机器人学中的状态估计 - 作业 6

peng00bo00

June 5, 2020

1. $\forall x \in \mathbb{R}^3$ 满足

$$\begin{aligned}
(Cu)^\wedge x &= (Cu) \times x \\
&= (Cu) \times (CC^{-1}x) \\
&= C(u \times C^{-1}x) \\
&= Cu^\wedge C^T x
\end{aligned} \tag{1}$$

因此

$$(Cu)^\wedge = Cu^\wedge C^T \tag{2}$$

2. $\forall C \in SO(3), u \in \mathbb{R}^3$ 满足如下性质:

$$(Cu)^\wedge = u^\wedge(\mathrm{tr}(C)I - C) - C^T u^\wedge \tag{3}$$

$$\mathrm{tr}(C) = 2\cos\phi + 1 \tag{4}$$

因此

$$\begin{aligned}(Cu)^\wedge &= u^\wedge(\mathrm{tr}(C)I - C) - C^T u^\wedge \\ &= u^\wedge((2\cos\phi + 1)I - C) - C^T u^\wedge \\ &= (2\cos\phi + 1)u^\wedge - u^\wedge C - C^T u^\wedge\end{aligned} \tag{5}$$

3.

$$
\begin{aligned}
\exp\{(Cu)^\wedge\} &= \exp\{Cu^\wedge C^T\} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(Cu^\wedge C^T)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} C(u^\wedge)^n C^T \\
&= C \sum_{n=0}^{\infty} \frac{1}{n!}(u^\wedge)^n C^T \\
&= C \exp\{u^\wedge\} C^T
\end{aligned}
\tag{6}
$$

4.

$$
\begin{aligned}
(\mathcal{T}x)^\wedge &= \left(\begin{bmatrix} C & r^\wedge C \\ 0 & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right)^\wedge \\
&= \begin{bmatrix} Cu + r^\wedge Cv \\ Cv \end{bmatrix}^\wedge \\
&= \begin{bmatrix} (Cv)^\wedge & Cu + r^\wedge Cv \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} Cv^\wedge C^T & Cu + r^\wedge Cv \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} v^\wedge C^T & u + C^{-1} r^\wedge Cv \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} v^\wedge C^T & u + (C^T r)^\wedge v \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} v^\wedge C^T & u - v^\wedge (C^T r) \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} v^\wedge & u \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} C^T & -C^T r \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= T x^\wedge T^{-1}
\end{aligned} \tag{7}
$$

5.

$$
\begin{aligned}
\exp\{(\mathcal{T}x)^\wedge\} &= \exp\{Tx^\wedge T^{-1}\} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(Tx^\wedge T^{-1})^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} T(x^\wedge)^n T^{-1} \\
&= T \sum_{n=0}^{\infty} \frac{1}{n!}(x^\wedge)^n T^{-1} \\
&= T \exp\{x^\wedge\} T^{-1}
\end{aligned} \tag{8}
$$

6.

$$
\begin{aligned}
x^\wedge p &= \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \\
&= \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \\
&= \begin{bmatrix} \phi^\wedge \varepsilon + \eta\rho \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\varepsilon^\wedge \phi + \eta\rho \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \eta I & -\varepsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \rho \\ \phi \end{bmatrix} \\
&= p^\odot x
\end{aligned} \tag{9}
$$

7.

$$
\begin{aligned}
p^T x^\wedge &= \begin{bmatrix} \varepsilon^T & \eta \end{bmatrix} \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge \\
&= \begin{bmatrix} \varepsilon^T & \eta \end{bmatrix} \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon^T \phi^\wedge & \varepsilon^T \rho \end{bmatrix} \\
&= \begin{bmatrix} (-\phi^\wedge \varepsilon)^T & \rho^T \varepsilon \end{bmatrix} \\
&= \begin{bmatrix} (\varepsilon^\wedge \phi)^T & \rho^T \varepsilon \end{bmatrix} \\
&= \begin{bmatrix} -\phi^T \varepsilon^\wedge & \rho^T \varepsilon \end{bmatrix} \\
&= \begin{bmatrix} \rho^T & \phi^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \varepsilon \\ -\varepsilon^\wedge & \mathbf{0} \end{bmatrix} \\
&= x^T p^\circledcirc
\end{aligned} \tag{10}
$$

8.

$$
\begin{aligned}
(Tp)^{\odot} &= \left(\begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \right)^{\odot} \\
&= \begin{bmatrix} C\varepsilon + \eta r \\ \eta \end{bmatrix}^{\odot} \\
&= \begin{bmatrix} \eta I & -(C\varepsilon + \eta r)^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
Tp^{\odot}\mathcal{T}^{-1} &= \begin{bmatrix} C & r \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} C^T & (-C^T r)^{\wedge} C^T \\ \mathbf{0} & C^T \end{bmatrix} \\
&= \begin{bmatrix} \eta C & -C\varepsilon^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} C^T & (C^T r^{\wedge} C) C^T \\ \mathbf{0} & C^T \end{bmatrix} \\
&= \begin{bmatrix} \eta C & -C\varepsilon^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} C^T & C^T r^{\wedge} \\ \mathbf{0} & C^T \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \eta r^{\wedge} - C\varepsilon^{\wedge} C^T \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \eta r^{\wedge} - (C\varepsilon)^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
&= \begin{bmatrix} \eta I & -(C\varepsilon + \eta r)^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}
\end{aligned} \tag{12}
$$

因此

$$
(Tp)^{\odot} = Tp^{\odot}\mathcal{T}^{-1} \tag{13}
$$

9.

$$
\begin{aligned}
(Tp)^{\odot T}(Tp)^{\odot} &= \left(Tp^{\odot}\mathcal{T}^{-1}\right)^{T} Tp^{\odot}\mathcal{T}^{-1} \\
&= \mathcal{T}^{-T} p^{\odot T} T^{T} T p^{\odot} \mathcal{T}^{-1}
\end{aligned} \tag{14}
$$

其中

$$
\begin{aligned}
p^{\odot T} T^{T} T p^{\odot} &= \begin{bmatrix} \eta I & \mathbf{0} \\ \varepsilon^{\wedge} & \mathbf{0}^{T} \end{bmatrix} \begin{bmatrix} C^{T} & \mathbf{0} \\ r^{T} & 1 \end{bmatrix} \begin{bmatrix} C & r \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ \mathbf{0}^{T} & \mathbf{0}^{T} \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \mathbf{0} \\ \varepsilon^{\wedge} & \mathbf{0}^{T} \end{bmatrix} \begin{bmatrix} I & C^{T} r \\ r^{T} C & r^{T} r + 1 \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ \mathbf{0}^{T} & \mathbf{0}^{T} \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \eta C^{T} r \\ \varepsilon^{\wedge} & \varepsilon^{\wedge} C^{T} r \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ \mathbf{0}^{T} & \mathbf{0}^{T} \end{bmatrix} \\
&= \begin{bmatrix} \eta^{2} I & -\eta\varepsilon^{\wedge} \\ \eta\varepsilon^{\wedge} & -\varepsilon^{\wedge}\varepsilon^{\wedge} \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \mathbf{0} \\ \varepsilon^{\wedge} & \mathbf{0}^{T} \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ \mathbf{0}^{T} & \mathbf{0}^{T} \end{bmatrix} \\
&= p^{\odot T} p^{\odot}
\end{aligned} \tag{15}
$$

因此

$$
(Tp)^{\odot T}(Tp)^{\odot} = \mathcal{T}^{-T} p^{\odot T} p^{\odot} \mathcal{T}^{-1} \tag{16}
$$