SEMIDEFINITE AND SECOND-ORDER CONE PROGRAMMING

Semidefinite programming (SDP)

Semidefinite programming (SDP) is a branch of convex programming (CP).many important classes of optimization problems such as linear-programming (LP) and convex quadratic-programming (QP) problems can be viewed as SDP problems, and many CP problems of practical usefulness that are neither LP nor QP problems can also be formulated as SDP problems.

second-order cone programming (SOCP) can also be viewed as SDP problems.

Primal and Dual SDP Problems

Notation and definitions:

Let $\mathcal{S}^n$ be the space of real symmetric $n \times n$ matrices. The standard inner product on $\mathcal{S}^n$ is defined by

$$\mathbf{A} \cdot \mathbf{B} = \text{trace}(\mathbf{AB}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}$$

where $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ are two members of $\mathcal{S}^n$.

Primal SDP problem:

The primal SDP problem is defined as

$$\text{minimize } \mathbf{C} \cdot \mathbf{X} \tag{14.1a}$$

$$\text{subject to: } \quad \mathbf{A}_i \cdot \mathbf{X} = b_i \qquad \text{for } i = 1, 2, \ldots, p \tag{14.1b}$$

$$\mathbf{X} \succeq \mathbf{0} \tag{14.1c}$$

Consider an SDP problem in inequality form:

$$\begin{array}{ll} \min & C \bullet X \\ & A_i \bullet X \le b_i, \quad i = 1, \cdots, m \\ & X \succeq 0. \end{array}$$

Add slack variables $\xi = (\xi_i)_{i=1}^m$ and write the problem as

$$\begin{array}{ll} \min & \hat{C} \bullet \hat{X} \\ & \hat{A}_i \bullet \hat{X} = b_i, \quad i = 1, \cdots, m \\ & \hat{X} \succeq 0, \end{array}$$

where

$$\hat{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}^{n+m}$$

and

$$\hat{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & e_i e_i^T \end{bmatrix}, \ i = 1, \cdots, m$$

$$\left(\text{and without loss of generality } \hat{X} = \begin{bmatrix} X & 0 \\ 0 & \mathrm{Diag}\,(\xi) \end{bmatrix} \right).$$

the dual SDP problem with respect to the primal SDP problem

$$\text{maximize } \mathbf{b}^T\mathbf{y} \tag{14.4a}$$

$$\text{subject to: } \sum_{i=1}^{p} y_i\mathbf{A}_i + \mathbf{S} = \mathbf{C} \tag{14.4b}$$

$$\mathbf{S} \succeq \mathbf{0} \tag{14.4c}$$

等价于

$$\text{maximize } \mathbf{b}^T\mathbf{y}$$

$$\text{subject to: } \mathbf{C} - \sum_{i=1}^{p} y_i\mathbf{A}_i \succeq \mathbf{0}$$

linear matrix inequality (LMI) constraints

等价于

$$\text{minimize } \mathbf{c}^T\mathbf{x} \tag{14.9a}$$

$$\text{subject to: } \mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \tag{14.9b}$$

where $\mathbf{c} \in R^{p\times 1}$, $\mathbf{x} \in R^{p\times 1}$, and

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{p} x_i\mathbf{F}_i$$

SDP KKT

$$\sum_{i=1}^{p} y_i^* \mathbf{A}_i + \mathbf{S}^* = \mathbf{C} \tag{14.24a}$$

$$\mathbf{A}_i \cdot \mathbf{X}^* = b_i \qquad \text{for } 1 \le i \le p \tag{14.24b}$$

$$\mathbf{S}^* \mathbf{X}^* = \mathbf{0} \tag{14.24c}$$

$$\mathbf{X}^* \succeq \mathbf{0},\ \mathbf{S}^* \succeq \mathbf{0} \tag{14.24d}$$

the duality gap

$$\begin{aligned} \delta[\mathbf{X}(\tau),\ \mathbf{y}(\tau)] &= \mathbf{C} \cdot \mathbf{X}(\tau) - \mathbf{b}^T \mathbf{y}(\tau) \\ &= \left[\sum_{i=1}^{p} y_i(\tau)\mathbf{A}_i + \mathbf{S}(\tau)\right] \cdot \mathbf{X}(\tau) - \mathbf{b}^T \mathbf{y}(\tau) \\ &= \mathbf{S}(\tau) \cdot \mathbf{X}(\tau) = \text{trace}[\mathbf{S}(\tau)\mathbf{X}(\tau)] \\ &= \text{trace}(\tau \mathbf{I}) = n\tau \end{aligned} \tag{14.26}$$

Central path

$$\sum_{i=1}^{p} y_i(\tau)\mathbf{A}_i + \mathbf{S}(\tau) = \mathbf{C} \tag{14.25a}$$

$$\mathbf{A}_i \cdot \mathbf{X}(\tau) = b_i \qquad \text{for } 1 \le i \le p \tag{14.25b}$$

$$\mathbf{X}(\tau)\mathbf{S}(\tau) = \tau \mathbf{I} \tag{14.25c}$$

$$\mathbf{S}(\tau) \succeq \mathbf{0},\ \mathbf{X}(\tau) \succeq \mathbf{0} \tag{14.25d}$$

SDP 内点法(略)

SDP Example

1. LP
2. QP
3. QCQP

LP:

$$\text{minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} \geq \mathbf{b}, \ \mathbf{A} \in R^{p\times n}$$

$$\downarrow$$

$$\mathbf{F}_0 = -\text{diag}\{\mathbf{b}\}, \quad \mathbf{F}_i = \text{diag}\{\mathbf{a}_i\} \qquad \text{for } \ i = 1, 2, \ldots, n$$

$$\begin{aligned}
\text{minimize } f(\mathbf{x}) &= -x_1 - 4x_2 \\
\text{subject to:} \quad x_1 &\geq 0 \\
-x_1 &\geq -2 \\
x_2 &\geq 0 \\
-x_1 - x_2 + 3.5 &\geq 0 \\
-x_1 - 2x_2 + 6 &\geq 0
\end{aligned}$$

```
import cvxpy as cvx
import numpy as np
x = cvx.Variable(shape=(2,1))

c = np.array([[-1],[-4]],dtype=np.float64)

A = np.array([[-1,0],[-1,-1],[-1,-2]],dtype=np.float64)
b = np.array([[-2],[-3.5],[-6]],dtype=np.float64)
objective =c.T*x

# Create two constraints.
constraints = [A*x>=b,x>=0]
# Form objective.
obj = cvx.Minimize(objective)
```

```
solvers=[cvx.CVXOPT,cvx.SCS,cvx.GUROBI,cvx.MOSEK]
# Form and solve problem.
prob = cvx.Problem(obj, constraints)
for solver in solvers:

    prob.solve(solver=solver,verbose=True)
    print('{} result:'.format(solver))
    print("status:", prob.status)
    print("optimal value", prob.value)
    print("optimal var", x.value)

solvers=[cvx.CVXOPT,cvx.MOSEK]
constraints = [cvx.diag(A*x-b)>>0,x>=0]
prob = cvx.Problem(obj, constraints)
for solver in solvers:

    prob.solve(solver=solver)
    print('{} result:'.format(solver))
    print("status:", prob.status)
    print("optimal value", prob.value)
    print("optimal var", x.value)
```

A block diagonal matrix is symmetric positive (semi-) definite iff all its blocks are symmetric positive (semi-) definite, i.e., for $X_\kappa \in \mathbb{R}^{n_\kappa \times n_\kappa}$, $1 \leq \kappa \leq k$,

$$X := \begin{pmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & X_k \end{pmatrix} \succeq 0 \ (\succ 0) \iff X_1, \dots, X_k \succeq 0 \ (\succ 0).$$

The Schur complement. Let

$$\boldsymbol{A} \in \mathbb{S}^k, \text{ positive definite}, \quad \boldsymbol{X} \in \mathbb{R}^{k\times\ell},\ \boldsymbol{Y} \in \mathbb{S}^\ell.$$

Then

$$\boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \succeq \boldsymbol{O} \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.$$

Proof: If $A \succ 0$, then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^TA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C - B^TA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

since $\begin{bmatrix} I & 0 \\ B^TA^{-1} & I \end{bmatrix}$ is a nonsingular matrix. $\square$

- When $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{X} \succeq \boldsymbol{O} \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.$
- When $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{X} = \boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{Y} = \boldsymbol{y} \in \mathbb{R}$,

$$\boldsymbol{y} - \boldsymbol{x}^T\boldsymbol{x} \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{I} & \boldsymbol{x} \\ \boldsymbol{x}^T & \boldsymbol{y} \end{pmatrix} \succeq \boldsymbol{O}.$$

- When $\boldsymbol{A} = \boldsymbol{I}\boldsymbol{y}$, $\boldsymbol{X} = \boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{Y} = \boldsymbol{y} \in \mathbb{R}$,

$$\boldsymbol{y} - \sqrt{\boldsymbol{x}^T\boldsymbol{x}} \geq 0 \Leftrightarrow \boldsymbol{y}^2 - \boldsymbol{x}^T\boldsymbol{x} \geq 0,\ \boldsymbol{y} \geq 0 \Leftrightarrow \begin{pmatrix} \boldsymbol{I}\boldsymbol{y} & \boldsymbol{x} \\ \boldsymbol{x}^T & \boldsymbol{y} \end{pmatrix} \succeq \boldsymbol{O}.$$

$(y - \boldsymbol{x}^T\boldsymbol{x}/y \geq 0$ if $y > 0)$

$$\text{minimize } \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{p}^T\mathbf{x} \qquad \text{with } \mathbf{H} \succeq \mathbf{0} \tag{14.12a}$$

$$\text{subject to: } \quad \mathbf{A}\mathbf{x} \geq \mathbf{b} \tag{14.12b}$$

$$\text{minimize } \delta \tag{14.13a}$$

$$\text{subject to: } \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{p}^T\mathbf{x} \leq \delta \tag{14.13b}$$

$$\mathbf{A}\mathbf{x} \geq \mathbf{b} \tag{14.13c}$$

$$\boldsymbol{H} = \widehat{\boldsymbol{H}}^T \widehat{\boldsymbol{H}}$$

$$\boldsymbol{H} = \boldsymbol{G}\boldsymbol{D}\boldsymbol{G}^T, \widehat{\boldsymbol{H}} = \left(\boldsymbol{G}\boldsymbol{D} \hat{} (1/2)\right)^T$$

$\delta - \mathbf{p}^T\mathbf{x} - (\hat{\mathbf{H}}\mathbf{x})^T(\hat{\mathbf{H}}\mathbf{x}) \geq 0$ 等价于

$$\mathbf{G}(\delta, \mathbf{x}) = \begin{bmatrix} \mathbf{I}_n & \hat{\mathbf{H}}\mathbf{x} \\ (\hat{\mathbf{H}}\mathbf{x})^T & \delta - \mathbf{p}^T\mathbf{x} \end{bmatrix} \succeq \mathbf{0} \tag{14.15}$$

$$\begin{bmatrix} diag(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) & \\ & G\left(\delta, \boldsymbol{x}\right) \end{bmatrix} \succeq \boldsymbol{0}$$

QP problems with quadratic constraints(QCQP)

$$\text{minimize } \delta \tag{14.21a}$$

$$\text{subject to: } \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{p}^T\mathbf{x} \le \delta \tag{14.21b}$$

$$\mathbf{x}^T\mathbf{Q}_i\mathbf{x} + \mathbf{q}_i^T\mathbf{x} + r_i \le 0 \qquad \text{for } 1 \le i \le p \tag{14.21c}$$

↓

$$\mathbf{F}_i(\mathbf{x}) = \begin{bmatrix} \mathbf{I}_n & \hat{\mathbf{Q}}_i\mathbf{x} \\ (\hat{\mathbf{Q}}_i\mathbf{x})^T & -\mathbf{q}_i^T\mathbf{x} - r_i \end{bmatrix} \succeq \mathbf{0} \qquad \text{for } 1 \le i \le p$$

↓

$$\text{minimize } \hat{\mathbf{c}}^T\hat{\mathbf{x}}$$

$$\text{subject to: } \quad \mathbf{E}(\hat{\mathbf{x}}) \succeq \mathbf{0}$$

where

$$\mathbf{E}(\hat{\mathbf{x}}) = \text{diag}\{\mathbf{G}(\delta,\ \mathbf{x}),\ \mathbf{F}_1(\mathbf{x}),\ \mathbf{F}_2\,(\mathbf{x}),\ \ldots,\ \mathbf{F}_p(\mathbf{x})\}$$

$$\text{minimize } \delta$$

subject to:

$$[(x_1 - x_3)^2 + (x_2 - x_4)^2]^{1/2} \leq \delta$$

$$[x_1\ x_2]\begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [x_1\ x_2]\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \leq \frac{3}{4}$$

$$[x_3\ x_4]\begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 5/8 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - [x_3\ x_4]\begin{bmatrix} 11/2 \\ 13/2 \end{bmatrix} \leq -\frac{35}{2}$$

$$\mathrm{Q}_0 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \mathrm{Q}_1 = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}, \mathrm{Q}_2 = \begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 5/8 \end{bmatrix},$$

$$\mathrm{P}_0 = \mathrm{zeros}(4,0), \mathrm{P}_1 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}, \mathrm{P}_2 = \begin{bmatrix} -11/2 \\ -13/2 \end{bmatrix} \qquad \mathrm{Q}_0 = \mathrm{Q}_{00}^T \mathrm{Q}_{00}, \mathrm{Q}_1 = \mathrm{Q}_{11}^T \mathrm{Q}_{11}, \mathrm{Q}_2 = \mathrm{Q}_{22}^T \mathrm{Q}_{22}$$

$$\gamma_1 = 3/4, \gamma_2 = -35/2$$

```
Q0= np.array([[1,0,-1,0],[0,1,0,-1],[-
1,0,1,0],[0,-1,0,1]])
D,G=np.linalg.eig(Q0);
Q00=G.dot(np.diag(np.sqrt(D))).T
Q1=np.array([[1/4,0],[0,1]])
D,G=np.linalg.eig(Q1);
Q11=G.dot(np.diag(np.sqrt(D))).T
Q2=np.array([[5/8,3/8],[3/8,5/8]])
D,G=np.linalg.eig(Q2);
Q22=G.dot(np.diag(np.sqrt(D))).T
P1=np.array([[-1/2],[0]])
P2=np.array([[-11/2],[-13/2]])
b1=3/4
b2=-35/2
c0 = np.array([[0],[0],[0],[0]])
x = cvx.Variable((4, 1))
t = cvx.Variable()
x1 = x[0]
x2 = x[1]
x3 = x[2]
x4 = x[3]
y = cvx.vstack([x1,x2])
z = cvx.vstack([x3,x4])
X1 = cvx.vstack([cvx.hstack([np.eye(4),Q00*x]),cvx.hstack([(Q00*x).T,t -
c0.T*x])])
X2 = cvx.vstack([cvx.hstack([np.eye(2),Q11*y]),cvx.hstack([(Q11*y).T,b1-
P1.T*y])])
X3 = cvx.vstack([cvx.hstack([np.eye(2),Q22*z]),cvx.hstack([(Q22*z).T,b2-
P2.T*z])])
constraints = [X1>>0,X2>>0,X3>>0]
objective = cvx.Minimize(t)
prob = cvx.Problem(objective, constraints)
prob.solve(solver=cvx.CVXOPT)
print("status:", prob.status)
print("optimal value", prob.value)
print("optimal var", x.value)
```

$$\begin{array}{ll} \textbf{Min.} & \boldsymbol{x}^T\boldsymbol{Q}_0\boldsymbol{x} + 2\boldsymbol{q}_0^T\boldsymbol{x} + \gamma_0 \\ \textbf{sub.to} & \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + 2\boldsymbol{q}_i^T\boldsymbol{x} + \gamma_i \le 0 \ (1 \le i \le m) \end{array}$$

Here $\boldsymbol{Q}_i \in \mathcal{S}^n$ **(the set of** $n \times n$ **symmetric matrices)**
$\boldsymbol{q}_i \in \mathbb{R}^n$ **(the** n **dimensional Euclidean space)**
$\gamma_i \in \mathbb{R}$ **(the set of real numbers)**

Let $\boldsymbol{M}_i = \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \in \mathcal{S}^{1+n}$. **Then we can rewrite**

$$\begin{aligned} \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + 2\boldsymbol{q}_i^T\boldsymbol{x} + \gamma_i &\equiv \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \\ &\equiv \boldsymbol{M}_i \bullet \boldsymbol{X} \equiv \sum_{j=0}^{n}\sum_{k=0}^{n}[M_i]_{jk}X_{jk}. \end{aligned}$$

Here

$$\boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}_+^{1+n}, \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T.$$

A quasi-convex optimization problem

$$\min \frac{(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})^T(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})}{\boldsymbol{d}^T\boldsymbol{x}} \text{ sub.to } \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}.$$

Here $\boldsymbol{L} \in \mathbb{R}^{k\times n}$, $\boldsymbol{c} \in \mathbb{R}^{k}$, $\boldsymbol{d} \in \mathbb{R}^{n}$, $\boldsymbol{A} \in \mathbb{R}^{\ell\times n}$, $\boldsymbol{b} \in \mathbb{R}^{\ell}$, and $\boldsymbol{d}^T\boldsymbol{x} > 0$ for $\forall$ feasible $\boldsymbol{x} \in \mathbb{R}^n$.

$$\Updownarrow$$

$$\min \zeta \text{ sub.to } \zeta \geq \frac{(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})^T(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})}{\boldsymbol{d}^T\boldsymbol{x}},\ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}.$$

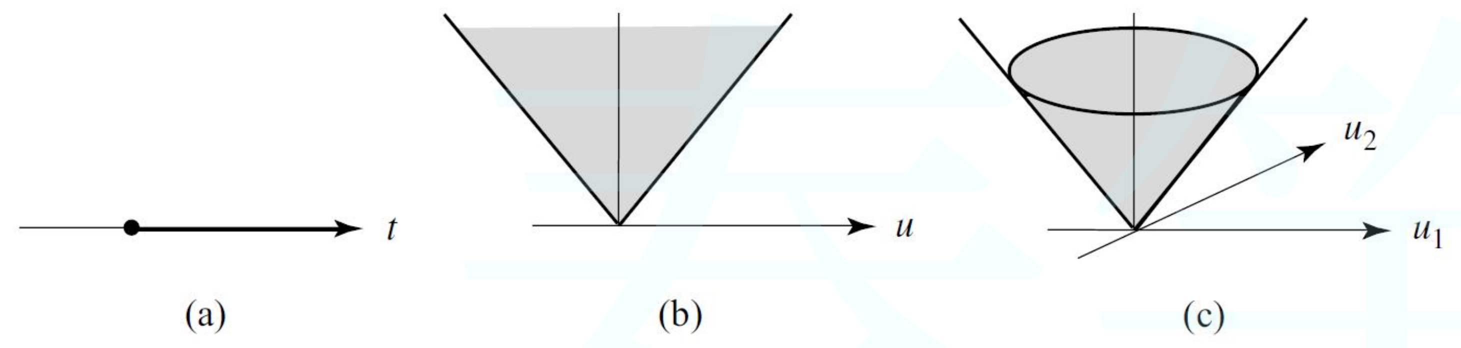
$$\Updownarrow\ \zeta - \frac{(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})^T(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})}{\boldsymbol{d}^T\boldsymbol{x}} \geq 0 \Leftrightarrow \begin{pmatrix} (\boldsymbol{d}^T\boldsymbol{x})\boldsymbol{I} & \boldsymbol{L}\boldsymbol{x}-\boldsymbol{c} \\ (\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})^T & \zeta \end{pmatrix} \succeq \boldsymbol{O}.$$

$$\text{SDP: } \min \zeta \text{ sub.to } \begin{pmatrix} \boldsymbol{d}^T\boldsymbol{x}\boldsymbol{I} & \boldsymbol{L}\boldsymbol{x}-\boldsymbol{c} \\ (\boldsymbol{L}\boldsymbol{x}-\boldsymbol{c})^T & \zeta \end{pmatrix} \succeq \boldsymbol{O},\ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}.$$

Second-Order Cone Programming (SOCP)

A second-order cone is also called quadratic or Lorentz cone

$$\mathcal{K} = \left\{ \begin{bmatrix} t \\ \mathbf{u} \end{bmatrix} : \ t \in R, \ \mathbf{u} \in R^{n-1} \text{ for } \|\mathbf{u}\| \leq t \right\}$$



the second-order cone K is a convex set

$$\lambda \begin{bmatrix} t_1 \\ \mathbf{u}_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} t_2 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \lambda t_1 + (1-\lambda) t_2 \\ \lambda \mathbf{u}_1 + (1-\lambda) \mathbf{u}_2 \end{bmatrix}$$

where

$$\|\lambda \mathbf{u}_1 + (1-\lambda)\mathbf{u}_2\| \leq \lambda \|\mathbf{u}_1\| + (1-\lambda)\|\mathbf{u}_2\| \leq \lambda t_1 + (1-\lambda) t_2$$

the *second-order cone program* (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \le c_i^T x + d_i \quad i = 1, \ldots, N, \end{array}$$

$$\|u\| \le t \Longleftrightarrow \begin{bmatrix} tI & u \\ u^T & t \end{bmatrix} \succeq 0,$$

SOCP (1) can be expressed as an SDP

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, N. \end{array}$$

QP problems with quadratic constraints(QCQP)

$$\begin{array}{ll} \text{minimize} & x^T P_0 x + 2q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + 2q_i^T x + r_i \leq 0 \quad i = 1, \ldots, p, \end{array}$$

all P must be postive

$$\begin{array}{ll} \text{minimize} & \left\| P_0^{1/2} x + P_0^{-1/2} q_0 \right\|^2 + r_0 - q_0^T P_0^{-1} q_0 \\ \text{subject to} & \left\| P_i^{1/2} x + P_i^{-1/2} q_i \right\|^2 + r_i - q_i^T P_i^{-1} q_i \leq 0, \quad i = 1, \ldots, p, \end{array}$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \| P_0^{1/2} x + P_0^{-1/2} q_0 \| \leq t, \\ & \| P_i^{1/2} x + P_i^{-1/2} q_i \| \leq \left(q_i^T P_i^{-1} q_i - r_i \right)^{1/2}, \quad i = 1, \ldots, p, \end{array}$$

$$\text{minimize } \delta$$

$$\text{subject to: } \quad [(x_1 - x_3)^2 + (x_2 - x_4)^2]^{1/2} \le \delta$$

$$[x_1\ x_2]\begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [x_1\ x_2]\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \le \frac{3}{4}$$

$$[x_3\ x_4]\begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 5/8 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - [x_3\ x_4]\begin{bmatrix} 11/2 \\ 13/2 \end{bmatrix} \le -\frac{35}{2}$$

```
Q0= np.array([[1,0,-1,0],[0,1,0,-1],[-
1,0,1,0],[0,-1,0,1]])
D,G=np.linalg.eig(Q0);
Q00=G.dot(np.diag(np.sqrt(D))).T
Q1=np.array([[1/4,0],[0,1]])
D,G=np.linalg.eig(Q1);
Q11=G.dot(np.diag(np.sqrt(D))).T
Q2=np.array([[5/8,3/8],[3/8,5/8]])
D,G=np.linalg.eig(Q2);
Q22=G.dot(np.diag(np.sqrt(D))).T
P1=np.array([[-1/2],[0]])
P2=np.array([[-11/2],[-13/2]])
b1=3/4
b2=-35/2
c0 = np.array([[0],[0],[0],[0]])
x = cvx.Variable((4, 1))
t = cvx.Variable()
```

```
x1 = x[0]
x2 = x[1]
x3 = x[2]
x4 = x[3]

y = cvx.vstack([x1,x2])
z = cvx.vstack([x3,x4])

constraints = [cvx.norm(Q00*x)<=cvx.sqrt(t - c0.T*x)]
constraints += [cvx.norm(Q11*y)<=cvx.sqrt(b1-P1.T*y)]
constraints += [cvx.norm(Q22*z)<=cvx.sqrt(b2-P2.T*z)]
constraints += [b1-P1.T*y>=0]
constraints += [b2-P2.T*z>=0]
objective = cvx.Minimize(t)
prob = cvx.Problem(objective, constraints)
prob.solve(solver=cvx.CVXOPT)
print("status:", prob.status)
print("optimal value", prob.value)
print("optimal var", x.value)
```

Note that

$$\begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}_+^n \iff X_{jj} \geq 0,\ X_{kk} \geq 0,\ X_{jj}X_{kk} - X_{jk}^2 \geq 0$$

$$\iff \left\| \begin{pmatrix} X_{jj} - Xkk \\ 2X_{jk} \end{pmatrix} \right\| \leq X_{jj} + X_{kk} \text{ \textbf{(an SOCP constraint)}}$$

where each $f_i(y) = y^T C_i y - d_i^T y - \epsilon_i$ with C_i psd. We can write $C_i = G_i^T G_i$ where $G_i \in \mathbb{R}^{r_i \times m}$. Then

$$f_i(y) \le 0 \iff d_i^T y + \epsilon_i \ge (G_i y)^T (G_i y)$$
$$\iff M_i = \begin{bmatrix} d_i^T y + \epsilon & (G_i y)^T \\ G_i y & I \end{bmatrix} \succeq 0$$

Alternatively

$$f_i(y) \le 0 \iff (d_i^T y + \epsilon_i + 1)^2 \ge (d_i^T + \epsilon_i - 1)^2 + (2G_i y)^T (2G_i y)$$
$$\iff (d_i^T y + \epsilon_i + 1) \ge \left\| \begin{matrix} d_i^T y + \epsilon_i - 1 \\ 2G_i y \end{matrix} \right\|_2$$
$$\iff d_i^T y + \epsilon_i + 1 \ge \frac{1}{d_i^T y + \epsilon_i + 1} \left\| \begin{matrix} d_i^T y + \epsilon_i - 1 \\ 2G_i y \end{matrix} \right\|_2^2$$
$$\iff W_i = \begin{bmatrix} d_i^T y + \epsilon_i + 1 & d_i^T y + \epsilon_i - 1 & (2G_i y)^T \\ d_i^T y + \epsilon_i - 1 & d_i^T y + \epsilon_i + 1 & 0 \\ 2G_i y & 0 & (d_i^T y + \epsilon_i + 1) I \end{bmatrix} \succeq 0$$

Problems involving sums of norms

$$\text{minimize} \quad \sum_{i=1}^{p} \|F_i x + g_i\|$$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} t_i \\ \text{subject to} & \|F_j x + g_j\| \le t_j, \ \ j = 1, \ldots, p. \end{array}$$

problems involving a maximum of norms

$$\text{minimize} \quad \max_{i=1,\ldots,p} \|F_i x + g_i\|$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|F_i x + g_i\| \le t, \quad i = 1, \ldots, p, \end{array}$$

Problems with hyperbolic constraints

$$w^T w \le xy,\ x \ge 0,\ y \ge 0 \Longleftrightarrow \left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \le x + y.$$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} 1/(a_i^T x + b_i) \\ \text{subject to} & a_i^T x + b_i > 0, \quad i = 1, \ldots, p \\ & c_i^T x + d_i \ge 0, \quad i = 1, \ldots, q, \end{array}$$

$$\downarrow$$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} t_i \\ \text{subject to} & t_i(a_i^T x + b_i) \ge 1, \quad t_i \ge 0, \quad i = 1, \ldots, p \\ & c_i^T x + d_i \ge 0, \quad i = 1, \ldots, q. \end{array}$$

$$\rightarrow$$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} t_i \\ \text{subject to} & \left\| \begin{bmatrix} 2 \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \le a_i^T x + b_i + t_i, \quad i = 1, \ldots, p \\ & c_i^T x + d_i \ge 0, \quad i = 1, \ldots, q. \end{array}$$

the quadratic/linear fractional problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{p} \frac{\|F_i x + g_i\|^2}{a_i^T x + b_i} \\
\text{subject to} & a_i^T x + b_i > 0, \quad i = 1, \ldots, p,
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{p} t_i \\
\text{subject to} & (F_i x + g_i)^T (F_i x + g_i) \le t_i (a_i^T x + b_i), \quad i = 1, \ldots, p \\
& a_i^T x + b_i > 0, \quad i = 1, \ldots, p,
\end{array}
$$

Matrix-fractional problems

$$\begin{array}{ll}\text{minimize} & (Fx+g)^T\left(P_0+x_1P_1+\cdots+x_pP_p\right)^{-1}(Fx+g)\\ \text{subject to} & P_0+x_1P_1+\cdots+x_pP_p \succ 0\\ & x \geq 0,\end{array}$$

$$\downarrow$$

$$\begin{array}{ll}\text{minimize} & t\\ \text{subject to} & \begin{bmatrix} P(x) & Fx+g \\ (Fx+g)^T & t \end{bmatrix} \succeq 0,\end{array}$$

附录：

$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$: semidefinite constraint.

- Definition: $\boldsymbol{X} \succeq \boldsymbol{O}$ if
  $\boldsymbol{u}^T \boldsymbol{X} \boldsymbol{u} = \sum_{i=1}^n \sum_{j=1}^n X_{ij} u_i u_j \geq 0$ for $\forall \boldsymbol{u} \in \mathbb{R}^n$.
- Definition: $\boldsymbol{X} \succ \boldsymbol{O}$ if
  $\boldsymbol{u}^T \boldsymbol{X} \boldsymbol{u} = \sum_{i=1}^n \sum_{j=1}^n X_{ij} u_i u_j > 0$ for $\forall \boldsymbol{u} \neq \mathbf{0}$.
- $\boldsymbol{X} \in \mathbb{S}^n \Rightarrow$ all n e.values are real.
- $\boldsymbol{X} \succeq \boldsymbol{O}$ $(\succ \boldsymbol{O}) \Leftrightarrow$ all n e.values ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ $(\succ \boldsymbol{O}) \Leftrightarrow$ all principal minors ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ $(\succ \boldsymbol{O}) \Rightarrow$ all diagonal X_{ii}'s ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ and $X_{ii} = 0 \Rightarrow X_{ij} = 0$ $(\forall j)$.

- $\mathbb{S}^n_+$ is a cone; $\alpha \boldsymbol{X} \in \mathbb{S}^n_+$ if $\alpha \geq 0$ and $\boldsymbol{X} \in \mathbb{S}^n_+$.
- $\mathbb{S}^n_+$ is convex;
  $\lambda \boldsymbol{X} + (1-\lambda)\boldsymbol{Y} \in \mathbb{S}^n_+$ if $0 \leq \lambda \leq 1$ and $\boldsymbol{X},\ \boldsymbol{Y} \in \mathbb{S}^n_+$.
- self-dual;
  $(\mathbb{S}^n_+)^* \equiv \left\{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{Y} \bullet \boldsymbol{X} \geq 0 \text{ for } \forall \boldsymbol{X} \in \mathbb{S}^n_+\right\} = \mathbb{S}^n_+$.
- $\boldsymbol{X},\ \boldsymbol{Y} \in \mathbb{S}^n_+$ and $\boldsymbol{X} \bullet \boldsymbol{Y} = 0 \Longrightarrow \boldsymbol{X}\boldsymbol{Y} = \boldsymbol{O}$.

- $\boldsymbol{X} \succeq \boldsymbol{O}$ $(\succ \boldsymbol{O}) \Leftrightarrow \exists n \times n$ (nonsingular) $\boldsymbol{B}$; $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{B}^T$ (factorization).
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists n \times n$ lower triang. $\boldsymbol{L}$; $\boldsymbol{X} = \boldsymbol{L}\boldsymbol{L}^T$ (Cholesky factorization).
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists n \times n$ orthogonal P and $\exists n \times n$ diagonal $\boldsymbol{D}$;
  $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T$ (orthogonal decompostion).

  Here each diagonal element $\lambda_i = D_{ii}$ of D is an eigenvalue of $\boldsymbol{X}$ and each ith column p_i of P an eigenvector corresponding to λ_i.
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists \boldsymbol{C} \in \mathbb{S}^n_+$; $\boldsymbol{X} = \boldsymbol{C}^2 \Leftarrow$ Take $\boldsymbol{C} = \boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T$;

  $$\boldsymbol{C}^2 = \left(\boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T\right)\left(\boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T\right) = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T = \boldsymbol{X}.$$

  We will write $\boldsymbol{X} = \left(\sqrt{X}\right)^2$.