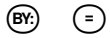
PRACTICAL CONSIDERATIONS IN EXPERIMENTAL COMPUTATIONAL SENSING

by

Phillip K. Poon



A Dissertation Submitted to the Faculty of the

COLLEGE OF OPTICAL SCIENCES

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

2016

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Phillip K. Poon titled Practical Considerations in Experimental Computational Sensing and recommend that it be accepted as fulfilling the dissertation requirement for the degree of Doctor of Philosophy.

Amit Ashok — Date: 9 December 2016

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director: Amit Ashok — Date: 9 December 2016

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ACKNOWLEDGEMENTS

Graduate school is an arduous and enlightening experience. It is not difficult by design but by nature it forces one into a state of mind which embraces the edge of knowledge and trek into the unknown. I was fortunate to have many guides who showed me the path, even when there were times when I wandered off to get my bearings. Along the way I encountered many people who not only helped me with the journey but bestowed kindness and friendship, asking for nothing in return.

My main guide along the journey was Professor Michael Gehm. I first met him when I took a graduate level Linear Algebra course which I found particularly challenging. I often went to his office hours asking for help and his ability to be patient and explain concepts from different perspectives is a gift few teachers have. As an advisor, I would like to thank him for all of the help and guidance he has given me over the years. His generosity for funding my graduate studies as well trips to conferences is appreciated. He believed in me more than I believed in myself. I consider him not only as a mentor but as a father figure. If I can become half the scientist that he is, I would consider that a successful career.

I especially want to thank Professor Esteban Vera, who I first met as a postdoctoral researcher in the Laboratory for Engineering Non-Traditional Sensors (LENS) and supervised me for the majority of my graduate studies. Much of the work and results in this disseration is due to his guidance. Even after he started his professorship in Chile, he was willing to review my data and suggest different methods of analysis. Professor Vera is directly responsible for much of my training as an experimentalist. I consider Professor Vera as an older brother who was always there to protect me from the pitfalls of the graduate school journey.

I also thank Doctor Dathon Golish. His approach to work and life was a calming effect in often stressful times. He made major contributions to the Adaptive Feature Specific Spectral Imaging-Classifier (AFSSI-C) and provided valuable feedback on various research projects and conference presentations.

Thank you Professor Mark Neifeld and Professor Amit Ashok for being my advisor and supervisor during my first year as a PhD student. They were the first to introduce me to many of the techniques and subjects related to computational sensing. They taught me fundamental concepts in optics, statistical signal processing and programming. Many of the results in this dissertation would not have been possible without their teachings.

I've also had many other supervisors along the way whose effort must also be acknowledged: My undergraduate advisor at San Diego State University, Professor Matthew Anderson. Doctor John Crane, who was my supervisor during my internship at the Lawrence Livermore National Laboratory. Professor Joseph Eberly and Professor Gary Wicks who were my advisors at the Institute of Optics.

I would like to formally express gratitude to a number of exceptional teachers throughout my life. Professor Tom Milster who taught Diffraction and Interference and allowed me to be a teaching assistant for that course. Professor Masud Mansuripur, whose course in Electromagnetic Waves was the most elegant and well taught version of the classical nature of light that I have ever had the pleasure to

experience. Professor Jeff Davis, who first ignited my passion for optics while I was an undergraduate physics student at San Diego State University.

I also want to thank several faculty members who committed time from their busy schedules to help with several milestones of my graduate school experience. Special thanks to Professor Julie Bentley, Doctor James Oliver, and Professor Richard Morris who wrote letters of recommendation for me. Appreciation goes to Professor Tom Milster, Professor Harrison Barrett, Professor Russell Chipman, and Professor John Greivenkamp who formed my oral comprehensive exam committee. Thank you to Professor Rongguang Liang who served on my doctoral dissertation committee.

I would like to thank several members of the Duke Imaging and Spectroscopy Program (DISP) laboratory for their friendship: Patrick Llull, Mehadi Hassan, and Evan Chen. Tsung Han Tsai was not only a colleague but his work on computational polarimetry and spectroscopy using an Liquid Crystal on Silicon (LCOS) Spatial Light Modulator (SLM) was the inspiration which directly lead my idea of using the same architecture for computational spectral unmixing.

Other graduate students, colleagues, and faculty must also be thanked, for at one time or another they all helped me: Basel Salahieh, Vicha Treeaporn, John Hughes, Steve Feller, Myungjun Lee, Sarmad H. Albanna, Professor Lars Furenlid, Doctor Joseph Dagher, Professor Daniel Marks, Professor Janick Roland-Thompson, Mary Pope, Mark Rodriguez, and Amanda Ferris.

I've had the good fortune to form friendships with an amazing set of groupmates as part of the LENS. David Coccarelli invited our family to spend our first Thanksgiving in North Carolina with him and we had many discussions about college basketball and life. I would express my sincere gratitude to Matthew Dunlop-Gray, who designed and constructed the AFSSI-C which is the foundation for much the work in this disseration. I learned so much from Matthew especially much of my practical skills. Tariq Osman constructed the Static Computational Optical Undersampled Tracker (SCOUT) which is also a major part of this dissertation. Alyssa Jenkins whose combination of sense of humor and intelligence is unmatched. Thank you to Qian Gong for your kindness, positivity, and generousity. Thank you Xiaohan Li for keeping me company that final year of graduate school, conversations about basketball and helping me with my math. Thank you David Landry for helping me with all software and computer programming related issues. Thank you Joel Greenberg for your help and advice. Thank you Kevin Kelly, Adriana DeRoos, Andrew Stevens and Dineshbabu Dinakarababu for your friendship. Finally, I consider Wei-Ren Ng as one of my best friends and as a brother. Our time in the LENS group was marked by many late nights spent working in the lab and office and mornings in the gym. He was generous in sharing his knowledge and gave me the advice that I often did not want to hear but was true.

Appreciation goes to the all the staff at the College of Optical Sciences at the University of Arizona. It is one of the most friendly and well run academic departments I have ever had the fortune to be a part of. I hope my career will reflect well upon the college.

Finally, I would like to thank my closest friends that I've met throughout the years. They often provided much needed respites during my journey—Christopher MacGahan, Ricky Gibson, Krista MacGahan, Kristi Behnke, Michael Gehl, Carlos Montances, Matthew Reaves, Vijay Parachuru, Eric Vasquez. Thank you for letting me into your lives and being part of mine.

Last but not least, to my family. You make me happy.

DEDICATION

For my wife. We moved from city to city. You stuck with me through the highs and lows. You cooked dinner for me when I came home from a long day. You did the chores so I could concentrate on research. You acted as both mother and father to our son while I wrote. You believed in me even when I did not. You sacrificed your dreams and goals so I could accomplish mine.

You're the real Ph.D.

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ABSTRACT

Implementing computational optical sensors often comes with various issues that many traditional sensors may not encounter.

CHAPTER 1

Introduction

This chapter introduces the reader to the concepts of *isomorphic sensing*, *multiplex sensing*, *indirect-imaging*, *task-specific sensing*, *compressive sensing* and *computational sensing*. It also provides the motivation for the need to address the practical issues in experimental computational sensing.

A *measurement* is a process that converts a physical phenomena to a collection of data. The signal-of-interest is the physical phenomena that we are interested in quantifying. We will call the collection of data the measurement data. Often the measurement is *isomorphic*. Isomorphic sensing is the concept that a sensor's measurement data resembles the signal-of-interest. Isomorphic sensing is called *traditional sensing*. In isomorphic sensing the analog hardware, analog-to-digital converter (ADC), and processing algorithms are all seperate components, see Figure 1.1. Computational sensing is the concept that a joint design of the sensor, often though *coding* of the analog signal, with inversion algorithms can exceed the performance of an isomorphic sensor. *Computational sensors* exist at the intersection of these processes, see Figure 1.2 [1]. While isomorphic sensors can provide flexible sensing in multiple applications, a computational sensor's joint design can lead to performance increases. Throughout this chapter and the rest of this dissertation, we will provide many examples that highlight the differences between computational and isomorphic sensing.

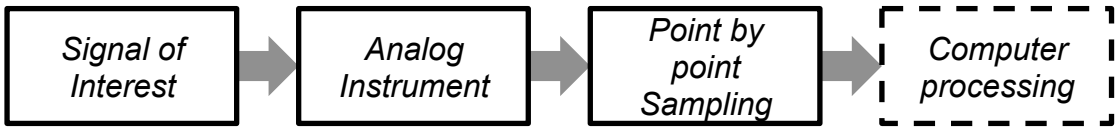

Figure 1.1: The signal-of-interest is incident upon the analog instrument. The analog instrument forms an isomorphism of the signal which is then periodically sampled point-by-point through an analog-to-digital converter (ADC) device. Once the signal is in digital form, post-processing algorithms are often used to perform various tasks such as noise reduction, detection, and classification. Notice that the analog instrument, sampling scheme, and processing are all seperated.

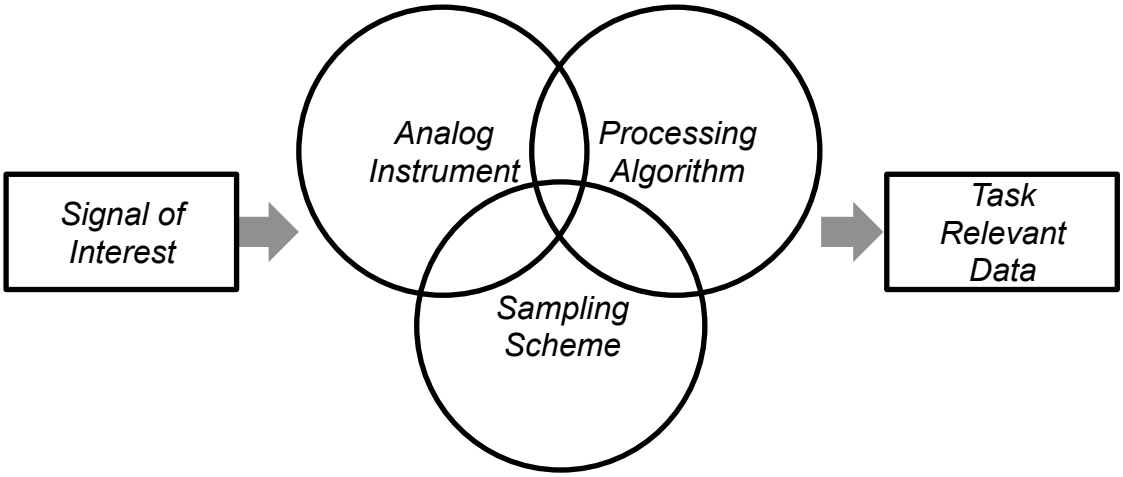

Figure 1.2: In a computational sensor, the analog instrument, the algorithms, and sampling scheme are often combined. Computational sensors are at the intersection of these traditional disparate systems.

Rather than a rigorous discussion, this chapter will discuss some of the major developments and concepts in the field of computational sensing on an intuitive level. This will familiarize the reader with important terminology and techniques common in the field of computational sensing. A rigorous discussion with mathematical formalism of the concepts is presented in chapter 2. This chapter will also discuss some of the challenges I and many other experimentalists and engineers have faced when developing computational sensing prototypes. I will close this chapter with a brief look ahead to the rest of the dissertation.

1.1 Isomorphic Sensing

In Greek, the word isomorphic loosely translates to equal in form. Traditional sensors perform isomorphic sensing. In the context of this dissertation, an isomorphic sensor is any sensor which attempts to produces measurement data that resembles the signal-of-interest. In this paradigm, the analog instrument, sampling scheme, and post-processing algorithms are seperate components and processes.

We will discuss three important examples of isomorphic sensors: the pinhole camera, the photographic camera and the optical spectrometer (which I will just call a spectrometer from now on, even though there are many instruments called spectromers that not concerned with optical spectra). These sensors have had major roles throughout the history of optics and in the physical sciences, so it is natural

to use them as examples of computational optical sensing. Therefore it is important we first understand the isomorphic version of these sensors.

In the photographic camera, the signal-of-interest is the object's intensity distribution. This can be the scatter or emitted light from a person, a tree or a distant group of stars. The analog instrument consists of the lenses which are designed and fabricated to produce an intensity disibution (optical image) that looks like the object at the focal-plane array (FPA). The more that the image resembles the object the better the optics. The FPA then samples and quantizes the image and produces a digital representation of the object's intensity distribution, the measurement data. If one is interested in performing a task such as detection or classification, the measurement data is often is post-processed to perform those tasks.

There are two major sub-systems in the photographic camera which determine how well it performs: the optics and the FPA. Ideally, the optics (the analog instrument in this case) will produce a point spread function (PSF) which is infinitely small in diameter. For example, in a task such as the detection of a star from several neighboring stars in the night sky, if the PSF is much larger than the center to center seperation of the two stars in the optical image, it will be quite difficult to detect. A careful reader will note that this is the same argument used by Lord Raleigh in proposing his resolution criterion [2]. Even if the PSF is small enough, the FPA must sample at a fine enough pixel-to-pixel spacing, called the *pixel pitch*, to accurately reproduce the intensity variations at the scale which is pertinant to the task. Intuitively, this makes sense because if the image of the stars are imaged onto a single pixel, then one cannot ever hope to be able to accurately the detect the star without some other prior or side information.

The pinhole camera consists of a small hole and a box which prevents any light except from the pinhole to enter, see Figure 1.3. The pinhole camera is useful for imaging in parts of the electromagnetic spectrum and particles for which there is no direct analog to refractive lens or reflective mirror. Like the photographic camera, the pinhole camera also attempts to produce a small PSF for better spatial resolution and a human observer, film, or FPA can be used to measure the object scene. As the hole size decreases, the spatial resolution increase at the cost of reducing the signal strength.

In the spectrometer, the signal of interest is the spectrum of the object. The optics are designed to take the incoming light and seperate various wavelength components, see Figure 1.4. The part of the spectrometer which is used to physically

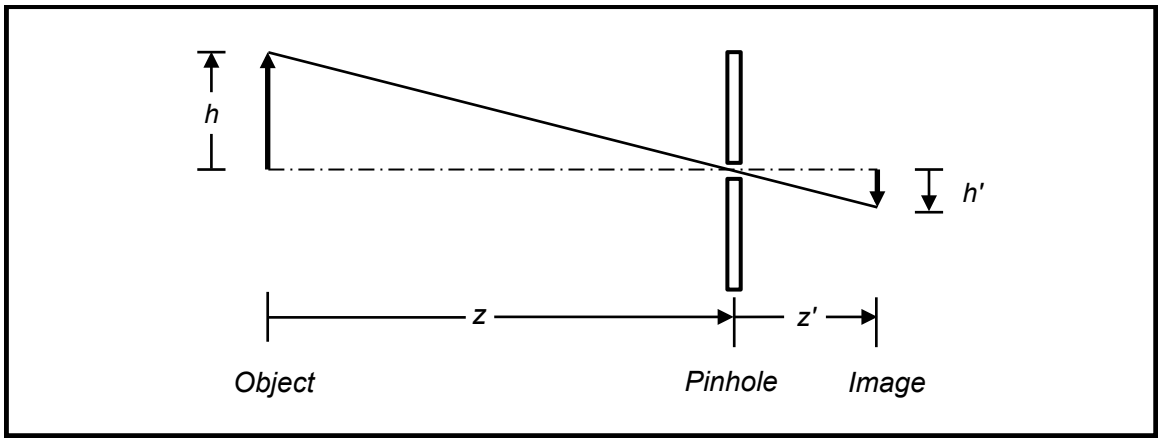

Figure 1.3: A pinhole camera is a simple imaging system that forms an image without a lens or mirror. This is due to the ray nature of light. A small hole will only admit a small amount of rays from an object point that is radiating light. Each point on a object emits light at different angles, and the image formed is a superposition of different rays. The smaller the hole, the less blurry the image. However, small holes also limit the amount of light

isolate the wavelengths is called a *monochromator*. The result is a spectral intensity as a function of position at the FPA. The FPA and post-processing algorithms are used in the same manner as the photographic camera, which is to sample the optical spectrum creating a digital version of it and to perform various tasks on the measurement data. For now, we will concentrate on the slit spectrometer, which measures the spectrum at a single point on the object.

In the spectrometer, one of the important performance metrics is *spectral resolution*, which we denote δ_λ. The spectral resolution is the smallest difference in wavelength the instrument can discern. Large spectral resolutions can degrade the spectrometers ability to discern important parts of the spectrum. Similarly with the camera, the FPA must have a pixel pitch which is small enough in order to correctly sample the variations in the spectrum.

The point-by-point nature of isomorphic sensing is both a strength and a source of weakness.

The strength comes from the straightforward and intuitive architecture of the isomorphic sensor. Each subsystem: the optics, the focal-plane array (FPA), and the post-processing can be designed and constructed seperately as long as they meet their individual specifications. As long as the signal-to-noise ratio (SNR) is sufficient and the sampling rate is high enough, we are guaranteed to recover the signal.

One of the weaknessess of the isomorphic approach is the the ability to measure low SNR signals. Because the signal-of-interest is sampled in a completely parallel

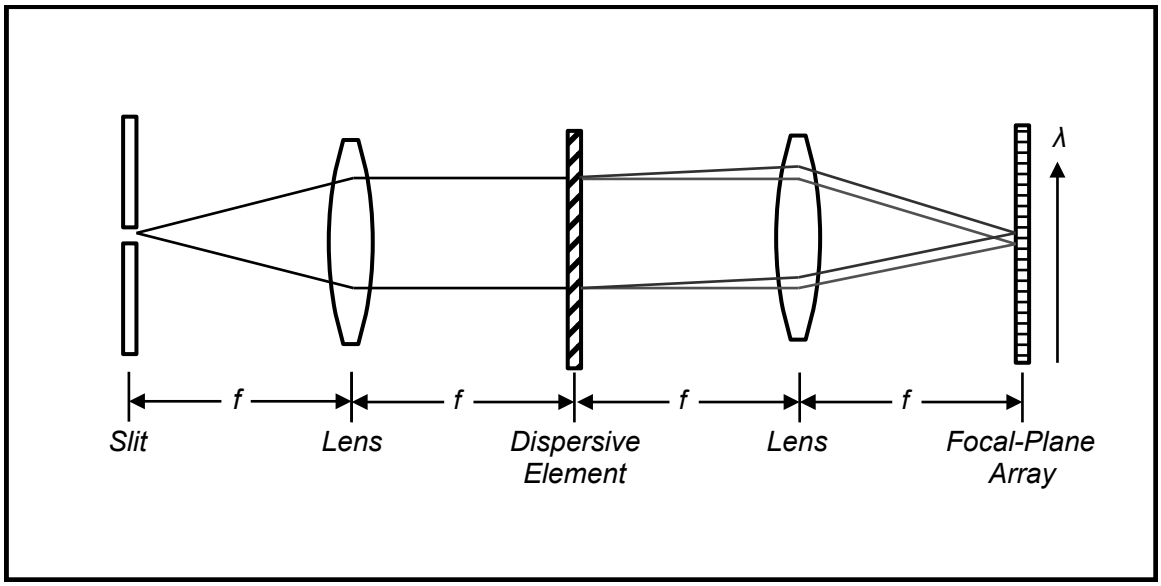

Figure 1.4: An isomorphic slit spectrometer with a 4F configuration. The slit limits the lateral extent of the object (or intermediate image). The light from the slit is collimated and seperated into different angles based on the wavelength. A second lens then images wavelength shifted copies of the slit on the image plane where the detector is. As the slit size decreases, less light is allowed, but we gain spectral resolution by rejected light from neighboring locations on the object.

fashion at each exposure, each pixel contributes a certain amount of noise. If the noise dominates, the measurement fidelity decreases often forcing the operator to increase the exposure time. For weak signals, the exposure time can become prohibitive and for temporally dynamic signals this may lead to a loss of resolution. Indeed, one of the major engineering trade-offs faced by traditional spectrometer designers is that when one attempts to increase the light collection (increased slit-width) the spectral resolution δ_λ degrades. Similarly, in the pinhole camera, there is a throughput versus spatial resolution trade-off, increasing the size of the pinhole degrades the PSF.

It would be easy to assume that with the recent revolution in machine learning and statistical signal processing combined with the dramatic increase in computing power that we could simply post-process poor measurments and obtain useful data. However, this isn't possible due to the an important theorem in information theory called the *data processing inequality* [3]. In layman's terms, it means "garbage in, garbage out".

Another weakness of isomorhic sensing is that the seperation of the analog instrument, the sampling scheme, and the data processing algorithms lead to increased size, weight and power-cost (SWAP-C). As we mentioned in the photographic cam-

era, the optics must be designed to produce a small PSF. For demanding applications, the optical design and fabrication can be the most expensive component of the sensor. While FPA prices in the visible have fallen, FPAs in certain parts of the electromagnetic spectrum can be quite expensive or non-existant [4, 5].

In many cases, the signal is redundant and high resolution sampling becomes a waste of resources, such as data storage and communications bandwidth. A good example is in photography where often the post-processing takes the digital image and applies a compression algorithm which looks for patterns in the signal and reduces the file size, discarding much of the sample data [6].

The isomorphic sensor has served humanity well, however with all the weakness that I have discussed, I will now begin to discuss some of major techniques in computational sensing that can be used to address some or all of the issues that I just stated.

1.2 Development of Multiplexing in Sensing

Multiplexing in sensing allows each measurement sample to be a combination of multiple points of the signal-of-interest. Multiplexing is a powerful tool that can be exploited by the sensor designer to eliminate or relax design trade-offs.

A simple example which illustrates the usefulness of multiplex sensing is weighing objects. In this example, we are given a 100 sheets of paper. Let's assume that the scale's measurement error is insignificant. Isomorphic measurement sensing means one would need to measure each sheet of paper individually. Requiring 100 measurements.

Now, let's say the scale's measurement error is on the order of the weight of the sheet. Measuring each sheet invidiually produces a large measurement error. In order to reduce the error to an acceptable SNR we need to make several measurements per sheet to reduce the error to an acceptable level.

However, we can be a little smarter. We can measure all 100 sheets at the same time. Since the weight of all 100 sheets is much larger than the measurement error of the scale, we can dramatically increase the precision of the measurement. If we can assume that each sheet is the same weight, then we are done.

The weighing problem is analogous to the spectroscopy example. As discussed earlier in section 1.1, there is trade-off between light collection and spectral resolution. Increasing the slit-width to increase the amount of light has the effect of

degrading the spectral resolution δ_λ. Around the late 1940's and early 1950's, several important papers and inventions demonstrated the effectiveness of multiplexing in spectroscopy. At the time the FPA was non-existant, so in the slit spectrometer shown in Figure 1.4, where the FPA is pictured, there was actually another slit. To record the intensity at each spectral channel, either the dispersive element or the exit slit had to be mechanically translated, making the measurements even slower by a factor of N_λ, the number of spectral channels of interest.

Golay was the first to propose multiplexing the slit spectrometer by creating a pattern of binary (1's and 0's) entrance and exit slits [7]. The idea borrowed heavily from communications theory which is concerned with the reliable transmission of information over a noisey channel. In the Golay multi-slit spectrometer, the patterns of entrance and exit slits are matched based on mathematically useful properties, this is similar to coding and decoding signals in commucations. In communications theory the process of structuring the data from the source to the receiver is referred to as *coding*. Similarly, in computational sensing, the transmission of information between a object signal-of-interest and the sensor is considered a coding problem [8]. In the multi-slit spectrometer, the entrance slits act to code the spectrum while the exit slits decoded the coded spectrum. Intuitively, the ability to use multiple entrance and exit slits increases the optical throughput of the spectrometer. Golay's idea dramatically increased the optical throughput without degrading the spectral resolution.

Another example that is pertinant to this dissertation is coded aperture imaging. Coded aperture imaging can be thought of as the multiplexed version of a pinhole camera. As mentioned earlier in section 1.1, there is a trade-off between the throughput and spatial resolution. However in many fields, such as high-energy particle imaging, refractive lenses and reflective mirrors are non-existant or underdeveloped. By using multiple pinholes the throughput is increased without sacraficing spatial resolution. However, the pattern of the pinholes, which is the code, in this case must be carefully designed in order to the reconstruction to be feasible. Fenimore, Canon, and Gottesman were the first to create an elegant solution to coded aperture design called uniformly redundant arrays [9, 10].

In summary, multiplexing has the ability to eliminate classic trade-offs in isomoprhic sensors: signal strength or resolution. Modern researchers are still actively developing novel ways to implement multiplexing to increase resolution and senstivity in the spatial domain [11, 12], spectral domain [13, 14], and temporal domain

[15, 16]. However, multiplexing is not without its own set of challenges. As we mentioned, the coding must often be designed to obtain feasible signal reconstruction. We now discuss inverse problems in computational sensing.

1.3 Forward Models and Inverse Problems

In the computational sensing community, a model explains the mapping of the signal-of-interest to the measurement data is called the *forward model*. The problem of taking the observed data and calculating a reconstruction of the signal-of-interst or task-specific parameters is called the *inverse problem*.

As you can imagine, solving inverse problems of isomorphic measurements, when one is concerned with reconstruction of the signal-of-interest, tend to be straight forward. In the weighing problem, the measurement is also the reconstruction. In the slit spectromer, where the forward model can be simply the continuous to discrete mapping of the spectrum. The spectrum is the interpolated measurement.

Of course, we can begin to add various levels of complexity to the forward model to account for various physical aspects of the sensor, such as the fact the FPA can't measure certain wavelength regions or the noise in our measurements. But again, assuming proper sampling and enough SNR, the reconstruction of the isomorphic signal is the measurement. This simplicity is one reason why isomorphic sensing still dominates at the consumer level despite all of the drawbacks I discussed earlier in section 1.1.

However, the multiplexing of signal information forces us to develop computational steps to solving the inverse problem. In the multiplexed weighing problem, a significant complication occurs when each sheet of paper has a different weight. Now solving the inverse problem is not as straight forward. A single measurement of all 100 sheets in this case is an underdetermined problem, since we have 1 equation and 100 unknowns. What we can do is try measuring different combinations of the 100 sheets, each new combination provides us with a new equation to work with reducing the error. Niavely, we might assume that we can randomly choose 100 unique combinations and solve 100 equations using the algebra we were taught in high school. This works fine when there is no measurement error. However, in the presense of noise, in many applications including the weighing problem, random combinations are not the best way to conduct the coding. They are sub-optimal in terms of reconstruction error. This lead many to begin working on optimal coding

strategies of signals for sensing and is major topic in this disseration.

In fact, the idea of multiplexing was not invented by Golay or Fenimore, what they contibuted was the creation of effective ways to code and decode their signal-of-interest, making their sensors more appealling to a broader community of scientists and engineers.

In summary, the forward model of a sensor is essentially accounting for the physics which govern the measurement. While the solving the inverse problem is a mathematical problem which attempts to either reconstruct the object or to calculate task-specific data from the measurement data. Unfortunately, not all multiplexing forward models codes have mathematically elegent inversion steps. Often the physics of the situation force non-isomorphic measurements which require a computational step to solve the inverse problem.

1.4 Indirect Imaging

While Golay, Fennimore, and others were levering multiplexing to eliminate tradeoffs in traditional sensors, an entirely disparate group of researchers were working on imaging techniques for which there was no isomorphic analog. In these cases the physics of the sensing modality prevents a point-by-point sampling of the signal-of-interest. Indirect imaging refer to sensing schemes which include X-ray Computed Tomography (CT), Single-Photon Emission Computed Tomography (SPECT), Positron Emission Tomography (PET), Magnetic Resonance Imaging (MRI) and certain forms of sonic and radio wave imaging all require a data-processing or reconstruction step to solve an inverse problem [17].

Perhaps one of the most successful early examples of indirect imaging and the rise of inverse problems in sensing is the development of radar. While early radar was concerned with the detection and distance of an object, development of imaging radar began after World War II. Imaging radar and specifically Synthetic Aperture Radar (SAR) can use time delay information combined with the doppler effects and interference patterns of coherent radio waves to create high resolutions of terrain and buildings.

In medicine, a common imaging modality is X-Ray CT. In X-Ray CT, computational inversion is required to reconstruct a 2 or 3-dimensional function from 1 or 2-dimensional measurement data. The forward model can be simple: In a collimated beam architecture with a 1-dimensional detector array, we say that each

sample from each pixel on the array is proportional to the total number of x-ray photons that have not been absorped by the object [18] plus noise. The inversion of course is not straight forward. The culmination of the work related to the inversion techniques and the actual prototype resulted in the Nobel Prize for Physiology or Medicine in 1979 [19].

Indirect imaging is a subfield of computational sensing. Due to the medical or military applications of these computational sensors, there has been an intense push to reduce measurement time and improve task-specific and reconstruction results. Many of the techniques from other subfields of computational sensing have been brought to bear for indirect imaging [20, 21].

We have discussed the development of multiplex sensing and indirect imaging and how the ideas from both subfields are analogous in terms of producing a non-isomorphic measurement. However a major step in practical implementation of computational sensing is being able to obtain the measurements in a quick, reliable and efficient manner. Computational sensing as a field would not exist without the most important invention in optics and photonics of the 20th century.

1.5 The Digital Imaging Revolution

In 1969 Boyle and Smith invented the Charge-Coupled Device (CCD) [22]. The CCD is the first integrated circuit device which using a 2-dimensional arrangement of pixels which could reliably convert an intensity distribution to a digital signal. The CCD is a type of FPA.

The CCD was a major breakthrough for entire fields and industries who depended on the reliable sampling, storage and transmission of optical signals. Until then one either had to use film or bulky tubes that required an electron beam to be scanned across an image scene, such as the Image orthicon [23]. For their invention, Boyle and Smith both received the Nobel Prize in Physics in 2009 [24]

The invention of the digital camera by Sasson followed shortly after [25]. Several years later the first digital spectrometer was invented. The exit slit was replacing by the CCD, whiched allowed for instant and simultaenous measurement of the entire spectrum in a compact architecture [26].

The development of the Complementary Metal–Oxide–Semiconductor (CMOS) FPA was also important. While in scientific settings, it could not rival the quality of the CCD, its cheaper cost brought digital imaging to the consumer level. Other

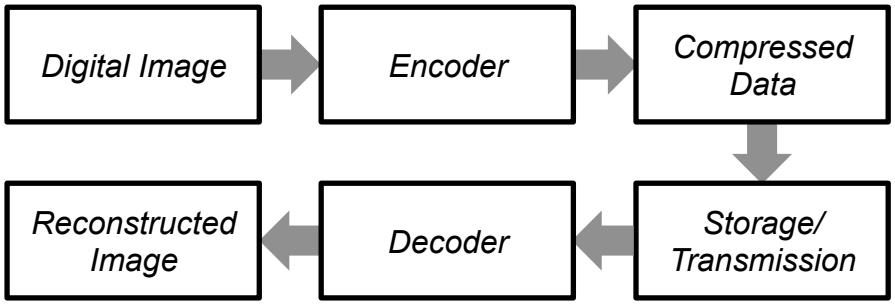

Figure 1.5: A general flowchart for image and data compression techniques. The digital image is analyzed by the encoder and compressed into a compressed form where it can be efficiently stored or transmitted. The decoder takes the compressed image and converts it back into an image that resembles the original digital image.

technology like the digital computers and computer networking also provided major contributions to the democratization of imaging and optical sensing. While scientific grade optical instrumentation was and is still expensive, the researcher could at least capture, process, and share measurement data with significantly less effort. Without it, the field of computational sensing would not exist.

Algorithms for efficient and reliable storage and transmission of digital images became more important. Over time the pixel count continued to increase and the sheer volume of digtal image and video data being generated and transmitted over networks began to outpaced improvements in storage and transmission capacity. While many engineers developed new technology to combat the hardware limitations of storage and transmission. This also led to a renewed effort by researchers to develop more efficient image and video compression algorithms [27, 28].

While a multitude of compression techniques exist, they all follow the same basic process, see Figure 1.5. Once the CCD samples the optical signal and produes the measurement data, the encoder usings the compression algorithm to look for redundancies in the data and produce a lower dimensional representation of the image. The compressed data can then either be stored or transmitted or both. The decoder solves the inverse problem of reconstructing the image.

As mentioned in at the beginning of this chapter, computational sensing lies at the intersection of the design of the analog hardware, sampling schemes, and processing algorithms. Many of the algorithms used in computational sensing are the same as or inspired by the techniques used by the image processing community. This is because a major effort of computational sensing is the desire to make more resource efficient measurements.

1.6 Compressive Sensing

Traditionally, in order to increase the resolution of a sensor, one had to increase the number of measurements. This means that the SWAP-C must also increase. A camera with just a few megapixels FPA costs less than one with hundreds of megapixels. The cost of designing the optics will also need to scale to provide enough optical resolution. In a perfect world, we could capture all the information we need from just a few measurements.

Conventional signal processing dictates that accurate reconstruction of the signal-of-interest is highly improbable. If we have a discrete signal we need at least as many measurements as there are signal elements to solve the inverse problem. If the number of measurements is less, then the inverse problem is underdeterimined. Fortunately, a signal acquisition technique called compressive sensing allows us to design sensors that solve these types of highly underdetermined inverse problems.

As discussed earlier, much of the data being generated by sensors are redundant. Images, spectra, video, and audio data of real-world signals tend to exhibit patterns or redundancies that can be exploited. This allows a compression algorithm to significantly reduce the amount of data needed to represent the signal.

There is a class of compression algorithms called lossy [29]. In lossy compression, not only are redundancies exploited but data that is deemed insignificant to the signal quality is discarded. Only the most important part of the signal is kept as part of the compressed representation of the original signal. When the signal is uncompressed, the amount of data is less that the original measured data. The difference in quality is often unnoticeable to a human observer. In both lossless and lossy compression, the goal is to obtain a *sparse* representation of the signal. A sparse representation means that the signal can be well approximated with only a few non-zero elements in a representation basis. A representation basis is a basis in which the signal-of-interest is sparse. For example, most natural images are sparse in the Fourier basis. The representation basis is typically not the native basis of the signal-of-interest, i.e. pixel number or spectral channel.

In 2006, David Donoho argued that traditional sensors tend to produce vast amounts of measurement data, but often the majority of data is redundant and discarded in the compression step. He proposed that sensors can be designed to directly measure the most relavent data in a signal, suggesting a measurement scheme that can measure a compressed form of the signal [30]. This is the idea behind com-

pressive sensing sometimes known as *compressive sampling*. If the measurements are compressive then it should be possible to significantly reduce the number of measurements to accurately reconstruct the signal.

Note that there is a subtle but powerful distinction between compressive sensing and the traditional approach of sensing and then compressing. In the traditional approach, compression algorithms operate as a post-processing step. Therefore, a traditional compression algorithm will have access to the entire signal to look for redundancies and convert it into a sparse representation. In compressive sensing, we attempt the compression directly and therefore do not have access to the entire uncompressed signal. The algorithms must assume that the signal has a sparse representation.

The question of how to actually measure or code the analog signal to directly obtain compressed data is also important. Fortunately, random coding tends to work well in many instances when the signal has a sparse representation. Much of the work in this dissertation will discuss other types of coding schemes that can be used to outperform random codes.

The idea of compressive sensing seems to be similar to the concept of multiplex sensing. However, there is an important distinction to be made. In compressive sensing, the aim is to obtain the relavent information in as few measurements as possible. In multiplexing, the goal is to overcome limitations mainly due to lack of SNR. Many compressive sensing schemes also employ multiplexing.

One useful example of compressive sensing versus traditional sensing is the single pixel camera [11]. The single pixel camera is a multiplexing camera architecture that uses time sequential random measurements and recovers the image in significantly less measurements (=number of exposures $\times$ pixels) than the conventional camera, see Figure 1.6. Another example is the Coded Aperture Snapshot Spectral Imaging (CASSI) architecture [31], which can reconstruct a spectral data cube in significantly less FPA exposures than a traditional spectral imaging architecture.

Another important distinction is between reconstruction and task-specific sensing. Task-specific sensing tends to refer to measurement techniques that attempt to directly perform tasks such as detection, classification, and estimation without the intermiediate step of reconstructing the high-dimensional signal. Compressive sensing is useful not just of overcoming resolution limitations in reconstruction but for reducing measurement resources for task-specific sensing. For example, in facial recognition the goal is detection of an individual person. Reconstruction of the

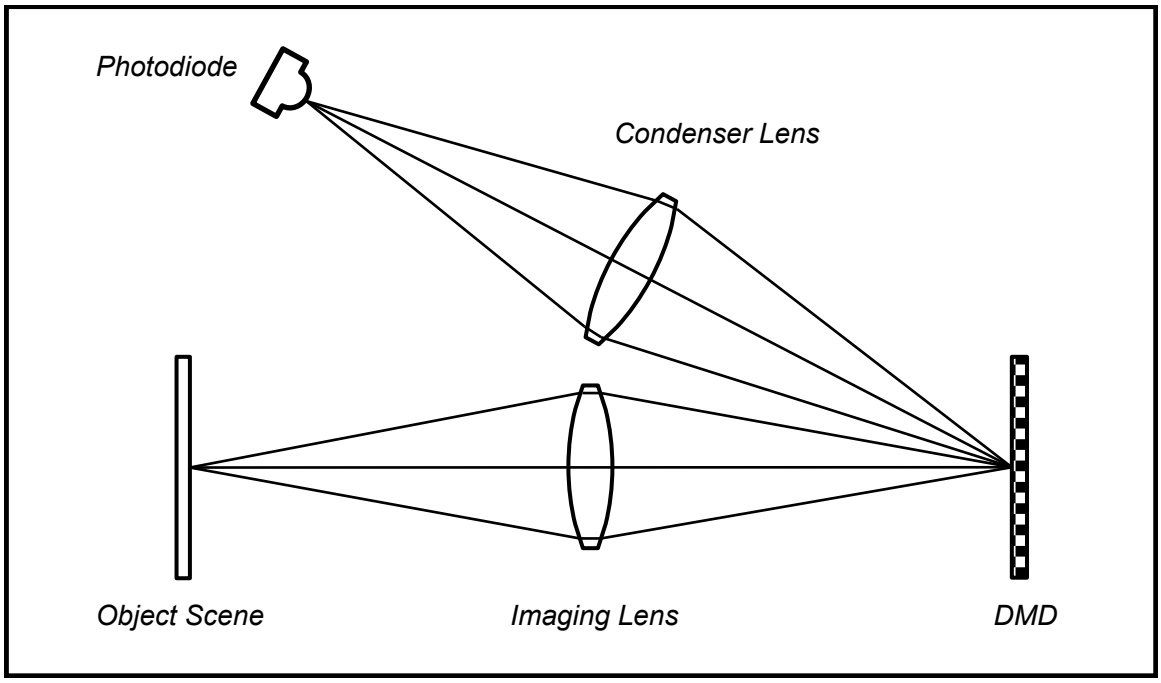

Figure 1.6: A single pixel camera architecture. The object scene (or intermediate image) is imaged onto a Digital Micro-Mirror Display (DMD). The each micro-mirror of DMD reflects light towards the photodiode or to another direction. This acts as a point-by-point multiplication of the discrete image with the DMD pattern. The condenser lens sums any light reflected by the DMD and focuses it onto the photodiode. This can be described mathematically as a vector multiplication of the image with the DMD pattern.

face image is simply an intermediate step, therefore, one can develop a compressive sensing scheme that is optimal for direct facial detection, skipping the step of image reconstruction [32].

Multiplex sensing, compressive sensing, and task-specific sensing are considered subfields of computational sensing. Computational sensors offer significant advantages that allow us to overcome classic engineering trade-offs in sensor design. However, computational sensing has its own unique set of engineering problems that we will now discuss.

1.7 Practical Considerations in Computational Sensing

So far we have the discussed the development of computational sensing techniques and how they are used to ameliorate trade-offs in traditional sensor design. Computational sensing as a field is continuing to grow at a rapid pace. The number of journal publications related to computational sensings has steadily increases every year since 2008 [33]. There is now a major OSA meeting dedicated to computational sensing called Computational Optical Sensing and Imaging (COSI) [34] and

textbooks dedicated to its study and development [8, 35]. It is important to recognize is that while it is a powerful approach to radically new sensor architectures, challenges remain in our ability to design and demonstrate experimental practical computational sensors.

One of the major obstables in computational sensing is calibration. Calibration is the act of quanitying the sensor's measurements in order to produce an accurate forward model. While many traditional sensors also require calibration, computational sensors tend to be more sensitive to calibration error.

There are two main reasons for this. The first reason is simply due to the fact that non-isomorphic measurements require a computational step to solve an inverse problem. The algorithms rely on accurate knowledge of the forward model to seperate measurement data due to the instrument and data due to the signal-of-interest.

The second reason is due to the lack of redundancy in a compressive sensor's measurement. The redundancies that are typically deemed wasteful in traditional sensing, can also be used by post-processing algorithms to solve the inverse problem in a robust fashion. If the calibration is wrong for a few measurement sames, one can still use error-correcting algorithms to recover the comprimised data. In a compressive measurement, these redunancies are exploited in the physical measurement process [36]. Only a few numbers are used to repesent many. If the few numbers are misinterrepted due to poor calibration, the inversion algorithm's performance will suffer. We will illustrate in this dissertation the calibration challenges in several computational sensors.

Since, calibration has become a major drawback in *Compressive sensors*. A consumer cannot be expected to spend time calibrating everytime the instrument is physically bumped or the air pressure changes. In high dimensional compressive sensors like hyperspectral imagers, the calibration time can last hours.

Another hurdle in the implementation of practical computational sensing is the need for prior knowledge. For example, in compressive imaging we have to assume the signal is sparse in some basis. Fortunately most realistic objects are sprase or compressible with modern representation bases, such as the wavelet basis. However, sometimes one needs to image something that is difficult to represent with any current basis, such as the intensity pattern due to laser speckle. One needs an alternative representation basis, so training sets are often used to generate a new basis in which the signal can be represented as sparse. One must generate many

different example training data so a particular signal-of-interest can be have a sparse representation in that basis. This becomes time and compuationally expensive. Many algorithms require some prior knowledge of the statistics of the signal and noise ahead of time for optimal performance. A good example of this is the AFSSI-C, a computational spectral classifier which requires knowledge of standard deviation of the noise's probability density function to perform spectral classification in the least number of measurements as possible.

Reducing the number of detector elements is often the goal in computational sensing. A notable example is the single pixel camera is an architecture. However, the single pixel camera requires several time sequential measurements. Each measurement displays a different DMD pattern to create a randomly encoded measurements [11]. The drawback to this architecture is that one must point the camera at the object scene until enough measurements have been collected for proper reconstruction. A complication arises when this architecture is used to image temporally varying object scenes. One must display the DMD patterns even faster and reduce the exposure time to keep up. As a result the SNR may begin to degrade.

A possible way to mitigate this issue is to do all the encoding in parallel. However, a completely parallel approach would require a lens, a DMD (or coded aperture), and a detector pixel for each measurement. Since each lens uses a different entrance pupil, this will means that each will have a slightly different view of the object scene. This drastically scales the complexity of the architecture and algorithms. In this dissertation, I will discuss a compromise to parallel coding, in two different computational sensors, by using a common entrance pupil and a CCD.

Much of the Often, theorectically optimal measurements take advantage of codes which are simultaneously positive and negative. In reality, with incoherent light we are unable to make negative measurements. One is often forced into situations where one must record two sets of measurements and subtract one set of measurements from the positive set of measurements. This means an additional noise term is added to each effective measurement.

Algorithms engineered to solve the inverse problems often do not account for the non-negativity of many physical situations. For example, in spectral unmixing where the problem is to solve for concentration of each material's spectrum. A non-negative fractional abundance that sums to one is a physical requirement. However, there is a serious lack of sparsity promitting algorithms that are able to enforce both the non-negativity and the sum to one constraint.

A major issue in both the theoretical and experimental compressive sensing community is a lack of code design schemes. For the most part random measurements techniques are dominant because they obey a key theorectical result which guarantees reconstruction with high probability. Intuitively, designed, meaning non-random, measurement codes which can take into account prior knowledge of the physical limitations of the sensing task and statistical assumptions of the signal-of-interest should be able to outperform random measurements. For example, we will show that in certain applications we can use Principal Component Analysis (PCA) to outperform random measurements. In one application, we will demonstrate that simply designed codes outperform random codes in low SNR environments. However, as of yet there is no adaptive measurement package one can simply take "out-of-the-box" and begin applying them to compuational sensing problems and outperform random measurements on a consistent basis.

These issues and others will be discussed in a case study manner throughout this dissertation. I will now discuss the organization of this dissertation and the three seperate computational sensors I have built and how I have tried to migitate and resolve some of the practical issues associated with computational sensing.

1.8 Dissertation Overview

I used this chapter to cover the major concepts of computational sensing. More importantly, I provided the major topic of discussion for this dissertation, the challenges that are remain towards developing practical computational sensors. I will use the next chapter as an opportunity to provide a more detailed and mathematically rigorous look at the various coding schemes that are popular in computational sensing, as well as the ones I have used. This includes a discussion of the mathematical methods that I and my collaborators developed and deployed in the AFSSI-C: Bayes rule, Log-Likelihood Ratios, and the Maximum a Posteriori decision technique. A more rigourous treatment of compressive sensing will also be provided which includes several important results from the compressive sensing community. I will also talk about several optimization algorithms and estimation techniques that have been used during the development of the three computational sensors in my work.

Chapter 3 will introduce a new system architecture for compressive target tracking—the Static Computational Optical Undersampled Tracker (SCOUT). This

system is designed to overcome a variety of challenges that typically arise in traditional optical sensing approaches. It provides several advantages over the single-pixel camera architecture, notably it can capture many simultaneous encodings of the field-of-view with a single camera exposure. I will present experimental results that validate the performance of the proposed architecture.

Chapter 4 will discuss an important experimental protoype which demonstrates adaptive spectral image classification. The system is a major experimental advancement compared to current computational spectral imaging architectures which focus on reconstruction. Furthermore, the ability to perform adaptive measurements, where each code is designed based on the history of prior measurement data allows this instrument to outperform traditional spectral imaging architectures by a factor of 100 in low SNR scenarios.

Chapter 5 will discuss current efforts to experimentally demonstrate compressive spectral unmixing. Often in spectral imaging, the spectra present in the field-of-view of each pixel is actual a mixture of several spectra, a mixed spectrum. Spectral unmixing is the task of inferring the fraction of each constituent spectrum in the mixed spectrum. I employ a unique architecture for computational spectral sensing which uses no diffraction grating or refracting prism but relies on the wavelength dependent nature of the birefringence in an LCOS. We will demonstrate the effectiveness of various compressive sensing measurement schemes to outperform a traditional sensor in significantly less measurements.

Chapter 6 will summarize the dissertation and provide my perspective on how the field should the approach the issues in practical computational optical sensing.

CHAPTER 2

Formalism

This chapter introduces the reader to the more rigorous concepts and mathematical background that will required to fully understand the material presented in the later chapters of this dissertation.

A discussion of multiplexing and signal-to-noise ratio will be discussed, as well are various coding schemes used in various notable computational sensors as well as the ones in this disseration.

The vast majority of modern optical sensing involves an analog-to-digital converter (ADC) step, which creates discrete digital values from a physical phenomena. Therefore, we will concentrate on continuous-to-discrete and discrete-to-discrete measurements. This not only makes the formalism we will discuss more relavent but in many cases it will simplify the mathematics.

As described earlier in Chapter 1, a measurement is a map from the physical signal-of-interest $\mathbf{f}$ to the measurement data $\mathbf{g}$. The solutions to the electromagnetic wave equation are linear in free space so the propagation of electromagnetic radiation is linear. We can also approximately model the response of our sensors as linear. Thus we can write the measurement as an integral

$$g_m = \int \mathbf{f}(\mathbf{x}) h_m(\mathbf{x}) d\mathbf{x} \tag{2.1}$$

where $h_m(\mathbf{x})$ is the continuous-to-discrete measurement process from point $\mathbf{x}$ to discrete measurement index m. We can write the continuous signal-of-interest as a superposition over a basis

$$f(\mathbf{x}) = \sum_n f_n \psi_n(\mathbf{x}) \tag{2.2}$$

This allows us to express the measurement of any optical phenomena as a matrix multiplication

$$\mathbf{g} = \mathbf{H}\mathbf{f} \tag{2.3}$$

where $\mathbf{g}$ is now a measurement data vector and $\mathbf{f}$ is the discrete representation of the object signal-of-interest and $\mathbf{H}$ is the matrix which describes measurement process. For brevity we will refer to the $\mathbf{f}$ as the object and $\mathbf{H}$ as either the sensing matrix

or the measurement matrix. Equation (2.3) represents the forward model in a wide variety of computational sensors. The object $\mathbf{f}$ is a vector in n dimensional vector space and the measurement $\mathbf{g}$ is a vector in m dimensional vector space. In general $m \neq n$. Note that Equation (2.3) is an extremely useful way to represent the forward model in optics, since it allows us invoke a vast amount of computationally attractive numerical techniques which are dedicated to linear systems.

In the real-world noise degrades the measurements

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{e} \tag{2.4}$$

where $\mathbf{e}$ is additive noise. Additive noise is noise which is independent of the signal. An example of additive noise includes the thermal noise generated by the random fluctuations of the charge carriers from CCD electronics. A different type of noise called multiplicative noise also exists, but will not be considered in this dissertation.

2.1 Isomorphic Sensing

In an isomorphic measurement, where the goal is a one-to-one mapping of object points to measurement points, the measurement matrix is often modeled with the identity matrix

$$\mathbf{H} = \mathbf{I} \tag{2.5}$$

We can get an idea of how much error exists in an isomorphic measurement through invoking the weighing example again. This example was originally discussed in book [37] but it is so useful in the context of this dissertation I will briefly summarize it here. In the weighing example, let's say we have 4 objects with true unknown weights $f_1, f_2, \cdots, f_4$. We record the measured weights $g_1, g_2, \cdots, g_4$ with additive noise (random error) $e_1, e_2, \cdots, e_4$. The forward model with noise is

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{2.6}$$

The estimated weights $\hat{\mathbf{f}}$ are the measurements $\mathbf{g}$

$$\hat{\mathbf{f}} = \mathbf{g} \tag{2.7}$$

Where the error between the estimated weights and the actual weights is the error $\mathbf{n} = \hat{\mathbf{f}} - \mathbf{f}$. For our purposes the assumption of a zero mean distributed noise is

a good assumption. This allows us to assume the the isomorphic measurements are unbiased. For an unbiased estimator, the mean squared error of the estimated weight is the variance

$$E[(\mathbf{e})^2] = E[(\hat{\mathbf{f}} - \mathbf{f})^2] = \sigma^2. \tag{2.8}$$

The smallest possible mean square error is limited to the variance of the noise. Now, I will show how multiplexing codes can be used to reduce the mean square error of the estimate.

2.2 Multiplexing

As we discussed in chapter 1, multiplexing is an extremely useful technique for overcoming trade-offs in traditional optical sensing. Now it is time to discuss and demonstrate the quantitative advantages provided by multiplexing.

In a multiplexed measurement, each element in the measurement vector $\mathbf{g}$ is a weighted linear combination of the elements in the object vector. Therefore the measurement matrix $\mathbf{H}$ is no longer an identity matrix.

2.2.1 Coding Schemes

We will now discuss formally the properties, advantages and disadvantages of several popular multiplex coding schemes used in computational sensing especially in dispersive spectroscopy such as the Hadamard, S-Matrices, and random coding.

A Hadamard matrix of order n is defined as a matrix $\mathbf{H}_n$ as the $n \times n$ matrix whose elements consist of $+1$'s and -1's and satisfies the following property:

$$\mathbf{H}_n^T \mathbf{H}_n = \mathbf{H}_n \mathbf{H}_n^T = nI_n \tag{2.9}$$

where $\mathbf{I}_n$ is an $n \times n$ indentity matrix.

In multiplex spectroscopy and imaging, Hadamard codes are extremely popular for a variety of reasons. Hadamard codes are provably optimal in the case where we are allowed to take a full set of measurements, meaning that $\mathbf{f}$ and $\mathbf{g}$ from eq. (2.3) are both vectors in n dimensional space and when one uses both $+1$'s and -1's in the code [37]. In this case, Hadamard codes achieves the minimal the mean square error defined as

$$\text{MSE} = \frac{1}{n}(e_1 + e_2 + \cdots e_n)^2 \tag{2.10}$$

In the weighing example, we can get negative measurements by using a balancing scale instead of a spring scale. An Hadamard multiplexed measurement would look

like

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} +1+1+1+1 \\ +1-1+1+1 \\ +1-1-1+1 \\ +1-1-1+1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2.11}$$

This means that in the first measurement all 4 items at placed in the same pan. In the second measurement items 1 and 3 are in the same pan while items 2 and 4 are in the opposite pan, etc [37]. We have four equations and four unknowns so we can solve for the estimates using algebra

$$\begin{aligned} \hat{f}_1 &= \frac{1}{4}(g_1 + g2 + g2 + g4) \\ &= f_1 + \frac{1}{4}(e_1 + e2 + e3 + e4) \\ &\vdots \\ \hat{f}_4 &= \frac{1}{4}(g_1 - g2 - g2 + g4) \\ &= f_4 + \frac{1}{4}(e_1 - e2 - e3 + e4) \end{aligned}$$

The mean square error of the m^{th} measurement

$$E[(\hat{f}_m - f_m)^2] = \frac{1}{4}\sigma^2. \tag{2.12}$$

which is 4 times lower than using an isomorphic measurement scheme. In general, the Mean Squared Error (MSE) of a Hadamard measurement is

$$\text{MSE} = \frac{\sigma^2}{N_\lambda} \tag{2.13}$$

where N_λ is the number of spectral channels, which is equal to the number of measurements N_m. Hotelling proved in 1944 that for a measurement matrix with elements $h_{mn} \in [-1, +1]$ the lowest possible MSE for the case of a full set of measurements is with a linear unbiased estimator is $\text{MSE} = \frac{\sigma^2}{N_m}$ [8]. So, one cannot possibly do better than Hadamard coding in this case. We will see later in other special cases that we can beat Hadamard codes.

In many practical cases in computational sensing, making a code with simultaneous positive and negative modulation of the signal is not possible. For incoherent light, the response is linear in intensity. In the case where one has the ability to make a full set of measurements but is limited to elements of +1's and 0's, an S-Matrix code minimizes the MSE [37].

In the weighing example, a spring balance rather than a two pan scale analogous to this situations.

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & +1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ +1 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2.14}$$

So items 2, 3, and 4 are weighed together, then 1 and 2, and so on. Solving the system of equations in a silimar fashion as before in the Hadamard weighing example we find that the mean square error for the m^{th} measurement when weighing 4 items is

$$\text{MSE} = E[(\hat{f}_m - f_m)^2] = \frac{7}{9}\sigma^2. \tag{2.15}$$

Which reduced the MSE compared to the isomorophic measurement scheme but it is less of a reduction compared to the Hadamard measurement scheme. The MSE of the S-Matrix is approximately a factor of 4 increase compared to the Hadamard matrix coding scheme.

In the case when the full set of measurements are availible, random coding schemes are provably sub-optimal to compared to Hadamard and S-Matrix codes. However, they should not be ignored because in compressive sensing, certain theoretical garuntees exist for random coding that do not exist for Hadamard and S-Matrix codes. Sometimes, the physics of the situation forces a random coding scheme. There is little literature on the performance of random coding schemes. At the time of this writing (2016), I am unaware of an optimality proofs for random codes when one has a full set of measurements availible.

2.2.2 The Fellgett Advantage

The *Fellgett advantage* is the improvement in SNR that occurs when an instrument takes multiplexed measurements compared to isomorphic measurements [38, 39]. Physically the Fellgett advantage occurs because a single detector element produces a noise contribution whether it's sampling a single part of the object or multiple parts of the object. Maximizing the signal-to-noise ratio (SNR) of the estimated object signal-of-interest for a given system throughput and detector noise is major design consideration in computational optical sensing particularly in the area of spectroscopy. In spectroscopy, there are two well known multiplex schemes, the Hadamard multiplexing in dispersive spectrometers and the interferometric spectrometer .

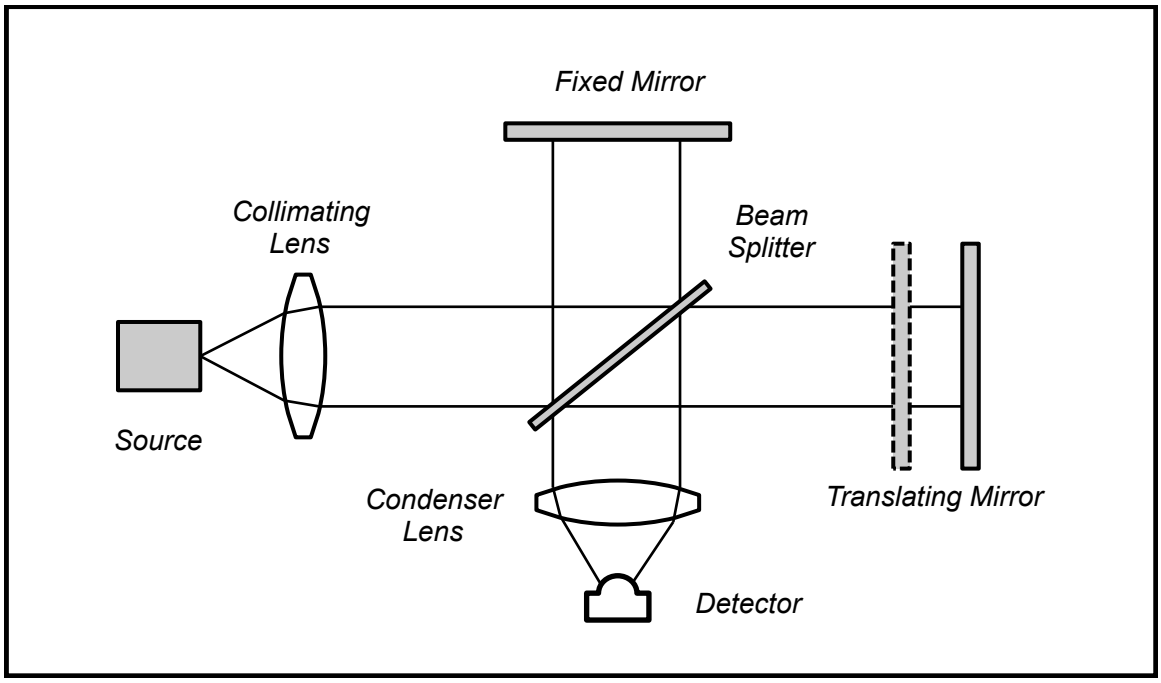

Figure 2.1: The architecture of the Fourier Transform Spectrometer resembles the Michelson interferometer. One of the mirrors is translated back and forth. The interferogram is the detected intensity versus mirror delay which is related to the autocorrelation of signal. The Fourier transform of the autocorrelation provides the spectrum.

The Fourier Transform Spectrometer (FTS) architecture is a similar to the Michelson interferometer, see Figure 2.1.The FTS operates by taking the autocorrelation of the complex electric field as a function of time delay by moving one of the mirrors in the interferometer [39]. The Wiener-Khinchin theorem says that for a wide-sense stationary random process, the Fourier transform of the autocorrelation is the power spectral density. Thus a computational post-processing step is required to reconstruct the spectrum from the measurement data. Since the FTS measures combinations of multiple wavelengths at each detector readout it also exhibits the Fellgett advantage.

The advantage that multiplexing provides over an isomorphic measurement depends on the coding sceheme. We've just seen that for a Hadamard coding scheme the MSE is given by Equation (2.13). It turns out that for a FTS the MSE obtained is a factor of two greater than the Hadamard multiplexing scheme [40].

$$\text{MSE} = 2\frac{\sigma^2}{N_\lambda} \tag{2.16}$$

2.3 Principal Component Analysis

Principal component analysis (PCA) is a technique that attempts to recast a dataset in manner which allows us to maximally discriminate the data with just a few vectors. The first vector points in the direction of maximum variance. The second vector points in a direction that also maximizes variance but is orthogonal to the first vector and so on. These new vectors are called the *principal components*. We will say the *rows* of matrix $\mathbf{P}$ are the principal components.

Let's say we have a dataset $\mathbf{S}$ which consists of N spectra with N_λ spectral channels. Instead of looking at the data as just intensity versus spectral channel, PCA attempts to construct a set of new vectors (also called features) that show as much variation in the spectra as possible. In other words, first direction (principal component) is used to recast the data to look as different (uncorrelated) as possible. This allows us to discriminate the data, as best we can with just one direction. The second principal component is the direction that provides the second most ability to discriminate the data, and so on.

The covariance matrix is defined as

$$\mathbf{C_X} = \frac{1}{N}\mathbf{S}\mathbf{S}^T. \tag{2.17}$$

Each element in the covariance matrix C_{Xmn} is the covariance of the m^{th} spectrum $\mathbf{s}_m$ and the n^{th} spectrum $\mathbf{s}_n$.

$$C_{Xmn} = \frac{1}{N}\mathbf{s}_m\mathbf{s}_n^T. \tag{2.18}$$

Note that large covariance means they look quite alike and therefore are difficult to disambiguate.

If the entire collection of spectra $\mathbf{S}$ were mutually orthogonal, being able to tell one spectrum apart from another would be easy. You would just have a collection of spikes at different spectral channels. The covariance matrix in this case would be a diagonal matrix.

Since there is typically some redundancy between spectra, the off-diagonal elements of the covariance matrix will be non-zero. The principal components allow us to recast the data to make it as uncorrelated as possible in a new basis.

$$\mathbf{Y} = \mathbf{P}\mathbf{X} \tag{2.19}$$

where $\mathbf{Y}$ is the data projected onto the principal component basis $\mathbf{P}$. The covariance of the projected data

$$\mathbf{C_Y} = \frac{1}{N}\mathbf{Y}\mathbf{Y}^T \tag{2.20}$$

is a now diagonal matrix. Indeed, the principal components are the optimal way to discriminate the spectra in the dataset [].

It turns out there is a way to calculate the principal components in closed form. The *eigenvectors of the covariance matrix are the principal components* [41].

Since the full set of principal components forms a basis, each spectra $\mathbf{s}$ in $\mathbf{S}$ can be written as a superposition of principal components without any error

$$\mathbf{s} = \sum_{\lambda=1}^{N_\lambda} y_\lambda \mathbf{v}_\lambda \tag{2.21}$$

In many cases, only a few of the first principal components are needed in the summation to approximate the original data well.

$$\mathbf{s} \approx \sum_{\lambda=1}^{M} y_\lambda \mathbf{v}_\lambda \tag{2.22}$$

Where $M \ll N_\lambda$. Note that each eigenvector has an associated eigenvalue. The eigenvalues are also informative because they can tell us how many principal components are really needed to discriminate all of the spectra. The magnitude of the eigenvalue tells us how well it's associated eigenvector is at discriminating the data.

This is another reason why PCA is so useful. It can be used as a type of lossy compression code and as a measurement matrix for compressive sensing. Simply project the data onto the first several principal compoenents associated with the largest M eigenvalues.

The AFSSI-C relies on a variation of Principal Component Analysis (PCA) for discriminating between spectra. In addition to PCA the AFSSI-C uses a Bayesian probability to create adaptive codes. We will now discuss some of fundamentals of Bayesian probability and the Log-likelihood ratios.

2.4 Bayesian statistics and Maximum a Posteriori

In the real world, measurements are corrupted by noise. The random nature of noise corrupted measurements lends itself to a stochastic perspective of signals. A hypothesis can be associated with various parameters or aspects of a signal. A hypothesis is nothing more than a claim or premise that one is interested in verifying. In imaging and spectroscopy, one example is that at a certain location in the field of view, the hypothesis is that a spectrum is present or not present. Another hypothesis is that the mean value of the signal is some value. Instead of attempting to determine

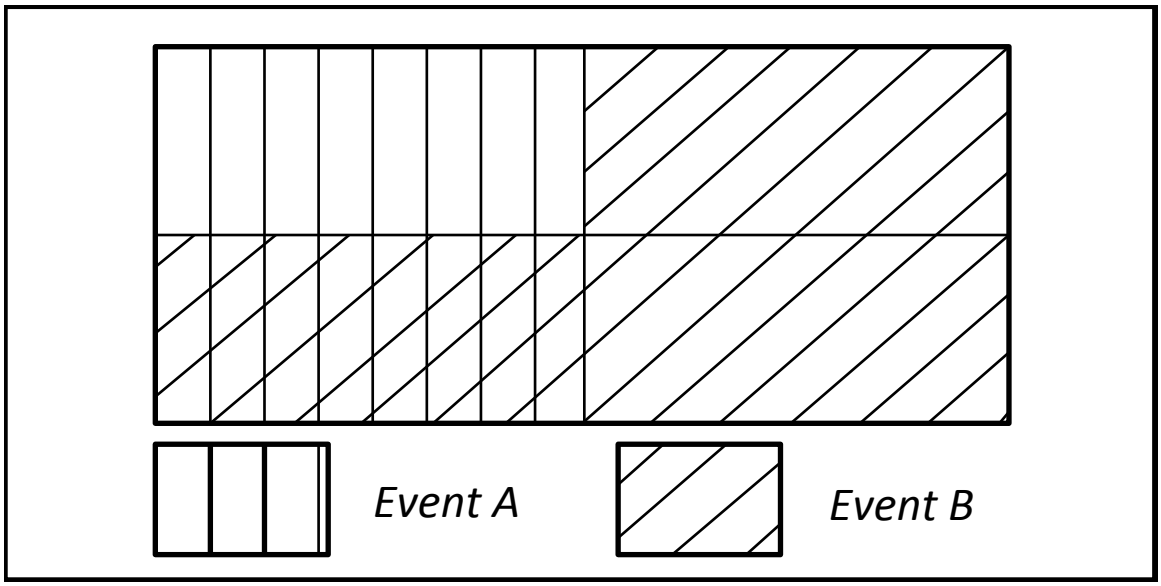

Figure 2.2: Graphical demonstration of joint probability. The probability of event A is $P(A) = 3/4$. The probability of event B is $P(B) = 3/4$. The joint probability of events A and B is $P(A \cap B) = 1/4$. The probability of event A occuring given that event B has occured is $P(A \,|\, B) = 1/3$. This is consistent with the equation $P(A \cap B) = P(A \,|\, B)P(B)$

whether a hypothesis is true, often times we are interested in estimating parameters of stochastic processes, which we denote θ.

Bayesian statistics allows one to treat the hypothesis or parameters as random variables rather than deterministric constants. A wide variety of Bayesian approaches exist and each require a heavy reliance on Bayes's theorem. Bayes' theorem is a way of computing probabilities of a hypothesis given some evidence which are related to the hypothesis. For example, Bayes' theorem can be used to decide which of two bags of candy has been opened or if a spectrum is present. The idea is that we can make a more informed calculation of probability if we are able to account or update the probability given some new piece of evidence that we may have not had at the beginning.

The derivation of Bayes' theorem follows from the definition of conditional probability, the conditional probability of event A occuring given that B occured is:

$$P\left(A \,\middle|\, B\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)} \tag{2.23}$$

this can be seen graphically in Figure

solving for the joint probability $P\left(A \cap B\right)$ gives

$$P\left(A \cap B\right) = P\left(A \,\middle|\, B\right) P\left(B\right) \tag{2.24}$$

since the joint probability commutes $P(A \cap B) = P(B \cap A)$, we can also write

$$P(A \cap B) = P\left(B \mid A\right) P(A) \tag{2.25}$$

equatin the right hand side of Equation 2.24 and Equation 2.25 gives use Bayes' theorem (also called Bayes' rule)

$$P\left(A \mid B\right) = \frac{P(A)\, P\left(B \mid A\right)}{P(B)} \tag{2.26}$$

One interpretation of Bayes' theorem is called the Diachronic interpretation, which says that conditional probability of the hypothesis or parameter θ given knowledge of some evidence or measurement data g is given by

$$P\left(\theta \mid g\right) = \frac{P(\theta)\, P\left(g \mid \theta\right)}{P(g)} \tag{2.27}$$

The term $P\left(\theta \mid g\right)$ is called the posterior. It represents our belief in the hypothesis given the data. The term $P(\theta)$ is called the prior. $P\left(g \mid \theta\right)$ is called the Likelihood. $P(\theta)$ is called the normalizing constant. In general the normalizing constant can be written as

$$P(\theta) = \sum_i P(\theta_i)\, P\left(g_i \mid \theta_i\right) \tag{2.28}$$

One can repeatly apply Bayes' theorem given new measurement data. We will use a simple example to illustrate how to update the belief. Imagine we have two bags of candy. Bag 1, which we denote B_1, has 10 pieces of cherry flavored candy, denoted as C, and has 30 pieces of strawberry flavored candy, denoted as S. Bag 2, B_2, has 20 pieces of cherry candy and 20 pieces of strawberry candy. At the beginning, the prior probability of selecting bag 1 or bag 2 is both equal

$$P(B_1) = P(B_2) = \frac{1}{2} \tag{2.29}$$

Someone then picks a bag at random and takes out a piece of candy that turns out to be strawberry flavor. What is the probability that bag 1 was selected? We can use Bayes' theorem to compute this

$$P\left(B_1 \mid S\right) = \frac{P(B_1)\, P\left(S \mid B_1\right)}{P(S)} \tag{2.30}$$

Where $P\left(S \mid B_1\right)$ means the probability of selecting a strawberry candy assuming we have selected bag 1, which is 3/4. The $P(S)$ is the probability of selecting a strawberry candy from either bag 1 or bag 2, which 5/8. Thus

$$P\left(B_1 \mid S\right) = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)} = \frac{3}{5} \tag{2.31}$$

Similarly, the probability that the person chosen bag 2 is $P\left(B_2 \mid S\right) = 2/5$. Qualatatively, this makes sense, since bag 1 contained more strawberry flavored candy, the belief that bag 1 was chosen should increase and the belief that bag 2 was not chosen should decrease.

Now let's continue the example. Say we put the first piece of candy back in the bag and draw another piece of candy, which turns out to be cherry flavor. Now we must update the beliefs with this new piece of information. We can keep using Baye's theorem, but the trick is that the posterior from the last draw is now used as the prior for the current update.

$$P\left(B_1 \mid C\right) = \frac{P\left(B_1 \mid S\right) P\left(C \mid B_1\right)}{P\left(C\right)} \tag{2.32}$$

$$P\left(B_2 \mid C\right) = \frac{P\left(B_2 \mid S\right) P\left(C \mid B_2\right)}{P\left(C\right)} \tag{2.33}$$

The probability of drawing a cherry flavored candy assuming bag 1 was chosen is 1/4 and the probability of drawing a cherry flavored candy assuming bag 2 was chosen is 1/2. We now must use Equation 2.28 to compute the normalizing constant, otherwise the posterior probabilities will not sum to 1. In this case $P(C) = 7/20$. Plugging these numbers into Equations 2.32 and 2.33 gives the updated posterior probabilities

$$P\left(B_1 \mid C\right) = \frac{3}{7} \approx 0.43 \tag{2.34}$$

$$P\left(B_2 \mid C\right) = \frac{4}{7} \approx 0.57 \tag{2.35}$$

Intuitively, drawing a cherry flavored candy has reduced our belief that bag 1 was chosen since bag 1 consist of only 1/4 cherry flavor candies while bag 2 consisted of 1/2 cherry candy.

Sometimes one is given a set of possible parameters that we are interested in estimating, θ, given some measurement data g. The method of Maximum a posteriori (MAP) says the parameters θ which maximize the posterior probability are the most likely parameters.

$$\theta_{map} = \arg\min_{\theta} \; p\left(\theta \mid g\right) \tag{2.36}$$

Using Bayes' theorem we can rewrite Equation 2.36 as

$$\theta_{map} = \arg\min_{\theta} \; \frac{p\left(g \mid \theta\right) p\left(\theta\right)}{p\left(g\right)} \tag{2.37}$$

Maximizing the posterior is now equal to maximizing the likelihood and prior. In certain cases, one needs to decide between two sets of parameters or hypotheses. One can do an analogous technique of comparing the posteriors by using a ratio

$$\frac{p(\theta_i|g)}{p(\theta_j|g)} = \frac{p(g|\theta_i)}{p(g|\theta_j)} \frac{p(\theta_i)}{p(\theta_j)} \tag{2.38}$$

If the ratio is larger than some threshold value then one choses parameter θ_i and if the ratio is less than the threshold then one choses θ_j. Similar to the earlier example of updating the posterior based on new data, one can update the Maximum a posteriori (MAP) decision based on new data. Define the likelihood ratio of the m^{th} measurement as

$$\Lambda_m = \frac{p(g_m|\theta_i)}{p(g_m|\theta_j)} \tag{2.39}$$

After each new set of measurement data g_m is collected, one can update the posterior ratios by multiplying the likelihood ratio from the old set of data with the likelihood ratio of the new set of data. The ratio which includes all the updates from measurement $m = 1$ to measurement $m = N_m$ is written as

$$\frac{p(\theta_i|\{g\}_{N_m})}{p(\theta_j|\{g\}_{N_m})} = \prod_{m=1}^{N_m} \Lambda_m \frac{p(\theta_i)}{p(\theta_j)} \tag{2.40}$$

where the notation $\{g\}_{N_m}$ represents the set of all data from measurement m to N_m.

Bayesian statistics is a useful way to update one's belief in a hypothesis or estimate a set of parameters. The Bayesian perspective is a different from the classical perspective. A classical approach to statistical estimation views parameters of interest as deterministic but unknown constants. The Bayesian approach assumes that θ is a random variable whose particular realization we must estimate.

As we've just seen through the Bayesian approach, the ability to take advantage of new measurement data and prior knowledge is a powerful concept. We will continue on this theme as we discuss compressive sensing which relies heavily on the prior assumption of *sparsity*.

2.5 Compressive Sensing

The fundamental approach of compressive sensing is that rather than sampling at a high rate and then compressing the sampled data, one can dramatically reduce the number of samples by sampling the signal in a compressed form. I will now discuss some of the more formal concepts of compressive sensing and some techniques to compressively sampling signals. This allows one to build some intuition about compressive sensing. In order to understand why compressive sensing is so powerful we should first discuss the conventional sampling strategy.

2.5.1 The Nyquist-Shannon Sampling Theorem

One of the most important results concerning the *sampling* of continuous signals is the Nyquist-Shannon sampling theorem (often referred to as the *sampling theorem* for short). The sampling theorem says that exact reconstruction of a continuous *bandlimited signal* $f(x)$ is possible if the sampling frequency f_s is atleast twice the maximum frequency B of the signal [42]. Assuming that a bandlimited signal $f(x)$ has been sampled according to the sampling theorem, then exact recovery from the discrete samples f_n is guaranteed.

However, if the sampling frequency is less than twice the maximum bandwidth $f_s < 2B$, then aliasing may occur in the reconstruction. Aliasing is the effect that high frequencies in the original signal will represented as lower frequencies after reconstruction and information contained in the high frequencies will be potentially lost [43].

Before we continue, it's important to clarify a small but important distinction between the meaning of sampling and the meaning of a measurement. A measurement is any process that maps physical phenomena which contains a signal-of-interest into measurement data. The measurement data may or may not be discrete. Sampling has a more precise mathematical definition. It is the process of mapping a continuous signal into a sequence of discrete numbers which are called the samples.

2.5.2 Sparsity, Incoherence, and the Restricted Isometry Property

At first glance, compressive sensing seems to go against the Nyquist-Shannon sampling theorem, however the sampling theorem's guarantee of exact reconstruction of a continuous signal relies on the assumption of a bandlimited signal and uniform periodic sampling, which leads to isomorphic sampling. Compressive sensing makes

different assumptions: sparsity and *incoherence*. Combined with non-linear optimization techniques, compressive sensing allows one to take far less samples without significant loss of information. Before we discuss how to actually do compressive sensing and some of its drawbacks, it is important to have a formal discussion of these concepts.

Most continuous signals $f(x)$ can be writen as a discrete summation of orthonormal basis functions

$$f(x) = \sum_{n=1}^{N} a_n \Psi_n(x), \tag{2.41}$$

where a_n are the coefficients. I will call the vector $\mathbf{a} = [a_1\ a_2\ \dots\ a_N]^T$ the representation vector of the signal and $\mathbf{\Psi}$ the representation basis. This allows us to rewrite the signal as a matrix multiplication

$$\mathbf{f} = \mathbf{\Psi a}. \tag{2.42}$$

As we discussed in Chapter 1, any vector $\mathbf{a}$ is sparse when all but a few of its entries are zero. A vector is called K-sparse when it has at most K non-zero entries. In real situations, signals with strictly sparse representation vectors are unlikely. Fortunately, it is possible to have approximately sparse representation vectors, which are called *compressible*. In other words, the sorted magnitudes of the coefficients $|a_n|$ quickly decay. When a signal has an expansion in terms of a compressible representation vector, we can intuitively understand Equation (2.41). Discarding smaller coefficients will not significantly degrade the information in the signal [44]. [1]

The concept of sparsity is important in compressive sensing. Sparsity determines how efficiently one can acquire the signals. All things being equal, if K decreases, then the reconstruction error tends to decrease. Sparse representations of the signal are not the only important prequisite for high probability reconstruction.

The random sensing basis is useful because of it's mathematical convience however, we shall show heuristically that incorporating additional prior knowledge of the statistics of the signal can reduce reconstruction error in reconstruction but other computational sensing tasks such as spectral classification. However, the options for compressive sensing bases are still quite limited.

In practice, the signal is sampled in a different basis than the representation basis. For example, while many natural signals have a sparse or compressible representation

[1] From now on, I will use the words sparse and compressible interchangeably.

in various wavelet bases or Fourier bases, sampling using the representation basis is not practical or desirable in many cases. Often a sensing basis $\mathbf{H}$ is used to perform the sampling. In traditional Nyquist-Shannon sampling, the sensing basis is a collection of delta functions. In compressive sensing for example, the sensing basis is the binary random coded aperture mask in a compressive imaging system [11]. In an LCOS based compressive sensing spectrometer, the sensing basis is a finite set of spectral filters [45, 46]. In short, the representation basis allows a signal to be represented as a sparse vector, while the sensing basis is composed of the functionals that one samples with.

The equation which combines these concepts to model a compressive measurement is

$$\mathbf{g} = \mathbf{H}\mathbf{\Psi}\mathbf{a} = \mathbf{H}\mathbf{f} \tag{2.43}$$

where $\mathbf{g}$ is a $N_m \times 1$ measurement vector, $\mathbf{H}$ is a $N_m \times N$ measurement vector, and $\mathbf{a}$ is an $N \times 1$ vector. A forward model in compressive sensing obeys Equation (2.43) when $N_m \ll N$.

One important concept in compressive sensing is coherence. The coherence between the sensing basis $\mathbf{H}$ and the representation basis $\mathbf{\Psi}$ is

$$\mu\left(\mathbf{H}, \mathbf{\Psi}\right) = \sqrt{n} \max_{1 \leq k, 1 \leq n} |\langle h_k, \psi_j \rangle|, \tag{2.44}$$

which defines coherence as a measure of the largest correlation between any two elements of $\mathbf{H}$ and $\mathbf{\Psi}$. There is an alternate definition of coherence which is concerned with only the sensing matrix $\mathbf{H}$ (the sensing matrix is also called the measurement matrix or the system matrix).

$$\mu\left(\mathbf{H}\right) = \max_{k \neq j} \frac{|\langle h_k, h_j \rangle|}{\|h_k\|_2 \|h_j\|_2}. \tag{2.45}$$

In this definition, the coherence is the maximum correlation between columns of the system matrix.

Ideally, one would like to use as less measurements N_m as possible without degrading the reconstructed signal. In compressive sensing, the amount of measurements needed is a function of the sparsity and the coherence. Given a coefficient sequence $\mathbf{a}$ that is K-sparse then one needs

$$N_m \geq C \cdot \mu^2(\mathbf{H}, \mathbf{\Psi}) \cdot K \cdot \log n \tag{2.46}$$

for a high probability of reconstruction, where C is just a constant and n is the dimensionality of the original signal.

Equation (2.46) shows the importance of sparsity and coherence in compressive sensing. Lower coherence and sparsity allows one to use less measurements [11]. The more incoherent—lower correlation—the two bases are, the higher the probability of succesful reconstruction of compressive measurements. Random matrices have a high probability of being incoherent with any basis [44]. Using the alternate definition of coherence from Equation (2.45) produces similar qualitative results [47].

The isometery constant δ_K of a matrix $\mathbf{A} = \mathbf{H\Psi}$ is the smallest number such that

$$(1 - \delta_K) \|\mathbf{a}\|_{\ell_2}^2 \leq \|\mathbf{Aa}\|_{\ell_2}^2 \leq (1 + \delta_K) \|\mathbf{a}\|_{\ell_2}^2 \tag{2.47}$$

for all K-sparse vectors $\mathbf{a}$. The matrix $\mathbf{A}$ obeys the *restricted isometry property* (RIP) of order K if δ_K is not too close to one [44]. If RIP is satisfied, the sensing matrix $\mathbf{A}$ approximately preserves the Euclidean length of signals. The Restricted isometry property (RIP) is an important theorectical result which allows robust compressive sensing where signals are corrupted by noise. Sensing matrices which have random entries obey the RIP with high probability [44, 11, 35].

All of the concepts I have just discussed can be understood at an intuitive level. Sparsity is the idea that the information content of a signal is relatively small compared to the amount of data which originally described the signal. Coherence extends that concept of how invertable a matrix is, the more linearly independent the system of equations, the easier it is to invert the matrix. The restricted isometry property is basically saying that any matrix with a small isometry constant will keep the distance between signal vectors the same. Why is this important? Think of a geometric picture, imagine noise as a sphere of uncertainty around the signal vectors. When the noise is small, two signal vectors can be mapped to a small part of the measurement space and still fit in that space. When one extracts the signal from the measurements, one can tell them apart. As the noise increases one wants the distance between measured signal vectors to at least stay the same and certainly not dramatic decrease, otherwise it will be difficult to tell the signals apart. The RIP is basically a way of seeing if a measurement matrix will pack the signal vectors with the space distance between them. A small amount of noise will lead to a small amount of reconstruction error instead of a small amount of noise leading to a large

reconstruction error. **A**

2.5.3 Solving Inverse Problems For Compressive Sensing

I will now discuss inverse problems and several general strategies which allows one to find solutions for the compressive sensing problem when $N_m \ll N$.

The least squares estimator (LSE) solves the problem of

$$\mathbf{g} = \mathbf{A}\mathbf{x} + \mathbf{n} \tag{2.48}$$

by minimizing the objective function

$$\arg\min_{\mathbf{x}} \sum_{n=1}^{N} (\mathbf{g} - \mathbf{T}\mathbf{a})^2 = \|\mathbf{g} - \mathbf{T}\mathbf{a}\|_2^2. \tag{2.49}$$

which is the ℓ_2 norm. The goal is to minimize the square difference between the given data $\mathbf{g}$ and the forward signal model for the data $\mathbf{T}\mathbf{a}$. There are no probabilistic assumptions about the measurement data. It turns out that a close form version of the LSE exist

$$\hat{\mathbf{a}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{g} \tag{2.50}$$

solves Equation (2.43) when $\mathbf{A}$ is linearly independent and when $N_m > N$ then can solve for $\hat{\mathbf{a}}$

The most straight forward technique of estimating or reconstructing a signal from its measurement data given a linear system is through the least squares estimator.

$$\mathbf{g} = \mathbf{H}\mathbf{r} \tag{2.51}$$

Given $\mathbf{g}$ and $\mathbf{H}$ we want to solve for $\mathbf{r}$. If the matrix is full rank then we can simply multiply both sides of equation 2.51 by $\mathbf{H}^{-1}$

If $\mathbf{H}$ is not full rank then its inverse does not exist. However we can try to find a solution $\hat{\mathbf{r}}$ that minimizes the least squared error. This is called the *Least Squares Solution* also known as the *Least Squares Estimator*, *Ordinary Least Squares* and by many other names. We define the squared error as

$$\|\mathbf{e}\|^2 = \|\mathbf{H}\mathbf{r} - \mathbf{g}\|^2 \tag{2.52}$$

To minimize the error, we take the derivative of equation 2.52 with respect to $\mathbf{r}$ and set it equal to zero and solve for $\mathbf{r}$. The full derivation which shows each step is given in Appendix A. The least squares estimate:

$$\hat{\mathbf{r}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{g} \tag{2.53}$$

Given measurements $\mathbf{g}$ and knowledge that $\mathbf{a}$ is sparse or compressible, solving an optimization problem of the form

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \|x\|_0 \text{ subject to } \mathbf{Ta} = \mathbf{g} \tag{2.54}$$

where $\|x\|_0$ is the ℓ_0 norm, which is equal to the number of nonzero entries in vector $\mathbf{a}$.[2] Equation (2.54) return $\hat{\mathbf{a}}$ that resembles $\mathbf{a}$ as long as the measurement noise is small. This is referred to as ℓ_0-minimization. The "subject to" constraint ensures the solution agrees with the observed measurement data. Unfortunately, ℓ_0-minimization is nonconvex which means that iterative methods may not converge to a solution. On the other hand, a convex problem has the property that any local minimum is also a global minimum. It has been shown that for general matrices $\mathbf{A}$, ℓ_0- minimization is computationally intractable [48].

Fortunately, one can use practical optimization techniques to find sparse solutions of using

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \|x\|_1 \text{ subject to } \mathbf{Ta} = \mathbf{g} \tag{2.55}$$

where $\|\mathbf{a}\|_1 = \sum_n^N |a_n|$. In other words, ℓ_1-minimization is a convex problem which allows for fast and accurate reconstruction [35]. A more practical version of Equation (2.55) is written as

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \|\mathbf{Ta} = \mathbf{g}\|_2^2 + \tau\|\mathbf{a}\|_1 \tag{2.56}$$

where τ is a non-negative number. τ is called the regularization parameter and serves to change the sparsity level of the solution and can be used to account for noise in the vector which increases the robustness of the optimization. In this form, the problem is called ℓ_1 regularized least squares (LS). ℓ_1 regularization also appears in the context of basis pursuit denoising [49]. In statistics, ℓ_1 regularization is used in the LASSO algorithm for feature selection, therefore one often refers to Equation (2.56) as the LASSO problem [50].

is also called Basis Pursu

$\mathfrak{F}\{\int_a^b\int g(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta\}$

[2]The ℓ_0 norm is not strictly a norm and is actually a quasi-norm.

CHAPTER 3

Static Computational Optical Undersampled Tracker

This chapter introduces the reader to the Static Computational Optical Undersampled Tracker.

CHAPTER 4

Adaptive Feature Specific Spectral Imaging-Classifier

This chapter introduces the reader the AFSSI-C.

CHAPTER 5

Computational Spectral Unmixing

This chapter introduces the reader the computational spectral unmixer.

CHAPTER 6

Conclusion

This chapter concludes the dissertation.

APPENDIX A

Derivation of the Least Squares Estimator

Suppose

$$\mathbf{g} = \mathbf{H}\mathbf{r} \tag{A.1}$$

Given $\mathbf{g}$ and $\mathbf{H}$ we want to solve for $\mathbf{r}$. If the matrix is full rank then we can simply multiply both sides of equation A.1 by $\mathbf{H}^{-1}$

$$\mathbf{H}^{-1}\mathbf{g} = \mathbf{H}^{-1}\mathbf{H}\mathbf{r} = \mathbf{I}\mathbf{r} = \mathbf{r} \tag{A.2}$$

If $\mathbf{H}$ is not full rank then its inverse does not exist. However we can try to find a solution $\hat{\mathbf{r}}$ that minimizes the squared error. This is called the *Least Squares Solution* also known as the *Least Squares Estimator*, *Ordinary Least Squares* and by many other names. We define the squared error as

$$\|\mathbf{e}\|^2 = \|\mathbf{H}\mathbf{r} - \mathbf{g}\|^2 \tag{A.3}$$

To minimize the error, we take the derivative of equation A.3 with respect to $\mathbf{r}$ and set it equal to zero and solve for $\mathbf{r}$. Note that the equation A.3 can be expanded in terms of an inner product

$$\|\mathbf{e}\|^2 = \|\mathbf{H}\mathbf{r} - \mathbf{g}\|^2 = \sum_{i=1}^{N} e_i^2 = \mathbf{e}^T\mathbf{e} = (\mathbf{H}\mathbf{r} - \mathbf{g})^T(\mathbf{H}\mathbf{r} - \mathbf{g}) \tag{A.4}$$

The transpose is distributive

$$(\mathbf{H}\mathbf{r} - \mathbf{g})^T = (\mathbf{H}\mathbf{r})^T - \mathbf{g}^T \tag{A.5}$$

The transpose of a product of matrices equals the product of their transposes in reverse order

$$(\mathbf{H}\mathbf{r})^T = \mathbf{r}^T\mathbf{H}^T \tag{A.6}$$

So equation A.4 becomes

$$\begin{aligned} \|\mathbf{e}\|^2 &= (\mathbf{r}^T\mathbf{H}^T - \mathbf{g}^T)(\mathbf{H}\mathbf{r} - \mathbf{g}) \\ &= \mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} - \mathbf{r}^T\mathbf{H}^T\mathbf{g} - \mathbf{g}^T\mathbf{H}\mathbf{r} + \mathbf{g}^T\mathbf{g} \end{aligned} \tag{A.7}$$

We can see that the two middle terms $\mathbf{r}^T\mathbf{H}^T\mathbf{r} = \mathbf{g}^T\mathbf{H}\mathbf{r}$ because they are just scalars.

$$\|\mathbf{e}\|^2 = \mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} - 2\mathbf{g}^T\mathbf{H}\mathbf{r} + \mathbf{g}^T\mathbf{g} \tag{A.8}$$

To find the least squares solution, take the gradient with respect to $\mathbf{r}$ and set it equal to zero.

It should be noted that there are two different notations for writing the derivative of a vector with respect to a vector $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$. If the numerator $\mathbf{y}$ is of size m and the denominator $\mathbf{x}$ of size n, then the result can be laid out as either an $m \times n$ matrix or $n \times m$ matrix, i.e. the elements of $\mathbf{y}$ laid out in columns and the elements of $\mathbf{x}$ laid out in rows, or vice versa. They are both correct and equal, which leads to confusion when switching back in forth. I will write both to reduce confusion.

Clearly the gradient of the third term in equation A.8 w.r.t $\mathbf{r}$ is 0, so it goes away. We first tackle the first term on the right hand side in equation A.8

$$\frac{\partial}{\partial \mathbf{r}}\mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} \tag{A.9}$$

Let $\mathbf{K} = \mathbf{H}^T\mathbf{H}$. Since $\mathbf{K}$ is symmetric, we can use the identity

$$\frac{\partial}{\partial \mathbf{r}}\mathbf{r}^T\mathbf{K}\mathbf{r} = 2\mathbf{r}^T\mathbf{K} = 2\mathbf{K}^T\mathbf{r} \tag{A.10}$$

since $\mathbf{K} = \mathbf{K}^T$ then

$$\frac{\partial}{\partial \mathbf{r}}\mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} = 2\mathbf{r}^T\mathbf{H}^T\mathbf{H} = 2\mathbf{H}^T\mathbf{H}\mathbf{r} \tag{A.11}$$

and the gradient of the middle term in equation A.8 is simply $-2\mathbf{H}^T\mathbf{g}$ so

$$\frac{\partial}{\partial \mathbf{r}}\|\mathbf{e}\|^2 = 2\mathbf{H}^T\mathbf{H}\mathbf{r} - 2\mathbf{g}^T\mathbf{H} \tag{A.12}$$

setting it equal to zero and solving for $\mathbf{r}$ gives the least squares estimate

$$\hat{\mathbf{r}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{g} \tag{A.13}$$

Glossary

Bandlimited signal A bandlimited signal is any signal $g(x)$ that whose Fourier transform $G(f_x)$ is zero and remains zero past a certain frequency $|f_x| \geq B$.. 43

Coding In the context of computational sensing, coding is the process of modifying or modulating an analog signal during measurement. Coding is often used to reduce degeneracy in the measurement data. In the context of spectroscopy, the spectral filters act to code the spectrum. . 19

Compressible The idea that . 44

Compressive sampling *See* compressive sensing. 25

Compressive sensing A sensing technique that attempts to directly a compressive or sparse representation of the signal-of-interest during the measurement. Compressive sensors attempt to uses significantly less measurements than the dimensionality of the signal-of-interest. Compressive sensing relies on nonlinear optimization algorithms to perform reconstruction or task-specific sensing from highly underdetermined inverse problems. These algorithms often rely on sparsity to avoid overfitting.. 13, 24, 42–45, 56

Compressive sensors *See* compressive sensing. 27

Computational sensing Any sensing technique in which the sensor uses indirect-imaging, multiplex sensing, compressive sensing, or task-specific sensing.. 13, 56

Computational sensor *See* computational sensing. 13

Data processing inequality An theorem from information theory that proves the information of a signal cannot be increased via a local physical operation.. 17

Fellgett advantage *See multiplex advantage*. 35, 36

Forward model A numerical model, typically an equation, that explains the mapping of the analog signal-of-interest to the measurement data.. 20

Incoherence The idea that . 44

Indirect-imaging An imaging sensor that attempts to reconstruct an image of an object using non-isomorphic measurements. A computational step is required to reconstruct the object signal-of-interest. X-Ray CT and SAR are examples types of indirect-imaging. . 13, 56

Inverse problem The problem of taking the measurement data and calculating a reconstruction of the signal-of-interst or task-specific parameters. In computational sensing, computer algorithms are used to solve inverse problems.. 20

Isomorphic *See* isomorphic sensing. 13

Isomorphic sensing An isomorphic sensor is any sensor that attempts to produce measurement data that resembles the signal-of-interest. An isomorphic measurement is a measurement that resembles the signal-of-interest. An isomorphic measurement can be described as a one-to-one mapping from the signal-of-interest to the measurement, and is represented in matrix notation with an identity matrix. Isomorphic sensing is synonomous with traditional sensing.. 13, 57

Measurement A process that converts a physical phenomena to single datum or a set of data. In the context of this dissertation it used synonomously with the detector readout.. 13, 31, 43

monochromator An optical instrument that transmits a selectable narrow wavelength band of light chosen from a wider range of wavelengths available at the input.. 16

Multiplex advantage The improvement in SNR that is due to multiplexed measurements rather than isomorphic measurements. This is often referred to as the Fellgett advantage since it was first discovered by Peter Fellgett. *See* multiplex sensing. 56

multiplex sensing A multiplexing sensor is any sensor that attempts to combine the physical phenomena of the analog signal-of-interest in to a few or one analog-to-digital sample to overcome limits due to signal-to-noise ratio. The measurement data that does resemble the signal-of-interest. A matrix representation of a multiplex measurement will not look like an identity matrix. 13, 56, 57

Multiplexing *See* multiplex sensing. 18, 33

Pixel pitch The center to center distance between pixels on a focal-plane array such as a CCD or CMOS image sensor.. 15, 16

Sample The process of mapping a continuous signal to a discrete signal. . 57

Sampling *See sample* . 43

Sparse *See* sparsity. 24, 44

sparsity A set, vector, or matrix which contains an overwhelming majority of zeros relative to the size of the set, vector, or matrix. . 42, 44, 57

Spectral resolution The smallest the smallest difference in wavelength the instrument or sensor can discern.. 16

Task-specific sensing A sensor that does not attempt to reconstruct the signal-of-interest to perform a signal processing task such as detection, estimation, and classification. The AFSSI-C is an example of a task-specific sensor. . 13, 56

Traditional sensing *See* isomorphic sensing. 13

Acronyms

ADC analog-to-digital converter. 13, 31

AFSSI-C Adaptive Feature Specific Spectral Imaging-Classifier. 4, 5, 28, 29, 38, 51, 57

CASSI Coded Aperture Snapshot Spectral Imaging. 25

CCD Charge-Coupled Device. 22, 23, 28, 32

CMOS Complementary Metal–Oxide–Semiconductor. 22

COSI Computational Optical Sensing and Imaging. 26

CT Computed Tomography. 21, 56

DISP Duke Imaging and Spectroscopy Program. 5

DMD Digital Micro-Mirror Display. 26, 28

FPA focal-plane array. 15, 16, 18–20, 22, 24, 25

FTS Fourier Transform Spectrometer. 36

LCOS Liquid Crystal on Silicon. 5, 30, 45

LENS Laboratory for Engineering Non-Traditional Sensors. 4, 5

MAP Maximum a posteriori. 41, 42

MRI Magnetic Resonance Imaging. 21

MSE Mean Squared Error. 34–36

PCA Principal Component Analysis. 29, 37, 38

PET Positron Emission Tomography. 21

PSF point spread function. 15, 17, 18

RIP Restricted isometry property. 46, 47

SAR Synthetic Aperture Radar. 21, 56

SCOUT Static Computational Optical Undersampled Tracker. 5

SLM Spatial Light Modulator. 5

SNR signal-to-noise ratio. 16, 18, 20, 25, 28–30, 35, 57

SPECT Single-Photon Emission Computed Tomography. 21

SWAP-C size, weight and power-cost. 17, 24

Symbols

δ_λ Spectral resolution. 16, 17, 19

$\mathbf{e}$ Additive noise vector. 32

$\mathbf{f}$ Object signal-of-interest. 31–33, 59

$\hat{\mathbf{f}}$ Estimated object. 32

$\mathbf{g}$ Measurement vector. 31–33, 45, 47, 48

$\mathbf{H}$ The system matrix which captures all of the physical phenomena in a measurement. Also called the measurement matrix.. 45

$\mathbf{H}_n$ A Hadamard matrix of size $n \times n$. 33

ℓ_1 The ℓ_1 is a norm which is defined as the sum of the absolute values of the entries $\sum_{i=1}^{N} |a_i|$. 48, 59

m measurement number/index. 31, 32

n The number of elements in the object $\mathbf{f}$. 32

N_λ Number of spectral channels. 19, 34, 37

N_m Total number of measurements. 34

. 46

τ The regularization parameter for ℓ_1 norm minimization. 48

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