51st IEEE Conference on Decision and Control
December 10-13, 2012. Maui, Hawaii, USA

Nonlinear Fixed-Time Control Protocol for Uniform Allocation of Agents on a Segment

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Abstract— **The paper addresses the problem of row straightening of agents via local interactions. A nonlinear control protocol that ensures finite-time equidistant allocation on a segment is proposed. With the designed protocol, any settling time can be guaranteed regardless of the initial conditions. A robust modification of the control algorithm based on sliding mode control technique is presented. The case of multidimensional agents is also considered. The theoretical results are illustrated via numerical simulations.**

I. Introduction

In the last two decades, various problems related to multi-agent and networked nature of systems are in the focus of the control community, e.g., see [18], [26], [24]. This interest is motivated by numerous real-life applications; it turns out that use of simple rules of local communication in the absence of centralized control may lead to many attractive features such as robustness, adaptivity, and flexibility. Multi-agent systems are usually high-order systems due to a large number of cooperating agents.

The problem of formation of geometric patterns is of particular interest; it has numerous engineering applications in the area of mobile robot control, spacecraft flying formations, UUVs, UAVs, etc. One of the typical problems of formation control is row straightening of agents via local interactions [29], [16], [27]. The averaging principle underlying the corresponding control protocol was first introduced as early as in 1878 by Jean Gaston Darboux in [9]. This principle suggests that each agent in the group tends to locate itself toward the middle of the segment connecting its indexed neighbors. The existed row straightening protocols provide only *asymptotic* convergence property. Moreover, the convergence (settling) time essentially depends on the initial positions of the agents. On the other hand, practical considerations typically require that the time of the transient process be predefined.

The *main objective* of this paper is to design a new row straightening protocol which provides the system with the *finite-time* convergence property; on top of that, *any guaranteed settling time can be specified in advance regardless of the initial positions of the agents*. It is also desired that this protocol be *robust* against bounded exogenous disturbances affecting the agents.

Since recently, finite-time stability and stabilization problems have been a subject of intensive research; e.g., see [13], [5], [17], [20]. The control theory encounters many systems that exhibit finite-time convergence to the equilibrium. Often, such systems appear in observation problems when finite-time convergence of the observed states to the real ones is required [3]. High order sliding mode control algorithms also ensure finite-time convergence to the origin [28], [15], [19], [21]; typically, the associated controllers have mechanical and electromechanical applications [2], [11], [6]. For use of finite-time control ideology in multi-agent systems see [7], [14], [30], [31]. Finite-time stability analysis usually exploits the theory of non-smooth Lyapunov functions and involves such concepts as weak and strong stability, differential inclusions, generalized gradients and derivatives [25], [14].

The *finite-time stability* concept was then extended to pave the road to the design of algorithms that guarantee any predefined settling time *regardless of the initial conditions*; these results are presented in [8], [22]. The corresponding property is referred to as *fixed-time stability*. Robust fixed-time algorithms based on sliding mode control technique can be found in [23].

Usually the finite-time stability is closely related to the homogeneity property of the system. If the system is asymptotically stable and homogeneous of negative degree, the system is shown to attain the equilibrium point in finite time [15], [19]. The concept of *homogeneity in bi-limit* introduced in [1] generalizes this property by ensuring that an asymptotically stable system is *fixed-time stable* if it is homogeneous of negative degree in 0-limit and homogeneous of positive degree in ∞-limit. Unfortunately, with the homogeneity approach, the settling time cannot be specified in advance or even estimated.

Control laws that provide a system with the fixed-time stability property are typically of polynomial-like form [22], [23]. Polynomial state feedback control systems have attracted considerable attention in nonlinear control [10]. This class of control systems appears in models of a wide range of applications such as chemical processes, electronic circuits and mechatronics, biological systems, etc.

This paper presents new control protocols of a polynomial type which guarantee fixed-time equidistant allocation of agents on a segment and demonstrate the robustness in the presence of bounded exogenous disturbances.

The paper is organized as follows. Section II presents the statement of the problem and basic assumptions. Preliminary facts related to the row straightening problem and the fixed-time stability concept are discussed in Section III. The nonlinear fixed-time control protocol together with its robust modification and a multidimensional generalization are given in Section IV. In the last two sections, we present the results of numerical simulations and brief concluding remarks.

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II. PROBLEM STATEMENT

We consider a group of n numbered mobile agents. Let their positions at time $t \geq 0$ be denoted by $x_i(t) \in \mathbb{R}$, $i = 1,2,...,n$, and let x_0, x_{n+1} denote the fixed endpoints of the segment. The dynamical model of each agent is described by a simple integrator:

$$\dot{x}_i = u_i + d_i(t,x), \quad i = 1,2,\ldots,n, \tag{1}$$

where $u_i \in \mathbb{R}$ is the control input, $x = [x_1, x_2, \ldots, x_n]^\top$, $d_i(t,x)$ is a bounded exogenous disturbance

$$|d_i(t,x)| \leq d_{\max}, \tag{2}$$

and the nonnegative number $d_{\max}$ is assumed to be given.

The objective in this paper is to design a feedback control protocol which:

- guarantees the equidistant allocation of the agents on the segment in a fixed time for any initial conditions;
- exploits the information only about the distances between each of the agents and its two indexed neighbors so that

$$u_i = u_i(x_{i-1} - x_i, x_{i+1} - x_i), \quad i = 1,\ldots,n, \tag{3}$$

- is robust against bounded exogenous disturbances.

The extension of the protocol to the case of multidimensional agents $x \in \mathbb{R}^m$ is also of interest.

III. PRELIMINARIES

A. Uniform allocation on a segment: The linear approach

Consider the following linear control protocol devised in [29], [27]:

$$u_i = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i), \quad i = 1,\ldots,n. \tag{4}$$

To show the idea of the protocol, we rewrite it in the form $u_i = \frac{x_{i+1}+x_{i-1}}{2} - x_i$, $i = 1,\ldots,n$, which is interpreted to mean that each agent in the group tends to locate itself toward the middle of the segment that links its indexed neighbors.

Let $x = [x_1, x_2, \ldots, x_n]^\top$ be the state vector of the multi-agent system; then the dynamics of the overall system can be written in compact form as

$$\dot{x} = Ax + b, \tag{5}$$

where the matrix A and the vector b are of the form

$$A = \begin{bmatrix} -1 & 0.5 & 0 & \ldots & 0 \\ 0.5 & -1 & 0.5 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{6}$$

$$b = [0.5x_0, 0, \ldots, 0.5x_{n+1}]^\top \in \mathbb{R}^n. \tag{7}$$

The matrix A is a tridiagonal matrix having eigenvalues [4]

$$\lambda_k = -2\sin^2\frac{k\pi}{2(n+1)}, \quad k = 1,2,...,n. \tag{8}$$

Since $\lambda_k < 0, k = 1,2,...,n$, the state $x^* \in \mathbb{R}^n : x^* = -A^{-1}b$ is a stable equilibrium point of system (5), so that as $t \to \infty$, we have $x_i \to x_0 + \frac{i}{n+1}(x_0 - x_{n+1})$, $i = 1,2,\ldots,n$. This exactly means that, no matter what the initial conditions are, the agents tend to locate themselves uniformly on the segment with the fixed endpoints x_0 and x_{n+1}.

It can be shown that the following estimate

$$\|x(t) - x^*\| \leq e^{\hat{\lambda}}\|x(0) - x^*\|$$

is true for system (5), where $x(0)$ is the vector of the initial positions and $\hat{\lambda}$ is of the form

$$\hat{\lambda} = \max_k \lambda_k = -2\sin^2\frac{\pi}{2(n+1)}. \tag{9}$$

Obviously, the control protocol (4) satisfies condition (3) but it provides only asymptotic convergence, i.e., the agents *will not* attain the final positions on the segment in any finite time. In this paper, we develop a nonlinear control protocol that guarantees convergence in a prescribed finite time; moreover, the obtained upper bound on the settling time does not depend on initial conditions of agents.

Below we consider some helpful notions, definitions and auxiliary lemmas needed for further discussion.

B. Fixed-time convergence

Consider the following system:

$$\dot{z} = g(t,z), \quad z(0) = z_0, \tag{10}$$

where $z \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a possibly discontinuous nonlinear function. In this case, the solutions of (10) are understood in the sense of Filippov [12]. Assume that the origin is an equilibrium point of system (10).

Definition 1 ([5]): The origin is said to be a *globally finite-time stable* equilibrium point for system (10) if it is globally asymptotically stable and any solution $z(t,z_0)$ of (10) attains it in finite time, i.e., $z(t,z_0) = 0 \forall t \geq T(z_0)$, where $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

The finite-time stability property is typical, e.g., to homogeneous systems with negative degree [15], [19]. For example, any solution of the system $\dot{z} = -z^{\frac{1}{3}}, z \in \mathbb{R}$ converges to the origin in finite time $T(z_0) := \frac{2}{3}\sqrt[3]{|z_0|^2}$.

Definition 2 ([22]): The origin is said to be a *fixed-time stable* equilibrium point of system (10) if it is globally finite-time stable and the settling-time function $T(z_0)$ is bounded, i.e., there exists $T_{\max} > 0\colon\ T(z_0) \leq T_{\max}\ \forall z_0 \in \mathbb{R}^n$.

For the system $\dot{z} = -z^{\frac{1}{3}} - z^3, z \in \mathbb{R}$, the origin is fixed-time stable, since it is globally finite-time stable and $z(t,z_0) = 0$ for $\forall t \geq 2.5$ and $\forall z_0 \in \mathbb{R}$.

Denote by $D^*\varphi(t)$ the upper right-hand derivative of the function $\varphi(t)$, $D^*\varphi(t) := \lim_{h\to+0}\sup\frac{\varphi(t+h)-\varphi(t)}{h}$.

Lemma 1 ([22]): If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

1) $V(z) = 0 \Leftrightarrow z = 0$;
2) any solution $z(t)$ of (10) satisfies the inequality $D^*V(z(t)) \leq -(\alpha V^p(z(t)) + \beta V^q(z(t)))^k$ for some $\alpha, \beta, p, q, k > 0$: $pk < 1$, $qk > 1$,

then the origin is globally fixed-time stable for system (10) and the following estimate holds:

$$T(z_0) \le \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)} \quad \forall z_0 \in \mathbb{R}^n.$$

This lemma together with its refinement (Lemma 2), which will be presented shortly, are the cornerstones for the design of the nonlinear fixed-time control protocol aimed at allocating the agents equidistantly on the segment. This is the subject of the rest of the paper.

IV. Nonlinear Fixed-time Control Protocol

A. Disturbance-free Case

We first concentrate on the case $d_i(t,x_i) \equiv 0$. Prior to designing the nonlinear control protocol, let us introduce the function $\phi(s), s \in \mathbb{R}$ defined by

$$\phi(s) := \alpha s^{[p]} + \beta s^{[q]}, \quad 0 < p < 1, \quad q > 1, \tag{11}$$

where α, β are some positive constants and

$$s^{[k]} := \operatorname{sign}(s)|s|^k. \tag{12}$$

Consider now the following nonlinear control law for each individual agent:

$$u_i = \phi\left(\frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i)\right). \tag{13}$$

Then the overall dynamics of the compound nonlinear system of n agents has the form

$$\dot{x} = \bar{\phi}(Ax + b), \tag{14}$$

where the matrix A and the vector b are defined by (6) and (7), and the vector-valued function $\bar{\phi}$ is given by

$$\bar{\phi}(z) := [\phi(z_1), \phi(z_2), \dots, \phi(z_n)]^\top, \\ z = [z_1, z_2, \dots, z_n]^\top \in \mathbb{R}^n.$$

Theorem 1: Assume $d_i(t,x_i) \equiv 0$ and let the control protocol u_i be defined by (13) with $\alpha > 0$, $\beta > 0$, $0 < p < 1$, $q > 1$. Then the agents of the multi-agent system (1) allocate equidistantly on the segment in a *fixed time* and the settling time function is globally (over all initial conditions) bounded by $T_{\max}$ given by

$$T_{\max} := \frac{2}{\alpha(1-p)(2|\hat{\lambda}|)^{\frac{p+1}{2}}} + \frac{2n^{\frac{q-1}{2}}}{\beta(q-1)(2|\hat{\lambda}|)^{\frac{q+1}{2}}}, \tag{15}$$

where $\hat{\lambda}$ is defined in (9).

Proof: Introduce the new variable

$$z = Ax + b$$

for simplicity of analysis. Clearly, we have

$$\dot{z} = A\bar{\phi}(z), \tag{16}$$

and the origin is an equilibrium point for system (16). Consider now the following Lyapunov function candidate:

$$V(z) = \frac{1}{2} z^\top P z \tag{17}$$

with the matrix P of the form $P = -A^{-1}$. It is easy to see that P is a positive-definite matrix, since all eigenvalues of A are negative, see (8). We then compute the derivative of V along the trajectories of the system:

$$\dot{V}(z) = \frac{1}{2}(\dot{z}^\top P z + z^\top P \dot{z}) = \frac{1}{2}(-\phi^\top(z) z - z^\top \phi(z)).$$

Taking into account (12) and

$$z_i = \operatorname{sign}(z_i)|z_i|,$$

we easily obtain

$$\begin{aligned} \dot{V} &= -\alpha \sum_{i=1}^n z_i \cdot z_i^{[p]} - \beta \sum_{i=1}^n z_i \cdot z_i^{[q]} \\ &= -\alpha \sum_{i=1}^n |z_i|^{p+1} - \beta \sum_{i=1}^n |z_i|^{q+1}. \end{aligned}$$

Hence, we conclude that the origin of the system (16) is stable. Obviously, for $z = 0$, the equality $x = -A^{-1}b$ holds, or equivalently, $x_i = x_0 + \frac{i}{n+1}(x_0 - x_{n+1})$, $i = 1, 2, \dots, n$. This shows that the proposed nonlinear control protocol solves the problem of equidistant allocation of agents on the segment $[x_0,\ x_{n+1}]$.

To obtain an estimate for the settling time we use the inequality

$$V \le \frac{1}{2}\lambda_{\max}(P)\|z\|_2^2$$

for the Lyapunov function (17) and the norm equivalence property, namely,

$$\|z\|_l \le \|z\|_r \le n^{\frac{1}{r}-\frac{1}{l}}\|z\|_l$$

for any $z \in \mathbb{R}^n$ and $l > r > 0$.

Then the inequality $\|z\|_2 \le \|z\|_{p+1}$ implies

$$\sum_{i=1}^n |z_i|^{p+1} \ge (\|z\|_2^2)^{\frac{p+1}{2}} \ge \left(\frac{2V}{\lambda_{\max}(P)}\right)^{\frac{p+1}{2}},$$

where $\lambda_{\max}(P)$ denotes the maximum eigenvalue of the matrix P. In the same way we derive

$$\sum_{i=1}^n |z_i|^{q+1} \ge n^{\frac{1-q}{2}} (\|z\|_2^2)^{\frac{q+1}{2}} \ge n^{\frac{1-q}{2}} \left(\frac{2V}{\lambda_{\max}(P)}\right)^{\frac{q+1}{2}}.$$

Hence for the total derivative of the Lyapunov function V we have the following estimate:

$$\dot{V} = -\alpha \sum_{i=1}^n |z_i|^{p+1} - \beta \sum_{i=1}^n |z_i|^{q+1} \le -\bar{\alpha} V^{\frac{p+1}{2}} - \bar{\beta} V^{\frac{q+1}{2}}, \tag{18}$$

where

$$\bar{\alpha} := \alpha \left(\frac{2}{\lambda_{\max}(P)}\right)^{\frac{p+1}{2}}, \quad \bar{\beta} := \beta n^{\frac{1-q}{2}} \left(\frac{2}{\lambda_{\max}(P)}\right)^{\frac{q+1}{2}}.$$

Therefore we showed that the Lyapunov function (17) satisfies the conditions of Lemma 1, and the following settling time estimate can be found as

$$T_{\max} := \frac{2}{\bar{\alpha}(1-p)} + \frac{2}{\bar{\beta}(q-1)}. \tag{19}$$

Taking into account

$$\lambda_{\max}(P) = \lambda_{\max}(-A^{-1}) = |\hat{\lambda}|^{-1}$$

and substituting $\bar{\alpha}$ and $\bar{\beta}$ in (19), we arrive at (15). ■

The theorem presents quite a conservative settling time estimate, since its proof is based on the results of Lemma 1. A more accurate estimate can be derived with the use of the lemma below.

Specifically, consider a case where the constants p and q are of the form $p = 1 - \frac{1}{2\mu}$ and $q = 1 + \frac{1}{2\mu}$, $\mu > 1$.

Lemma 2: If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

1) $V(z) = 0 \Leftrightarrow z = 0$;
2) any solution $z(t)$ of (10) satisfies the inequality $D^* V(z(t)) \leq -\alpha V^p(z(t)) - \beta V^q(z(t))$ for some $\alpha, \beta > 0$, $p = 1 - \frac{1}{2\mu}$, $q = 1 + \frac{1}{2\mu}$, $\mu > 1$,

then the origin is globally fixed-time stable for system (10) and the following estimate of the settling time function holds:

$$T(z_0) \leq T_{\max} := \frac{\pi\mu}{\sqrt{\alpha\beta}}, \forall z_0 \in \mathbb{R}^n.$$

Proof: Consider an auxiliary differential equation of the form

$$\dot{y} = -\alpha y^{1-\frac{1}{2\mu}} - \beta y^{1+\frac{1}{2\mu}}, \quad y \geq 0, \ \alpha, \beta > 0, \ \mu > 1, \quad (20)$$

with initial conditions $y_0 = y(0) \geq 0$. Obviously, $y = 0$ is the equilibrium point of this system.

Using separation of variables, rewrite equation (20) in the form

$$t = -\int \frac{dy}{\alpha y^{1-\frac{1}{2\mu}} + \beta y^{1+\frac{1}{2\mu}}}.$$

Change of variables $w = y^{\frac{1}{2\mu}}$, or equivalently, $y = w^{2\mu}$ gives

$$t = -2\mu \int \frac{w^{2\mu-1} dw}{\alpha w^{2\mu-1} + \beta w^{2\mu+1}} = -2\mu \int \frac{dw}{\alpha + \beta w^2}.$$

Hence, the solution of (20) for $t \geq 0$ has the form

$$\frac{2\mu}{\sqrt{\alpha\beta}} \arctan\left(\sqrt{\frac{\beta}{\alpha}} y^{\frac{1}{2\mu}}(t)\right) = -t + C_0,$$

where

$$C_0 = \frac{2\mu}{\sqrt{\alpha\beta}} \arctan\left(\sqrt{\frac{\beta}{\alpha}} y_0^{\frac{1}{2\mu}} > 0\right).$$

For $t = C_0$ we have $y(t) = 0$, so that any solution of (20) reaches the equilibrium in finite time. Moreover, since arctangent is a bounded function, the maximum settling time can be estimated by

$$T_{\max} = \frac{\pi\mu}{\sqrt{\alpha\beta}}.$$

This implies the settling time estimate for the Lyapunov function $V(z(t))$. ■

With this lemma, a less conservative estimate of the settling time can be obtained.

Corollary 1: If, under the conditions of Theorem 1, the constants p and q of system (14) are chosen as $p = 1 - \frac{1}{\mu}$ and $q = 1 + \frac{1}{\mu}$, $\mu > 1$, then the settling time estimate can be found as

$$T_{\max} := \frac{\pi\mu n^{\frac{1}{4\mu}}}{2|\hat{\lambda}|\sqrt{\alpha\beta}}. \quad (21)$$

The proof of this corollary immediately follows from inequality (18) and Lemma 2.

Estimate (21) yields an extremely important conclusion; namely, the system can be forced to have any *a priori* specified settling time by properly choosing the parameters μ, α, and β.

B. Robustification of Control Algorithm

To provide the presented control protocol with robustness property against bounded disturbances, we introduce the following simple modification of the function ϕ:

$$\varphi(s) := \alpha s^{[p]} + \beta s^{[q]} + d_{\max} \operatorname{sign}(s),$$

where $0 < p < 1, q > 1, \alpha > 0, \beta > 0$.

Corollary 2: Let the control protocol u_i be defined by

$$u_i = \varphi\left(\frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i)\right). \quad (22)$$

Then Theorem 1 and Corollary 1 remain valid in the presence of bounded disturbances (2), $d_i(t,x) \not\equiv 0$.

Proof: Similarly to the proof of Theorem 1 we introduce the variable $z = Ax + b$ and the Lyapunov function candidate $V(z) = 0.5 z^\top(-A^{-1})z$. The total derivative of V calculated along the trajectories of system (1) has the form

$$\dot{V} = -\alpha \sum_{i=1}^n |z_i|^{p+1} - \beta \sum_{i=1}^n |z_i|^{q+1} - d_{\max} \sum_{i=1}^n |z_i| - \sum_{i=1}^n z_i d_i$$

Taking into account $|d_i(t,x)| \leq d_{\max}$, we obtain

$$\dot{V} \leq -\alpha \sum_{i=1}^n |z_i|^{p+1} - \beta \sum_{i=1}^n |z_i|^{q+1}$$

All other considerations repeat the proof of Theorem 1. ■

C. A Generalization to the Multidimensional Case

So far, the individual dynamics of an agent was described by a scalar differential equation. We now discuss a generalization of the proposed control protocol to the case where the state of each agent is multidimensional; for brief, we deal with "multidimensional agents." In this setting, a coordinate-wise analysis is valid, so that the overall dynamics of the system can be written as

$$\dot{x} = \bar{\phi}\left((A \otimes I_m)x + \hat{b}\right), \quad (23)$$

where $x = \left[x_1^\top, x_2^\top, \ldots, x_n^\top\right]^\top \in \mathbb{R}^{nm}$, the symbol $\otimes$ denotes the Kronecker product, $\hat{b} = \left[0.5 x_0^\top, 0, \ldots, 0.5 x_{n+1}^\top\right]^\top \in \mathbb{R}^{nm}$, and $\bar{\phi}$ is the vector-valued function

$$\bar{\phi}(z) := [\phi(z_1), \phi(z_2), \ldots, \phi(z_{nm})]^\top,$$
$$z = [z_1, z_2, \ldots, z_{nm}]^\top \in \mathbb{R}^{nm},$$

with ϕ defined by (11).

Due to the properties of the Kronecker product, the matrix $A \otimes I_m$ is positive definite and its eigenvalues are of the form (8), hence, all the previous results remain valid. Moreover, since system (23) can be considered as an aggregate of m independent subsystems having the same settling time estimate, relations (15) and (21) hold no matter what m is.

V. Numerical Examples

To demonstrate the efficiency of the proposed fixed-time control protocol we consider the multi-agent system (1) with the following parameters:

$$n = 3, \quad x_0 = 0, \quad x_{n+1} = 16, \quad d_i(\cdot,\cdot) = 2\sin(5t).$$

For the same initial conditions

$$x_1(0) = -5, \quad x_2(0) = -5\sqrt{2}, \quad x_3(0) = -5,$$

Fig. 1 presents the results of simulations for the linear and the proposed nonlinear control protocols in the disturbance-free case ($d_i \equiv 0$); the values of the parameters of the fixed-time control law (13) were chosen as

$$\alpha = 1, \quad \beta = 1, \quad p = 1 - \frac{1}{\mu}, \quad q = 1 + \frac{1}{\mu}, \quad \mu = 2.$$

The results obtained confirm the theoretical conclusions

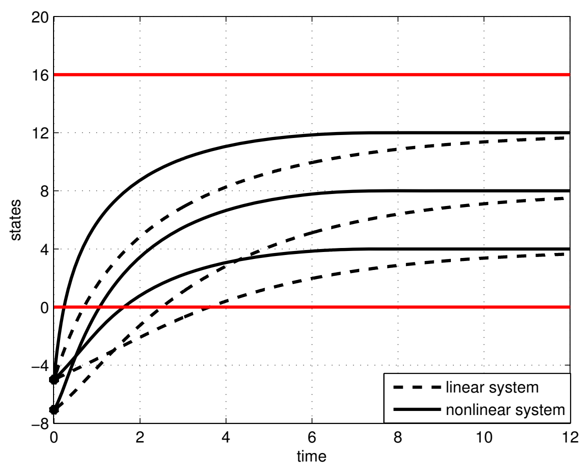

Fig. 1. Trajectories of the system subject to the linear and nonlinear protocols (disturbance-free case)

of Theorem 1 showing finite-time equidistant allocation of agents on a segment in the nonlinear case. The settling time is seen to be less than the theoretical estimate

$$T_{\max} = \frac{\pi \mu n^{\frac{1}{4\mu}}}{2|\hat{\lambda}|\sqrt{\alpha\beta}} = \frac{\pi n^{\frac{1}{8}}}{|\hat{\lambda}|} \approx 12.3 \tag{24}$$

of Corollary 1.

Figure 2 depicts the settling time as function of the distance between the initial position and the equilibrium point $x^* = [4, 8, 12]^\top$. The settling time remains bounded even for large initial conditions, testifying to the fixed-time nature of the developed control protocol. It is seen that estimate (24) of the settling time is of moderate conservatism. Notably, for a large number of agents, $n > 100$, calculations show that the estimate obtained in Corollary 1 is twice less conservative than estimate (15).

We make the following important comment at this point. Clearly, the nonlinear fixed-time protocol might require higher control resources, since it forces a system to enter

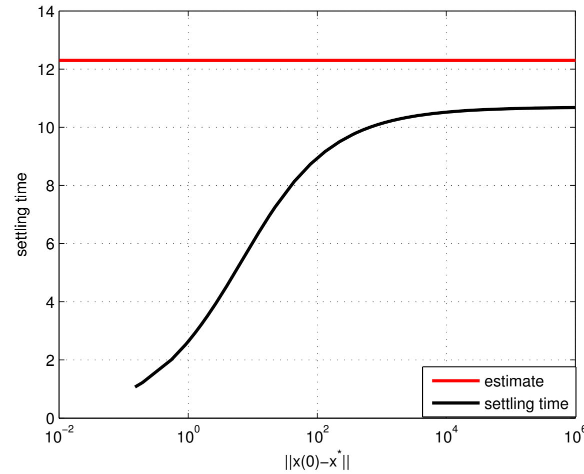

Fig. 2. Settling time as function of the distance between the initial position and the equilibrium

an ε-neighborhood of the equilibrium much faster than the linear control does. By way of comparison, Fig. 3 depicts the control signals for both the linear and nonlinear protocols. It is seen that in the nonlinear case, the peak magnitudes of u are much greater than those in the linear case. There is nothing surprising in this observation: A smaller settling time for the nonlinear algorithm is obtained at the expense of applying a more powerful control. On the other hand,

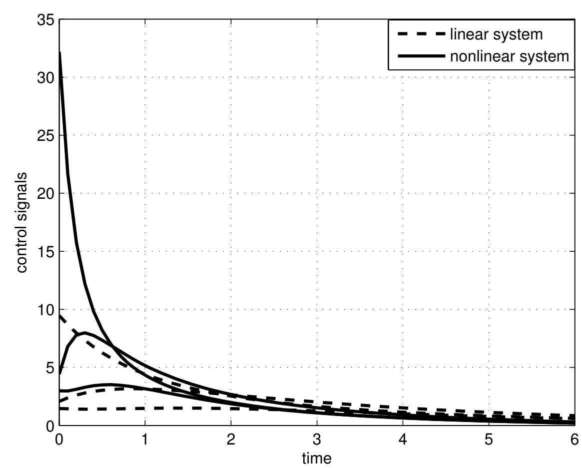

Fig. 3. Magnitudes of control signals for linear and nonlinear systems

computing the *energy* of component-wise control signals

$$E(u_i(\cdot)) = \int_0^{T(\varepsilon)} |u_i(\tau)| d\tau$$

associated with this kind of behavior for the two types of control, we obtain the nearly coinciding values

$$\begin{array}{ll} E(u_1^{lin}(\cdot)) \approx 8.9599 & E(u_1^{nonlin}(\cdot)) \approx 8.9785 \\ E(u_2^{lin}(\cdot)) \approx 15.0159 & E(u_2^{nonlin}(\cdot)) \approx 15.0142 \\ E(u_3^{lin}(\cdot)) \approx 16.9411 & E(u_3^{nonlin}(\cdot)) \approx 17.0939 \end{array}$$

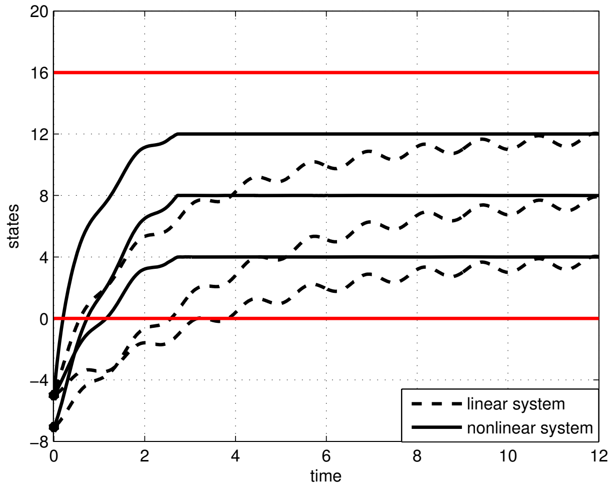

Fig. 4. Comparison of the linear and nonlinear protocols (disturbed case)

for $\varepsilon = 0.03$, $T^{lin}(\varepsilon) \approx 15$ and $T^{nonlin}(\varepsilon) \approx 6$.

The simulation results for the disturbed case are depicted in Fig. 4. They demonstrate the robustness of the nonlinear control protocol (22) against exogenous disturbances bounded by $d_{\max} = 2$. Fixed-time stability property holds.

VI. Conclusions

The contribution of the paper is the following:

- a nonlinear control protocol for finite-time equidistant allocation of agents is developed;
- it is proved that the guaranteed settling time of the system can be specified in advance regardless of the initial positions of the agents (fixed-time convergence);
- to provide the system with robustness against bounded exogenous disturbances, a sliding mode based modification of the control protocol is presented;
- a generalization of the proposed nonlinear control protocol to the case of multidimensional agents is given;
- the prespecified settling time is shown to be independent of the dimension of the agents.

The theoretical results were successfully tested through several numerical experiments.

The fixed-time stability framework developed in the paper looks promising in applications to other problems related to multi-agent systems and formation control. Of a special interest is the extension of the proposed algorithms to consensus-like problems. But this is another story.

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