8. QUADRATIC PROGRAMMING

Quadratic programming (QP) is a family of methods, techniques, and algorithms that can be used to minimize quadratic objective functions subject to linear constraints

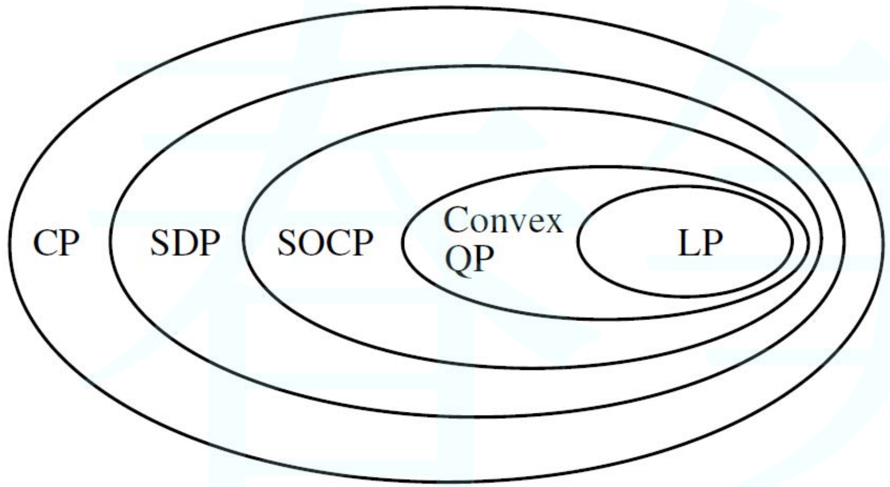

Figure 14.4. Relations among LP, convex QP, SOCP, SDP, and CP problems.

Convex QP Problems with Equality Constraints

$$\text{minimize } f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{13.1a}$$

$$\text{subject to: } \mathbf{A}\mathbf{x} = \mathbf{b} \tag{13.1b}$$

H is positive definite

$$\begin{bmatrix} \mathbf{H} & -\mathbf{A}^T \\ -\mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = - \begin{bmatrix} \mathbf{p} \\ \mathbf{b} \end{bmatrix} \leftarrow \begin{aligned} \mathbf{H}\mathbf{x}^* + \mathbf{p} - \mathbf{A}^T\boldsymbol{\lambda}^* &= \mathbf{0} \\ -\mathbf{A}\mathbf{x}^* + \mathbf{b} &= \mathbf{0} \end{aligned} \leftarrow$$

KTT

$$\downarrow$$

$$\boldsymbol{\lambda}^* = (\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{H}^{-1}\mathbf{p} + \mathbf{b}) \tag{13.11a}$$

$$\mathbf{x}^* = \mathbf{H}^{-1}(\mathbf{A}\boldsymbol{\lambda}^* - \mathbf{p}) \tag{13.11b}$$

Interior-Point Methods

Pimal

$$\text{minimize } f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{13.28a}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{13.28b}$$

$$\mathbf{x} \geq \mathbf{0} \tag{13.28c}$$

Dual

$$\text{maximize; } h(\mathbf{x},\ \boldsymbol{\lambda},\ \boldsymbol{\mu}) = -\tfrac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{b} \tag{13.29a}$$

$$\text{subject to:} \quad \mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} - \mathbf{H}\mathbf{x} = \mathbf{p} \tag{13.29b}$$

$$\boldsymbol{\mu} \geq \mathbf{0} \tag{13.29c}$$

KKT conditions

$$\mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0} \quad \text{for } \mathbf{x} \geq \mathbf{0} \tag{13.30a}$$

$$\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} - \mathbf{H}\mathbf{x} - \mathbf{p} = \mathbf{0} \quad \text{for } \boldsymbol{\mu} \geq \mathbf{0} \tag{13.30b}$$

$$\mathbf{X}\boldsymbol{\mu} = \mathbf{0} \tag{13.30c}$$

Dual gap

$$\begin{aligned}\delta(\mathbf{x},\ \boldsymbol{\lambda},\ \boldsymbol{\mu}) &= f(\mathbf{x}) - h(\mathbf{x},\ \boldsymbol{\lambda},\ \boldsymbol{\mu}) = \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} - \boldsymbol{\lambda}^T\mathbf{b} \\ &= \mathbf{x}^T(\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu}) - \boldsymbol{\lambda}^T\mathbf{b} = \mathbf{x}^T\boldsymbol{\mu} \end{aligned} \tag{13.31}$$

which is always nonnegative and is equal to zero at solution $\{\mathbf{x}^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*\}$

central path

$$\begin{aligned} \mathbf{A}\mathbf{x} - \mathbf{b} &= \mathbf{0} \quad \text{for } \mathbf{x} > \mathbf{0} && (13.32a) \\ \mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} - \mathbf{H}\mathbf{x} - \mathbf{p} &= \mathbf{0} \quad \text{for } \boldsymbol{\mu} > \mathbf{0} && (13.32b) \\ \mathbf{X}\boldsymbol{\mu} &= \tau\mathbf{e} && (13.32c) \end{aligned}$$

$$\downarrow$$

$$\delta[\mathbf{x}(\tau),\ \boldsymbol{\lambda}(\tau),\ \boldsymbol{\mu}(\tau)] = \mathbf{x}^T(\tau)\boldsymbol{\mu}(\tau) = n\tau \qquad (13.33) \longrightarrow \tau \to 0 \longrightarrow \mathbf{w}^* = \{\mathbf{x}^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*\}$$

将13.28变换为罚函数形式

$$\begin{aligned} \text{minimize } \hat{f}(\mathbf{x}) &= \tfrac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} - \tau\sum_{i=1}^{n}\ln x_i && (13.34a) \\ \text{subject to:} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} && (13.34b) \end{aligned}$$

$$\downarrow$$

（13.34） KKT conditions

$$\begin{aligned} \mathbf{A}\mathbf{x} - \mathbf{b} &= \mathbf{0} \quad \text{for } \mathbf{x} > \mathbf{0} && (13.35a) \\ \mathbf{A}^T\boldsymbol{\lambda} + \tau\mathbf{X}^{-1}\mathbf{e} - \mathbf{H}\mathbf{x} - \mathbf{p} &= \mathbf{0} && (13.35b) \end{aligned}$$

$$\downarrow$$

$$\boldsymbol{\mu} = \tau\mathbf{X}^{-1}\mathbf{e},$$

A primal-dual path-following method

let $\mathbf{w}_k = \{\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k\}$ $\quad \boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\lambda, \boldsymbol{\delta}_\mu\}$

$$\mathbf{w}_{k+1} = \{\mathbf{x}_{k+1}, \boldsymbol{\lambda}_{k+1}, \boldsymbol{\mu}_{k+1}\} = \mathbf{w}_k + \boldsymbol{\delta}_w$$

代入（13.32）

$$-\mathbf{H}\boldsymbol{\delta}_x + \mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu = \mathbf{0} \tag{13.37a}$$

$$\mathbf{A}\boldsymbol{\delta}_x = \mathbf{0} \tag{13.37b}$$

$$\Delta\mathbf{X}\boldsymbol{\mu}_k + \mathbf{X}\boldsymbol{\delta}_\mu + \Delta\mathbf{X}\boldsymbol{\delta}_\mu = \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \tag{13.37c}$$

where $\Delta\mathbf{X} = \mathrm{diag}\{(\boldsymbol{\delta}_x)_1, (\boldsymbol{\delta}_x)_2, \ldots, (\boldsymbol{\delta}_x)_n\}$

$$-\mathbf{H}\boldsymbol{\delta}_x + \mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu = \mathbf{0} \tag{13.38a}$$

$$\mathbf{A}\boldsymbol{\delta}_x = \mathbf{0} \tag{13.38b}$$

$$\mathbf{M}\boldsymbol{\delta}_x + \mathbf{X}\boldsymbol{\delta}_\mu = \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \tag{13.38c}$$

where $\mathbf{M} = \mathrm{diag}\{(\boldsymbol{\mu}_k)_1, (\boldsymbol{\mu}_k)_2, \ldots, (\boldsymbol{\mu}_k)_n\}$

$$\begin{bmatrix} -\mathbf{H} & \mathbf{A}^T & \mathbf{I} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{X} \end{bmatrix} \boldsymbol{\delta}_w = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \end{bmatrix} \tag{13.39}$$

$$\tau_{k+1} = \frac{\mathbf{x}_k^T\boldsymbol{\mu}_k}{n+\rho} \quad \text{with } \rho \geq \sqrt{n}$$

The solution of Eq. (13.38) can be obtained as

$$\boldsymbol{\delta}_\lambda = \mathbf{Y}\mathbf{y} \tag{13.42a}$$

$$\boldsymbol{\delta}_x = \boldsymbol{\Gamma}\mathbf{X}\mathbf{A}^T\boldsymbol{\delta}_\lambda - \mathbf{y} \tag{13.42b}$$

$$\boldsymbol{\delta}_\mu = \mathbf{H}\boldsymbol{\delta}_x - \mathbf{A}^T\boldsymbol{\delta}_\lambda \tag{13.42c}$$

where

$$\boldsymbol{\Gamma} = (\mathbf{M} + \mathbf{X}\mathbf{H})^{-1} \tag{13.42d}$$

$$\mathbf{Y} = (\mathbf{A}\boldsymbol{\Gamma}\mathbf{X}\mathbf{A}^T)^{-1}\mathbf{A} \tag{13.42e}$$

and

$$\mathbf{y} = \boldsymbol{\Gamma}(\mathbf{X}\boldsymbol{\mu}_k - \tau_{k+1}\mathbf{e}) \tag{13.42f}$$

Algorithm 13.2 Primal-dual path-following algorithm for convex QP problems

Step 1

Input a strictly feasible $\mathbf{w}_0 = \{\mathbf{x}_0,\ \boldsymbol{\lambda}_0,\ \boldsymbol{\mu}_0\}$.

Set $k = 1$ and $\rho \geq \sqrt{n}$, and initialize the tolerance ε for duality gap.

Step 2

If $\mathbf{x}_k^T \boldsymbol{\mu}_k \leq \varepsilon$, output solution $\mathbf{w}^* = \mathbf{w}_k$, and stop; otherwise, continue with Step 3.

Step 3

Set τ_{k+1} using Eq. (13.40) and compute $\boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x,\ \boldsymbol{\delta}_\lambda,\ \boldsymbol{\delta}_\mu\}$ using Eqs. (13.42a) to (13.42c).

Step 4

Compute α_k using Eq. (13.44) and update $\mathbf{w}_{k+1}$ using Eq. (13.43).

Set $k = k + 1$ and repeat from Step 2.