Exercise: Vector-Valued Function

1. Let C be a smooth curve whose parametric equation of this form:

$$x = 3t^3, \quad y = te^{-2t}, \quad z = \sin t$$

   a) Find the position vector $\vec{r}(t)$ of this curve.

   b) Find $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

2. Find the arc length of curve $\vec{r}(t) = 3\cos(2t)\vec{i} - 3\sin(2t)\vec{j} + 8t\vec{k}, \quad 0 \le t \le 2\pi$.

3. Find the arc length parameterization of the curve $\vec{r}(t) = \cos(3t)\vec{i} + \sin(3t)\vec{j} + 6t^{\frac{3}{2}}\vec{k}$.

4. Find the equation of the line tangent to $\vec{r}(t)$ at t_0, then sketch the graph of $\vec{r}(t)$ and draw the tangent vecter $\vec{r}\,'(t_0)$.

   a) $\vec{r}(t) = \langle t, t^2 \rangle; \quad t_0 = 2$

   b) $\vec{r}(t) = \sec(t)\vec{i} + \tan(t)\vec{j}; \quad t_0 = 0$

   c) $\vec{r}(t) = 2\sin(t)\vec{i} + \vec{j} + 2\cos(t)\vec{k}; \quad t_0 = \dfrac{\pi}{2}$

5. Find a vector equation of the line tangent to the graph of $\vec{r}(t)$ at the point P_0 on the curve.

   a) $\vec{r}(t) = (3t-1)\vec{i} + \sqrt{3t+4}\vec{j}; \quad P_0(-1, 2)$

   b) $\vec{r}(t) = 4\cos(t)\vec{i} - 3t\vec{j}; \quad P_0(2, -\pi)$

   c) $\vec{r}(t) = t^2\vec{i} - \dfrac{1}{t+3}\vec{j} + (4-t^2)\vec{k}; \quad P_0(4, -1, 0)$

6. Let $\vec{r}(t)$ be the position vector of particle moving in the plane. Find the velocity, acceleration, and speed at an arbitrary time t.

   a) $\vec{r}(t) = 3\cos(t)\vec{i} + 3\sin(t)\vec{j}; \quad t = \dfrac{\pi}{3}$

   b) $\vec{r}(t) = e^t\vec{i} + e^{-t}\vec{j}; \quad t = 0$

   c) $\vec{r}(t) = t\vec{i} + \dfrac{1}{2}t^2\vec{j} + \dfrac{1}{3}t^3\vec{k}; \quad t = 2$

7. Find the unit vector: $\left(\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right)$

   a) $\vec{r}(t) = \ln(2t)\vec{i} + t\vec{j}; \quad t = e$

   b) $\vec{r}(t) = 4\cos(t)\vec{i} + 4\sin(t)\vec{j} + t\vec{k}; \quad t = \dfrac{\pi}{2}$

   c) $\vec{r}(t) = t\vec{i} + \dfrac{1}{2}t^2\vec{j} + \dfrac{1}{3}t^3\vec{k}; \quad t = 0$

8. Find the curvature as follows.

   a) Line vector equation in the form $\vec{r}(t) = \vec{r_0} + t\vec{v}$ passing through the terminal point of the position vector $\vec{r_0}$ and paralle to vector $\vec{v}$. Find the line equation $\vec{r}(s)$ with arc length parameter s, and find the curvature at any point.
   $\left(\text{Using } \kappa(s) = \left\|\dfrac{d\vec{T}}{ds}\right\| = \|\vec{T}\,'(s)\|\right)$

   b) Find the curvature κ of $\vec{r}(t) = t\vec{i} + \ln(\cos t)\vec{j}$.
   $\left(\text{Using } \kappa(t) = \dfrac{\|\vec{T}(t)\|}{\|\vec{r}\,'(t)\|}\right)$

   c) Find the curvature κ of a circle of radius 2 with the center is (x_0, y_0).
   $\left(\text{Using } \kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}\right)$

9. Find the unit normal vector as follows.

   a) $\vec{r}(t) = 2\sin(2t)\vec{i} + 2\cos(2t)\vec{j} + 4\vec{k}$
   $\left(\text{Using } \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right)$

   b) $\vec{r}(t) = t\vec{i} + 3t\vec{j} + \dfrac{1}{2}t^2\vec{k}$ at $t = 0$
   $\left(\text{Using } \vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}\right)$

10. Find the center of circle and equation for the osculating cicle at the origin on the parabola $y = -2x^2$.

11. Find unit binormal vector $\vec{B}$ of the position of a moving particle is given by $\vec{r}(t) = (e^t\cos t)\vec{i} + (e^t\sin t)\vec{j} + 2\vec{k}$.

12. Determine the torsion of $\vec{r}(t) = (6\sin t)\vec{i} + (6\cos t)\vec{j} + 8t\vec{k}$.

13. Find $T(t)$, $N(t)$, and $B(t)$ for the given value of t. Then find equations for the osculating, normal, and rectifying planes at the point that corresponds to that value of t when $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + \vec{k}; \quad t = \dfrac{\pi}{4}$.