MTH10101

Mathematics I (International)

Module 1 Limit, Continuity and Derivatives

Functions

Definition 1

A **function** is a rule that takes certain numbers as inputs and assigns exactly one output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

Note: A function can be considered as a set of ordered pairs (x, y).

Notations:

Let f be a function from A to B ($f : A \to B$)

- D_f represents domain of function f
- R_f represents range of function f
- Image of x is y since $f(x) = y$

$f : A \to B$ is called a function from A **onto** B if $R_f = B$

Normally, we may present a function via four common ways:

1) Description (words)

2) Numeric (tables)

3) Visual (graphs)

4) Algebra (formulas)

Example 1

Consider a set $\{(-3,1),(0,2),(3,-1),(5,4)\}$. Is it a function?

Domain:

Range:

Example 2

Let $f = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = x^2 - 2\}$.

So $D_f = \mathbb{R}$ and $R_f = [-2, +\infty)$. We usually write $f(x) = x^2 - 2$.

The values of f at some points are as follow.

$f(0) = (0)^2 - 2 = -2$

$f(-1) = (-1)^2 - 2 = -1$

$f(\sqrt{3}) = (\sqrt{3})^2 - 2 = 1$

$f(c) = c^2 - 2$

$f(x+h) = (x+h)^2 - 2 = x^2 + 2hx + h^2 - 2$

$f(x+h) - f(x) = (x^2 + 2hx + h^2 - 2) - (x^2 - 2) = 2hx + h^2$

and $\dfrac{f(x+h) - f(x)}{h} = 2x + h, \quad h \neq 0$

Example 3

Let $f = \left\{(x, y) : x^2 + y^2 = 1^2\right\}$. Is f a function?

Example 4 Find the domain of the following functions.

(1) $f(x)=\dfrac{4}{x-1}$

(2) $f(x)=\dfrac{x}{x^2-9}$

(3) $f(x)=\dfrac{\sqrt{4-x}}{x}$

(4) $f(x)=\sqrt{4-x^2}$

Example 5

1) $y=\sin x$ has the set of all real numbers as its domain and the interval $[-1,1]$ as its range.

2) $y=\sqrt{x^2+4}$ has the set of all real numbers as a domain and the interval $[2,+\infty)$ as its range.

Example 6

$$h(x)=\begin{cases}\dfrac{2x^2-9x+4}{x-4} & ,x\neq 4\\ 5 & ,x=4\end{cases} \quad \text{or} \quad h(x)=\begin{cases}2x-1 & ,x\neq 4\\ 5 & ,x=4\end{cases}$$

$D_f =$

$R_f =$

Definition 2 The function f equals to the function g if and only if

1. $D_f = D_g$
2. $f(x)=g(x)$ for all $x\in D_f$.

Example 7 Check if the following functions are equal.

1) Let $f(x)=\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ and $g(x)=\dfrac{1}{\sqrt{2+x}+\sqrt{2}}$

2) Let $f(x)=x+3$ and $g(x)=\begin{cases} \dfrac{2x^2+7x+3}{2x+1} & ,x\neq -\dfrac{1}{2} \\ \dfrac{5}{2} & ,x=-\dfrac{1}{2} \end{cases}$

Definition 3

Let f and g be functions and $R_g \cap D_f \neq \varnothing$.

A composite function of f and g (denoted by $f \circ g$) is a function $(f \circ g)(x) = f(g(x))$ whose domain is $\{x : x \in D_g \text{ and } g(x) \in D_f\}$.

Example 8 Let $f(x)=\sqrt{x-3}$ and $g(x)=2x-1$

a) Let $F = f \circ g$ Find $F(x)$ and domain of F

b) Let $G = g \circ f$ Find $G(x)$ and domain of G

c) Let $H = f \circ f$ Find $H(x)$ and domain of H

Solutions

a) The domain of g is $(-\infty,\infty)$ and domain of f is $[3,\infty)$.

To find the domain of $F = f \circ g$, we consider only x where $g(x)$ is in domain of f. That is, $2x-1 \geq 3$.

Thus domain of F is a set of x where $x \geq 2$ i.e. $[2,\infty)$.

Then, the function $F = f \circ g$ can be found by

$$F(x) = f \circ g(x) = f(g(x)) = f(2x-1) = \sqrt{(2x-1)-3} = \sqrt{2x-4}.$$

<u>**Symmetry**</u>

Definition 4 Let f be a function.

a. If $f(-x) = -f(x)$, f is called an **odd function** whose graph is symmetric about the origin.

b. If $f(-x) = f(x)$, f is called an **even function** whose graph is symmetric about the y-axis.

Example 9

a) Let $f(x) = x^3$.

Consider $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Thus f is an odd function and it graph is shown in figure 1 below.

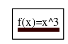


Figure 1

b) Let $f(x) = 3x^2 - 1$.

Consider $f(-x) = 3(-x)^2 - 1 = 3x^2 - 1 = f(x)$.

Thus f is an even function whose graph shown in Figure 2.

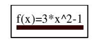


Figure 2

Inverse function

Definition 5 The function f is called a **one-to-one** function if and only if for all x, y, z if (x, y) and $(z, y) \in f$ then $x = z$.

Definition 6 Let f be a one-to-one function from A onto B.

An inverse function of f is defined by $f^{-1} = \{(b,a) \mid (a,b) \in f\}$

which is also a one-to-one function from B to A.

Remark Graphs of f and f^{-1} are symmetric about the line $y = x$ as shown in Figure 3 below.

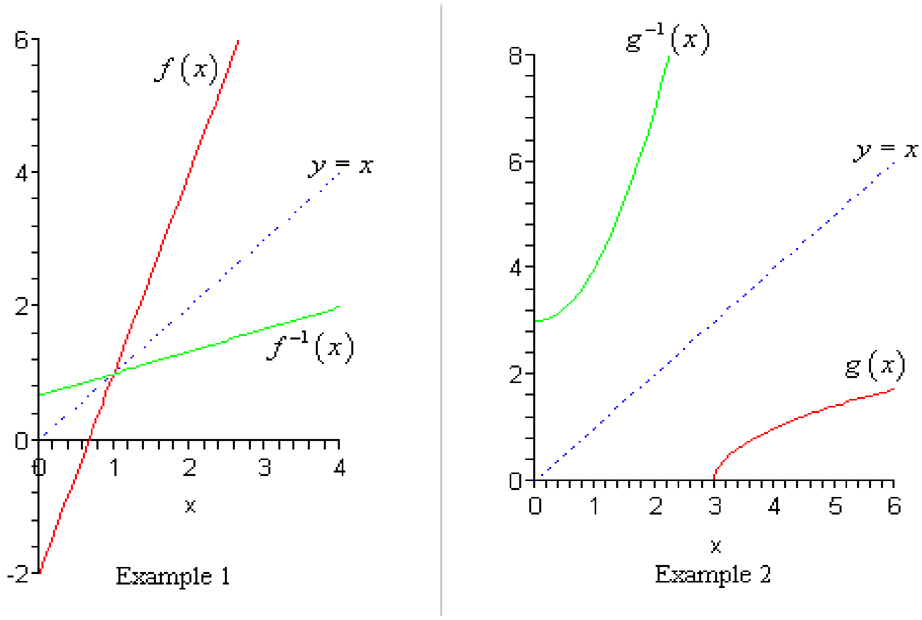


Figure 3

Example10 Find an inverse of f where $f(x) = x^3 - 1$.

<u>**Solution**</u> From $y = f(x) = x^3 - 1$ (i.e. $x = \sqrt[3]{y+1}$), we have that $f^{-1} = \left\{(y,x) \middle|\ y = x^3 - 1\right\}$ or $f^{-1} = \left\{(x,y) \middle|\ y = \sqrt[3]{x+1}\right\}$

We normally write $f^{-1}(x) = \sqrt[3]{x+1}$ so that we can easily draw graphs of both functions f and f^{-1} as follows

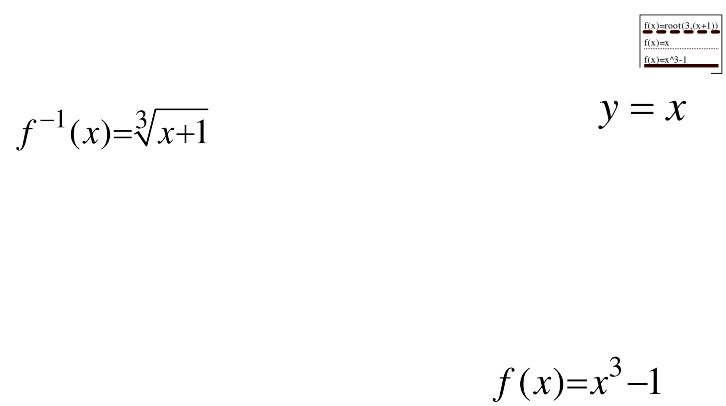


Figure 4

<u>Other Interesting Functions</u>

All functions here will be useful in the next sections.

Algebraic Function

a. Polynomial Functions are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where a_i is a real number for each $i = 0,1,2,\ldots,n$ and

n is a non-negative integer.

If n is the largest number such that $a_n \neq 0$, we call f a polynomial function of degree n such as $f(x) = 3x^3 - 5x^2 + x + 4$ is a polynomial function of degree 3.

Normally, if there is nothing specific, the domain of a polynomial function is the set of all real numbers.

b. Rational Functions are functions formed by a ratio between two polynomial functions.

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0},$$

Note that, if there is nothing specific, the domain of this rational function is $\left\{ x \in \mathbb{R} \;\middle|\; b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0 \neq 0 \right\}$

Example 11 Let $y = f(x) = \dfrac{x^2 + x}{x}$

Rewrite function f : $f(x) = x + 1$ where $x \neq 0$

Thus graph of $f(x)$ is the graph of $y = x + 1$, but undefined at $x = 0$

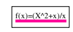


Figure 5

c. Functions of the form $\sqrt[n]{f(x)}$; $n \in \mathbb{N}$ where the function $f(x)$ is either a polynomial or a rational function.

The domain of this type of functions can be considered as follows

<u>Case1</u> n is odd

The domain of $\sqrt[n]{f(x)}$ is exactly the domain D_f of $f(x)$

<u>Case2</u> n is even

The domain of $\sqrt[n]{f(x)}$ is $D_f \cap \{x \mid f(x) \geq 0\}$

d. Functions formed by summation, multiplication and division of functions in part a. to c.

Below are some examples of functions in part c. and d.

1) $f(x) = x^{\frac{2}{3}}$ 2) $f(x) = \sqrt[4]{\dfrac{x}{x+1}}$

3) $f(x) = \dfrac{\sqrt{x}}{\sqrt{x}+1}$

<u>**Transcendental Functions**</u>

a. Exponential Functions are functions of the form

$$y = a^x, \text{ where } a > 0 \text{ and } a \neq 1$$

When $a > 1$, its graph can be shown in Figure 6 below.

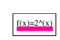


Figure 6

When $0 < a < 1$, its graph can be shown in figure 7 below

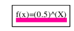


Figure 7

b. Logarithmic Function

Logarithmic function is an inverse of exponential function. Given an exponential function $y = a^x$. Then its inverse function is $x = a^y$ or we can rewrite it as $y = \log_a x$.

If $y = \log_a x$, $a > 1$, then its graph is shown in Figure 8.

If $y = \log_a x$, $0 < a < 1$, then its graph is shown in Figure 9.

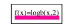


Figure 8

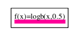


Figure 9

Some facts about logarithmic functions

1 . Domain of a logarithmic function is $\{x : x > 0\}$ and its range is $\{y : y \in \mathbb{R}\}$

2. A logarithmic function is a one-to-one function.

3. $\log_a 1 = 0$

4. Graph of $y = \log_a x$ is a reflection of the graph $y = a^x$ across the line $y = x$.

Remark: When $a = e$ (where $e = 2.71818...$= natural number) $y = e^x$ has the inverse $y = \log_e x$ which is normally written as $y = \ln x$ and it is called a natural logarithm.

The properties of $y = e^x$ and $y = \ln x$ are the same as of the following properties of $y = a^x$ and $y = \log_a x$ $(a > 0)$, respectively

Properties of logarithmic and exponential functions

Given positive numbers a,b where $a \neq 1, b \neq 1$ and $x, y \in R$

1. $a^x \cdot a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $a^x \cdot b^x = (ab)^x$ and $\dfrac{a^x}{b^x} = \left[\dfrac{a}{b}\right]^x$

4. $\left(a^x\right)^y = a^{xy}$

5. $a^{-x} = \dfrac{1}{a^x}$

6. If $x > 0, y > 0$, then $\log_a(xy) = \log_a x + \log_a y$

$$\log_a(\frac{x}{y}) = \log_a x - \log_a y$$

7. $\log_a x^r = r\log_a x$

8. $\log_a x = \dfrac{\log_b x}{\log_b a}$

9. $\log_a a = 1$

10. $\ln e^x = x$ and $e^{\ln x} = x \quad , x > 0$

11. $a^x = y$ and $x = \log_a y \quad , y > 0$

Example 12 Find the values of x

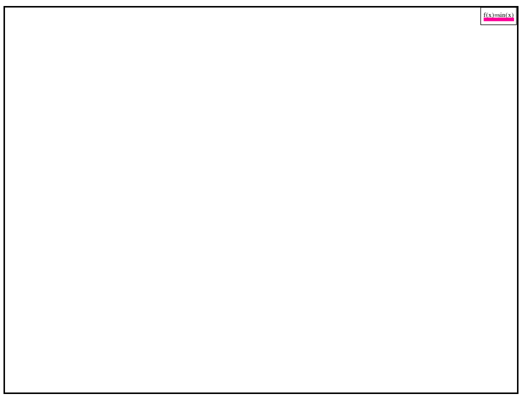
(a) $4 \cdot 3^x = 8 \cdot 6^x$

(b) $7^{x+2} = e^{17x}$

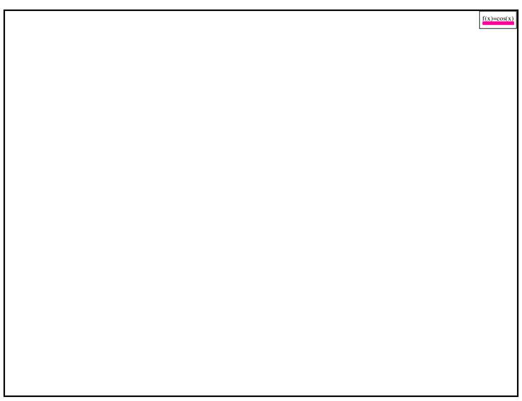
c. Trigonometric Function

$y = \sin x \qquad y = \cos x \qquad y = \tan x = \dfrac{\sin x}{\cos x}$

$y = \csc x = \dfrac{1}{\sin x} \qquad y = \sec x = \dfrac{1}{\cos x} \qquad y = \cot x = \dfrac{\cos x}{sinx}$

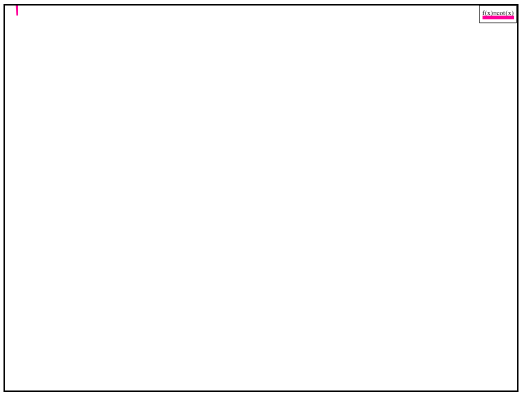


Graph of $y = \sin x$

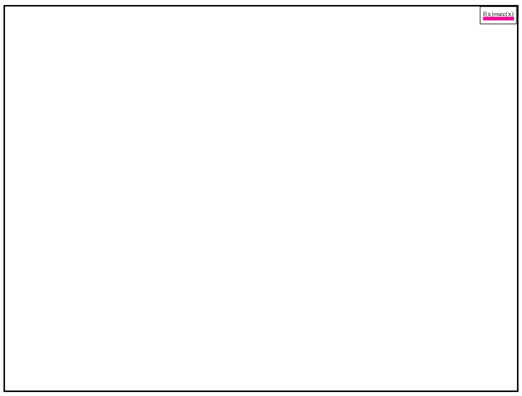


Graph of $y = \cos x$

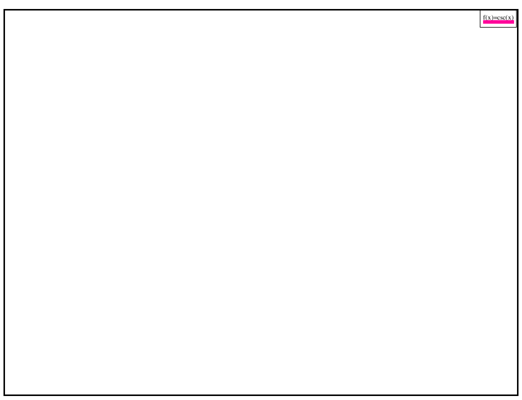
Graph of $y = \tan x$



Graph of $y = \cot x$



Graph of $y = \sec x$



Graph of $y = \csc x$

Normally, the inverse of a trigonometric function is not a function since each trigonometric function is not one-to-one. However, if we restrict the domain, we can make a one-to-one trigonometric function and define an inverse function as follows.

1) Restrict the domain of $y = \sin x$ to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Its inverse function is $y = \arcsin x$.

$y = \sin x$ $\qquad\qquad$ $y = \arcsin x$

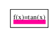
2) Restrict domain of $y = \cos x$ to $[0, \pi]$

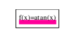
Its inverse function is $y = \arccos x$.

$y = \cos x$ $\qquad\qquad$ $y = \arccos x$

3) Restrict domain of $y = \tan x$ to $\left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$

Its inverse function is $y = \arctan x$.

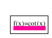


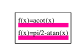


$y = \tan x$ $\qquad\qquad$ $y = \arctan x$

4) Restrict domain of $y = \cot x$ to $(0, \pi)$

Its inverse function is $y = \operatorname{arccot} x$.

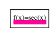


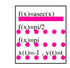


$y = \cot x$ $\qquad\qquad$ $y = \operatorname{arccot} x$

5) Restrict domain of $y = \sec x$ to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Its inverse function is $y = \operatorname{arcsec} x$.

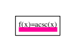


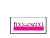


$y = \sec x$ $\qquad\qquad$ $y = \operatorname{arcsec} x$

6) Restrict domain of $y = \csc x$ to $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Its inverse function is $y = \operatorname{arccsc} x$.





$y = \csc x$ $\qquad\qquad$ $y = \operatorname{arccsc} x$

Exercises on Functions

1. Determine if the following are functions. Locate domain and range.

(a) $\{(1,3),(2,3),(3,4),(4,5)\}$

(b) $\{(x,y): y > 4x-1\}$

(c) $y = x^4 - 1$

(d) Let

x	y
15	2
2	13
13	13
5	3

2. Determine if each following function is either even or odd or neither.

(a) $f(x) = x^3 + 2x$

(b) $g(x) = \dfrac{8}{x^2 - 2}$

(c) $h(x) = 3x|x|$

(d) $k(x) = x + |x|$

3. What is the difference of $\sin x^2$, $\sin^2 x$ and $\sin(\sin x)$? Show in terms of composite functions.

Answers to Function Exercises

1. (a) yes $D = \{1, 2, 3, 4\}$ and $R = \{3, 4, 5\}$

   (b) no $D = R =$ all real numbers

   (c) yes $D = \mathbb{R}$ and $R = \{y : y \geq -1\}$

   (d) yes $D = \{2, 5, 13, 15\}$ and $R = \{2, 3, 13\}$

2. (a) odd (b) even

   (c) odd (d) neither

3. Let $f(x) = \sin x$ and $g(x) = x^2$

$\sin x^2 = f(g(x))$, $\sin^2 x = g(f(x))$, while

$\sin(\sin x) = f(f(x))$.

Limit and Continuity of Function with one variable

2.1 Limit of function

Let f be a function. The limit of $f(x)$ when x approaches to a is not the value of $f(a)$ but it is a value that $f(x)$ is approaching to (as x approaches to a). There are two types of the limit.

2.1.1 Limit of function as $x \to a$ (a is a real number.)

Suppose that $f(x) = 5x - 1$ and $g(x) = \ x$ defined by the largest integer which is less than or equal to x. For example,

$$g(4) = \ 4 \ = 4,\ g(3.8) = \ 3.8 \ = 3,\ g(-1.2) = \ -1.2 \ = -2.$$

For some values of x which approaches to $a = 1$, the value $f(x)$ and $g(x)$ are shown in Table 1.

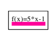
x	0.5	0.9	0.99	0.999	...	1.001	1.01	1.1
$f(x)$	1.5	3.5	3.95	3.995	...	4.005	4.05	4.5
$g(x)$	0	0	0	0	...	1	1	1

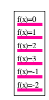
Table 1

We can see that when x approaches to $a = 1$, $f(x)$ gets closer and closer to the value 4. However, $g(x) = 1$ when $x \geq 1$ and $g(x) = 0$ when $x < 1$. Thus $g(x)$ does not approach to one number.

Therefore, we say that $f(x)$ has the limit equal to 4 as x approaches to 1 and $g(x)$ does not have a limit when x approaches to 1. We may write them as

$$\lim_{x \to 1} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1} g(x) \text{ does not exist.}$$





$f(x) = 5x - 1$ $\qquad\qquad$ $g(x) = \ x$

The graph of the function f shows that the value of $f(x)$ gets closer to 4 when x approaches to 1. But the graph of the function g jumps from $y = 0$ to $y = 1$ at $x = 1$. Thus $g(x) = \ x$ has no limit at $x = 1$.

Using this concept, one can define the limit as follows:

Definition If $f(x)$ gets closer to L when x approaches to a, we say that L is the limit of $f(x)$ when x approaches to a, denoted by $\lim_{x\to a} f(x) = L$.

The values of x approaches to a from two sides:

- x **approaches to** a **from the right side** is denoted by $x \to a^+$. In this case, we focus on x when $x > a$.
- x **approaches to** a **from the left side** is denoted by $x \to a^-$. In this case, we focus on x when $x < a$.

From the above example, we have $\lim_{x\to 1^+} x = 1$ but $\lim_{x\to 1^-} x = 0$ and $\lim_{x\to 1^+} 5x-1 = \lim_{x\to 1^-} 5x-1 = 4$.

We see that the function f has the same limit from both sides when x approaches to 1 and

(Right limit) $\lim_{x\to a^+} f(x) =$ (Left limit) $\lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$.

The following theorem guarantees the above remark.

Theorem 1 $\lim_{x \to a} f(x)$ exists and equals to L if

(1) both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and

(2) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$

Example 1 Compare $\lim_{x \to 0} \frac{x}{|x|}$ and $\lim_{x \to 0} \frac{x^2}{|x|}$.

Solution

Properties of limits

Let a, k, L and M be real numbers. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then,

1. $\lim_{x \to a} kf(x) = kL$,
2. $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$,
3. $\lim_{x \to a} f(x)g(x) = LM$,
4. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$,
5. If f is a polynomial function, then for any number a

   $\lim_{x \to a} f(x) = f(a)$,
6. $\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$ where n is a natural number.

Example 2 Evaluate $\lim_{x\to 0} \dfrac{x^3 - 3x^2 + 4}{\cos x}$.

Solution

Example 3 Let f be a function defined by $f(x) = \begin{cases} 2x^2 & , x < 0, \\ x & , 0 \le x < 1, \\ x+1 & , x \ge 1. \end{cases}$

Find the limits of $f(x)$ when x approaches 0 and 1.

Solution

Example 4 Evaluate $\lim\limits_{x \to 9} \left(2x^{\frac{3}{2}} - 9\sqrt{x} \right)^{\frac{1}{3}} \sin 2x.$

Solution

Sometimes, we find the limit by replacing x by a and may get the result in the form of $\frac{0}{0}$. So, we can use these two techniques to find the limit.

1) Factoring
2) Conjugating

Example 5 Calculate $\lim\limits_{x \to 3} \dfrac{x^3 - x^2 - 9x + 9}{x^2 - x - 6}$.

Solution

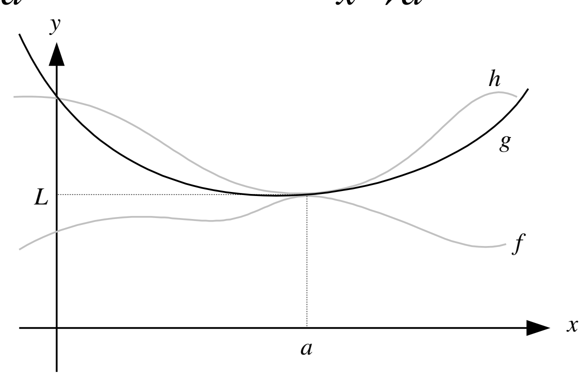
Example 6 Calculate $\lim\limits_{x \to 0^+} \dfrac{2\sqrt{x}}{\sqrt{16 + 2\sqrt{x}} - 4}$.

Solution

The following theorem is one of an important theorem that helps us to find the limit. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all values of x, $x \ne a$ at some points a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.



Example 7 Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{x^2}{1+\left(1+x^4\right)^{\frac{5}{2}}} = 0.$$

Example 8

1. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x\to 1} f(x)$.

2. Calculate $\lim_{x\to 0} x^2 \sin\dfrac{2}{x}$.

Solution

Theorem

1. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0$

Example 9 Use $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ to show that $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0.$

Proof $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = \lim_{x \to 0} \dfrac{\cos x - 1}{x} \left(\dfrac{\cos x + 1}{\cos x + 1} \right)$

$$= \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0.$$

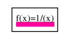
Example 10 Evaluate $\lim_{x \to 1} \dfrac{\sin(x-1)}{x^2 + x - 2}.$

Solution

2.1.2 Limit of function as $x \to \infty$ (infinity)

When the domain of a function f is unbounded, the values of $f(x)$ may get closer to one value when x increases unboundedly (written as $x \to +\infty$) or x decreases unboundedly (written as $x \to -\infty$).

Let $f(x) = \dfrac{1}{x}$. Its graph can be shown here.



Consider the value of $f(x)$ in the following table.

x	100	1000	10000	Increases unboundedly
$f(x) = \dfrac{1}{x}$	0.01	0.001	0.0001	$\ldots \to 0$
x	-100	-1000	-10000	Decreases unboundedly
$f(x) = \dfrac{1}{x}$	-0.01	-0.001	-0.0001	$\ldots \to 0$

Table 2

We see that, when $x \to +\infty$, the values of $f(x)$ get closer to 0 and $f(x) > 0$. So, we say that limit of $f(x)$ equals 0 as $x \to +\infty$, denoted by $\lim_{x \to +\infty} \frac{1}{x} = 0$. Also, when $x \to -\infty$, the values of $f(x)$ get closer to 0 as well, but $f(x) < 0$. We say that limit of $f(x)$ equals 0 as $x \to -\infty$ and denote it by $\lim_{x \to -\infty} \frac{1}{x} = 0$.

The above graph shows that $f(x) = \frac{1}{x}$ gets closer to x-axis as x increases to infinity and decreases to negative infinity, but it never hit the x-axis. We call a line that the graph gets closer to as an **asymptote** of the function.

Properties of infinite limits

Many properties of infinite limits are the same as those of limits at a finite number a.

Let k, L and M be real numbers. Suppose that $\lim\limits_{x\to+\infty} f(x) = L$ and $\lim\limits_{x\to+\infty} g(x) = M$. Then,

1. $\lim\limits_{x\to+\infty} k = k,$
2. $\lim\limits_{x\to+\infty} [f(x) \pm g(x)] = L \pm M$,
3. $\lim\limits_{x\to+\infty} f(x)g(x) = LM$,
4. $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0,$
5. $\lim\limits_{x\to+\infty} [f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}$ where n is positive and $L \geq 0,$
6. $\lim\limits_{x\to+\infty} \dfrac{1}{x^n} = 0$ where n is a positive integer.

All 6 properties are the same when we replace $x \to +\infty$ by $x \to -\infty$

Example 1 Calculate

a) $\lim\limits_{x\to+\infty} \dfrac{5}{x^3}$, b) $\lim\limits_{x\to-\infty} \dfrac{-3}{x^{\frac{2}{3}}}$, c) $\lim\limits_{x\to\infty} \dfrac{4^x-4^{-x}}{4^x+4^{-x}}$.

Solution

Example 2 Evaluate $\lim\limits_{x\to+\infty}\dfrac{\sqrt{3x^4+7x^2+6}}{4x^2-3x-6}$.

Solution

Example 3 Evaluate $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+3}}{x+3}$.

Solution

Example 4 Calculate $\lim_{x \to 2^+} \frac{x-3}{x-2}$.

Solution

Example 5 Calculate $\lim_{x \to 0^+} (x-1)\ln x$.

Solution

Limit of a function associating with the number e

For any constant a,

$$\lim_{x\to 0}(1+ax)^{1/x} = e^a \quad \text{and} \quad \lim_{x\to\infty}(1+\frac{a}{x})^x = e^a.$$

Example 6 Calculate $\lim_{x\to\infty}\left(\frac{x+4}{x+1}\right)^{x+1}$.

Solution

2.2 Continuity of Function

Definition A function f is continuous at $x=a$ if all of the three following conditions are satisfied:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists, (That is, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.)
3. $\lim_{x\to a} f(x) = f(a)$.

Remark: If at least one of the above conditions is not satisfied, then the given function is discontinuous at $x=a$.

Example 1 Let $f(x) = x^2 + 2x + 1$

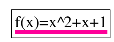
Consider the continuity of this function at $x = 0$:

1. $f(0) = 1$ exists,

2. $\lim\limits_{x \to 0} f(x) = 1$ exists, and

3. $\lim\limits_{x \to 0} f(0) = f(0) = 1$.

Thus, $f(x)$ is continuous at $x = 0$. Its graph is here.



Example 2 Let f be a function defined by

$$f(x)=\begin{cases}\dfrac{1-x^2}{1-x} & ,x\neq 1,\\ 3 & ,x=1.\end{cases}$$

Determine if this function is continuous at $x=1$.

Solution

Example 3 Let f be a function defined by

$$f(x) = \begin{cases} bx^2 + 1 & , x < -2, \\ x & , x \geq -2. \end{cases}$$

Find b that makes this function continuous at $x = -2$.

Solution

Three Types of Discontinuities

Consider the continuity of $f(x)$ at $x = a$.

1. Removable discontinuity

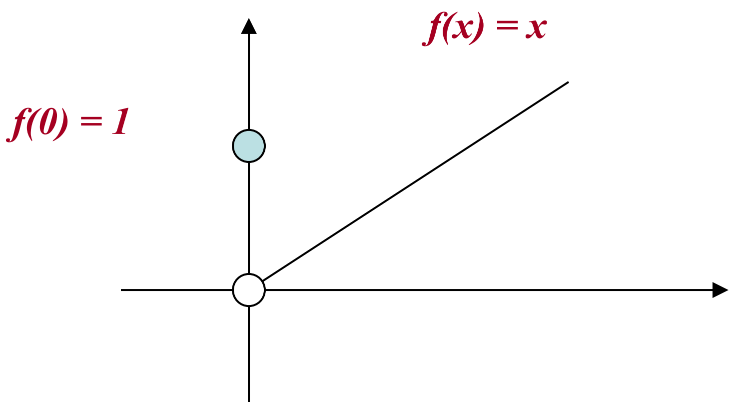
It occurs when

(i) $\lim_{x \to a} f(x)$ exists, but not equal to $f(a)$ or

(ii) $f(a)$ is undefined.

For example, $f(x) = \begin{cases} 1 & , x = 0 \\ x & , x \neq 0 \end{cases}$ has a removable discontinuity

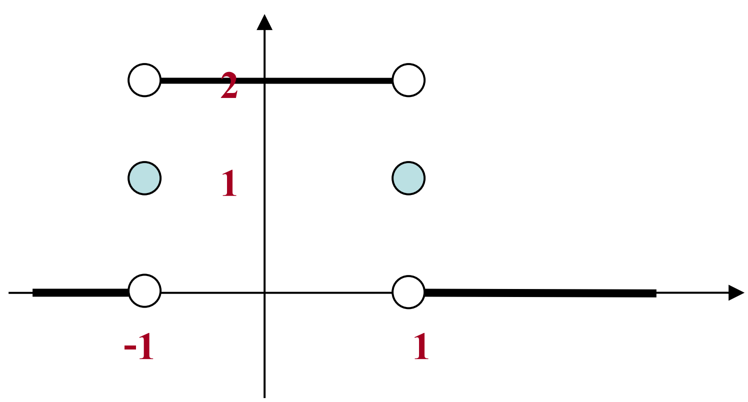
at $x = 0$ as show in the Figure below.



2. Jump discontinuity or Ordinary discontinuity

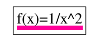
It occurs when $\lim_{x\to a} f(x)$ does not exist due to the **unequal** existence of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. For example, the function

$$f(x)=\begin{cases} 2 & ,|x|<1 \\ 1 & ,|x|=1 \\ 0 & ,|x|>1 \end{cases}$$ has a jump discontinuity at $x=1,-1$.



3. Infinite discontinuity

It occurs when at least one of the left limit or the right limit does not exist. For example, $f(x) = \dfrac{1}{x^2}$ has an infinite discontinuity at $x = 0$ as shown here.



Algebraic properties of functions on the continuity

1. If f and g are continuous at $x = a$, then $f \pm g$, $f.g$, $\dfrac{f}{g}$ $(g(a) \neq 0)$ and kf (k is a constant) are also continuous at $x = a$.

2. If f is continuous at $x = b$ and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} (f \circ g)(x) = f(b)$.

3. If g is continuous at $x = a$ and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at $x = a$.

Example 4 Let f be a function defined by

$$f(x) = \frac{2(x^2 + 4x + 2)}{(x^2 - 9)(x - 1)}.$$

Locate where this function is continuous.

Definition If the function f is continuous everywhere in the interval (a,b), we say that f is continuous on (a,b).

Definition A function f is continuous in $[a,b]$ where $a < b$ if

1. $f(x)$ is continuous on (a,b),
2. $\lim\limits_{x \to a^+} f(x) = f(a)$ and
3. $\lim\limits_{x \to b^-} f(x) = f(b)$.

Example 5 Let g be a function defined by $g(x) = \sqrt{\dfrac{3-x}{4+x}}$.

Locate where this function is continuous.

Solution

Limit and Continuity Exercises

1. Find the limits of the following functions.

(a) Let $f(x) = \dfrac{x^3}{|x-1|}$. Find $\lim_{x\to 1} f(x)$.

(b) $\lim_{x\to 1} 3x \quad x$

(c) Let $g(x) = \begin{cases} x^2 - 2; & x > 0 \\ -2 - x; & x < 0 \end{cases}$. Calculate $\lim_{x\to 0} g(x)$.

(d) $\lim_{x\to\infty} \dfrac{6\sqrt{x^2-3}}{2x-1}$

(e) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+7}}{2x-4}$

2. Make the following functions continuous at $x = a$.

(a) $f(x) = \dfrac{\sqrt{3x^2}}{2|x|}$, $a = 0$

(b) $g(x) = \dfrac{x^n - 1}{x-1}, \quad n \in \mathbb{Z}^+, \quad a = 1$

3. Locate domain that makes the following function continuous.

(a) $h(x) = \dfrac{2}{x^2 + 3x - 28}$

(b) $k(x) = \sqrt[3]{(x-a)(x-b)}$

4. Find k that makes $f(x)=\begin{cases}\dfrac{x^2-5x+6}{x-2}; & x\neq 2\\ kx-3 \quad ; & x=2\end{cases}$ continuous everywhere.

5. Find k that makes each following limit exists.

(a) $\lim\limits_{x\to 1}\dfrac{x^2-kx+4}{x-1}$

(b) $\lim\limits_{x\to\infty}\dfrac{x^4+3x-5}{2x^2-1+x^k}$

(c) $\lim\limits_{x\to-\infty}\dfrac{e^{2x}-5}{e^{kx}+4}$

6. Compute the following limits.

(a) $\lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$

(b) $\lim\limits_{h\to 0}\dfrac{1/(1+h)-1}{h}$

(c) $\lim\limits_{h\to 0}\dfrac{\sqrt{4+h}-2}{h}$

7. Compute the following limits.

(a) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\sin x}$

(b) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

(c) $\lim\limits_{x\to\infty} x\sin\dfrac{\pi}{x}$

Answers to limit and continuity exercises

1. (a) $+\infty$

   (b) Does not exist

   (c) –2

   (d) 3

   (e) –1 / 2

2. (a) add $f(0) = \dfrac{\sqrt{3}}{2}$

   (b) add $g(1) = n$

3. (a) $x \neq -7, 4$

   (b) $(-\infty, \infty)$

4. 1

5. (a) 5

   (b) greater than or equal to 4

   (c) less than or equal to 2

6. (a) 6

   (b) –1

   (c) –1/16

7. (a) 0

   (b) 3/5

   (c) π

The Derivative

Definition of Derivative

Let L be a line connecting points P and Q on the curve of $y = f(x)$ as shown in Figure 1 here.

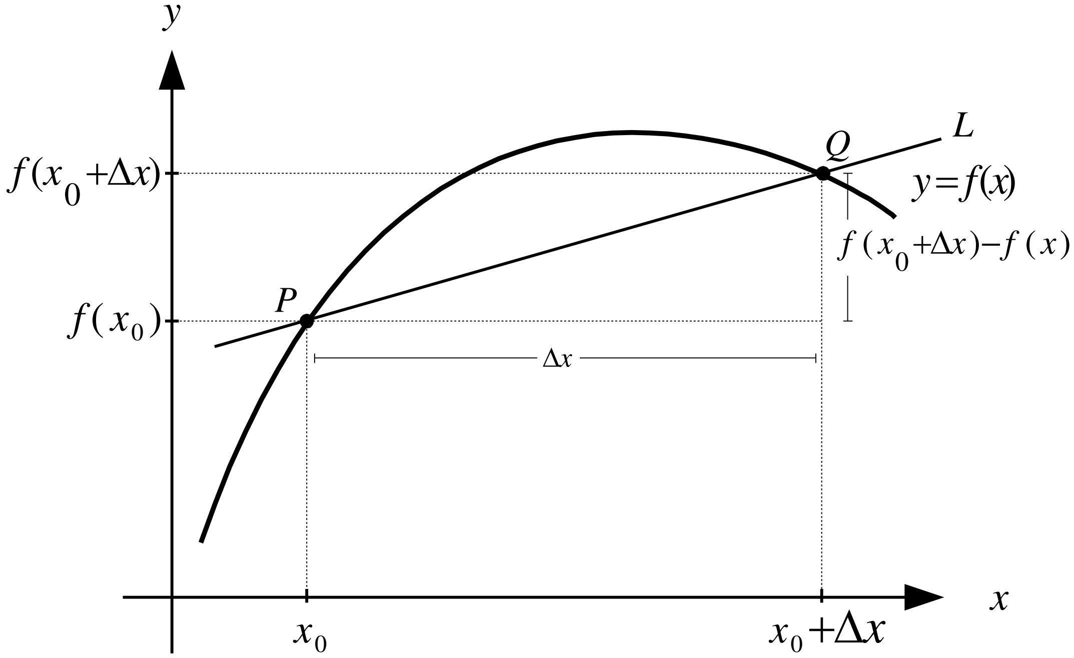


Figure 1

From Figure 1, consider the slope of line L:

$$\text{Slope of the line } L = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{\text{Changed values of } f}{\text{Changed values of } x} \qquad [\text{From } x \text{ to } x+\Delta x]$$

$=$ Average rate of change of f from x to $x+\Delta x$.

Next, consider the slope of line L when point Q is moved closer and closer to point P along the curve of $y = f(x)$ as in Figure 2.

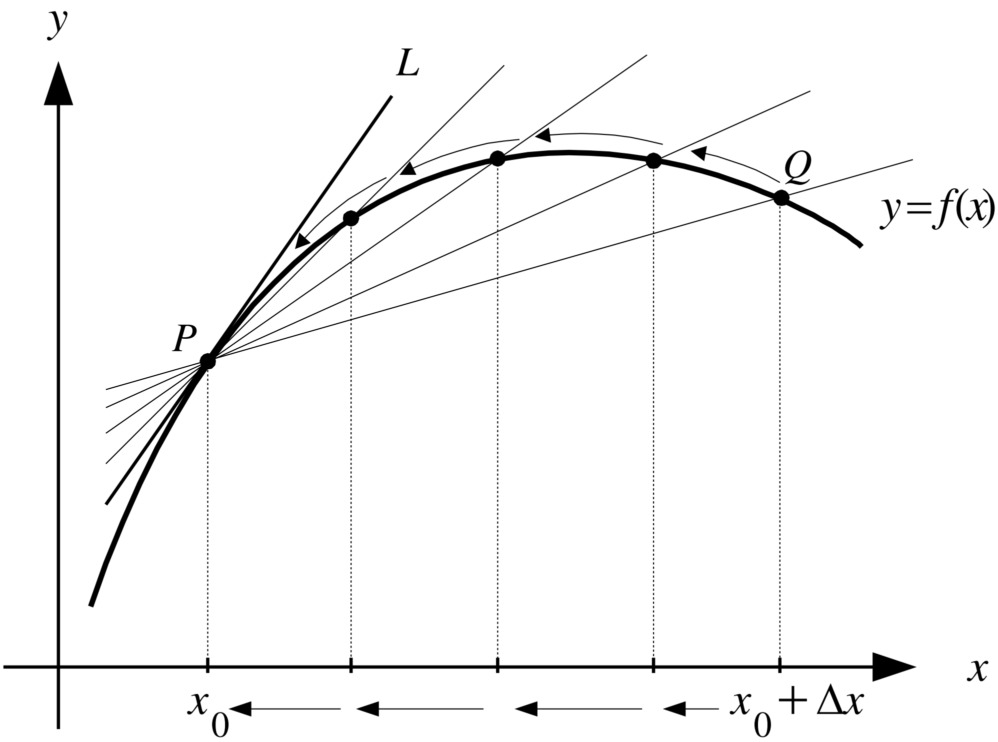


Figure 2

As Q is moved closer to P or as Δx gets smaller to 0, the slope of line L gets closer to the slope of a tangent line of the curve $y = f(x)$ at the point P.

Slope of the tangent line of $y = f(x)$ at the point P

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

= Instantaneous rate of change of f at $x = x_0$.

Definition 1:

Let f be a function defined on an open interval containing x.

Then the **derivative of** f **at** x**,** denoted by $f'(x)$, is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Remark

- Beside the notation $f'(x)$, the following notations are also used to denote the derivative of f at x:

$$\frac{dy}{dx}, \ \frac{d}{dx} f(x) \ \text{ or just } \ y'.$$

For the derivative of f at $x = a$, we may write it as

$$f'(a) \ \text{ or } \ \left.\frac{dy}{dx}\right|_{x=a}.$$

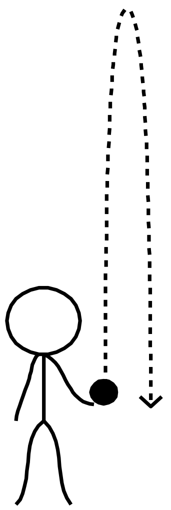
- If f is continuous at $x = a$, then

$f'(a) =$ The slope of tangent line of $y = f(x)$ at $x = a$

$$= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

◈ Throw an object vertically. At time t seconds, the object is at the position $s(t) = -4.9t^2 + 49t$.



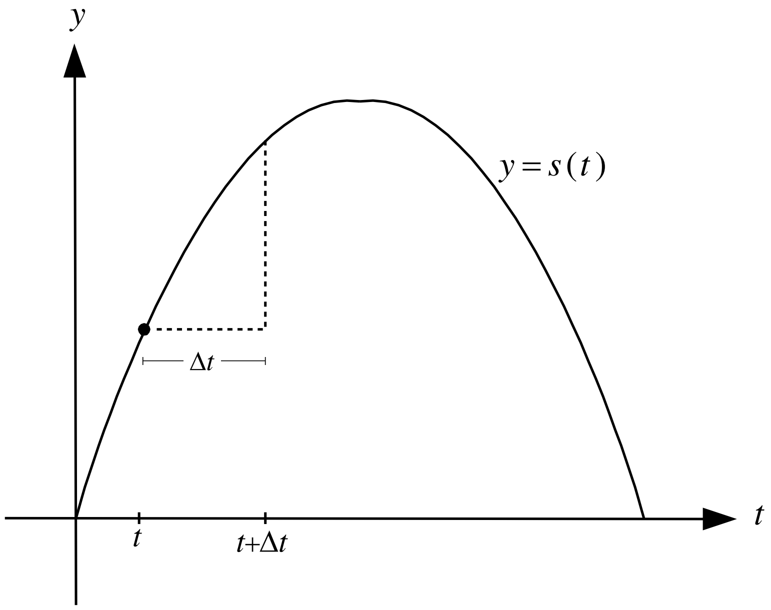


Figure 3 shows the object's position at time t

■ **Average velocity of the object from time t to $t + \Delta t$** is the average rate of distance change from time t to $t + \Delta t$ seconds.

Average velocity from time t **to** $t + \Delta t$ $= \dfrac{s(t + \Delta t) - s(t)}{\Delta t}$.

■ **Instantaneous velocity of the object at time t** is the instantaneous rate of distance change at time t seconds as $\Delta t \to 0$

$$\text{Instantaneous velocity at time } t = \lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{ds}{dt}.$$

Thus, at time t seconds, the object is thrown from the ground with the velocity $v(t) = \dfrac{ds}{dt} = -9.8t+49$.

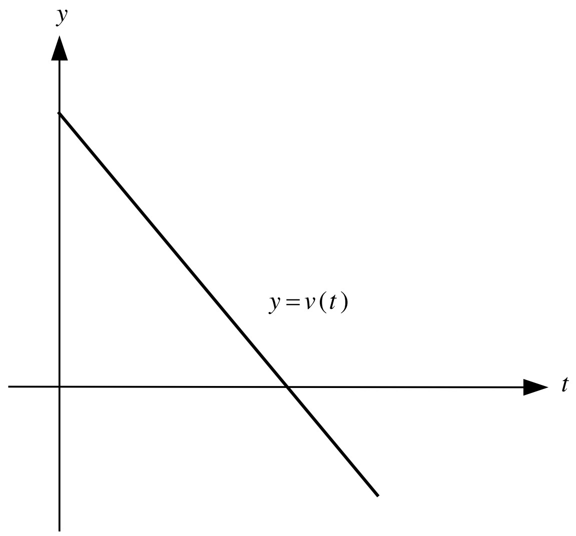


Figure 4 shows the velocity of the object at time t seconds

<u>Example</u> Let $f(x) = x^2 + 4$.

(a) Find derivative of f.

(b) Find derivative of f at $x = 5$.

Example Find the slope of the tangent line of $y = \dfrac{1}{x^2}$ at the point $(1,1)$.

Example An object moves horizontally. At time t seconds, the object has distance $s = 5 - 2t + t^2$ meters. Compute

a. the average velocity of the object from 1 to 3 seconds,

b. the instantaneous velocity of the object at t seconds,

c. the instantaneous velocity of the object at $t = 1$ second.

Theorem

If a function f has derivative (or say "is differentiable") at $x = a$ ($f'(a)$ exists as a real number), then f is continuous at $x = a$.

Proof Consider

$$\begin{aligned}\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} f(x) - \lim_{x\to a} f(a)\\ &= \lim_{x\to a}\left[f(x) - f(a)\right]\\ &= \lim_{x\to a}\frac{(x-a)\left[f(x)-f(a)\right]}{x-a}\\ &= \lim_{x\to a}(x-a)\cdot\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= 0\cdot f'(a)\\ &= 0\,.\end{aligned}$$

That is, $\lim_{x\to a} f(x) = f(a)$.

Thus, f is continuous at $x = a$.

$\square$

Remark: The converse may not be true. That is, if f is continuous at $x = a$, then f **may or may not** be differentiable at $x = a$.

Example Find the derivative of $f(x) = |x|$ at $x = 0$.

<u>Example</u> Let $f(x)=\sqrt{x}$. Compute $f'(x)$.

Derivative Formulas

Finding a derivative by using the definition is quite complicated and time consuming. However, there are several theorems and formulas to help us calculating derivatives easier and faster. Consider the following formulas.

Formulas

Let u, v be functions of x and c, n are some constants.

1. $$\frac{dc}{dx} = 0 \quad \text{and} \quad \frac{dx}{dx} = 1$$

2. $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

3. $$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

4. $$\frac{du^n}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

$$\frac{dx^n}{dx} = nx^{n-1} \cdot \frac{dx}{dx} = nx^{n-1}$$

5. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

6. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example Find the derivatives of the following functions .

(a) $f(x) = 2x^9 - \dfrac{5}{x} + 7x - 1$

(b) $g(x) = \sqrt[3]{x^4} + \dfrac{1}{\sqrt[3]{x^4 + 2}}$

(c) $p(x) = \left(2x^7 - x^{-1}\right)\left(5x^9 - 10\right)$

Example Find the derivatives of the following functions.

(a) $y = \dfrac{(2x-1)^2}{x^2+7}$

Solution $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\dfrac{(2x-1)^2}{x^2+7}\right]$

$$= \frac{(x^2+7)\dfrac{d}{dx}(2x-1)^2 - (2x-1)^2\dfrac{d}{dx}(x^2+7)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\dfrac{d}{dx}(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\cdot 2 - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{4(x^2+7)(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

(b) $f(x)=\dfrac{x^7-4\sqrt{x}-2}{x^2}$

Solution $f'(x)=\dfrac{d}{dx}\left[\dfrac{x^7-4\sqrt{x}-2}{x^2}\right]$

$$=\frac{d}{dx}\left(x^5-4x^{-\frac{3}{2}}-2x^{-2}\right)$$

$$=\frac{dx^5}{dx}-4\frac{dx^{-\frac{3}{2}}}{dx}-2\frac{dx^{-2}}{dx}$$

$$=5x^4-4\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}-2\left(-2\right)x^{-3}$$

$$=5x^4+6x^{-\frac{5}{2}}+4x^{-3}$$

(c) $r(t)=\left(4t^3-7t^{-6}\right)^{12}$

Solution

$$r'(t)=\frac{d}{dt}\left(4t^3-7t^{-6}\right)^{12}$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\cdot\frac{d}{dt}\left(4t^3-7t^{-6}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\cdot\frac{d}{dt}\left(4t^3-7t^{-6}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\left(4\frac{dt^3}{dt}-7\frac{dt^{-6}}{dt}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\left(12t^2+42t^{-7}\right)$$

$$=72\left(4t^3-7t^{-6}\right)^{11}\left(2t^2+7t^{-7}\right)$$

(d) $y=\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$$

$$=\left(5x^{10}-8\right)\frac{d}{dx}\left(x^2+9\right)^{\frac{1}{4}}+\left(x^2+9\right)^{\frac{1}{4}}\frac{d}{dx}\left(5x^{10}-8\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\frac{d}{dx}\left(x^2+9\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\left(2x\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\frac{1}{2}x\left(5x^{10}-8\right)\left(x^2+9\right)^{-\frac{3}{4}}+50x^9\left(x^2+9\right)^{\frac{1}{4}}$$

(e) $y=\dfrac{4x^2+5}{3x-1}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{4x^2+5}{3x-1}\right]$$

$$=\frac{(3x-1)\frac{d}{dx}\left(4x^2+5\right)-(4x^2+5)\frac{d}{dx}\left(3x-1\right)}{(3x-1)^2}$$

$$=\frac{(3x-1)\left(8x\right)-(4x^2+5)\left(3\right)}{(3x-1)^2}$$

$$=\frac{12x^2-8x-15}{(3x-1)^2}$$

(f) $g(x)=\sqrt{\dfrac{8x^4}{2-x^7}}$

Solution

<u>Example</u> Find an equation of the tangent line of $y = \dfrac{1}{\sqrt{x^4 + 8x}}$ at the point $x = 1$.

The Chain Rule

If functions f and g are differentiable and $F = f \circ g$ is a composite function given by $F(x) = f(g(x))$, then F is also differentiable at x and

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In other words, if $y = f(u)$ and $u = g(x)$ are differentiable, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Remark:

In case of more than two functions composed, we can extend the chain rule as follows.

Let $y = f(u)$, $u = g(x)$ and $x = h(t)$ be differentiable, then

$$\boxed{\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}}$$

Example Find $\dfrac{dy}{dx}$ where $y=\sqrt{x^{\frac{2}{3}}+x^{\frac{4}{3}}-1}$

Solution From $y = \sqrt{x^{\frac{2}{3}}+x^{\frac{4}{3}}-1} = \sqrt{x^{\frac{2}{3}}+(x^{\frac{2}{3}})^2-1}$,

we consider $y=\sqrt{u}$, $u=v+v^2-1$ and $v=x^{\frac{2}{3}}$.

Apply the following chain rule: $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$

where
$$\frac{dy}{du} = \frac{du^{\frac{1}{2}}}{du} = \frac{1}{2}u^{-\frac{1}{2}},$$
$$\frac{du}{dv} = \frac{d}{dv}(v+v^2-1) = 1+2v,$$

and
$$\frac{dv}{dx} = \frac{dx^{\frac{2}{3}}}{dx} = \frac{2}{3}x^{-\frac{1}{3}}.$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}}\cdot(1+2v)\cdot\frac{2}{3}x^{-\frac{1}{3}}\\ &= \frac{1}{3}\cdot(x^{\frac{2}{3}}+x^{\frac{4}{3}}-1)^{-\frac{1}{2}}\cdot(1+2x^{\frac{2}{3}})\cdot x^{-\frac{1}{3}}\\ &= \frac{1}{3}\cdot(x^{\frac{2}{3}}+x^{\frac{4}{3}}-1)^{-\frac{1}{2}}\cdot(x^{-\frac{1}{3}}+2x^{\frac{1}{3}})\ .\end{aligned}$$

Example Let $y = \dfrac{u^2}{u^3 - 16}$, $u = 3x^2 - 8$ and $x = \sqrt[4]{t+5}$.

Find $\dfrac{dy}{dt}$ at $t = 11$.

Derivatives of Trigonometric Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$$
2. $$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$$
3. $$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$$
4. $$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$$
5. $$\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$$
6. $$\frac{d}{dx}\csc u = -\csc u \cot u \cdot \frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = (x^4 + 1)\tan x$,

<u>Solution</u>

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^4 + 1)\tan x \\
&= (x^4 + 1)\frac{d}{dx}\tan x + \tan x \frac{d}{dx}(x^4 + 1) \\
&= (x^4 + 1)\sec^2 x + \tan x(4x^3) \\
&= (x^4 + 1)\sec^2 x + 4x^3 \tan x
\end{aligned}$$

(b) $y = \dfrac{\sin(2x)}{7 - \cos(3x)}$,

(c) $y = \cot \sqrt[3]{4 - x^3}$,

(d) $\frac{d}{dx}\left[\sec(2x) + \tan(2x)\right]^3$,

(e) $y = 8 + 3\cos(x^4)\sec(7x)$.

<u>Example</u> Find the tangent line equation of $y = \cos(x)$ at $x = \frac{3\pi}{2}$.

<u>Solution</u> At $x = \frac{3\pi}{2}$, we have $y = 0$.

Consider $\frac{dy}{dx} = \frac{d}{dx}\cos(x) = -\sin(x)$.

Thus, the slope of the tangent line at $x = \frac{3\pi}{2}$ equals to

$$\left.\frac{dy}{dx}\right|_{x=\frac{3\pi}{2}} = -\sin(\frac{3\pi}{2}) = 1.$$

Therefore, the equation of the tangent line at $x = \frac{3\pi}{2}$ is

$$(y-0) = 1(x - \frac{3\pi}{2}) \quad \text{or}$$

$$x - y = \frac{3\pi}{2}.$$

Derivatives of Logarithmic Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}$$

2. $$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = \log_3(7x^4+1)$,

(b) $y = [\ln(3 - x^2)]^4$,

(c) $y = \ln\left[\dfrac{(8x-9)^4}{\sqrt{1+x^6}}\right]$.

Derivatives of Exponential Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}a^u = a^u \ln a \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

<u>Example</u> Find $f'(x)$ where

(a) $f(x) = 10^{\sin(4x)}$,

(b) $f(x) = e^{5x} \sin(\ln x)$,

(c) $f(x) = e^{(x^2-3)\tan x}$.

<u>Example</u> Let $y = \sqrt{e^{8x} + 3e^{-8x}}$. Find $\dfrac{dy}{dx}$ at $x = 0$.

Derivatives of Inverse Trigonometric Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

2. $$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

3. $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

4. $$\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

5. $$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

6. $$\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

Proof 1. We want to show that $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$.

Let $\quad y = \sin^{-1} u$.

That is, $\quad u = \sin y$

$$\frac{du}{dx} = \frac{d}{dx}\sin y$$

$$\frac{du}{dx} = \cos y \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = \cos y \cdot \frac{dy}{dx} \ .$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\cos y} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} \cdot \frac{du}{dx} \ .$$

Therefore, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$.

* Formulas 2-6 can be proven analogously to the formula 1 above.

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = \sin^{-1}(1 + x^2)$,

(b) $y = [1 + \cos^{-1}(\sqrt{x})]^6$,

(c) $y = e^{\sec^{-1}(\sqrt{x})}$,

$$\frac{dy}{dx} = \frac{d}{dx} e^{\sec^{-1}(\sqrt{x})}$$

$$= e^{\sec^{-1}(\sqrt{x})} \frac{d}{dx} \sec^{-1}(\sqrt{x})$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{(\sqrt{x})^2 - 1}} \cdot \frac{d}{dx}\sqrt{x}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{dx^{\frac{1}{2}}}{dx}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\sec^{-1}(\sqrt{x})}}{2x\,\sqrt{x-1}}$$

(d) $y = \tan^{-1}\left[\dfrac{1-x}{2+x}\right]$.

Solution

Derivatives of Hyperbolic Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sinh u = \cosh u \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}\cosh u = \sinh u \cdot \frac{du}{dx}$$

3. $$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

4. $$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$

5. $$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \cdot \frac{du}{dx}$$

6. $$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \cdot \frac{du}{dx}$$

Example Find the following derivatives.

(a) $\dfrac{d}{dx}\cosh(9x^2 - 2) =$

(b) $\dfrac{d}{dx}\ln(\tanh(x^3)) =$

(c) $\dfrac{d}{dx}\left(e^{7x}\sinh^3(5x)\right)=$

(d) $\frac{d}{dx}\left[\frac{\sinh x}{\cosh x-1}\right]$

$$= \frac{(\cosh x-1)\frac{d}{dx}\sinh x-\sinh x\frac{d}{dx}(\cosh x-1)}{(\cosh x-1)^2}$$

$$= \frac{(\cosh x-1)\cdot\cosh x-\sinh x(\sinh x)}{(\cosh x-1)^2}$$

$$= \frac{\cosh^2 x-\cosh x-\sinh^2 x}{(\cosh x-1)^2}$$

$$= \frac{1-\cosh x}{(\cosh x-1)^2}$$

$$= \frac{1}{1-\cosh x}$$

(e) $\frac{d}{dx}\sinh^5(e^x+1)$

Solution

(f) $\frac{d}{dx}\left(x^9+\sin(\coth 2x)\right)$.

Solution

Let a, b be constants, and u, v functions of x.

Consider the following derivatives:

1. $\frac{d}{dx}(a^b) \quad = \quad 0$

2. $\frac{d}{dx}[u]^a \quad = \quad a\,u^{a-1}\cdot\frac{du}{dx}$

3. $\frac{d}{dx}[a^u] \quad = \quad a^u \ln a\cdot\frac{du}{dx}$

4. $\frac{d}{dx}[u]^v \quad = \quad \ldots \qquad .$

Logarithmic Derivative

It is used to find derivative of a function in a form of $[u(x)]^{v(x)}$ and when the function consists of several products or quotients of functions. The process is as follows.

1. Take natural log both sides.
2. Apply properties of logarithm.
3. Find derivatives of both sides.
4. Solve equation for $\frac{dy}{dx}$.

Example Find $\dfrac{dy}{dx}$ where $y = x^{\sin(3x)}, \quad x > 0.$

Example Find $\dfrac{dy}{dx}$ where $y = (\sin^2 x + 4)^x$.

Solution

Take $\ln$ both sides of the equation

$$\ln y = \ln(\sin^2 x + 4)^x$$

$$\ln y = x \ln(\sin^2 x + 4)$$

Take derivative with respect to x both sides

$$\frac{d}{dx} \ln y = \frac{d}{dx} x \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = x \frac{d}{dx} \ln(\sin^2 x + 4) + \ln(\sin^2 x + 4) \frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\sin^2 x + 4} \cdot \frac{d}{dx}(\sin^2 x + 4) + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^2 x + 4} \cdot 2 \sin x \cos x + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)$$

Thus,

$$\frac{dy}{dx} = y\left[\frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right]$$

$$= (\sin^2 x + 4)^x \left[\frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right].$$

<u>Example</u> Find derivative of $y = x^{3x}\sqrt{\dfrac{(x^2+3)(6+x^4)}{3x+4}}$.

Solution

Exercise 1

1. Use definition to find $f'(x)$ where

(1.1) $f(x) = \pi$ (1.2) $f(x) = 4x - 3$

(1.3) $f(x) = 2 - x^2$ (1.4) $f(x) = (2 - x)^2$

(1.5) $f(x) = x^3 - 9$ (1.6) $f(x) = \dfrac{1}{(5x-1)^2}$

(1.7) $f(x) = \sqrt{x}$ (1.8) $f(x) = \dfrac{1}{\sqrt{x}}$

(1.9) $f(x) = \dfrac{2x-1}{2x+1}$ (1.10) $f(x) = \dfrac{1}{\sqrt{1+2x}}$.

Answers

(1.1) 0 (1.5) $3x^2$ (1.8) $-\dfrac{1}{2x\sqrt{x}}$

(1.2) 4 (1.6) $-10(5x-1)^{-3}$ (1.9) $\dfrac{4}{(2x+1)^2}$

(1.3) $-2x$ (1.7) $\dfrac{1}{2\sqrt{x}}$ (1.10) $-\dfrac{1}{2(2+x)^{\frac{3}{2}}}$

(1.4) $2x - 4$

2. Find $f'(a)$ if it exists where

(2.1) $f(x) = |x^2 - 9|$; $a = 3$

(2.2) $f(x) = \dfrac{x-4}{|x-4|}$; $a = 4$

(2.3) $f(x) = \begin{cases} x^2 , & x \geq 0 \\ 2x , & x < 0 \end{cases}$; $a = 0$

(2.4) $f(x) = \begin{cases} 1 - x^2 , & x \leq 1 \\ 2 - 2x , & x > 1 \end{cases}$; $a = 1$.

Answers (2.1) doesn't exist (2.3) doesn't exist

(2.2) 0 (2.4) -2

3. A ball is being inflated. Let V be the volume of the ball in cm^3 and r be the ball's radius in cm such that $V = \dfrac{4}{3}\pi r^3$. Find

(3.1) The average rate of volume's change with respect to radius when the radius changes from 6 cm to 9 cm.

(3.2) The instantaneous rate of volume's change with respect to radius when the radius is 9 cm.

Answers (3.1) 228π (3.2) 324π

Exercise 2

1. Find the derivatives of the following functions.

(1.1) $y = 7 + 9x - 7x^3 + 4x^7$ (1.2) $y = \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3}$

(1.3) $y = 2\sqrt{x} + 6\sqrt[3]{x} - 2\sqrt{x^3}$ (1.4) $y = \sqrt[3]{3x^2} - \frac{1}{\sqrt{5x}}$

(1.5) $y = (x^5 - 4x)^{43}$ (1.6) $y = \frac{3}{(a^2 - x^2)^2}$

(1.7) $y = \sqrt{x^2 + 6x + 3}$ (1.8) $y = \frac{3 - 2x}{3 + 2x}$

(1.9) $y = (x^2 + 4)^2 (2x^3 - 1)^3$ (1.10) $y = \frac{x^2}{\sqrt{4 - x^2}}$

(1.11) $y = \frac{x^3 - 2x\sqrt{x}}{x}$ (1.12) $y = \frac{1}{x^4 + x^2 + 1}$

(1.13) $y = \frac{x}{x + \frac{a}{x}}$

Answers

(1.1) $9 - 21x^2 + 28x^6$ (1.2) $-\frac{1}{x^2} - \frac{6}{x^3} - \frac{6}{x^4}$

(1.3) $\frac{1}{\sqrt{x}} + \frac{2}{\sqrt[3]{x^2}} - \frac{3}{\sqrt{x}}$ (1.4) $\frac{2}{\sqrt[3]{9x}} + \frac{1}{2x\sqrt{5x}}$

(1.5) $43(5x^4 - 4)(x^5 - 4x)^{42}$ (1.6) $\frac{12x}{(a^2 - x^2)^3}$

(1.7) $\frac{x+3}{\sqrt{x^2 + 6x + 3}}$ (1.8) $\frac{-12x}{(3 + 2x)^2}$

(1.9) $2x(x^2 + 4)(2x^3 - 1)^2 (13x^3 + 36x - 2)$

(1.10) $\dfrac{8x - x^3}{(4 - x^2)^{\frac{3}{2}}}$

(1.11) $2x - \dfrac{1}{\sqrt{x}}$

(1.12) $\dfrac{-(4x^3 + 2x)}{(x^4 + x^2 + 1)^2}$

(1.13) $\dfrac{2ax}{(x^2 + a)^2}$

2. Find the equation of a tangent line of $y = \dfrac{2x}{x+1}$ at $(1,1)$.

Answer $y = \dfrac{x+1}{2}$

3. Find the equation of a tangent line of $y = x + \sqrt{x}$ at $(1,2)$.

Answer $y = \dfrac{3}{2}x + \dfrac{1}{2}$

Exercise 3

1. Find the derivatives of the following functions.

(1.1) $f(x) = x - 3\sin x$

(1.2) $y = \sin x + 10\tan x$

(1.3) $g(t) = t^3 \cos t$

(1.4) $h(\theta) = \theta \csc\theta - \cot\theta$

(1.5) $y = \dfrac{x}{\cos x}$

(1.6) $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta}$

(1.7) $y = \sqrt{\sin x}$

(1.8) $y = \cos(a^3 + x^3)$

(1.9) $y = \cot(\dfrac{x}{2})$

(1.10) $y = \sin(x\cos x)$

(1.11) $y = \tan(\cos x)$

(1.12) $y = (1 + \cos^2 x)^6$

(1.13) $y = \sec^2 x + \tan^2 x$ (1.14) $y = \cot^2(\sin\theta)$

(1.15) $y = \sin(\tan\sqrt{\sin x})$

Answer

(1.1) $f'(x) = 1 - 3\cos x$ (1.2) $y' = \cos x + 10\sec^2 x$

(1.3) $g'(t) = 3t^2 \cos t - t^3 \sin t$

(1.4) $h'(\theta) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

(1.5) $y' = \dfrac{\cos x + x\sin x}{\cos^2 x}$ (1.6) $f(\theta) = \dfrac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$

(1.7) $y' = \dfrac{\cos x}{2\sqrt{\sin x}}$ (1.8) $y' = 3x^2 \sin(a^3 + x^3)$

(1.9) $y' = \dfrac{1}{2}\csc^2(\dfrac{x}{2})$

(1.10) $y' = (\cos x - x\sin x)\cos(x\cos x)$

(1.11) $y' = -\sin x \sec^2(\cos x)$

(1.12) $y' = -12\cos x\sin x(1+\cos^2 x)^5$

(1.13) $y' = 4\sec^2 x\tan x$

(1.14) $y' = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$

(1.15) $y' = \cos(\tan\sqrt{\sin x})(\sec^2\sqrt{\sin x})(2\sqrt{\sin x})(\cos x)$

2. Find the equation of a tangent line of each function at a given point.

(2.1) $y = \tan x$ at $(\frac{\pi}{4}, 1)$

(2.2) $y = x + \cos x$ at $(0,1)$

(2.3) $y = x\cos x$ at $(\pi, -\pi)$

(2.4) $y = \sin(\sin x)$ at $(\pi, 0)$

(2.5) $y = \tan(\frac{\pi x^2}{4})$ at $(1,1)$

Answer

(2.1) $y = 2x + 1 - \frac{\pi}{2}$ (2.2) $y = x + 1$ (2.3) $y = -x$

(2.4) $y = -x + \pi$ (2.5) $y = \pi x - \pi + 1$

3. Find the value of x such that there exists a tangent line of $f(x) = x + 2\sin x$ to be a horizontal line.

Answer $(2n+1)\pi \pm \frac{\pi}{3}$ where n is an integer

Exercise 4

1. Find the derivatives of the following functions.

(1.1) $y = \log_a(3x^2 - 5)$

(1.2) $y = \ln(x+3)^2$

(1.3) $y = \ln^2(x+3)$

(1.4) $y = \ln(x^3+2)(x^2+3)$

(1.5) $y = \ln\dfrac{x^4}{(3x-4)^2}$

(1.6) $y = \ln(\sin 3x)$

(1.7) $y = \ln(x + \sqrt{1+x^2})$

(1.8) $y = x\ln x - x$

(1.9) $y = \ln(\sec x + \tan x)$

(1.10) $y = \ln(\ln\tan x)$

(1.11) $y = \dfrac{(\ln x^2)}{x^2}$

(1.12) $y = \dfrac{1}{5}x^5(\ln x - 1)$

(1.13) $y = x(\sin\ln x - \cos\ln x)$

(1.14) $y = \dfrac{\ln x}{1+\ln(2x)}$

(1.15) $y = \ln(e^{-x} + xe^{-x})$

(1.16) $y = \ln(\dfrac{1}{x}) + \dfrac{1}{\ln x}$

Answer

(1.1) $\dfrac{6x}{(3x^2-5)\ln a}$

(1.2) $\dfrac{2}{x+3}$

(1.3) $\dfrac{2\ln(x+3)}{x+3}$

(1.4) $\dfrac{3x^2}{x^3+2} + \dfrac{2x}{x^2+3}$

(1.5) $\dfrac{4}{x} - \dfrac{6}{3x-4}$

(1.6) $3\cot 3x$

(1.7) $\dfrac{1}{\sqrt{1+x^2}}$

(1.8) $\ln x$

(1.9) $\sec x$

(1.10) $\dfrac{2}{\sin(2x)\cdot\ln(\tan x)}$

(1.11) $\dfrac{2-4\ln x}{x^3}$ (1.12) $x^4 \ln x$

(1.13) $2\sin\ln x$ (1.14) $\dfrac{1+\ln 2}{x[1+\ln(2x)]^2}$

(1.15) $\dfrac{-x}{1+x}$ (1.16) $-\dfrac{1}{x}[1+\dfrac{1}{(\ln x)^2}]$

2. Let $f(x)=\dfrac{x}{\ln x}$. Find $f'(e)$.

Answer 0

3. Find the equation of a tangent line of $y=\ln(\ln x)$ at $(e,0)$.

Answer $x-ey=e$

Exercise 5

1. Find the derivatives of the following functions

(1.1) $f(x)=x^2e^x$ (1.2) $y=3^{ax^3}$

(1.3) $f(u)=e^{1/u}$ (1.4) $f(t)=e^{t\sin 2t}$

(1.5) $y=\sqrt{1+2e^{3x}}$ (1.6) $y=e^{e^x}$

(1.7) $y=\dfrac{ae^x+b}{ce^x+d}$ (1.8) $f(t)=\cos(e^{-t\ln t})$

(1.9) $y=\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$ (1.10) $y=7^{x^3+8}$

(1.11) $y = 7^{x^3+8}\ (x^4 - x)$ (1.12) $h(t) = (\ln t + 1)10^{\ln t}$

(1.13) $g(x) = \dfrac{\ln x}{e^{x^2} - e^x}$ (1.14) $y = \tan^2(e^{3x})$

(1.15) $f(x) = e^{\sin^3(\ln(x^2+1))}$

Answer

(1.1) $f'(x) = x(x+2)e^x$

(1.2) $y' = 3^{ax^3}\ \ln 3 \cdot (3ax^2)$

(1.3) $f(u) = (-1/u^2)e^{1/u}$

(1.4) $f'(t) = e^{t\sin 2t}\ (2t\cos 2t + \sin 2t)$

(1.5) $y' = 3e^{3x} / \sqrt{1+2e^{3x}}$

(1.6) $y' = e^{e^x}\ e^x$

(1.7) $y' = \dfrac{(ad-bc)e^x}{(ce^x+d)^2}$

(1.8) $f'(t) = \sin(e^{-t\ln t}) \cdot e^{-t\ln t}(1+\ln t)$

(1.9) $y' = -\dfrac{1}{2}a^{\sqrt{\cos x}} \cdot \sin x\ (\ln a - \dfrac{1}{\sqrt{\cos x}})$

(1.10) $y' = 3x^2 \ln 7 \cdot 7^{x^3+8}$

(1.11) $y' = 7^{x^3+8}[(4x^3-1)+(3x^6-3x^3)\ln 7]$

(1.12) $h'(t) = \dfrac{10^{\ln t}}{t}[\ln 10(\ln t+1)+1]$

(1.13) $g'(x) = \dfrac{1}{x(e^{x^2}-e^x)} - \dfrac{\ln x(2xe^{x^2}-e^x)}{(e^{x^2}-e^x)^2}$

(1.14) $y' = 6e^{3x}\tan(e^{3x})\ \sec^2(e^{3x})$

(1.15) $f'(x) = \dfrac{6x}{x^2+1}\ \sin^2(\ln(x^2+1))\cdot\cos(\ln(x^2+1))\cdot e^{\sin^3(\ln(x^2+1))}$

Exercise 6

1. Find the derivatives of the following functions.

(1.1) $y = \arcsin(2x-3)$

(1.2) $y = \arccos(x^2)$

(1.3) $y = \arctan 3x^2$

(1.4) $y = \operatorname{arc}\cot\dfrac{1+x}{1-x}$

(1.5) $f(x) = x\csc^{-1}(\dfrac{1}{x}) + \sqrt{1+x^2}$

(1.6) $y = \dfrac{1}{ab}\arctan(\dfrac{b}{a}\tan x)$

(1.7) $y = x\ln(4+x^2) + 4\arctan\dfrac{x}{2} - 2x$

(1.8) $h(t) = \cot^{-1}(t) + \cot^{-1}(\dfrac{1}{t})$

(1.9) $y = \cos^{-1}\left[\dfrac{b+a\cos x}{a+b\cos x}\right]$

Answer

(1.1) $y' = \dfrac{1}{\sqrt{3x-x^2-2}}$

(1.2) $y' = -\dfrac{2x}{\sqrt{1-x^4}}$

(1.3) $y' = \dfrac{6x}{1+9x^4}$

(1.4) $y' = -\dfrac{1}{1+x^2}$

(1.5) $f'(x) = \csc^{-1}(\dfrac{1}{x})$

(1.6) $y' = \dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$

(1.7) $y' = \ln(4+x^2)$

(1.8) $h'(t) = 0$

(1.9) $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$

2. Show that

$$\frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right] = \frac{1}{(1+x)(1+x^2)}\ .$$

Exercise 7

1. Evaluate $\dfrac{dy}{dx}$ of the following functions.

(1.1) $y = \sinh 3x$

(1.2) $y = \tanh(1+x^2)$

(1.3) $y = \coth(\frac{1}{x})$

(1.4) $y = x\,\mathrm{sech}\,x^2$

(1.5) $y = \mathrm{csch}^2(x^2+1)$

(1.6) $y = \ln(\tanh(2x))$

(1.7) $y = \sinh(\tan^{-1} e^{3x})$

Answer

(1.1) $3\cosh 3x$

(1.2) $2x\,\mathrm{sech}^2(1+x^2)$

(1.3) $\dfrac{1}{x^2}\mathrm{csch}^2(\frac{1}{x})$

(1.4) $-2x^2\,\mathrm{sech}\,x^2\tanh x^2 + \mathrm{sech}\,x^2$

(1.5) $-4x\,\mathrm{csch}^2(x^2+1)\coth(x^2+1)$

(1.6) $4\,\mathrm{csch}\,4x$

(1.7) $\dfrac{3e^{3x}\cosh(\tan^{-1} e^{3x})}{1+e^{6x}}$

Exercise 8

1. Use logarithmic derivative to find $\dfrac{dy}{dx}$ where

(1.1) $y=(x^2+2)(1-x^3)^4$

(1.2) $y=\dfrac{x(1-x^2)^2}{\sqrt{1+x^2}}$

(1.3) $y=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

(1.4) $y=(2x+1)^5(x^4-3)^6$

(1.5) $y=\dfrac{\sin^2 x\tan^4 x}{(x^2+1)^2}$

(1.6) $y=x^2e^{2x}\cos 3x$

(1.7) $y=x^x$

(1.8) $y=x^{\ln x}$

(1.9) $y=x^{\sin x}$.

Answer

(1.1) $y'=6x(x^2+2)^2(1-x^3)^3(1-4x-3x^3)$

(1.2) $y'=\dfrac{(1-5x^2-4x^4)(1-x^2)}{(1+x^2)^{3/2}}$

(1.3) $y'=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\dfrac{3}{4x}+\dfrac{x}{x^2+1}-\dfrac{15}{3x+2}\right]$

(1.4) $y'=(2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1}+\dfrac{24x^3}{x^2+1}\right)$

(1.5) $y' = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

(1.6) $y' = x^2 e^{2x} \cos 3x[2/x + 2 - 3\tan 3x]$

(1.7) $y' = x^x(1+\ln x)$

(1.8) $y' = 2x^{\ln x - 1}\ln x$

(1.9) $y' = x^{\sin x}[(\sin x)/x + \ln x \cos x]$

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Implicit Differentiation

Definition Implicit function

Implicit function of one independent variable function is a function written in a form of $F(x, y) = 0$. For example, $x^2 + 3xy^2 + 2y - 5 = 0$.

Implicit Differentiation

Since we start with $F(x, y) = 0$, we can find $\frac{dy}{dx}$ by taking derivatives with respect to x both sides separately. Then we solve the equation for $\frac{dy}{dx}$. This method is called "implicit differentiation."

Example 1 Find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by $y^2 + xy - 6x = 0$.

Example 2 Find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by

$$x = 2\sin^{-1}\left(\frac{y}{2}\right) - \sqrt{4 - y^2}\,.$$

Exercise 9

Find $f'(x)$ of each function defined by

1. $2x^4 - 3x^2y^2 + y^4 = 0$

2. $(x+y)^2 - (x-y)^2 = x^4 + y^4$

3. $x^2 + xy + y^2 - 3 = 0$

4. $x\sec 5x = 4y - y\tan 8x$

5. $\sqrt{x} + \sqrt{\sqrt{x} + \cos y} = 1$

Answer

1. $\dfrac{3xy^2 - 4x^3}{2y^3 - 3x^2y}$ **2.** $\dfrac{x^3 - y}{x - y^3}$ **3.** $\dfrac{2x + y}{x^2 + xy - x}$

4. $\dfrac{8y\sec^2 8x + \sec 5x + 5x\sec 5x\tan 5x}{4 - \tan 8x}$

5. $\dfrac{\sqrt{\sqrt{x} + \cos y}}{\sin y\sqrt{x}} + \dfrac{1}{2\sin y\sqrt{x}}$

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Derivatives of Higher Order

Definition Derivatives of Higher Order

Derivatives of higher order refer to finding derivatives several times. So, we call them second order and third order derivatives depending on the number of times taking derivatives.

For example, let $y = f(x)$. By taking derivative one time, we obtain $\frac{dy}{dx} = f'(x)$ and it is called the *first order derivative of* f.

If we find the derivative of $\frac{dy}{dx} = f'(x)$ one more time, the derivative of $\frac{dy}{dx} = f'(x)$ is called the *second (order) derivative of* f and is denoted by $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x) = f''(x)$.

Similarly, if we find the derivative of $\frac{d^2y}{dx^2} = f''(x)$ one more time, we will obtain the *third (order) derivative of* f and is denoted by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^3}\right) = \frac{d}{dx} f''(x) = f'''(x).$$

Continue to do these steps, we may define the n^{th} *(order) derivative* for any positive integers n as follows :

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d}{dx} f^{(n-1)}(x) = f^{(n)}(x).$$

<u>Remark</u>: The notation $f^{(n)}(x)$ is used when $n \geq 4$.

Example 3 Find $f^{(n)}(x)$ where $y = f(x) = \dfrac{1}{(x-1)}$.

Example 4 Find $f^{(n)}(x)$ where $y = \ln(1-x)$.

Exercise 10

1. Find $\dfrac{d^5y}{dx^5}$ **of** $y=3^x$.
2. Find $f'''(x)$ **of** $f(x)=e^{3x+1}$.
3. Find $f^{(n)}(x)$ **of** $f(x)=(ax+b)^n$.

Answer

1. $3^x\ln^5 3$ **2.** $27e^{3x+1}$ **3.** $a^n n!$

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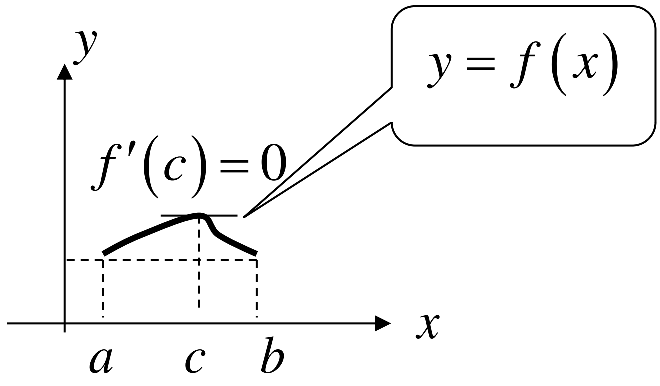
1. Some Useful Theorems

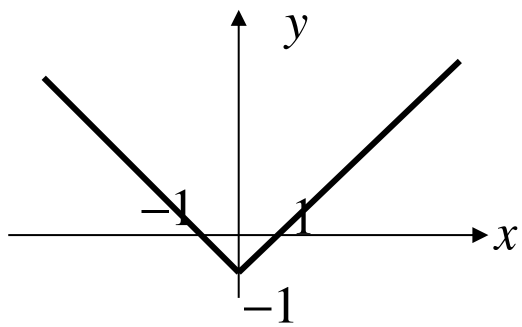
Roll's Theorem

If $f(x)$ is a continuous function on the interval $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$, then there exist at least one point $x=c$ in the interval (a,b) such that $f'(c)=0$.



Remark

Absolute functions such as $y=|x|-1$ is continuous everywhere but not differentiable at $x=0$. Thus, the Roll's theorem does not apply to this function, i.e., there is no c such that $f'(c)=0$.



Graph of $y=|x|-1$

Mean - Value Theorem

If f is a continuous function on the interval $[a,b]$, and differentiable on (a,b), then there exists at least one point $x=c$ in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

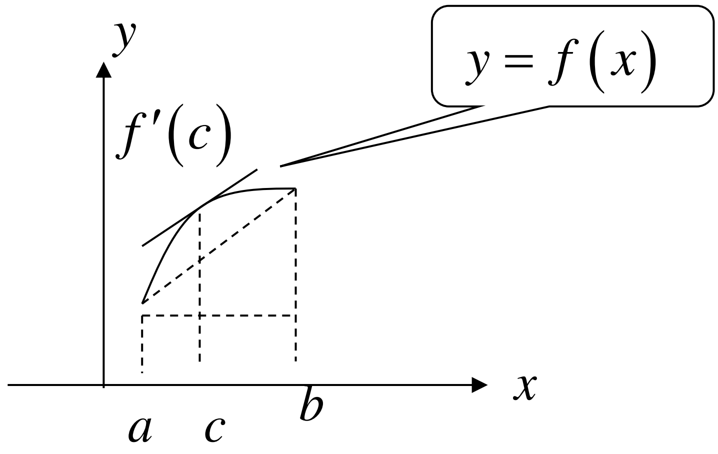


Figure shows the Mean – Value Theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{\Delta y}{\Delta x}.$$

Example 6 Verify the Roll's Theorem to the given function: $f(x)=x^2-6x+8$ on the interval $[2,4]$.

Solution

Example 7 Let $f(x)=2x^3-6x^2+6x-3$. Find x such that $f'(x)$ equals to the average rate of change of f over $0\le x\le 2$.

Solution

Exercise of Mean-Value Function

Find a point c on $[a,b]$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ where the functions and intervals are given as follow.

1. $f(x)=x^6$; $x\in[-3,3]$
2. $f(x)=\sin x$; $x\in[0,2\pi]$
3. $f(x)=x^3-2x^2+x+1$; $x\in[0,1]$

Answers

1. 0
2. $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$
3. $\dfrac{1}{3}$

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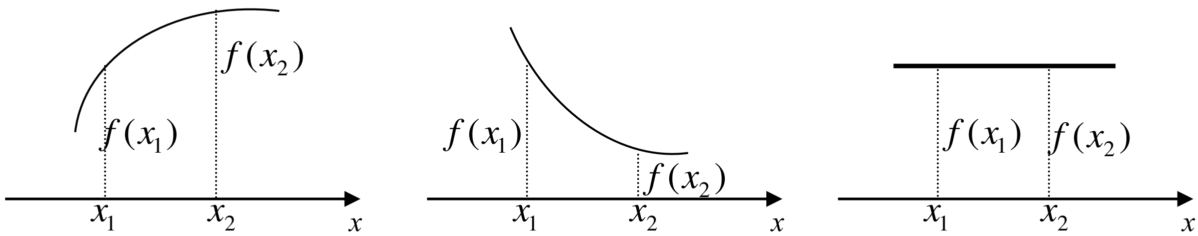
2. Increasing and Decreasing Functions

Definition: (Increasing, decreasing and constant functions)

Let f be a function defined on the interval I, and x_1, x_2 are two points in I.

1. The function f is an increasing function on the interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. The function f is a decreasing function on the interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. The function f is a constant function on the interval I if $f(x_1) = f(x_2)$ for any x_1, x_2 in I.



Increasing	Decreasing	Constant
$f(x_1) < f(x_2)$	$f(x_1) > f(x_2)$	$f(x_1) = f(x_2)$
when $x_1 < x_2$.	when $x_1 < x_2$.	for all x_1, x_2.

<u>Theorem</u>

Let f be a continuous function on $[a,b]$ and differentiable on (a,b).

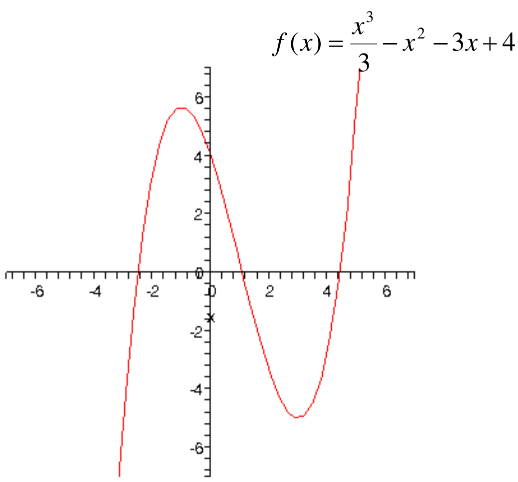
1. If $f'(x) > 0$ for all $x \in (a,b)$, then f increases on $[a,b]$.
2. If $f'(x) < 0$ for all $x \in (a,b)$, then f decreases on $[a,b]$.
3. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Example 8 Let f be a function defined by

$$f(x) = \frac{x^3}{3} - x^2 - 3x + 4 \ .$$

Find the intervals of x for which f is increasing and is decreasing.

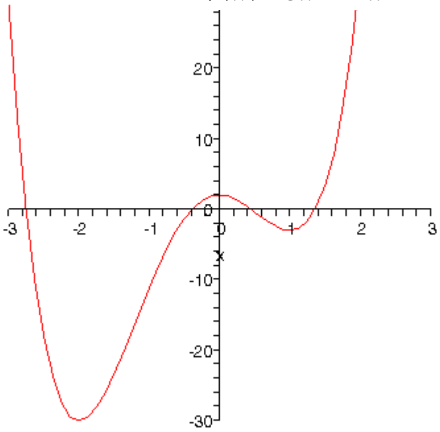
<u>Solution</u>



Example 9 Identify the intervals of x where the given function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and decreasing.

<u>Solution</u>

$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

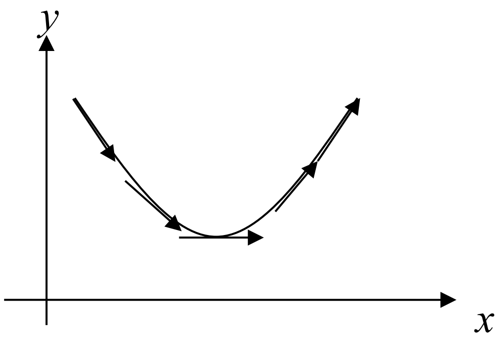


3. Concavity and Point of Inflection

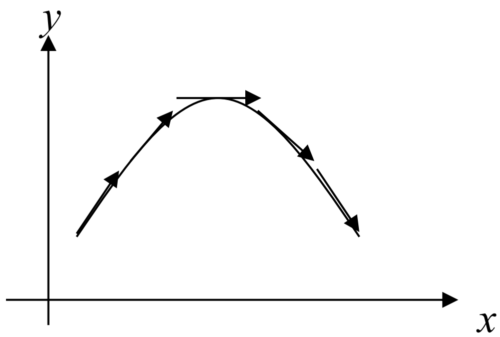
<u>Definition</u> (Concave Up and Concave Down)

Let f be a differentiable function on an open interval I. Then

1. The function f is called concave up on I if f' increases on I.
2. The function f is called concave down on I if f' decreases on I.



concave up



concave down

<u>Theorem</u>

Let f be a function such that $f''(x)$ exists on an open interval I.

1. If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.
2. If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Example 10 Identify the intervals where the following functions are concave up and where they are concave down.

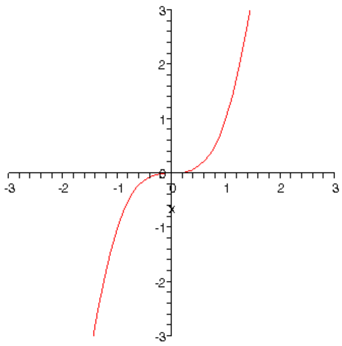
a. $f(x) = x^3$

b. $f(x) = x^3 - 3x^2 + 1$

<u>Solution</u>

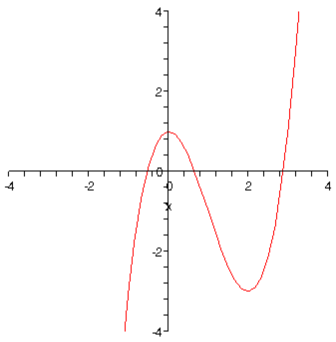
a.

$f(x) = x^3$



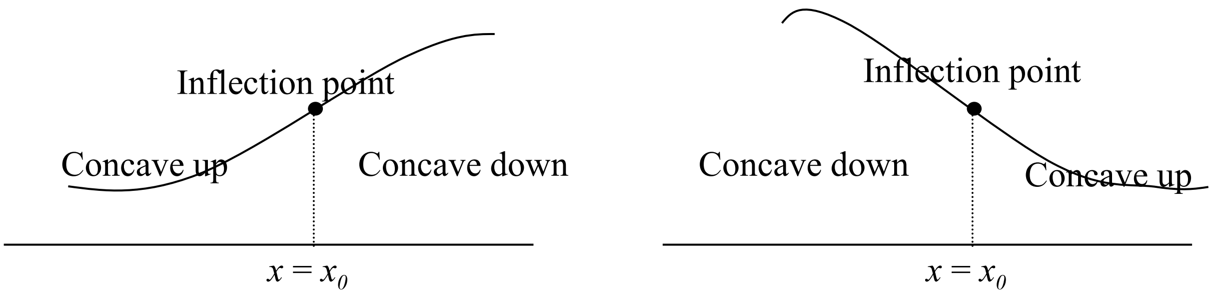
b.

$f(x) = x^3 - 3x^2 + 1$



<u>Definition</u> (Inflection Point)

A point $(x_0, f(x_0))$ is called an inflection point if the graph of f changes the concavity at $x = x_0$.



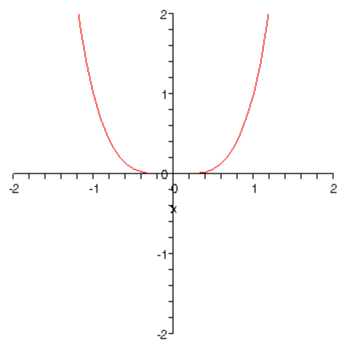
<u>Theorem</u>

If the point $(x_0, f(x_0))$ is an inflection point, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

<u>Remark</u> The converse of this theorem is not true.

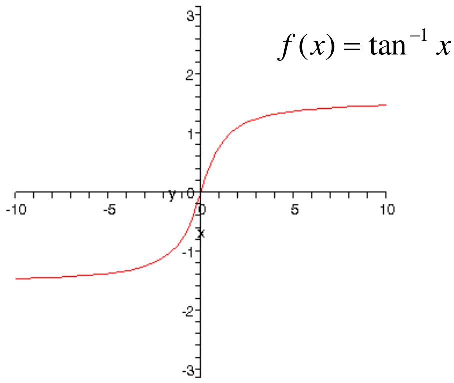
That is, if $f''(x_0) = 0$ or $f''(x_0)$ does not exist, the point $(x_0, f(x_0))$ may or may not be an inflection point.

For example, $f(x) = x^4$ has $f''(x) = 12x^2$ and $f''(0) = 0$, but $x = 0$ is not an inflection point.

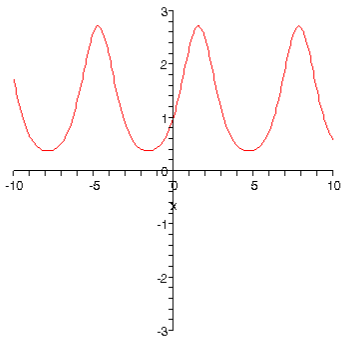


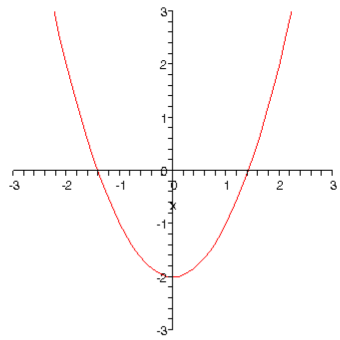
Example 11 Find all inflection points of $f(x) = \tan^{-1} x$.

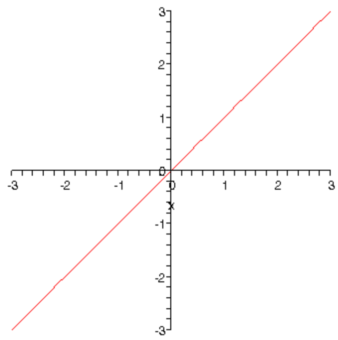
<u>Solution</u>



4. Maximum Value and Minimum Value of Function







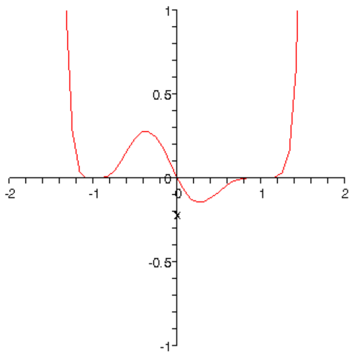
<u>Definition</u> (Relative Maximum and Relative Minimum)

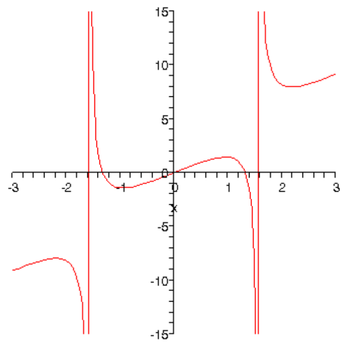
Let f be a real valued function. We say that

1. f has a relative maximum at $x = x_0$ if $f(x) \le f(x_0)$ for all x in an open interval containing x_0. The point $(x_0, f(x_0))$ is called a relative maximum point of f.

2. f has a relative minimum at $x = x_0$ if $f(x) \ge f(x_0)$ for all x in an open interval containing x_0. The point $(x_0, f(x_0))$ is called a relative minimum point of f.

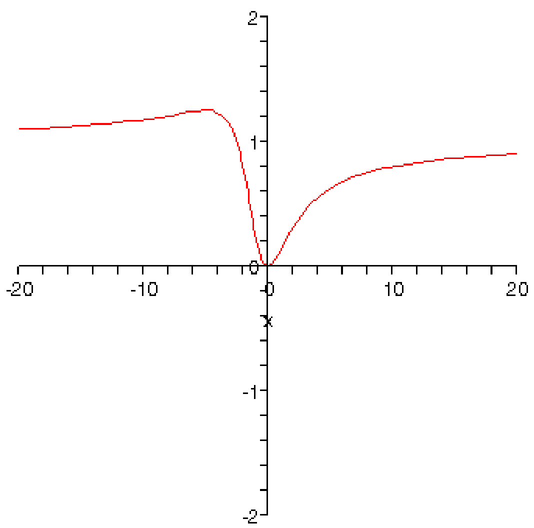
We may refer to a relative maximum and a relative minimum of a function as its relative extrema.

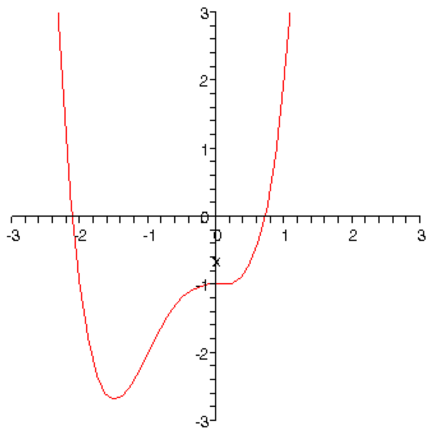




> **<u>Definition</u>** (Absolute Maximum and Absolute Minimum)
>
> Let f be a real valued function. Then
>
> 1. f has an absolute maximum at $x = x_0$ if $f(x_0) \geq f(x)$ for all $x \in D_f$. We call $(x_0, f(x_0))$ an absolute maximum point of f.
> 2. f has an absolute minimum at $x = x_0$ if $f(x_0) \leq f(x)$ for all $x \in D_f$. We call $(x_0, f(x_0))$ an absolute minimum point of f.

Also, an absolute max and an absolute min of a function can be referred as its absolute extrema.



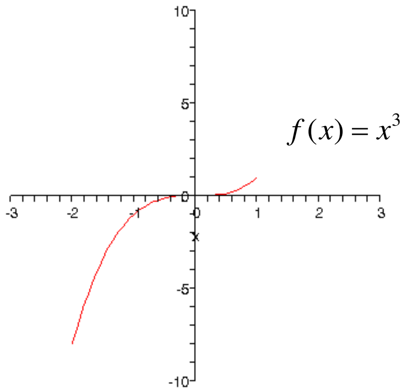


Extreme Value Theorem

If f is a continuous function on $[a,b]$, then f has both absolute minimum and absolute maximum values on $[a,b]$.

Example 12 Find all absolute extreme values of $f(x)=x^3$ where $x\in[-2,1]$.

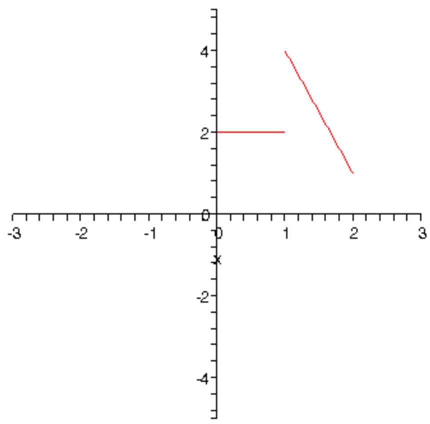
Solution



Example 13 Find all absolute extreme values of the function

$$f(x)=\begin{cases} 2 & , \quad 0\le x\le 1 \\ -3x+7 & , \quad 1<x\le 2 \end{cases}.$$

Solution



From the above example, we can see that if a given function is not continuous on a closed interval $[a,b]$, that function may or may not have absolute extreme values.

<u>Theorem</u>

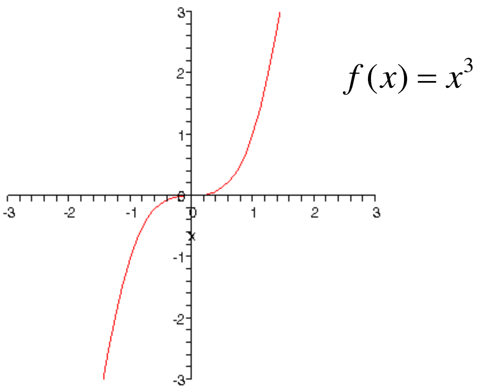
If f has a relative extreme value at $x = x_0$, then either

$f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Note:

1. If $f'(x_0) = k \neq 0$, then $x = x_0$ is not a relative extreme point.
2. If $f'(x_0) = 0$ or undefined, then $x = x_0$ is not necessarily a relative extreme point.

In example 5, we see that $f'(x) = 0$ at $x = 0$, but f does not have a relative max and relative min at $x = 0$.



<u>Procedure of finding maximum and minimum values</u>

<u>Definition</u>

A point $x = x_0$ is called a **critical point** of a function f if either

1. $f'(x_0) = 0$ or
2. $f'(x_0)$ does not exist.

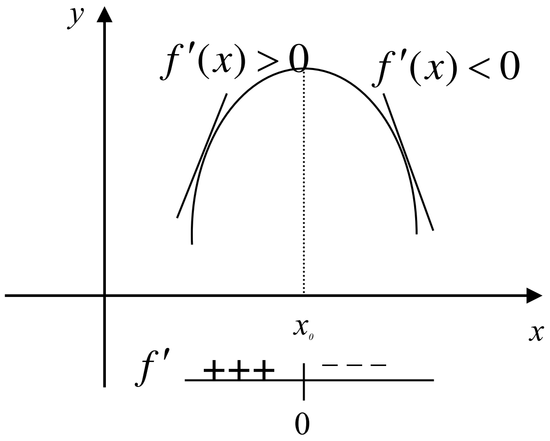
We call $f(x_0)$ a critical value of f.

<u>Theorem</u> (The First Derivative Test for Relative Extreme Points.)

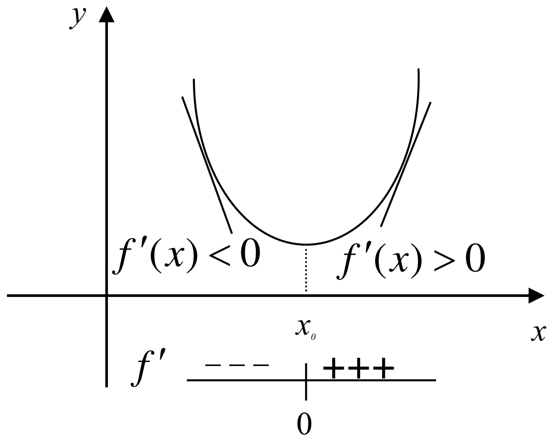
Suppose $x = x_0$ is a critical point of a continuous function f.

Consider the sign of the derivative of f around x_0.

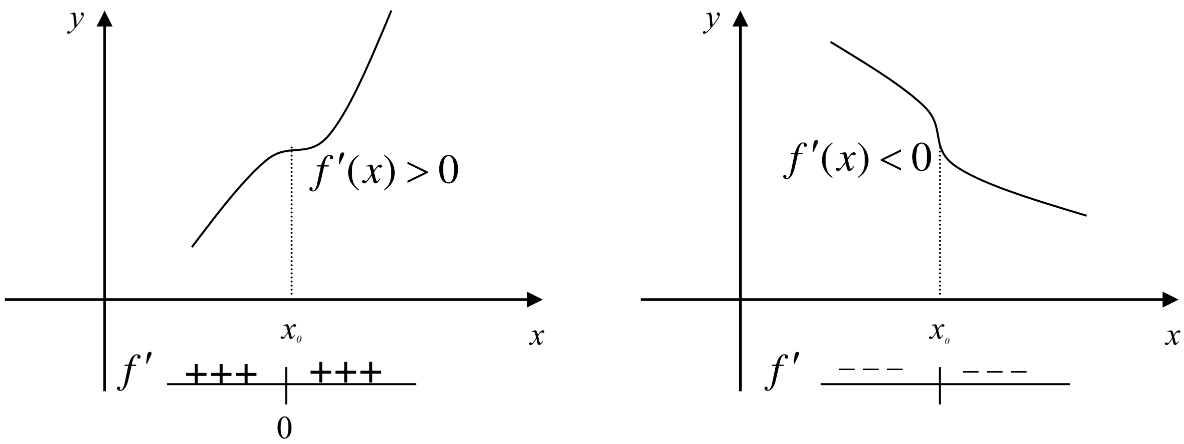
1. If the sign of $f'(x)$ changes from positive to negative at x_0, then f has a relative maximum at $x = x_0$.
2. If the sign of $f'(x)$ changes from negative to positive, then f has a relative minimum at $x = x_0$.



Relative Maximum



Relative Minimum

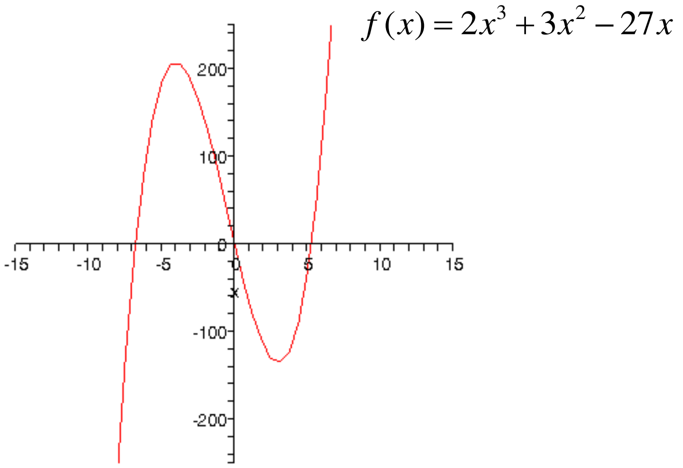


Neither Relative Maximum nor Minimum

Example 14 Find relative max and relative min values of

$$f(x) = 2x^3 + 3x^2 - 27x \ .$$

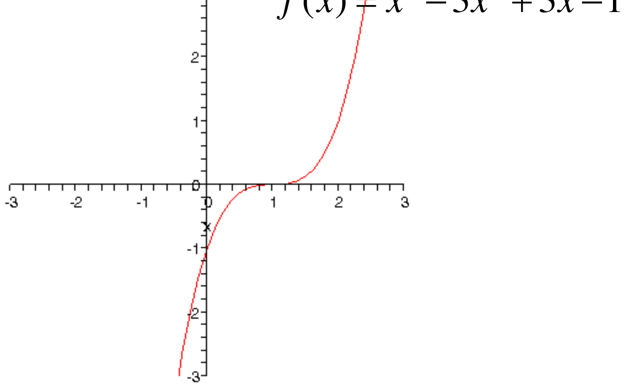
Solution



Example 15 Find relative max and relative min values of

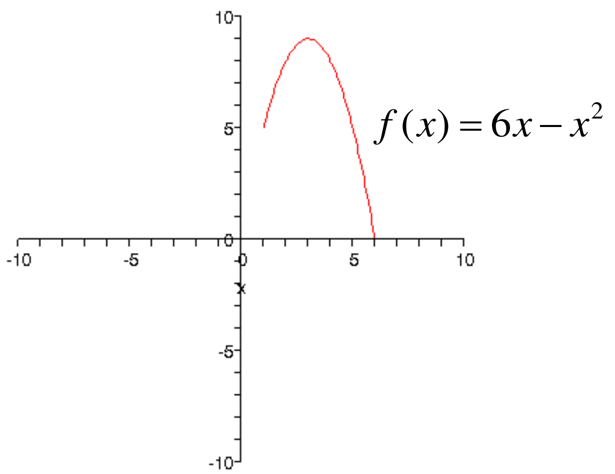
$$f(x) = x^3 - 3x^2 + 3x - 1 \, .$$

Solution

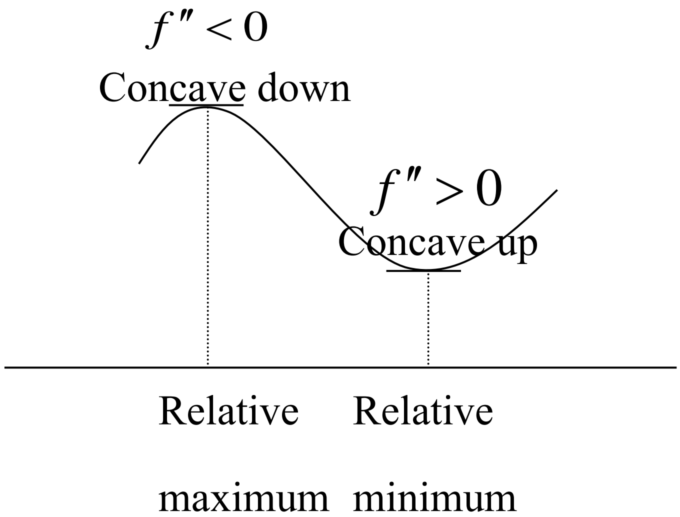


Example 16 Find all relative extrema and absolute extrema of the function $f(x) = 6x - x^2$ on the interval [1,6].

<u>Solution</u>



We may use the second derivative to identify relative max and relative min points of a function via the concavity concepts as follows:



Theorem (The Second Derivative Test for Relative Extremum)

Let f be a differentiable function such that $f''(x_0)$ exists and $f'(x_0) = 0$.

1. If $f''(x_0) > 0$, then f has a relative minimum at x_0.
2. If $f''(x_0) < 0$, then f has a relative maximum at x_0.
3. If $f''(x_0) = 0$, the test fails. We have no conclusions.
   (That is, x_0 may or may not be a relative extreme point.)

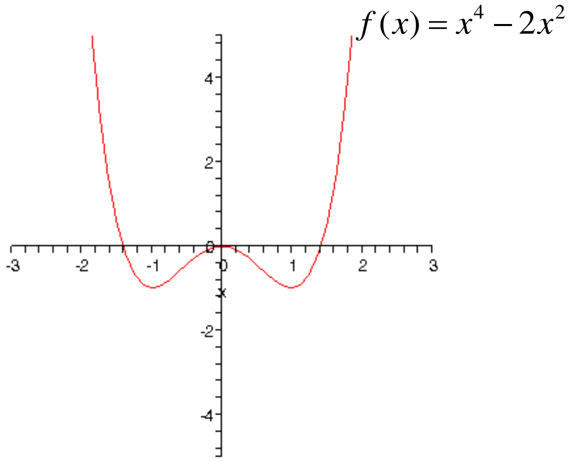
Example 17 Let $f(x) = x^4 - 2x^2$.

(a) Identify the intervals of x for which f is concave up and concave down.

(b) Find all inflection points of f.

(c) and both relative extreme points.

<u>Solution</u>



Example18 Find a and b so that $f(x)=x^3+ax^2+bx$ has a relative maximum at $x=-1$ and a relative minimum at $x=3$.

<u>Solution</u>

5. **Sketching a graph of rational function**

Let f be a rational function. To sketch the graph of f, we do the following procedure:

1. Find basic properties of function f such as
   a. domain and range
   b. x-intercept and y-intercept
   c. symmetry
   d. asymptotes.
2. Apply the first derivative f' to find
   a. critical points
   b. intervals of x for which f is increasing and decreasing
   c. relative extreme points.
3. Apply the second derivative f'' to find
   a. inflection points
   b. intervals of x for which f is concave up and concave down.

Asymptotes

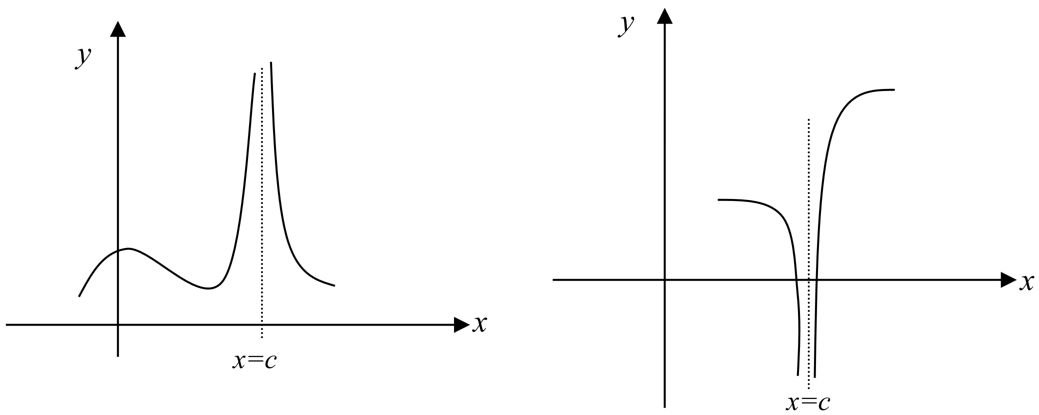
An asymptote is a line which a graph of a function gets arbitrarily close to as x or y or both increases (or decreases) unboundedly. In general, there are 3 types of asymptotes.

1. **Vertical Asymptote**: $x = c$

A line $x = c$ is called a vertical asymptote of a function f if

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = \pm\infty.$$

In case of a rational function, its vertical asymptote can be easily identified by only considering all the points $x = c$ where the function is undefined ($x = c$ which make the denominator of the function equal zero). Then the lines $x = c$ will automatically be its asymptotes.

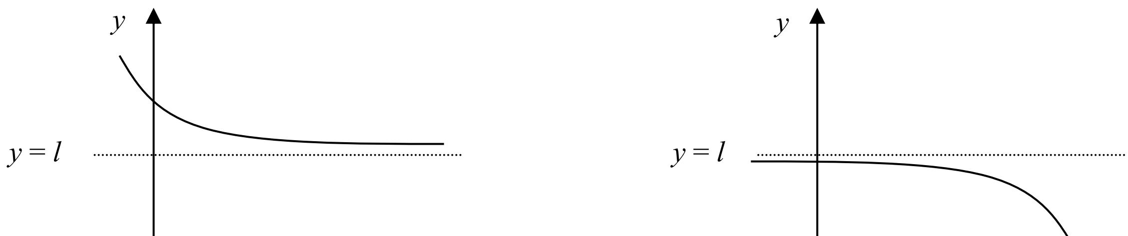


2. **Horizontal Asymptote**: $y = l$

A line $y = l$ is called a horizontal asymptote of a function f if

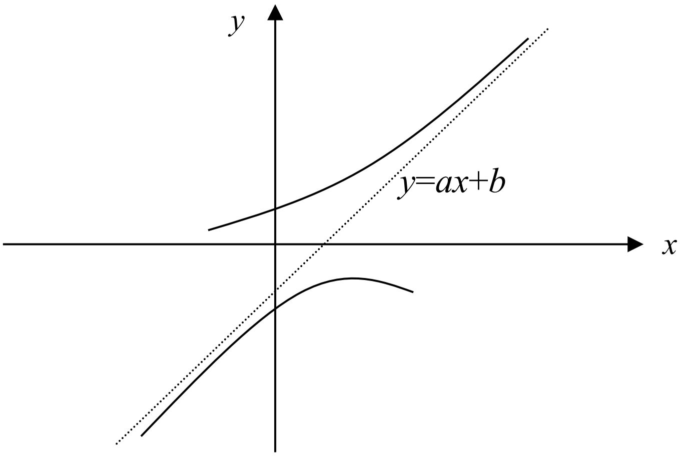
$$\lim_{x \to +\infty} f(x) = l \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l.$$

From the above definition, a horizontal asymptote can be found by evaluating the limits of function as $x \to \pm\infty$.



3. **Oblique Asymptote**: $y = ax + b$

A line $y = ax + b$ is called an oblique asymptote of a function f if both $a = \lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ and $b = \lim\limits_{x\to\infty}[f(x) - ax]$ exist.

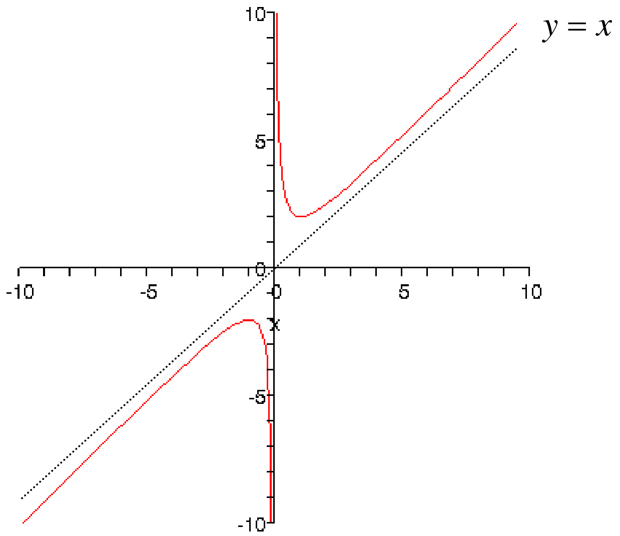


Example19 Find all asymptotes of the following functions.

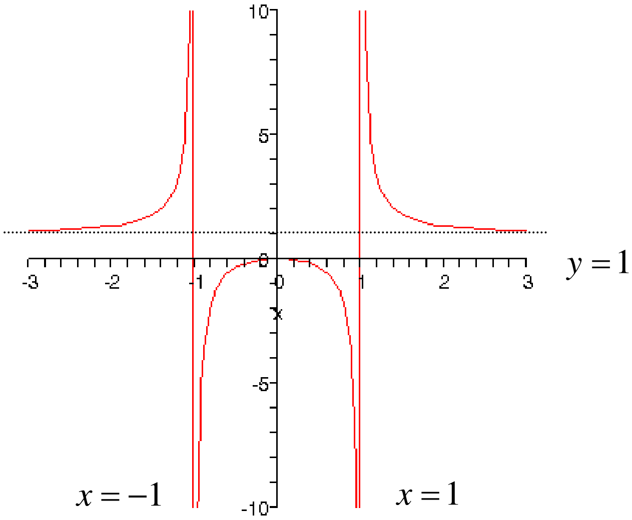
a. $f(x)=\dfrac{x^2+1}{x}$ b. $f(x)=\dfrac{x^2}{x^2-1}$

<u>Solution</u>

a. $f(x)=\dfrac{x^2+1}{x}$



b. $f(x)=\dfrac{x^2}{x^2-1}$

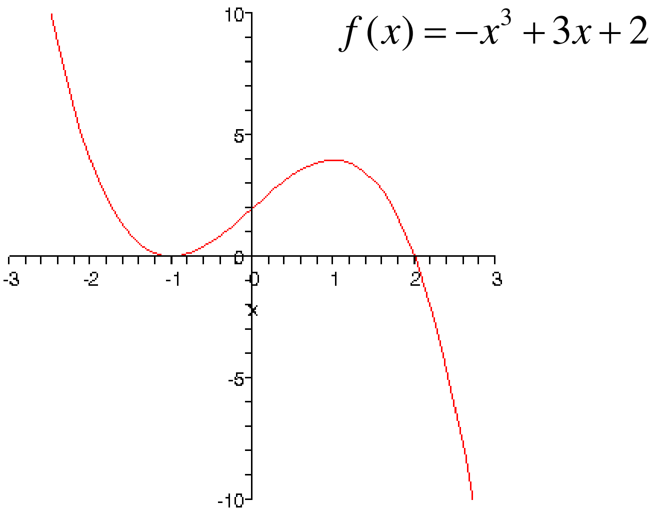


Example20 Analyze and sketch graphs of the following functions.

a. $f(x) = -x^3 + 3x + 2$ b. $f(x) = \dfrac{2x^2 - 8}{x^2 - 16}$

<u>Solution</u>

a.



b. Domain and range: $D_f = \mathbb{R} - \{-4, 4\}$,

$$R_f = \{y \,|\, y \le \frac{1}{2} \cup y > 2\}$$

x-intercept: $(-2,0)$ and $(2,0)$

y-intercept: $(0, \frac{1}{2})$

Symmetry: symmetric about the y-axis

Asymptote:

Vertical asymptote: $x = -4$ and $x = 4$

Horizontal asymptote: $y = 2$

Oblique asymptote: none

$$f'(x) = \frac{4x}{(x^2 - 16)} - \frac{2x(2x^2 - 8)}{(x^2 - 16)^2} = -\frac{48x}{(x^2 - 16)^2}$$

$$f''(x) = -\frac{48}{(x^2 - 16)^2} + \frac{192x^2}{(x^2 - 16)^3}$$

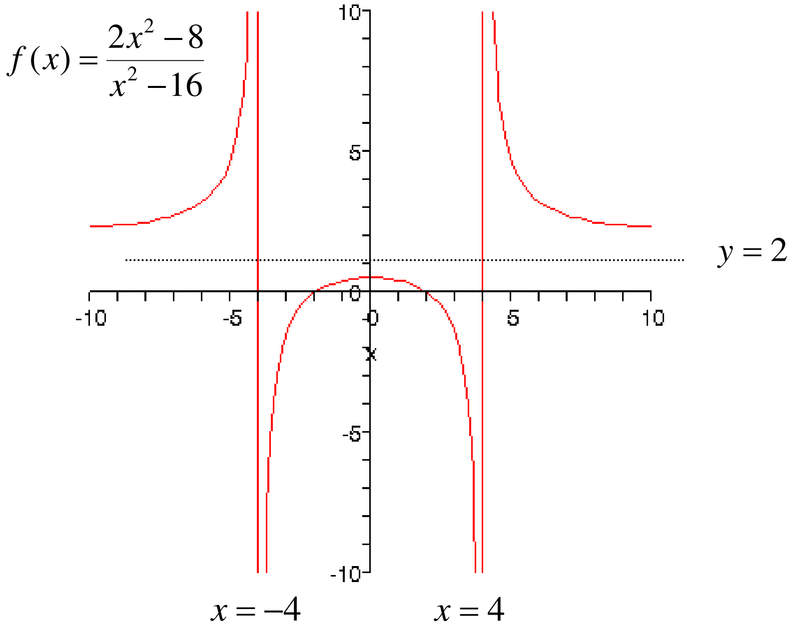
Critical point: $x = 0$

$f'(x)$	(+)		(+)		(–)		(–)
x		-4		0		4	

Thus, f is decreasing on $[0,4)\cup(4,\infty)$ and

increasing on $(-\infty,-4)\cup(-4,0]$.

Also, f is concave up on $(-\infty,-4)\cup(4,\infty)$ and concave down on $(-4,4)$.



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Applications of Derivative

1. Differentials

Definition (Differential)

If f is a differentiable function with $\dfrac{dy}{dx} = f'(x)$, then $dy = f'(x)dx$ is called a **differential** of y (dependent variable) at some point x.

The following are differential formulas of some basic functions.

1. $da = 0$; a is a constant.
2. $dx^n = nx^{n-1}dx$
3. $d\sin x = \cos x dx$
4. $d\cos x = -\sin x dx$
5. $da^x = a^x \ln a \, dx$ where a is a positive constant.
6. $de^x = e^x dx$
7. $d\ln x = \dfrac{1}{x}dx$
8. $d\log_a x = \dfrac{1}{x\ln a}dx$ where a is a positive constant with $a \neq 1$.

Example 1 Find dy where $y = x^2(x+1)$.

Solution

Remark

For parametric functions, we may use

$$\frac{\text{differential of } y}{\text{differential of } x} = \frac{dy}{dx}$$

where $y = f\left(x\right)$ and x, y are parametric functions in term of t.

Example 2 Evaluate $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ where $y = f\left(x\right)$ is defined by $y = t^2 + 1$ and $x = t^3 + 3$.

Solution

Exercise of Differential

Find differentials of the following functions.

1. $y = \dfrac{x}{\sin x}$

2. $x^2 y - y^2 x + 2 = 0$

3. $x + xy\sin x + \dfrac{y^2 \cos x}{x} = 1$

Answers

1. $\dfrac{\sin x - x\cos x}{\sin^2 x} dx$ **2.** $\dfrac{y^2 - 2xy}{x^2 - 2xy} dx$

3. $\left(\dfrac{\dfrac{x\sin x + \cos x}{x^2} y^2 - xy\cos x - y\sin x - 1}{x\sin x + 2y\dfrac{\cos x}{x}} \right) dx$

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2. Linear Approximation

From the definition of differential: $dy = f'(x)dx$ and the change of dependent variable: $\Delta y = y_2 - y_1$, $\Delta x = x_2 - x_1$, we may consider the geometric meaning as follows.

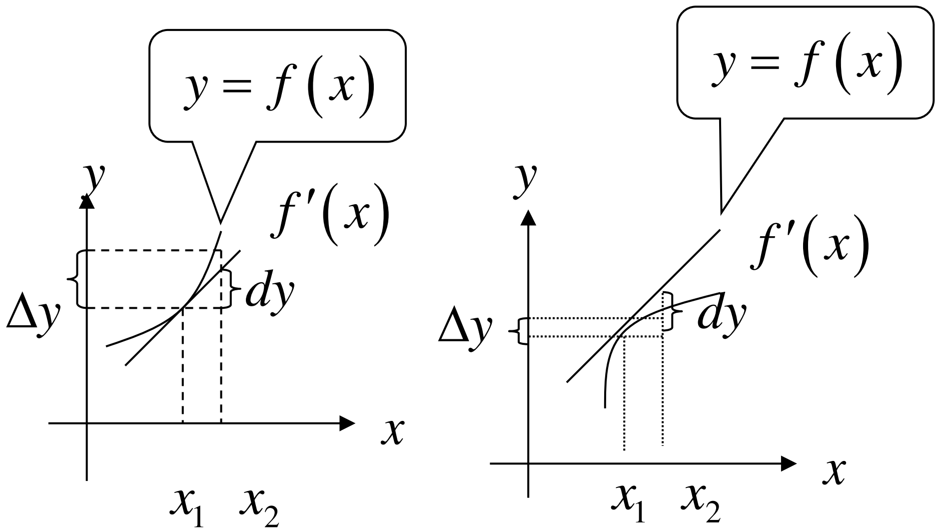

Figure shows $dy, \Delta y$.

The differential of y (dy) at some small interval of x is calculated from the different values on the tangent line to the curve, while the difference of y (Δy) is computed from the real values of the curve. So we may say $dy \approx \Delta y$. However, $dx = \Delta x$ and when $\Delta x \to 0$ we have $dy = \Delta y$.

To find the linear approximation at some point x, we use the following approximation: $\Delta y \approx dy$

$$y_2 - y_1 \approx f'(x)dx$$

$$f(x+\Delta x) - f(x) \approx f'(x)dx$$

$$f(x+\Delta x) \approx f(x) + f'(x)dx.$$

The above equation shows that we may approximate $f(x+\Delta x)$ by adding function $f(x)$ to its differential where $\Delta x = dx$.

Example 3 Use the differential to find a linear approximation of $f(x) = \sqrt{1+x}$ at $x = 0$ when $\Delta x = a$.

Solution

Example 4 Approximate $\sqrt{1.1}$ linearly.

Solution

Example 5 Linearly approximate $\cos 62^{\circ}$.

solution

Exercise of Linear Approximation

Use differential to find linear approximations of the following:

1. $\sqrt{63.999}$ 2. $(31)^{\frac{3}{5}}$ 3. $\cos\left(44^{\circ}\right)$

4. $(1.01)^{5}+3(1.01)^{\frac{3}{2}}-1.$

Answer

1. 7.9999 2. 7.85 3. 0.7194 4. 3.095

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Indeterminate forms

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist such that $\lim_{x\to a} g(x) \neq 0$. We have that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ exist.

However, if $\lim_{x\to a} g(x) = 0$, we are not able to say anything about $\lim_{x\to a} \frac{f(x)}{g(x)}$. There are two cases as follow:

1. If $\lim_{x\to a} f(x)$ equals some nonzero number and $\lim_{x\to a} g(x) = 0$, then we can conclude that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist (DNE).

2. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form, namely $\frac{0}{0}$.

Besides the indeterminate form $\frac{0}{0}$, the indeterminate forms also include the forms $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \pm\infty$, $\pm\infty \pm \infty$, 0^0, $\pm\infty^0$, and $1^{\pm\infty}$.

To find the limits of these indeterminate forms, a French mathematician named *L'Hopital* made the following rule:

For any real number a and two functions $f(x)$ and $g(x)$ which are differentiable on the interval $0<|x-a|<\delta$ for some $\delta>0$, if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{0}{0}$, then we may apply *L'Hopital rule* until $\lim_{x\to a} f^{(n)}(x)$ and $\lim_{x\to a} g^{(n)}(x)$ are not zero simultaneously.

Example 1: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{4+\cos\theta}-2}{\theta-\frac{\pi}{2}}$.

Example 2: Evaluate $\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-\left(1+\dfrac{t}{2}\right)}{t^2}$.

Example 3: Evaluate $\lim\limits_{x \to 0} \dfrac{e^x(1-e^x)}{(1+x)\ln(1-x)}$.

In the case of $\frac{\pm\infty}{\pm\infty}$, we can do one of the following:

(1) Eliminate the terms of ∞ by dividing every term by the highest term as we do for polynomial functions in the chapter 2.

(2) Apply the *L'Hopital's* rule by rewriting the functions:

Suppose $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}\left(=\frac{\infty}{\infty}\right)$. Then, we rewrite as $\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{\frac{1}{\lim_{x\to a}g(x)}}{\frac{1}{\lim_{x\to a}f(x)}}\left(=\frac{0}{0}\right)$ before applying the *L'Hopital*'s rule.

Theorem1: Suppose $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$, and both functions $f(x)$, $g(x)$ are differentiable. Then, we also have $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{\infty}{\infty}$, then we can apply *L'Hopital rule* until $\lim_{x\to a}f^{(n)}(x)$ and $\lim_{x\to a}g^{(n)}(x)$ do not approach infinity simultaneously.

Example 4: Evaluate $\lim\limits_{x\to\infty} \dfrac{5x+2\ln x}{x+3\ln x}$.

Example 5: Evaluate $\lim\limits_{x\to 0} \dfrac{\cot x}{\cot 2x}$.

Example 6: Evaluate $\lim_{x\to 0^+} \dfrac{e^{-3/x}}{x^2}$.

For those indeterminate forms $0 \cdot \pm\infty, \pm\infty \pm \infty$, we have to convert them to either $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ before applying L'Hoptial's rule.

Example 7: Evaluate $\lim_{x\to 0^+} x^3 \ln x$.

Example 8: Evaluate $\lim_{x\to 0^+} \left(\csc x - \frac{1}{x} \right)$.

Example 9: Evaluate $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$.

For the rest of indeterminate forms: $0^0, \pm\infty^0, 1^{\pm\infty}$, we take the natural log (ln) to the function so that it has the form $0\cdot\pm\infty$. Then, we continue just like what we do in the last section.

__Note:__ 1^∞ is not always equal to 1. For example, $\lim_{x\to 0} 1^{1/x} = 1$, but $\lim_{x\to 0}(1+x)^{1/x} = e$.

Example 10: Evaluate $\lim\limits_{x\to 0}\left(\sec^3 2x\right)^{\cot^2 3x}$.

Example 11: Evaluate $\lim\limits_{x\to 0^+}\left(1+\dfrac{5}{x}\right)^{2x}$.

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3. Applications of Maxima and Minima

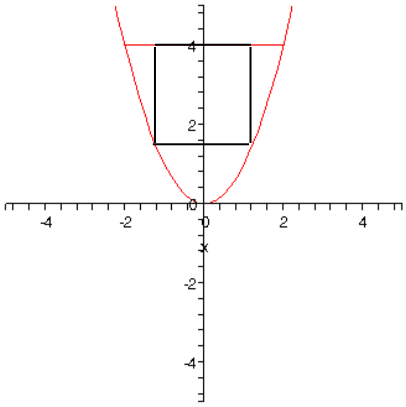
Here is the procedure on how to solve the maximum and minimum problems.

1. Draw a picture (if possible).
2. Define relevant variables.
3. Write down a function we want to optimize.
4. If the function has more than one independent variable, reduce number of variables by applying the conditions given in the problem.
5. Use the derivative to find the extreme points.

Example 21 We want to make a rectangular box with open top from a square paper. Each side of this square paper is 12 cm long. We cut out all of its four corners as shown below. To obtain the highest volume of the box, how long should we cut at each corner?

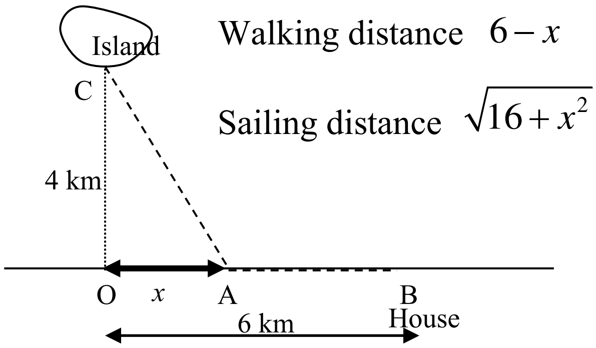
Example 22 Find the dimension of a rectangle which occupies the largest area between a parabola of $y = x^2$ and a line $y = 4$.

<u>Solution</u>



Example 23 A man is on a deserted island which is 4 km vertically far from the seashore. He wants to sail back to his house 6 km away from the given point O as shown below. If he can sail with speed 4 km per hour and walk by 5 km per hour. How should he travel so that he reaches his house fastest?

Solution Let A be a point where the man reaches seashore, and x be the distance from point O to A.



Let T = total travel time

= sailing time + walking time

$$T = \frac{\sqrt{16+x^2}}{4} + \frac{6-x}{5}, \quad 0 \le x \le 6$$

4. Related Rates

Related rate is the rate of change of some quantity compared to time. It can be found by computing the derivative with respect to time. The following is the procedure.

1. Draw a picture (if possible).
2. Define relevant variables.
3. Write down a function and a rate of change we want to optimize in terms of time.
4. Find the derivative of the function with respect to time.
5. Evaluate the rate of change by using the given quantities and their rates of changes.

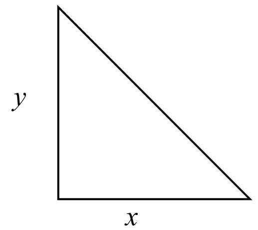
Note: The rate of change has a sign.

If t increases and the value of x also increases, we have $\dfrac{dx}{dt} = +$.

If t increases but the value of x decreases, we then have $\dfrac{dx}{dt} = -$.

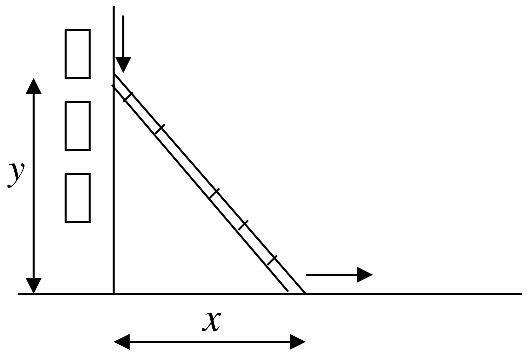
Example 24 Assume that the area of a right triangle is fixed to be 6 square inch. Suppose that it has 4 inches long base at the beginning. If its height increases by 0.5 inches per minute, what is the rate of change of the base of this triangle?

<u>Solution</u>

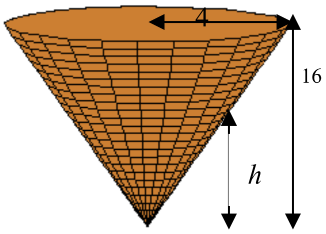


Example 25 A 13 meters long ladder is put up against the wall. Its top end of the ladder is moving down along the wall by 5 meters per minute. This makes the other end of the ladder on the ground moves horizontally away from the wall. Calculate the rate of change of the ground distance between the wall and the ladder when the ladder end is 5 meters away from the wall.

<u>Solution</u>



Example 26 A circular cone has top radius 4 cm. and height 16 cm. Water is pouring into this cone by the rate of 10 cm^3 per minute. Find the rate of change of the water's height in the cone when water is 6 cm high from the bottom.



Let

More Exercises on Applications of Derivatives

1. Locate the intervals of x where each function is increasing and where it is decreasing.

1.1. $f(x) = 6x^2 - 2x^3 - 3$

1.2. $f(x) = x^3 - 6x^2 + 9x - 5$

1.3 $f(x) = \ln(1 + x^2)$

2. Find all the critical points of the following functions.

2.1 $y = x^3 - 2x^2$

2.2 $y = x^2 + \dfrac{2}{x}$

2.3 $y = \dfrac{x-1}{x^2}$

3. Show that these functions have no absolute extreme points.

3.1 $y = 2x^3 - 9x^2 + 12x$

3.2 $y = x + \sin x$

4. Locate the intervals of x where each function's graph is concave up and where it is concave down. Identify the inflection points and calculate relative extreme values.

4.1 $f(x) = x^4 - 4x^3 + 8x - 2$

4.2 $f(x) = 5 + 12x - x^3$

4.3 $f(x) = 2x^3 - 9x^2 + 12x$

5. Find all extreme values of the following functions.

5.1 $f(x) = \tan^2 3x$

5.2 $f(x) = 2x^3 + 3x^2 - 72x$, $x \in [-10,5]$

5.3 $f(x) = \dfrac{ax}{x^2 + a^2}$

6. Analyze and sketch a graph of each function.

6.1 $y = x^4 - 4x^3 + 8x - 2$

6.2 $y = \dfrac{8}{4 - x^2}$

6.3 $y = \dfrac{3x^2 - 4x - 4}{x^2}$

7. Find the maximum volume of a cylinder inscribed in a sphere whose radius is r.

8. An area of 14,4000 m^2 is required to construct one 7-Eleven shop in Bangkok. Its floor plan has a rectangular shape. The shop has three brick walls and one glass wall in the front. The cost of the material is calculated by the length. Suppose glass wall costs 1.88 times as much as the brick wall costs. Find the dimension of this shop so that the material cost is minimized.

9. Identify the point on the curve of $xy^2 = 128$ which is closest to the origin.

10. A rectangular bucket has the dimension: width$\times$length$\times$height = $x \times y \times x$ inch^3. It is made of a piece of tin with area 1350 inch^2. Calculate the possible maximum volume of this bucket.

11. A six-foot tall man walks along the road toward the lamp pole with speed 5 feet per second. The lamp is 16 feet above ground. Find the velocity of his shadow's tip and how the shadow's length changes when he is 10 feet away from the lamp pole.

12. Suppose the volume of a symmetric cube increases by 4 cm^3 per second. Find the rate of change of the cube's surface area when the surface area 24 cm^2.

13. A boy is flying a kite 300 feet high above the ground. If the wind pushes the kite away from the boy by horizontal speed of 25 feet per second, then how fast does the boy release the kite's rope when the kite is 500 feet far from him?

14. Two sailors sails two ships from the same position. The first ship starts sailing at noon and sail toward the east by 20 miles per hour. The second one starts sailing at 1 pm and sail toward the south by 25 miles per hour. Find the rate of change of the distance between these two ships at 2 pm.

Answers

1.1 decrease $(-\infty,0]\cup[2,\infty)$ / increase $[0,2]$

1.2 decrease $[1,3]$ / increase $(-\infty,1]\cup[3,\infty)$

1.3 decrease $(-\infty,0]$ / increase $[0,\infty)$

2.1 $(0,0)$, $(\frac{4}{3},-\frac{37}{27})$

2.2 $(1,3)$

2.3 $(2,\frac{1}{4})$

4.1 concave down $(0,2)$ / concave up $(-\infty,0)\cup(2,\infty)$ /

inflection point $(0,-2)$ / relative max 3 / relative min -6

4.2 concave down $(0,\infty)$ / concave up $(-\infty,0)$ /

inflection point $(0,5)$ / relative max 21 / relative min -11

4.3 concave down $(0,\frac{3}{2})$ / concave up $(\frac{3}{2},\infty)$ /

Inflection point $(\frac{3}{2},\frac{9}{2})$ / relative max 5 / relative min 4

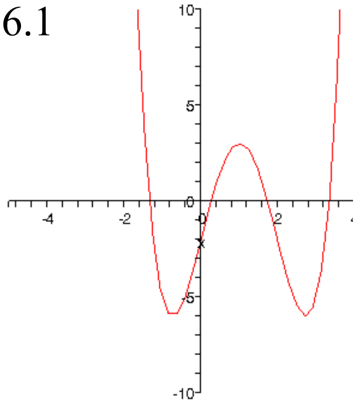
5.1 relative min 0

5.2 absolute max $f(-4)=208$/ relative min $f(3)=-135$

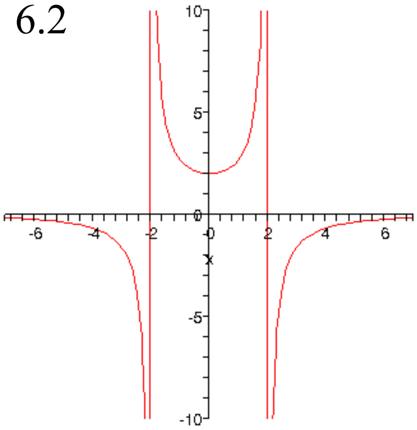
absolute min $f(-10)=-980$

5.3 relative max $\frac{1}{2}$ / relative min $-\frac{1}{2}$

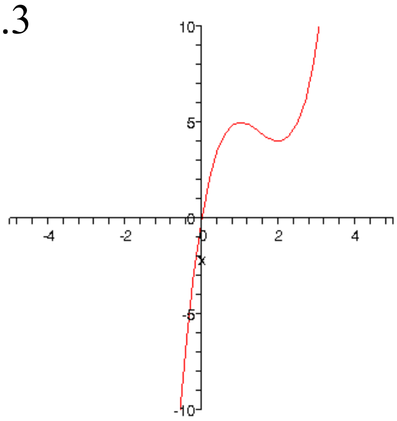
6.1



6.2



6.3



7. $\frac{4}{9}r^3\sqrt{3}$

8. 144 meters wide, 100 meters long

9. $(4, \pm 4\sqrt{2})$

10. 4500 inch^3

11. 8 ft/sec, decrease by 3 ft/sec

12. 8 cm^2/sec

13. 20 ft/sec

14. $\dfrac{285}{\sqrt{89}}$ mph (miles per hour)

Multivariable Functions

(Functions of several variables)

1. Functions of several variables

For examples:

The area of a triangle has the formula: $A = \frac{1}{2}bh$

where b = length of the base and h = height .

The volume of a rectangular box: $V = Lwh$

where L = length , w = width , h = height .

The arithmetic mean: $\bar{x} = \frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right)$

where $x_1, x_2, \ldots, x_n$ are n real numbers.

From the above example , we have that

A is a function of 2 variables: b and h,

V is a function of 3 variables: L, w, h,

$\bar{x}$ is a function of n variables: $x_1, x_2, \ldots, x_n$.

Notation: $z = f(x, y)$

It means that z is a function of x and y. The variables x and y are called *independent variables* or inputs while the variable z is called a *dependent variable* or output. Analogously, $w = f(x, y, z)$ means w is a function of x, y and z. Also, $u = f(x_1, x_2, \ldots, x_n)$ refers to u as a function in terms of variables $x_1, x_2, \ldots, x_n$.

Definition 1: A function f of two variables x and y is the assignment of each point (x, y) in its domain $D \subseteq \mathbb{R}^2$ to some real number $f(x, y)$.

Definition 2: A function f of three variables x, y and z is the assignment of a point (x, y, z) in its domain $D \subseteq \mathbb{R}^3$ to some real number $f(x, y, z)$.

Example 1 Find the domain and draw the graph of the domain of the functions below:

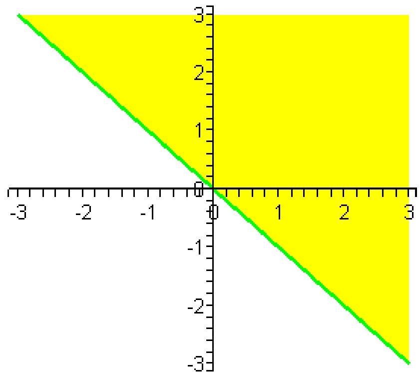
1.1 $f(x, y) = \sqrt{x+y}$.

1.2 $f(x, y) = \sqrt{x}+\sqrt{y}$.

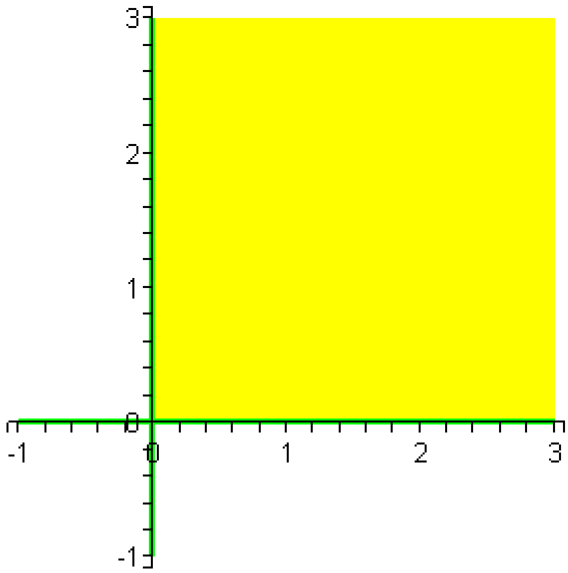
1.3 $f(x, y) = \ln(9-x^2-9y^2)$.

Solution

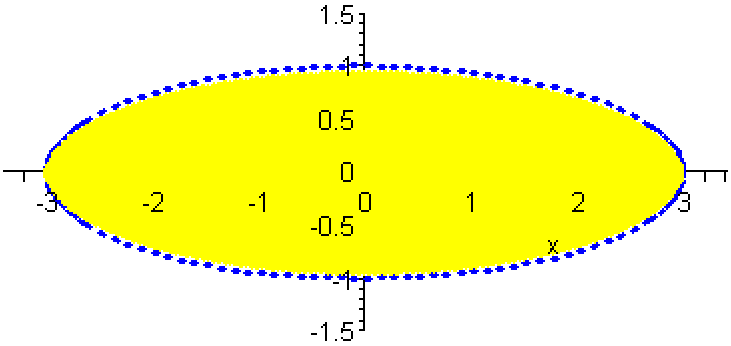
1.1



1.2



1.3



Example 2 Let $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2-16}}$. Find its domain.

Solution

Three Dimensional Spaces

1. Rectangular Coordinate System in Three-dimensional Space

It consists of three orthogonal axes called x-axis, y-axis and z-axis. The intersection point of all axes is called the *origin*.

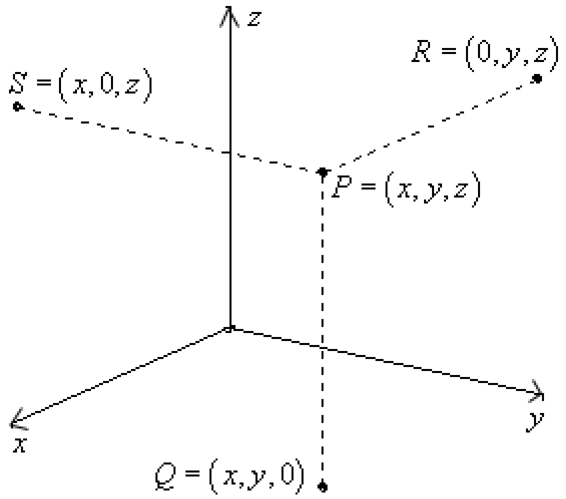


Figure 1 Three dimensional space

Every pair of (x, y, z) forms a coordinate plane: xy-plane, xz-plane and yz-plane. These three planes divide the space into eight parts. Each of which is called *octant*. The first octant is the octant that all x, y, z are positive. The other octants show different signs of x, y, z.

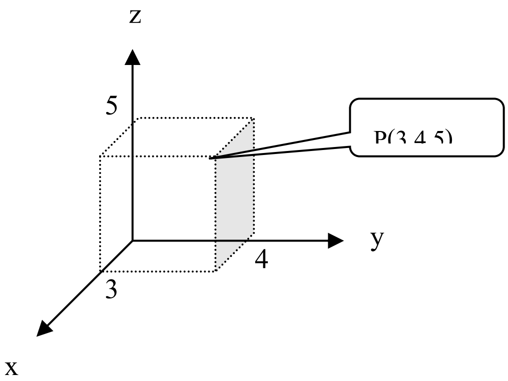
The rectangular coordinate system (x, y, z) refers to the distance from the point to each plane. In particular, the x-coordinate measures the distance from the point to yz-plane while the y-coordinate and z-coordinate indicate the distance from the point to the xz-plane and the xy-plane, respectively.

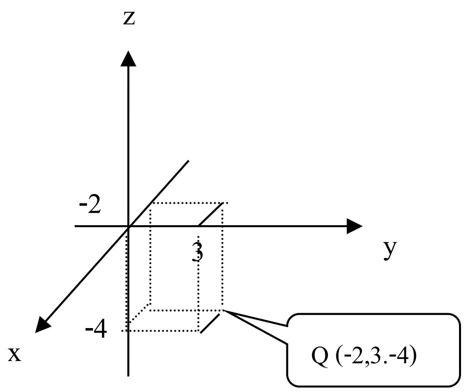
Example1 Locate the following points in the three-dim space.

a. $P(3,4,5)$

b. $Q(-2,3,-4)$

Solution

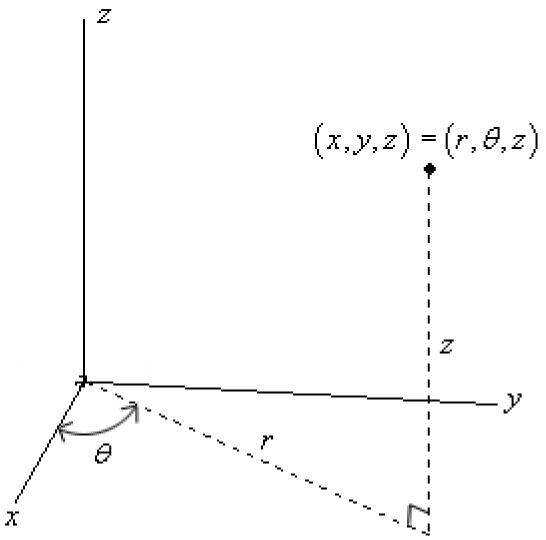




2. Cylindrical and Spherical Coordinates

Beside the rectangular coordinate system, we can locate a point in three dimensional space by cylindrical and/or spherical coordinates as follow:

- Cylindrical coordinate, we use the coordinate (r, θ, z).
- Spherical coordinate, we use the coordinate (ρ, θ, ϕ).



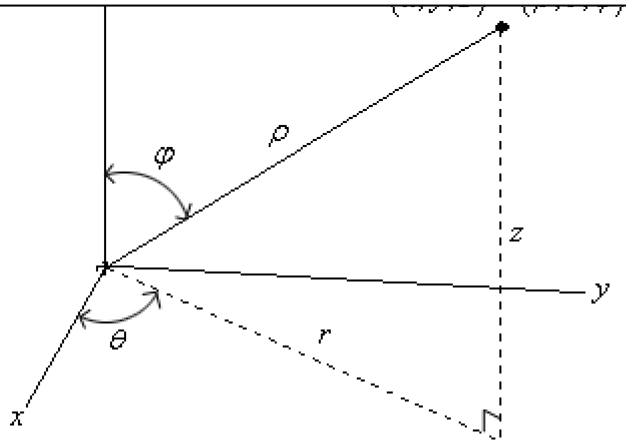
Cylindrical coordinate

Relationship between rectangular and cylindrical coordinates:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

$$x^2 + y^2 = r^2\ ,\ \tan\theta = \frac{y}{x}$$

where $\quad r \geq 0,\ \ 0 \leq \theta \leq 2\pi$.



Spherical coordinate

The relationship between rectangular and spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2 \,, \; \tan\theta = \frac{y}{x}, \; \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

where $\rho \ge 0, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$.

Example 1 Find the rectangular coordinates of these two points.

a. $(r, \theta, z) = \left(4, \frac{\pi}{3}, -3\right)$ b. $(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$

Solution a.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$x^2 + y^2 = r^2 \,, \; \tan\theta = \frac{y}{x}$$

b.

$$\boxed{\begin{gathered} x=\rho\sin\phi\cos\theta,\;\; y=\rho\sin\phi\sin\theta,\;\; z=\rho\cos\phi \\ x^2+y^2+z^2=\rho^2\;,\;\tan\theta=\frac{y}{x},\;\cos\phi=\frac{z}{\sqrt{x^2+y^2+z^2}} \end{gathered}}$$

Example 2 Covert the following equations:

a. $z = x^2 + y^2 - 2x + y$ to cylindrical coordinate system.

b. $z = x^2 + y^2$ to spherical coordinate system.

Solution: a.

$$\boxed{\begin{gathered} x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ x^2 + y^2 = r^2\ ,\ \tan\theta = \frac{y}{x} \end{gathered}}$$

b.

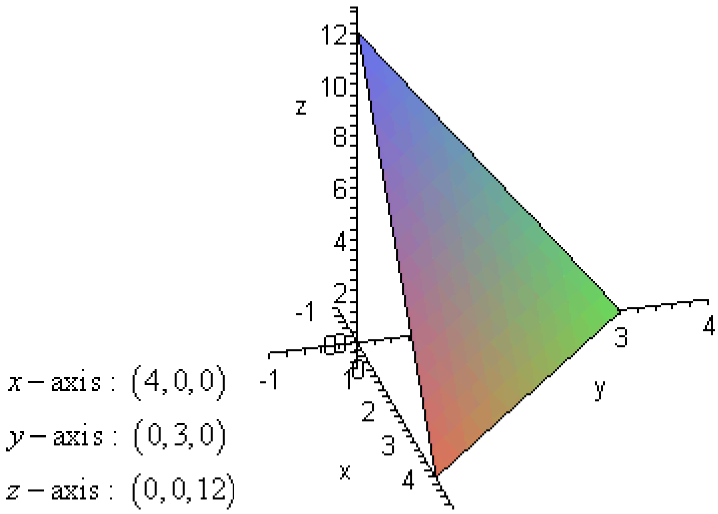
$$\boxed{\begin{gathered} x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi \\ x^2 + y^2 + z^2 = \rho^2\ ,\ \tan\theta = \frac{y}{x},\ \ \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{gathered}}$$

2. Graphs of multivariable functions

A graph of two variable functions $z = f(x, y)$ on three dimensional space is just a surface.

Example 3 Draw a graph of $f(x, y) = 12-3x-4y$ on XYZ-space.

Solution



2.1 Level Curves

By using the same method, we are not able to draw a graph of $w = f(x, y, z)$ since its graph will be in four dimensions. How can we solve this problem?

Let us go back to a function of two variables. Consider a geological map which is a picture of area in 2 and 3 dimensions. Figures A and C below show the three dimensional pictures of mountain area with contour lines (traces). Figure B shows a 2-dimensional picture with different heights indicated.

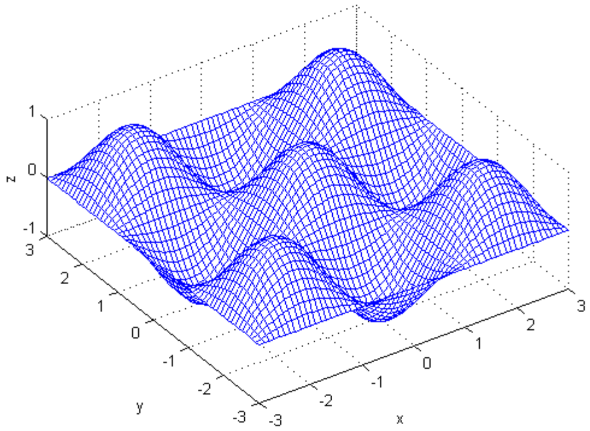


Figure A

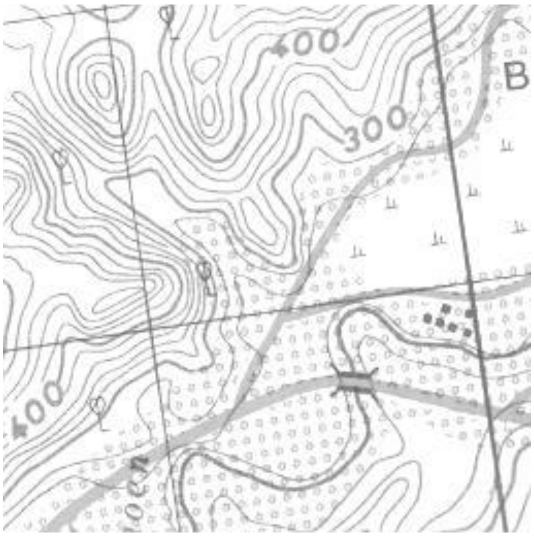


Figure B

Figure D shows several curves of area at different heights in 3 dimensions. Figure E is the projection of curves in figure D onto the xy-plane. This shows how we obtain a map as in figure B.

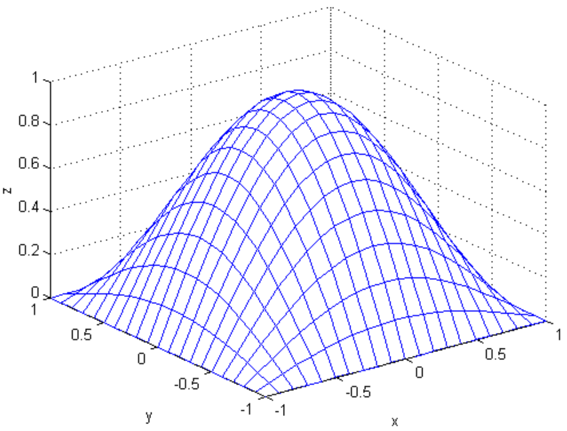


Figure C

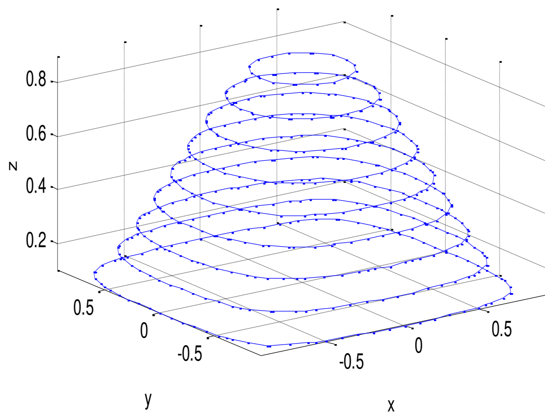


Figure D

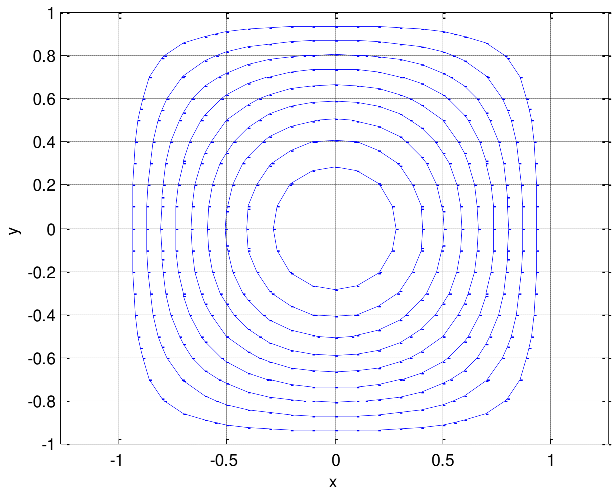
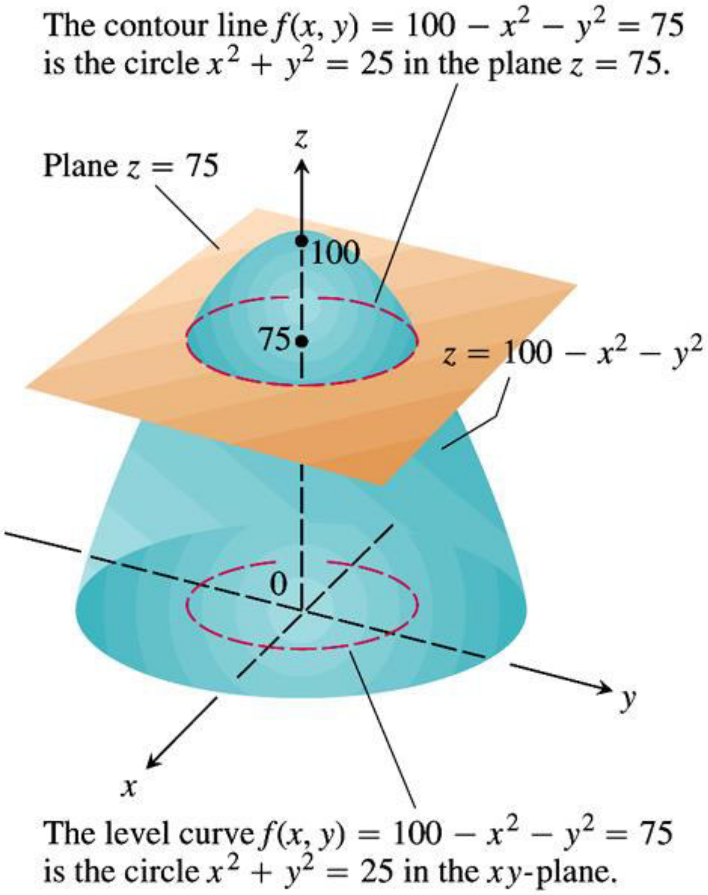


Figure E

The curves in figure E are called *level curves*. Each curve represents $f(x, y) = k$ on the xy-plan.

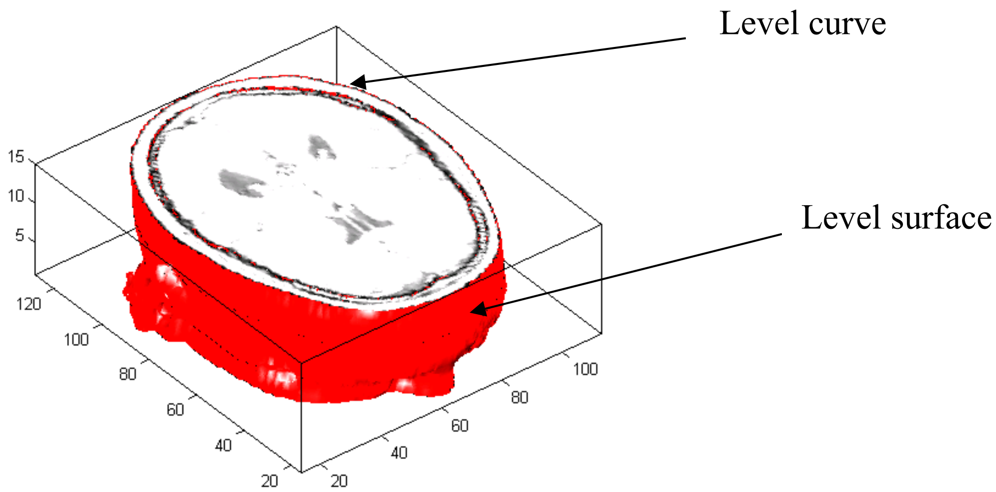
Level curves have been widely used in the atmospheric map to indicate the area with some fixed condition such as constant pressure level and constant temperature. We call the level curve for a constant pressure "isobar," and call the level curve for a constant temperature "isotherm."

Example 4 Draw several level curves of $f(x, y) = 100 - x^2 - y^2$ when $k = 100$, $k = 75$, $k = 0$.



2.2 Level Surface

In the case of 4 variables function $w = f(x, y, z)$, we are not able to draw a graph in 3 dimensional space. For example, we take the MRI picture of someone's brain at a given time. Here, the time becomes another variable. We can see the change by looking at the brain at several different times. At time c, we have $f(x, y, z) = c$. This given equation forms the area called "level surface," as shown here.



Exercise

1. Let $f(x, y) = 3x + y^2$. Evaluate the following:

   (a) $f(2, 3)$ (b) $f(2, \sqrt{2})$ (c) $f(0,0)$.

2. Find the domain of the following functions and draw a graph of its domain.

   (a) $f(x, y) = \dfrac{x^2 + y^2 + 8}{(x-4)(y-3)}$

   (b) $g(x, y) = e^{\sqrt{x}} + \ln y$

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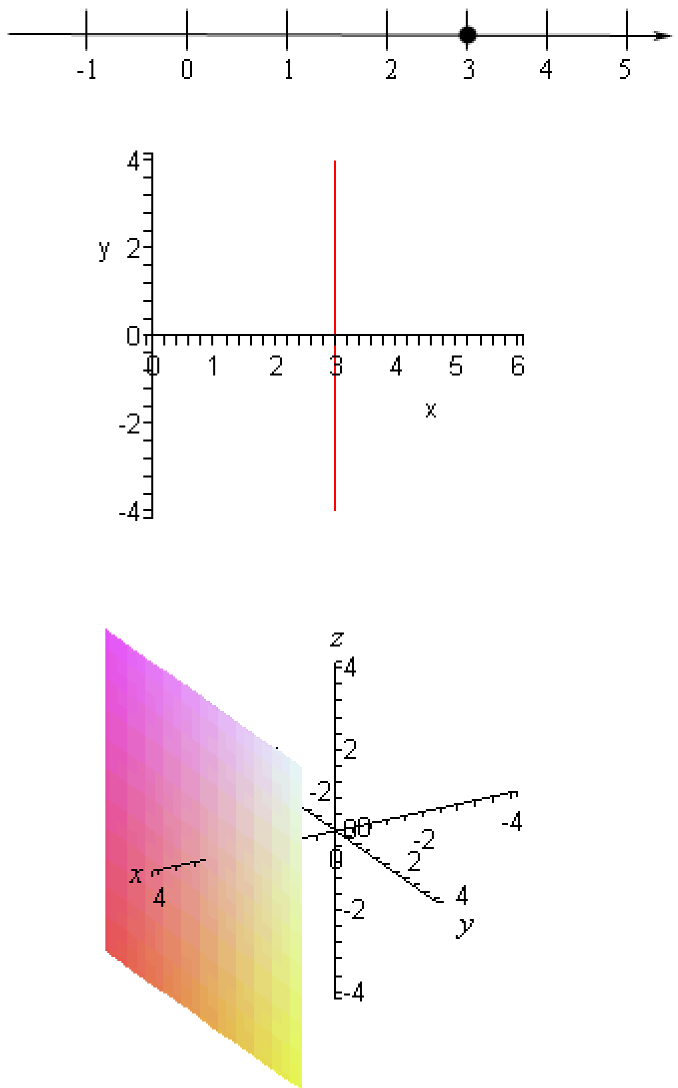
3. Graph of Planar Surface

Let A, B, C, D be some constants.

Definition: A plane is a set of all points in the three dimensional space satisfying the equation $Ax + By + Cz + D = 0$.

Example 1 Draw the graphs of $x=3$ in one-, two- and three-dimensional spaces.

Solution:



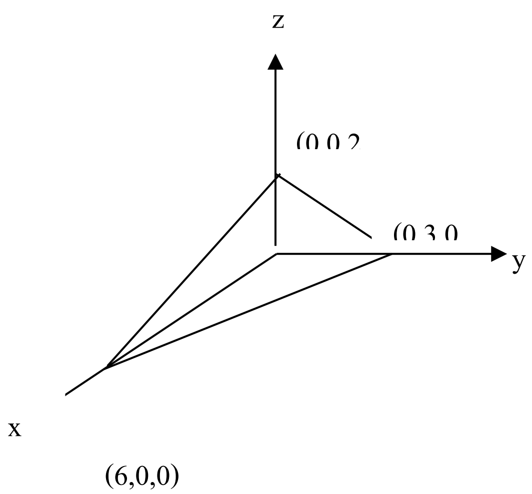
To sketch a planar graph of $Ax + By + Cz + D = 0$, we generally draw a plane only on the octant where the plane lies on by finding the intercepts of the plane and each axis. Then connect all intercepted points with lines.

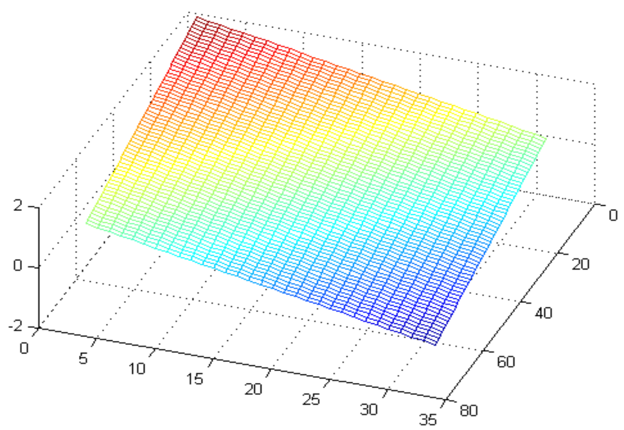
Three intercepted points:

1. If $x = 0,\ y = 0$, the point $(0, 0, z)$ is the z-intercept.
2. If $x = 0,\ z = 0$, the point $(0, y, 0)$ is the y-intercept.
3. If $y = 0,\ z = 0$, the point $(x, 0, 0)$ is the x-intercept.

Example 2 Draw a graph of $x + 2y + 3z = 6$.

Solution:





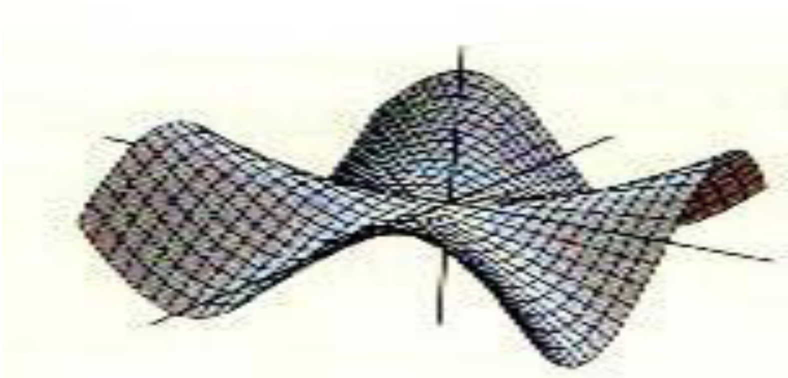
4. Graphs of Quadratic Surfaces

A graph of quadratic surface is a set of all points in three dimensional space satisfying the following equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, I, J$ are constants.

To draw a graph of quadratic surface, we cut the surface by the planes parallel to xy-, yz- and xz-planes making several traces on the surface. These traces form the graph of a surface.



The figure above shows traces of function: $z = \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}$.

The important quadratic surfaces you should know are the following:

1. Sphere

2. Ellipsoid

3. Hyperboloid of one sheet

4. Hyperboloid of two sheets

5. Elliptic cone

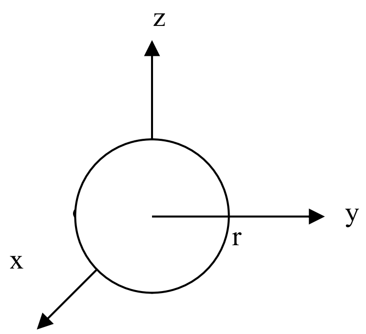
6. Elliptic paraboloid

7. Hyperbolic paraboloid

8. Cylinder

4.1 Sphere

The equation of a sphere centered at (0,0,0) with radius r has the form: $x^2 + y^2 + z^2 = r^2$.

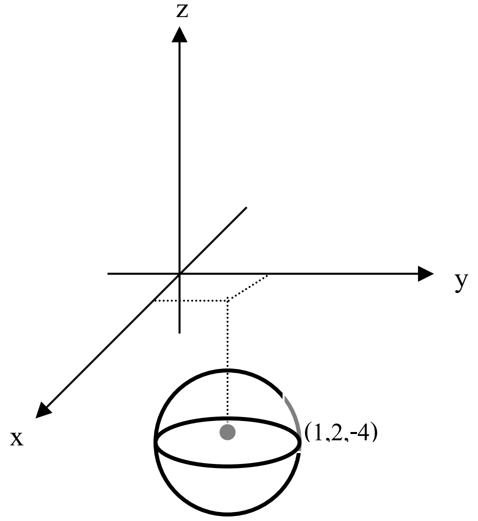


Similarly, the equation of a sphere centered at (x_0, y_0, z_0) with radius r has the form:

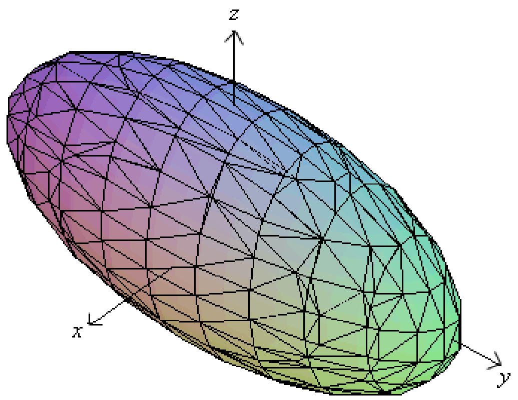
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.$$

Example 1: Draw a graph of

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0.$$



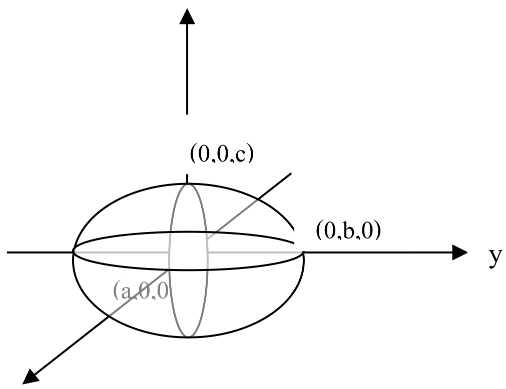
4.2 Ellipsoid



The equation of an ellipsoid centered at $(0,0,0)$ has the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a, b, c are some constants.



Similarly, the equation of an ellipsoid centered at (x_0, y_0, z_0) has the form:

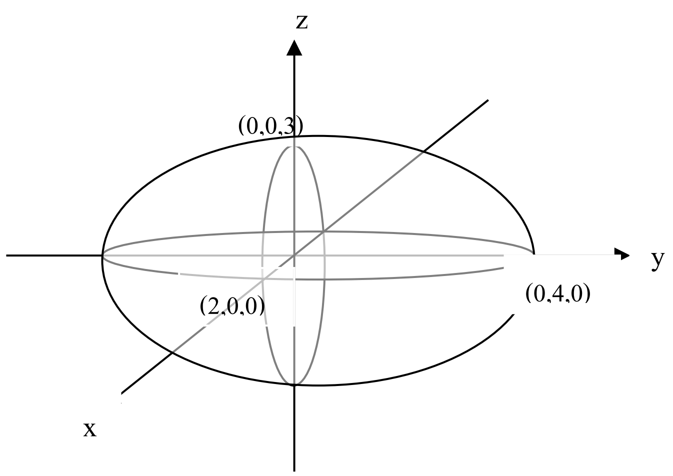
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1.$$

Example Draw a graph of quadratic surface of

$$\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

and find the equation of a graph after cutting the surface by the plane $x=k$, where k is some constant.

Solution



The plane $x=k$ is parallel to the yz-plane, after cutting the surface by this plane, we get the equation: $\frac{y^2}{16}+\frac{z^2}{9}=1-\frac{k^2}{4}$ which is an ellipse on the plane $x=k$; $-2\le k\le 2$.

4.3 Hyperboloid of one sheet

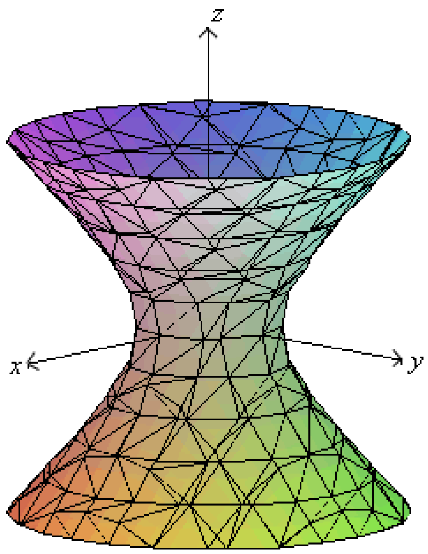
The equation of a hyperboloid of one sheet centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



The graph of quadratic surface of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Remark: If the hyperboloid of one sheet lies along the line parallel to the z-axis and centered at (x_0,y_0,z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1.$$

Similarly, if it lies along the line parallel to the y-axis, its equation has the form: $\dfrac{(x-x_0)^2}{a^2}-\dfrac{(y-y_0)^2}{b^2}+\dfrac{(z-z_0)^2}{c^2}=1$.

If it lies along the line parallel to the x-axis, its equation has the form: $-\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}+\dfrac{(z-z_0)^2}{c^2}=1$.

4.4 Hyperboloid of two sheets

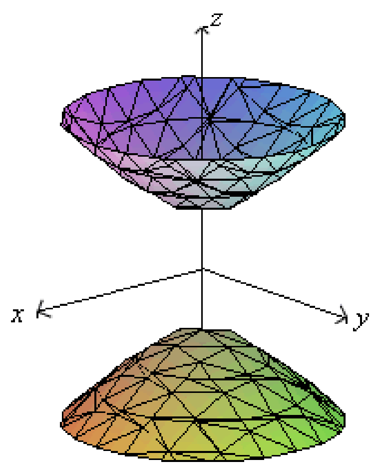
The equation of a hyperboloid of two sheets centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along y-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Lie along x-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



The graph of quadratic surface $-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

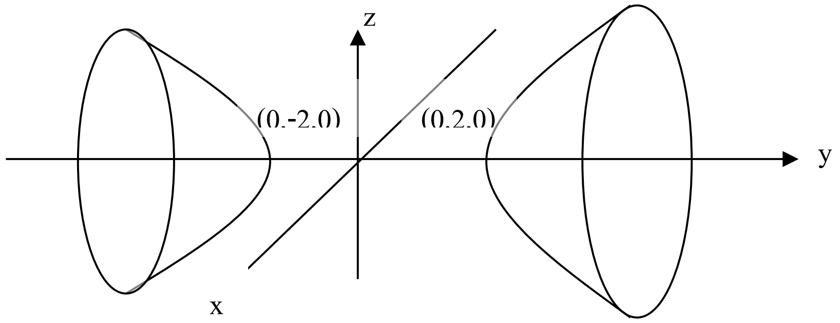
Remark:

If the hyperboloid of two sheets lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$-\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1.$$

Similarly, in the case of hyperboloid of two sheets lying along the y-axis or x-axis, we get the similar form of the equation.

Example: Draw a surface of $4x^2 - y^2 + 2z^2 + 4 = 0$ and find the equations of the traces after cutting the surface by each plane.



4.5 Elliptic cones

The equation of an elliptic cone centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

where a, b, c are some constants.

Remark:

If the elliptic cone lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=0.$$

Similarly, in the case of elliptic cone lying along the y-axis or the x-axis, we get the similar form of the equation.

Example: Draw a graph of the quadratic surface

$$x^2 + y^2 - z^2 - 2x + 6z - 8 = 0.$$

4.6 Paraboloid

The equation of a paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis and open on positive z: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

open on negative z: $z = -\frac{x^2}{a^2} - \frac{y^2}{b^2}$

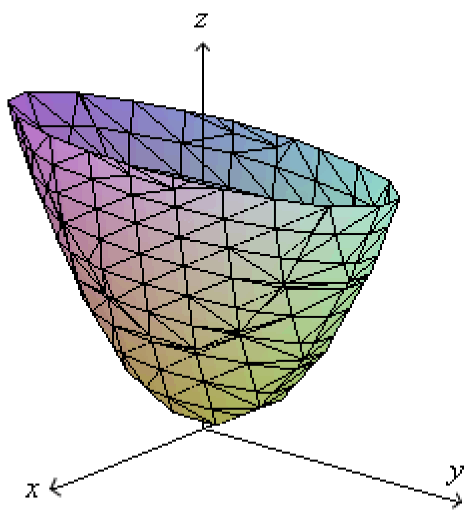
Lie along y-axis and open on positive y: $y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$

open on negative y: $y = -\frac{x^2}{a^2} - \frac{z^2}{c^2}$

Lie along x-axis and open on positive x: $x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$

open on negative x: $x = -\frac{y^2}{b^2} - \frac{z^2}{c^2}$

where a, b, c are some constants.



The graph of quadratic surface $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

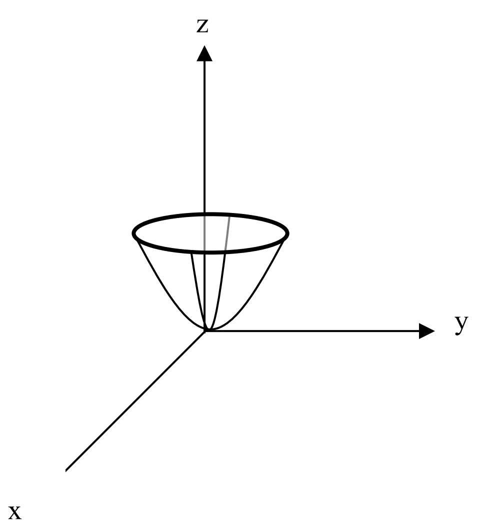
Remark:

If the paraboloid lies along the line parallel to z-axis, centered at (x_0, y_0, z_0) and opens on positive z, its equation is of the form:

$$(z - z_0) = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}$$

Similarly, in the case of paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Example Draw a graph of $z = \frac{x^2}{4} + \frac{y^2}{9}$.



4.7 Hyperbolic Paraboloid

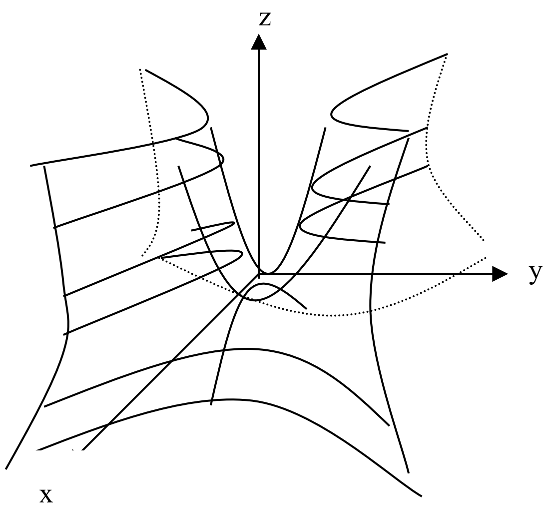
The equation of a hyperbolic paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis : $z = -\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Lie along y-axis : $y = -\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$, $y = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$

Lie along x-axis : $x = -\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$, $x = \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$

where a, b, c are some constants.



The graph of quadratic surface $z = -\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Remark: If the hyperbolic paraboloid lies along the line parallel to z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

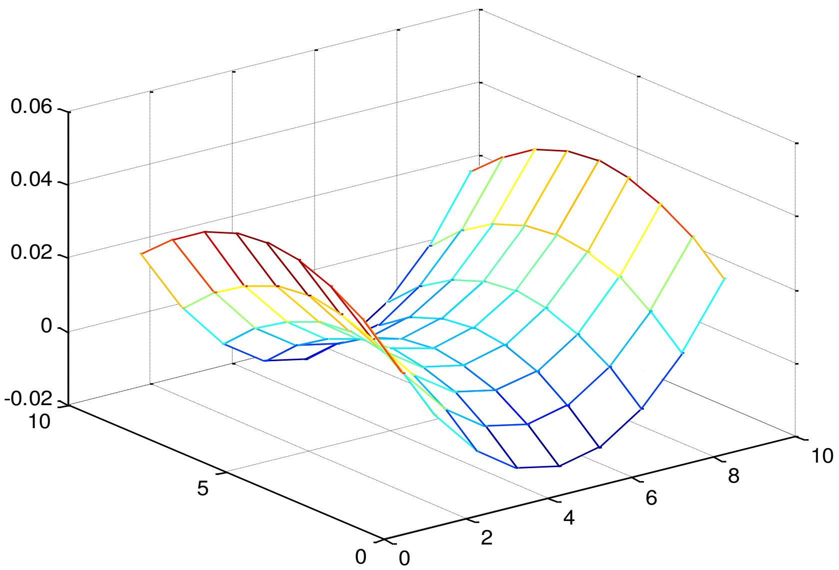
$$(z-z_0) = -\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

or $$(z-z_0) = \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2}$$

Analogously, in the case of hyperbolic paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Note that in the case of a hyperbolic paraboloid, its center can also be called as a *saddle point*.

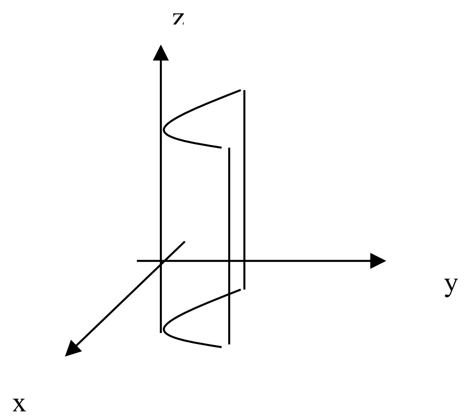
Example: Draw the quadratic surface of $z = \frac{x^2}{4} - \frac{y^2}{9}$.


0.06
0.04
0.02
0
-0.02
10
5
0
0
2
4
6
8
10

4.8 Cylinders

There are several types of cylinders we need to learn such as cyclic cylinders, elliptic cylinders, hyperbolic cylinders, and parabolic cylinders.

The equation forms of each cylindrical surface are in 2-variables. Since we consider a graph in 3-dimensional space, the missing variable has values in $(-\infty, \infty)$. Thus, the cylinder will lie along the axis of missing variable. For example, the function $y = ax^2$ where a is a positive constant. This equation forms a parabolic cylinder lying along z-axis as shown in the figure below:



Parabolic cylinder of $y = ax^2$ when $a > 0$.

Problem: Draw the following graphs in the 3-dimensional space.

a. $z = \sqrt{y^2 + 1}$

b. $y - z^2 = 0$

c. $y^2 - x^2 = 1$

d. $25x^2 + 9z^2 = 1$

Solution:

Partial Derivatives of a Two Variable Function

In case of one variable function $y = f(x)$, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

But it does not apply to the case of two variable function $z = f(x, y)$ since there are two independent variables.

To find the derivatives of $f(x, y)$, we need to calculate the derivatives with respect to each independent variable separately. We call it ***"partial derivative"***.

Partial Derivatives

How to calculate the partial derivative?

To find a partial derivative of $f(x, y)$ with respect to one variable, we consider another input variable as a constant. For example, to find the partial derivative of $f(x, y)$ with respect to x, we considers y as a constant. Then, we take an ordinary derivative of $f(x, y)$ with respect to x as in the case of one variable function.

Notations for Partial Derivatives

Let $z = f(x, y)$.

The partial derivative of $f(x, y)$ with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

If we want to evaluate the partial derivative at (x_0, y_0), we use the notations:

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0).$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y.$$

Partial Derivative with Respect to x

Definition The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is defined as

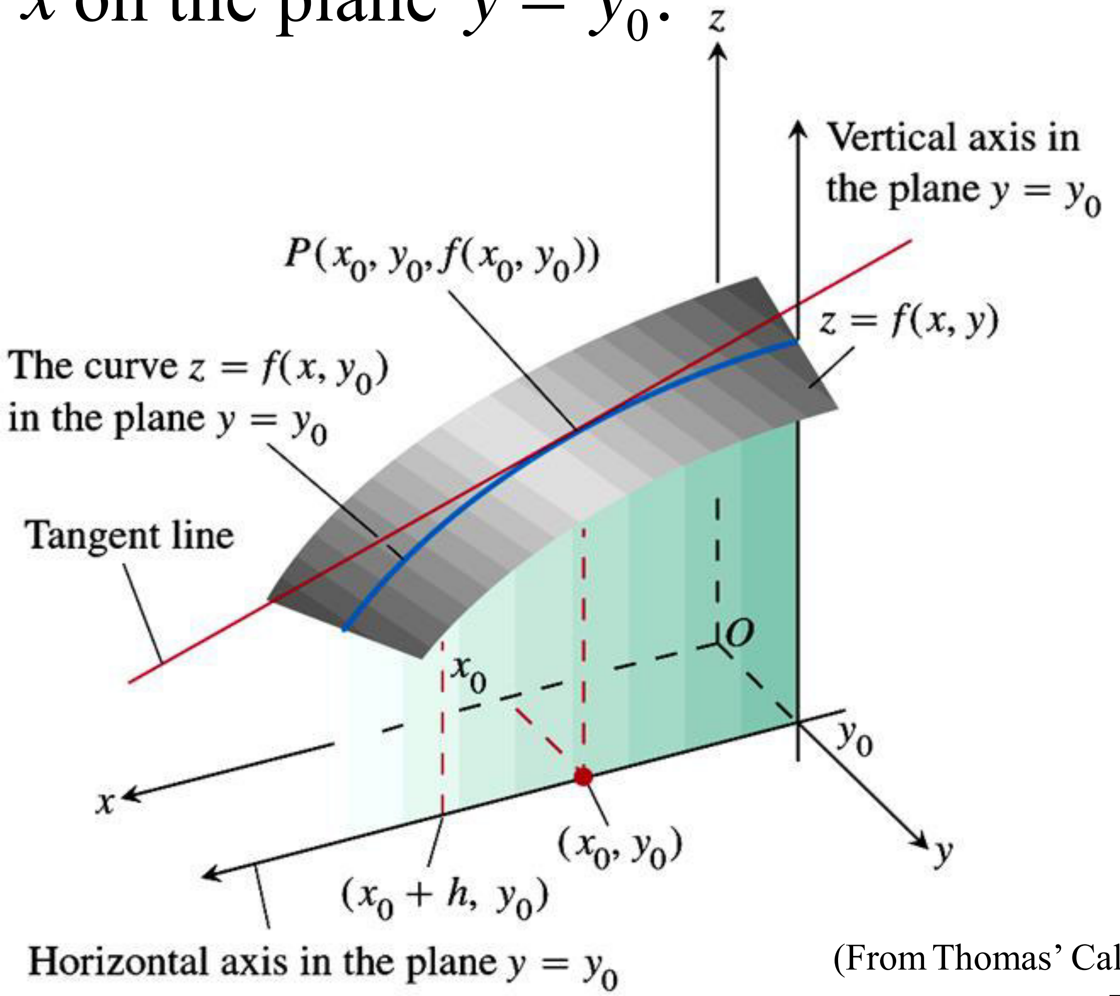
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_x(1,0)$.

Geometrical Interpretation of f_x

Let y_0 be a constant. The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $y = y_0$. Note that curve $(x, y_0, f(x, y_0))$ is a function of only one variable x on the plane $y = y_0$.

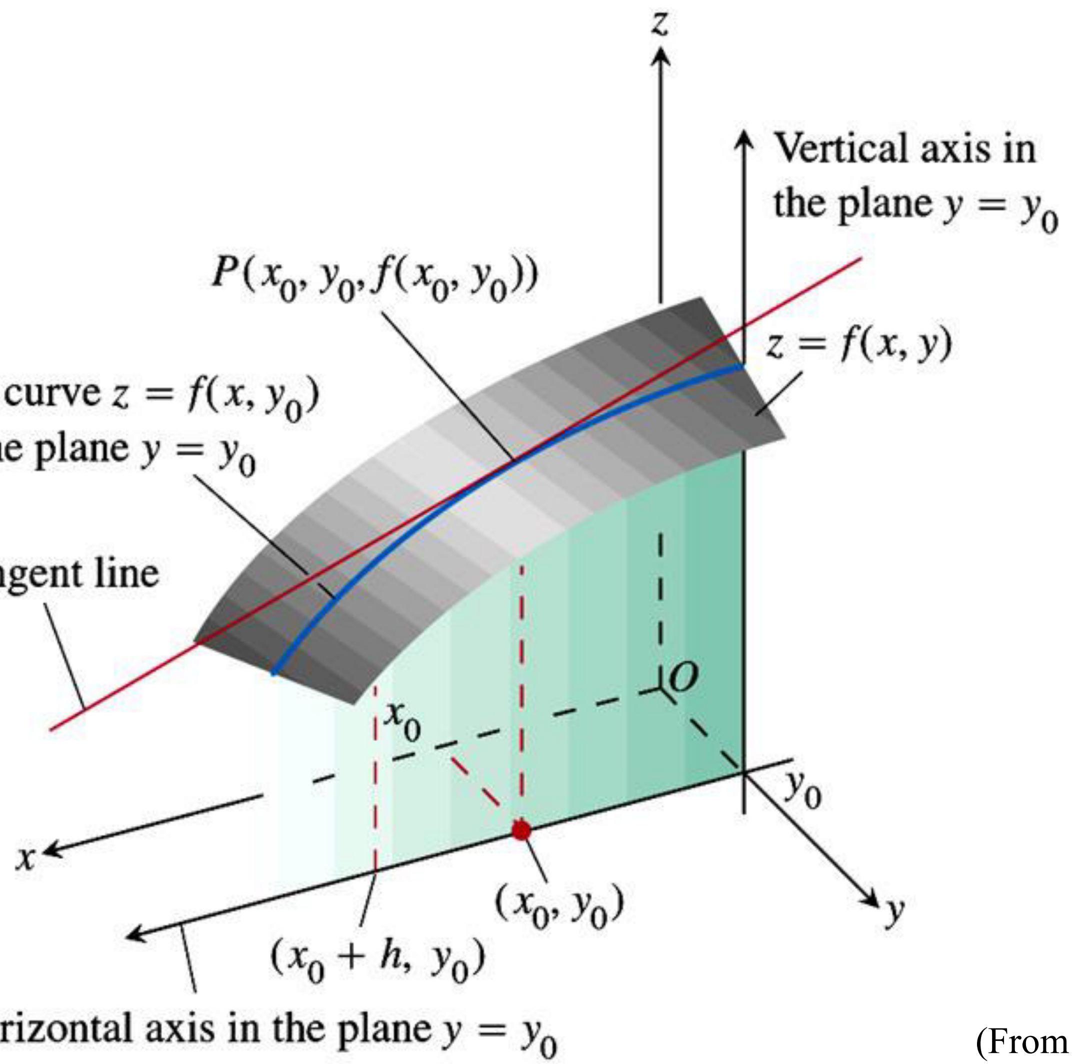
Set $y = y_0$ to be a constant.



(From Thomas' Calculus)

Geometrical Interpretation of f_x

$f_x(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).



(From Thomas' Calculus)

Partial Derivative with Respect to y

Definition The partial derivative of $f(x, y)$ with respect to y at (x_0, y_0) is defined as

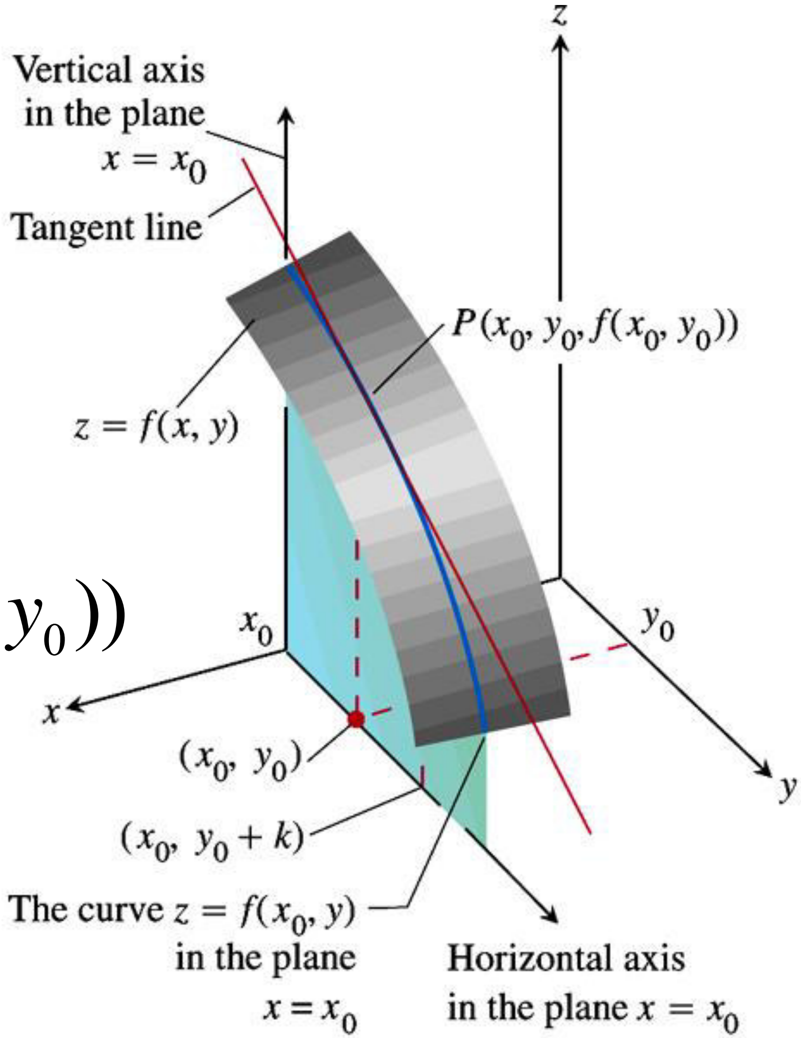
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_y(1, 0)$.

Geometrical Interpretation of f_y

$f_y(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).

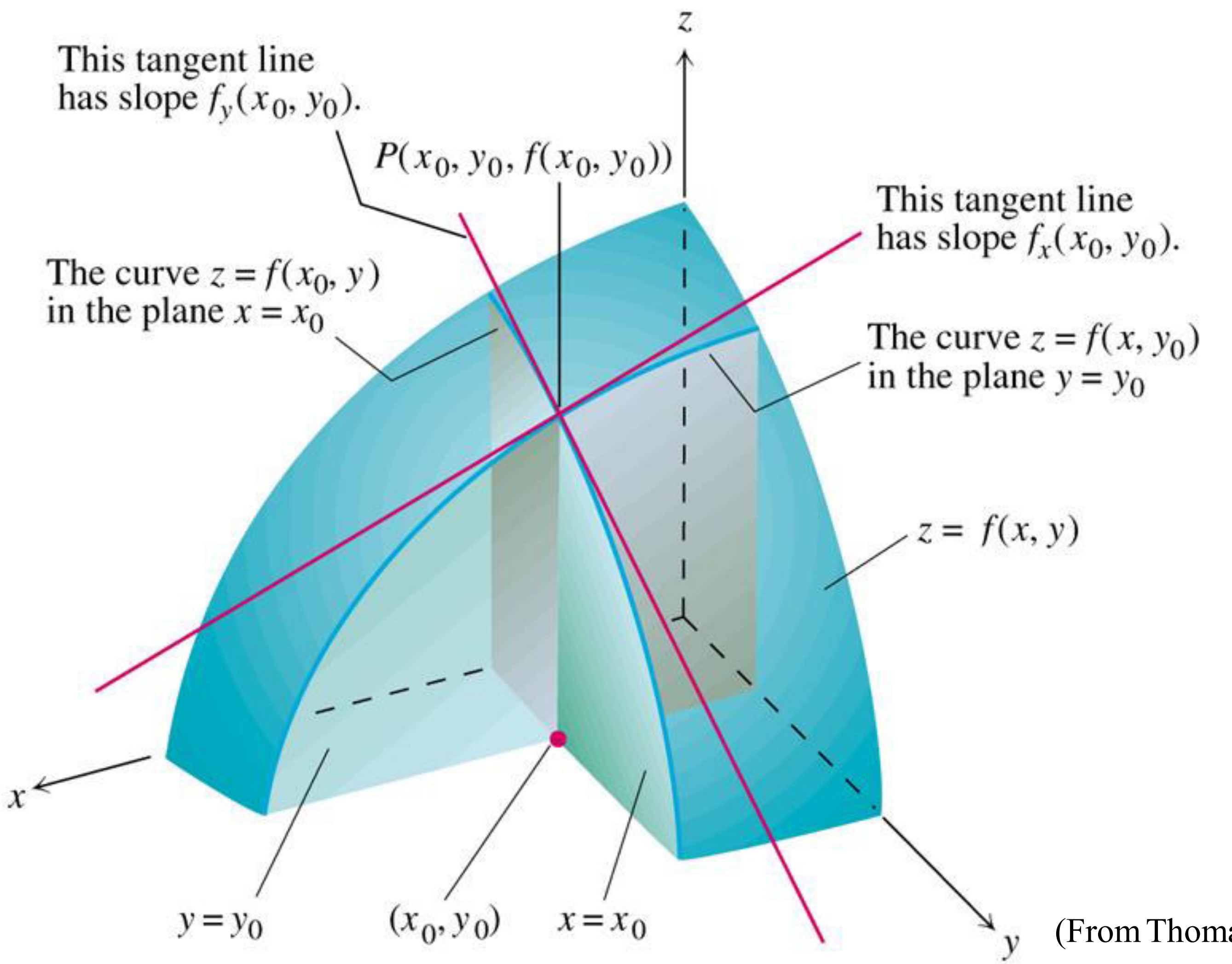
The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $x = x_0$.
Note that the curve $(x, y_0, f(x, y_0))$ is a function of one variable y.



(From Thomas' Calculus)

Partial Derivatives

There are many tangent lines to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ depending on the directions.



(From Thomas' Calculus)

Example: Partial Derivatives

Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Example: Partial Derivatives

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Implicit Partial Differentiation

In some case, for example $z^2+2z+1=x^2+2xy+y^2$, it's not easy to write $z=f(x,y)$ before calculating the derivatives. In this situation, we can find the partial derivative by taking partial derivative operator $\dfrac{\partial}{\partial x}$ on both sides of the equation:

$$\frac{\partial}{\partial x}(z^2+2z+1)=\frac{\partial}{\partial x}(x^2+2xy+y^2).$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(z^2)=\frac{\partial(z^2)}{\partial z}\frac{\partial z}{\partial x}=2z\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(2z)=2\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(x^2+2xy+y^2)=2x+2y
\end{aligned}\right\}\quad (2z+2)\frac{\partial z}{\partial x}=2x+2y$$

Thus, $\dfrac{\partial z}{\partial x}=\dfrac{x+y}{z+1}$.

Example: Implicit Partial Differentiation

Let $xz^2 + e^z = \sin(x^2 + y^2)$. Find $\dfrac{\partial z}{\partial x}$.

Functions of More Than 2 Variables

To find partial derivatives of multivariable functions, we use the same method as in the case of two-variable functions. For example, the case of the partial derivative of $f(x, y, z)$ with respect to x, we consider y and z as constants. Then, we take a usual derivative of $f(x, y, z)$ with respect to x as in the case of one variable function.

Example Let $f(x, y, z) = x\cos(3y - z^2)$. Then

$$\frac{\partial f}{\partial z} =$$

Second Order Partial Derivatives

We just take partial derivative twice which consists of four possibilities.

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx} \qquad\qquad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx} \qquad\qquad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Then

$$\frac{\partial^2 f}{\partial x^2} =$$

$$\frac{\partial^2 f}{\partial y^2} =$$

$$\frac{\partial^2 f}{\partial x \partial y} = \qquad \frac{\partial^2 f}{\partial y \partial x} =$$

Example: Second Order Partial Derivatives

Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Then

$$\frac{\partial f}{\partial x} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}, \qquad \frac{\partial f}{\partial y} = \frac{-2xy}{\left(x^2 + y^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2x\left(x^2 + y^2\right)^2 - 2\left(x^2 + y^2\right)(2x)(y^2 - x^2)}{\left(x^2 + y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x\left(x^2 + y^2\right)^2 - 2\left(x^2 + y^2\right)(2y)(-2xy)}{\left(x^2 + y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2y\left(x^2 + y^2\right)^2 - 2\left(x^2 + y^2\right)(2x)(-2xy)}{\left(x^2 + y^2\right)^4} \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2y\left(x^2 + y^2\right)^2 - 2\left(x^2 + y^2\right)(2y)(y^2 - x^2)}{\left(x^2 + y^2\right)^4}.$$

The Mixed Derivative Theorem

The 2nd partial derivatives $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ contain $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.

They are called "Mixed partial derivatives".

Theorem:

If f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in open region containing (a,b) and all are continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

This theorem implies that if f, f_x, f_y, f_{xy} and f_{yx} are all continuous, then the order of $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in mixed partial derivative does not matter.

Example: The Mixed Derivative Theorem

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3\cos\left(3x - y^2\right)(-2y) = 6y\cos\left(3x - y^2\right),$$

$$\frac{\partial f}{\partial y} = 2y\sin\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\cos\left(3x - y^2\right)(3) = 6y\cos\left(3x - y^2\right).$$

In this case, $\boxed{\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}}$.

Partial Derivatives of Higher Order

Partial derivative operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ may be applied several times such as

$$\frac{\partial^3 f}{\partial y \partial x^2} \quad \text{or} \quad f_{xxy}, \qquad\qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} \quad \text{or} \quad f_{xxyy}.$$

Example Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial x^2} = -9\cos\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 9\sin\left(3x - y^2\right)\left(-2y\right) = -18y\sin\left(3x - y^2\right).$$

Differentiability of a Functions of Two Variables

Definition A function f is said to be *differentiable* at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and it is called a *differentiable function* if it is differentiable everywhere.

Continuity of Partial Derivatives

If f_x and f_y are both continuous in open region *R*, then f is differentiable everywhere in *R*.

Differentiability Implies Continuity

If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Composite Functions in Higher Dimensions

Let $w = f(x, y)$, $y = h(t)$ and $x = g(t)$.

Thus,

$$w = f(g(t), h(t))$$

We say that w is a composite function in terms of t,
x and y are called "intermediate variables",
w is called a "dependent variable",
and t is called an "independent variable".

Chain Rule for Functions of Two Independent Variables

Derivative of a composite function can be calculated by applying the chain rule. In case of one-variable functions $y = f(x)$ and $x = g(t)$, we have

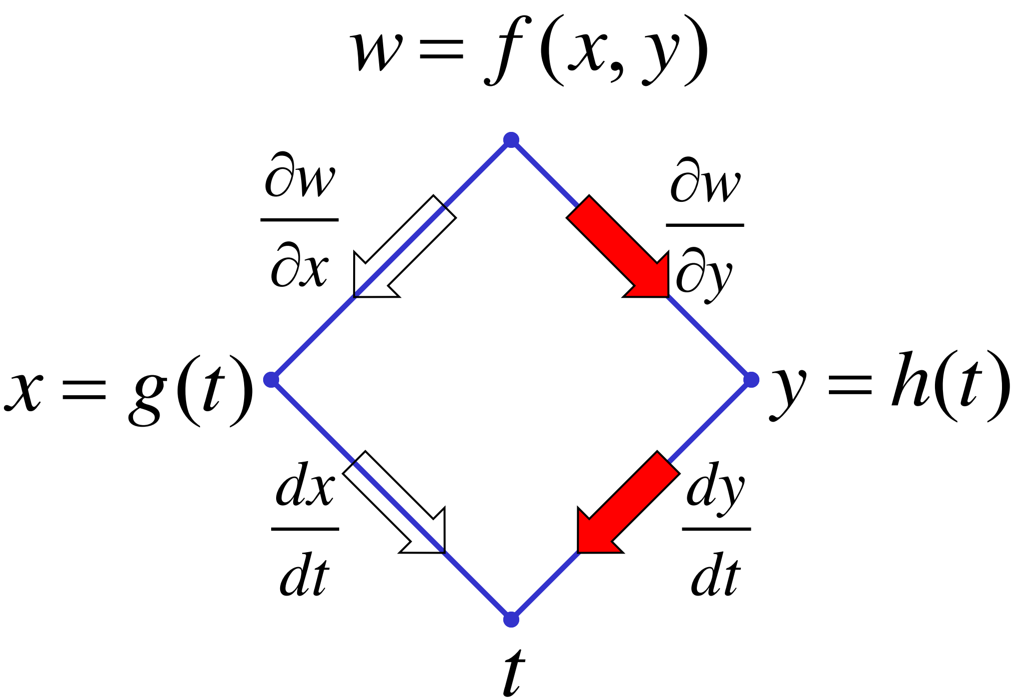
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In case of two-variable functions, let $w = f(x, y)$ and $y = h(t)$ and $x = g(t)$ where f, g and h are differentiable. We have

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Tree Diagram

This diagram shows how to find the derivatives of a composite function.



1. Each node represents each variable.
2. Arrow is the derivative of beginning node with respect to the ending node.
3. Start with the dependent variable w and then walk along all branches to t.
4. Add them all up.

Get $\frac{\partial w}{\partial x}, \frac{dx}{dt}$

Get $\frac{\partial w}{\partial y}, \frac{dy}{dt}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example: Chain Rule for Functions of 2 Variables

Let $w = x^2 + 2xy + y^2$, $x = \cos(t)$ and $y = \sin(t)$. Find $\dfrac{dw}{dt}$.

Chain Rule for Functions of 3 Independent Variables

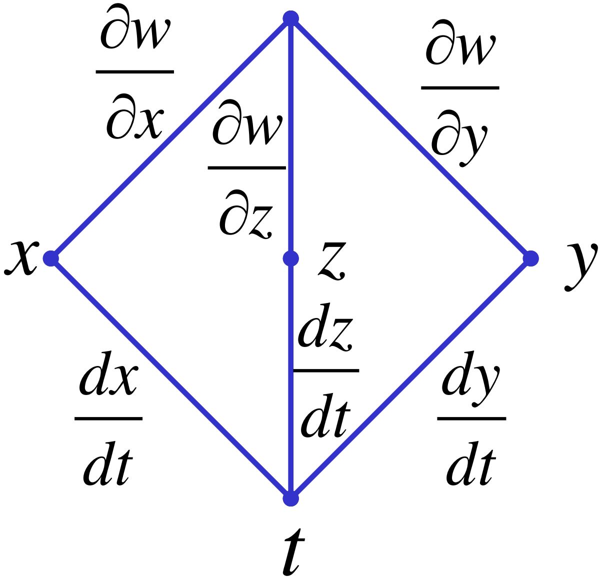
In case of three-variable functions

$$w = f(x, y, z),\ x = g(t),\ y = h(t) \quad \text{and} \quad z = k(t)$$

where f, g, h and k are differentiable. We have

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.}$$

$$w = f(x, y, z)$$



$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Chain Rule for Functions of 3 Variables

Let $w = xy + yz + zx,\ x = \cos(t),\ y = \sin(t)$ and $z = t^2$. Then

$$\frac{\partial w}{\partial x} = y + z, \qquad \frac{\partial w}{\partial y} = x + z, \qquad \frac{\partial w}{\partial z} = y + x,$$

$$\frac{dx}{dt} = -\sin(t), \qquad \frac{dy}{dt} = \cos(t), \qquad \frac{dz}{dt} = 2t,$$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\
&= (y+z)\big(-\sin(t)\big) + (x+z)\big(\cos(t)\big) + (y+x)(2t) \\
&= (\sin t + t^2)\big(-\sin t\big) + (\cos t + t^2)\big(\cos t\big) + (\sin t + \cos t)(2t).
\end{aligned}$$

Functions Defined on Surfaces

Suppose that we have several two-variable functions as intermediate variables

$$w = f(x, y, z),\ x = g(r, s),\ y = h(r, s),\ z = k(r, s).$$

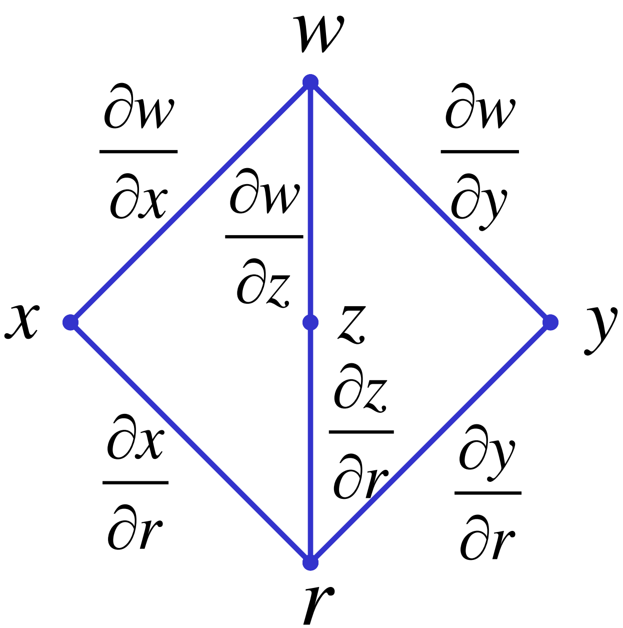
All x, y, z are considered as surfaces while w is a function of all three surfaces. Its partial derivatives are

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

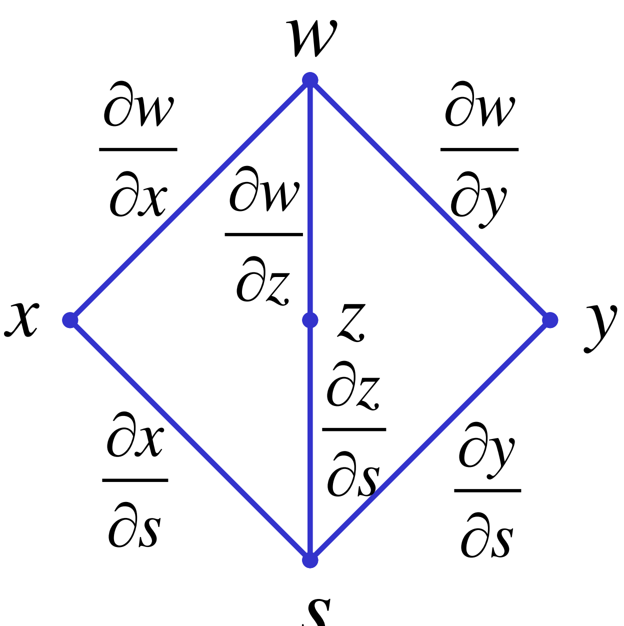
and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Tree Diagram for $f(g(r,s),h(r,s),k(r,s))$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example

Let $w = xz + y^2$, $x = \dfrac{r}{s}$, $y = r^2 + \ln(s)$ and $z = r^2$. Then

$$\frac{\partial w}{\partial x} = z, \qquad \frac{\partial w}{\partial y} = 2y, \qquad \frac{\partial w}{\partial z} = x,$$

$$\frac{\partial x}{\partial r} = \frac{1}{s}, \qquad \frac{\partial y}{\partial r} = 2r, \qquad \frac{\partial z}{\partial r} = 2r,$$

$$\frac{\partial x}{\partial s} = \frac{-r}{s^2}, \qquad \frac{\partial y}{\partial s} = \frac{1}{s}, \qquad \frac{\partial z}{\partial s} = 0,$$

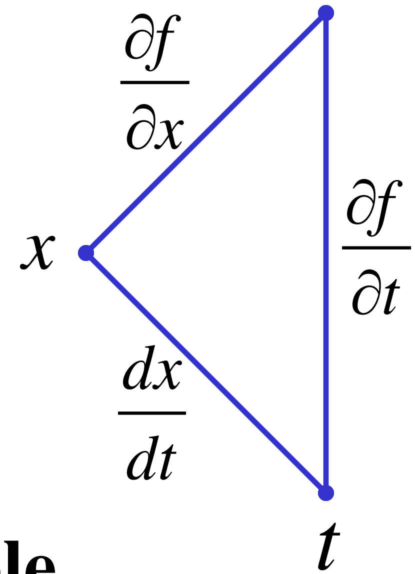
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{z}{r} + 2y(2r) + x(2r)$$

$$= r + 4r(r^2 + \ln s) + 2\frac{r^2}{s},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{-zr}{s^2} + 2\frac{y}{s} + x(0) = \frac{-r^3}{s^2} + 2\frac{r^2 + \ln s}{s}.$$

Functions in a Form of $w = f(t, x(t))$

Suppose that w is a function of one independent and one intermediate variables: $w = f(t, x)$



$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}}$$

Example

Let $w = xt + x^2$ where $x = (t-2)^2$. Find $\frac{dw}{dt}$.

$$\frac{\partial f}{\partial x} = t + 2x, \quad \frac{\partial f}{\partial t} = 2(t-2) \quad \text{and}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} = (t+2x)2(t-2) = 2\left(t + 2(t-2)^2\right)(t-2).$$

Implicit Functions

Explicit function is a function in a form $y = h(x)$. Then $\dfrac{dy}{dx}$ can be calculated easily. **Implicit function** is a function in a form $F(x, y) = 0$. We may compute $\dfrac{dy}{dx}$ by 2 methods:

<u>Method 1</u>

Rewrite $F(x, y) = 0$ into $y = h(x)$ before computing $\dfrac{dy}{dx}$.

This is sometimes difficult.

<u>Method 2</u>

Use the chain rule for multivariable functions.

Implicit Differentiation

Let y be a function defined implicitly in term of x.

We can find $\dfrac{dy}{dx}$ by following this procedure.

1. Set up $F(x, y) = 0.$

2. Differentiate $F(x, y) = 0$ with respect to x on both sides:

$$\frac{d}{dx} F(x, y) = \frac{d}{dx} 0.$$

Then, we get

$$0 = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example: Implicit Differentiation

Suppose that $x^2 + 2xy + y^2 = \sin(xy)$. Find $\dfrac{dy}{dx}$.

Implicit differentiations of a system of equations

- **Example** Let u and v be functions x and y such that

$$uv = x + y \qquad \text{and} \qquad u - v^2 = x - y.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}.$

Definition Let F and G be functions of u, v. We call

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = F_u G_v - F_v G_u$$

as a Jacobian determinant of $F(u,v)$ and $G(u,v)$.

Jacobian formulas:

Now, we have $F(u,v,x,y)=0$ and $G(u,v,x,y)=0$.

By Cramer's rule: If $F_uG_v - F_vG_u \neq 0$, then

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \qquad u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}},$$

$$v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}.$$

Example Let u and v be functions x and y such that

$$uv^2 + xy = x^2 + y \quad \text{and} \quad u^2 - 3v = x^2 + y^2.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}.$

Applications of Multivariable Functions

1. The Chain Rule

We apply the chain rule to related rate problems which are unable to get solved by using derivatives of one variable functions.

Example 1 A cone pile of sand increases by 2 inches/second in height and by 1 inch/second in base radius. Find the rate of volume's change of this sand pile when its cone is 30 inches in height and has base radius 20 inches.

Example 2: An airplane flies from the west to the east of an observer on the ground. Suppose the plane flies with horizontal speed 440 feet/second and vertical speed 10 feet/second. What is the rate of distance's change between the plane and the observer when the plane is 12000 feet above ground and 16000 feet to the west of the observer?

2. **Total Differential**

In the case of one variable functions, we have the differential:

$$df = f'(x)\Delta x.$$

Similarly, in the case of two variable functions, this df is called the total differential of f and is defined as follows.

Definition: The total differential of $f(x, y)$, denoted by df , is defined to be

$$df = f_x(x, y)dx + f_y(x, y)dy$$

Example Find dz where $z = \ln(x^3 y^2)$.

Definition: The difference of function between two points

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x, y).$$

<u>Application of total difference/differential</u>

We know that $\Delta f \approx df$ when $\Delta x, \Delta y \to 0$.

Thus, $f(x+\Delta x, y+\Delta y) - f(x, y) \approx f_x(x, y)dx + f_y(x, y)dy$.

i.e. $f(x+\Delta x, y+\Delta y) \approx f(x, y) + f_x(x, y)dx + f_y(x, y)dy$.

This relation can be used to approximate the value of the function f at some points $(x+\Delta x, y+\Delta y)$ near the point (x, y) when the value of $f(x, y)$ is much easier to compute. It is called the *linear approximation* of two variable functions.

Example Estimate $\sqrt{(5.98)^2 + (8.01)^2}$.

3. Max and Min of multivariable functions

Definition

If a function f of two variable has a point $(a,b) \in D_f$ such that $f(x,y) \le f(a,b)$ for all points (x,y) in an open disk centered at (a,b), then the point $(a,b,f(a,b))$ is called a *local maximum point* of f, and the value $f(a,b)$ is called a *local maximum value* of f.

If $f(x,y) \le f(a,b)$ for all points (x,y) in domain of f, we call the point $(a,b,f(a,b))$ as an *absolute maximum point* of f and call the value $f(a,b)$ as an *absolute maximum value* of f.

If a function f of two variables has a point $(c,d) \in D_f$ such that $f(x,y) \ge f(c,d)$ for all points (x,y) in an open disk centered at (c,d), then the point $(c,d,f(c,d))$ is called as a *local minimum point* of f and the value $f(c,d)$ is called a *local minimum value* of f.

If $f(x,y) \ge f(c,d)$ for all points (x,y) in domain of f, we call the point $(c,d,f(c,d))$ as an *absolute minimum point* of f and called the value $f(c,d)$ as an *absolute minimum value*.

How to find local maximum and minimum

Theorem: If f has a local max or local min at (a,b) and its first partial derivatives exist at (a,b), we would have both

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0.$$

Definition: A point (a,b) in domain of f is called a critical point of f if either both $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or at least one partial derivative of f does not exist at (a,b).

<u>Remark</u>: A critical point may not be a local max or local min; it could be a saddle point.

Sometimes we may use a graph of f to locate a local max and a local min.

Example: Find all critical points and locate local max and local min of $f(x,y) = x^2 - 6x + y^2 - 4y$.

The test of local max and local min

Theorem: Let $z = f(x, y)$. Suppose f has continuous second partial derivatives in an open disk around (x_0, y_0) such that both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. There are three possibilities:

1. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$,
   - and $f_{xx}(x_0, y_0) > 0$, then f has a local min at (x_0, y_0).
   - and $f_{xx}(x_0, y_0) < 0$, then f has a local max at (x_0, y_0).
2. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$, then the point (x_0, y_0) is a saddle point of f.
3. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 = 0$, then the test fails. We have no conclusion about the point (x_0, y_0). It may be the local max, the local min, the saddle point or none of these.

Example Suppose $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. Find the local max and local min of f.

Maximum and Minimum of a Function in a Closed Domain.

Theorem: Suppose $f : D \to R$ where D is a closed plane. If the function f is a continuous function and bounded on D, then f always has the absolute maximum and absolute minimum points in D.

Example Find the absolute maximum and absolute minimum points of

$$f(x, y) = x^2 - 6x + y^2 - 4y$$

on the area bounded by x-axis, y-axis and the line $x + y = 7$.

Problem Find the size of a rectangular box whose volume $V = 1000$ cubic feet and has the least surface area A.

Maximum and minimum of a function with boundary conditions

Example Find the size and the volume of a rectangular box with the maximum volume where the box is located in the first octant so that one boxes' corner is at the origin and the opposite corner is on the paraboloid $z = 4 - x^2 - 4y^2$.

The following discussion is optional.

Most of the time, the given conditions are not easy to put in the functions directly. For example, we want to find a local max and a local min of $f(x,y)=xy$ with the condition: $g(x,y)=\dfrac{x^2}{8}+\dfrac{y^2}{2}-1=0$.

Hence, we may need the following method:

Method of Lagrange Multipliers

This method can be used to find the local max or local min of two variable functions (in general, multivariable functions) f with the given condition g . This can be done by following these steps.

1. Form a function F with 3 variables: x, y, λ such that $F(x,y,\lambda)=f(x,y)-\lambda g(x,y)$ where f and g have continuous first partial derivatives.
2. Find all critical points of F . That is, we find x, y, λ such that $F_x(x,y,\lambda)=0$, $F_y(x,y,\lambda)=0$, $F_\lambda(x,y,\lambda)=0$.
3. Calculate the value of f at all critical points (x,y) found in 2.

Example Find maximum and minimum of $f(x, y) = xy$ with the condition: $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$.

Exercises

1. Estimate the value of $\sqrt{8.99}+\cos\left(0.02\right)$.
2. Find all local extreme (maximum and/or minimum), and saddle points of $f(x,y)=x^2y+2y^2-2xy-15y$.
3. Find the maximum temperature defined by

$$T(x,y)=x^2+y^2+4x-4y+3$$

on the disk circumference $x^2+y^2=2$.

Answer 1. $4-\frac{1}{600}=3.99833...$

2. Saddle points at $(-3,0)$ and $(5,0)$

   Local minimum at $(1,4)$

3. Maximum temperature = 13