Marked Ruler and Compass Constructions

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1 Introduction

Straightedge and compass constructions have existed since the times of the ancient Greeks. The straightedge was considered to be infinitely long and without markings. The compass could open as wide or as narrow as desired, but also had no markings.

Naively, we may define the conditions of constructibility recursively, based on existing objects that we have already decided are constructible. For a polygon to be constructible with straightedge and compass, its vertices must all be constructible points. For a line to be constructible, it must contain two constructible points. For a circle to be constructible, its center must be a constructible point and it must contain a constructible point on its boundary. For an angle included in two lines to be constructible, its vertex must be the center of a constructible circle, and the two lines must intersect that circle at constructible points. Constructible points are constructed in a finite amount of steps using repeated applications of the following five basic constructions, given any objects that have already been constructed:
1) Using the straightedge to create the unique line through two existing points, extended indefinitely
2) Using the compass to create the unique circle through an existing point with its center on another existing point
3) Creating the unique point which is the intersection of two existing, non-parallel lines
4) Creating the one or two points in the intersection of an existing line and an existing circle (if they intersect)
5) Creating the one or two points in the intersection of two existing circles (if they intersect)

For simplicity, we can consider objects such that they are in Euclidean 2-space R^2, and the only objects that we may begin with are the points $(0, 0)$ and $(0, 1)$. From these basic constructions, we know that a point P is constructible if and only if there is some finite tower of fields

$$Q = K_0 \subseteq K_1 \subseteq ... \subseteq K_n$$

where both coordinates of P are in K_n and for all $0 \leq i < n$, $[K_i : K_{i+1}] = 2$. However, we may replace $(0,0)$ and $(0,1)$ with any points in R^2, and Q will be replaced with a field generated by the distances between each of the points. Then the condition on both coordinates of P being in the finite tower of fields is replaced by the condition that P be both a constructible distance from any of the existing points, and that the angle P makes with any two other points be generated by a finite number of constructions.

The ancient Greeks considered neusis, or using a marked ruler in constructions to be an inferior, last resort, form of construction. They wondered if, using compass and unmarked straightedge, they could:

1) Trisect any given angle
2) Double the cube
3) Square the circle

Another interesting problem is to construct the regular heptagon. All these are impossible with compass and straightedge. However, we will see that trisecting the angle, doubling the cube, and constructing the regular heptagon are possible if the ruler is marked appropriately, but since π is transcendental, squaring the circle is not(unless π is marked on the ruler, which we will not allow).

2 The Precise Meaning of Marked Ruler and Straightedge Constructions

Given a marked ruler and a compass, we may use the marked ruler as a straightedge, and more. Given lines L and M, a distance d, and a point O, we can draw line OAB, with A in L, B in M, and AB of length d (provided lines L and M are not both parallel and too far away from each other). For a visual representation of this, refer to $(figure3)$ d must be in the field F we begin with, or d must be a constructible length from F, or d must be a length that can be constructed by marked ruler and compass in a finite number of steps. Note that other meanings of neusis may allow L and M to be circles. These are stronger constructions and will not be analyzed in this paper.

3 Trisecting the angle

It is possible to trisect a given angle, using Hartshorne's construction and proof (See $(figure1)$:

Let $\angle AOB$ be the given angle. Drop a perpendicular AC from A to OB. Draw a line L through A parallel to OB. Now use the marked ruler to draw a line ODE such that D in AC, E in L, and $DE = 2AO$. This line will trisect the original angle.

Proof: Let F be the midpoint of DE, and let G be the midpoint of AE. Then FG is perpendicular to AE, so the triangles EFG and AFG are congruent by

(SAS). Now the new angle $\angle EOB = x$ is equal to $\angle AEO$ by parallel lines and to $\angle EAF$ by congruent triangles. So $\angle AFO = 2x$, since it is an exterior angle to the triangle AEF. But $DE = 2AO$, so $AO = EF = AF$. Hence the triangle AOF is isosceles, and so $\angle AOD = 2x$. Thus the original angle $\angle AOB = 3x$, and $\angle EOB = x$ is one-third of it, as required.

As an aside, we note that if we can construct a regular n-gon, then we can use angle trisection to construct a regular $3n$-gon. This is because the regular n-gon has an internal angle of

$$\frac{(n-2)\pi}{n} = \pi - \frac{2\pi}{n}$$

and the regular $3n$-gon has an internal angle of

$$\frac{(3n-2)\pi}{3n} = \pi - \frac{2\pi}{3n}$$

4 Doubling the Cube

Given the edge length of a cube, we may construct the length of an edge of a cube with double the volume by using Hartshorne's construction of cube roots. Given segments of lengths 1 and a, we first construct the cube root of a by the following method, using $(figure 2)$ as a reference:

Let AB be the given segment of length a. Using the segment of length 1, choose $b = 2^{3k-1}$ with k an integer such that $b > a$. Make an isosceles triangle ABC with $CA = CB = b$, and extend CA to D with $AD = b$. Draw the line DB. Now use the marked ruler to draw CEF with E in DB, F in AB, and $EF = b$. Let $BF = y$. Then

$$a^{1/3} = \frac{y}{2^{2k}}$$

Hartshorne gives proof that this construction does in fact produce a cube root, so in the interest of space, I will not provide a proof here. In order to complete the problem of doubling the cube, simply multiply the given edge of the cube by $2^{1/3}$.

5 The Field Theory behind Neusis

To find the degree of field extensions generated by unmarked straightedge and compass constructions, Hartshorne tells us to examine the intersections of lines and circles in the field. Refer to $(figure 3)$. For neusis constructions, we will examine the intersections of lines L and M with line OAB. The field extensions generated by points A and B will be the degree of field extension of each use of neusis. For simplicity, we will assume that O is at the origin, and L is the line $y = b$. We do not lose generality by doing this, as one can easily see by looking at the objects from a different perspective that this is a linear change of coordinates. We will consider the locus of points P such that OP cuts L at a point Q and $PQ = d$. This is called the Conchoid of Nicomedes. To find the line OAB is equivalent to finding the intersection of line M with the conchoid.

First, we need to find the equation of the conchoid. Take an arbitrary line $y = ax$ through O. This line meets L at the point $Q = (b/a, b)$. Let P have coordinates $P = (x, y)$. Then the condition $PQ = d$ gives

$$(x - b/a)^2 + (y - b)^2 = d$$

But P also lies on the line $y = ax$, so we can eliminate the variable a by substituting $a = y/x$ and simplifying to obtain the equation of the conchoid:

$$(x^2 + y^2)(y - b)^2 = (dy)^2$$

To find the intersection B of the conchoid with the line M, we substitute the linear equation of M in the equation of the conchoid. This gives a quartic equation in x. If r is a root of that equation, the coordinates of B are then $x = r$, and y is a linear expression in r. From there we get the line OB and the coordinates of A.

The general formula for the roots of a quartic equation is massive. However, it is solvable in terms of only sums, products, differences, and quotients of square and cube roots of elements in the starting field. One can verify this fact by using a very lengthy substitution technique, which can be found here: http://www.sosmath.com/algebra/factor/fac12/fac12.html

Thus, since we can take the general cube root, and the coordinates for A and B are found by solving a quartic equation, a point P is constructible with marked ruler and compass if and only if there is some finite tower of fields

$$Q = K_0 \subseteq K_1 \subseteq ... \subseteq K_n$$

where both coordinates of P are in K_n and for all $0 \leq i < n$, $[K_i : K_{i+1}] = 2^z 3^w$. And of course, we may start with a field F other than Q, and the condition on coordinates of P is replaced by the condition that P be a constructible (with marked ruler and compass) distance from the field generated by the distances between each of the initial points, and that the angle P makes with any existing points be generated by a finite number of constructions.

6 Constructing the Regular Heptagon

As we know from the Gauss-Wantzel Theorem, the regular heptagon is the first regular n-gon that is not constructible with compass and straightedge. However, it is constructible with marked ruler and compass. We know that the internal angle of any regular n-gon is

$$\frac{(n-2)\pi}{n}$$

so we must construct an angle measuring $5\pi/7$ to construct the regular 7-gon. We start with the observation that $2cos(2\pi/7)$ is a solution to the cubic equation

$$x^3 + x^2 - 2x - 1 = 0$$

From this length we can construct the desired angle measure of $5\pi/7$. But first, we must construct $2cos(2\pi/7)$ from the cubic equation. We will solve $x^3 + x^2 - 2x - 1 = 0$ using trigonometry. First, we must put the cubic equation into an alternate form, by making the substitution

$$x = y - 1/3$$

This puts the equation into the form $y^3 + py + q = 0$ where $p = -7/3$ and $q = -7/27$. We can then use the following formula to find the three roots of the cubic:

$$y_k = 2\sqrt{\frac{-p}{3}}cos(\frac{1}{3}arccos(\frac{3q}{2p}\sqrt{\frac{-3}{p}}) - \frac{2k\pi}{3})$$

for $k = 0, 1$, or 2. We pick $k = 0$ to obtain the only positive root, as our final answer should be positive. Substituting in p and q, we get

$$y = y_0 = \frac{2\sqrt{7}}{3}cos(\frac{1}{3}arccos(\frac{\sqrt{7}}{14}))$$

Now, to construct the heptagon, we must work our way outwards from the inside of our formula. First, we need $\theta = arccos(\sqrt{7}/14)$ We construct a line segment FG of length $\sqrt{7}$ and a perpendicular ray L from G. We then set our compass to length 14, centered at F, and draw an arc which intersects L. This creates a right triangle, where θ is the included angle in L and FG. Now use neusis to trisect θ. We then create a new right triangle ABC, with hypotenuse $AB = 1$, and $\angle A = \frac{\theta}{3}$. Now

$$AC = cos(\frac{1}{3}arccos(\frac{\sqrt{7}}{14}))$$

Multiply AC by $2\sqrt{7}/3$, subtract $1/3$, and divide by 2, and we have the length $cos(2\pi/7)$. Now, let $cos(2\pi/7)$ be the length of leg TU of right triangle TUV with hypotenuse TV of length 1. Bisect $\angle VTU$ to get $\alpha = \pi/7$, and quintuple the resulting angle to form the angle $\beta = 5\pi/7$. Finally, we construct the regular heptagon by drawing an angle $\angle HIJ = \beta$ with sides HI and IJ of length s, and copying β and s in succession until the regular heptagon is complete.

7 Works Cited

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