2. Technological change, population and economic growth

I. Exercise Questions

Readings:
Lecture slide set: #2
The Economy: *Production function (2.7.1), input factors (3), average & marginal product (3.1.1)*
https://policonomics.com: *marginal rate of technical substitution (MRTS), isoquant, cost minimisation*

Problem 1 (*The production function*)

In our model of an agricultural economy, the production function shows how the output of grain depends on the input of labour - the number of farmers working the land. The production function for grain is represented graphically in Figure 1.

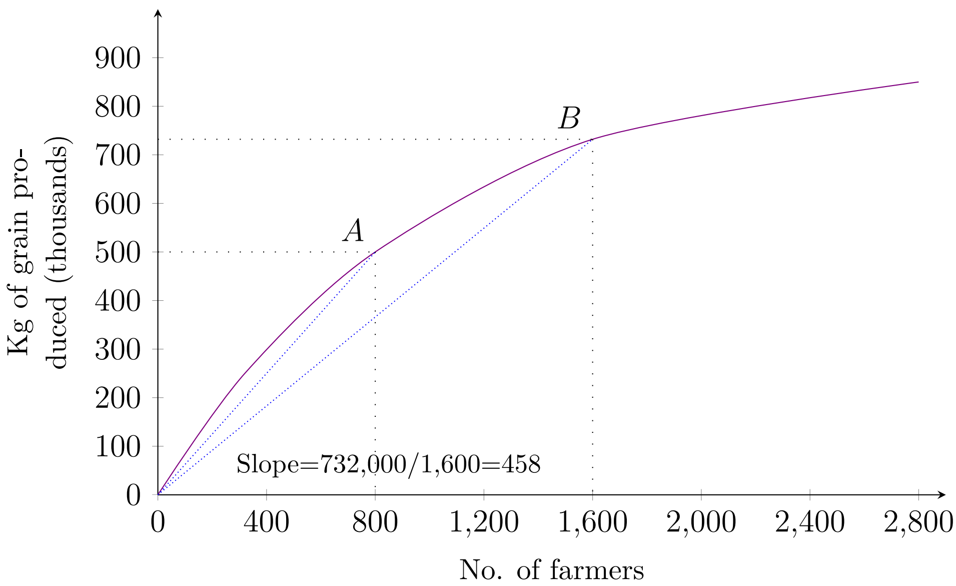


Figure 1

If we let X be the labour input (number of farmers) and Y be the amount of grain produced (in kilogrammes), we can write the production function as

$$Y = f(X)$$

f could be any function, but if it is to represent a production function like the one in Figure 1, it must have certain properties.

(a) Which assumptions must be made to arrive at the production function in Figure 1?

(i) First, we can see that if the input is zero, no grain is produced, and if the input is greater than zero the amount of grain is strictly positive: $f(0) = 0$ and $f(X) > 0$ if $X > 0$.

(ii) Second, the function is increasing: that is, as X increases, so does Y. So its slope, which is given by the derivative of the function, is positive: $dY/dX > 0$.

(iii) These two properties are typical of most production functions. Another property of the production function in the figure is that it gets gradually less steep: As X increases, its slope dY/dX decreases, which means that its second derivative is negative: $d/dX(dY/dX) = d^2Y/dX^2 < 0$.

(b) In the following, we consider the general ($Y = F(K, L) = L^\alpha \cdot K^\beta$, $\alpha, \beta \in (0, 1)$) and the special Cobb-Douglas production function ($Y = F(K, L) = L^\alpha \cdot K^{1-\alpha}, \alpha \in (0, 1)$), where Y describes the total output of the considered economy. Compute and interpret for the general as well as for the special Cobb-Douglas production function

(i) the marginal product of the input factors labour L and capital K

(ii) the marginal rate of technical substitution $MRTS_{L,K}$

(i) Marginal product
The marginal product of labour, and respectively capital, describes the change of the macroeconomic output Y due to a marginal change of the respective input factors.
Example: How much does the macroeconomic output increase, if one more worker (one monetary unit) is hired (deployed) in the economy?
The marginal product of labour (MP_L) and respectively capital (MP_K) is determined mathematically through the partial derivative of the production function with respect to the respective factor:

$$MP_L = \frac{\partial F(K, L)}{\partial L} = \frac{\partial Y}{\partial L}$$

$$MP_K = \frac{\partial F(K, L)}{\partial K} = \frac{\partial Y}{\partial K}$$

Thus, for the general as well as the special Cobb-Douglas production function holds:
1) General Cobb-Douglas production function

$$MP_L = \frac{\partial Y}{\partial L} = \frac{\partial (L^\alpha K^\beta)}{\partial L} = \alpha \cdot L^{\alpha-1} \cdot K^\beta = \alpha \cdot \frac{K^\beta}{L^{1-\alpha}} > 0$$

$$MP_K = \frac{\partial Y}{\partial K} = \frac{\partial (L^\alpha K^\beta)}{\partial K} = \beta \cdot L^\alpha \cdot K^{\beta-1} = \beta \cdot \frac{L^\alpha}{K^{1-\beta}} > 0$$

2) Special Cobb-Douglas production function

$$\begin{aligned} MP_L &= \frac{\partial Y}{\partial L} = \frac{\partial (L^\alpha K^{1-\alpha})}{\partial L} = \alpha \cdot L^{\alpha-1} \cdot K^{1-\alpha} = \alpha \cdot \frac{K^{1-\alpha}}{L^{1-\alpha}} \\ &= \alpha \cdot \left(\frac{K}{L}\right)^{1-\alpha} > 0 \end{aligned}$$

$$\begin{aligned} MP_K &= \frac{\partial Y}{\partial K} = \frac{\partial (L^\alpha K^{1-\alpha})}{\partial K} = (1-\alpha) \cdot L^\alpha \cdot K^{-\alpha} = (1-\alpha) \cdot \frac{L^\alpha}{K^\alpha} \\ &= (1-\alpha) \cdot \left(\frac{L}{K}\right)^{\alpha} > 0 \end{aligned}$$

Both the marginal product of labour as well as the marginal product of capital are positive. This denotes that an additional (marginal) unit of the respective factor leads to an increase of the macroeconomic output Y (and respectively leads to a decrease if one removes one marginal unit from the economy).
It further holds:
1) General Cobb-Douglas production function

$$\frac{\partial^2 Y}{\partial L^2} = \frac{\partial MP_L}{\partial L} = \alpha \cdot (\alpha - 1) \cdot L^{\alpha-2} \cdot K^\beta = \alpha \cdot (\alpha - 1) \frac{K^\beta}{L^{2-\alpha}} < 0$$

$$\frac{\partial^2 Y}{\partial K^2} = \frac{\partial MP_K}{\partial K} = \beta \cdot (\beta - 1) \cdot L^\alpha \cdot K^{\beta-2} = \beta \cdot (\beta - 1) \cdot \frac{L^\alpha}{K^{2-\beta}} < 0$$

2) Special Cobb-Douglas production function

$$\frac{\partial^2 Y}{\partial L^2} = \frac{\partial MP_L}{\partial L} = \alpha \cdot (\alpha - 1) \cdot L^{\alpha-2} \cdot K^{1-\alpha} = \alpha \cdot (\alpha - 1) \cdot \frac{K^{1-\alpha}}{L^{2-\alpha}} < 0$$

$$\frac{\partial^2 Y}{\partial K^2} = \frac{\partial MP_K}{\partial K} = -\alpha \cdot (1 - \alpha) \cdot L^{\alpha} \cdot K^{-\alpha-1} = -\alpha \cdot (1 - \alpha) \cdot \frac{L^{\alpha}}{K^{1+\alpha}} < 0$$

The second partial derivatives (respectively the partial derivatives of the marginal products) are both negative. Therefore, the production function has decreasing marginal products: One addidtional marginal unit of an input factor leads indeed to a continous increase of the macroeconomic output, however the latter steadily declines with every further unit.

(ii) Marginal rate of technical substitution
The marginal rate of technical substitutiton (MRTS) describes how much of one factor (the microeconomic equivalent is commodity) we have to additionally deploy in order to compensate the (marginal) decrease of another factor such that the macroeconomic output remains the same. The MRTS is defined as

$$MRTS_{L,K} = \frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial K}} = -\frac{dK}{dL}. \tag{1}$$

Here the last equalitiy sign holds on grounds of the implicit function theorem (see Econ I exercise).
For the general and special Cobb-Douglas production function, it holds therefore

$$MRTS_{L,K} = \frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial K}} = \frac{\alpha \cdot \frac{K^{\beta}}{L^{1-\alpha}}}{\beta \cdot \frac{L^{\alpha}}{K^{1-\beta}}} = \frac{\alpha}{\beta} \cdot \frac{K^{\beta} \cdot K^{1-\beta}}{L^{1-\alpha} \cdot L^{\alpha}} = \frac{\alpha}{\beta} \cdot \frac{K}{L}$$

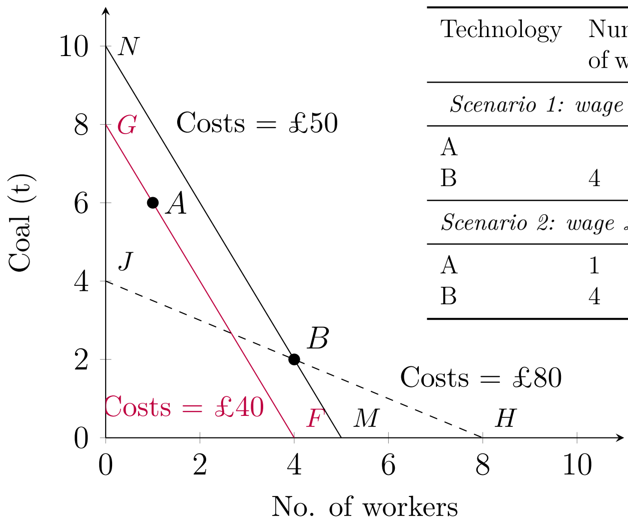
$$MRTS_{L,K} = \frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial K}} = \frac{\alpha \cdot \frac{K^{1-\alpha}}{L^{1-\alpha}}}{(1-\alpha) \cdot \frac{L^{\alpha}}{K^{\alpha}}} = \frac{\alpha}{1-\alpha} \cdot \frac{K^{1-\alpha} \cdot K^{\alpha}}{L^{1-\alpha} \cdot L^{\alpha}} = \frac{\alpha}{1-\alpha} \cdot \frac{K}{L}.$$

II. Multiple Choice

Readings:
The Economy: *Technology costs (2.4), Malthusian trap (2.7-2.10)*

Select one answer.

1. Technology choice: In the diagramme in Figure 2, you are given two technologies, A and B, which can produce 100 meters of cloth. You have two price scenarios. In Scenario 1, the wage and the coal price are £10 and £20, respectively. In Scenario 2, the wage and the coal price are £10 and £5, respectively. Based on this information, which of the following statements is correct?



Technology	Number of workers	Coal (tons)	Total costs (£)
Scenario 1: wage £10, cost of coal £20 per ton			
A			
B	4	2	80
Scenario 2: wage £10, cost of coal £5 per ton			
A	1	6	40
B	4	2	50

Figure 2

(A) In Scenario 1, the isocost line going through technology A would be parallel to isocost HJ with the associated cost of £90.

(B) In Scenario 1, technology B is the optimal choice.

(C) Going from Scenario 1 to Scenario 2, the cost of using technology B falls from £80 to £50. Therefore, it is optimal to stick with technology B after the change.

(D) As isocost FG is below isocost MN, the former represents lower cost than the latter. Therefore, A is preferred to B in both price scenarios.

Feedback:
(A) No: The associated cost is $£10 \times 1 + £20 \times 6 = £130$.
(C) No: It is true that the cost of production with technology B falls from $£80$ to $£50$ after the scenario change. However in scenario 2 you can do better with technology A, which would be cheaper at $£40$.
(D) No: It is true that under scenario 2, technology A would be the preferred choice. However in scenario 1 the isocost line going through A would be parallel to isocost HJ but above isocost HJ. Therefore in this case technology B is preferred.

2. You are given the following different isocost lines, FG, HJ and MN in Figure 2. Based on this information, we can say that:

(A) Isocost FG represents all points that can produce 100 metres of cloth at a particular price ratio.

(B) Isocost HJ represents a higher wage to coal price-ratio than isocost FG.

(C) Along isocost MN, it is more costly to produce at point N than at point B.

(D) Isocosts FG and MN represent the same price ratio (wage to coal price) but different total costs of production.

Feedback:
(A) No: An isocost represents all combinations of number of workers and tons of coal for which the total cost of production is the same (different price ratios). Along isocost HJ we know that at point B (i.e. 4 workers and 2 tons of coal) the technology can produce 100 metres of cloth. This does not mean that all technologies along the isocost can produce 100 metres of cloth.

(B) No: Isocost FG has slope -2 (i.e. replacing two tons of coal with one worker leaves the total cost of production the same), while isocost HJ has slope -0.5 (i.e. replacing 1 ton of coal with 2 workers leaves the total cost the same). This means that labour is relatively cheaper along HJ, or isocost HJ has a lower wage/coal price ratio.

(C) No: An isocost represents all combinations of number of workers and tons of coal for which the total cost of production is the same. Therefore using 10 tons of coal and no workers (point N), and 2 tons of coal and 4 workers (point B), both cost a total of Â£50.

3. Average product of labour

The diagramme in Figure 3 depicts the production function of the farmers, where diminishing average product of labour is assumed. At A, the average product of labour is 500,000kg / 800 = 625kg of grain per farmer. At B, the average product of labour is 458kg of grain per farmer. If you know that for 2,800 farmers the grain output is 894,000kg, then which of the following statements is correct?

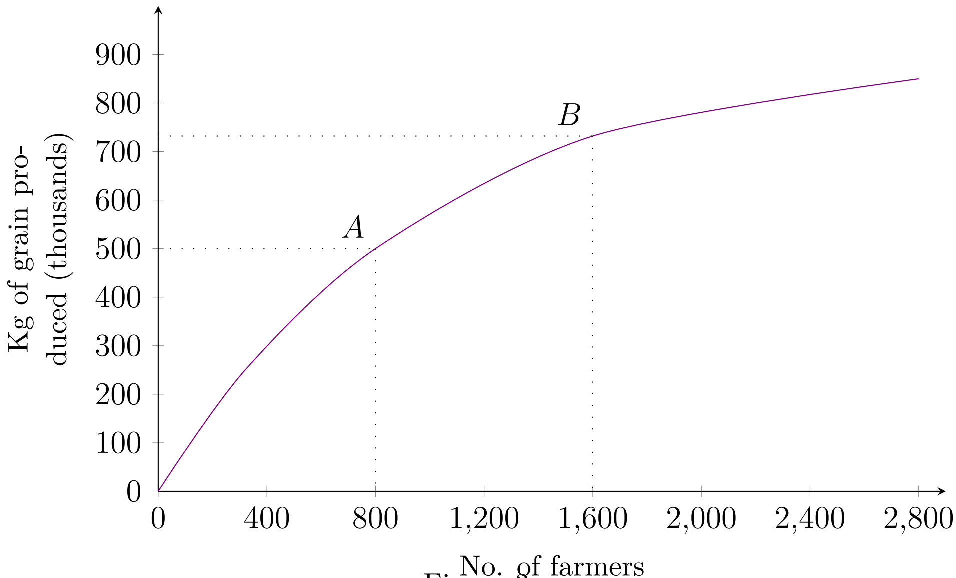


Figure 3

(A) It is possible that initially, there are economies of scale: For example, going from one farmer to two, the average output increases as they efficiently share the workload.

(B) The average product of labour when the labour input is 2,800 is 300kg/farmer.

(C) If the production function curve is a downward-sloping straight line, then there is no diminishing average product of labour.

(D) The decreasing slopes of the rays from the origin to the production function along the curve indicate the decreasing average product of labour.

Feedback:

(A) No: The diminishing average product of labour assumption rules out this possibility.

(B) No: APL = 894,000 / 2,800 = 319.

(C) No: It needs to be upward sloping to be correct.

4. The Malthusian trap

Figure 4 plots an index of real wages against population in England from the 1280s to the 1880s. According to Malthus, with diminishing average product of labour in production and population growth in response to increases in real wages, an increase in productivity will result in increased population but not increased real wages in the long run. Based on this information, which of the following statements is correct?

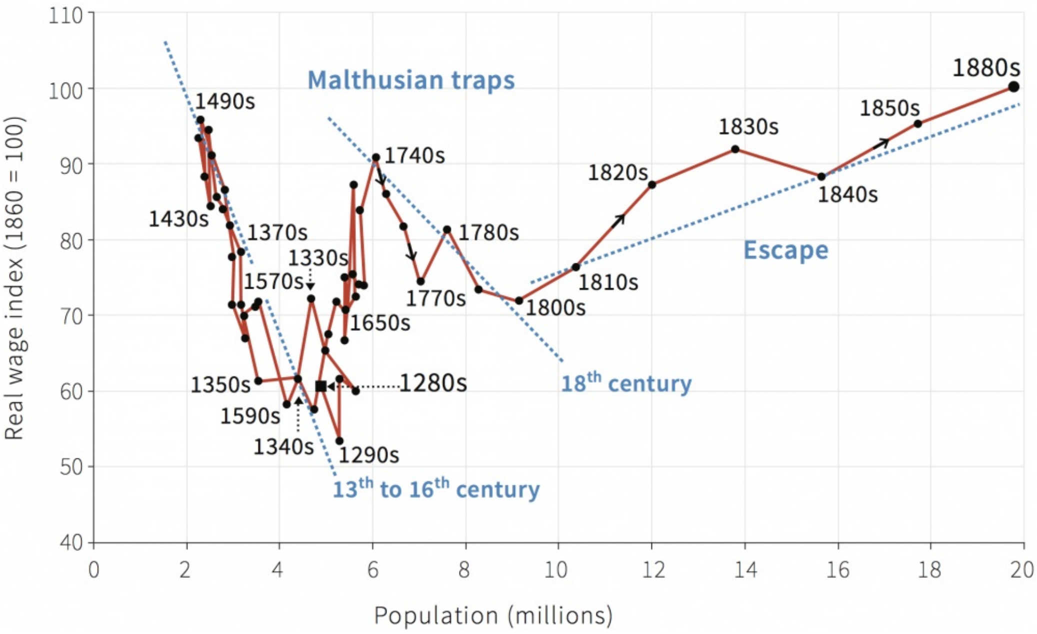


Figure 4

(A) There is clear evidence of a continuous Malthusian trap between 1280s and 1880s.

(B) The Malthusian traps seem to occur in a cycle of 60 years.

(C) Between the 1800s and the 1880s, the population grows as real wages increase. This is entirely in line with Malthus's description of economic growth.

(D) The Malthusian model does not take into account the possibility of persistent technological improvements that may offset the diminishing average product of labour.