机器人学中的状态估计 - 作业 5

peng00bo00

May 23, 2020

1.

$$
\begin{aligned}
\mathbf{u}^\wedge\mathbf{v} &= \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= \begin{bmatrix} -u_3v_2 + u_2v_3 \\ u_3v_1 - u_1v_3 \\ -u_2v_1 + u_1v_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
&= -\mathbf{v}^\wedge\mathbf{u}
\end{aligned} \tag{1}
$$

2.

$$\begin{aligned}
\mathbf{C}\mathbf{C}^T &= [\cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{a}\mathbf{a}^T + \sin\theta\mathbf{a}^\wedge][\cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{a}\mathbf{a}^T + \sin\theta\mathbf{a}^\wedge]^T \\
&= \cos^2\theta\mathbf{I} + \cos\theta(1-\cos\theta)\mathbf{a}\mathbf{a}^T + \sin\theta\cos\theta\mathbf{a}^\wedge \\
&+ \cos\theta(1-\cos\theta)\mathbf{a}\mathbf{a}^T + (1-\cos\theta)^2\mathbf{a}\mathbf{a}^T\mathbf{a}\mathbf{a}^T + \sin\theta(1-\cos\theta)\mathbf{a}^\wedge\mathbf{a}\mathbf{a}^T \\
&- \sin\theta\cos\theta\mathbf{a}^\wedge - \sin\theta(1-\cos\theta)\mathbf{a}\mathbf{a}^T\mathbf{a}^\wedge - \sin^2\theta\mathbf{a}^\wedge\mathbf{a}^\wedge \\
&= \cos^2\theta\mathbf{I} + \sin^2\theta\mathbf{a}\mathbf{a}^T - \sin^2\theta\mathbf{a}^\wedge\mathbf{a}^\wedge
\end{aligned} \tag{2}$$

注意到

$$\begin{aligned}
\mathbf{a}^\wedge\mathbf{a}^\wedge &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -a_2^2 - a_3^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & -a_1^2 - a_3^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & -a_1^2 - a_2^2 \end{bmatrix} \\
&= \begin{bmatrix} a_1^2 - 1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 - 1 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 - 1 \end{bmatrix} \\
&= \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix} - \mathbf{I} \\
&= \mathbf{a}\mathbf{a}^T - \mathbf{I}
\end{aligned} \tag{3}$$

因此

$$\begin{aligned}
\mathbf{C}\mathbf{C}^T &= \cos^2\theta\mathbf{I} + \sin^2\theta\mathbf{a}\mathbf{a}^T - \sin^2\theta\mathbf{a}^\wedge\mathbf{a}^\wedge \\
&= \cos^2\theta\mathbf{I} + \sin^2\theta\mathbf{I} \\
&= \mathbf{I}
\end{aligned} \tag{4}$$

即

$$\mathbf{C}^{-1} = \mathbf{C}^T \tag{5}$$

3. 记旋转矩阵 $\mathbf{C}$ 为:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{6}$$

对于任意向量 $\mathbf{v}$ 有:

$$\begin{aligned} (\mathbf{C}\mathbf{v})^\wedge &= \begin{bmatrix} C_{11}v_1 + C_{12}v_2 + C_{13}v_3 \\ C_{21}v_1 + C_{22}v_2 + C_{23}v_3 \\ C_{31}v_1 + C_{32}v_2 + C_{33}v_3 \end{bmatrix}^\wedge \\ &= \begin{bmatrix} 0 & -C_{31}v_1 - C_{32}v_2 - C_{33}v_3 & C_{21}v_1 + C_{22}v_2 + C_{23}v_3 \\ C_{31}v_1 + C_{32}v_2 + C_{33}v_3 & 0 & -C_{11}v_1 - C_{12}v_2 - C_{13}v_3 \\ -C_{21}v_1 - C_{22}v_2 - C_{23}v_3 & C_{11}v_1 + C_{12}v_2 + C_{13}v_3 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{v}^\wedge \mathbf{C}^T &= \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \\ &= \begin{bmatrix} -v_3C_{12} + v_2C_{13} & -v_3C_{22} + v_2C_{23} & -v_3C_{32} + v_2C_{33} \\ v_3C_{11} - v_1C_{13} & v_3C_{21} - v_1C_{23} & v_3C_{31} - v_1C_{33} \\ -v_2C_{11} + v_1C_{12} & -v_2C_{21} + v_1C_{22} & -v_2C_{31} + v_1C_{32} \end{bmatrix} \end{aligned} \tag{8}$$

由于 $(\mathbf{C}\mathbf{v}^\wedge\mathbf{C}^T)^T = \mathbf{C}(\mathbf{v}^\wedge)^T\mathbf{C}^T = -\mathbf{C}\mathbf{v}^\wedge\mathbf{C}^T$, 即 $\mathbf{C}\mathbf{v}^\wedge\mathbf{C}^T$ 是反对称矩阵，故只需验证上三角 3 个元素即可。

$$\begin{aligned} (\mathbf{C}\mathbf{v}^\wedge\mathbf{C}^T)_{12} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix} \begin{bmatrix} -v_3C_{22} + v_2C_{23} \\ v_3C_{21} - v_1C_{23} \\ -v_2C_{21} + v_1C_{22} \end{bmatrix} \\ &= -v_3C_{11}C_{22} + v_2C_{11}C_{23} + v_3C_{12}C_{21} - v_1C_{12}C_{23} - v_2C_{13}C_{21} + v_1C_{13}C_{22} \\ &= -\begin{vmatrix} v_1 & v_2 & v_3 \\ C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{vmatrix} \\ &= -\mathbf{v} \cdot (\mathbf{C}_{1,} \times \mathbf{C}_{2,}) \\ &= -\mathbf{v} \cdot \mathbf{C}_{3,} \\ &= -C_{31}v_1 - C_{32}v_2 - C_{33}v_3 \\ &= (\mathbf{C}\mathbf{v})^\wedge{}_{12} \end{aligned} \tag{9}$$

类似的可以验证上三角的另外 2 个元素也相等, 因此:

$$\mathbf{C}\mathbf{v}^\wedge\mathbf{C}^T = (\mathbf{C}\mathbf{v})^\wedge \tag{10}$$