Modelling of Ball and Plate System Based on First Principle Model and Optimal Control

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***Abstract*—This paper presents modelling of ball and plate systems based on first principles by considering balance of forces and torques. A non-linear model is derived considering the dynamics of motors, gears, ball and plate. The non-linear model is linearized near the operating region to obtain a standard state space model. This linear model is used for discrete optimal control of the ball and plate system – the trajectory of the ball is controlled by control voltages to the motor.**

***Keywords*—Ball and plate system; first principle model; optimal control; LQ control**

I. Introduction

Laboratory experiments are an integral part of control education. There are lots of educational platforms available e.g. inverted pendulum, magnetic levitation, ball and beam system etc. Ball and plate system is an upgraded version of ball and beam system where the position of the ball can be manipulated in two directions [1]. The educational model ball and plate, consists of a plate, pivoted at its center, such that the slope of the plate can be manipulated in two perpendicular directions. The ball and plate is a non-linear, multi-variable and open-loop unstable system. There are basically two control problems: point stabilization and trajectory tracking. In point stabilization, the aim is to carry the ball to a specific position and hold it there. In trajectory tracking control, the goal is to make the ball follows a predefined trajectory (linear, square, circle and Lissajous curve) [2-5].

The first step is finding out a mathematical model which describes the system. There are basically two modelling approaches for the ball and plate system in literature: the Lagrangian method and the Newton-Euler method. The derivation of dynamical equations of ball and plate system by Lagrangian method can be seen in [6]. The modelling based on Newton-Euler method is quite rare in the literature. Even though, Newton-Euler method is quite cumbersome, the variables and equations have physical meaning which is suitable for control educational purpose. The balance of forces and torques are considered in the Newton-Euler method to derive the mathematical model.

In this paper, a non-linear mathematical model of ball and plate system is derived by considering the dynamics of the ball and plate system, DC motors and gear system, based on balance of forces and torques. The model is linearized around equilibrium points to arrive at a linear state space model. Infinite horizon optimal linear quadratic (LQ) control is applied to the trajectory tracking problem by penalizing the state and control effort. Simulation results of model verification and trajectory tracking control are also provided.

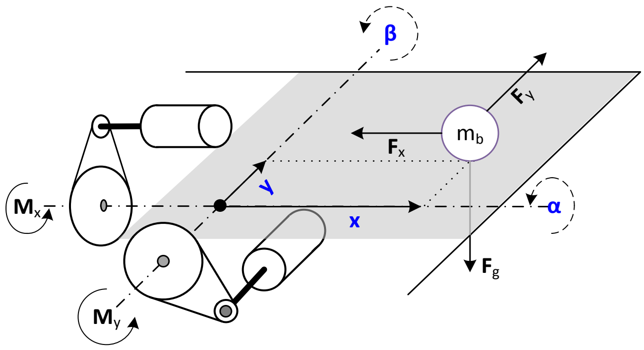

Fig. 1. Ball and plate system

II. Mathematical Modelling

Mathematical model of nonlinear dynamics takes account of the position of the ball on the plate depending on the voltage of the motors that control the tilt of the plate, in two perpendicular axes (see Fig. 1). A ball of mass m_b, moment of inertia J_b and radius r_b is located on a square surface (plane), tilting in the two perpendicular axes x and y. The origin of the axis is located at the intersection of coordinate axes. The moment of inertia of the plate is J_p (relative to each axis of rotation). On each axis, the torques M_x and M_y are operated. Moments are created by two DC geared motors via two cable systems with the same gear ratio of G, but with different moments of inertia J_{Gx} and J_{Gy}.

The mathematical model is based on the balance of forces and torques acting on the ball, and the dynamic model of DC series motor. The real behaviour is taken into account by including an approximation of linear mechanical losses, depending on the speed of the rotational motion. In the case of moving balls, the mechanical losses are proportional to the square of opposition translational speed of movement.

The model is built on the following assumptions:

a) there is no loss of contact area with the ball
b) the ball is hollow ball (ping-pong ball)
c) an infinitely large area (not considering the rebound)
d) connection of the motor to tilt the axis is perfectly rigid

A. *Balance of forces - ball*

Since the ball is placed on a surface rotating in two axes, we need to consider, in addition to the inertia of the translational movement, the influence of the apparent forces (Euler, centrifugal and Coriolis). These forces are caused by the rotational movement i.e. all the forces in curvilinear motion consisting of the movement of translation and rotation. A general representation of the resultant forces acting upon the curvilinear motion on the mass point in vector form ($\vec{v}$ is the translational velocity vector and $\vec{\omega}$ is the angular velocity vector) is then described [7],

$$\vec{F}_{ext} = \underbrace{m\frac{d\vec{v}}{dt}}_{Inerita\ force\ \vec{F}_{acc}} + \underbrace{m\frac{d\vec{\omega}}{dt}\times r}_{Euler\ force\ \vec{F}_{Eul}} + \underbrace{2m\vec{\omega}\times\frac{dr}{dt}}_{Coriolis\ force\ \vec{F}_{Cor}} + \underbrace{m\vec{\omega}\times(\vec{\omega}\times r)}_{Centrifugal\ force\ \vec{F}_{cen}}. \qquad (1)$$

All the apparent forces acting in a plane are perpendicular to the axis of rotation. Fig. 2 shows a situation where all the forces are acting in the plane of motion. The mass point is at a perpendicular distance r from the rotational axis, and the vector (line connecting the point and the axis of rotation) forms an angle α from the selected x axis. The Euler force F_{Eul} (also apparent inertia force) acts only when there is a change in the speed of rotation ω and its direction is perpendicular to the vector. Centrifugal force F_{cen} acts at non-zero rotation speed and its direction is in the direction of the vector. Coriolis force F_{Cor} acts when the velocity of motion is not perpendicular to the vector (i.e. there is a change in the size of the vector). Its direction is perpendicular to the direction of translational speed.

In the arrangement (rotational axis in the coordinate axes), the forces are decomposed in the directions of axes x and y directions. The translational velocity v_x, v_y and rotation speed ω_α and ω_β are either parallel or perpendicular to each other. Decomposition of the axes describes the movement in one direction only i.e. the apparent Euler and Coriolis force is applied only in the balance of moments. The situation of moving in the x-axis is shown in Fig. 3. Since it is not a fixed point we need to consider more ball spin (moment of inertia) with speeds of rotation ω_x and ω_y loss due to rolling resistance and environment resistance. Rolling resistance is proportional to the rotational speed of the ball and resistance is proportional to the square of environment translational velocity. The overall balance of forces can be expressed as,

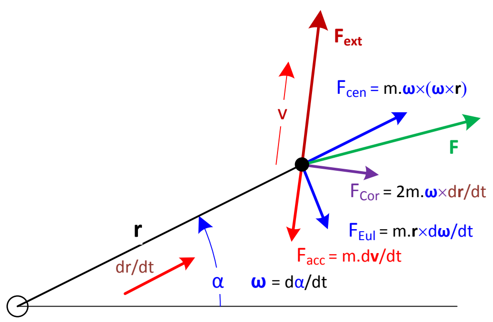

Fig. 2. Forces acting on ball and plate

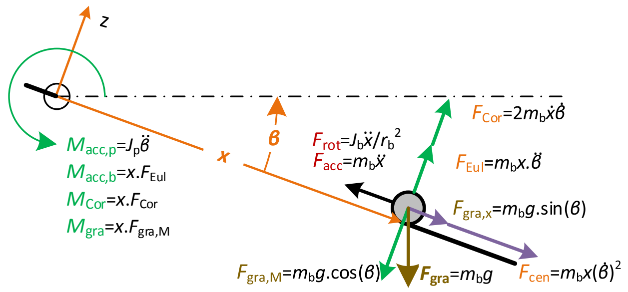

Fig. 3. Ball and plate – movement in the x-axis (x-z plane) and rotation (torque) acting in y-axis

$$F_{acc} + F_{rot} + F_{los} + F_{Eul} + F_{Cor} + F_{cen} = F_{gra}(\alpha,\beta) \quad [kg.m.s^{-2}]$$

Translational force $\quad F_{acc} = m_b\frac{d^2x}{dt^2}$

Rotational force $\quad F_{rot} = \frac{J_b}{r_b}\frac{d\omega_x}{dt} = \frac{J_b}{r_b^2}\frac{d^2x}{dt^2}$

Rolling resistance $\quad F_{los} = k_b\omega_x = \frac{k_b}{r_b}\frac{dx}{dt}$

Environmental resistance $F_{los} = c_x\frac{1}{2}\rho_{air}S_b\frac{dx}{dt}\left|\frac{dx}{dt}\right| = k_c\frac{dx}{dt}\left|\frac{dx}{dt}\right|$

Centrifugal force $\quad F_{cen} = m_b\omega_\beta^2 x = m_b x\left(\frac{d\beta}{dt}\right)^2$

External force (gravity) $\quad F_{gra,x} = -m_b g.\sin(\beta)$

where,

m_b	Mass of the ball
r_b	Radius of the ball
S_b	Total area $S_b = \pi r_b^2$
Δ	Thickness of the ball
J_b	Torque of ball $\quad J_b = \frac{2}{5}m_b(2r_b - \Delta)\Delta$
ω_x, ω_y	Angular velocity x-axis $\frac{dx}{dt} = r_b\omega_x$
k_b	Coefficient approximation rolling losses
c_x=0.5	Coefficient of aerodynamic resistance ball
ρ_{air}=1.2	Density of dry air at 20 ° C and a pressure of 101.4 kPa
$k_c = c_x\rho_{air}\pi r_b^2/2$	Coefficient of resistance of the environment
g=9.81	Gravitational constant

By considering balance of forces on x axis we get,

$$m_b\frac{d^2x}{dt^2} + \frac{J_b}{r_b^2}\frac{d^2x}{dt^2} + \frac{k_b}{r_b}\frac{dx}{dt} + k_c\frac{dx}{dt}\left|\frac{dx}{dt}\right| + m_b x\left(\frac{d\beta}{dt}\right)^2 = = -m_b g.\sin(\beta)$$

$$\left(1+\frac{J_b}{m_b r_b^2}\right)\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(\frac{k_b}{m_b r_b} + \frac{k_c x}{m_b}\left|\frac{dx}{dt}\right|\right) + x\left(\frac{d\beta}{dt}\right)^2 = -g.\sin(\beta) \qquad (2)$$

Similarly, by considering balance of forces on y axis we get,

$$m_b\frac{d^2y}{dt^2} + \frac{J_b}{r_b^2}\frac{d^2y}{dt^2} + \frac{k_b}{r_b}\frac{dy}{dt} + k_c r_b^2\frac{dy}{dt}\left|\frac{dy}{dt}\right| + m_b y\left(\frac{d\alpha}{dt}\right)^2 = = -m_b g.\sin(\alpha)$$

$$\left(1+\frac{J_b}{m_b r_b^2}\right)\frac{d^2y}{dt^2} + \frac{dy}{dt}\left(\frac{k_b}{m_b r_b} + \frac{k_c x}{m_b}\left|\frac{dy}{dt}\right|\right) + y\left(\frac{d\alpha}{dt}\right)^2 = -g.\sin(\alpha) \qquad (3)$$

B. *Balance of moments - plate with ball*

Overall the balance of moments can be expressed as,

$$M_{acc,p} + M_{acc,b} + M_{Cor,b} + M_{los} = M_{mot} - M_{gra} \quad [kg.m^2.s^{-2}]$$

Torque of plate $M_{acc,p} = \left(J_p + J_{Gx}\right)\frac{d^2\alpha}{dt^2}$

Torque of ball $M_{acc,b} = y.F_{Eul} = m_b y^2 \frac{d^2\alpha}{dt^2}$

Coriolis moment $M_{Cor,b} = y.F_{Cor} = 2m_b y \frac{dy}{dt}\frac{d\alpha}{dt}$

Torque losses $M_{los} = k_{px}\frac{d\alpha}{dt}$

Gravitational moment $M_{gra} = -m_b y.g\cos(\alpha)$

where,

J_p	Plate moment of inertia (MI) $J_p = \frac{1}{12}m_p a^2$
m_p	Mass of the plate
a	Pivot length, passes through the center axis of
J_{Gx}, J_{Gy}	Pivot length, passes through the center axis
k_{px}, k_{py}	Coefficient of approximation of rotational losses
x, y	Current position of the ball
M_{Gx}, M_{Gy}	Actual moments of the drives
α, β	Current the angle of the platform according to the respective axes

By considering balance of forces on x axis we get,

$$\left(J_p + J_{Gx}\right)\frac{d^2\alpha}{dt^2} + m_b y^2\frac{d^2\alpha}{dt^2} + 2m_b y\frac{dy}{dt}\frac{d\alpha}{dt} + k_{px}\frac{d\alpha}{dt} = \\ = M_{Gx} - m_b y.g\cos(\alpha)$$

$$\left(\frac{J_p + J_{Gx}}{m_b} + y^2\right)\frac{d^2\alpha}{dt^2} + \left(2y\frac{dy}{dt} + \frac{k_{px}}{m_b}\right)\frac{d\alpha}{dt} = \frac{M_{Gx}}{m_b} - y.g\cos(\alpha) \quad (4)$$

Similarly, by considering balance of torques on y axis we get,

$$\left(J_p + J_{Gy}\right)\frac{d^2\beta}{dt^2} + m_b x^2\frac{d^2\beta}{dt^2} + 2m_b x\frac{dx}{dt}\frac{d\beta}{dt} + k_{py}\frac{d\beta}{dt} = \\ = M_{Gy} - m_b x.g\cos(\beta)$$

$$\left(\frac{J_p + J_{Gy}}{m_b} + x^2\right)\frac{d^2\beta}{dt^2} + \left(2x\frac{dx}{dt} + \frac{k_{py}}{m_b}\right)\frac{d\beta}{dt} = \frac{M_{Gy}}{m_b} - x.g\cos(\beta) \quad (5)$$

C. Gear system

The gear box reduces the angular velocities of motors to output angular velocities with respect to the gear ratio. Similarly, the torques of motors are increased to output torques.

$$\alpha = \frac{1}{G}\varphi_x \qquad \beta = \frac{1}{G}\varphi_y \\ M_{Gx} = G.M_x \qquad M_{Gy} = G.M_x \quad (6)$$

where,

G	Gear transmission ratio of the drive
φ_x, φ_y	Angle of rotation of the rotor
α, β	Angle of rotation of the plate with respect to the relevant axis

D. Balance of energy and moment – motor

An equivalent circuit of an ideal DC series motor, is shown in Fig. 4. It consists of resistance R, inductance L and magnetic field M. Each motor is independently controlled by its own supply voltage U_x, U_y taken from a common voltage source U_0 through control signal u_x, u_y. The rotor generated back electromotive force (EMF) is in reverse polarity and is proportional to the rotor angular velocity. The torque of the motor is proportional to the current i.

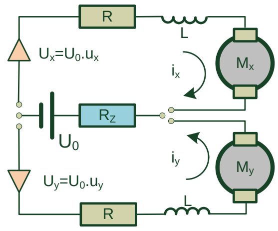

Fig. 4. Equivalent circuit of DC motor

By considering the balance of voltages (Kirchhoff's law) of the motor connected to x axis,

$$R.i_x + L\frac{di_x}{dt} + k_u\Phi\frac{d\varphi_x}{dt} + R_z\left(i_x + i_y\right) = u_x.U_0 \\ L\frac{di_x}{dt} + k_u\Phi.G\frac{d\alpha}{dt} + (R_z + R)i_x + R_z i_y = u_x.U_0 \quad (7)$$

Similarly, by considering balance of voltages on motor connected to y axis,

$$R.i_y + L\frac{di_y}{dt} + k_u\Phi\frac{d\varphi_y}{dt} + R_z\left(i_x + i_y\right) = u_y.U_0 \\ L\frac{di_y}{dt} + k_u\Phi.G\frac{d\beta}{dt} + (R_z + R)i_y + R_z i_x = u_y.U_0 \quad (8)$$

where,

R	Resistance of motor winding
R_z	Internal source resistance
L	Inductance of motor winding
$\Phi = 1$	Magnetic flux constant
k_u	Speed constant (voltage) of the motor
i_x, i_y	Current of the motor

By considering the balance of moments - moment of inertia M_s, rotational resistance proportional to rotational speed (mechanical losses) M_o and load torque M_x caused by magnetic field which is proportional to current.

$$J_m\frac{d^2\varphi_x}{dt^2} + k_o\frac{d\varphi_x}{dt} + M_x = k_m\Phi.i_x$$

Substituting (6) to the above equation gives,

$$\frac{M_{Gx}}{m_b} = \frac{G.k_m\Phi}{m_b}i_x - \frac{J_m G^2}{m_b}\frac{d^2\alpha}{dt^2} - \frac{k_o G^2}{m_b}\frac{d\alpha}{dt} \quad (9)$$

Similarly, considering balance of torques on motor connected to y axis,

$$J_m\frac{d^2\varphi_y}{dt^2} + k_o\frac{d\varphi_y}{dt} + M_y = k_m\Phi.i_y$$

Substituting (6) gives to the above equation gives,

$$\frac{M_{Gy}}{m_b} = \frac{G.k_m\Phi}{m_b}i_y - \frac{J_m G^2}{m_b}\frac{d^2\beta}{dt^2} - \frac{k_o G^2}{m_b}\frac{d\beta}{dt} \quad (10)$$

where,

J_m	Moment of inertia of rotating parts of the motor
k_m	Torque constant of motor
k_o	Coefficient of rotational loss of motor
M_x, M_y	Current load torque of motors
φ_x, φ_y	Current angle of rotors

E. Combined model

The dynamical equations of ball and plate, gears and DC motors, by substituting (9-10) to (4-5), are given by,

$$\left(1+\frac{J_b}{m_b r_b^2}\right)\frac{d^2x}{dt^2}+\frac{dx}{dt}\left(\frac{k_b}{m_b r_b}+\frac{k_{cx}}{m_b}\left|\frac{dx}{dt}\right|\right)+x\left(\frac{d\beta}{dt}\right)^2=-g.\sin(\beta)$$
$$\left(1+\frac{J_b}{m_b r_b^2}\right)\frac{d^2y}{dt^2}+\frac{dy}{dt}\left(\frac{k_b}{m_b r_b}+\frac{k_{cx}}{m_b}\left|\frac{dy}{dt}\right|\right)+y\left(\frac{d\alpha}{dt}\right)^2=-g.\sin(\alpha) \tag{11}$$

$$\left(\frac{J_p+J_{Gx}+J_mG^2}{m_b}+y^2\right)\frac{d^2\alpha}{dt^2}+\left(2y\frac{dy}{dt}+\frac{k_{px}}{m_b}+\frac{k_oG^2}{m_b}\right)\frac{d\alpha}{dt}= =\frac{G.k_m\Phi}{m_b}i_x-y.g\cos(\alpha)$$
$$\left(\frac{J_p+J_{Gy}+J_mG^2}{m_b}+x^2\right)\frac{d^2\beta}{dt^2}+\left(2x\frac{dx}{dt}+\frac{k_{py}}{m_b}+\frac{k_oG^2}{m_b}\right)\frac{d\beta}{dt}= =\frac{G.k_m\Phi}{m_b}i_y-x.g\cos(\beta) \tag{12}$$

$$L\frac{di_x}{dt}+k_u\Phi.G\frac{d\alpha}{dt}+(R_z+R)i_x+R_zi_y=u_x.U_0$$
$$L\frac{di_y}{dt}+k_u\Phi.G\frac{d\beta}{dt}+(R_z+R)i_y+R_zi_x=u_y.U_0 \tag{13}$$

By substituting the following parameters,

$$a_1=1+\frac{J_b}{m_br_b^2}\qquad a_2=\frac{k_b}{m_br_b}\qquad a_3=\frac{k_{cx}}{m_b}$$
$$b_{1x}=\frac{J_p+J_{Gx}+J_mG^2}{m_b}\qquad b_{2x}=\frac{k_{px}}{m_b}+\frac{k_oG^2}{m_b}\qquad b_3=\frac{G.k_m\Phi}{m_b}$$
$$b_{1y}=\frac{J_p+J_{Gy}+J_mG^2}{m_b}\qquad b_{2y}=\frac{k_{py}}{m_b}+\frac{k_oG^2}{m_b}\qquad d_0=\frac{U_0}{L}$$
$$d_1=\frac{R_z}{L}\qquad d_2=\frac{R+R_z}{L}\qquad d_3=\frac{k_u}{L}\Phi.G$$

The dynamics governing the ball and plate system becomes,

$$a_1\frac{d^2x}{dt^2}+\frac{dx}{dt}\left(a_2+a_3\left|\frac{dx}{dt}\right|\right)+x\left(\frac{d\beta}{dt}\right)^2=-g.\sin(\beta) \tag{14}$$
$$a_1\frac{d^2y}{dt^2}+\frac{dy}{dt}\left(a_2+a_3\left|\frac{dy}{dt}\right|\right)+y\left(\frac{d\alpha}{dt}\right)^2=-g.\sin(\alpha) \tag{15}$$

$$(b_{1x}+y^2)\frac{d^2\alpha}{dt^2}+\left(2y\frac{dy}{dt}+b_{2x}\right)\frac{d\alpha}{dt}=b_3i_x-y.g\cos(\alpha) \tag{16}$$
$$(b_{1y}+x^2)\frac{d^2\beta}{dt^2}+\left(2x\frac{dx}{dt}+b_{2y}\right)\frac{d\beta}{dt}=b_3i_y-x.g\cos(\beta) \tag{17}$$
$$\frac{di_x}{dt}+d_3\frac{d\alpha}{dt}+d_2i_x+d_1i_y=d_0u_x \tag{18}$$
$$\frac{di_y}{dt}+d_3\frac{d\beta}{dt}+d_2i_y+d_1i_x=d_0u_y \tag{19}$$

III. Linear State Space Model

The non-linear dynamic equations (14-19) can be linearized around operating points (x_0, y_0) by assuming the following approximation:

1. At small angles of plate inclination: $\sin\theta\approx\theta, \cos\theta\approx 1$
2. At small rate of change and at initial conditions: $v_x=\alpha=\omega_x=i_x=v_y=\beta=\omega_y=i_y=0$

The linearized model can be represented in standard state space model in the form of,

$$\frac{d\bar{\mathbf{x}}}{dt}=\bar{\mathbf{A}}\bar{\mathbf{x}}+\bar{\mathbf{B}}\mathbf{u}$$
$$\mathbf{y}=\bar{\mathbf{C}}\bar{\mathbf{x}} \tag{20}$$

where, $\bar{\mathbf{x}}=[\ x\quad v_x\quad \alpha\quad \omega_x\quad i_x\quad y\quad v_y\quad \beta\quad \omega_y\quad i_y\ ]^T$

$$\bar{\mathbf{B}}=\begin{bmatrix}0&0&0&0&d_0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&d_0\end{bmatrix}^T$$
$$\bar{\mathbf{C}}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\end{bmatrix}$$

TABLE I. PARAMETERS OF BALL AND PLATE SYSTEM

	Symbol	*Unit*	*Value*	*Description*
	g	m.s^{-2}	9.81	Gravitational constant
ball	m_b	kg	0.01	Mass of the ball (ping pong)
	r_b	m	0.02	Radius of the ball
	Δ	m	0.001	Thickness of the ball
	J_b	kg.m^2	$J_b=\frac{2}{5}m_b(2r_b-\Delta)\Delta$	MI of hollow ball
	k_b	kg.m.s^{-1}	0.01	Coefficient of friction (ball) $F=k_b\omega_x$
	c_x	---	0.5	Coefficient of aerodynamic resistance ball
	ρ_{air}	kg.m^{-3}	1.2	Air density
	k_c	kg.m^{-1}	$k_x=\frac{1}{2}c_x\rho_{air}\pi r_b^2$	Approximation of resistance of environment
Plate	m_p	kg	0.4	Mass of plate
	a	m	0. 5	Length of the pivot
	J_p	kg.m^2	$J_p=\frac{1}{12}m_pa^2$	MI of the plate
	k_{px}	kg.m^2.s^{-1}	0.1	Approx of loss (platform x-axis) $M=k_{px}\frac{d\alpha}{dt}$ (estimate)
	k_{py}	kg.m^2.s^{-1}	0.1	Approx of loss (platform y-axis) $M=k_{py}\frac{d\beta}{dt}$ (estimate)
Drive	J_{Gx}	kg.m^2	$J_p/3$	MI of drive (x axis) (estimate)
	J_{Gy}	kg.m^2	$J_p/3$	MI of drive (y axis) (estimate)
	G	---	10	Gear ratio
Src	U_0	V	12	Nominal voltage of motor = Source voltage
	R_z	Ω	0.05	Internal source resistance
Motor	J_m	kg.m^2	45e-7	MI of rotor
	L	H	1.2e-3	Inductance of motor
	ω_0	rad.s^{-1}	4550*π/30	Motor ideal speed
	i_0	A	0.15	No load current of motor
	M_s	kg. m^2.s^{-2}	0.13	Moment of motor at still
	i_s	A	2.5	Current of motor at still
	R	Ω	$R=\frac{U_0}{i_s}$	Winding resistance
	k_m	kg. m^2.s^{-2}.A^{-1}	$k_m\emptyset=\frac{M_s}{i_s}$	Torque constant of motor $M=k_m\emptyset i$
	k_u	V.s.rad^{-1}	$k_u\emptyset=\frac{U_0(i_s-i_0)}{i_s\omega_0}$	Rate constant of motor $U=k_u\emptyset\frac{d\varphi}{dt}=k_u\emptyset\omega$
	k_o	kg. m^2.s^{-1}.rad^{-1}	$k_0=\frac{M_s}{i_s}\frac{i_0}{\omega_0}$	Approximation of loss (motor) $M=k_o\frac{d\varphi}{dt}=k_o\omega$

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a_2}{a_1} & 0 & 0 & 0 & 0 & 0 & -\frac{g}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-b_{2x}}{b_{1x}+y_0^2} & \frac{b_3}{b_{1x}+y_0^2} & \frac{-g(b_{1x}-y_0^2)}{(b_{1x}+y_0^2)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_3 & -d_2 & 0 & 0 & 0 & 0 & -d_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g}{a_1} & 0 & 0 & 0 & -\frac{a_2}{a_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-g(b_{1y}-x_0^2)}{(b_{1y}+x_0^2)^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-b_{2y}}{b_{1y}+x_0^2} & \frac{b_3}{b_{1y}+x_0^2} \\ 0 & 0 & 0 & 0 & -d_1 & 0 & 0 & 0 & -d_3 & -d_2 \end{bmatrix}$$

The model parameters used in the simulation is listed in Table 1. The linearized model, (20) is discretized with a sampling time of $T_s = 0.1s$ and compared with the continuous time dynamic model (15-19) by applying a series of step control voltages. Fig. 5 shows the control voltages and ball positions in x and y axes of linear and non-linear model. Plate angles and motor currents are shown in Fig. 6. Since the system is open loop unstable and has integrating character, the quality of linearization has to be finally checked by closed loop experiments.

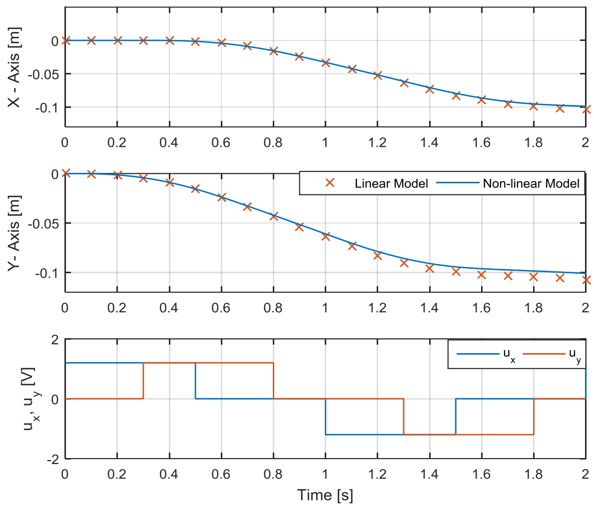

Fig. 5. Non-linear vs Linear model: outputs and inputs in open loop verification

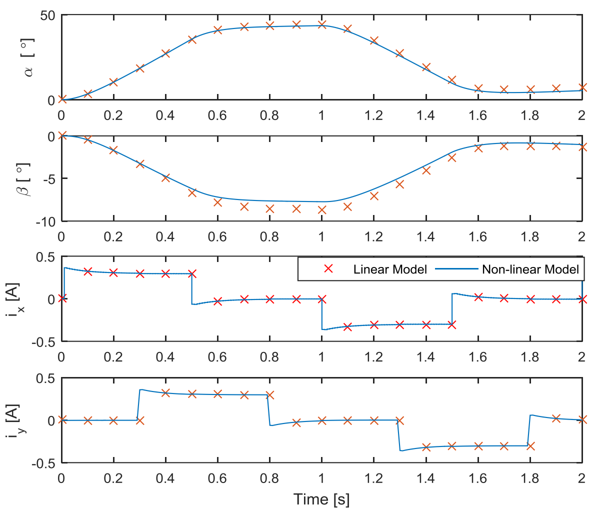

Fig. 6. Plate angles and motor currents in open loop verification

IV. Optimal Control of Ball And Plate System

Discretizing the state space model (20) with a sampling time T_s we get,

$$\begin{aligned} \mathbf{x}(\mathrm{k}+1) &= \mathbf{A}\mathbf{x}(\mathrm{k}) + \mathbf{B}\mathbf{u}(\mathrm{k}) \\ \boldsymbol{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \tag{21}$$

With the linear state space model, an optimal LQ controller can be designed for the ball and plate system. The aim of the controller is generating optimal control voltages by minimizing the following criteria,

$$J_\infty = \sum_{i=1}^{\infty} [\mathbf{x}^T(k+i)\mathbf{Q}.\mathbf{x}(k+i) + \mathbf{u}^T(k+i)\mathbf{R}.\mathbf{u}(k+i)]$$

The cost function consists of penalization (weighting matrix $\mathbf{Q}$) of state variables and control effort (weighting matrix $\mathbf{R}$). If the state variables are able to be estimated, the optimal control actions can be calculated by,

$$\mathbf{u}(k) = -\mathbf{K}[\mathbf{x}(k) - \mathbf{x_w}(k)] \tag{22}$$

Where $\mathbf{x_w}$ is the desired state variable for reference point at time $t = t_k$ and $\mathbf{K}$ is feedback gain matrix obtained by the following equation,

$$\mathbf{K} = (\mathbf{B^T P B} + \mathbf{R})^{-1}\mathbf{B^T P A}$$

The matrix $\mathbf{P}$ is the solution of discrete Riccati equation which is given by,

$$\mathbf{P} = \mathbf{A^T P A} + \mathbf{Q} - \mathbf{A^T P B}(\mathbf{B^T P B} + \mathbf{R})^{-1}\mathbf{B^T P A}$$

In MATLAB, the feedback gain can be obtained by,

```
[K,~,~,]=dlqr(A,B,Q,R)
```

Simulation experiments were conducted on two different trajectories: square shaped and Lissajous curve shaped trajectory. The model parameters used in the simulation are as listed in Table I, with a sampling period of $T_s = 0.1s$. The weighting matrices are chosen as follows,

Q=eye(10)/10 **R**=eye(2)/10
Q(1,1)=100 **Q**(6,6)=100
Q(2,2)=100 **Q**(7,7)=10

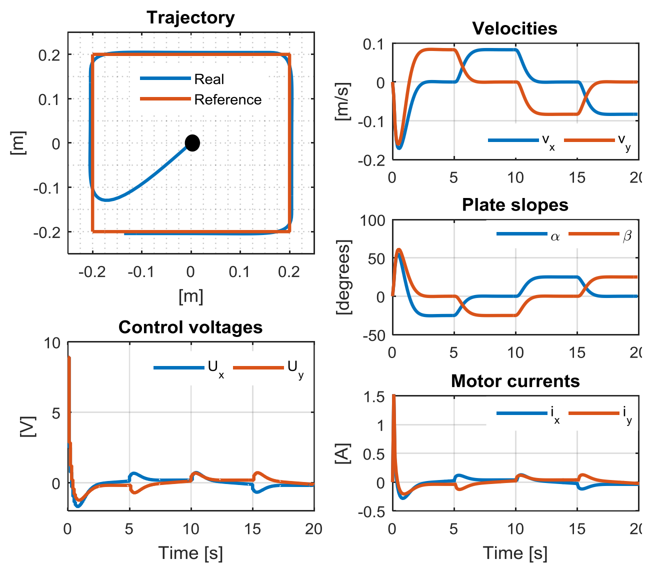

Fig. 7. Square trajectory: control voltages, velocities, plate slopes and currents

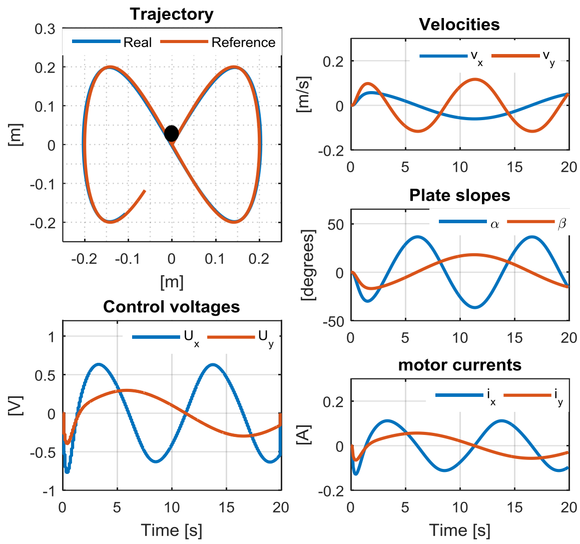

Fig. 8. Lissajous curve trajectory: control voltages, velocities, plate slopes and currents

Fig. 7 shows the simulation results of LQ control trajectory tracking, with a trajectory in the shape of square. Control voltages, ball velocities, plate slopes and motor currents are also shown. The initial location of the ball was at origin and was different from initial reference point. The controller was able to track the ball to the reference trajectory points. Fig. 8 shows the simulation results of the Lissajous curve shaped trajectory. The simulation experiments with both the trajectories show the quality of linearized model, which is derived from the non-linear model, is good for control purposes.

V. Conclusion

The mathematical model of ball and plate system is derived by Newton-Euler method – considering balance of forces and torques of ball and plate, motors and gears. The non-linear model is linearized around operating points following some approximation. Simulation of open loop model verification is performed. The linearized model is used to discrete optimal LQ control of the trajectory tracking problem of ball and plate system. The simulation result proves the quality of linearization of non-linear model.

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