1\sin x+b_2\sin 2x+b_3\sin 3x+\cdots\right]$$
$$=\frac{\pi}{4}+\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\cdots\right]+\left[\sin x+\frac{2}{2}\sin 2x+\frac{1}{3}\sin 3x+\cdots\right]$$
$$=\frac{\pi}{4}+\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\cdots\right]+\left[\sin x+\frac{1}{3}\sin 3x+\cdots\right]+\left[\frac{2}{2}\sin 2x+\frac{2}{6}\sin 6x+\cdots\right]$$
$$=\frac{\pi}{4}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{2n-1}+\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}+2\sum_{n=1}^{\infty}\frac{\sin(4n-2)x}{(4n-2)}$$

for $x\in(-\pi,\pi),\quad x\neq 0,\dfrac{\pi}{2}$

Convergent values when $x=-\pi,-\dfrac{\pi}{2},0,\dfrac{\pi}{2}$

$x=-\pi$, series converges to

$x=-\dfrac{\pi}{2}$, series converges to

$x=0$, series converges to

$x=\dfrac{\pi}{2}$, series converges to

2. The sum of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}=-1+\frac{1}{3}-\frac{1}{5}+\cdots$

$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{2n-1}+\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}+2\sum_{n=1}^{\infty}\frac{\sin(4n-2)x}{(4n-2)}$$

At $x=\dfrac{\pi}{2}$, $\quad f\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{4}+\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}=\frac{\pi}{4}-\left(-1+\frac{1}{3}-\frac{1}{5}+\cdots\right)$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}=-1+\frac{1}{3}-\frac{1}{5}+\cdots=\frac{\pi}{4}-f\left(\frac{\pi}{2}\right)=\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}$.

3. Sine Series and Cosine Series

Sine and cosine series are Fourier series created by some periodic function f(x). Sine series is an odd function, while cosine series is even on each defined period.

Definition: Odd and Even Functions

$f(x)$ is an odd function if $f(-x) = -f(x)$ for all x. For example: $x^3, x^5 - 3x^3 + 2x, \sin x, \tan 3x$

$f(x)$ is an even function if $f(-x) = f(x)$ for all x. For example: $x^4, 2x^6 - 4x^2 + 5, \cos x, e^x + e^{-x}$.

Properties of odd and even functions

1. If f is an even function on $(-L, L)$: $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$

2. If f is an odd function on $(-L, L)$: $\int_{-L}^{L} f(x)dx = 0$

3. The product of two odd functions is even.

   The product of two even functions is even.

   The product of even and odd functions is odd.

Theorem 2: Let $f(x)$ be a 2π period function and continuous or piecewise continuous on $[-\pi,\pi]$, then

1. If $f(x)$ is **odd** on $(-\pi,\pi)$, then Fourier series of $y=f(x)$ has $a_n=0$ for all *n*, and $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$ for $n=1,2,3,\ldots$.

Thus, we have $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$ and it's called **sine series**.

2. If $f(x)$ is **even** on $(-\pi,\pi)$, then Fourier series of $y=f(x)$ has $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$ for $n=0,1,2,\ldots$ and $b_n=0$ for all *n*.

Thus, we have $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$ and it's called **cosine series**.

Theorem 3: Let $f(x)$ be a $2L$ period function and continuous or piecewise continuous on $[-L,L]$, then

1. If $f(x)$ is **odd** on $(-L,L)$, then Fourier series of $y=f(x)$ has $a_n=0$ for all n and $b_n=\frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}dx$ for $n=1,2,3,\ldots$.

Thus, we have the **sine series**: $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

2. If $f(x)$ is **even** on $(-L,L)$, then Fourier series of $y=f(x)$ has $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$ for $n=0,1,2,\ldots$ and $b_n=0$ for all *n*.

Thus we have the **cosine series**: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$.

Example 8: Write Fourier series of $f(x)=|x| \quad ; x\in(-\pi,\pi)$

<u>Solution</u> $f(x)$ is a periodic function on $x\in(-\pi,\pi)$ with period 2π

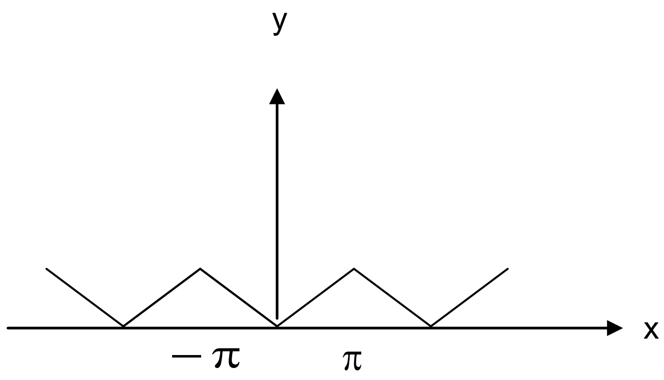


Figure 12: Graph of periodic function $f(x)=|x|$

Since $f(-x)=|-x|=x=f(x)$, then $f(x)=|x|$ is an even function. Thus its Fourier series is **cosine series**:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}dx$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1)$$

So $\quad a_1, a_2, a_3 \ldots = -\frac{4}{\pi}, 0, -\frac{4}{3^2\pi}, 0, -\frac{4}{5^2\pi}, \cdots$

and its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} + \left(-\frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x - \cdots\right)$$

Remark:

From $|x| = \frac{\pi}{2} + \left(-\frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x - \cdots\right) \quad ; x \in (-\pi, \pi)$

When $x = 0$, we have $0 = \frac{\pi}{2} + \left(-\frac{4}{\pi}\cos 0 - \frac{4}{3^2\pi}\cos 0 - \frac{4}{5^2\pi}\cos 0 - \cdots\right)$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Example 9: Write Fourier series of $f(x)=\begin{cases} x+1 & ;-1<x<0 \\ x-1 & ;0<x<1 \end{cases}$

<u>Solution</u> $f(x)$ has period $2L=2$

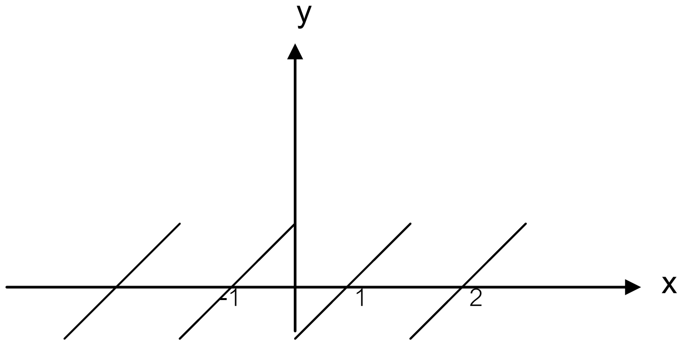


Figure 13: graph of $f(x)$ with period $2L=2$

Since $f(-x)=-f(x)$, function $f(x)=\begin{cases} x+1 & ;-1<x<0 \\ x-1 & ;0<x<1 \end{cases}$ is odd.

Fourier series is **sine series**:

Example 10 $f(x) = \sin 2x \quad ; -\pi < x < \pi$

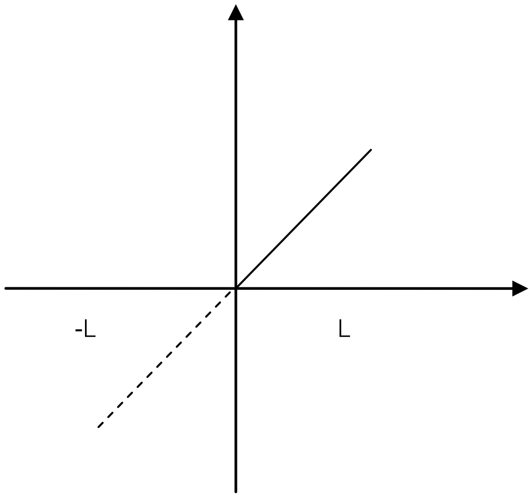
Solution $f(x)$ has period $2L = 2\pi$

Since $f(-x) = -f(x)$, the function $f(x) = \sin 2x$ is odd. Its

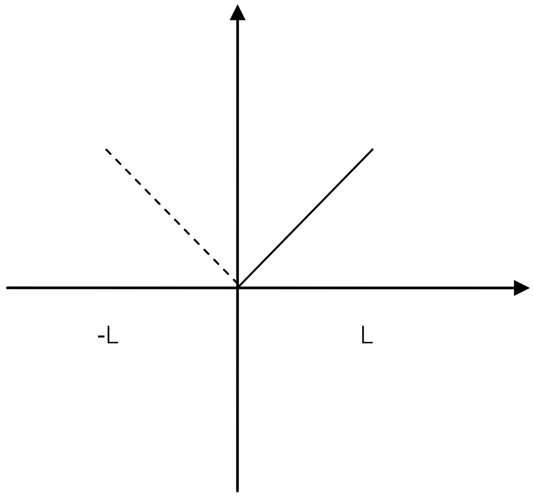
Fourier series is $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

4. Half Range Expansion

Suppose we know only a half period of function f(x). We may make Fourier of f(x) by extending function f(x). In general, we make it either even or odd for simplicity.



Odd extension



Even extension

The extended function is either Half Range Fourier Sine or Cosine function according to odd or even extension, respectively.

Example11 Write the half range Fourier Sine series expansion of

$$f(x)=\begin{cases}0 & ;0\le x\le \frac{\pi}{2}\\ 1 & ;\frac{\pi}{2}<x\le \pi\end{cases}$$

We extend $f(x)$ to an **odd** function with period 2π

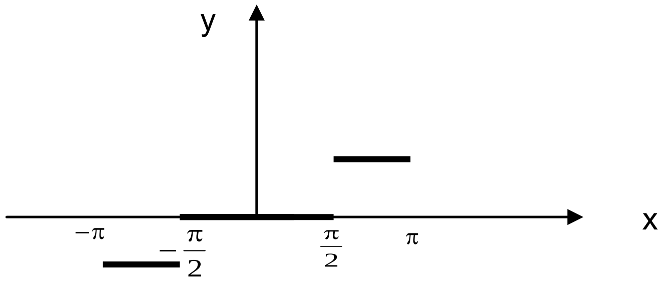


Figure 14 Graph of extension of $f(x)$ to be odd on $[-\pi,\pi]$

Half range sine series of $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin nx dx=-\frac{2}{n\pi}\left(\cos\frac{n\pi}{2}-\cos n\pi\right)$$

So, $\quad b_1,b_2,b_3,...=\frac{2}{\pi},-\frac{2}{\pi},\frac{2}{3\pi},0,\frac{2}{5\pi},...$

The half range Fourier Sine expansion is

$$\sum_{n=1}^{\infty} b_n \sin nx=\frac{2}{\pi}\left(\sin x-\sin 2x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)\quad ;x\in[0,\pi],x\ne\frac{\pi}{2},\pi$$

Example 12 Write the half range cosine series expansion of

$$f(x) = x \quad ; 0 < x < 4$$

<u>Solution</u> Extend $f(x)$ to be even on $(-4,4)$ with period $2L = 8$

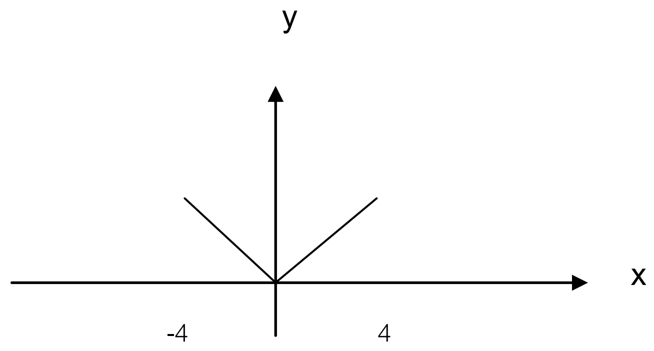


Figure 15: Graph of extension of $f(x)$

Half range cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$

5. Differentiation and Integration of Fourier Series

Differentiation and integration of Fourier series can be done by term by term.

Theorem 5.1

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$ be Fourier series.

Its derivative on $[-L,L]$ may not be Fourier series, but converges to $\frac{1}{2}\left[\lim_{x\to x_0^-} f'(x)+\lim_{x\to x_0^+} f'(x)\right]$ at any x_0.

Theorem 5.2

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$ be Fourier series.

Its integral from *a* to *x* may not be Fourier series, but converges to $\int_a^x f(t)dt$ where $f(x)$ is piecewise continuous on $-L\le x\le L$ and both *a* and *x* are in this interval.

Example 13 Given $x=\frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\cdots\right)$

Find 1. Series of $f(x)=x^2$ with period $0<x<2$

2. The sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$

Solution From $x=\frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\cdots\right)$

Integrate: $\int x\,dx=\int\left[\frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\cdots\right)\right]dx$

$$x^2=c-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}-\frac{1}{2^2}\cos\frac{2\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}-\cdots\right)$$

$$x^2=c-\frac{16}{\pi^2}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{2}$$

To make it a Fourier series, we need

$$c=\frac{a_0}{2}=\frac{1}{2}\int_0^2 x^2dx=\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2=\frac{4}{3}$$

2. The sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$

From $x^2=c-\frac{16}{\pi}\left(\cos\frac{\pi x}{2}-\frac{1}{2^2}\cos\frac{2\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}-\cdots\right)$

When $x=0$, we get $c=\frac{16}{\pi^2}\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)$

Thus, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots=c\frac{\pi^2}{16}=\frac{4}{3}\frac{\pi^2}{16}=\frac{\pi^2}{12}$

Example 14: Let $x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin nx}{n}$ be Fourier series on $-\pi < x < \pi$. Write Fourier series of $f(x) = x^2$ and $f(x) = \frac{\pi^2 x - x^3}{3}$ with period 2π.

Exercises:

Write Fourier series of the following functions

1. $f(x)=\begin{cases} -\dfrac{\pi}{4} & ;-\pi < x < 0 \\ \dfrac{\pi}{4} & ;0 < x < \pi \end{cases}$

2. $f(x)=|\cos x| \quad ;-\pi < x < \pi$

3. $f(x)=\sin\dfrac{x}{2} \quad ;0 \le x \le 2\pi$

4. $f(x)=1-|x| \quad ;-1 \le x \le 1$

5. $f(x)=\begin{cases} 0 & ;-5 < x < 0 \\ 3 & ;0 < x < 5 \end{cases}$

6. $f(x)=\sin x \quad ;0 \le x \le \pi$

7. Let $\sin x = \dfrac{2}{\pi}-\dfrac{4}{\pi}\left(\dfrac{\cos 2x}{2^2-1}+\dfrac{\cos 4x}{4^2-1}+\dfrac{\cos 6x}{6^2-1}+\cdots\right)$ be Fourier series on $0 \le x \le \pi$. Show that $\dfrac{1}{1^2\cdot 3^2}+\dfrac{1}{3^2\cdot 5^2}+\dfrac{1}{5^2\cdot 7^2}+\cdots=\dfrac{\pi^2-8}{16}$.

Answer to questions

1. $f(x)=\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\cdots$

2. $f(x)=\dfrac{2}{\pi}+\dfrac{4}{\pi}\left(\dfrac{\cos 2x}{2^2-1}-\dfrac{\cos 4x}{4^2-1}+\dfrac{\cos 6x}{6^2-1}-\cdots\right)$

3. $f(x)=\sin\dfrac{x}{2} \quad ;0 \le x \le 2\pi$

4. $f(x)=\dfrac{1}{2}+\dfrac{4}{\pi^2}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\cdots\right)$

5. $f(x)=\dfrac{3}{2}+\dfrac{6}{\pi}\left(\sin\dfrac{\pi x}{5}+\dfrac{1}{3}\sin\dfrac{3\pi x}{5}+\dfrac{1}{5}\sin\dfrac{5\pi x}{5}+\cdots\right)$

6. $f(x)=\dfrac{2}{\pi}-\dfrac{4}{\pi}\left(\dfrac{\cos 2x}{2^2-1}+\dfrac{\cos 4x}{4^2-1}+\dfrac{\cos 6x}{6^2-1}+\cdots\right)$