

单纯形法及应用练习

DOCTYPE - md & Katex. 使用基于Pandoc的渲染器输出PDF文档

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单纯形法的知识前提：图片从右向左看

Basic Feasible solution



$$\begin{aligned} \min f(\mathbf{x}) &= \mathbf{c}^T \mathbf{x} \\ \text{s.t. } A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Assumption:

$$m < n$$

$$\text{rank}(A) = m$$

$$(B, N) \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b}$$

$$B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$$

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N$$

$$A = (\underbrace{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m}_{\text{linearly independent}}, \mathbf{p}_{m+1}, \mathbf{p}_{m+2}, \dots, \mathbf{p}_n) = (B, N)$$

$$B = (\underbrace{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m}_{\text{Base vectors}}) \quad N = (\underbrace{\mathbf{p}_{m+1}, \mathbf{p}_{m+2}, \dots, \mathbf{p}_n}_{\text{Non-base vectors}})$$

$$\mathbf{x} = (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n)^T = (\mathbf{x}_B, \mathbf{x}_N)$$

$$\mathbf{x}_B = (\underbrace{x_1, x_2, \dots, x_m}_{\text{Base variables}})^T \quad \mathbf{x}_N = (\underbrace{x_{m+1}, x_{m+2}, \dots, x_n}_{\text{Non-base variables}})^T$$

Let $\mathbf{x}_N = \mathbf{0}$ \Rightarrow $\mathbf{x}_B = B^{-1}\mathbf{b}$

Basic solution $\mathbf{x} = (B^{-1}\mathbf{b}, \mathbf{0})^T$

If $B^{-1}\mathbf{b} \geq \mathbf{0}$, \mathbf{x} is a basic feasible solution.

当解出 $\vec{x} = (\vec{B}^{-1} b, 0)^T$ 得到基本解(Basic solution).

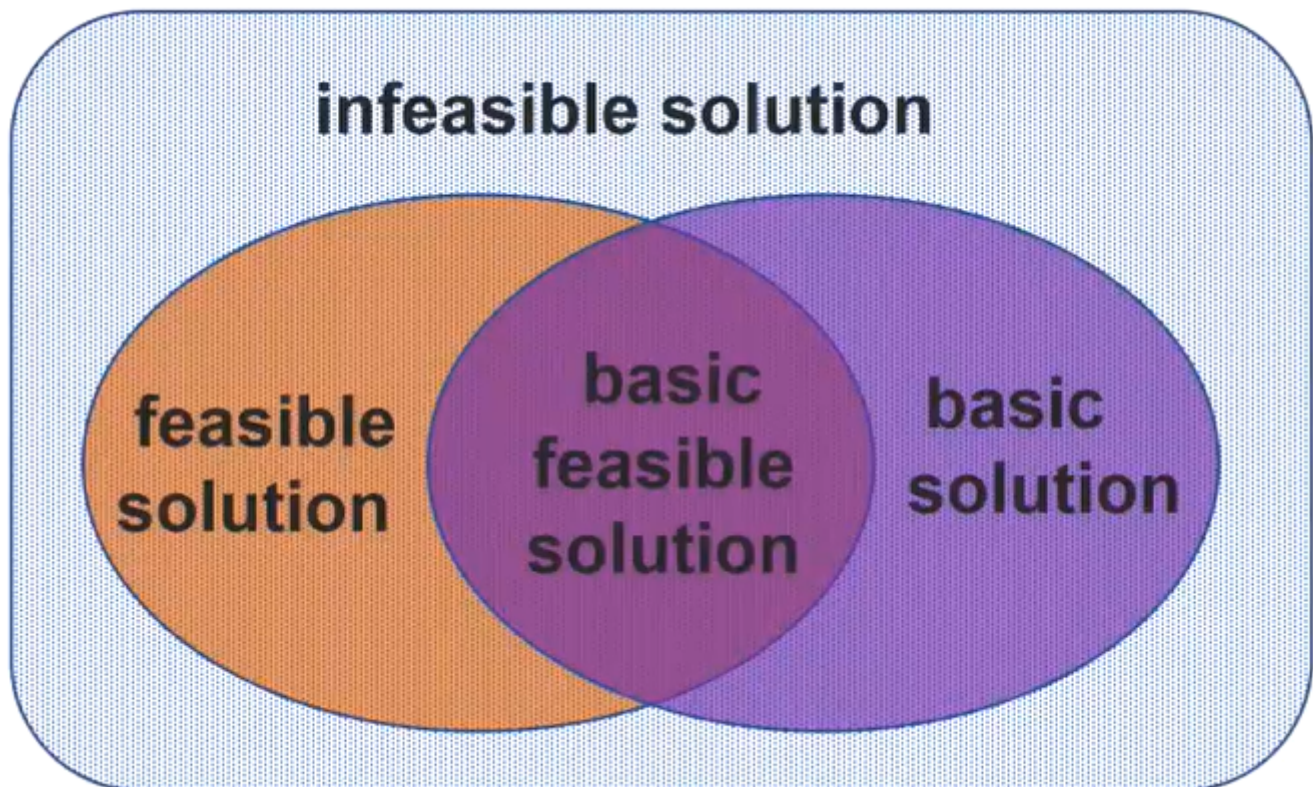
当满足 $\vec{B}^{-1} b \geq 0$, 称 x 为一个基本可行解(Basic feasible solution).

Basic Feasible Solution

- Consider a LP problem in the standard form that contains n variables and m constraints (assume $n \geq m$)
- A **basic solution** is obtained by setting $n - m$ variables (NBV) equal to 0 and solving for the remaining m variables (BV)

$$\binom{n}{m} = \binom{n}{n-m} = \frac{n!}{m!(n-m)!}$$

- Any basic solution in which all variables are nonnegative is called a **basic feasible solution** (or **BFS**)



什么是“凸 Convex”和“凹 Concave”？

参考资料：

- 什么是凸组合(Convex combination)?
- Convex combination的一个特殊情况: 两个向量的convex combination
- Operation Research Course 这个的课程蛮好, 口音清晰, 讲解清楚, 比.....

凸组合 Convex Combination的定义是:

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k \\ \lambda_1, \lambda_2, \dots, \lambda_k \in [0, 1] \\ \lambda_1 + \lambda_2 + \dots + \lambda_k = 1 \quad or \quad \sum_{i=1}^N \lambda_i = 1 \end{cases}$$

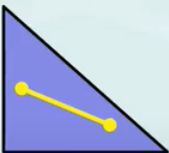


这里还有一个严格凸组合(Strict convex combination)的定义, 仅仅需要将 Convex combination's definition约束到:

$$\lambda_1, \lambda_2, \dots, \lambda_k \in (0, 1)$$

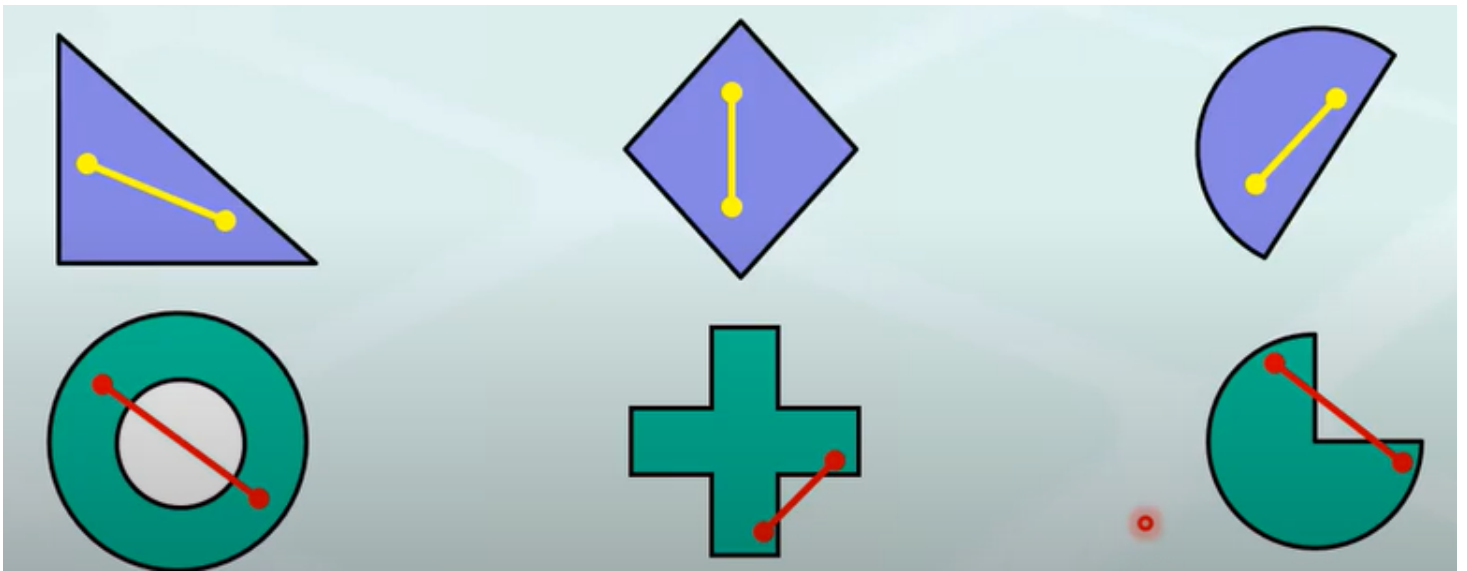
什么是凸集 Convex Set -

Convex Set

- A set of $S \subseteq \mathbb{R}^n$ is a **convex set** if it contains all convex combinations of any two points within it
- Graphically: A set of points S is a convex set if **the line segment joining any two points in S is wholly contained in S**

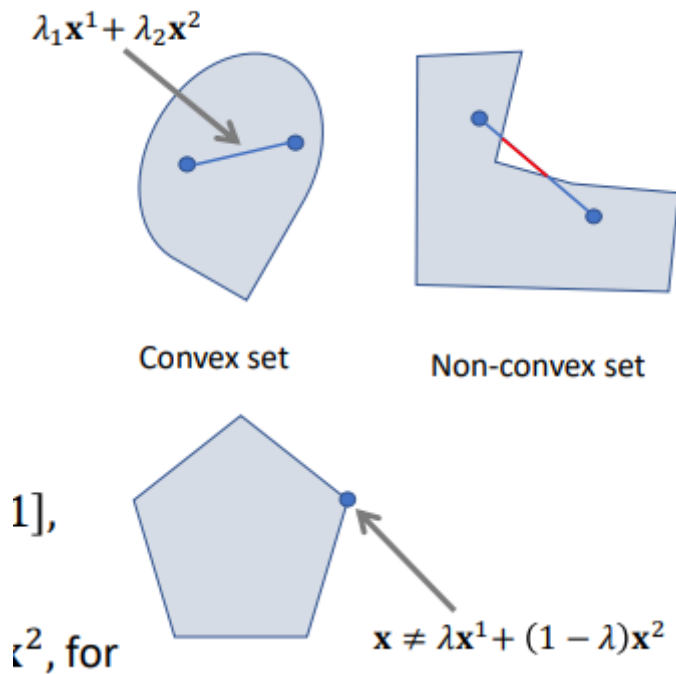




上排为Convex Set, 下排为 Non-Convex Set:



补充 -

- Given a convex set S and $x \in S, x_1 \in S, x_2 \in S, x_1 \neq x_2$, for any $\lambda \in (0, 1), x \neq \lambda x_1 + (1 - \lambda) x_2$, then S is a vertex of S

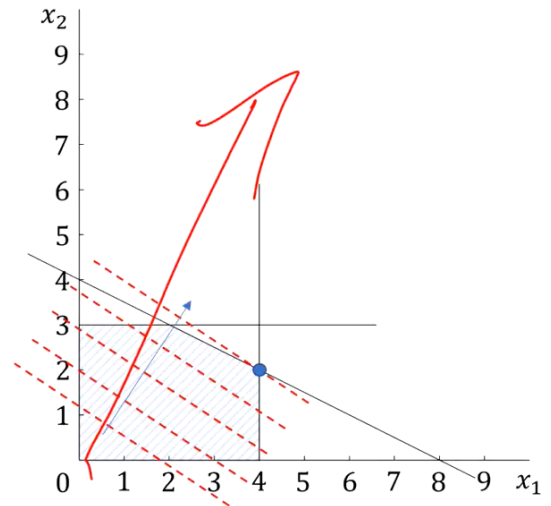


Two special cases:

- A empty set is a convex set.
- A single point set is a convex set.

Basic Theories of LP

- The feasible region of the LP problem is a convex set.
- One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem.
- The basic feasible solution of the linear programming problem lies at the vertex of its feasible region.



Theorem - The feasible region of a LP problem is a convex set

证明也很简单:

Every linear constraint will split the space in half. In each half space, it is a convex set. It is easy to prove that the intersection of convex sets is still a convex set.

Theorem - One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem

Theorem - The basic feasible solution of the linear programming problem lies at the vertex of its feasible region

证明第三条theorem:

第三条Theorem的数学表达是 -

$$\text{For any } \lambda \in (0, 1), x \neq \lambda x_1 + (1 - \lambda) x_2$$

那么要说明其不成立, 可以利用反证法: 假设 x 在 $x_1, x_2 (x_1 \neq x_2)$ 的line segment上, 那么有 $x = \lambda x_1 + (1 - \lambda) x_2$.

存在如下关系:

$$\begin{cases} x_1 = (x_B^1, x_N^1)^T \\ x_2 = (x_B^2, x_N^2)^T \\ x_B = \lambda x_B^1 + (1 - \lambda) x_B^2 \\ x_N = \lambda x_N^1 + (1 - \lambda) x_N^2 \end{cases}$$

由于我们知道，由于加入了slack variables，我们令 $x_N = 0$ 且 $\lambda \in (0, 1)$ ，从而很显然地我们得到

$$\begin{cases} x_N = x_N^1 = x_N^2 = 0 \\ x_1 = (x_B^1, 0)^T \\ x_2 = (x_B^2, 0)^T \end{cases}$$

那么我们从最基础的条件出发 $Ax = (B, N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = B$ 从而有

$$Bx_B = Bx_B^1 = Bx_B^2 = b$$

上面这个式子有同学问我为什么能够成立，其实很简单，因为我们的前提是 x_1, x_2 都是Feasible solutions.最后我们得到的结果就很容易了：由于 B 是一个空间的基，显然是一个非奇异矩阵(Non-Singular Matrix).因此我们得到了这样的结论—— $x_B = x_B^1 = x_B^2$, i.e. $x = x_1 = x_2$.

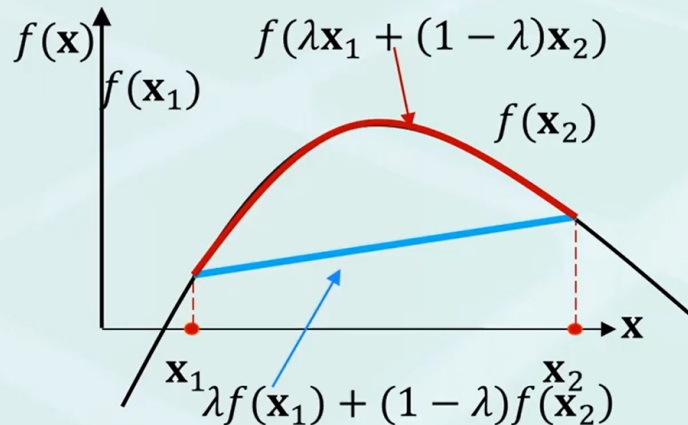
什么是凸函数Convex Function 和 凹函数Concave Funtion

Concave Functions

- Let S be a **convex set**. The function $f(x): S \rightarrow \mathbb{R}$ is a **concave function** if for any two points x_1, x_2 in S

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2), \lambda \in [0,1]$$

- $f(x)$ is concave if its value is **above the interpolation** formed between any two points



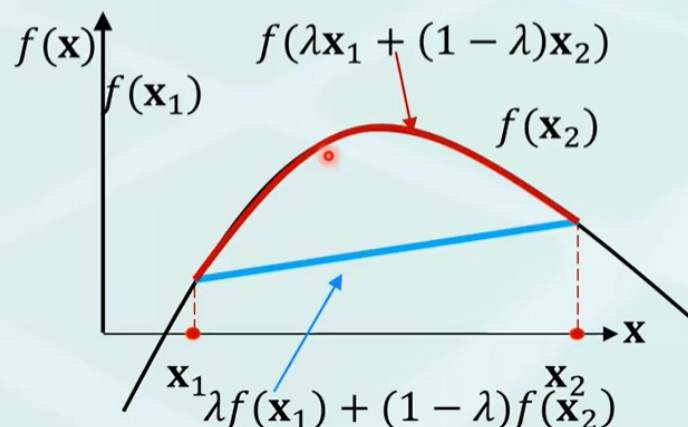
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Concave Functions

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单纯形法 - 步骤与讲解

这一部分内容参考了textbook - Introduction to Operations Research (Frederick S. Hillier & Gerald J. Lieberman) 第十版 (英文版) ;

其他参考内容将在内容下注明.

TODO: 期末复习单纯形法时回顾并把书上的笔记摘录

TODO: 单纯形法优越性与成立的数学证明