Automatica 54 (2015) 305–309

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Nonsingular fixed-time consensus tracking for second-order multi-agent networks☆

Zongyu Zuo [1]

The Seventh Research Division, and Science and Technology on Aircraft Control Laboratory, Beihang University (BUAA), Beijing 100191, China

ARTICLE INFO

Article history:
Received 7 May 2014
Received in revised form
12 November 2014
Accepted 3 January 2015
Available online 27 February 2015

Keywords:
Directed graph
Fixed-time stability
Multi-agent systems
Network consensus
Nonsingular
Sliding mode

ABSTRACT

This paper investigates the fixed-time consensus tracking problem for second-order multi-agent systems in networks with directed topology. Global well-defined nonlinear consensus protocols are constructed with the aid of a newly-designed sliding surface for each double-integrator agent dynamics. In particular, the proposed framework eliminates the singularity and the settling time is assignable for any initial conditions. This makes it possible for network consensus problems to design and estimate the convergence time off-line. Finally, simulation is included to demonstrate the performance of the new protocols.

1. Introduction

Network *Consensus* (Ren, 2008) means that a group of agents reaches an agreement upon a common value by local interaction. In some applications, groups of agents are required to track a dynamic leader (time-varying reference information), which is called *consensus tracking*. In the analysis of consensus problems, an important performance index for a proposed consensus protocol is convergence rate (Wang & Xiao, 2010). Olfati-Saber and Murray (2004) proposed a linear consensus protocol and demonstrated that the *algebraic connectivity* of a interaction graph qualified the convergence rate. On the other hand, *finite-time consensus* problem has been promoted to achieve high-speed convergence. The benchmark work due to Bhat and Bernstein (2000) related the regularity properties of the Lyapunov functions to the settling time function in finite-time stability analysis for autonomous systems. Xiao and Wang (2007) and Xiao, Wang, and Chen (2011) expanded the finite-time control idea (Bhat & Bernstein, 2000) to multi-agent systems with single-integrator kinematics. Chen, Cao, and Ren (2012) and Cortés (2006) proposed discontinuous finite-time protocols for first-order agent networks based on different ways of applying the signum function. However, it is nontrivial to extend the finite-time consensus algorithm from the first-order case to the second-order case straightforwardly. Despite the difficulties, still some progress on this issue has been made recently. The works due to Wang and Hong (2008) and Zhao, Duan, Wen, and Zhang (2013) presented the finite-time consensus and tracking control respectively for second-order multi-agent systems via using homogeneity with dilation. Khoo, Xie, and Man (2009) expanded the terminal sliding mode technique (Feng, Yu, & Man, 2002) to the finite-time consensus problem of second-order multi-robot systems. The work due to Li, Du, and Lin (2011) proposed continuous finite-time consensus algorithms for leaderless and leader–follower second-order multi-agent systems respectively by adding a power integrator method.

However, the settling time functions derived in the finite-time consensus depend on initial states of the agents, which prohibits their practical applications if the knowledge of initial conditions is unavailable in advance. Recently, Zuo and Tie (2014a,b) present a novel class of nonlinear consensus protocols for single-integrator multi-agent networks, called fixed-time consensus which assumes uniform boundedness of a settling time regardless of the initial conditions, i.e. fixed-time stability (Polyakov, 2012). Unfortunately, the direct extension to second-order case has come across a challenging issue: the singularity in the controller (Feng et al., 2002; Feng, Yu, & Man, 2013; Wu, Yu, & Man, 1998). Motivated

☆ This work was supported by the National Natural Science Foundation of China (61203022) and Aeronautic Science Foundation of China (2012CZ51029). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Zhihua Qu under the direction of Editor André L. Tits.

E-mail address: zzybobby@buaa.edu.cn.

[1] Tel.: +86 10 82315584; fax: +86 10 82315584.

http://dx.doi.org/10.1016/j.automatica.2015.01.021
0005-1098/

by this, this paper extends the idea in Zuo and Tie (2014a,b) and constructs global well-defined fixed-time consensus protocols for second-order multi-agent networks with directed information flow. The key contributions include: (i) a guaranteed settling time independent of initial conditions is obtained, i.e., fixed-time consensus; (ii) a continuous sinusoid function is introduced into the protocols to eliminate the singularity. To the author's knowledge, no results on fixed-time consensus tracking with assignable settling time for second-order multi-agent networks are available till now.

2. Preliminaries

2.1. Graph theory notions

A weighted graph $\mathcal{G}(A) = \{\mathcal{V}, \mathcal{E}, A\}$ consists of a node set $\mathcal{V}(\mathcal{G}) = \{\pi_1, \pi_2, \ldots, \pi_n\}$, an edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$ and an adjacent matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. An edge (π_i, π_j) on $\mathcal{G}(A)$ denotes the state of node π_i is available to node π_j, but not necessarily vice versa. If $(\pi_i, \pi_j) \in \mathcal{E}$, then node π_i is called a *neighbor* of node π_j. The index set of all neighbors of π_j is denoted by $\mathcal{N}_j = \{i : (\pi_i, \pi_j) \in \mathcal{E}\}$. The weighted adjacent matrix A of a directed graph is defined such that $a_{ij} = 1$ for $(\pi_j, \pi_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Let $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ be the graph Laplacian of $\mathcal{G}$. A graph has a directed spanning tree if a subset of the edges forms a spanning tree (Ren & Beard, 2005).

2.2. Some lemmas

Lemma 1 (*Zuo & Tie, 2014a*). *Let $\xi_1, \xi_2, \ldots, \xi_N \geq 0$. Then*

$$\sum_{i=1}^{N} \xi_i^p \geq \left(\sum_{i=1}^{N} \xi_i\right)^p \quad \text{if } 0 < p \leq 1 \tag{1}$$

$$\sum_{i=1}^{N} \xi_i^p \geq N^{1-p} \left(\sum_{i=1}^{N} \xi_i\right)^p \quad \text{if } 1 < p < \infty. \tag{2}$$

Lemma 2 (*Zuo & Tie, 2014b*). *Consider a scalar system*

$$\dot{y} = -\alpha y^{\frac{m}{n}} - \beta y^{\frac{p}{q}}, \qquad y(0) = y_0 \tag{3}$$

where $\alpha > 0$, $\beta > 0$, m, n, p, q are positive ***odd*** *integers satisfying $m > n$ and $p < q$. The equilibrium of (3) is globally fixed-time stable with settling time T bounded by*

$$T < T_{max} := \frac{1}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}. \tag{4}$$

If $\varepsilon \triangleq [q(m-n)]/[n(q-p)] \leq 1$, a less conservative estimation of the settling time can be obtained instead as

$$T < T_{max} := \frac{q}{q-p}\left(\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\sqrt{\frac{\alpha}{\beta}} + \frac{1}{\alpha\varepsilon}\right). \tag{5}$$

2.3. Problem formulation

Consider a group of N continuous-time agents as

$$\dot{x}_i(t) = v_i(t), \qquad \dot{v}_i(t) = u_i(t) \tag{6}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ denote the position, velocity and input (*protocol*) of the ith agent respectively, and $i \in \mathcal{I}_N \triangleq \{1, 2, \ldots, N\}$.

Let the consensus reference state evolution be

$$\dot{x}_0(t) = v_0(t), \qquad \dot{v}_0(t) = u_0(t) \tag{7}$$

where $x_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}$ denote the position, velocity and input of the reference system. We introduce a virtual leader π_0 with the states $\xi_0 = [x_0, v_0]^T$ for the multi-agent system in (6). Note, however, that ξ_0 is available not to all agents but to only a portion of agents. Here, define a nonnegative diagonal matrix $B = \text{diag}\{b_1, b_2, \ldots, b_N\}$ to indicate the accessibility of ξ_0 by the agents, where $b_i = 1$ if ξ_0 is accessible by the ith agent, and $b_i = 0$ otherwise. The directed graph incorporating π_0 into $\mathcal{G}$ is denoted by $\mathcal{G}^e$.

Assumption 3. $\mathcal{G}^e$ has a spanning tree with π_0 being its root vertex, i.e., $B \neq 0$.

Assumption 4. The input $u_0(t)$ of the leader is unavailable to the group members, but its upper-bound u_0^{M} is accessible by its neighboring agents.

With a given protocol u_i, the closed-loop system in (6) is said to reach or achieve *fixed-time consensus tracking*, if, for $\forall \xi_i(0)$ and $\forall i, j \in \mathcal{I}_N$ there exists a constant $T_{\max} > 0$ such that the *settling time* $T < T_{\max}$ and

$$\begin{cases} \lim\limits_{t\to T} |\xi_i(t) - \xi_0(t)| \to 0 \\ \xi_i(t) = \xi_0(t), \quad \forall t \geq T \end{cases} \tag{8}$$

where $\xi_i \triangleq [x_i, v_i]^T$, $|\cdot|$ is defined componentwise.

3. Nonsingular fixed-time control

To clarify the core idea, we first consider the fixed-time control of a single system defined by

$$\dot{z}_1(t) = z_2(t), \qquad \dot{z}_2(t) = u(t) \tag{9}$$

where $z = [z_1, z_2]^T \in \mathbb{R}^2$ denotes the system state vector, $u \in \mathbb{R}$ the control input.

Let m_k, n_k, p_k, q_k be positive odd integers satisfying $m_k > n_k$, $p_1 < q_1 < 2p_1$, $p_2 < q_2$ and $m_1/n_1 - p_1/q_1 > 1$, and α_k, β_k be positive constants, where $k = 1, 2$. Define a $\mathcal{C}^1$ function $\mu_\tau(\cdot) : [0, +\infty) \to [0, 1]$ as

$$\mu_\tau(x) = \begin{cases} \sin\left(\dfrac{\pi}{2}\cdot\dfrac{x}{\tau}\right) & \text{if } x \leq \tau \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

where τ is a positive constant.

To circumvent the singularity problem (Feng et al., 2013), a new sliding surface is proposed:

$$s = z_1 + [\kappa(z_1)\cdot z_2]^{\frac{q_1}{p_1}} \tag{11}$$

where $\kappa(\cdot) : \mathbb{R} \to \mathbb{R}^+$ denotes a scalar positive function, given by

$$\kappa(x) = \frac{1}{\alpha_1 x^{m_1/n_1 - p_1/q_1} + \beta_1}. \tag{12}$$

In the sequel, the parameter in $\kappa(\cdot)$ will be omitted for simplicity after its first definition. It can be straightforwardly verified that $s = 0$ implies

$$z_2 = -\alpha_1 z_1^{\frac{m_1}{n_1}} - \beta_1 z_1^{\frac{p_1}{q_1}}.$$

A new fixed-time nonlinear control law is defined as

$$u = \frac{1}{\kappa}\left[\alpha_1\left(\frac{m_1}{n_1} - \frac{p_1}{q_1}\right) z_1^{\frac{m_1}{n_1} - \frac{p_1}{q_1} - 1}(\kappa z_2)^2 - \frac{p_1}{q_1}\kappa^{1-\frac{q_1}{p_1}} z_2^{2-\frac{q_1}{p_1}}\right.$$
$$\left. - \frac{p_1}{q_1}\kappa^{-\frac{q_1}{p_1}}\cdot\mu_\tau\left(z_2^{\frac{q_1}{p_1}-1}\right)\cdot z_2^{1-\frac{q_1}{p_1}}\left(\alpha_2 s^{\frac{m_2}{n_2}} + \beta_2 s^{\frac{p_2}{q_2}}\right)\right] \tag{13}$$

where $\mu_\tau(\cdot)$ is defined in (10) with $x = z_2^{q_1/p_1 - 1}$. The fact $\sin\omega x \leq x$, $\forall\omega > 0$ ensures the input (13) is always well-defined. The following theorem presents the novel *fixed-time* stability property (Polyakov, 2012).

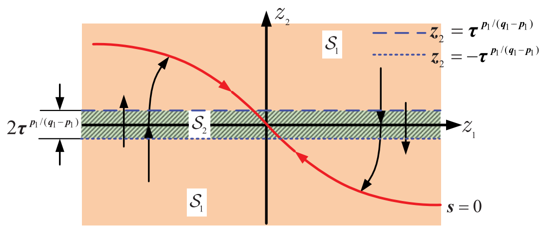


Fig. 1. The phase plot of the system.

Theorem 5. *Consider the second-order system in (9) with the control feedback law designed as (13). Then the state $z = [z_1, z_2]^T$ is globally fixed-time stable and the settling time estimate is derived as*

$$T < T_{\max} := T_1 + T_2 + \epsilon(\tau) \tag{14}$$

where $T_k \triangleq n_k/[\alpha_k(m_k - n_k)] + q_k/[\beta_k(q_k - p_k)]$, $(k = 1, 2)$, and $\epsilon(\tau)$ denotes a small time margin related to τ.

Proof. Differentiating s in (11) against time yields

$$\dot{s} = z_2 - \frac{q_1(\kappa z_2)^{\frac{q_1}{p_1}-1}}{p_1}\left[\alpha_1\left(\frac{m_1}{n_1} - \frac{p_1}{q_1}\right) z_1^{\frac{m_1}{n_1}-\frac{p_1}{q_1}-1}(\kappa z_2)^2 - \kappa u\right].$$

Substituting (13) into the preceding equation obtains

$$\dot{s} = -\mu_\tau\left(z_2^{\frac{q_1}{p_1}-1}\right)\cdot\left(\alpha_2 s^{\frac{m_2}{n_2}} + \beta_2 s^{\frac{p_2}{q_2}}\right). \tag{15}$$

Observe that $\mu_\tau(\cdot) > 0$ if $z_2 \neq 0$. For the convenience of the proof, the state space $z \in \mathbb{R}^2$ is divided into two different areas, as shown in Fig. 1,

$$\mathcal{S}_1 = \left\{(z_1, z_2) | z_2^{\frac{q_1}{p_1}-1} \geq \tau\right\}, \qquad \mathcal{S}_2 = \left\{(z_1, z_2) | z_2^{\frac{q_1}{p_1}-1} < \tau\right\}$$

(i) When the system states (z_1, z_2) belong to $\mathcal{S}_1$, the function $\mu_\tau(\cdot)$ takes value one. Applying Lemma 2, the states (z_1, z_2) will reach the sliding surface $s = 0$ or enter $\mathcal{S}_2$ within fixed-time.

(ii) In $\mathcal{S}_2$, $0 < \mu_\tau < 1$ when $z_2 \neq 0$. Applying Lemma 2 for (15), the sliding surface $s = 0$ is still an attractor. What remains is to prove that the z_1-axis in Fig. 1 is not attractive except for the origin $(z_1, z_2) = (0, 0)$. It can be shown that in close proximity to z_1-axis the control input (13) degenerates into

$$u = -\frac{p_1}{q_1}\kappa^{-\frac{p_1}{q_1}}\left(\alpha_2 s^{\frac{m_2}{n_2}} + \beta_2 s^{\frac{p_2}{q_2}}\right) \tag{16}$$

where the fact $\mu_\tau(z_2^{q_1/p_1-1}) \cdot z_2^{1-p_1/q_1} \to 1$ as $z_2 \to 0$ is used. In view of $\kappa^{-p_1/q_1} > 0$, we have $\dot{z}_2 < 0$ for $s > 0$ and $\dot{z}_2 > 0$ for $s < 0$. Thus, $z(t)$ will transgress $\mathcal{S}_2$ into $\mathcal{S}_1$ monotonically within finite time $\epsilon(\tau)$ (refer to Fig. 1).

It is thus concluded that the sliding surface $s = 0$ can be reached from anywhere in the phase plane within the time $t_1 < T_2 + \epsilon(\tau)$. Once the sliding surface $s = 0$ is reached, it follows from Lemma 2 that (z_1, z_2) will reach the origin within the period $t_2 < T_1$. Hence, the uniform bound (14) follows.

Remark 6. Note that the traveling time $\epsilon(\tau)$ across $\mathcal{S}_2$ cannot be estimated precisely. However, for sufficiently small τ and $z_1 \neq 0$, integrating both sides of $\dot{z}_2 = u$ subject to (16) with $s \to z_1$ obtains

$$\epsilon(\tau) = \frac{q_1}{p_1}[\kappa(|z_1|)]^{\frac{p_1}{q_1}} \frac{1}{\alpha_2|z_1|^{m_2/n_2} + \beta_2|z_1|^{p_2/q_2}} \cdot 2\tau^{\frac{p_1}{q_1-p_1}}$$

where $p_1/(q_1 - p_1) > 1$. This implies that $\epsilon(\tau)$ can be made very small by choosing sufficiently small τ. For very small z_1 including the case $z_1 = 0$, $z_2 \to 0$ in $\mathcal{S}_2$ implies the system trajectory gets very close to the sliding surface $s = 0$ where the fixed-time convergence is guaranteed, and thus $\epsilon(\tau) \to 0$. It makes sense to neglect $\epsilon(\tau)$ in practice for small τ due to the estimation conservativeness of T_2.

Remark 7. It can be verified that the parameter constraints $m_1/n_1 - p_1/q_1 > 1$ and $p_1/q_1 > 1/2$ lead to $\varepsilon_1 > 1$. Thus, the estimation (5) is only suitable for computing tighter T_2 since the case $\varepsilon_2 \leq 1$ is possible.

4. Nonsingular fixed-time consensus tracking

Let $e^p = [e_1^p, e_2^p, \ldots, e_N^p]^T$ and $e^v = [e_1^v, e_2^v, \ldots, e_N^v]^T$ be the position and velocity disagreement vectors, respectively, with their elements defined by $e_i^p = \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + b_i(x_i - x_0)$ and $e_i^v = \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) + b_i(v_i - v_0)$, taking time derivatives of which yields

$$\dot{e}^p(t) = e^v(t), \qquad \dot{e}^v(t) = (L + B)u(t) - B\mathbf{1}u_0(t) \tag{17}$$

where $u = [u_1, u_2, \ldots, u_N]^T$.

To solve the fixed-time consensus tracking problem, a new nonsingular distributed protocol may be derived as

$$u_i = \left(\sum_{j\in\mathcal{N}_i} a_{ij} + b_i\right)^{-1}\left[u_i^d + \sum_{j\in\mathcal{N}_i} a_{ij}u_j - b_i u_0^{\mathrm{M}}\mathrm{sgn}(s_i)\right] \tag{18}$$

where s_i denotes a sliding surface, defined by

$$s_i = e_i^p + [\kappa_i(e_i^p) \cdot e_i^v]^{\frac{q_1}{p_1}} \tag{19}$$

with the virtual control signal u_i^d defined by

$$\begin{aligned} u_i^d = &\frac{1}{\kappa_i}\left[\alpha_1\left(\frac{m_1}{n_1} - \frac{p_1}{q_1}\right)(e_i^p)^{\frac{m_1}{n_1}-\frac{p_1}{q_1}-1}(\kappa_i e_i^v)^2 \right.\\ &\left. - \frac{p_1}{q_1}\kappa_i^{1-\frac{q_1}{p_1}}(e_i^v)^{2-\frac{q_1}{p_1}}\right] - \frac{p_1}{q_1}\kappa_i^{-\frac{q_1}{p_1}}\mu_i^\tau\left((e_i^v)^{\frac{q_1}{p_1}-1}\right)\\ &\cdot(e_i^v)^{1-\frac{q_1}{p_1}}\left(\alpha_2 s_i^{\frac{m_2}{n_2}} + \beta_2 s_i^{\frac{p_2}{q_2}}\right) \end{aligned} \tag{20}$$

where $\mu_i^\tau(\cdot)$ is defined in (10) with $x = (e_i^v)^{q_1/p_1-1}$. It is noted that the threshold parameter τ can be different for each μ_i^τ. Without loss of generality, we choose the same τ for simplicity in this paper.

Theorem 8. *Consider a multi-agent system (6) and (7) in networks with interaction topology satisfying Assumptions 3 and 4. The nonsingular distributed protocol proposed in (18) achieves fixed-time consensus tracking with the settling time T bounded by*

$$T < T_{\max} := T_1 + T_2 + \epsilon(\tau) \tag{21}$$

where $T_k = [N^{(m_k-n_k)/2n_k} n_k]/[\alpha_k(m_k - n_k)] + q_k/[\beta_k(q_k - p_k)]$, $(k = 1, 2)$, and $\epsilon(\tau)$ denotes a small time margin.

Proof. (i) Substituting (18) into (17) obtains

$$\dot{e}_i^v = u_i^d - b_i u_0^{\mathrm{M}}\mathrm{sgn}(s_i) - b_i u_0. \tag{22}$$

Differentiating (19) against time and using (22), we have

$$\begin{aligned} \dot{s}_i = &-\mu_i^\tau\left((e_i^v)^{\frac{q_1}{p_1}-1}\right)\cdot\left(\alpha_2 s_i^{\frac{m_2}{n_2}} + \beta_2 s_i^{\frac{p_2}{q_2}}\right)\\ &- \frac{q_1}{p_1}\kappa_i^{\frac{q_1}{p_1}}(e_i^v)^{\frac{q_1}{p_1}-1}\left[b_i u_0^{\mathrm{M}}\mathrm{sgn}(s_i) + b_i u_0\right]. \end{aligned} \tag{23}$$

Consider Lyapunov function $V_2 = (1/2)\sum_{i=1}^{N} s_i^2$ and its time derivative is

$$\begin{aligned}
\dot{V}_2 &= -\sum_{i=1}^{N}\mu_i^\tau\left((e_i^v)^{\frac{q_1}{p_1}-1}\right)\left(\alpha_2 s_i^{\frac{m_2}{n_2}+1}+\beta_2 s_i^{\frac{p_2}{q_2}+1}\right)\\
&\quad -\sum_{i=1}^{N}\kappa_i^{\frac{q_1}{p_1}}(e_i^v)^{\frac{q_1}{p_1}-1}(b_i u_0^{\mathrm{M}}|s_i|+b_i u_0 s_i)\\
&\le -\sum_{i=1}^{N}\mu_i^\tau\left((e_i^v)^{\frac{q_1}{p_1}-1}\right)\left(\alpha_2(s_i^2)^{\frac{m_2+n_2}{2n_2}}+\beta_2(s_i^2)^{\frac{p_2+q_2}{2q_2}}\right)\\
&\quad -\sum_{i=1}^{N}\kappa_i^{\frac{q_1}{p_1}}(e_i^v)^{\frac{q_1}{p_1}-1}b_i(u_0^{\mathrm{M}}-|u_0|)|s_i|\\
&\le -\mu_{\mathrm{m}}^\tau\sum_{i=1}^{N}\left(\alpha_2(s_i^2)^{\frac{m_2+n_2}{2n_2}}+\beta_2(s_i^2)^{\frac{p_2+q_2}{2q_2}}\right)\\
&\le -\mu_{\mathrm{m}}^\tau\left[\alpha_2 N^{\frac{n_2-m_2}{2n_2}}(2V_2)^{\frac{m_2+n_2}{2n_2}}+\beta_2(2V_2)^{\frac{p_2+q_2}{2q_2}}\right]
\end{aligned}$$

where $\kappa_i^{q_1/p_1}\cdot(e_i^v)^{q_1/p_1-1}\ge 0$, $\mu_{\mathrm{m}}^\tau \triangleq \min\{\mu_1^\tau,\dots,\mu_N^\tau\}$, Lemma 1 is used in the last inequality. If $V_2\neq 0$, then let $y_2=\sqrt{2V_2}$ be the solution to the differential equation

$$\dot{y}_2(t) = -\mu_{\mathrm{m}}^\tau\left(\alpha_2 N^{\frac{n_2-m_2}{2n_2}}y_2^{\frac{m_2}{n_2}}(t)+\beta_2 y_2^{\frac{p_2}{q_2}}(t)\right)$$

where $\dot{y}_2=\dot{V}_2/\sqrt{2V_2}$ is used. Since $\mu_{\mathrm{m}}^\tau = 1$ if $\min\{|e_1^v|,|e_2^v|,\dots,|e_N^v|\} > \tau^{p_1/(q_1-p_1)}$, by Theorem 5 and *Comparison Principle* of differential equations Khalil (2005), we obtain $\lim_{t\to t_2}V_2(s_i)=0$, implying $\lim_{t\to t_2}s_i=0$ for all $i\in\mathcal{I}_N$, where the settling time t_2 is bounded by $t_2<T_2+\epsilon(\tau)$.

(ii) Note that on the sliding surface $s_i=0$, (19) becomes

$$\dot{e}_i^p = -\alpha_1(e_i^p)^{\frac{m_1}{n_1}}-\beta_1(e_i^p)^{\frac{p_1}{q_1}} \tag{24}$$

for all $i\in\mathcal{I}_N$. Consider Lyapunov function $V_1=(1/2)\sum_{i=1}^{N}(e_i^p)^2$ and its time derivative is

$$\begin{aligned}
\dot{V}_1 &= -\alpha_1\sum_{i=1}^{N}(e_i^p)^{\frac{m_1+n_1}{n_1}}-\beta_1\sum_{i=1}^{N}(e_i^p)^{\frac{p_1+q_1}{q_1}}\\
&\le -\alpha_1 N^{\frac{n_1-m_1}{2n_1}}(2V_1)^{\frac{m_1+n_1}{2n_1}}-\beta_1(2V_1)^{\frac{p_1+q_1}{2q_1}}
\end{aligned}$$

where Lemma 1 is inserted. If $V_1\neq 0$, then let $y_1=\sqrt{2V_1}$ be the solution to the differential equation

$$\dot{y}_1(t) = -\alpha_1 N^{\frac{n_1-m_1}{2n_1}}y_1^{\frac{m_1}{n_1}}(t)-\beta_1 y_1^{\frac{p_1}{q_1}}(t).$$

Similarly as (i), we have $\lim_{t\to t_1}e_i^p = 0$ with settling time t_1 bounded by $t_1<T_1$ for all $i\in\mathcal{I}_N$.

Thus, e^p converges to zero within the period $T<T_1+T_2+\epsilon(\tau)$. Applying Theorem 4 in Khoo et al. (2009), (8) follows, i.e., the fixed-time consensus tracking is achieved.

To incorporate the interaction topology into the sliding surface, we may modify the sliding variables as

$$\bar{s}_i = \bar{e}_i^p + [\kappa_i(\bar{e}_i^p)\cdot e_i^v]^{\frac{q_1}{p_1}} \tag{25}$$

where $\bar{e}_i^p = \sum_{j\in\mathcal{N}_i}\hat{a}_{ij}(e_i^p-e_j^p)+b_i e_i^p$ with $\hat{a}_{ij}=(a_{ij}+a_{ji})/2$ for all $i,j\in\mathcal{I}_N$. Let $\hat{L}=(L+L^T)/2$ be the Laplacian matrix of $\mathcal{G}(\hat{A})$ with $\hat{A}\triangleq[\hat{a}_{ij}]_{n\times n}$. Thus, the corresponding protocol for fixed-time consensus tracking is then obtained:

$$u_i = \left(\sum_{j\in\mathcal{N}_i}a_{ij}+b_i\right)^{-1}\left[\bar{u}_i^d+\sum_{j\in\mathcal{N}_i}a_{ij}u_j-b_i u_0^{\max}\mathrm{sgn}(\bar{s}_i)\right] \tag{26}$$

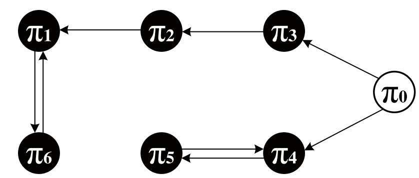


Fig. 2. The extended information flow $\mathcal{G}^e$.

where the virtual signal $\bar{u}_i^d$ is given by

$$\begin{aligned}
\bar{u}_i^d &= \frac{1}{\kappa_i}\left[\alpha_1\left(\frac{m_1}{n_1}-\frac{p_1}{q_1}\right)(\bar{e}_i^p)^{\frac{m_1}{n_1}-\frac{p_1}{q_1}-1}(\kappa_i e_i^v)^2\right.\\
&\quad\left.-\frac{p_1}{q_1}\kappa_i^{1-\frac{q_1}{p_1}}(e_i^v)^{2-\frac{q_1}{p_1}}\right]-\frac{p_1}{q_1}\kappa_i^{-\frac{q_1}{p_1}}\mu_i^\tau\left((e_i^v)^{\frac{q_1}{p_1}-1}\right)\\
&\quad\cdot(e_i^v)^{1-\frac{q_1}{p_1}}\left(\alpha_2\bar{s}_i^{\frac{m_2}{n_2}}+\beta_2\bar{s}_i^{\frac{p_2}{q_2}}\right)
\end{aligned} \tag{27}$$

Theorem 9. *Consider a multi-agent system (6) and (7) in networks with interaction topology satisfying Assumptions 3 and 4. The nonsingular distributed protocol proposed in (26) achieves fixed-time consensus tracking with the settling time bounded by*

$$T < T_{\max} := T_1+T_2+\epsilon(\tau) \tag{28}$$

where

$$T_1 = \frac{N^{(m_1-n_1)/2n_1}}{\underline{\lambda}^{(m_1+n_1)/2n_1}\alpha_1}\frac{n_1}{m_1-n_1}+\frac{1}{\underline{\lambda}^{(p_1+q_1)/2q_1}\beta_1}\frac{q_1}{q_1-p_1}$$

$$T_2 = \frac{N^{(m_2-n_2)/2n_2}}{\alpha_2}\frac{n_2}{m_2-n_2}+\frac{1}{\beta_2}\frac{q_2}{q_2-p_2},$$

$\underline{\lambda}$ denotes the smallest eigenvalue of matrix $(\hat{L}+B)$, $\epsilon(\tau)$ denotes a small time margin.

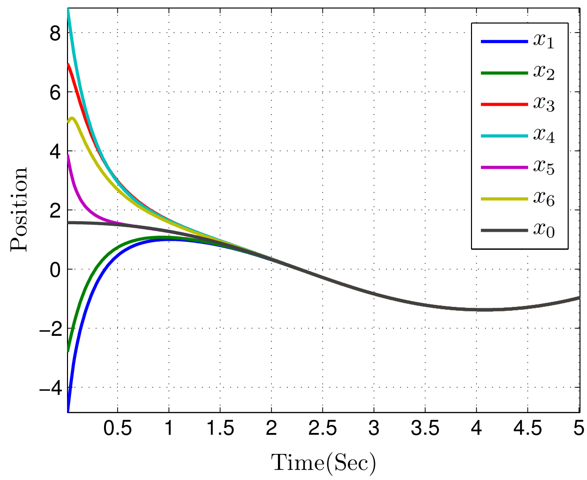
The proof follows the same lines of Theorem 8.

Remark 10. It is worthwhile noticing that the settling time estimation (28) also relies on the interaction topology property (i.e. $\underline{\lambda}$). In addition, the introduction of the interaction topology into the definition of the sliding variables guarantees the consensus during the transition. However, the protocol in (18) only achieves the reference goal seeking of the multiple agents within fixed-time.

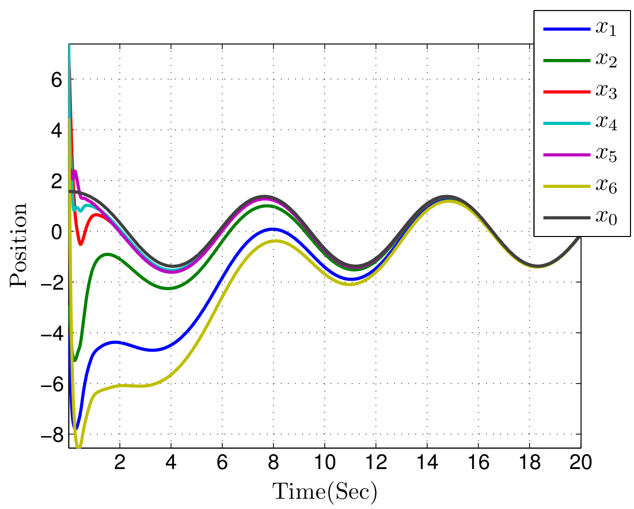
5. Simulation example

Consider a group of six agents in a network with the interaction graph shown in Fig. 2. The time-varying control input of the virtual leader (7) is designed as $u_0 = -\sin(x_0)/(1+\exp(-t))$ verifying $u_0^M = 1$. Consider the initial scenario that $x(0)=[-5,-3,7,9,4,5]^T$ and $v(0)=0$ are set for the group of agents (6) and $x_0(0)=\pi/2$ and $v_0(0)=0$ for the virtual leader (7). The design parameters $\alpha_1=\beta_1=\alpha_2=\beta_2=2$, $m_1=9$, $n_1=5$, $p_1=7$, $q_1=9$, $m_2=11$, $n_2=9$, $p_2=5$, $q_2=7$ and $\tau=0.1$ are set for protocols (18) and (26). It can be verified that the parameter constraints $m_1/n_1-p_1/q_1>1$ and $p_1/q_1>1/2$ are satisfied. The bounds for settling time T in (21) and (28) are 8.025 s and 24.985 s, respectively. Since $\varepsilon_2=7/9<1$, a less conservative bounds can be calculated as 7.889 and 24.425 s.

The simulation results in Fig. 3 show that the consensus tracking objective is achieved within fixed time. Note that the settling times under the two protocols are about respectively 2 s and 20 s, which demonstrate the performance claimed in Theorems 8 and 9. It can



(a) Protocol (18).



(b) Protocol (26).

Fig. 3. Consensus tracking results.

be further observed from Fig. 3(a) that each agent in the group achieves the goal state seeking directly, while from Fig. 3(b) the group seeks consensus among neighbors during transition and reaches the goal state together.

References

Bhat, S. P., & Bernstein, D. S. (2000). Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization*, *38*(3), 751–766.

Chen, F., Cao, Y., & Ren, W. (2012). Distributed average tracking of multiple time-varying reference signals with bounded derivatives. *IEEE Transactions on Automatic Control*, *57*(12), 3169–3174.

Cortés, J. (2006). Finite-time convergent gradient flows with applications to network consensus. *Automatica*, *42*(11), 1993–2000.

Feng, Y., Yu, X., & Man, Z. (2002). Non-singular terminal sliding mode control of rigid manipulators. *Automatica*, *28*(11), 2159–2167.

Feng, Y., Yu, X., & Man, Z. (2013). On nonsingular terminal sliding-mode control of nonlinear systems. *Automatica*, *49*(6), 1715–1722.

Khalil, H. K. (2005). *Nonlinear systems* (3rd ed.). Upper Saddle River, NJ: Prentice Hall.

Khoo, S., Xie, L., & Man, Z. (2009). Robust finite-time consensus tracking algorithm for multirobot systems. *IEEE/ASME Transactions on Mechatronics*, *14*(2), 219–228.

Li, S., Du, H., & Lin, X. (2011). Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica*, *47*(8), 1706–1712.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, *57*(8), 2106–2110.

Ren, W. (2008). On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, *53*(6), 1503–1509.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Wang, X., & Hong, Y. (2008). Finite-time consensus for multi-agent networks with second-order agent. In *Proc. 17th IFAC world congress. Seoul, Korea, 6–11 July 2008* (pp. 15185–15190).

Wang, L., & Xiao, F. (2010). Finite-time consensus problem for networks of dynamic agents. *IEEE Transactions on Automatic Control*, *55*(4), 950–955.

Wu, Y., Yu, X., & Man, Z. (1998). Terminal sliding mode control design for uncertain dynamic systems. *Systems & Control Letters*, *34*(5), 281–288.

Xiao, F., & Wang, L. (2007). Reaching agreement in finite time via continuous local state feedback. In *Proc. 26th Chinese control conference. Hunan, China, July 2007* (pp. 711–715).

Xiao, F., Wang, L., & Chen, T. (2011). Finite-time consensus of multi-agent systems with directed and intermittent links. In *Proc. 30th Chinese control conference. Yantai, China, July 2011* (pp. 6539–6543).

Zhao, Y., Duan, Z., Wen, G., & Zhang, Y. (2013). Distributed finite-time tracking control for multi-agent systems: an observer-based approach. *Systems & Control Letters*, *62*(1), 22–28.

Zuo, Z., & Tie, L. (2014a). A new class of finite-time nonlinear consensus protocols for multi-agent systems. *International Journal of Control*, *87*(2), 363–370.

Zuo, Z., & Tie, L. (2014b). Distributed robust finite-time nonlinear consensus protocols for multi-agent systems. *International Journal of Systems Science*, Published online.