

PRACTICAL CONSIDERATIONS IN EXPERIMENTAL COMPUTATIONAL SENSING

by

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

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SIGNED: Phillip K. Poon

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DEDICATION

For my wife. We moved from city to city. You stuck with me through the highs and lows. You cooked dinner for me when I came home from a long day. You did the chores so I could concentrate on research. You acted as both mother and father to our son while I wrote. You believed in me even when I did not. You sacrificed your dreams and goals so I could accomplish mine.

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ABSTRACT

Implementing computational optical sensors often comes with various issues that many traditional sensors may not encounter.

CHAPTER 1

Introduction

This chapter provides the motivation for the need to address the practical issues in experimental computational sensing. While computational sensing has the ability to ameliorate or eliminate trade-offs in many traditional isomorphic sensors, the sensor engineer is faced with a new set of challenges when designing a computational sensor. For example, while both computational and traditional isomorphic sensors often require calibration, a computational sensor can be more sensitive to calibration error due to the lack of redundancy in the measurement data. It is the author's hope that a full discussion on these various challenges will encourage more research on these issues.

Isomorphic sensing is the concept that the measurement data of a sensor resembles the signal-of-interest [1]. Isomorphic sensing is defined as sensing that attempts to produce measurement data which resembles the signal-of-interest. For example, in a camera, the digital image looks like the object. In isomorphic sensing the analog hardware, analog-to-digital converter (ADC), and processing algorithms are separate components, see Figure 1.1.

Computational sensing is the concept that a joint design of the sensor, often through active *coding* (often called structured illumination) or passive *coding* of the analog signal, with inversion algorithms will outperform the isomorphic sensor, see Figure 1.2 [2]. An example of computational sensing is the Magnetic Resonance Imaging (MRI) technique, where the measurements are in Fourier space. While *isomorphic* sensors can provide flexible sensing in multiple applications, joint design of a *computational sensor* will lead to performance increases often in resource constrained scenarios. Throughout this chapter and the rest of this dissertation, I will provide many examples that highlight the differences between computational and isomorphic sensing.

Rather than a rigorous discussion, this chapter will discuss some of the major developments and concepts in the field of computational sensing on an intuitive level. This will familiarize the reader with important terminology and techniques common in the field of computational sensing. This chapter will also discuss some

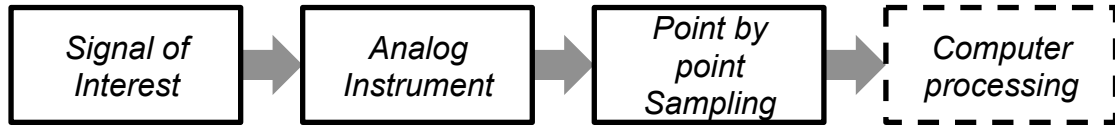


Figure 1.1: The signal-of-interest is incident upon the analog instrument. The analog instrument forms an isomorphism of the signal which is then periodically sampled point-by-point through an analog-to-digital converter (ADC) device. Once the signal is in digital form, post-processing algorithms are often used to perform various tasks such as noise reduction, detection, and classification. Notice that the analog instrument, sampling scheme, and processing are all separated.

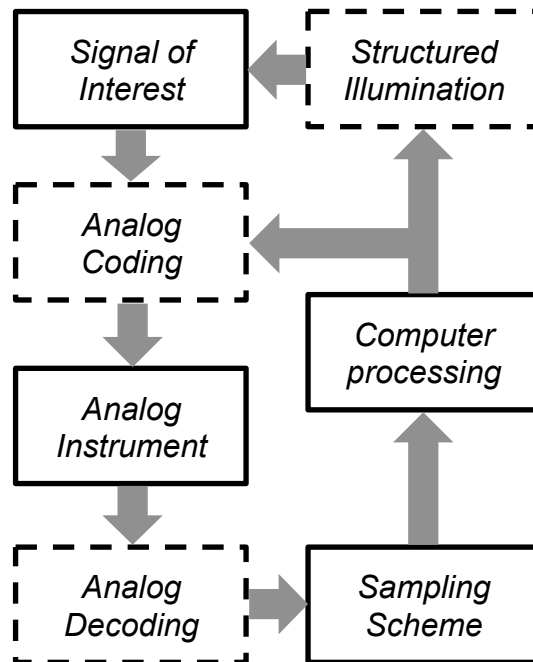


Figure 1.2: In a computational sensor, coding, decoding, and structured illumination are used in addition to the analog instrument, the algorithms, and sampling scheme. The dashed boxes represent optional processes that may or may not be needed.

of the challenges I and many other experimentalists and engineers have faced when developing computational sensing prototypes. I will close this chapter with a brief look ahead to the rest of the dissertation.

1.1 Isomorphic Sensing

In Greek, the word isomorphic loosely translates to “equal in form.” Traditional sensors perform isomorphic sensing. In the context of this dissertation, an isomorphic sensor is any sensor which attempts to produce measurement data that resembles the signal-of-interest. In this paradigm, the analog instrument, sampling scheme, and post-processing algorithms are separate components and processes.

I will discuss three important examples of isomorphic sensors: the pinhole camera, the photographic camera and the optical spectrometer¹. These sensors have inspired many other optical and non-optical sensors throughout history, so it is natural to use them as examples for comparison when discussing computational sensing.

Before I continue, I want to define *measurement* because it can often be used in an informal manner. In this dissertation a measurement has a very specific meaning. A measurement is a process that converts a physical phenomena to a collection of data. The signal-of-interest is the physical phenomena that one is interested in quantifying. I will call the collection of data the measurement data. The word sensing has a less precise meaning and I will use it in an informal manner. Sensing is any act that uses techniques, instruments, or processes that produces measurement data.

In the photographic camera, the signal-of-interest is the intensity distribution of the object. The analog instrument consists of the lenses which are designed and fabricated to produce an image that looks like the object at the focal-plane array (FPA). The more the image resembles the object the better the optics. The FPA then samples the image and produces a digital representation of the intensity distribution of the object, the measurement data. If one is interested in performing a task such as detection or classification, the measurement data sent to a post processing algorithm to perform those tasks.

There are two major sub-systems in the photographic camera which determine how well it performs: the optics and the FPA. Ideally, the optics (the analog in-

¹I will call the optical spectrometer just a spectrometer from now on, even though there are many instruments called spectrometers that not concerned with optical spectra

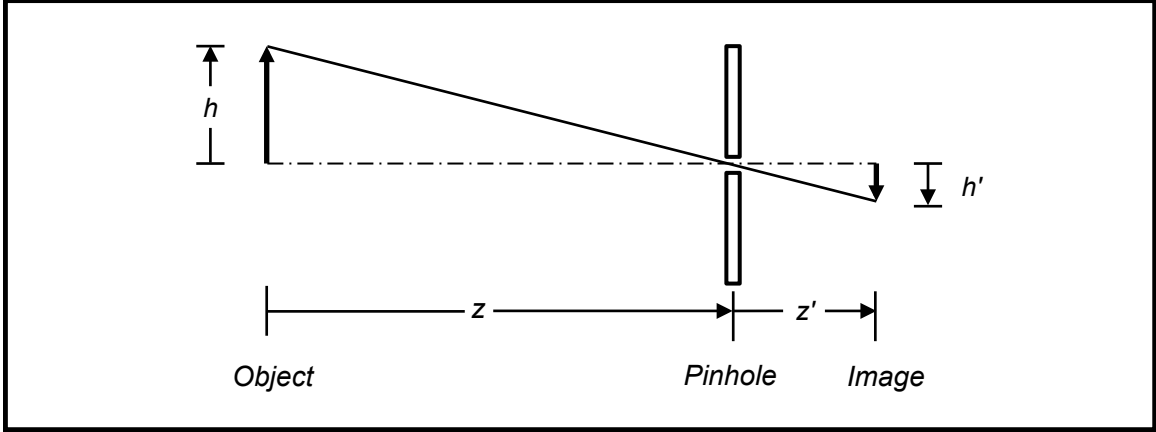


Figure 1.3: A pinhole camera is a simple imaging system that forms an image without a lens or mirror. This is due to the ray nature of light. A small hole will only admit a small amount of rays from an object point that is radiating light. Each point on a object emits light at different angles, and the image formed is a superposition of different rays. The smaller the hole, the less blurry the image. However, small holes also limit the amount of light

strument in this case) will produce a point spread function (PSF) which is infinitely small in diameter. For example, in a task such as the detection of a star from several neighboring stars in the night sky, if the PSF is much larger than the center to center separation of the two stars in the optical image, it will be difficult to detect. Even if the PSF is small enough, the FPA must sample at a fine enough pixel-to-pixel spacing, called the *pixel pitch*, to accurately reproduce the intensity variations at the scale which is pertinent to the task. Intuitively, this makes sense because if the stars are imaged onto a single pixel, then one cannot ever hope to be able to accurately detect the star without some other prior or side information.

The pinhole camera consists of a small hole and a box which prevents any light except from the pinhole to enter, see Figure 1.3. The pinhole camera is useful for imaging in parts of the electromagnetic spectrum and particles for which there is no direct analog to the refractive lens or reflective mirror. Like the photographic camera, the smaller the PSF diameter, the better the spatial resolution. Unfortunately, in the traditional pinhole camera, the only way to reduce the PSF diameter is by decreasing the diameter of the pinhole which reduces the amount of light.

In the spectrometer, the signal of interest is the spectrum of the object. The optics are designed to take the incoming light and separate various wavelength components, see Figure 1.4. The part of the spectrometer which is used to physically isolate the wavelengths is called a *monochromator*. The monochromator contains a prism or diffraction grating which creates a wavelength dependent angular separa-

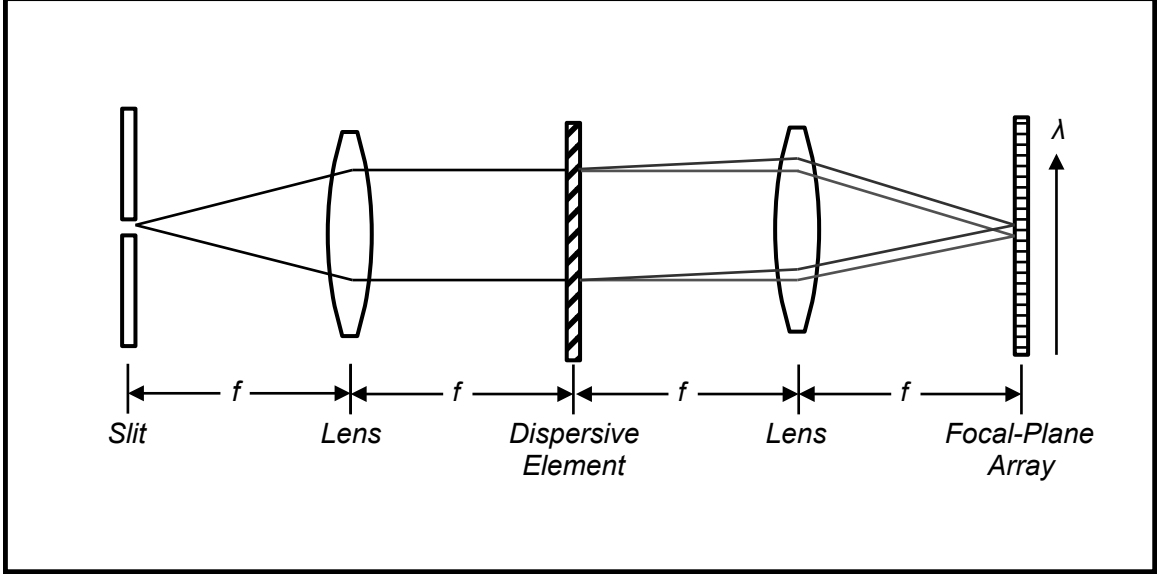


Figure 1.4: An isomorphic slit spectrometer with a 4F configuration. The slit limits the lateral extent of the object (or intermediate image). The light from the slit is collimated and separated into different angles based on the wavelength. A second lens then images wavelength shifted copies of the slit on the image plane where the detector is. As the slit size decreases, less light is allowed, but one gains spectral resolution by rejected light from neighboring locations on the object.

tion. The result is a spectral intensity as a function of position at the FPA. The FPA and post-processing algorithms are used in the same manner as the photographic camera, which is to sample the spatially varying intensity (which is now encoding spectral information) creating a digital version of it and to perform various tasks on the measurement data. For now, I will concentrate on the slit spectrometer, which measures the spectrum at a single spatial location on the object.

In the spectrometer, one of the important performance metrics is *spectral resolution*, which I denote δ_λ . The spectral resolution is the smallest difference in wavelength the instrument can discern. Large spectral resolutions can degrade the spectrometers ability to discern important parts of the spectrum. Similarly with the camera, the FPA must have a pixel pitch which is small enough to correctly sample the variations in the spectrum.

The point-by-point nature of isomorphic sensing is both a strength and a source of weakness. The strength comes from the straightforward and intuitive architecture of the isomorphic sensor. Each subsystem: the optics, the focal-plane array (FPA), and the post-processing can be designed and constructed separately as long as they meet their individual specifications. As long as the signal-to-noise ratio (SNR) is sufficient and the sampling rate is high enough, one is guaranteed to recover the

signal.

One of the weaknesses of the isomorphic approach is the ability to measure low SNR signals. Because the signal-of-interest is sampled in a completely parallel fashion at each exposure, each pixel contributes a certain amount of noise. If the noise dominates, the measurement fidelity decreases forcing the operator to increase the exposure time. For weak signals, the exposure time can become prohibitive and for temporally dynamic signals this leads to a loss of resolution. Indeed, one of the major engineering trade-offs faced by traditional spectrometer designers is that when one attempts to increase the light collection (increased slit-width) the spectral resolution δ_λ degrades. Similarly, in the pinhole camera, there is a throughput versus spatial resolution trade-off, increasing the size of the pinhole degrades the PSF.

It would be easy to assume that recent progress in machine learning and statistical signal processing combined with the dramatic increase in computing power that one could simply post-process poor measurements and obtain useful data. However, this is not possible due to the an important theorem in information theory called the *data processing inequality* [3]. In layman’s terms, it means “garbage in, garbage out”.

Another weakness of isomorphic sensing is that the separation of the analog instrument, the sampling scheme, and the data processing algorithms lead to increased size, weight and power-cost (SWAP-C). As I mentioned in the photographic camera, the optics must be designed to produce a small PSF. For demanding applications, the optical design and fabrication can be the most expensive component of the sensor. While the price of FPAs sensitive to the visible wavelength region have fallen, FPAs sensitive to certain parts of the electromagnetic spectrum can be quite expensive or non-existent [4, 5].

In many cases, the signal is redundant and high resolution sampling becomes a waste of resources, such as data storage and communications bandwidth. A good example is in photography where often the post-processing takes the digital image and applies a compression algorithm which looks for patterns in the signal and reduces the file size, discarding much of the sample data [6].

The isomorphic sensor has served humanity well, however with all the weakness that I have discussed, there is a need for sensors which can operate in low-SNR conditions, with fewer measurement time, fewer measurements, or at lower SWAP-C while still producing useful information relevant to the sensor task. I will now begin to discuss some of major techniques in computational sensing that can be used

to address some or all of the issues that I just stated.

1.2 Development of Multiplexing in Sensing

Multiplexing in sensing is the idea that each measurement sample is a physical combination of various parts of the analog signal-of-interest. Multiplexing is a powerful tool that can be exploited by the sensor designer to eliminate or relax SNR related trade-offs.

A simple example which illustrates the usefulness of multiplex sensing is weighing objects. In this example, one needs to weight 100 sheets of paper. Assume that the measurement error of the scale is insignificant. Isomorphic sensing means one would need to measure each sheet of paper individually. Requiring 100 measurements.

If the measurement error of the scale is on the order of the weight of the sheet. Measuring each sheet individually produces a large measurement error. In order to reduce the error to an acceptable SNR one needs to make several measurements per sheet.

However, one can measure all 100 sheets at the same time. Since the weight of all 100 sheets is much larger than the measurement error of the scale, one can dramatically increase the precision of the measurement. If each sheet is the same weight, then the measurement process is finished.

The weighing problem is analogous to the spectroscopy example. As discussed earlier in section 1.1, there is trade-off between light collection and spectral resolution. Increasing the slit-width to increase the amount of light has the effect of degrading the spectral resolution δ_λ . Around the late 1940's and early 1950's, several important papers and inventions demonstrated the effectiveness of multiplexing in spectroscopy. At the time the FPA was non-existent, so in the slit spectrometer shown in Figure 1.4, where the FPA is pictured, there was actually another slit. To record the intensity at each spectral channel, either the dispersive element or the exit slit had to be mechanically translated, making the measurements even slower by a factor of N_λ , the number of spectral channels of interest.

Golay was the first to propose multiplexing the slit spectrometer by creating a pattern of binary (1's and 0's) entrance and exit slits [7]. In the Golay multi-slit spectrometer, the patterns of entrance and exit slits are matched based on mathematically useful properties. Intuitively, the ability to use multiple entrance and exit slits increases the optical throughput of the spectrometer. In communications

theory the process of structuring the data from the source to the receiver is referred to as coding. Similarly, in computational sensing, the transmission of information between an object signal-of-interest and the sensor is considered a coding problem [1]. In the multi-slit spectrometer, the entrance slits act to code the spectrum while the exit slits decode the coded spectrum. Golay’s idea dramatically increased the optical throughput without degrading the spectral resolution.

Another example that is pertinent to this dissertation is coded aperture imaging. Coded aperture imaging can be thought of as the multiplexed version of a pinhole camera. As mentioned earlier in section 1.1, there is a trade-off between the throughput and spatial resolution. However in many fields, such as high-energy particle imaging, refractive lenses and reflective mirrors are non-existent or underdeveloped. By using multiple pinholes the throughput is increased without sacrificing spatial resolution. However, the pattern of the pinholes, which is the code, in this case must be carefully designed in order for the reconstruction to be feasible. Fienman, Canon, and Gottesman were the first to create an elegant solution to coded aperture design called uniformly redundant arrays [8, 9]. The uniformly redundant array increases throughput without significantly degrading the spatial resolution.

In summary, multiplexing has the ability to eliminate classic trade-offs in isomorphic sensors: signal strength or resolution. Modern researchers are still actively developing novel ways to implement multiplexing to increase resolution and sensitivity in the spatial domain [10, 11], spectral domain [12, 13], and temporal domain [14, 15]. However, multiplexing is not without its own set of challenges. As I mentioned, the coding must often be designed to obtain feasible signal reconstruction. I now discuss inverse problems in computational sensing.

1.3 Forward Models and Inverse Problems

In the computational sensing community, a model that explains the mapping of the signal-of-interest to the measurement data is called the *forward model*. The problem of taking the observed data and calculating a reconstruction of the signal-of-interest or task-specific parameters is called the *inverse problem*.

As you can imagine, solving inverse problems of isomorphic measurements, when one is concerned with reconstruction of the signal-of-interest, tend to be straight forward. In the weighing problem, the measurement is also the reconstruction. In the slit spectrometer, where the forward model can be simply the continuous to

discrete mapping of the spectrum. The spectrum is the interpolated measurement.

Of course, one can begin to add levels of complexity to the forward model to account for various physical aspects of the sensor, such as the fact the FPA cannot measure certain wavelength regions or the noise in our measurements. But again, assuming proper sampling and enough SNR, the reconstruction of the isomorphic signal is the measurement. This simplicity is one reason why isomorphic sensing still dominates at the consumer level despite all of the drawbacks I discussed earlier in section 1.1.

However, the multiplexing of signal information forces one to develop computational steps to solving the inverse problem. In the multiplexed weighing problem, a significant complication occurs when each sheet of paper has a different weight. Now solving the inverse problem is not as straight forward. A single measurement of all 100 sheets in this case is an underdetermined problem, since one has 1 equation and 100 unknowns. What one can do is try measuring different combinations of the 100 sheets, each new combination provides a new equation to work with reducing the error. Naively, one might assume that randomly choosing 100 unique combinations and solve 100 equations using algebra. This works fine when there is no measurement error. However, in the presence of noise, in many applications including the weighing problem, random combinations are not the best way to conduct the coding. They are sub-optimal in terms of reconstruction error. This lead many to begin working on optimal coding strategies of signals for sensing and is major topic in this dissertation.

In summary, the forward model of a sensor is essentially accounting for the physics which govern the measurement. While the solving the inverse problem is a mathematical problem which attempts to either reconstruct the object or to calculate task-specific data from the measurement data. Unfortunately, not all multiplexing forward models codes have mathematically elegant inversion steps. Often the physics of the situation force non-isomorphic measurements which require a computational step to solve the inverse problem.

1.4 Indirect Imaging

While Golay, Fennimore, and others were leveraging multiplexing to eliminate trade-offs in traditional sensors, an entirely disparate group of researchers were working on imaging techniques for which there was no isomorphic analog. In these

cases the physics of the sensing modality prevents a point-by-point sampling of the signal-of-interest. Indirect imaging refers to sensing schemes which include X-ray Computed Tomography (CT), Single-Photon Emission Computed Tomography (SPECT), Positron Emission Tomography (PET), MRI and certain forms of sonic and radio wave imaging. All require a data-processing or reconstruction step to solve an inverse problem [16].

Perhaps one of the most successful early examples of indirect imaging which led to the rise of inverse problems in sensing is the development of radar. While early radar was concerned with the detection and distance of an object, development of imaging radar began after World War II. Imaging radar and specifically Synthetic Aperture Radar (SAR) can use time delay information combined with the Doppler effect and interference of coherent radio waves to create high resolution images of terrain and buildings.

In medicine, a common imaging modality is X-Ray CT. In X-Ray CT, computational inversion is required to reconstruct a 2 or 3-dimensional function from 1 or 2-dimensional measurement data. The forward model can be simple: In a collimated beam architecture with a 1-dimensional detector array, each sample from each pixel on the array is proportional to the total number of x-ray photons that have not been absorbed by the object [17]. The inversion relies on computing the inverse Radon transform [18].

Indirect imaging is a subfield of computational sensing. Due to the medical and military applications of these computational sensors, there has been an intense push to reduce measurement time and improve task-specific and reconstruction results. Many of the techniques from other subfields of computational sensing have been brought to bear for indirect imaging [19, 20].

I have discussed the development of multiplex sensing and indirect imaging and how the ideas from both subfields are analogous in terms of producing a non-isomorphic measurement. However a major step in practical implementation of computational sensing is being able to obtain the measurements in a quick, reliable and efficient manner. Computational sensing as a field would not exist without the most important invention in optics and photonics of the 20th century, the digital image sensor².

² staring array, staring-plane array, focal-plane array (FPA), focal-plane, and image sensor are all synonymous

1.5 The Digital Imaging Revolution

The invention of the Charge-Coupled Device (CCD) FPA by Boyle and Smith in 1969 was a major breakthrough for entire fields and industries who depended on the reliable sampling, storage and transmission of optical signals [21]. The CCD is the first integrated circuit device that could reliably convert an optical imaging to a digital signal. Until then one either had to use film or bulky tubes that required an electron beam to be scanned across an image scene, such as the Image orthicon [22].

The invention of the digital camera by Sasson followed shortly after [23]. Several years later the first digital spectrometer was invented. In the digital spectrometer the exit slit of the monochromator was replaced by the CCD, which allowed for instant and simultaneous measurement of the entire spectrum in a compact architecture [24].

The development of the Complementary Metal–Oxide–Semiconductor (CMOS) FPA was also important. While in scientific settings, it could not rival the quality of the CCD, its cheaper cost brought digital imaging to the consumer level. Other technology like the digital computer and computer networking also provided major contributions to the democratization of imaging and optical sensing. While scientific grade optical instrumentation was and is still expensive, the researcher could at least capture, process, and share measurement data with significantly less effort. Without it, the field of computational sensing would not exist.

Algorithms for efficient and reliable storage and transmission of digital images became more important. Over time the pixel count continued to increase and the sheer volume of digital image and video data being generated and transmitted over networks began to outpace improvements in storage and transmission capacity. While many engineers developed new technology to combat the hardware limitations of storage and transmission. This also led to a renewed effort by researchers to develop more efficient image and video compression algorithms [25, 26].

Compression techniques all follow the same basic process, see Figure 1.5. Once the CCD samples the optical signal and produces the measurement data, the encoder uses the compression algorithm to look for redundancies in the data and produce a compact representation of the signal-of-interest. The compressed data can then either be stored or transmitted or both. The decoder solves the inverse problem of reconstructing the image.

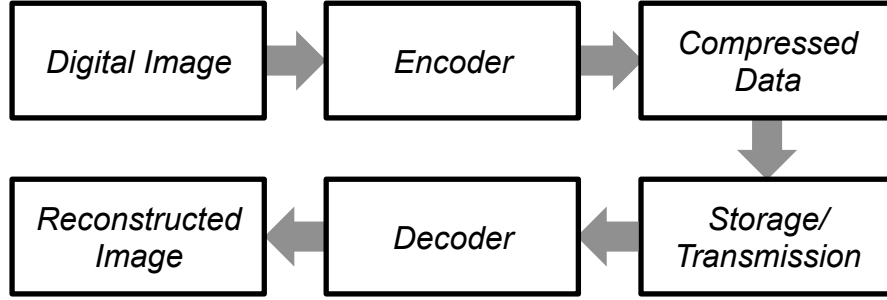


Figure 1.5: A general flowchart for image and data compression techniques. The digital image is analyzed by the encoder and compressed into a compressed form where it can be efficiently stored or transmitted. The decoder takes the compressed image and converts it back into an image that resembles the original digital image.

In the next section I will discuss how many of the techniques and algorithms used in computational sensing are inspired by the techniques used by the image processing and digital communications community. This is because one of the major efforts of computational sensing is to make resource efficient measurements to reduce the total amount of measurements, whereas in image processing and communications, measurement data is often corrupted, missing, or too large for efficient storage and transmission.

1.6 Compressive Sensing

Traditionally, in order to increase the resolution of a sensor, one had to increase the number of measurements. This means that the SWAP-C must also increase. A camera with just a few megapixels FPA costs less than one with hundreds of megapixels. The cost of designing the optics will also need to scale to provide enough optical resolution. In a perfect world, one could capture all the information one needs from just a few measurements.

With a discrete signal one needs at least as many measurements as there are signal elements to solve the inverse problem. If the number of measurements is fewer, then the inverse problem is underdetermined. Conventional signal processing dictates that accurate reconstruction of the signal-of-interest is highly improbable. Fortunately, a signal acquisition technique called *compressive sensing* allows one to design sensors that solve these types of highly underdetermined inverse problems.

As discussed earlier, much of the data being generated by sensors are redundant. Images, spectra, video, and audio data of real-world signals tend to exhibit patterns or redundancies that can be exploited. This allows a compression algorithm to

significantly reduce the amount of data needed to represent the signal.

There is a class of compression algorithms called lossy [27]. In lossy compression, not only are redundancies exploited but data that is deemed insignificant to the signal quality is discarded. Only the most important part of the signal is kept as part of the compressed representation of the original signal. When the signal is uncompressed, the amount of data is less than the original measured data. The difference in quality is often unnoticeable to a human observer. In both lossless and lossy compression, the goal is to obtain a *sparse* representation of the signal. A sparse representation means that the signal can be well approximated with only a few non-zero elements in a representation basis. A representation basis is a basis in which the signal-of-interest is sparse. For example, most natural images are sparse in the Fourier basis. The representation basis is typically not the native basis of the signal-of-interest, i.e. pixel number or spectral channel.

In 2006, David Donoho argued that traditional sensors tend to produce vast amounts of measurement data, but often the majority of data is redundant and discarded in the compression step. He proposed that sensors can be designed to directly measure the most relevant data in a signal, suggesting a measurement scheme that can measure a compressed form of the signal [28]. This is the idea behind compressive sensing sometimes known as *compressive sampling*. If the measurements are compressive then it should be possible to significantly reduce the number of measurements to accurately reconstruct the signal.

Note that there is a distinction between compressive sensing and the traditional approach of sensing and then compressing. In the traditional approach, compression algorithms operate as a post-processing step. Therefore, a traditional compression algorithm will have access to the entire signal to look for redundancies and convert it into a sparse representation. In compressive sensing, one attempts the compression directly and therefore do not have access to the entire uncompressed signal. The algorithms must assume that the signal has a sparse representation.

The question of how to actually measure or code the analog signal to directly obtain compressed data is also important. Fortunately, random coding tends to work well in many instances when the signal has a sparse representation. However in many cases, designed codes can significantly outperform random coding. I will discuss other types of coding schemes that can be used to outperform random codes.

The idea of compressive sensing seems to be similar to the concept of multiplex sensing. However, there is an important distinction to be made. In compressive

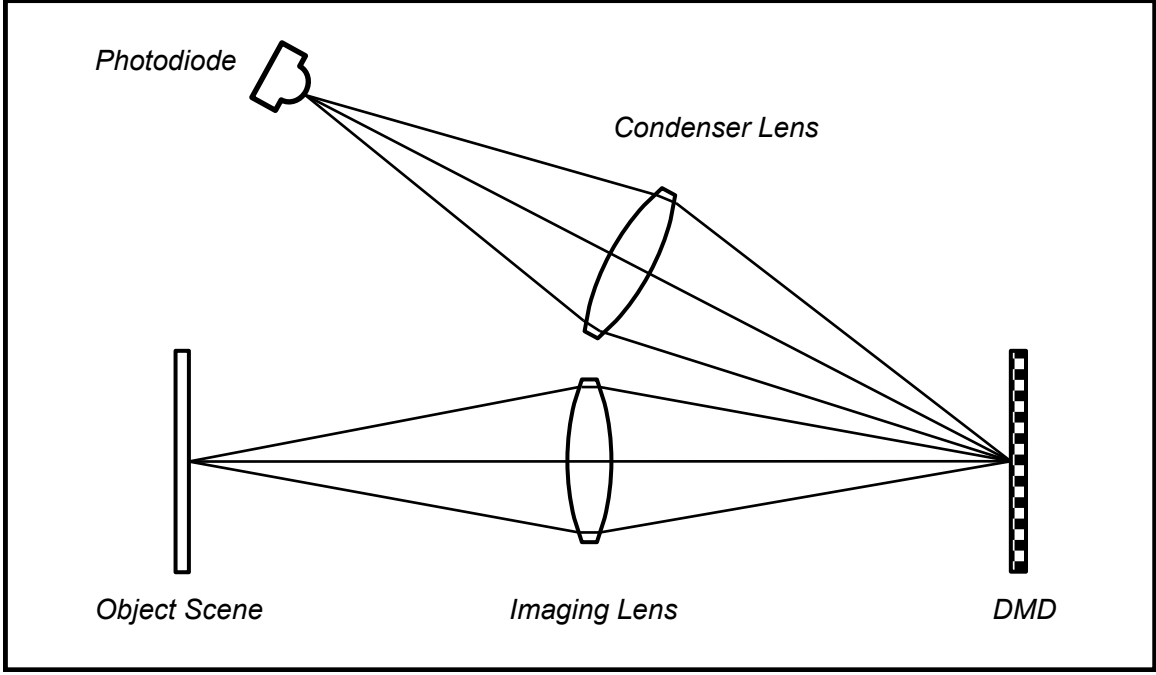


Figure 1.6: A single pixel camera architecture. The object scene (or intermediate image) is imaged onto a Digital Micro-Mirror Display (DMD). The each micro-mirror of DMD reflects light towards the photodiode or to another direction. This acts as a point-by-point multiplication of the discrete image with the DMD pattern. The condenser lens sums any light reflected by the DMD and focuses it onto the photodiode. This can be described mathematically as a vector multiplication of the image with the DMD pattern.

sensing, the aim is to obtain the relevant information in as few measurements as possible. In multiplexing, the goal is to overcome limitations mainly due to lack of SNR. Many compressive sensing schemes also employ multiplexing.

One useful example of compressive sensing versus traditional sensing is the single pixel camera [10]. The single pixel camera is a multiplexing camera architecture that uses time sequential random measurements and recovers the image in significantly fewer measurements (equal to number of exposures \times pixels) than the conventional camera, see Figure 1.6. Another example is the Coded Aperture Snapshot Spectral Imaging (CASSI) architecture [29], which can reconstruct a spectral data cube in significantly fewer FPA exposures than a traditional spectral imaging architecture.

Another important distinction is between reconstruction and task-specific sensing. Task-specific sensing tends to refer to measurement techniques that attempt to directly perform tasks such as detection, classification, and estimation without the intermediate step of reconstructing the high-dimensional signal. Compressive sensing is useful not just of overcoming resolution limitations in reconstruction but for task-specific sensing. For example, in facial recognition the goal is detection of an individual person. Reconstruction of the face image is simply an intermediate

step, therefore, one can develop a compressive sensing scheme that is optimal for direct facial detection, skipping the step of image reconstruction [30].

Computational sensors can overcome classic engineering trade-offs in sensor design. However, there is a unique set of challenges related to computational sensing.

1.7 Practical Considerations in Computational Sensing

So far I have discussed the development of computational sensing techniques and how they are used to ameliorate trade-offs in traditional sensor design. Computational sensing as a field is continuing to grow at a rapid pace. The number of journal publications related to computational sensing has steadily increased every year since 2008 [31]. There is now a major Optical Society of America (OSA) meeting dedicated to computational sensing [32] and textbooks dedicated to its study and development [1, 33]. While it is a powerful approach to radically new sensor architectures, textbooks and papers tend to focus on architecture concepts and positive results. Little attention is given towards the practical issues one faces when implementing computational sensors. For example, calibration is a major topic in this dissertation. In many talks and papers on computational sensing, calibration is barely mentioned or relegated to a minor paragraph.

Calibration is the process of quantifying the response of a sensor in order to produce an accurate forward model. While many traditional sensors also require calibration, computational sensors tend to be more sensitive to calibration error.

There are two main reasons for this. The first reason is simply due to the fact that non-isomorphic measurements require a computational step to solve an inverse problem. The algorithms rely on accurate knowledge of the forward model to separate measurement data due to the instrument and data due to the signal-of-interest.

The second reason is due to the lack of redundancy of measurement data in compressive sensing. The redundancies that are typically deemed wasteful in traditional sensing, can also be used by post-processing algorithms to solve the inverse problem in a robust fashion to correct for missing or corrupted data [34]. In compressive sensing, only a few numbers are used to represent many. If the few numbers are misinterpreted due to poor calibration, it can have a drastic impact on the performance of the estimation algorithm. I will illustrate in this dissertation the calibration challenges in several computational sensors.

Calibration has become a major drawback in compressive sensing. A consumer cannot be expected to spend time calibrating every time the instrument is physically bumped, the air temperature or pressure changes. In high dimensional compressive sensors like hyperspectral imagers, the calibration time can last hours.

Another issue facing sensors that rely on multiplexing is lack of dynamic range. In the context of optics, if more photons are being combined onto a single detector pixel, there is a higher chance of saturating the detector or operating in a non-linear regime. This is one of the potential issues faced by single-pixel compressive sensing architectures. The SCOUT architecture attempts to alleviate this by “spreading” the photons onto more pixels for a compressive measurement.

Another hurdle in the implementation of practical computational sensing is the need for prior knowledge. For example, in compressive imaging one must assume the signal is sparse in some basis. Fortunately most realistic objects can be treated as such with many commonly known bases, such as a wavelet basis. However, sometimes one needs to image something that is difficult to represent in any known basis. One needs to resort to generating training data. Training requires one to generate many different examples of the signal. This becomes time and computationally expensive. Another example for the need of prior knowledge is the AFSSI-C, a computational spectral classifier which requires knowledge of the standard deviation of the probability density function of the noise to perform spectral classification in the least number of measurements as possible.

Reducing the number of detector elements is often the goal in computational sensing. A notable example is the single pixel camera is an architecture. However, the single pixel camera requires several time sequential measurements. Each measurement displays a different DMD pattern to create a randomly encoded measurements [10]. The drawback to this architecture is that one must point the camera at the object scene until enough measurements have been collected for proper reconstruction. A complication arises when this architecture is used to image temporally varying object scenes. One must display the DMD patterns even faster and reduce the exposure time to keep up. As a result the SNR may begin to degrade.

A possible way to mitigate this issue is to do all the encoding in parallel. However, a completely parallel approach would require a lens, a DMD (or coded aperture), and a detector pixel for each measurement. Since each lens uses a different entrance pupil, this means that each detector pixel will have a different view of the object scene. This drastically scales the complexity of the architecture and algorithms.

In this dissertation, I will discuss a compromise to parallel coding, in two different computational sensors, by using a common entrance pupil and a CCD.

Much of the optimal measurement codes are simultaneously positive and negative measurement weights. In reality, with incoherent light one is unable to make negative measurements. One is often forced into situations where one must record two sets of measurements and subtract one set of measurements from the positive set of measurements. This means an additional noise term is added to each effective measurement.

Algorithms engineered to solve the inverse problems often do not account for the non-negativity of many physical situations. For example, in spectral unmixing where the problem is to solve for the concentration of each material given a mixed spectrum. A non-negative fractional abundance that sums to one is a physical requirement. However, there is a lack of sparsity promoting algorithms that are able to enforce both the non-negativity and the sum to one constraint.

A major issue in both the theoretical and experimental compressive sensing community is a lack of code design schemes. There are PCA, Random, and Hadamard based codes [1]. Relatively little theoretical results for other codes exist. For the most part random measurements techniques are dominant because they obey well known theoretical results which guarantees reconstruction with high probability [28, 35, 36]. Intuitively, designed codes which can take into account prior knowledge of the physical limitations of the sensing task and statistical assumptions of the signal-of-interest should be able to outperform random measurements. For example, I will show in the AFSSI-C, I will demonstrate that adaptively designed Principal Component Analysis (PCA) codes dramatically outperform random codes in low SNR environments.

These issues and others will be discussed in a case study manner throughout this dissertation. I will now discuss the organization of this dissertation and the three separate computational sensors I have built and how I have tried to mitigate and resolve some of the practical issues associated with computational sensing.

1.8 Dissertation Overview

I used this chapter to cover the major concepts of computational sensing. More importantly, I provided the major topic of discussion for this dissertation, the challenges that remain towards developing practical computational sensors. I will use

the next chapter as an opportunity to provide a more detailed and mathematically rigorous look at the various coding schemes that are popular in computational sensing, as well as the ones I have used. This includes a discussion of the mathematical methods that I and my collaborators developed and deployed in the AFSSI-C: Bayes rule, Log-Likelihood Ratios, and the Maximum a Posteriori decision technique. A more rigorous treatment of compressive sensing will also be provided which includes several important results from the compressive sensing community. I will also discuss several estimation and task-specific sensing algorithms that I used during the development of the three computational sensors in my work.

Chapter 3 will introduce a new system architecture for compressive target tracking—the Static Computational Optical Undersampled Tracker (SCOUT). This system is designed to overcome a variety of challenges that typically arise in traditional optical sensing approaches. It provides several advantages over the single-pixel camera architecture, notably it can capture many simultaneous encodings of the field-of-view with a single camera exposure. I will present experimental results that validate the performance of the proposed architecture.

Chapter 4 will discuss an important experimental prototype which demonstrates adaptive spectral image classification. The system is a major experimental advancement compared to current computational spectral imaging architectures which focus on reconstruction. Furthermore, the ability to perform adaptive measurements, where each code is designed based on the history of prior measurement data allows this instrument to outperform traditional spectral imaging architectures by a factor of 100 in low SNR scenarios.

Chapter 5 will discuss current efforts to experimentally demonstrate compressive spectral unmixing. Often in spectral imaging, the spectra present in the field-of-view of each pixel is actually a mixture of several spectra, a mixed spectrum. Spectral unmixing is the task of inferring the fraction of each constituent spectrum in the mixed spectrum. I employ a unique architecture for computational spectral sensing which uses no diffraction gratings or refracting prisms but relies on the wavelength dependent nature of the birefringence in an Liquid Crystal on Silicon (LCOS). I will demonstrate the effectiveness of various compressive sensing measurement schemes to outperform a traditional sensor in significantly fewer measurements.

Chapter 6 will summarize the dissertation and provide my perspective on how the field should approach the issues in practical computational optical sensing.

CHAPTER 2

Formalism

This chapter introduces the reader to the more rigorous concepts and mathematical background that will be required to fully understand the material presented in the later chapters of this dissertation.

A discussion of multiplexing and signal-to-noise ratio will be discussed, as well as various coding schemes used in various notable computational sensors as well as the ones in this dissertation.

The vast majority of modern optical sensing involves an analog-to-digital converter (ADC) step, which creates discrete digital values from a physical phenomena. Therefore, we will concentrate on continuous-to-discrete and discrete-to-discrete measurements. This not only makes the formalism we will discuss more relevant but in many cases it will simplify the mathematics.

As described earlier in Chapter 1, a measurement is a map from the physical signal-of-interest \mathbf{f} to the measurement data \mathbf{g} . The solutions to the electromagnetic wave equation are linear in free space so the propagation of electromagnetic radiation is linear. We can also approximately model the response of our sensors as linear. Thus we can write the measurement as an integral

$$g_m = \int \mathbf{f}(\mathbf{x}) h_m(\mathbf{x}) d\mathbf{x} \quad (2.1)$$

where $h_m(\mathbf{x})$ is the continuous-to-discrete measurement process from point \mathbf{x} to discrete measurement index m . We can write the continuous signal-of-interest as a superposition over a basis

$$f(\mathbf{x}) = \sum_n f_n \psi_n(\mathbf{x}) \quad (2.2)$$

This allows us to express the measurement of any optical phenomena as a matrix multiplication

$$\mathbf{g} = \mathbf{H}\mathbf{f} \quad (2.3)$$

where \mathbf{g} is now a measurement data vector and \mathbf{f} is the discrete representation of the object signal-of-interest and \mathbf{H} is the matrix which describes measurement process. For brevity we will refer to the \mathbf{f} as the object and \mathbf{H} as either the sensing matrix

or the measurement matrix. Equation (2.3) represents the forward model in a wide variety of computational sensors. The object \mathbf{f} is a vector in n dimensional vector space and the measurement \mathbf{g} is a vector in m dimensional vector space. In general $m \neq n$. Note that Equation (2.3) is an extremely useful way to represent the forward model in optics, since it allows us invoke a vast amount of computationally attractive numerical techniques which are dedicated to linear systems.

In the real-world noise degrades the measurements

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{e} \quad (2.4)$$

where \mathbf{e} is additive noise. Additive noise is noise which is independent of the signal. An example of additive noise includes the thermal noise generated by the random fluctuations of the charge carriers from CCD electronics. A different type of noise called multiplicative noise also exists, but will not be considered in this dissertation.

2.1 Isomorphic Sensing

In an isomorphic measurement, where the goal is a one-to-one mapping of object points to measurement points, the measurement matrix is often modeled with the identity matrix

$$\mathbf{H} = \mathbf{I} \quad (2.5)$$

We can get an idea of how much error exists in an isomorphic measurement through invoking the weighing example again. This example was originally discussed in book [37] but it is so useful in the context of this dissertation I will briefly summarize it here. In the weighing example, let's say we have 4 objects with true unknown weights f_1, f_2, \dots, f_4 . We record the measured weights g_1, g_2, \dots, g_4 with additive noise (random error) e_1, e_2, \dots, e_4 . The forward model with noise is

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad (2.6)$$

The estimated weights $\hat{\mathbf{f}}$ are the measurements \mathbf{g}

$$\hat{\mathbf{f}} = \mathbf{g} \quad (2.7)$$

Where the error between the estimated weights and the actual weights is the error $\mathbf{n} = \hat{\mathbf{f}} - \mathbf{f}$. For our purposes the assumption of a zero mean distributed noise is

a good assumption. This allows us to assume the the isomorphic measurements are unbiased. For an unbiased estimator, the mean squared error of the estimated weight is the variance

$$E[(\mathbf{e})^2] = E[(\hat{\mathbf{f}} - \mathbf{f})^2] = \sigma^2. \quad (2.8)$$

The smallest possible mean square error is limited to the variance of the noise. Now, I will show how multiplexing codes can be used to reduce the mean square error of the estimate.

2.2 Multiplexing

As we discussed in chapter 1, multiplexing is an extremely useful technique for overcoming trade-offs in traditional optical sensing. Now it is time to discuss and demonstrate the quantitative advantages provided by multiplexing.

In a multiplexed measurement, each element in the measurement vector \mathbf{g} is a weighted linear combination of the elements in the object vector. Therefore the measurement matrix \mathbf{H} is no longer an identity matrix.

2.2.1 Coding Schemes

We will now discuss formally the properties, advantages and disadvantages of several popular multiplex coding schemes used in computational sensing especially in dispersive spectroscopy such as the Hadamard, S-Matrices, and random coding.

A Hadamard matrix of order n is defined as a matrix \mathbf{H}_n as the $n \times n$ matrix whose elements consist of $+1$'s and -1 's and satisfies the following property:

$$\mathbf{H}_n^T \mathbf{H}_n = \mathbf{H}_n \mathbf{H}_n^T = n \mathbf{I}_n \quad (2.9)$$

where \mathbf{I}_n is an $n \times n$ identity matrix.

In multiplex spectroscopy and imaging, Hadamard codes are extremely popular for a variety of reasons. Hadamard codes are provably optimal in the case where we are allowed to take a full set of measurements, meaning that \mathbf{f} and \mathbf{g} from eq. (2.3) are both vectors in n dimensional space and when one uses both $+1$'s and -1 's in the code [37]. In this case, Hadamard codes achieves the minimal the mean square error defined as

$$\text{MSE} = \frac{1}{n}(e_1 + e_2 + \cdots e_n)^2 \quad (2.10)$$

In the weighing example, we can get negative measurements by using a balancing scale instead of a spring scale. An Hadamard multiplexed measurement would look

like

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad (2.11)$$

This means that in the first measurement all 4 items are placed in the same pan. In the second measurement items 1 and 3 are in the same pan while items 2 and 4 are in the opposite pan, etc [37]. We have four equations and four unknowns so we can solve for the estimates using algebra

$$\begin{aligned} \hat{f}_1 &= \frac{1}{4}(g_1 + g_2 + g_3 + g_4) \\ &= f_1 + \frac{1}{4}(e_1 + e_2 + e_3 + e_4) \\ &\vdots \\ \hat{f}_4 &= \frac{1}{4}(g_1 - g_2 - g_3 + g_4) \\ &= f_4 + \frac{1}{4}(e_1 - e_2 - e_3 + e_4) \end{aligned}$$

The mean square error of the m^{th} measurement

$$E[(\hat{f}_m - f_m)^2] = \frac{1}{4}\sigma^2. \quad (2.12)$$

which is 4 times lower than using an isomorphic measurement scheme. In general, the Mean Squared Error (MSE) of a Hadamard measurement is

$$\text{MSE} = \frac{\sigma^2}{N_\lambda} \quad (2.13)$$

where N_λ is the number of spectral channels, which is equal to the number of measurements N_m . Hotelling proved in 1944 that for a measurement matrix with elements $h_{mn} \in [-1, +1]$ the lowest possible MSE for the case of a full set of measurements is with a linear unbiased estimator is $\text{MSE} = \frac{\sigma^2}{N_m}$ [1]. So, one cannot possibly do better than Hadamard coding in this case. We will see later in other special cases that we can beat Hadamard codes.

In many practical cases in computational sensing, making a code with simultaneous positive and negative modulation of the signal is not possible. For incoherent light, the response is linear in intensity. In the case where one has the ability to make a full set of measurements but is limited to elements of $+1$'s and 0 's, an S-Matrix code minimizes the MSE [37].

In the weighing example, a spring balance rather than a two pan scale analogous to this situations.

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & +1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ +1 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad (2.14)$$

So items 2, 3, and 4 are weighed together, then 1 and 2, and so on. Solving the system of equations in a similar fashion as before in the Hadamard weighing example we find that the mean square error for the m^{th} measurement when weighing 4 items is

$$\text{MSE} = E[(\hat{f}_m - f_m)^2] = \frac{7}{9}\sigma^2. \quad (2.15)$$

Which reduced the MSE compared to the isomorphic measurement scheme but it is less of a reduction compared to the Hadamard measurement scheme. The MSE of the S-Matrix is approximately a factor of 4 increase compared to the Hadamard matrix coding scheme.

In the case when the full set of measurements are available, random coding schemes are provably sub-optimal compared to Hadamard and S-Matrix codes. However, they should not be ignored because in compressive sensing, certain theoretical guarantees exist for random coding that do not exist for Hadamard and S-Matrix codes. Sometimes, the physics of the situation forces a random coding scheme. There is little literature on the performance of random coding schemes. At the time of this writing (2016), I am unaware of an optimality proofs for random codes when one has a full set of measurements available.

2.2.2 The Fellgett Advantage

The *Fellgett advantage* is the improvement in SNR that occurs when an instrument takes multiplexed measurements compared to isomorphic measurements [38, 39]. Physically the Fellgett advantage occurs because a single detector element produces a noise contribution whether it's sampling a single part of the object or multiple parts of the object. Maximizing the signal-to-noise ratio (SNR) of the estimated object signal-of-interest for a given system throughput and detector noise is major design consideration in computational optical sensing particularly in the area of spectroscopy. In spectroscopy, there are two well known multiplex schemes, the Hadamard multiplexing in dispersive spectrometers and the interferometric spectrometer .

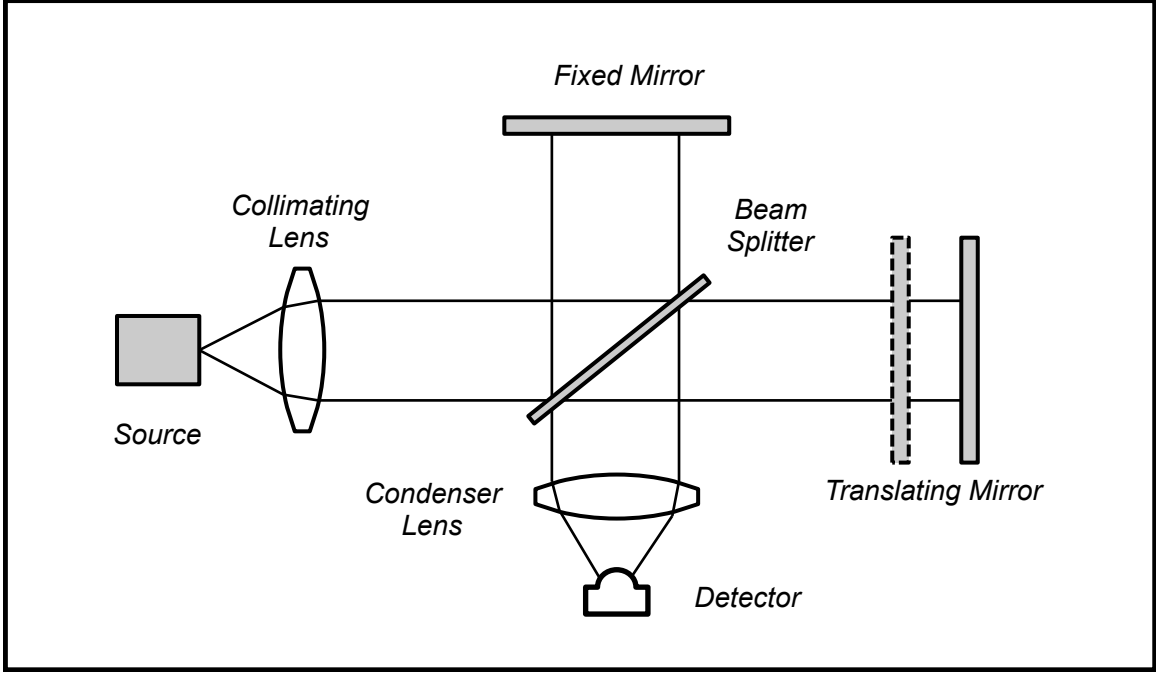


Figure 2.1: The architecture of the Fourier Transform Spectrometer resembles the Michelson interferometer. One of the mirrors is translated back and forth. The interferogram is the detected intensity versus mirror delay which is related to the autocorrelation of signal. The Fourier transform of the autocorrelation provides the spectrum.

The Fourier Transform Spectrometer (FTS) architecture is similar to the Michelson interferometer, see Figure 2.1. The FTS operates by taking the autocorrelation of the complex electric field as a function of time delay by moving one of the mirrors in the interferometer [39]. The Wiener-Khinchin theorem says that for a wide-sense stationary random process, the Fourier transform of the autocorrelation is the power spectral density. Thus a computational post-processing step is required to reconstruct the spectrum from the measurement data. Since the FTS measures combinations of multiple wavelengths at each detector readout it also exhibits the Fellgett advantage.

The advantage that multiplexing provides over an isomorphic measurement depends on the coding scheme. We've just seen that for a Hadamard coding scheme the MSE is given by Equation (2.13). It turns out that for a FTS the MSE obtained is a factor of two greater than the Hadamard multiplexing scheme [40].

$$\text{MSE} = 2 \frac{\sigma^2}{N_\lambda} \quad (2.16)$$

2.3 Principal Component Analysis

Principal component analysis (PCA) is a technique that attempts to recast a dataset in manner which allows us to maximally discriminate the data with just a few vectors. The first vector points in the direction of maximum variance. The second vector points in a direction that also maximizes variance but is orthogonal to the first vector and so on. These new vectors are called the *principal components*. We will say the *rows* of matrix \mathbf{P} are the principal components.

Let's say we have a dataset \mathbf{S} which consists of N spectra with N_λ spectral channels. Instead of looking at the data as just intensity versus spectral channel, PCA attempts to construct a set of new vectors (also called features) that show as much variation in the spectra as possible. In other words, first direction (principal component) is used to recast the data to look as different (uncorrelated) as possible. This allows us to discriminate the data, as best we can with just one direction. The second principal component is the direction that provides the second most ability to discriminate the data, and so on.

The covariance matrix is defined as

$$\mathbf{C}_\mathbf{X} = \frac{1}{N} \mathbf{S} \mathbf{S}^T. \quad (2.17)$$

Each element in the covariance matrix C_{Xmn} is the covariance of the m^{th} spectrum \mathbf{s}_m and the n^{th} spectrum \mathbf{s}_n .

$$C_{Xmn} = \frac{1}{N} \mathbf{s}_m \mathbf{s}_n^T. \quad (2.18)$$

Note that large covariance means they look quite alike and therefore are difficult to disambiguate.

If the entire collection of spectra \mathbf{S} were mutually orthogonal, being able to tell one spectrum apart from another would be easy. You would just have a collection of spikes at different spectral channels. The covariance matrix in this case would be a diagonal matrix.

Since there is typically some redundancy between spectra, the off-diagonal elements of the covariance matrix will be non-zero. The principal components allow us to recast the data to make it as uncorrelated as possible in a new basis.

$$\mathbf{Y} = \mathbf{P} \mathbf{X} \quad (2.19)$$

where \mathbf{Y} is the data projected onto the principal component basis \mathbf{P} . The covariance of the projected data

$$\mathbf{C}_\mathbf{Y} = \frac{1}{N} \mathbf{Y} \mathbf{Y}^T \quad (2.20)$$

is a now diagonal matrix. Indeed, the principal components are the optimal way to discriminate the spectra in the dataset \mathbb{J} .

It turns out there is a way to calculate the principal components in closed form. The *eigenvectors of the covariance matrix are the principal components* [41].

Since the full set of principal components forms a basis, each spectra \mathbf{s} in \mathbf{S} can be written as a superposition of principal components without any error

$$\mathbf{s} = \sum_{\lambda=1}^{N_\lambda} y_\lambda \mathbf{v}_\lambda \quad (2.21)$$

In many cases, only a few of the first principal components are needed in the summation to approximate the original data well.

$$\mathbf{s} \approx \sum_{\lambda=1}^M y_\lambda \mathbf{v}_\lambda \quad (2.22)$$

Where $M \ll N_\lambda$. Note that each eigenvector has an associated eigenvalue. The eigenvalues are also informative because they can tell us how many principal components are really needed to discriminate all of the spectra. The magnitude of the eigenvalue tells us how well it's associated eigenvector is at discriminating the data.

This is another reason why PCA is so useful. It can be used as a type of lossy compression code and as a measurement matrix for compressive sensing. Simply project the data onto the first several principal components associated with the largest M eigenvalues.

The AFSSI-C relies on a variation of Principal Component Analysis (PCA) for discriminating between spectra. In addition to PCA the AFSSI-C uses a Bayesian probability to create adaptive codes. We will now discuss some of fundamentals of Bayesian probability and the Log-likelihood ratios.

2.4 Bayesian statistics and Maximum a Posteriori

In the real world, measurements are corrupted by noise. The random nature of noise corrupted measurements lends itself to a stochastic perspective of signals. A hypothesis can be associated with various parameters or aspects of a signal. A hypothesis is nothing more than a claim or premise that one is interested in verifying. In imaging and spectroscopy, one example is that at a certain location in the field of view, the hypothesis is that a spectrum is present or not present. Another hypothesis is that the mean value of the signal is some value. Instead of attempting to determine

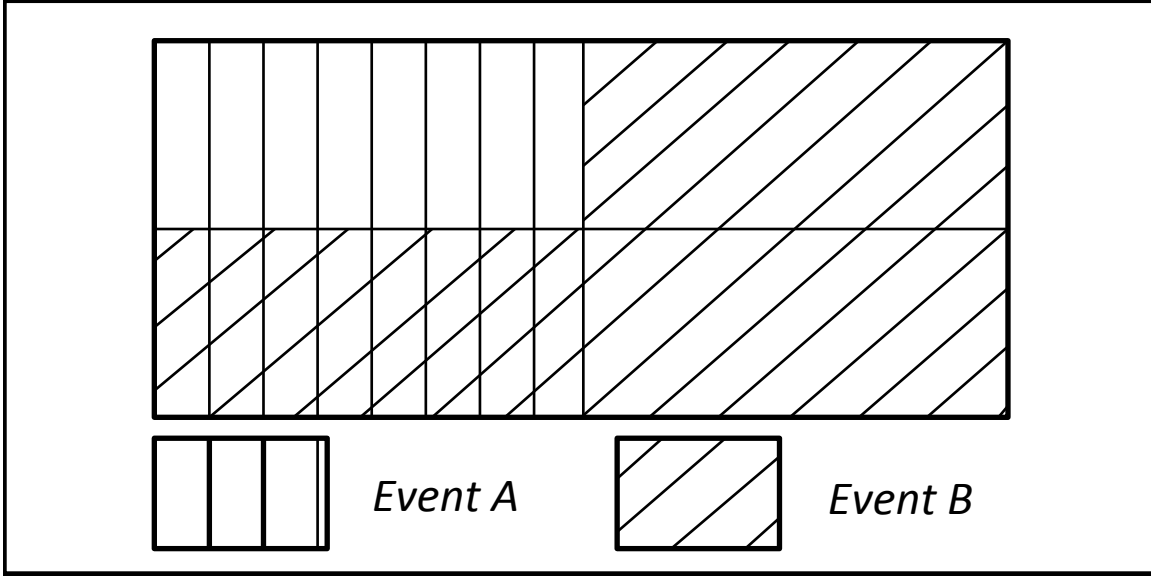


Figure 2.2: Graphical demonstration of joint probability. The probability of event A is $P(A) = 3/4$. The probability of event B is $P(B) = 3/4$. The joint probability of events A and B is $P(A \cap B) = 1/4$. The probability of event A occurring given that event B has occurred is $P(A|B) = 1/3$. This is consistent with the equation $P(A \cap B) = P(A|B)P(B)$

whether a hypothesis is true, often times we are interested in estimating parameters of stochastic processes, which we denote θ .

Bayesian statistics allows one to treat the hypothesis or parameters as random variables rather than deterministic constants. A wide variety of Bayesian approaches exist and each require a heavy reliance on Bayes's theorem. Bayes' theorem is a way of computing probabilities of a hypothesis given some evidence which are related to the hypothesis. For example, Bayes' theorem can be used to decide which of two bags of candy has been opened or if a spectrum is present. The idea is that we can make a more informed calculation of probability if we are able to account or update the probability given some new piece of evidence that we may have not had at the beginning.

The derivation of Bayes' theorem follows from the definition of conditional probability, the conditional probability of event A occurring given that B occurred is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (2.23)$$

this can be seen graphically in Figure

solving for the joint probability $P(A \cap B)$ gives

$$P(A \cap B) = P(A|B)P(B) \quad (2.24)$$

since the joint probability commutes $P(A \cap B) = P(B \cap A)$, we can also write

$$P(A \cap B) = P(B | A) P(A) \quad (2.25)$$

equating the right hand side of Equation 2.24 and Equation 2.25 gives us Bayes' theorem (also called Bayes' rule)

$$P(A | B) = \frac{P(A) P(B | A)}{P(B)} \quad (2.26)$$

One interpretation of Bayes' theorem is called the Diachronic interpretation, which says that conditional probability of the hypothesis or parameter θ given knowledge of some evidence or measurement data g is given by

$$P(\theta | g) = \frac{P(\theta) P(g | \theta)}{P(g)} \quad (2.27)$$

The term $P(\theta | g)$ is called the posterior. It represents our belief in the hypothesis given the data. The term $P(\theta)$ is called the prior. $P(g | \theta)$ is called the Likelihood. $P(\theta)$ is called the normalizing constant. In general the normalizing constant can be written as

$$P(\theta) = \sum_i P(\theta_i) P(g_i | \theta_i) \quad (2.28)$$

One can repeatedly apply Bayes' theorem given new measurement data. We will use a simple example to illustrate how to update the belief. Imagine we have two bags of candy. Bag 1, which we denote B_1 , has 10 pieces of cherry flavored candy, denoted as C , and has 30 pieces of strawberry flavored candy, denoted as S . Bag 2, B_2 , has 20 pieces of cherry candy and 20 pieces of strawberry candy. At the beginning, the prior probability of selecting bag 1 or bag 2 is both equal

$$P(B_1) = P(B_2) = \frac{1}{2} \quad (2.29)$$

Someone then picks a bag at random and takes out a piece of candy that turns out to be strawberry flavor. What is the probability that bag 1 was selected? We can use Bayes' theorem to compute this

$$P(B_1 | S) = \frac{P(B_1) P(S | B_1)}{P(S)} \quad (2.30)$$

Where $P(S | B_1)$ means the probability of selecting a strawberry candy assuming we have selected bag 1, which is $3/4$. The $P(S)$ is the probability of selecting a strawberry candy from either bag 1 or bag 2, which $5/8$. Thus

$$P(B_1 | S) = \frac{\left(\frac{1}{2}\right) \left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)} = \frac{3}{5} \quad (2.31)$$

Similarly, the probability that the person chosen bag 2 is $P(B_2 | S) = 2/5$. Qualitatively, this makes sense, since bag 1 contained more strawberry flavored candy, the belief that bag 1 was chosen should increase and the belief that bag 2 was not chosen should decrease.

Now let's continue the example. Say we put the first piece of candy back in the bag and draw another piece of candy, which turns out to be cherry flavor. Now we must update the beliefs with this new piece of information. We can keep using Baye's theorem, but the trick is that the posterior from the last draw is now used as the prior for the current update.

$$P(B_1 | C) = \frac{P(B_1 | S) P(C | B_1)}{P(C)} \quad (2.32)$$

$$P(B_2 | C) = \frac{P(B_2 | S) P(C | B_2)}{P(C)} \quad (2.33)$$

The probability of drawing a cherry flavored candy assuming bag 1 was chosen is $1/4$ and the probability of drawing a cherry flavored candy assuming bag 2 was chosen is $1/2$. We now must use Equation 2.28 to compute the normalizing constant, otherwise the posterior probabilities will not sum to 1. In this case $P(C) = 7/20$. Plugging these numbers into Equations 2.32 and 2.33 gives the updated posterior probabilities

$$P(B_1 | C) = \frac{3}{7} \approx 0.43 \quad (2.34)$$

$$P(B_2 | C) = \frac{4}{7} \approx 0.57 \quad (2.35)$$

Intuitively, drawing a cherry flavored candy has reduced our belief that bag 1 was chosen since bag 1 consist of only $1/4$ cherry flavor candies while bag 2 consisted of $1/2$ cherry candy.

Sometimes one is given a set of possible parameters that we are interested in estimating, θ , given some measurement data g . The method of Maximum a posteriori (MAP) says the parameters θ which maximize the posterior probability are the most likely parameters.

$$\theta_{map} = \arg \min_{\theta} p(\theta | g) \quad (2.36)$$

Using Bayes' theorem we can rewrite Equation 2.36 as

$$\theta_{map} = \arg \min_{\theta} \frac{p(g | \theta) p(\theta)}{p(g)} \quad (2.37)$$

Maximizing the posterior is now equal to maximizing the likelihood and prior. In certain cases, one needs to decide between two sets of parameters or hypotheses. One can do an analogous technique of comparing the posteriors by using a ratio

$$\frac{p(\theta_i | g)}{p(\theta_j | g)} = \frac{p(g | \theta_i) p(\theta_i)}{p(g | \theta_j) p(\theta_j)} \quad (2.38)$$

If the ratio is larger than some threshold value then one chooses parameter θ_i and if the ratio is less than the threshold then one chooses θ_j . Similar to the earlier example of updating the posterior based on new data, one can update the Maximum a posteriori (MAP) decision based on new data. Define the likelihood ratio of the m^{th} measurement as

$$\Lambda_m = \frac{p(g_m | \theta_i)}{p(g_m | \theta_j)} \quad (2.39)$$

After each new set of measurement data g_m is collected, one can update the posterior ratios by multiplying the likelihood ratio from the old set of data with the likelihood ratio of the new set of data. The ratio which includes all the updates from measurement $m = 1$ to measurement $m = N_m$ is written as

$$\frac{p(\theta_i | \{g\}_{N_m})}{p(\theta_j | \{g\}_{N_m})} = \prod_{m=1}^{N_m} \Lambda_m \frac{p(\theta_i)}{p(\theta_j)} \quad (2.40)$$

where the notation $\{g\}_{N_m}$ represents the set of all data from measurement m to N_m .

Bayesian statistics is a useful way to update one's belief in a hypothesis or estimate a set of parameters. The Bayesian perspective is different from the classical perspective. A classical approach to statistical estimation views parameters of interest as deterministic but unknown constants. The Bayesian approach assumes that θ is a random variable whose particular realization we must estimate.

As we've just seen through the Bayesian approach, the ability to take advantage of new measurement data and prior knowledge is a powerful concept. We will continue on this theme as we discuss compressive sensing which relies heavily on the prior assumption of *sparsity*.

2.5 Compressive Sensing

The fundamental approach of compressive sensing is that rather than sampling at a high rate and then compressing the sampled data, one can dramatically reduce the number of samples by sampling the signal in a compressed form. I will now discuss some of the more formal concepts of compressive sensing and some techniques to compressively sampling signals. This allows one to build some intuition about compressive sensing. In order to understand why compressive sensing is so powerful we should first discuss the conventional sampling strategy.

2.5.1 The Nyquist-Shannon Sampling Theorem

One of the most important results concerning the *sampling* of continuous signals is the Nyquist-Shannon sampling theorem (often referred to as the *sampling theorem* for short). The sampling theorem says that exact reconstruction of a continuous *bandlimited signal* $f(x)$ is possible if the sampling frequency f_s is atleast twice the maximum frequency B of the signal [42]. Assuming that a bandlimited signal $f(x)$ has been sampled according to the sampling theorem, then exact recovery from the discrete samples f_n is guaranteed.

However, if the sampling frequency is less than twice the maximum bandwidth $f_s < 2B$, then aliasing may occur in the reconstruction. Aliasing is the effect that high frequencies in the original signal will be represented as lower frequencies after reconstruction and information contained in the high frequencies will be potentially lost [43].

Before we continue, it's important to clarify a small but important distinction between the meaning of sampling and the meaning of a measurement. A measurement is any process that maps physical phenomena which contains a signal-of-interest into measurement data. The measurement data may or may not be discrete. Sampling has a more precise mathematical definition. It is the process of mapping a continuous signal into a sequence of discrete numbers which are called the samples.

2.5.2 Sparsity, Incoherence, and the Restricted Isometry Property

At first glance, compressive sensing seems to go against the Nyquist-Shannon sampling theorem, however the sampling theorem's guarantee of exact reconstruction of a continuous signal relies on the assumption of a bandlimited signal and uniform periodic sampling, which leads to isomorphic sampling. Compressive sensing makes

different assumptions: sparsity and *incoherence*. Combined with non-linear optimization techniques, compressive sensing allows one to take far less samples without significant loss of information. Before we discuss how to actually do compressive sensing and some of its drawbacks, it is important to have a formal discussion of these concepts.

Most continuous signals $f(x)$ can be written as a discrete summation of orthonormal basis functions

$$f(x) = \sum_{n=1}^N x_n \Psi_n(x), \quad (2.41)$$

where x_n is the coefficients. I will call the vector $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_N]^T$ the representation vector of the signal and Ψ the representation basis. This allows us to rewrite the signal as a matrix multiplication

$$\mathbf{f} = \Psi \mathbf{x}. \quad (2.42)$$

As we discussed in Chapter 1, any vector \mathbf{x} is sparse when all but a few of its entries are zero. A vector is called K -sparse when it has at most K non-zero entries. In real situations, signals with strictly sparse representation vectors are unlikely. Fortunately, it is possible to have approximately sparse representation vectors, which are called *compressible*. In other words, the sorted magnitudes of the coefficients $|x_n|$ quickly decay. When a signal has an expansion in terms of a compressible representation vector, we can intuitively understand Equation (2.41). Discarding smaller coefficients will not significantly degrade the information in the signal [35].¹

The concept of sparsity is important in compressive sensing. Sparsity determines how efficiently one can acquire the signals. All things being equal, if K decreases, then the reconstruction error tends to decrease. Sparse representations of the signal are not the only important prerequisite for high probability reconstruction.

The random sensing basis is useful because of its mathematical convenience however, we shall show heuristically that incorporating additional prior knowledge of the statistics of the signal can reduce reconstruction error in reconstruction but other computational sensing tasks such as spectral classification. However, the options for compressive sensing bases are still quite limited.

In practice, the signal is sampled in a different basis than the representation basis. For example, while many natural signals have a sparse or compressible representation

¹From now on, I will use the words sparse and compressible interchangeably.

in various wavelet bases or Fourier bases, sampling using the representation basis is not practical or desirable in many cases. Often a sensing basis \mathbf{H} is used to perform the sampling. In traditional Nyquist-Shannon sampling, the sensing basis is a collection of delta functions. In compressive sensing for example, the sensing basis is the binary random coded aperture mask in a compressive imaging system [10]. In an LCOS based compressive sensing spectrometer, the sensing basis is a finite set of spectral filters [44, 45]. In short, the representation basis allows a signal to be represented as a sparse vector, while the sensing basis is composed of the functionals that one samples with.

The equation which combines these concepts to model a compressive measurement is

$$\mathbf{g} = \mathbf{H}\Psi\mathbf{x} = \mathbf{H}\mathbf{f} \quad (2.43)$$

where \mathbf{g} is a $N_m \times 1$ measurement vector, \mathbf{H} is a $N_m \times N$ measurement vector, and \mathbf{x} is an $N \times 1$ vector. A forward model in compressive sensing obeys Equation (2.43) when $N_m \ll N$.

One important concept in compressive sensing is coherence. The coherence between the sensing basis \mathbf{H} and the representation basis Ψ is

$$\mu(\mathbf{H}, \Psi) = \sqrt{n} \max_{1 \leq k, 1 \leq n} |\langle h_k, \psi_j \rangle|, \quad (2.44)$$

which defines coherence as a measure of the largest correlation between any two elements of \mathbf{H} and Ψ . There is an alternate definition of coherence which is concerned with only the sensing matrix \mathbf{H} (the sensing matrix is also called the measurement matrix or the system matrix).

$$\mu(\mathbf{H}) = \max_{k \neq j} \frac{|\langle h_k, h_j \rangle|}{\|h_k\|_2 \|h_j\|_2}. \quad (2.45)$$

In this definition, the coherence is the maximum correlation between columns of the system matrix.

Ideally, one would like to use as less measurements N_m as possible without degrading the reconstructed signal. In compressive sensing, the amount of measurements needed is a function of the sparsity and the coherence. Given a coefficient sequence \mathbf{a} that is K -sparse then one needs

$$N_m \geq C \cdot \mu^2(\mathbf{H}, \Psi) \cdot K \cdot \log n \quad (2.46)$$

for a high probability of reconstruction, where C is just a constant and n is the dimensionality of the original signal.

Equation (2.46) shows the importance of sparsity and coherence in compressive sensing. Lower coherence and sparsity allows one to use less measurements [10]. The more incoherent—lower correlation—the two bases are, the higher the probability of successful reconstruction of compressive measurements. Random matrices have a high probability of being incoherent with any basis [35]. Using the alternate definition of coherence from Equation (2.45) produces similar qualitative results [36].

The isometry constant δ_K of a matrix $\mathbf{A} = \mathbf{H}\Psi$ is the smallest number such that

$$(1 - \delta_K) \|\mathbf{x}\|_{\ell_2}^2 \leq \|\mathbf{A}\mathbf{x}\|_{\ell_2}^2 \leq (1 + \delta_K) \|\mathbf{x}\|_{\ell_2}^2 \quad (2.47)$$

for all K -sparse vectors \mathbf{x} . The matrix \mathbf{A} obeys the *restricted isometry property* (RIP) of order K if δ_K is not too close to one [35]. If RIP is satisfied, the sensing matrix \mathbf{A} approximately preserves the Euclidean length of signals. The Restricted isometry property (RIP) is an important theoretical result which allows robust compressive sensing where signals are corrupted by noise. Sensing matrices which have random entries obey the RIP with high probability [35, 10, 33].

All of the concepts I have just discussed can be understood at an intuitive level. Sparsity is the idea that the information content of a signal is relatively small compared to the amount of data which originally described the signal. Coherence extends that concept of how invertible a matrix is, the more linearly independent the system of equations, the easier it is to invert the matrix. The restricted isometry property is basically saying that any matrix with a small isometry constant will keep the distance between signal vectors the same. Why is this important? Think of a geometric picture, imagine noise as a sphere of uncertainty around the signal vectors. When the noise is small, two signal vectors can be mapped to a small part of the measurement space and still fit in that space. When one extracts the signal from the measurements, one can tell them apart. As the noise increases one wants the distance between measured signal vectors to at least stay the same and certainly not dramatic decrease, otherwise it will be difficult to tell the signals apart. The RIP is basically a way of seeing if a measurement matrix will pack the signal vectors with the space distance between them. A small amount of noise will lead to a small amount of reconstruction error instead of a small amount of noise leading to a large

reconstruction error.

2.5.3 Solving Inverse Problems For Compressive Sensing

Now that I have discussed what compressive sensing is mathematically, it is time to address how to actually solve the problem of extracting \mathbf{x} from

$$\mathbf{g} = \mathbf{A}\mathbf{x} + \mathbf{e} \quad (2.48)$$

where \mathbf{g} is the measurement vector, \mathbf{x} is the sparse representation of the signal, \mathbf{A} is the linear mapping from \mathbf{x} to \mathbf{g} , \mathbf{e} is the noise vector, and the number of measurements N_m is much less than the dimensionality of the sparse representation vector N .

The least squares estimator (LS) estimator attempts to solve this inverse problem by minimizing the objective function

$$\sum_{n=1}^N (\mathbf{g} - \mathbf{A}\mathbf{x})^2 = \|\mathbf{g} - \mathbf{A}\mathbf{x}\|_2^2. \quad (2.49)$$

which is the ℓ_2 norm between the given data \mathbf{g} and the forward model for the data $\mathbf{A}\mathbf{x}$. Notice, there are no probabilistic assumptions about the measurement data. It turns out that a closed form version of the LS estimator exists as

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{g}. \quad (2.50)$$

where $\hat{\mathbf{x}}$ is the estimated value of \mathbf{x} .

The derivation of the closed form of the least squares estimator is given in Appendix A. Alternatively, if the vector \mathbf{x} is very large, solving the inverse problem in closed form maybe too computationally intensive, one may use the gradient descent algorithm or other types of iterative optimization algorithms to solve the least squares problem. While the Least squares (LS) estimator may provide a solution when $N_m \ll N$, these solutions tends to overfit the data in these situations and does not take advantage of the prior knowledge of sparsity in reduce the solution space. Therefore, the unconstrained LS approach tends to be sub-optimal for the compressive sensing problem.

There are a vast number of algorithms that claim to be designed for compressive sensing. A full discussion of each algorithm, estimator, and optimization technique is simply not feasible. The ones used by the author and notable ones in the community will be briefly discussed now and in the relavent chapters. Since this dissertation is

geared towards those who wish to construct experimental computational sensors, full rigour is omitted and only intuitive justification is given for each type of algorithm.

One can categorize compressive sensing algorithms in three groups: optimization methods, greedy methods, and thresholding-based methods.

Many of the algorithm for compressive sensing are optimization algorithms, specifically called *basis pursuit* algorithms [33]. Optimization algorithms are problems which serve to minimize (or maximize) an objective function F_0 .

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{R}^N} F_0(\mathbf{x}) \quad (2.51)$$

which is called an unconstrained optimization problem. Similarly, solving the problem

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{R}^N} F_0(\mathbf{x}) \quad \text{subject to} \quad F_n(\mathbf{x}) \leq b_n \quad \forall n \quad (2.52)$$

is a constrained optimization problem. If $F_n \forall n$ are all convex then the problem is a convex optimization problem. If $F_n \forall n$ are all linear functions, then it is called a linear program (program is synonymous with optimization) [33].

Given measurements \mathbf{g} and knowledge that \mathbf{x} is sparse or compressible, solving an optimization problem of the form

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{g} \quad (2.53)$$

where $\|x\|_0$ is the ℓ_0 norm, which is equal to the number of nonzero entries in vector \mathbf{x} .² Equation (2.53) return $\hat{\mathbf{x}}$ that resembles \mathbf{x} as long as the measurement noise is small. This is referred to as ℓ_0 -minimization. The constraint ensures the solution agrees with the observed measurement data. In other words, one wants the sparsest solution possible that agrees with the measurement data.

The ℓ_1 minimization or *basis pursuit* algorithm is

Given that one has prior knowledge of the existence of a sparse solution, a more intuitive approach is to rewrite the inverse problem to ensure that sparsity is part of the solution.

Unfortunately, ℓ_0 -minimization is nonconvex which means that iterative methods may not converge to a solution. This is an important feature for inverse problems. A convex problem has the property that any local minimum is also a global minimum. In fact, it has been shown that for a general matrix \mathbf{A} , ℓ_0 -minimization is intractable [46].

²The ℓ_0 norm is not strictly a norm and is actually a quasi-norm.

Fortunately, one can minimize the ℓ_1 norm

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{g} \quad (2.54)$$

where $\|\mathbf{x}\|_1 = \sum_n^N |x_n|$, and get excellent reconstruction in the case where Equation (2.46) and the Equation (2.47) (the RIP) are satisfied. Additionally, ℓ_1 -minimization is a convex problem which allows for fast and accurate reconstruction [28, 47, 33]. A more practical version of Equation (2.54) is written as

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{Tx} - \mathbf{g}\|_2^2 + \tau \|\mathbf{x}\|_1 \quad (2.55)$$

where τ is a non-negative number. τ is called the regularization parameter and serves to change the sparsity level of the solution and can be used to account for noise in the vector which increases the robustness of the optimization. In this form, the problem is called ℓ_1 regularized least squares (LS). ℓ_1 regularization also appears in the context of basis pursuit denoising [48]. In statistics, ℓ_1 regularization is used in the LASSO algorithm for feature selection, therefore one often refers to Equation (2.55) as the LASSO problem [49].

CHAPTER 3

Static Computational Optical Undersampled Tracker

This chapter introduces the reader to the Static Computational Optical Undersampled Tracker.

CHAPTER 4

Adaptive Feature Specific Spectral Imaging-Classifier

This chapter introduces the reader the AFSSI-C.

CHAPTER 5

Computational Spectral Unmixing

This chapter introduces the reader the computational spectral unmixer.

CHAPTER 6

Conclusion

This chapter concludes the dissertation.

APPENDIX A

Derivation of the Least Squares Estimator

Suppose

$$\mathbf{g} = \mathbf{A}\mathbf{x} \quad (\text{A.1})$$

Given \mathbf{g} and \mathbf{A} we want to solve for \mathbf{x} . If the matrix is full rank then we can simply multiply both sides of equation A.1 by \mathbf{A}^{-1}

$$\mathbf{A}^{-1}\mathbf{g} = \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x} \quad (\text{A.2})$$

where \mathbf{I} is the identity matrix.

If \mathbf{A} is not full rank then its inverse does not exist. However we can try to find a solution $\hat{\mathbf{x}}$ that minimizes the squared error. This is called the *Least Squares Solution* also known as the *Least Squares Estimator*, *Ordinary Least Squares* and by many other names. We define the squared error as

$$\|\epsilon\|^2 = \|\mathbf{A}\mathbf{x} - \mathbf{g}\|^2 \quad (\text{A.3})$$

To minimize the error, we take the derivative of equation A.3 with respect to \mathbf{x} and set it equal to zero and solve for \mathbf{x} . Note that the equation A.3 can be expanded in terms of an inner product

$$\|\epsilon\|^2 = \|\mathbf{A}\mathbf{x} - \mathbf{g}\|^2 = \sum_{i=1}^N \epsilon_i^2 = \epsilon^T \epsilon = (\mathbf{A}\mathbf{x} - \mathbf{g})^T (\mathbf{A}\mathbf{x} - \mathbf{g}) \quad (\text{A.4})$$

The transpose is distributive

$$(\mathbf{A}\mathbf{x} - \mathbf{g})^T = (\mathbf{A}\mathbf{x})^T - \mathbf{g}^T \quad (\text{A.5})$$

The transpose of a product of matrices equals the product of their transposes in reverse order

$$(\mathbf{A}\mathbf{x})^T = \mathbf{x}^T \mathbf{A}^T \quad (\text{A.6})$$

So equation A.4 becomes

$$\begin{aligned}\|\epsilon\|^2 &= (\mathbf{x}^T \mathbf{A}^T - \mathbf{g}^T)(\mathbf{A}\mathbf{x} - \mathbf{g}) \\ &= \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x} - \mathbf{x}^T \mathbf{A}^T \mathbf{g} - \mathbf{g}^T \mathbf{A}\mathbf{x} + \mathbf{g}^T \mathbf{g}\end{aligned}\tag{A.7}$$

We can see that the two middle terms $\mathbf{x}^T \mathbf{A}^T \mathbf{g} = \mathbf{g}^T \mathbf{A}\mathbf{x}$ because they are just scalars.

$$\|\epsilon\|^2 = \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x} - 2\mathbf{g}^T \mathbf{A}\mathbf{x} + \mathbf{g}^T \mathbf{g}\tag{A.8}$$

To find the least squares solution, take the gradient with respect to \mathbf{x} and set it equal to zero.

It should be noted that there are two different notations for writing the derivative of a vector with respect to a vector $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$. If the numerator \mathbf{y} is of size m and the denominator \mathbf{x} of size n , then the result can be laid out as either an $m \times n$ matrix or $n \times m$ matrix, i.e. the elements of \mathbf{y} laid out in columns and the elements of \mathbf{x} laid out in rows, or vice versa. They are both correct and equal, which leads to confusion when switching back in forth. I will write both to reduce confusion.

Clearly the gradient of the third term in equation A.8 w.r.t \mathbf{x} is 0, so it goes away. We first tackle the first term on the right hand side in equation A.8

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}\tag{A.9}$$

Let $\mathbf{K} = \mathbf{A}^T \mathbf{A}$. Since \mathbf{K} is symmetric, we can use the identity

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{K}\mathbf{x} = 2\mathbf{x}^T \mathbf{K} = 2\mathbf{K}^T \mathbf{x}\tag{A.10}$$

since $\mathbf{K} = \mathbf{K}^T$ then

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x} = 2\mathbf{x}^T \mathbf{A}^T \mathbf{A} = 2\mathbf{A}^T \mathbf{A}\mathbf{x}\tag{A.11}$$

and the gradient of the middle term in equation A.8 is simply $-2\mathbf{A}^T \mathbf{g}$ so

$$\frac{\partial}{\partial \mathbf{x}} \|\epsilon\|^2 = 2\mathbf{A}^T \mathbf{A}\mathbf{x} - 2\mathbf{A}^T \mathbf{g}\tag{A.12}$$

setting it equal to zero and solving for \mathbf{x} gives the least squares estimate

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{g}\tag{A.13}$$

Glossary

Bandlimited signal A bandlimited signal is any signal $g(x)$ that whose Fourier transform $G(f_x)$ is zero and remains zero past a certain frequency $|f_x| \geq B$.

Basis pursuit Algorithms which attempt to solve a compressive sensing problem by solving the convex optimization problem of

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{g} \quad (6.14)$$

Coding In the context of computational sensing, coding is the process of modifying or modulating an analog signal during measurement. Coding is often used to reduce degeneracy in the measurement data. In the context of spectroscopy, the spectral filters act to code the spectrum.

Compressible The idea that

Compressive sampling *See* compressive sensing.

Compressive sensing A sensing technique that attempts to directly a compressive or sparse representation of the signal-of-interest during the measurement. Compressive sensors attempt to uses significantly less measurements than the dimensionality of the signal-of-interest. Compressive sensing relies on non-linear optimization algorithms to perform reconstruction or task-specific sensing from highly underdetermined inverse problems. These algorithms often rely on sparsity to avoid overfitting.

Computational sensing Any sensing technique in which the sensor uses *indirect-imaging*, *multiplex sensing*, compressive sensing, or *task-specific sensing*.

Computational sensor *See* computational sensing.

Data processing inequality An theorem from information theory that proves the information of a signal cannot be increased via a local physical operation.

Fellgett advantage *See* *multiplex advantage*.

Forward model A numerical model, typically an equation, that explains the mapping of the analog signal-of-interest to the measurement data.

Incoherence The idea that

Indirect-imaging An imaging sensor that attempts to reconstruct an image of an object using non-isomorphic measurements. A computational step is required to reconstruct the object signal-of-interest. X-Ray CT and SAR are examples types of indirect-imaging.

Inverse problem The problem of taking the measurement data and calculating a reconstruction of the signal-of-interest or task-specific parameters. In computational sensing, computer algorithms are used to solve inverse problems..

Isomorphic *See* isomorphic sensing.

Isomorphic sensing An isomorphic sensor is any sensor that attempts to produce measurement data that resembles the signal-of-interest. An isomorphic measurement is a measurement that resembles the signal-of-interest. An isomorphic measurement can be described as a one-to-one mapping from the signal-of-interest to the measurement, and is represented in matrix notation with an identity matrix. Isomorphic sensing is synonymous with traditional sensing..

Measurement A process that converts a physical phenomena to single datum or a set of data. In the context of this dissertation it used synonymously with the detector readout..

monochromator An optical instrument that transmits a selectable narrow wavelength band of light chosen from a wider range of wavelengths available at the input..

Multiplex advantage The improvement in SNR that is due to multiplexed measurements rather than isomorphic measurements. This is often referred to as the Fellgett advantage since it was first discovered by Peter Fellgett. *See* multiplex sensing.

multiplex sensing A multiplexing sensor is any sensor that attempts to combine the physical phenomena of the analog signal-of-interest in to a few or one analog-to-digital sample to overcome limits due to signal-to-noise ratio. The measurement data that does resemble the signal-of-interest. A matrix representation of a multiplex measurement will not look like an identity matrix.

Multiplexing *See* multiplex sensing.

Pixel pitch The center to center distance between pixels on a focal-plane array such as a CCD or CMOS image sensor..

Sample The process of mapping a continuous signal to a discrete signal. .

Sampling *See sample* .

Sparse *See* sparsity.

sparsity A set, vector, or matrix which contains an overwhelming majority of zeros relative to the size of the set, vector, or matrix. .

Spectral resolution The smallest the smallest difference in wavelength the instrument or sensor can discern..

Task-specific sensing A sensor that does not attempt to reconstruct the signal-of-interest to perform a signal processing task such as detection, estimation, and classification. The AFSSI-C is an example of a task-specific sensor. .

Acronyms

- ADC** analog-to-digital converter.
- AFSSI-C** Adaptive Feature Specific Spectral Imaging-Classifier.
- CASSI** Coded Aperture Snapshot Spectral Imaging.
- CCD** Charge-Coupled Device.
- CMOS** Complementary Metal–Oxide–Semiconductor.
- CT** Computed Tomography.
- DISP** Duke Imaging and Spectroscopy Program.
- DMD** Digital Micro-Mirror Display.
- FPA** focal-plane array.
- FTS** Fourier Transform Spectrometer.
- LCOS** Liquid Crystal on Silicon.
- LENS** Laboratory for Engineering Non-Traditional Sensors.
- LS** Least squares.
- MAP** Maximum a posteriori.
- MRI** Magnetic Resonance Imaging.
- MSE** Mean Squared Error.
- OSA** Optical Society of America.
- PCA** Principal Component Analysis.
- PET** Positron Emission Tomography.
- PSF** point spread function.
- RIP** Restricted isometry property.
- SAR** Synthetic Aperture Radar.
- SCOUT** Static Computational Optical Undersampled Tracker.
- SNR** signal-to-noise ratio.
- SPECT** Single-Photon Emission Computed Tomography.
- SWAP-C** size, weight and power-cost.

Symbols

- A** The product of the sensing matrix and the representation matrix $\mathbf{A} = \mathbf{H}\Psi$.
- δ_λ Spectral resolution.
- e** Additive noise vector.
- f** Object signal-of-interest.
- $\hat{\mathbf{f}}$ Estimated object.
- g** Measurement vector.
- H** The system matrix which captures all of the physical phenomena in a measurement. Also called the measurement matrix..
- \mathbf{H}_n A Hadamard matrix of size $n \times n$.
- ℓ_1 The ℓ_1 is a norm which is defined as the sum of the absolute values of the entries a vector.
- m measurement number/index.
- n The number of elements in the object **f**.
- N_λ Number of spectral channels.
- N_m Total number of measurements.
- τ Notation for a regularization parameter in an objective function for any kind of optimization.

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