机器人学中的状态估计 - 作业 7

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1. 令 $\frac{\partial J}{\partial \mathbf{r}^T}=0$，得到

$$\frac{\partial J}{\partial \mathbf{r}^T}=\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j\left(\mathbf{y}_j^{\oplus}-(\mathbf{p}_j-\mathbf{r})^{+}\right)\mathbf{q}=0 \tag{1}$$

$$\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j\mathbf{y}_j^{\oplus}\mathbf{q}=\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j(\mathbf{p}_j-\mathbf{r})^{+}\mathbf{q} \tag{2}$$

对等式左边有

$$\begin{aligned}\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j\mathbf{y}_j^{\oplus}\mathbf{q}&=\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j\mathbf{q}^{+}\mathbf{y}_j\\&=\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+}\sum_{j=1}^{M}w_j\mathbf{y}_j\\&=w\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+}\mathbf{y}\end{aligned} \tag{3}$$

对等式右边有

$$\begin{aligned}\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j(\mathbf{p}_j-\mathbf{r})^{+}\mathbf{q}&=\mathbf{q}^{-1^{\oplus}}\sum_{j=1}^{M}w_j\mathbf{q}^{\oplus}(\mathbf{p}_j-\mathbf{r})\\&=\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{\oplus}\sum_{j=1}^{M}w_j(\mathbf{p}_j-\mathbf{r})\\&=\sum_{j=1}^{M}w_j(\mathbf{p}_j-\mathbf{r})\\&=w\mathbf{p}-w\mathbf{r}\end{aligned} \tag{4}$$

因此

$$\begin{aligned}w\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+}\mathbf{y}&=w\mathbf{p}-w\mathbf{r}\\\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+}\mathbf{y}&=\mathbf{p}-\mathbf{r}\\\mathbf{r}&=\mathbf{p}-\mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+}\mathbf{y}\\&=\mathbf{p}-\mathbf{q}^{+}\mathbf{q}^{-1^{\oplus}}\mathbf{y}\\&=\mathbf{p}-\mathbf{q}^{+}\mathbf{y}^{+}\mathbf{q}^{-1}\end{aligned} \tag{5}$$

2.

$$
\begin{aligned}
\frac{1}{w}\sum_{j=1}^{M} w_j z_j^{\odot^T} z_j^{\odot} &= \frac{1}{w}\sum_{j=1}^{M} w_j (\mathbf{T}_{\text{op}} p_j)^{\odot^T} (\mathbf{T}_{\text{op}} p_j)^{\odot} \\
&= \frac{1}{w}\sum_{j=1}^{M} w_j (\mathbf{T}_{\text{op}} p_j^{\odot} \mathcal{T}_{\text{op}}^{-1})^T (\mathbf{T}_{\text{op}} p_j^{\odot} \mathcal{T}_{\text{op}}^{-1}) \\
&= \frac{1}{w}\sum_{j=1}^{M} w_j \mathcal{T}_{\text{op}}^{-T} p_j^{\odot^T} \mathbf{T}_{\text{op}}^T \mathbf{T}_{\text{op}} p_j^{\odot} \mathcal{T}_{\text{op}}^{-1} \\
&= \frac{1}{w} \mathcal{T}_{\text{op}}^{-T} \Big(\sum_{j=1}^{M} w_j p_j^{\odot^T} \mathbf{T}_{\text{op}}^T \mathbf{T}_{\text{op}} p_j^{\odot} \Big) \mathcal{T}_{\text{op}}^{-1}
\end{aligned} \tag{6}
$$

$\forall T \in SE(3), p \in \mathbb{R}^3$ 满足

$$
\begin{aligned}
Tp^{\odot} &= \begin{bmatrix} R & t \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
&= \begin{bmatrix} \eta R & -R\varepsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
(Tp^{\odot})^T Tp^{\odot} &= p^{\odot^T} T^T T p^{\odot} \\
&= \begin{bmatrix} \eta R^T & \mathbf{0} \\ -\varepsilon^{\wedge^T} R^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta R & -R\varepsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
&= \begin{bmatrix} \eta^2 I & -\eta\varepsilon^\wedge \\ -\eta\varepsilon^{\wedge^T} & \varepsilon^{\wedge^T}\varepsilon^\wedge \end{bmatrix} \\
&= \begin{bmatrix} \eta I & \mathbf{0} \\ -\varepsilon^{\wedge^T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta I & -\varepsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
&= p^{\odot^T} p^{\odot}
\end{aligned} \tag{8}
$$

因此

$$
\begin{aligned}
\frac{1}{w}\sum_{j=1}^{M} w_j z_j^{\odot^T} z_j^{\odot} &= \frac{1}{w} \mathcal{T}_{\text{op}}^{-T} \Big(\sum_{j=1}^{M} w_j p_j^{\odot^T} \mathbf{T}_{\text{op}}^T \mathbf{T}_{\text{op}} p_j^{\odot} \Big) \mathcal{T}_{\text{op}}^{-1} \\
&= \frac{1}{w} \mathcal{T}_{\text{op}}^{-T} \Big(\sum_{j=1}^{M} w_j p_j^{\odot^T} p_j^{\odot} \Big) \mathcal{T}_{\text{op}}^{-1}
\end{aligned} \tag{9}
$$

3.

$$\begin{aligned}y_j - z_j &= y_j - T_{\mathrm{op}} p_j \\ &= \begin{bmatrix}\mathrm{y}_j \\ 1\end{bmatrix} - \begin{bmatrix}C_{\mathrm{op}} & -C_{\mathrm{op}}\mathrm{r}_{\mathrm{op}} \\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathrm{p}_j \\ 1\end{bmatrix} \\ &= \begin{bmatrix}\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ 0\end{bmatrix}\end{aligned} \tag{10}$$

$$\begin{aligned}z_j^{\odot} &= \begin{bmatrix}C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ 1\end{bmatrix}^{\odot} \\ &= \begin{bmatrix}I & -[C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})]^{\wedge} \\ \mathbf{0}^T & \mathbf{0}^T\end{bmatrix}\end{aligned} \tag{11}$$

$$\begin{aligned}z_j^{\odot T}(y_j - z_j) &= \begin{bmatrix}I & \mathbf{0} \\ [C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})]^{\wedge} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ 0\end{bmatrix} \\ &= \begin{bmatrix}\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ [C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})] \times [\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})]\end{bmatrix} \\ &= \begin{bmatrix}\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \times \mathrm{y}_j\end{bmatrix}\end{aligned} \tag{12}$$

因此

$$\frac{1}{w}\sum_{j=1}^{M} w_j z_j^{\odot T}(y_j - z_j) = \frac{1}{w}\sum_{j=1}^{M} w_j \begin{bmatrix}\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ [C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})] \times \mathrm{y}_j\end{bmatrix} \tag{13}$$

对于上半部分有

$$\begin{aligned}\frac{1}{w}\sum_{j=1}^{M} w_j[\mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}})] &= \frac{1}{w}\sum_{j=1}^{M} w_j\mathrm{y}_j - \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \\ &= \mathrm{y} - C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}})\end{aligned} \tag{14}$$

对于下半部分有

$$\begin{aligned}\frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{r}_{\mathrm{op}}) \times \mathrm{y}_j &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p} + \mathrm{p} - \mathrm{r}_{\mathrm{op}}) \times \mathrm{y}_j \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times \mathrm{y}_j + \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}}) \times \mathrm{y}_j \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times \mathrm{y}_j + C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}}) \times \mathrm{y} \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times \mathrm{y}_j - \mathrm{y}^{\wedge} C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}}) \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times (\mathrm{y}_j - \mathrm{y} + \mathrm{y}) - \mathrm{y}^{\wedge} C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}}) \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times (\mathrm{y}_j - \mathrm{y}) + \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times \mathrm{y} - \mathrm{y}^{\wedge} C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}}) \\ &= \frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j - \mathrm{p}) \times (\mathrm{y}_j - \mathrm{y}) - \mathrm{y}^{\wedge} C_{\mathrm{op}}(\mathrm{p} - \mathrm{r}_{\mathrm{op}})\end{aligned} \tag{15}$$

其中 $\frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p})\times(\mathrm{y}_j-\mathrm{y})$ 为一个 3 维列向量，第 i 位为

$$\begin{aligned}
\mathbf{1}_i^T\frac{1}{w}\sum_{j=1}^{M} w_j C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p})\times(\mathrm{y}_j-\mathrm{y}) &= \mathbf{1}_i^T\frac{1}{w}\sum_{j=1}^{M} w_j [C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p})]^\wedge(\mathrm{y}_j-\mathrm{y}) \\
&= \mathbf{1}_i^T\frac{1}{w}\sum_{j=1}^{M} -w_j(\mathrm{y}_j-\mathrm{y})^\wedge C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p}) \\
&= \frac{1}{w}\sum_{j=1}^{M} w_j(\mathrm{y}_j-\mathrm{y})^T\mathbf{1}_i^\wedge C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p}) \\
&= \mathrm{tr}\Big(\frac{1}{w}\sum_{j=1}^{M} w_j(\mathrm{y}_j-\mathrm{y})^T\mathbf{1}_i^\wedge C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{p})\Big) \\
&= \mathrm{tr}\Big(\mathbf{1}_i^\wedge C_{\mathrm{op}}\ \frac{1}{w}\sum_{j=1}^{M} w_j(\mathrm{p}_j-\mathrm{p})(\mathrm{y}_j-\mathrm{y})^T\Big) \\
&= \mathrm{tr}\Big(\mathbf{1}_i^\wedge C_{\mathrm{op}} W^T\Big) \\
&= b_i
\end{aligned} \tag{16}$$

最终

$$\begin{aligned}
\frac{1}{w}\sum_{j=1}^{M} w_j z_j^{\odot T}(y_j-z_j) &= \frac{1}{w}\sum_{j=1}^{M} w_j \begin{bmatrix} \mathrm{y}_j - C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{r}_{\mathrm{op}}) \\ [C_{\mathrm{op}}(\mathrm{p}_j-\mathrm{r}_{\mathrm{op}})]\times\mathrm{y}_j \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{y} - C_{\mathrm{op}}(\mathrm{p}-\mathrm{r}_{\mathrm{op}}) \\ \mathrm{b} - \mathrm{y}^\wedge C_{\mathrm{op}}(\mathrm{p}-\mathrm{r}_{\mathrm{op}}) \end{bmatrix}
\end{aligned} \tag{17}$$

4. 代码可参见./icp.cpp 和./pcl_icp.cpp，其中./icp.cpp 为手写的 ICP 算法而./pcl_icp.cpp 为调用 PCL 库的 ICP 算法。

./icp.cpp 进行点云配准后结果如 Fig.1所示，以第 1 个点云为基准第 2 个点云对应姿态为

$$
\begin{aligned}
T_{21} &= \begin{bmatrix} 0.999978 & 0.00717575 & 0.00229956 & -1.3049 \\ -0.0071792 & 0.999977 & 0.00154666 & -0.00251042 \\ -0.00228846 & -0.00156278 & 1.00001 & -0.0031377 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
P = T_{21}^{-1} &= \begin{bmatrix} 0.999966 & -0.00717924 & -0.00228835 & 1.30483 \\ 0.00717556 & 0.999969 & -0.0015631 & 0.0118688 \\ 0.00229956 & 0.00154628 & 0.999979 & 0.00614222 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{18}
$$

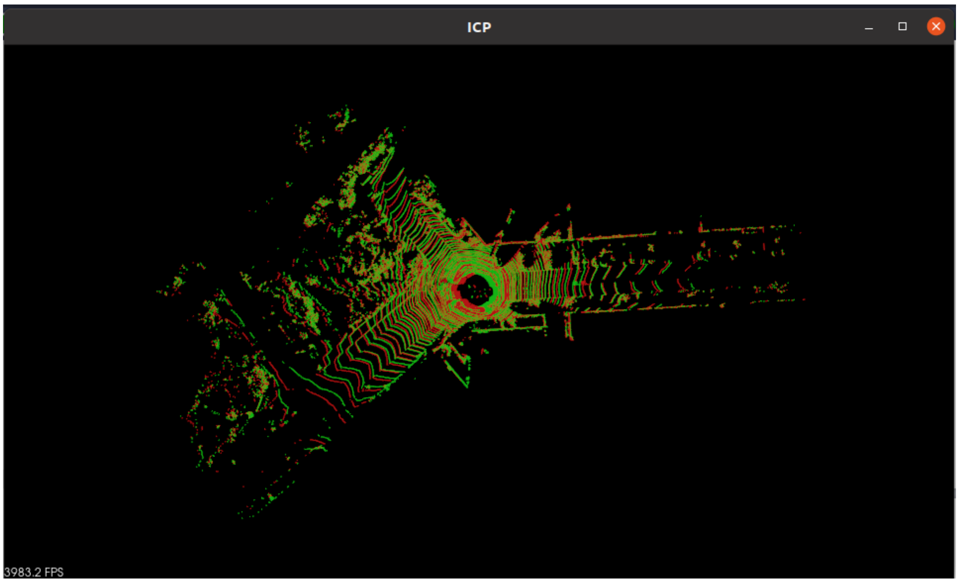

Figure 1: ICP

./pcl_icp.cpp 进行点云配准后结果如 Fig.2所示，对应姿态为

$$
\begin{aligned}
T_{21} &= \begin{bmatrix} 0.99975 & 0.0223736 & 0.000507309 & -1.03246 \\ -0.0223765 & 0.999725 & 0.00703656 & 0.0148587 \\ -0.000349739 & -0.00704614 & 0.999976 & -0.0223887 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
P = T_{21}^{-1} &= \begin{bmatrix} 0.999749 & -0.0223766 & -0.000349737 & 1.03253 \\ 0.0223735 & 0.999725 & -0.00704614 & 0.00808743 \\ 0.00050731 & 0.00703655 & 0.999974 & 0.0228074 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{19}
$$

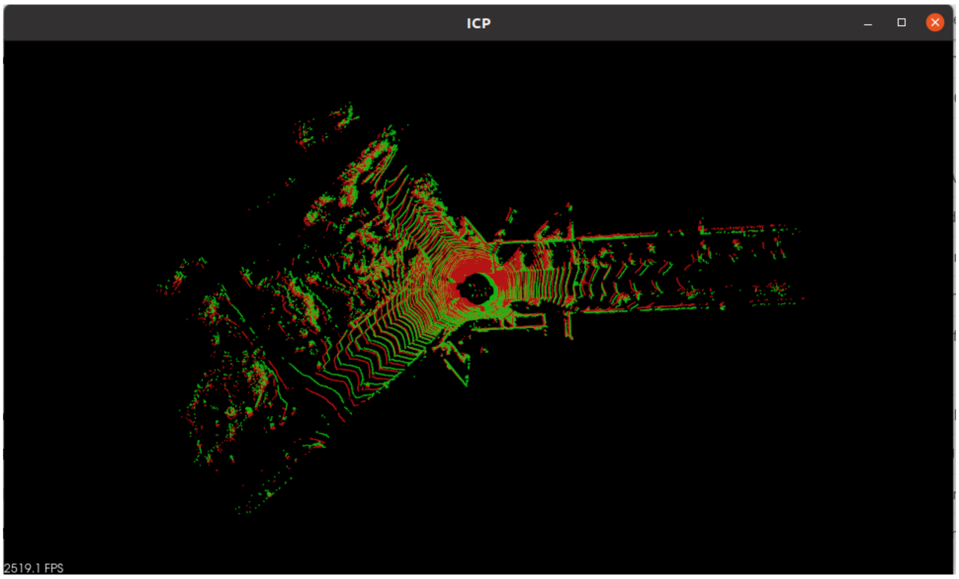

Figure 2: PCL-ICP