SVD разложение

$$A_{mn} = U_{mm} S_{mn} V^T_{nn} \qquad U \cdot U^T = I \qquad V \cdot V^T = I$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$$

Находим U:

$$AA^T = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}$$

$$A\bar{v} = \lambda \bar{v} \Rightarrow \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$11x_1 + x_2 = \lambda x_1$$
$$x_1 + 11x_2 = \lambda x_2 \Rightarrow$$

$$(11-\lambda)x_1 + x_2 = 0$$
$$x_1 + (11-\lambda)x_2 = 0$$

$$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0 \Rightarrow (11-\lambda)(11-\lambda) - 1 = 0$$

$$(\lambda-11)(\lambda-11) - 1 = 0$$

$$\lambda^2 - 22\lambda + 120 = 0$$

$$\lambda = 10 \quad \lambda = 12$$

собственные числа

① $\begin{cases} (11-10)x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$ собств. вектора $x_1 = 1$, $x_2 = -1$

② $\begin{cases} (11-12)x_1 + x_2 = 0 \\ x_1 = x_2 \end{cases}$ $x_1 = 1$, $x_2 = 1$

$\lambda = 12 \qquad \lambda = 10$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ нормализация Г.Ш.

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{[1, 1]}{\sqrt{1+1}} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$\bar{w}_2 = \bar{v}_2 - \bar{u}_1 \cdot \bar{v}_2 \times \bar{u}_1 = [1, -1] - \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \cdot [1, -1] \times \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$\bar{w}_2 = [1, -1] - 0 \times \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] = [1, -2]$$

$$\bar{u}_2 = \frac{w_2}{|\bar{w}_2|} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Находим V:

собственные числа

$$A^T A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$$

$$\begin{cases} 10x_1 + 2x_3 = \lambda x_1 \\ 10x_2 + 4x_3 = \lambda x_2 \\ 2x_1 + 4x_2 + 2x_3 = \lambda x_3 \end{cases} \Rightarrow \begin{cases} (10-\lambda)x_1 + 2x_3 = 0 \\ (10-\lambda)x_2 + 4x_3 = 0 \\ 2x_1 + 4x_2 + (2-\lambda)x_3 = 0 \end{cases}$$

$$\Leftarrow$$

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = -\lambda^3 + 22\lambda^2 - 120\lambda = 0$$

$$\Downarrow$$

$$\lambda = 0 \quad \lambda = 10 \quad \lambda = 12$$

$\underline{\lambda = 12}$:
$$\begin{cases} -2x_1 + 2x_3 = 0 \\ -2x_2 + 4x_3 = 0 \\ 2x_1 + 4x_2 - 10x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = x_3 = 1? \\ x_2 = 2x_3 = 2? \end{array}$$

$$\Downarrow$$

$$\cancel{2x_3 + 8x_3 - 10x_3 = 0}$$

$$\bar{v}_1 = (1,2,1) \Leftarrow$$

$\lambda = 10$:
$$\begin{cases} x_3 = 0 \\ 2x_1 + 4x_2 = 0 \Rightarrow x_1 = -2x_2 \end{cases}$$

$$\bar{v}_2 = (2,-1,0) \qquad x_2 = -1 \Rightarrow x_1 = 2$$

$\lambda = 0$:
$$\begin{cases} \cancel{x_3 = 0} \quad 10x_1 + 2x_3 = 0 \\ 10x_2 + 4x_3 = 0 \\ 2x_1 + 4x_2 + 2x_3 = 0 \end{cases} \quad \begin{cases} 5x_1 + x_3 = 0 \\ 5x_2 + 2x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

$$\bar{v}_3 = (1,2,-5)$$

$x_1 = 1 \Rightarrow x_3 = -5$

$5x_2 = 10 \Rightarrow \underline{x_2 = 2}$

собственные векторы

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

нормализация Г.Ш

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{[1,2,1]}{\sqrt{1+4+1}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \Rightarrow 0$$

$$\bar{w}_2 = \bar{v}_2 - \bar{u}_1 \cdot \bar{v}_2 \times \bar{u}_1 = [2,-1,0] - \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right] \cdot [2,-1,0]$$

$$0 \times u_1$$

$$\Rightarrow \bar{w}_2 = (2,-1,0)$$

$$\bar{u}_2 = \frac{w_2}{|\bar{w}_2|} = \frac{(2,-1,0)}{\sqrt{4+1}} = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right)$$

$$\bar{w}_3 = \bar{v}_3 - \bar{u}_1 \cdot \bar{v}_3 \times \bar{u}_1 - \bar{u}_2 \cdot \bar{v}_3 \times \bar{u}_2 = \left(-\frac{2}{3}, -\frac{4}{3}, \frac{10}{3}\right)$$

$$u_3 = \frac{\bar{w}_3}{|\bar{w}_3|} = \left[\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right]$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{30}} \end{pmatrix} \Rightarrow V^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{pmatrix}$$

$U \to 2\times2 \quad V \to 3\times3$

$S \qquad S \to 2\times3$

$$S \Rightarrow = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \quad (2)$$

$$A_{mn} = U_{mm} \cdot S_{mn} V_{nn}^{T} =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

LU разложение

v - столбец $n-1$

w^T строка $n-1$

A' $(n-1)\times(n-1)$

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & w^T \\ v & A' \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 0 \\ \frac{v}{a_{11}} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^T \\ 0 & A' - \frac{v w^T}{a_{11}} \end{pmatrix}$$ - комп. Шура

$$\begin{pmatrix} 2 & 3 & 1 & 5 \\ 6 & 13 & 5 & 19 \\ 2 & 19 & 10 & 23 \\ 4 & 10 & 11 & 31 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 & 5 \\ 3 & 4 & 2 & 4 \\ 1 & 16 & 9 & 18 \\ 2 & 4 & 9 & 21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 & 5 \\ 3 & 4 & 2 & 4 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 7 & 17 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 & 5 \\ 3 & 4 & 2 & 4 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 7 & 3 \end{pmatrix}$$

$$L \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 2 & 1 & 7 & 1 \end{pmatrix}$$

$$U \rightarrow \begin{pmatrix} 2 & 3 & 1 & 5 \\ 0 & 4 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

X_2

LUP разложение: $PA = LU$

$Ax = b \Rightarrow PAx = Pb \Rightarrow L\underbrace{Ux}_{y} = Pb$

$Ux = y \Leftarrow Ly = Pb \Leftarrow$

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 4 \\ 5 & 6 & 3 \end{pmatrix}}_{A} \underbrace{x}_{x} = \underbrace{\begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}}_{b}$$

$$\begin{array}{c|cc c} 5 & 6 & 3 & 8 \\ \hline 3 & 4 & 4 & 7 \\ 1 & 2 & 0 & 3 \end{array} \Rightarrow \begin{array}{cccc} 5 & 6 & 3 & 8 \\ 3/5 & 0,4 & 2,2 & 7 \\ 1/5 & 0,8 & -0,6 & 3 \end{array}$$

$$\begin{array}{cccc} 5 & 6 & 3 & 8 \\ 0,2 & 0,8 & -0,6 & 3 \\ 0,6 & 0,5 & 2,5 & 7 \end{array} \Leftarrow \begin{array}{c|ccc} 5 & 6 & 3 & 8 \\ \hline 0,2 & 0,8 & -0,6 & 3 \\ 0,6 & 0,4 & 2,2 & 7 \end{array}$$

$\Downarrow$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0,2 & 1 & 0 \\ 0,6 & 0,5 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 5 & 6 & 3 \\ 0 & 0,8 & -0,6 \\ 0 & 0 & 2,5 \end{pmatrix}$$

$$P = {}_{21}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{13} \; 32$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0{,}2 & 1 & 0 \\ 0{,}6 & 0{,}5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 7 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 8 \\ 1{,}4 \\ 1{,}5 \end{pmatrix}$$

$$Ux = y$$

$$\begin{pmatrix} 5 & 6 & 3 \\ 0 & 0{,}8 & -0{,}6 \\ 0 & 0 & 2{,}5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1{,}4 \\ 1{,}5 \end{pmatrix}$$

$$\Downarrow$$

$x_1 \quad x_2 \quad x_3$