6 Linear programming

The standard form of a linear programming (LP) problem

$$\text{minimize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$

$$\text{subject to: } \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

where $\mathbf{c} \in R^{n\times 1}$ with $\mathbf{c} \neq \mathbf{0}$, $\mathbf{A} \in R^{p\times n}$, and $\mathbf{b} \in R^{p\times 1}$

the p constraints can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_p^T \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

$$\mathbf{a}_i^T\mathbf{x} \geq b_i \qquad \text{for } i = 1, 2, \ldots, p$$

$$\text{minimize } f(\mathbf{x}) = x_1 + 1.5x_2 + x_3 + x_4$$

$$\begin{aligned} \text{subject to:} \quad x_1 + 2x_2 + x_3 + 2x_4 &= 3 \\ x_1 + x_2 + 2x_3 + 4x_4 &= 5 \\ x_i &\geq 0 \qquad \text{for } i = 1,\ 2,\ 3,\ 4 \end{aligned}$$

non-standard form

$$\text{minimize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} \geq \mathbf{b}$$

Transform to standard form: p-dimensional slack vector variable y

$$\mathbf{A}\mathrm{x} - \mathbf{y} = \mathbf{b} \qquad \text{for } \mathbf{y} \geq \mathbf{0}$$

$$\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^- \quad \text{with} \quad \mathbf{x}^+ \geq \mathbf{0} \quad \text{and} \quad \mathbf{x}^- \geq \mathbf{0}$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{y} \end{bmatrix}, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \\ \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{A}} = [\mathbf{A} \;\; -\mathbf{A} \;\; -\mathbf{I}_p]$$

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) = \hat{\mathbf{c}}^T \hat{\mathbf{x}} \\ &\text{subject to:} \quad \hat{\mathbf{A}}\hat{\mathbf{x}} = \mathbf{b} \\ &\qquad\qquad\qquad\;\; \hat{\mathbf{x}} \geq \mathbf{0} \end{aligned}$$

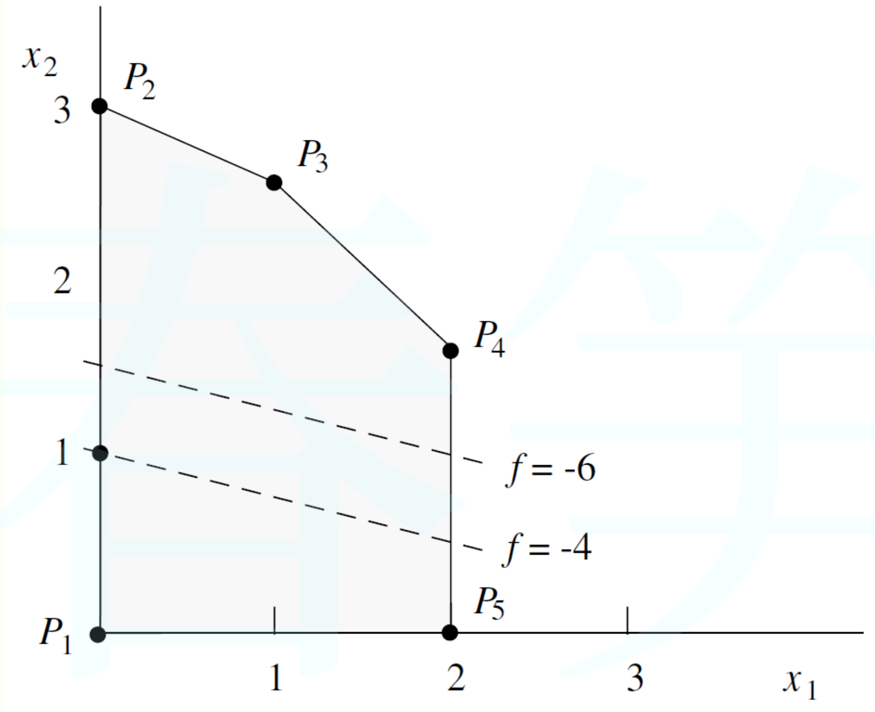
$$
\begin{aligned}
&\text{minimize } f(\mathbf{x}) = x_1 - 4x_2 \\
\text{subject to:}\quad & -x_1 + x_2 + 2 \geq 0 \\
& -x_1 - x_2 + 6 \geq 0 \\
& x_i \geq 0 \qquad \text{for } i = 1,\ 2
\end{aligned}
$$

the most general LP problem with both equality and inequality constraints

$$
\begin{aligned}
&\text{minimize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\
&\text{subject to: } \quad \mathbf{A}\mathbf{x} = \mathbf{b} \\
&\qquad\qquad\qquad\quad \mathbf{C}\mathbf{x} \geq \mathbf{d}
\end{aligned}
$$

Algorithms of LP

- Simplex Methods
- Interior-Point Methods


x_2
P_2
3
P_3
2
P_4
1
$f = -6$
$f = -4$
P_5
P_1
1
2
3
x_1

Optimality conditions

Theorem 11.1 ***Karush-Kuhn-Tucker conditions for standard-form LP problem*** *If* $\mathbf{x}^*$ *is regular for the constraints that are active at* $\mathbf{x}^*$*, then it is a global solution of the LP problem in Eq. (11.1) if and only if*

(a) $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, (11.5a)

(b) $\mathbf{x}^* \geq \mathbf{0}$, (11.5b)

(c) *there exist Lagrange multipliers* $\boldsymbol{\lambda}^* \in R^{p\times 1}$ *and* $\boldsymbol{\mu}^* \in R^{n\times 1}$ *such that* $\boldsymbol{\mu}^* \geq \mathbf{0}$ *and*

$$\mathbf{c} = \mathbf{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^* \tag{11.5c}$$

(d) $\mu_i^* x_i^* = 0$ *for* $1 \leq i \leq n$.

Interior-Point Methods

Karmarkar in 1984 interior-point methods

interior-point algorithms are much more efficient than simplex methods for large-scale LP problems

Duality is a concept of central importance in modern interior-point methods

interior-point algorithms:

- the primal **Newton barrier** method
- **primal-dual path-following** methods

Primal-Dual Solutions and Central Path

Primal problem

the standard-form LP problem

$$\begin{aligned} \text{minimize } & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ \text{subject to: } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize } & h(\boldsymbol{\lambda}) = \mathbf{b}^T\boldsymbol{\lambda} \\ \text{subject to: } & \mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c} \\ & \boldsymbol{\mu} \geq \mathbf{0} \end{aligned}$$

An LP problem is said to be **feasible** if its feasible region is not empty. **strictly feasible** if there exists an x that satisfies constraints with x > 0.

KKT:

$$
\begin{aligned}
\mathbf{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^* &= \mathbf{c} && (12.3a)\\
\mathbf{A}\mathbf{x}^* &= \mathbf{b} && (12.3b)\\
x_i^*\mu_i^* &= 0 \qquad \text{for } 1 \le i \le n && (12.3c)\\
\mathbf{x}^* &\ge \mathbf{0}, \quad \boldsymbol{\mu}^* \ge \mathbf{0} && (12.3d)
\end{aligned}
$$

A set {x□ , λ □ , μ□ } satisfying Eq. (12.3) is called a **primal-dual solution**

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \mathbf{b} \qquad \text{with } \mathbf{x} \ge \mathbf{0} && (12.8a)\\
\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} &= \mathbf{c} \qquad \text{with } \boldsymbol{\mu} \ge \mathbf{0} && (12.8b)\\
\mathbf{X}\boldsymbol{\mu} &= \mathbf{0} && (12.8c)
\end{aligned}
$$

where $\mathbf{X} = \text{diag}\{x_1, x_2, \ldots, x_n\}$

Theorem 12.1 ***Existence of a primal-dual solution*** *A primal-dual solution exists if the primal and dual problems are both feasible.*

Proof If point $\mathbf{x}$ is feasible for the LP problem in Eq. (12.1) and $\{\boldsymbol{\lambda},\ \boldsymbol{\mu}\}$ is feasible for the LP problem in Eq. (12.2), then set

$$\begin{aligned}\boldsymbol{\lambda}^T\mathbf{b} \le \boldsymbol{\lambda}^T\mathbf{b} + \boldsymbol{\mu}^T\mathbf{x} &= \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\mu}^T\mathbf{x} \\ &= (\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu})^T\mathbf{x} = \mathbf{c}^T\mathbf{x}\end{aligned} \tag{12.4}$$

Since $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ has a finite lower bound in the feasible region, there exists a set $\{\mathbf{x}^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*\}$ that satisfies Eq. (12.3). Evidently, this $\mathbf{x}^*$ solves the problem in Eq. (12.1). From Eq. (12.4), $h(\boldsymbol{\lambda})$ has a finite upper bound and $\{\boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*\}$ solves the problem in Eq. (12.2). Consequently, the set $\{\mathbf{x}^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*\}$ is a primal-dual solution.

Theorem 12.2 ***Strict feasibility of primal-dual solutions*** *If the primal and dual problems are both feasible, then*

(*a*) *solutions of the primal problem are bounded if the dual is strictly feasible;*
(*b*) *solutions of the dual problem are bounded if the primal is strictly feasible;*
(*c*) *primal-dual solutions are bounded if the primal and dual are both strictly feasible.*

proof: p375-376

$$\mathbf{c}^T\mathbf{x}^* = [(\boldsymbol{\mu}^*)^T + (\boldsymbol{\lambda}^*)^T\mathbf{A}]\mathbf{x}^* = (\boldsymbol{\lambda}^*)^T\mathbf{A}\mathbf{x}^* = (\boldsymbol{\lambda}^*)^T\mathbf{b} \qquad (12.5)$$

i.e.,

$$f(\mathbf{x}^*) = h(\boldsymbol{\lambda}^*)$$

If we define the *duality gap* as

$$\delta(\mathbf{x},\ \boldsymbol{\lambda}) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\boldsymbol{\lambda} \qquad (12.6)$$

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\boldsymbol{\lambda}^* \geq \mathbf{b}^T\boldsymbol{\lambda}$$

Central path

where $\mathbf{X} = \text{diag}\{x_1, x_2, \ldots, x_n\}$. The central path for a standard-form LP problem is defined as a set of vectors $\{\mathbf{x}(\tau), \boldsymbol{\lambda}(\tau), \boldsymbol{\mu}(\tau)\}$ that satisfy the conditions

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \text{with } \mathbf{x} > \mathbf{0} \tag{12.9a}$$

$$\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c} \qquad \text{with } \boldsymbol{\mu} > \mathbf{0} \tag{12.9b}$$

$$\mathbf{X}\boldsymbol{\mu} = \tau\mathbf{e} \tag{12.9c}$$

where τ is a strictly positive scalar parameter, and $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$

the set {x(τ), λ(τ), μ(τ)} satisfying Eq. (12.9), and when τ varies, the corresponding points form a set of trajectories called **the central path**.

every point on the central path is strictly feasible. Hence the central path lies in the interior of the feasible regions of the problems in Eqs. (12.1) and (12.2), and it approaches a primal-dual solution as $\tau \to 0$.

A more explicit relation of the central path with the primal-dual solution can be observed using the duality gap:

$$\begin{aligned}\delta[\mathbf{x}(\tau),\ \boldsymbol{\lambda}(\tau)] &= \mathbf{c}^T\mathbf{x}(\tau) - \mathbf{b}^T\boldsymbol{\lambda}(\tau)\\ &= [\boldsymbol{\lambda}^T(\tau)\mathbf{A} + \boldsymbol{\mu}^T(\tau)]\mathbf{x}(\tau) - \mathbf{b}^T\boldsymbol{\lambda}(\tau)\\ &= \boldsymbol{\mu}^T(\tau)\mathbf{x}(\tau) = n\tau\end{aligned}$$

Hence the duality gap along the central path converges linearly to zero as τ approaches zero. Consequently, as $\tau \to 0$ the objective function of the primal problem, $\mathbf{c}^T\mathbf{x}(\tau)$, and the objective function of the dual problem, $\mathbf{b}^T\boldsymbol{\lambda}(\tau)$, approach the same optimal value.

Primal Newton Barrier Method

the inequality constraints are incorporated in the objective function by adding a **logarithmic barrier function**

$$\text{minimize } f_\tau(\mathbf{x}) = \mathbf{c}^T\mathbf{x} - \tau \sum_{i=1}^{n} \ln\, x_i$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

where τ is a strictly positive scalar.

the PNB method solves the LP problem through the solution of a sequence of optimization problems.

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

$\longrightarrow$

Logarithmic barrier

reformulation of (1) via indicator function:

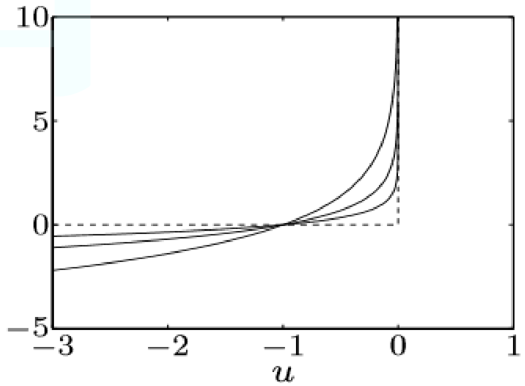
$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise (indicator function of $\mathbf{R}_-$)

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$



Lemma 3.1 (Barrier Lemma).

1. $r(c_k, x^k) \geq r(c_{k+1}, x^{k+1})$

2. $b(x^k) \leq b(x^{k+1})$

3. $f(x^k) \geq f(x^{k+1})$

4. $f(x^*) \leq f(x^k) \leq r(c_k, x^k)$.

1.
$$
\begin{aligned}
r(c_k, x^k) = f(x^k) + \frac{1}{c_k} b(x^k) &\geq f(x^k) + \frac{1}{c_{k+1}} b(x^k) \\
&\geq f(x^{k+1}) + \frac{1}{c_{k+1}} b(x^{k+1}) = r(c_{k+1}, x^{k+1})
\end{aligned}
$$

2.
$$f(x^k) + \frac{1}{c_k} b(x^k) \leq f(x^{k+1}) + \frac{1}{c_k} b(x^{k+1})$$

and

$$f(x^{k+1}) + \frac{1}{c_{k+1}} b(x^{k+1}) \leq f(x^k) + \frac{1}{c_{k+1}} b(x^k).$$

Summing and rearranging, we have

$$\left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) b(x^k) \leq \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) b(x^{k+1}).$$

Since $c_k < c_{k+1}$, it follows that $b(x^{k+1}) \geq b(x^k)$.

3. From the proof of (1.),

$$f(x^k) + \frac{1}{c_{k+1}} b(x^k) \geq f(x^{k+1}) + \frac{1}{c_{k+1}} b(x^{k+1}).$$

But from (2.), $b(x^k) \leq b(x^{k+1})$. Thus $f(x^k) \geq f(x^{k+1})$.

4. $f(x^*) \leq f(x^k) \leq f(x^k) + \frac{1}{c_k} b(x^k) = r(c_k, x^k)$.

logarithmic barrier function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \quad \mathbf{dom}\,\phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- convex (follows from composition rules)
- twice continuously differentiable, with derivatives

$$\begin{aligned}
\nabla \phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\
\nabla^2 \phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\end{aligned}$$

$$\text{minimize } f_\tau(\mathbf{x}) = \mathbf{c}^T\mathbf{x} - \tau \sum_{i=1}^{n} \ln\, x_i$$

$$\text{subject to: } \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

$\longrightarrow$

$$\nabla f_\tau(\mathbf{x}) = \mathbf{c} - \tau \mathbf{X}^{-1}\mathbf{e} \tag{12.27a}$$

$$\nabla^2 f_\tau(\mathbf{x}) = \tau \mathbf{X}^{-2} \tag{12.27b}$$

with $\mathbf{X} = \text{diag}\{x_1, x_2, \ldots, x_n\}$ and $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$

目标函数是下降的？

$$\mathbf{c}^T\mathbf{x}_k^* - \tau_k \sum_{i=1}^{n} \ln(\mathbf{x}_k^*)_i \le \mathbf{c}^T\mathbf{x}_{k+1}^* - \tau_k \sum_{i=1}^{n} \ln(\mathbf{x}_{k+1}^*)_i$$

and

$$\mathbf{c}^T\mathbf{x}_{k+1}^* - \tau_{k+1} \sum_{i=1}^{n} \ln(\mathbf{x}_{k+1}^*)_i \le \mathbf{c}^T\mathbf{x}_k^* - \tau_{k+1} \sum_{i=1}^{n} \ln(\mathbf{x}_k^*)_i$$

These equations yield (see Prob. 12.11(a))

$$f(\mathbf{x}_{k+1}^*) = \mathbf{c}^T\mathbf{x}_{k+1}^* \le \mathbf{c}^T\mathbf{x}_k^* = f(\mathbf{x}_k^*) \quad \downarrow$$

求解：

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

$$\mathbf{A}\mathbf{d}_k = \mathbf{0} \leftarrow \mathbf{A}\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \alpha_k \mathbf{A}\mathbf{d}_k = \mathbf{b}$$

12.7 $\mathbf{A}^T\boldsymbol{\lambda}_k + \tau_k \mathbf{X}^{-1}\mathbf{e} = \mathbf{c}$ ← KKT

$\boldsymbol{\mu}_k = \tau_k \mathbf{X}^{-1}\mathbf{e}$ where $\mathbf{X} = \text{diag}\{(\mathbf{x}_k^*)_1, (\mathbf{x}_k^*)_2, \ldots, (\mathbf{x}_k^*)_n\}$

$$\text{minimize } \tfrac{1}{2}\tau \mathbf{d}^T\mathbf{X}^{-2}\mathbf{d} + \mathbf{d}^T(\mathbf{c} - \tau\mathbf{X}^{-1}\mathbf{e}) \tag{12.36a}$$

$$\text{subject to: } \mathbf{A}\mathbf{d} = \mathbf{0} \tag{12.36b}$$

$$\tau\mathbf{X}^{-2}\mathbf{d}_k + \mathbf{c} - \tau\mathbf{X}^{-1}\mathbf{e} = \mathbf{A}^T\boldsymbol{\lambda} \tag{12.37a}$$

$$\mathbf{A}\mathbf{d}_k = \mathbf{0} \tag{12.37b}$$

From Eq. (12.37), we obtain

$$\mathbf{d}_k = \mathbf{x}_k + \frac{1}{\tau}\mathbf{X}^2(\mathbf{A}^T\boldsymbol{\lambda} - \mathbf{c}) \tag{12.38a}$$

and

$$\begin{aligned}\mathbf{A}\mathbf{X}^2\mathbf{A}^T\boldsymbol{\lambda} &= \tau\mathbf{A}\mathbf{d}_k + \mathbf{A}\mathbf{X}^2\mathbf{c} - \tau\mathbf{A}\mathbf{x}_k \\ &= \mathbf{A}(\mathbf{X}^2\mathbf{c} - \tau\mathbf{x}_k)\end{aligned} \tag{12.38b}$$

Algorithm 12.2 Primal Newton barrier algorithm for the standard-form LP problem

Step 1

Input $\mathbf{A}$, $\mathbf{c}$, and a strictly feasible initial point $\mathbf{x}_0$.

Set $l = 0$, initialize the barrier parameter such that $\tau_0 > 0$, and input the outer-loop tolerance $\varepsilon_{\text{outer}}$.

Step 2

Set $k = 0$ and $\mathbf{x}_0^{(l)} = \mathbf{x}_l$, and input the inner-loop tolerance $\varepsilon_{\text{inner}}$.

Step 3.1

Use Eq. (12.38) with $\tau = \tau_l$ to calculate $\mathbf{d}_k^{(l)}$ at $\mathbf{x}_k^{(l)}$.

Step 3.2

Use Eq. (12.39) to calculate $\bar{\alpha}_k$ where $\mathbf{x}_k = \mathbf{x}_k^{(l)}$ and $\mathbf{d}_k = \mathbf{d}_k^{(l)}$.

Step 3.3

Use a line search (e.g., a line search based on the golden-section method) to determine $\alpha_k^{(l)}$.

Step 3.4

Set $\mathbf{x}_{k+1}^{(l)} = \mathbf{x}_k^{(l)} + \alpha_k^{(l)} \mathbf{d}_k^{(l)}$.

Step 3.5

If $||\alpha_k^{(l)} \mathbf{d}_k^{(l)}|| < \varepsilon_{\text{inner}}$, set $\mathbf{x}_{l+1} = \mathbf{x}_{k+1}^{(l)}$ and go to Step 4; otherwise, set $k = k + 1$ and repeat from Step 3.1.

Step 4

If $||\mathbf{x}_l - \mathbf{x}_{l+1}|| < \varepsilon_{\text{outer}}$, output $\mathbf{x}^* = \mathbf{x}_{l+1}$, and stop; otherwise, choose $\tau_{l+1} < \tau_l$, set $l = l + 1$, and repeat from Step 2.

$$\bar{\alpha}_k = 0.99 \times \min_{i \text{ with } d_i < 0} \left[\frac{x_i}{(-d_i)} \right]$$

Primal-Dual Path-Following Interior-Point Algorithm

Let

$$\mathbf{w}_k = \{\mathbf{x}_k,\ \boldsymbol{\lambda}_k,\ \boldsymbol{\mu}_k\}$$

$$\mathbf{w}_{k+1} = \{\mathbf{x}_{k+1},\ \boldsymbol{\lambda}_{k+1},\ \boldsymbol{\mu}_{k+1}\} = \{\mathbf{x}_k + \boldsymbol{\delta}_x,\ \boldsymbol{\lambda}_k + \boldsymbol{\delta}_\lambda,\ \boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu\}$$

$$\mathbf{A}\boldsymbol{\delta}_x = \mathbf{0} \qquad (12.40\text{a})$$

$$\mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu = \mathbf{0} \qquad (12.40\text{b})$$

$$\Delta\mathbf{X}\boldsymbol{\mu}_k + \mathbf{X}\boldsymbol{\delta}_\mu + \Delta\mathbf{X}\boldsymbol{\delta}_\mu = \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \qquad (12.40\text{c})$$

where

$$\Delta\mathbf{X} = \text{diag}\{(\boldsymbol{\delta}_x)_1,\ (\boldsymbol{\delta}_x)_2,\ \ldots,\ (\boldsymbol{\delta}_x)_n\} \qquad (12.41)$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{with } \mathbf{x} > \mathbf{0} \qquad (12.9\text{a})$$

$$\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c} \quad \text{with } \boldsymbol{\mu} > \mathbf{0} \qquad (12.9\text{b})$$

$$\mathbf{X}\boldsymbol{\mu} = \tau\mathbf{e} \qquad (12.9\text{c})$$

$$\mathbf{A}\boldsymbol{\delta}_x = \mathbf{0} \qquad (12.42\text{a})$$

$$\mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu = \mathbf{0} \qquad (12.42\text{b})$$

$$\mathbf{M}\boldsymbol{\delta}_x + \mathbf{X}\boldsymbol{\delta}_\mu = \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \qquad (12.42\text{c})$$

where term $\Delta\mathbf{X}\boldsymbol{\mu}_k$ in Eq. (12.40c) has been replaced by $\mathbf{M}\boldsymbol{\delta}_x$ with

$$\mathbf{M} = \text{diag}\{(\boldsymbol{\mu}_k)_1,\ (\boldsymbol{\mu}_k)_2,\ \ldots,\ (\boldsymbol{\mu}_k)_n\} \qquad (12.43)$$

$$\tau_{k+1} = \left(1 - \frac{\delta}{\sqrt{n}}\right)\tau_k \qquad (12.47)$$

$\Delta\mathbf{X}\boldsymbol{\delta}_\mu$ is neglected

$$\mathbf{w}_{k+1} = \{\mathbf{x}_{k+1},\ \boldsymbol{\lambda}_{k+1},\ \boldsymbol{\mu}_{k+1}\}$$

解方程，得

$$\boldsymbol{\delta}_\lambda = \mathbf{YAy} \tag{12.44a}$$

$$\boldsymbol{\delta}_\mu = -\mathbf{A}^T\boldsymbol{\delta}_\lambda \tag{12.44b}$$

$$\boldsymbol{\delta}_x = -\mathbf{y} - \mathbf{D}\boldsymbol{\delta}_\mu \tag{12.44c}$$

where

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{X} \tag{12.44d}$$

$$\mathbf{Y} = (\mathbf{ADA}^T)^{-1} \tag{12.44e}$$

and

$$\mathbf{y} = \mathbf{x}_k - \tau_{k+1}\mathbf{M}^{-1}\mathbf{e} \tag{12.44f}$$

Algorithm 12.3 Primal-dual path-following algorithm for the standard-form LP problem

Step 1

Input $\mathbf{A}$ and a strictly feasible $\mathbf{w}_0 = \{\mathbf{x}_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0\}$ that satisfies Eq. (12.45). Set $k = 0$ and initialize the tolerance ε for the duality gap.

Step 2

If $\boldsymbol{\mu}_k^T \mathbf{x}_k \leq \varepsilon$, output solution $\mathbf{w}^* = \mathbf{w}_k$ and stop; otherwise, continue with Step 3.

Step 3

Set τ_{k+1} using Eq. (12.47) and compute $\boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\lambda, \boldsymbol{\delta}_\mu\}$ using Eq. (12.44).

Step 4

Set $\mathbf{w}_{k+1}$ using Eq. (12.46). Set $k = k + 1$ and repeat from Step 2.