南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Basic Concepts of Operational Research

Zhenkun Wang (王振坤)

系统设计与智能制造学院
School of System Design and Intelligent Manufacturing

What is Operational Research?

运筹学的起源

孙膑运用了一种科学合理的思想方法，就是我国数学家华罗庚先生在《统筹方法》中所介绍的对策论。

齐王 / 田忌	上	中	下
上	★	★	
中		★	★
下	★		★

知已知彼 百战不殆?
需要学好运筹学

田忌赛马的故事说明，在已有条件下，经过统筹安排，选择一个最好的方案，就会取得最好的效果。

中国早期运筹学思想的运用：田忌赛马

What is Operational Research?

School of System Design and Intelligent Manufacturing

SUSTech Southern University of Science and Technology

- Operational Research (also called Operations Research, OR) is a discipline that deals with the application of advanced analytical methods to help make better decisions.

- It is an applied mathematics subject that uses methods such as mathematical modeling, statistical analysis, and mathematical optimization to find the best or near best solution to complex engineering and management problems.

- 运筹学是一门以定量方法为管理决策提供科学依据的学科。

- 运筹学是一门应用科学，它广泛应用现有的科学技术和数学方法，解决实际中提出的专门问题，为决策者选择最优决策提供定量依据。

- 运筹学是为决策机构在对其控制下的业务活动进行决策时，提供以数量化为基础的科学方法。

What is Operational Research?

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- OR is a discipline that studies how to achieve the optimal arrangement.
- 日本译作“运用学”。
- 中国香港、中国台湾地区译作“作业研究”。
- 中国大陆1956年译作“运用学”，1957年之后定名“运筹学”。

History of OR

- 学界通常将运筹学起源定为第二次世界大战期间，英美两国为有效地配置各项资源，召集了一支包含数学家、物理学家甚至心理学家的综合团队(由于成员复杂，被戏称为布莱克特马戏团)，对军事作业规划进行研究。为了保密需要他们把这项研究成为“Operational Research”。

- 他们的研究成果帮助联军打赢了“不列颠空战” (Air Battle of Britain)、“北大西洋战争” (Battle of the North Atlantic)、“太平洋岛屿战争” (Island Campaign in the Pacific)。

History of OR

1935年

History of OR

1939年

History of OR

1942年

History of OR

管理运筹学阶段:

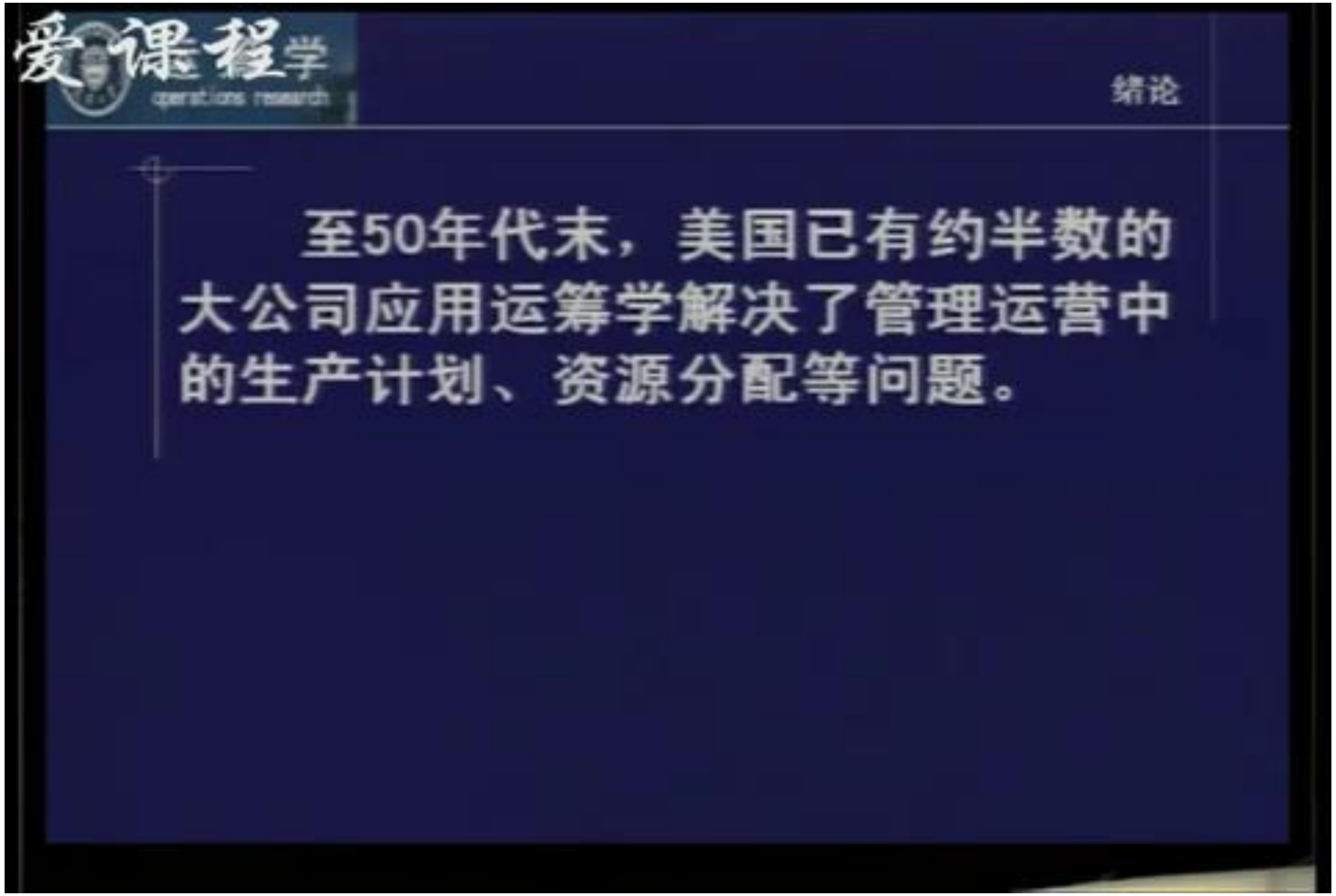
第二次世界大战以后的工业恢复繁荣时期，从事战时运筹学工作的许多专家致力于将战时形成的运筹学方法应用到工商企业、政府以及其他社会经济部门，运筹学作为一门学科逐步形成并得以迅速发展。

- 排队论的先驱者——丹麦工程师爱尔朗（A.K.Erlang）1917年在哥本哈根电话公司研究电话通信系统时，提出了排队论的一些著名公式。
- 20世纪30年代，荷兰人何雷斯列文生（Horace.C.Levenson）用运筹学思想分析商业广告和顾客心里，有次提出了存储论中著名的“经济批量公式”。
- 1939年前苏联学者康托洛维奇在解决工业生产组织和计划问题时，已提出类似线性规划的模型，并给出“解乘数法”的求解方法。出版了线性规划的第一部专著《生产组织与计划中的数学计算问题》、由于当时未被领导重视，直到1960年发表了《最佳资源利用的经济计算》一书后，才受到国内外的一致重视。康托洛维奇因此获得诺贝尔经济学奖。

History of OR

1950s

History of OR



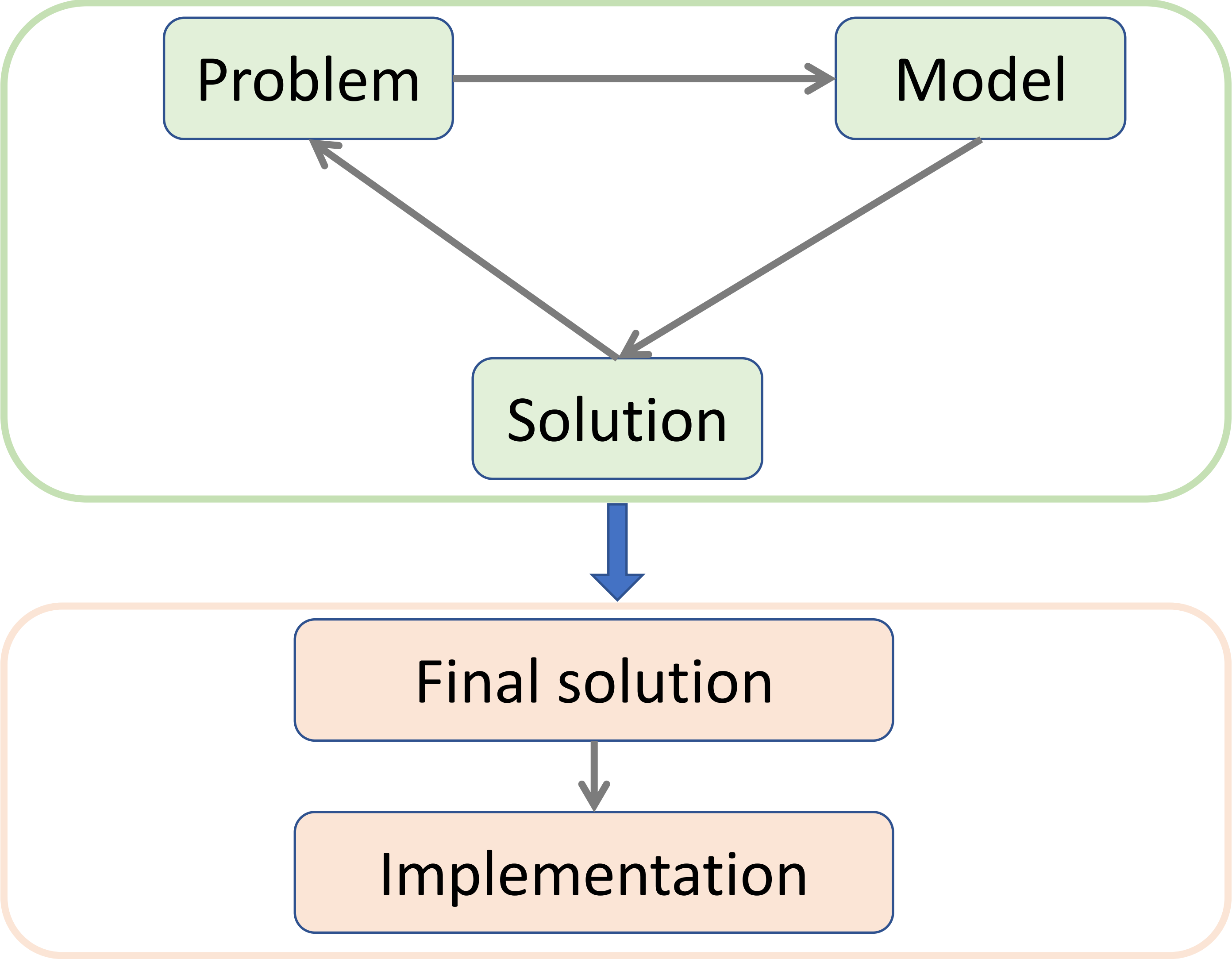
Application of OR

Application of OR
Production Planning
Facility Location
Job-Scheduling
Logistic Transportation
Bin-Packing
Inventory Management
Resource Allocation
Project Planning
Portfolio
Queuing

Procedures of OR

OR

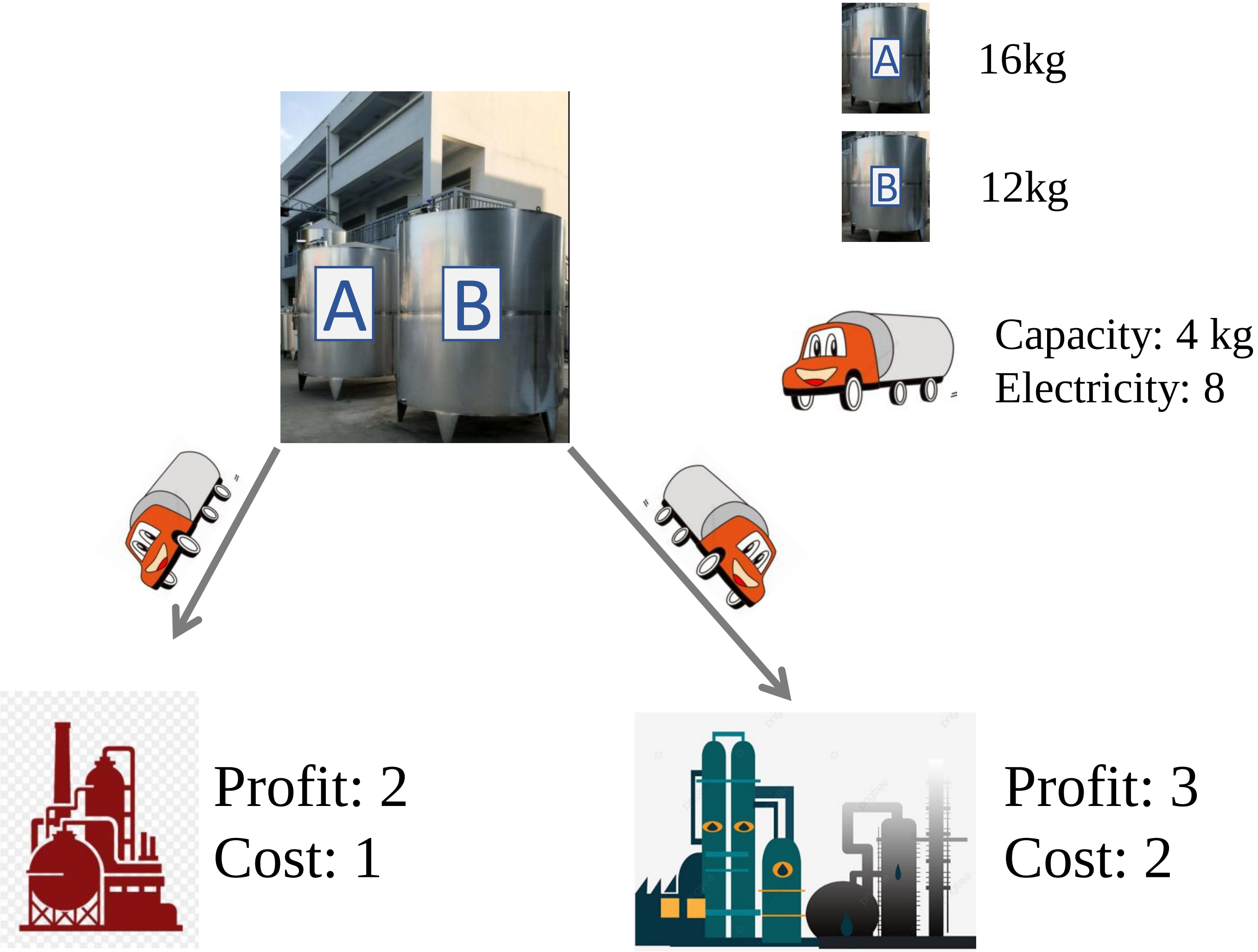
Decision-maker



Mathematical Model of OR

• Decision variables	$\mathbf{x} = x_1, x_2, \dots, x_n$	x_1, x_2
• Objective functions	minimize $f(\mathbf{x})$	$\min f(\mathbf{x}) = -2x_1 - 3x_2$
• Constraints	$g(\mathbf{x}) \geq 0$	$x_1 + 2x_2 \leq 8$ $0 \leq x_1 \leq 4$ $0 \leq x_2 \leq 3$
• Parameters	$\mu_1, \mu_2, \dots, \mu_k$	$0, -2, -3, 3, 4$

Linear Programming (LP)



Linear Programming (LP)

- Decision variables

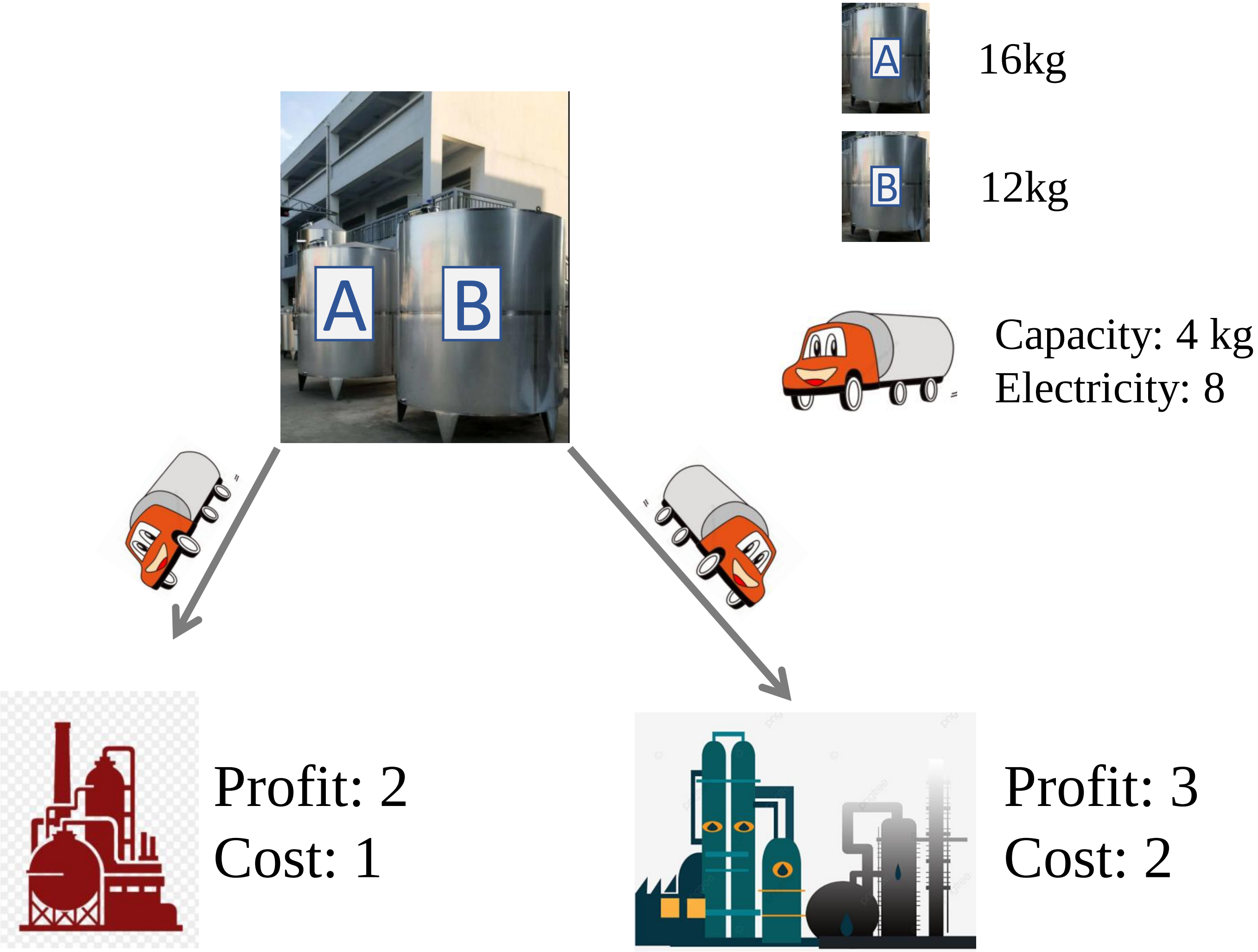
$$x_1, x_2$$

- Objective function

$$\max \ z = 2x_1 + 3x_2$$

- Constraints

$$x_1 + 2x_2 \leq 8$$
$$4x_1 \leq 16$$
$$4x_2 \leq 12$$
$$x_1, x_2 \geq 0$$



Graphical Method

- Decision variables

$$x_1, x_2$$

- Objective function

$$\max\ z = 2x_1 + 3x_2$$

- Constraints

$$x_1 + 2x_2 \leq 8$$
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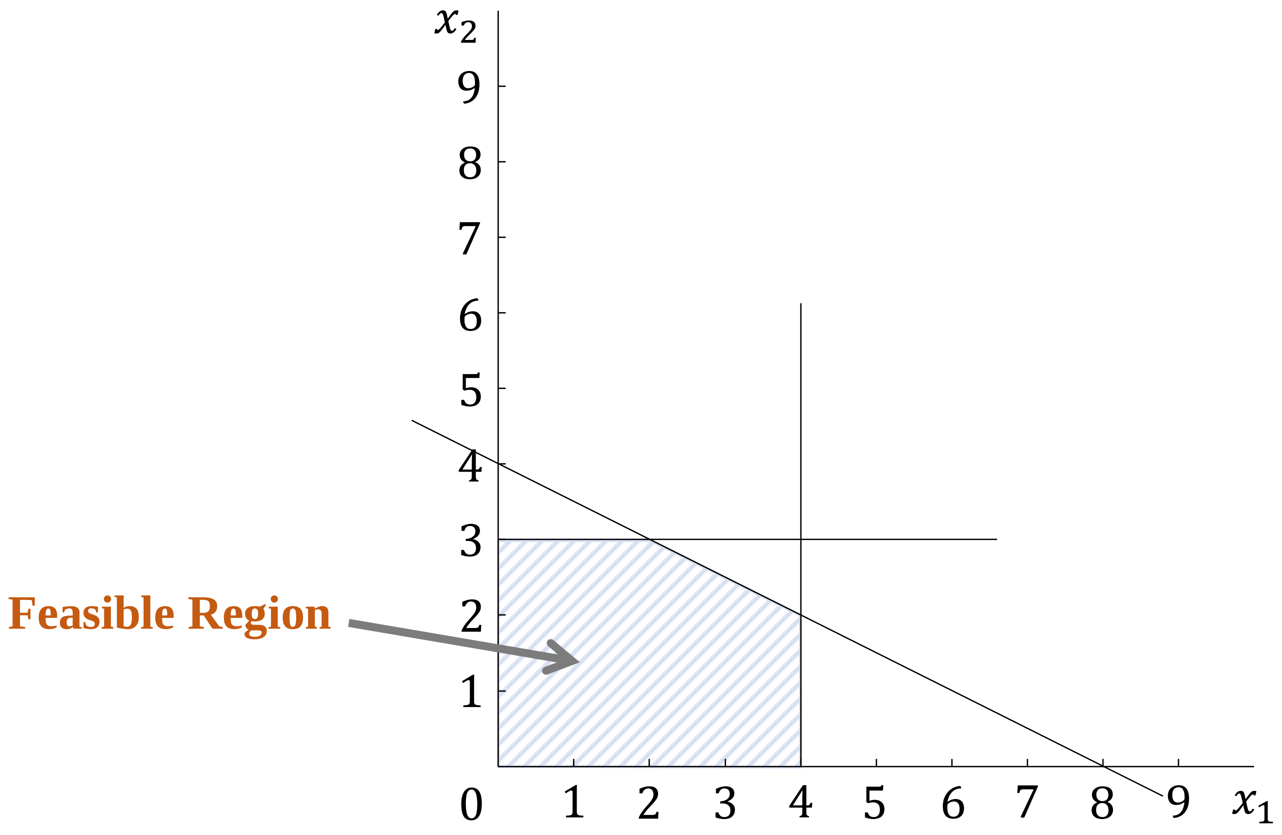

Graphical Method

- Decision variables

$$x_1, x_2$$

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$$\max \; z = 2x_1 + 3x_2$$

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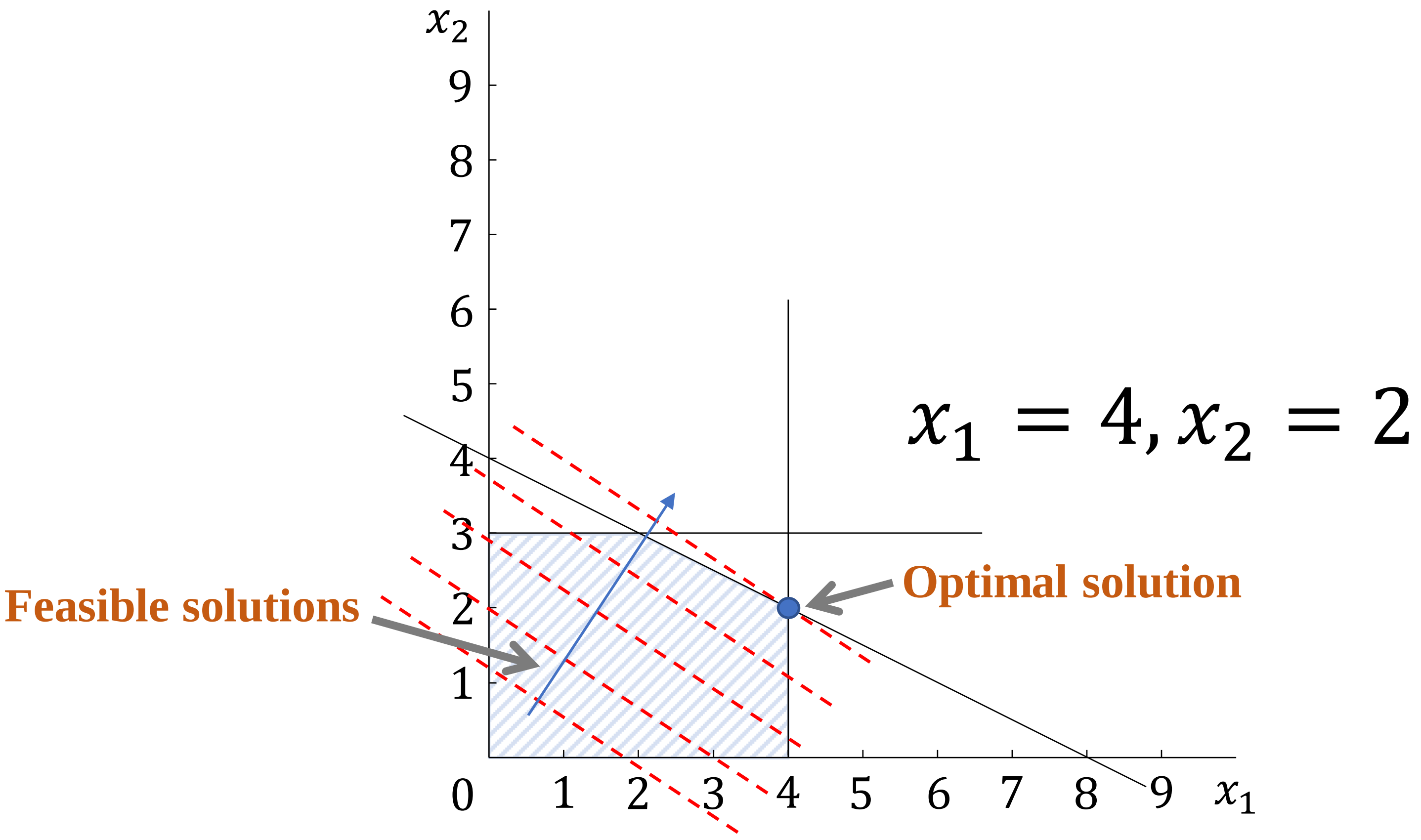
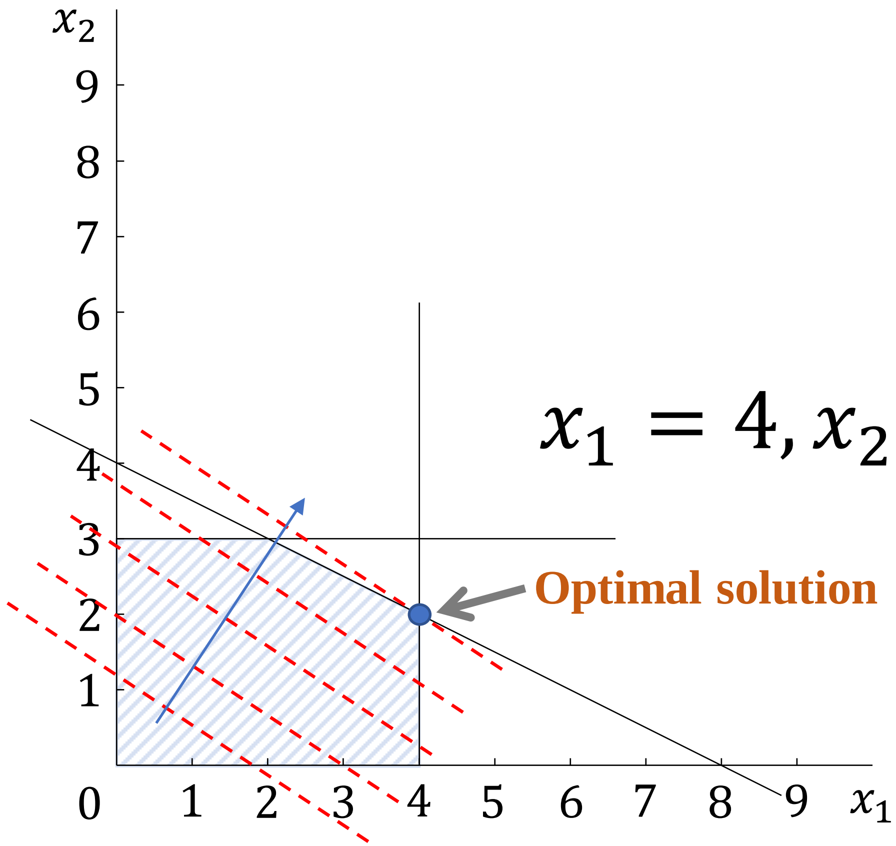
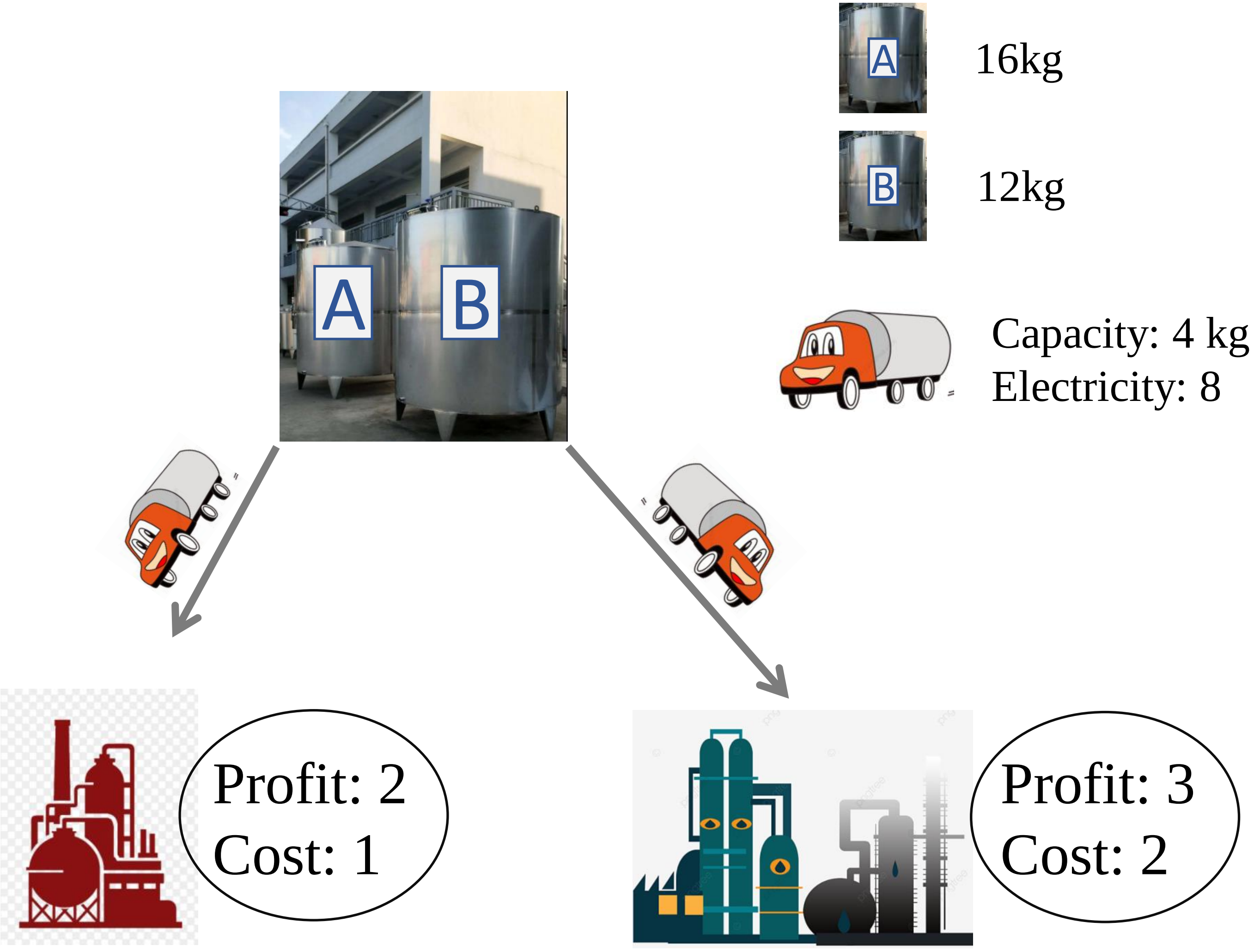


Linear Programming (LP)



$$x_1 = 4, x_2 = 2$$



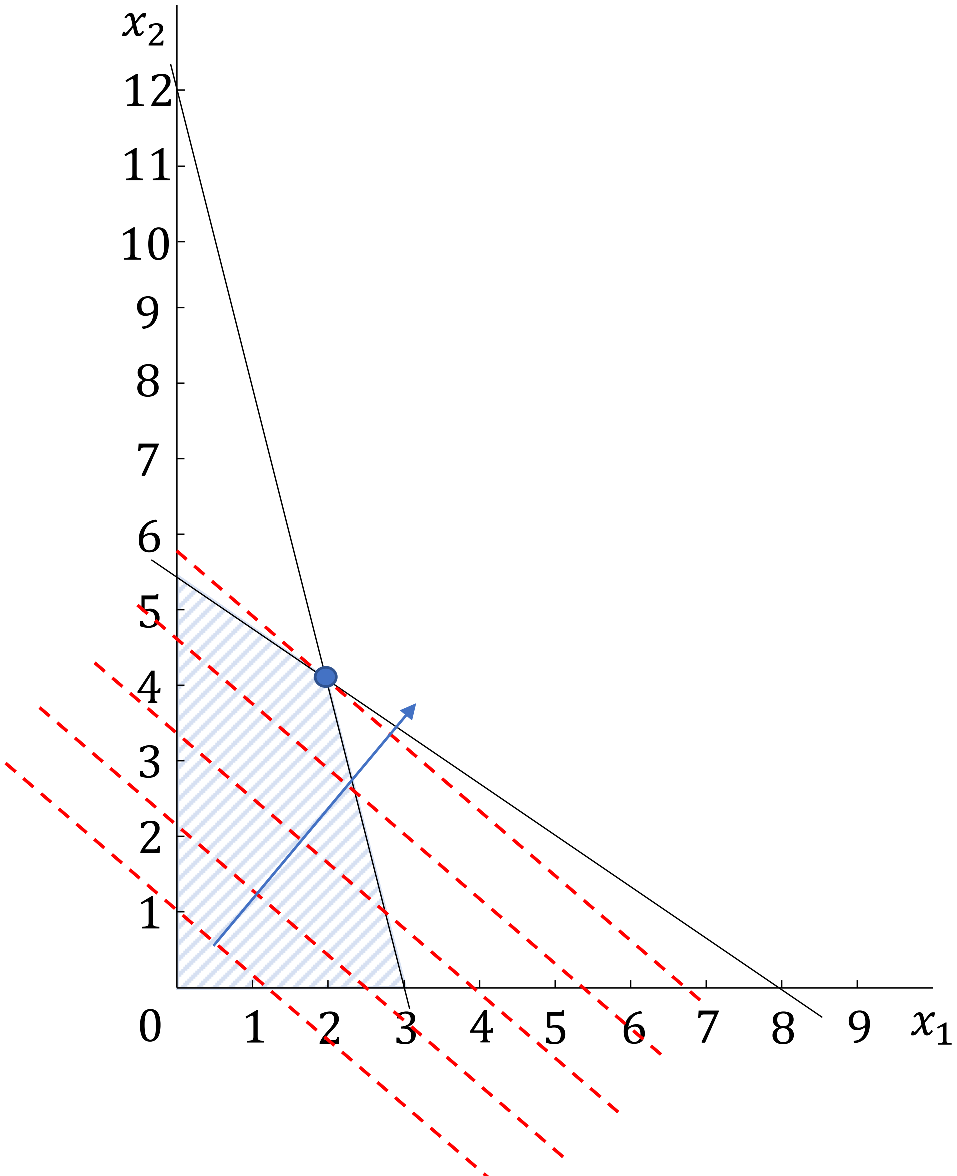
Practice

$$\begin{aligned} \max \quad & z = 6x_1 + 7x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \le 16 \\ & 4x_1 + x_2 \le 12 \\ & x_1, x_2 \ge 0 \end{aligned}$$

5 minutes

Practice

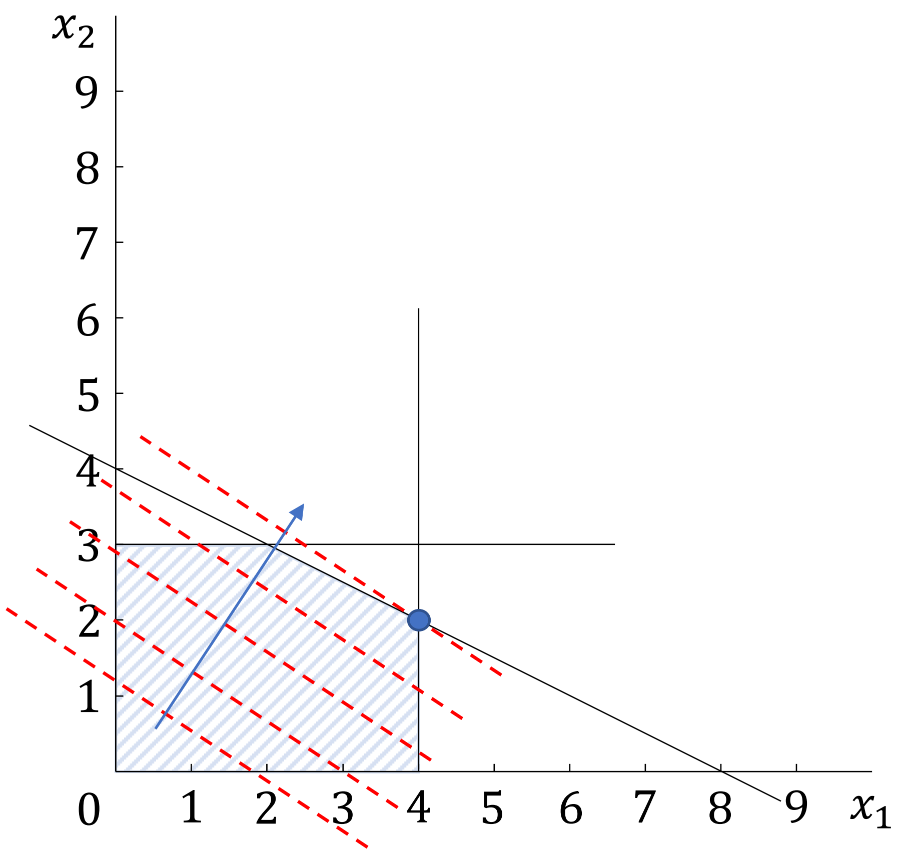
$$\max\ z = 6x_1 + 7x_2$$
$$\text{s.t.}\quad 2x_1 + 3x_2 \le 16$$
$$4x_1 + x_2 \le 12$$
$$x_1, x_2 \ge 0$$



Multiple optimal solutions

$$\max \ z = 2x_1 + 3x_2$$

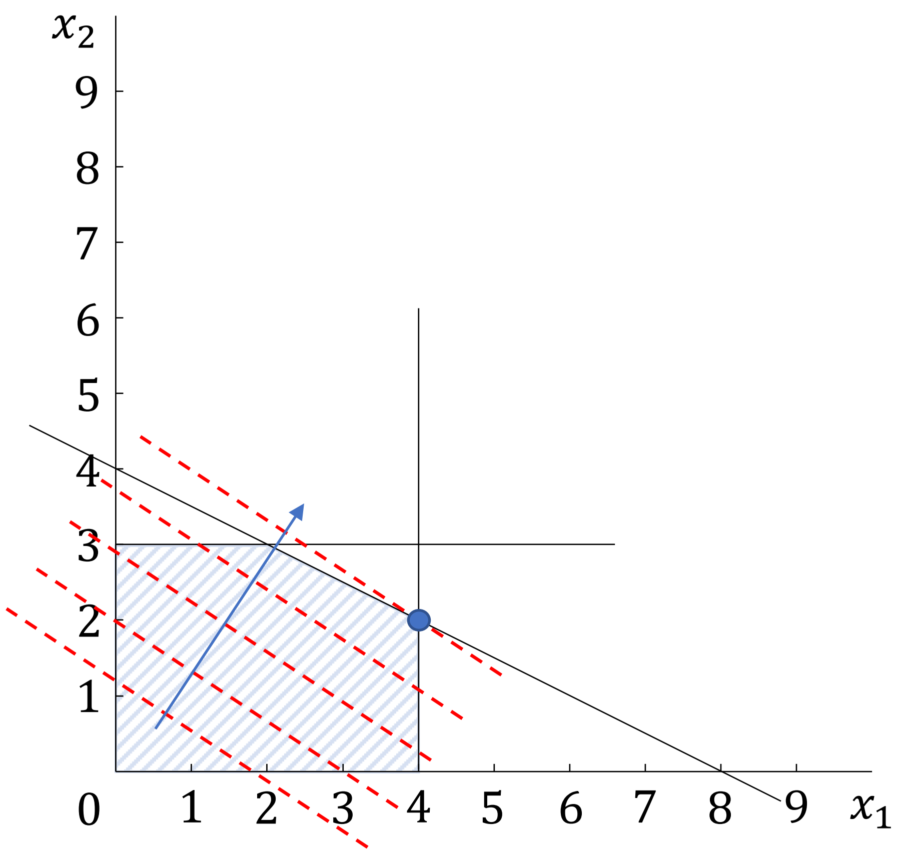
$$\text{s.t.} \quad x_1 + 2x_2 \leq 8$$
$$4x_1 \leq 16$$
$$4x_2 \leq 12$$
$$x_1, x_2 \geq 0$$



Multiple optimal solutions

$$\max \ z = x_1 + 2x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + 2x_2 &\leq 8 \\ 4x_1 &\leq 16 \\ 4x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Multiple optimal solutions

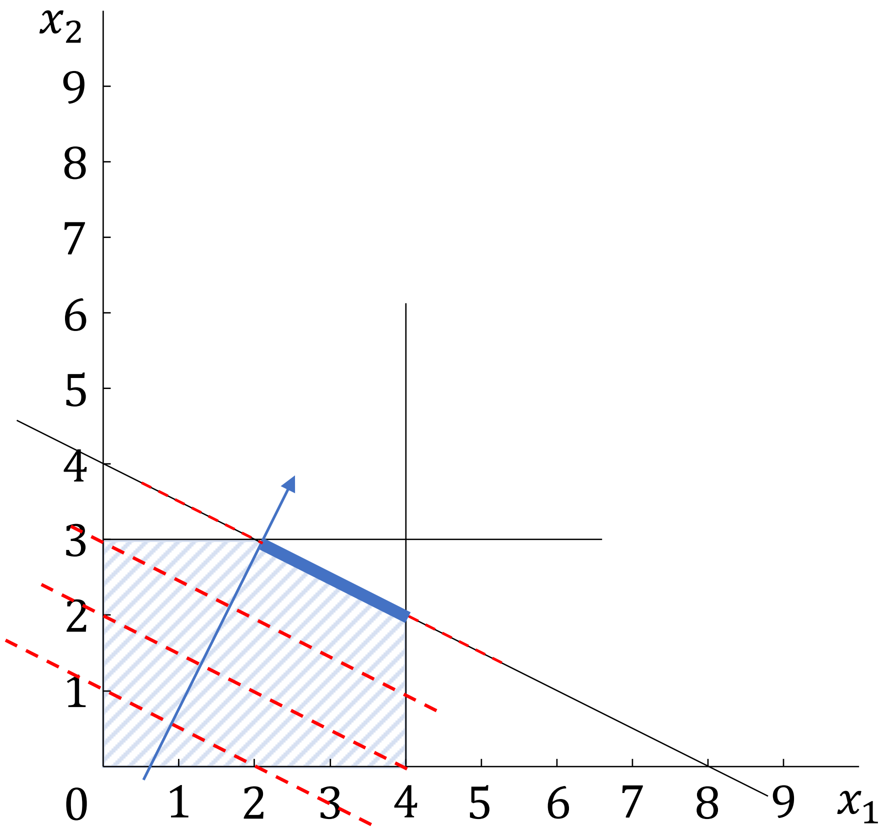
$$\max \ z = x_1 + 2x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \le 8$$

$$4x_1 \le 16$$

$$4x_2 \le 12$$

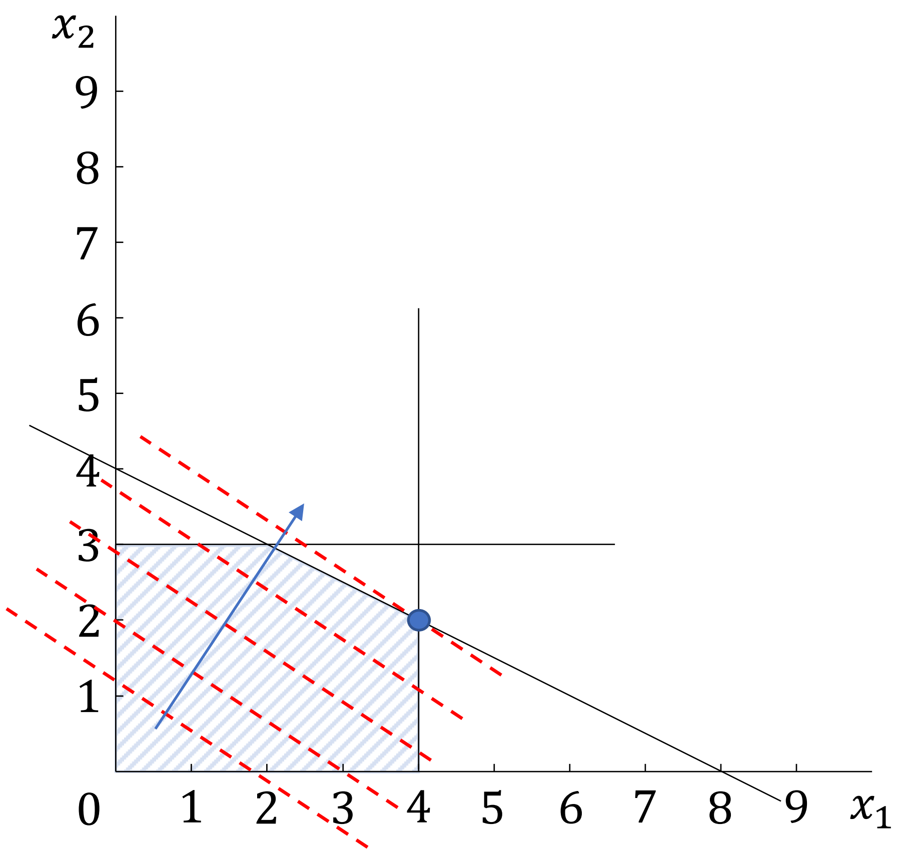
$$x_1, x_2 \ge 0$$



Unbounded solutions

$$\max \; z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + 2x_2 &\le 8 \\ 4x_1 &\le 16 \\ 4x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$



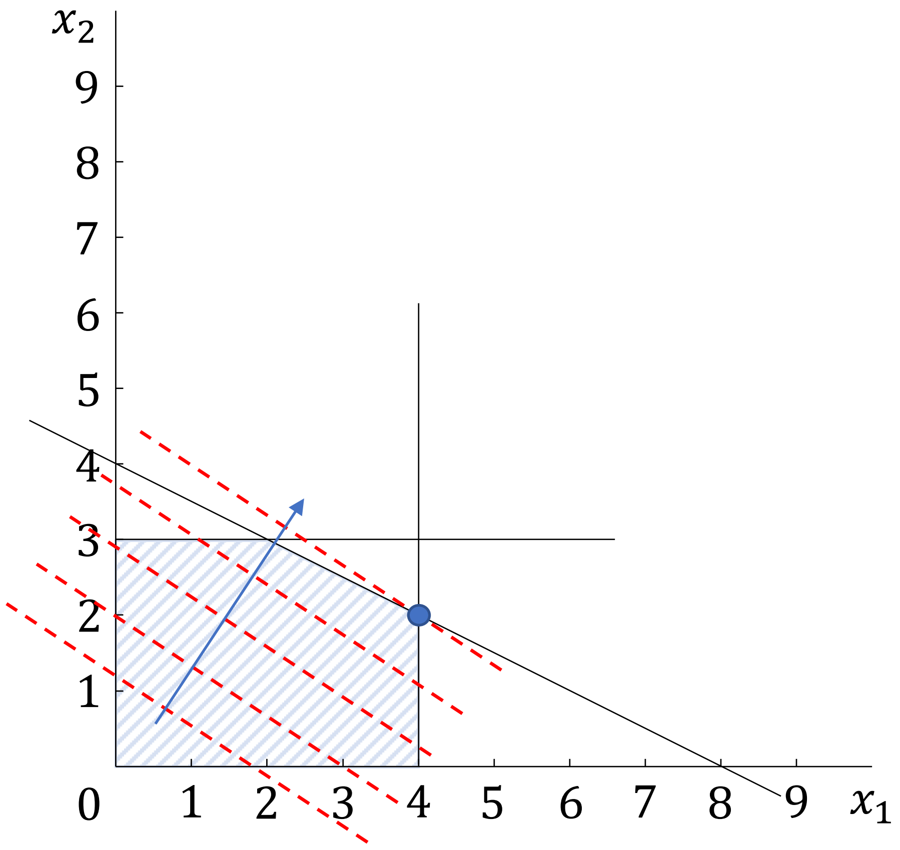
Unbounded solutions

$$\max\ z = 2x_1 + 3x_2$$

s.t.

$$4x_1 \leq 16$$

$$x_1, x_2 \geq 0$$



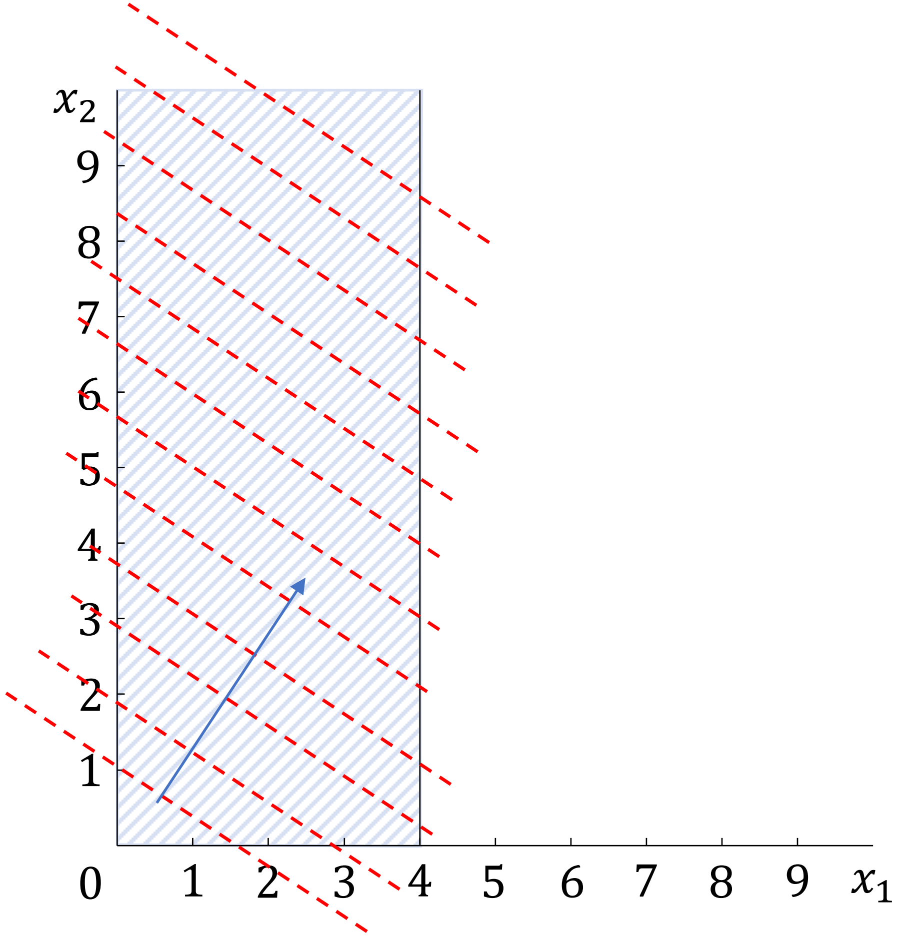
Unbounded solutions

$$\max \ z = 2x_1 + 3x_2$$

s.t.

$$4x_1 \leq 16$$

$$x_1, x_2 \geq 0$$

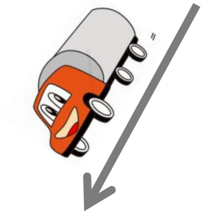
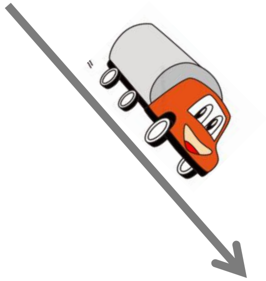


Unbounded solutions

16kg

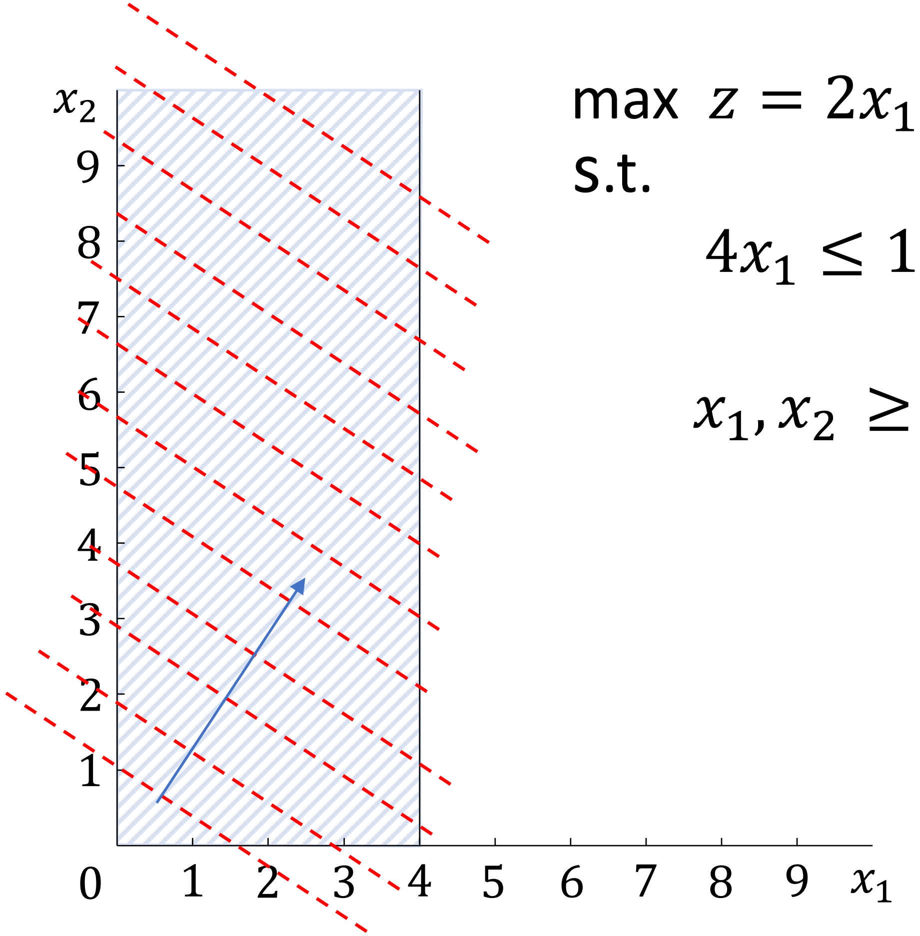
inf


Capacity: 4 kg
Electricity: inf


Profit: 2
Cost: 1


Profit: 3
Cost: 2



$$\max\ z = 2x_1 + 3x_2$$

s.t.

$$4x_1 \le 16$$

$$x_1, x_2 \ge 0$$

Practice

$$\max \ z = x_1 + x_2$$

$$\text{s.t.} \quad -2x_1 + x_2 \le 4$$

$$x_1 - x_2 \le 2$$

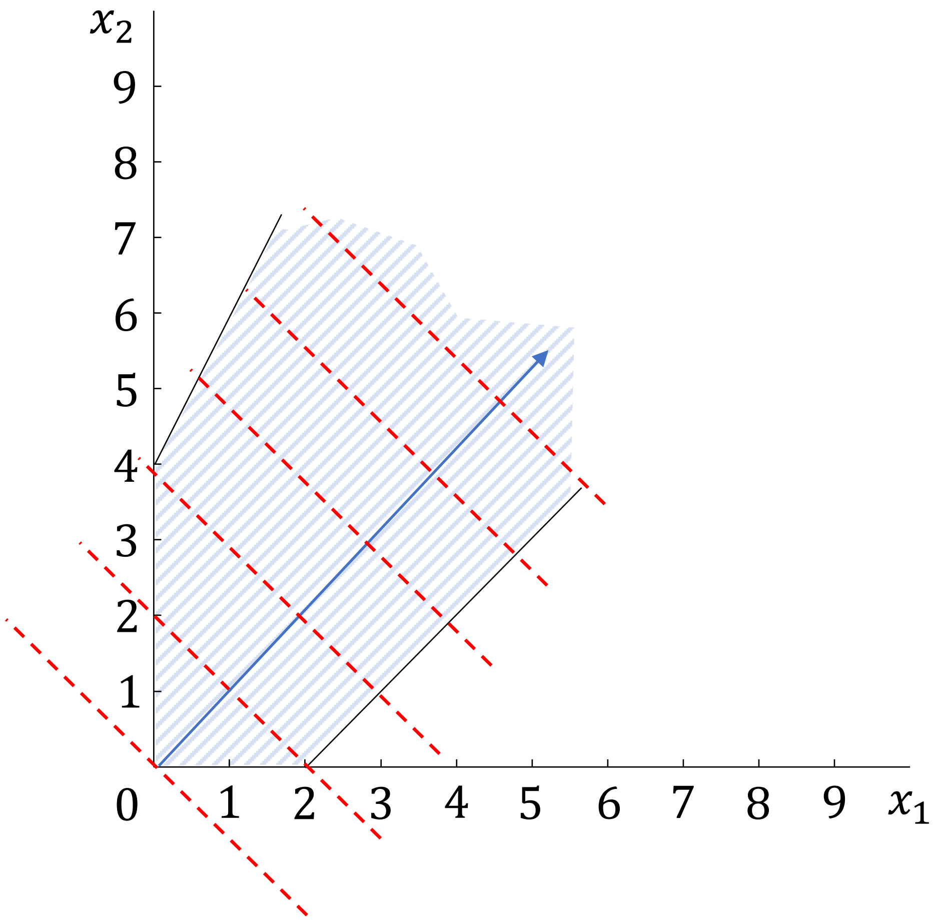
$$x_1, x_2 \ge 0$$

5 minutes

Practice

$$\max \ z = x_1 + x_2$$

$$\text{s.t.} \quad \begin{aligned} -2x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



General Form

$$\max(\min)\ z = c_1x_1 + c_1x_2 + \cdots + c_nx_n$$

s.t.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq (=, \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq (=, \geq) b_2 \\ \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq (=, \geq) b_m \\ x_i \geq 0 (i = 1, \ldots, n) \end{cases}$$

Standard Form

$$\min \ z = c_1x_1 + c_1x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ x_i \geq 0 (i = 1, \ldots, n) \end{cases}$$

Standard Form

$$\max z$$

$$-\min(-z)$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_s = b_i$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - x_s = b_i$$

x_i not constrained to be non-negative

$$x_i = x_i' - x_i'', \quad x_i', x_i'' \geq 0$$

Standard Form

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

s.t. $\quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$

$\qquad \mathbf{x} \geq \mathbf{0}$

$\mathbf{x} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ represents the vector of decision variables (to be determined).

$\mathbf{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}}$ and **b**$= (b_1, b_2, \dots, b_m)^{\mathrm{T}}$ are vectors (known) of coefficients.

$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ is a (known) matrix of coefficients.

$\mathbf{0} = (0, 0, \dots, 0)^{\mathrm{T}}$

Standard Form

$$\max\ z = 2x_1 + 3x_2$$

$$\text{s.t.} \begin{cases} x_1 + 2x_2 \le 8 \\ 4x_1 \le 16 \\ 4x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\min\ f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \ge \mathbf{0}$$

$$\min\ z = -2x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + x_3 &&= 8 \\ & 4x_1 + x_4 &&= 16 \\ & 4x_2 + x_5 &&= 12 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

$$\mathbf{c} = (-2, -3, 0, 0, 0)^{\mathrm{T}}$$

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = (8, 16, 12)^{\mathrm{T}}$$

Practice

$$\max\ z = 6x_1 + 7x_2$$

$$\text{s.t.} \quad \begin{cases} 2x_1 + 3x_2 \le 16 \\ 4x_1 + x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\max\ z = x_1 - 2x_2 + 3x_3$$

$$\text{s.t.} \quad \begin{cases} x_1 + x_2 + x_3 \le 7 \\ x_1 - x_2 + x_3 \ge 12 \\ -3x_1 + x_2 + 2x_3 = 5 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \ge \mathbf{0}$$

5 minutes

Basic Feasible solution

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Assumption:

$$m < n$$

$$\mathrm{rank}(\boldsymbol{A}) = m$$

$$\boldsymbol{A} = (\underbrace{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m}_{\text{linearly independent}}, \mathbf{p}_{m+1}, \mathbf{p}_{m+2}, \dots, \mathbf{p}_n) = (\boldsymbol{B}, \boldsymbol{N})$$

$$\boldsymbol{B} = (\underbrace{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m}_{\text{Base vectors}}) \qquad \boldsymbol{N} = (\underbrace{\mathbf{p}_{m+1}, \mathbf{p}_{m+2}, \dots, \mathbf{p}_n}_{\text{Non-base vectors}})$$

$$\mathbf{x} = (x_1, x_2, \dots, x_m, x_{m+1} \dots, x_n)^{\mathrm{T}} = (\mathbf{x}_B, \mathbf{x}_N)$$

$$\mathbf{x}_B = (\underbrace{x_1, x_2, \dots, x_m}_{\text{Base variables}})^{\mathrm{T}} \qquad \mathbf{x}_N = (\underbrace{x_{m+1}, x_{m+2}, \dots, x_n}_{\text{Non-base variables}})^{\mathrm{T}}$$

Basic Feasible solution

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$
$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Assumption:

$$m < n$$
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$$\boldsymbol{B} = (\underbrace{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m}_{\text{Base vectors}}) \qquad \boldsymbol{N} = (\underbrace{\mathbf{p}_{m+1}, \mathbf{p}_{m+2}, \dots, \mathbf{p}_n}_{\text{Non-base vectors}})$$

$$\mathbf{x} = (x_1, x_2, \dots, x_m, x_{m+1} \dots, x_n)^{\mathrm{T}} = (\mathbf{x}_B, \mathbf{x}_N)$$

$$\mathbf{x}_B = (\underbrace{x_1, x_2, \dots, x_m}_{\text{Base variables}})^{\mathrm{T}} \qquad \mathbf{x}_N = (\underbrace{x_{m+1}, x_{m+2}, \dots, x_n}_{\text{Non-base variables}})^{\mathrm{T}}$$

$$(\boldsymbol{B}, \boldsymbol{N}) \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b}$$

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \mathbf{b}$$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$$

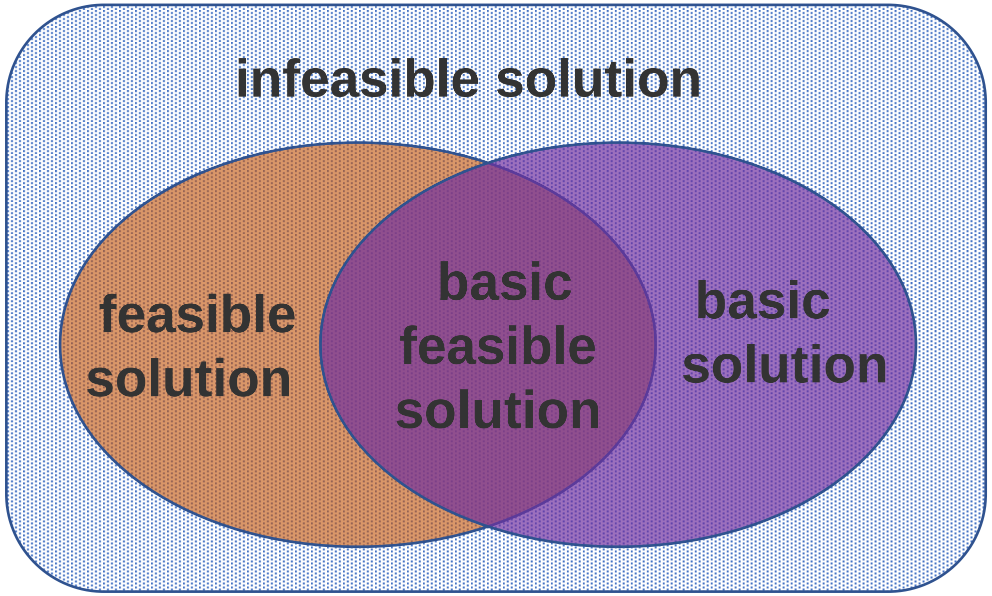
Let $\mathbf{x}_N = \mathbf{0}$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$
$$\mathbf{x} = (\boldsymbol{B}^{-1}\boldsymbol{b}, \mathbf{0})^{\mathrm{T}}$$

Basic solution

If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \mathbf{0}$, $\mathbf{x}$ is a basic feasible solution.

Basic Feasible solution



Basic Theories of LP

Convex combination

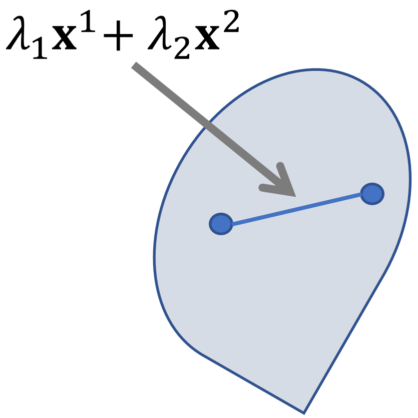
$$\mathbf{x} = \lambda_1\mathbf{x}^1 + \lambda_2\mathbf{x}^2 + \dots \lambda_k\mathbf{x}^k$$

$$\lambda_1, \lambda_2, \dots, \lambda_k \in [0, 1] \qquad \sum_{i=1}^{k} \lambda_i = 1$$

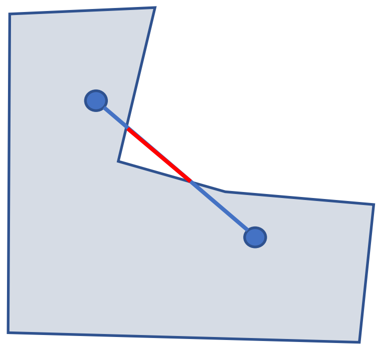
Strict convex combination

$$\mathbf{x} = \lambda_1\mathbf{x}^1 + \lambda_2\mathbf{x}^2 + \dots \lambda_k\mathbf{x}^k$$

$$\lambda_1, \lambda_2, \dots, \lambda_k \in (0, 1) \qquad \sum_{i=1}^{k} \lambda_i = 1$$

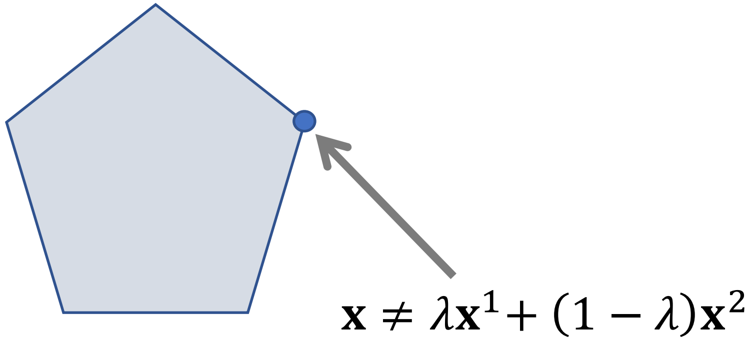


Convex set



Non-convex set

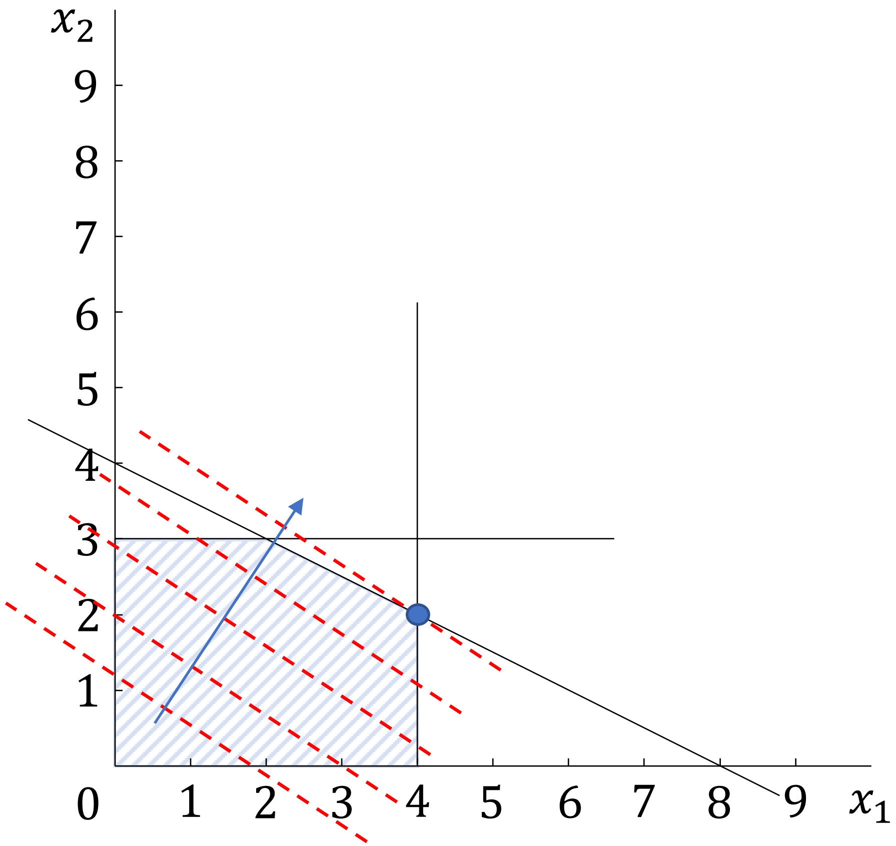
Given a set S, $\mathbf{x}^1 \in S$, $\mathbf{x}^2 \in S$, $\mathbf{x}^1 \neq \mathbf{x}^2$, for any $\lambda \in [0, 1]$, $\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2 \in S$, then S is a convex set.



Given a convex set S and $\mathbf{x} \in S$, $\mathbf{x}^1 \in S$, $\mathbf{x}^2 \in S$, $\mathbf{x}^1 \neq \mathbf{x}^2$, for any $\lambda \in (0,1)$, $\mathbf{x} \neq \lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$, then S is a vertex of S.

Basic Theories of LP

- The feasible region of the LP problem is a convex set.
- One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem.
- The basic feasible solution of the linear programming problem lies at the vertex of its feasible region.



Basic Theories of LP

- The feasible region of the LP problem is a convex set.
- One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem.
- The basic feasible solution of the linear programming problem lies at the vertex of its feasible region.

For any $\lambda \in (0,1), \mathbf{x} \neq \lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$

$$\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N) = (\boldsymbol{B}^{-1}\boldsymbol{b}, \mathbf{0})^{\mathrm{T}}$$

$$\mathbf{x} = \lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$$

$$\mathbf{x}^1 = (\mathbf{x}_N^1, \mathbf{x}_B^1)^{\mathrm{T}} \qquad \mathbf{x}^2 = (\mathbf{x}_N^2, \mathbf{x}_B^2)^{\mathrm{T}}$$

$$\mathbf{x}_B = \lambda\mathbf{x}_B^1 + (1-\lambda)\mathbf{x}_B^2$$

$$\mathbf{x}_N = \lambda\mathbf{x}_N^1 + (1-\lambda)\mathbf{x}_N^2$$

$\mathbf{x}_N = \mathbf{0}$ and $\lambda \in (0,1)$,

$$\mathbf{x}_N^1 = \mathbf{x}_N^2 = \mathbf{0}$$

$$\mathbf{x}^1 = (\mathbf{x}_N^1, \mathbf{0})^{\mathrm{T}} \qquad \mathbf{x}^2 = (\mathbf{x}_N^2, \mathbf{0})^{\mathrm{T}}$$

$$\boldsymbol{A}\mathbf{x} = (\boldsymbol{B}, \boldsymbol{N})\begin{pmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{pmatrix} = \mathbf{b}$$

$$\boldsymbol{B}\mathbf{x}_B = \boldsymbol{B}\mathbf{x}_B^1 = \boldsymbol{B}\mathbf{x}_B^2 = \mathbf{b}$$

$\boldsymbol{B}$ is consist of base vectors, it is a non-singular matrix $\Rightarrow$ $\mathbf{x}_B = \mathbf{x}_B^1 = \mathbf{x}_B^2$

Simplex Method

Simplex Method

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

s.t. $\boldsymbol{A}\mathbf{x} = \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

$$(\boldsymbol{B}, \boldsymbol{N})\begin{pmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{pmatrix} = \mathbf{b}$$

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \mathbf{b}$$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$$

$$\min \ z = -2x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\text{s.t.} \begin{cases} x_1 + 2x_2 + x_3 = 8 \\ 4x_1 + x_4 = 16 \\ 4x_2 + x_5 = 12 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

$$\begin{aligned} x_3 &= 8 - x_1 - 2x_2 \\ x_4 &= 16 - 4x_1 \\ x_5 &= 12 - 4x_2 \\ z &= -2x_1 - 3x_2 \end{aligned}$$

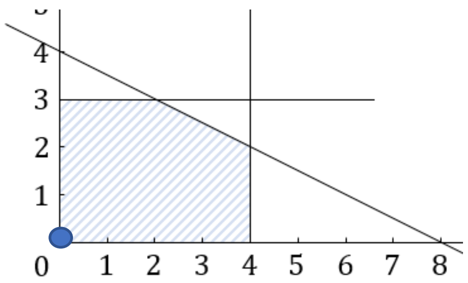
$$\boldsymbol{A} = (\mathbf{p}_1, \mathbf{p}_2, \underbrace{\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5}_{\text{Base vectors}}) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{p}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \ \mathbf{B} = (\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $x_1 = x_2 = 0$, we can get the basic feasible solution:

$$\mathbf{x} = (0,0,8,16,12)^{\mathrm{T}}$$

$$z = 0$$



Simplex Method

Let x_3, x_4, x_2 be basic variables

$$x_3 = 2 - x_1 + \frac{1}{2}x_5$$

$$x_4 = 16 - 4x_1$$

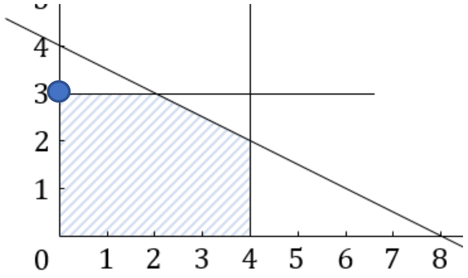
$$x_2 = 3 - \frac{1}{4}x_5$$

$$z = -9 - 2x_1 + \frac{3}{4}x_5$$

Let $x_1 = x_5 = 0$, we can get the basic feasible solution:

$$\mathbf{x} = (0,3,2,16,0)^{\mathrm{T}}$$

$$z = -9$$



Let $x_5 = 0$, $x_1 = \min\{2,4,-\} = 2$

$$x_3 = 2 - x_1 + \frac{1}{2}x_5 = 0 \qquad x_1 = 2 - x_3 + \frac{1}{2}x_5$$

$$z = -2x_1 - 3x_2$$

$$x_3 = 8 - x_1 - 2x_2 \geq 0$$

$$x_4 = 16 - 4x_1 \geq 0$$

$$x_5 = 12 - 4x_2 \geq 0$$

Let $x_1 = 0$, we can get:

$$x_3 = 8 - 2x_2 \geq 0$$

$$x_4 = 16 \geq 0$$

$$x_5 = 12 - 4x_2 \geq 0$$

$$x_2 = \min\{\frac{8}{2}, -, \frac{12}{4}\} = 3\,.$$

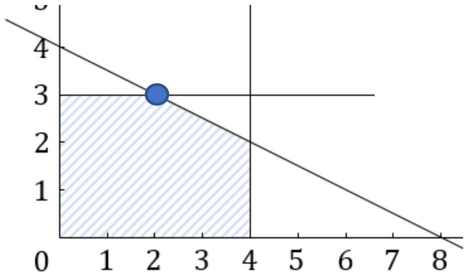
$$x_5 = 12 - 4x_2 = 0$$

$$x_2 = 3 - \frac{1}{4}x_5$$

Simplex Method

$$x_3 = 2 - x_1 + \frac{1}{2}x_5$$
$$x_4 = 16 - 4x_1$$
$$x_2 = 3 - \frac{1}{4}x_5$$

$$x_1 = 2 - x_3 + \frac{1}{2}x_5$$
$$x_4 = 8 + 4x_3 - 2x_5$$
$$x_2 = 3 - \frac{1}{4}x_5$$



Let $x_3 = x_5 = 0$, $\mathbf{x} = (2,3,0,8,0)^{\mathrm{T}}$, $z = -13$

$$z = -13 + 2x_3 - \frac{1}{4}x_5$$

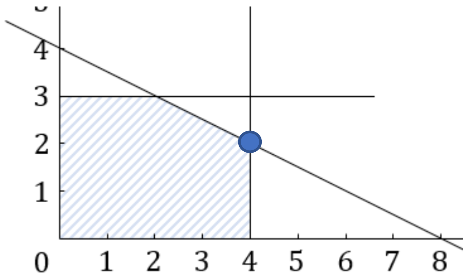
Let $x_3 = 0$, $x_5 = \min\{-,4,12\} = 2$, $x_4 = 0$.

$$x_1 = 4 - \frac{1}{4}x_4$$
$$x_5 = 4 + 2x_3 - \frac{1}{2}x_4$$
$$x_2 = 2 - \frac{1}{2}x_3 + \frac{1}{8}x_4$$

Let $x_3 = x_4 = 0$, $\mathbf{x} = (4,2,0,0,4)^{\mathrm{T}}$,

$$z = -14 + \frac{3}{2}x_3 + \frac{1}{8}x_4$$

$$z = -14$$
$$-z = 14$$



Simplex Method

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

$$(\boldsymbol{B}, \boldsymbol{N})\begin{pmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{pmatrix} = \mathbf{b}$$

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \mathbf{b}$$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$$

Simplex Method

$$\min f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \boldsymbol{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

$$(\boldsymbol{B}, \boldsymbol{N})\begin{pmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{pmatrix} = \mathbf{b}$$

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \mathbf{b}$$

$$\left.\begin{aligned}\mathbf{x}_B &= \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N \\ \mathbf{c}^{\mathrm{T}}\mathbf{x} &= \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N\end{aligned}\right\}$$

$$z + (\mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N} - \mathbf{c}_N^{\mathrm{T}})\mathbf{x}_N = \mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\mathbf{b}$$

$$(\mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N} - \mathbf{c}_N^{\mathrm{T}})\mathbf{x}_N = (\mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{A} - \mathbf{c}^{\mathrm{T}})\mathbf{x}$$

$$z + (\mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{A} - \mathbf{c}^{\mathrm{T}})\mathbf{x} = \mathbf{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b}$$

$$\boldsymbol{B}^{-1}\boldsymbol{B}\mathbf{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b}$$

$$\boldsymbol{B}^{-1}\boldsymbol{A}\mathbf{x} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

Simplex Method

$$z + (\mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{A} - \mathbf{c}^{\mathrm{T}})\mathbf{x} = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \mathbf{b}$$

$$\boldsymbol{B}^{-1} \boldsymbol{A}\mathbf{x} = \boldsymbol{B}^{-1} \boldsymbol{b}$$

Simplex Method

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$$(\gamma_1, \gamma_2, \dots, \gamma_n) = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{A} - \mathbf{c}^{\mathrm{T}}$$

$$\bar{z} = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{b}$$

$$z + \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n = \bar{z}$$

$$\begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \cdots & \bar{a}_{mn} \end{bmatrix} = \boldsymbol{B}^{-1} \boldsymbol{A}$$

$$\left(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m\right)^{\mathrm{T}} = \boldsymbol{B}^{-1}\ \boldsymbol{b}$$

$$\left.\begin{array}{l} \bar{a}_{11} x_1 + \bar{a}_{12} x_2 + \cdots + \bar{a}_{1n} x_n \quad = \bar{b}_1 \\ \bar{a}_{21} x_1 + \bar{a}_{22} x_2 + \cdots + \bar{a}_{2n} x_n \quad = \bar{b}_2 \\ \qquad \cdots\cdots \\ \bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 + \cdots + \bar{a}_{mn} x_n = \bar{b}_m \end{array}\right\}$$

$$z + (\mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{A} - \mathbf{c}^{\mathrm{T}})\mathbf{x} = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \mathbf{b}$$

$\mathbf{x}$	right hand
$\mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{A} - \mathbf{c}^{\mathrm{T}}$	$\mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \mathbf{b}$
$\boldsymbol{B}^{-1} \boldsymbol{A}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$

$$\boldsymbol{B}^{-1} \boldsymbol{A} \mathbf{x} = \boldsymbol{B}^{-1} \boldsymbol{b}$$

Simplex Method

$$(\gamma_1, \gamma_2, \dots, \gamma_n) = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{A} - \mathbf{c}^{\mathrm{T}}$$

$$\bar{z} = \mathbf{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{b}$$

$$z + \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n = \bar{z}$$

$$\begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \cdots & \bar{a}_{mn} \end{bmatrix} = \boldsymbol{B}^{-1} \boldsymbol{A}$$

$$\left(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m\right)^{\mathrm{T}} = \boldsymbol{B}^{-1} \ \boldsymbol{b}$$

$$\left.\begin{aligned} \bar{a}_{11} x_1 + \bar{a}_{12} x_2 + \cdots + \bar{a}_{1n} x_n &= \bar{b}_1 \\ \bar{a}_{21} x_1 + \bar{a}_{22} x_2 + \cdots + \bar{a}_{2n} x_n &= \bar{b}_2 \\ \dots\dots& \\ \bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 + \cdots + \bar{a}_{mn} x_n &= \bar{b}_m \end{aligned}\right\}$$

	x_1	x_2	…	x_n	
z	γ_1	γ_2	…	γ_n	$\bar{z}$
x_{B1}	$\bar{a}_{11}$	$\bar{a}_{12}$	…	$\bar{a}_{1n}$	$\bar{b}_1$
x_{B2}	$\bar{a}_{21}$	$\bar{a}_{22}$	…	$\bar{a}_{2n}$	$\bar{b}_2$
…	…	…	…	…	…
x_{Bm}	$\bar{a}_{m1}$	$\bar{a}_{m2}$	…	$\bar{a}_{mn}$	$\bar{b}_m$

Practice

$$\min \ f(\mathbf{x}) = x_1 + 4x_2 - 2x_3$$

$$\text{s.t.} \quad \begin{cases} x_1 + 3x_4 - x_5 = 7 \\ x_2 + 4x_4 + x_5 = 8 \\ x_3 + x_4 + 2x_5 = 3 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

5 minutes

Simplex Method

	x_1	x_2	…	x_n	
z	γ_1	γ_2	…	γ_n	$\bar{z}$
x_{B1}	$\bar{a}_{11}$	$\bar{a}_{12}$	…	$\bar{a}_{1n}$	$\bar{b}_1$
x_{B2}	$\bar{a}_{21}$	$\bar{a}_{22}$	…	$\bar{a}_{2n}$	$\bar{b}_2$
…	…	…	…	…	…
x_{Bm}	$\bar{a}_{m1}$	$\bar{a}_{m2}$	…	$\bar{a}_{mn}$	$\bar{b}_m$

$$\min \ f(\mathbf{x}) = -5x_1 - 10x_2$$

$$\text{s.t.} \quad \frac{1}{14}x_1 + \frac{1}{7}x_2 + x_3 = 1$$
$$\frac{1}{7}x_1 + \frac{1}{12}x_2 + x_4 = 1$$
$$x_1 + x_2 + x_5 = 8$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

	x_1	x_2	x_3	x_4	x_5	
z	5	10	0	0	0	0
x_3	$\frac{1}{14}$	$\frac{1}{7}$	1	0	0	1
x_4	$\frac{1}{7}$	$\frac{1}{12}$	0	1	0	1
x_5	1	1	0	0	1	8

$\gamma_2 = 10 > 0$

$\min\{\frac{\bar{b}_1}{\bar{a}_{12}}, \frac{\bar{b}_2}{\bar{a}_{22}}, \frac{\bar{b}_3}{\bar{a}_{32}}\}=\min\{7, 12, 8\}=7$

x_2 is entry variable

x_3 is exit variable

row1 ×7

row2 −row1×1/12

row3 −row1

	x_1	x_2	x_3	x_4	x_5	
z	0	0	-70	0	0	-70
x_2	$\frac{1}{2}$	1	7	0	0	7
x_4	$\frac{17}{168}$	0	$-\frac{7}{12}$	1	0	$\frac{5}{12}$
x_5	$\frac{1}{2}$	0	-7	0	1	1

- If $\gamma_i \leq 0 \ (i = 1,2,..,n)$, $\mathbf{x}$ is the optimal solution.
- If any $\gamma_d > 0$ and, $\bar{a}_{1d}, \bar{a}_{2d}, \dots, \bar{a}_{md} \leq 0$, there is no optimal solution.
- Otherwise, find $\gamma_k = \max\{\gamma_i | i = 1,2,..,n\}$, find $\frac{\bar{b}_r}{\bar{a}_{rk}} = \min\{\frac{\bar{b}_i}{\bar{a}_{ik}} | \bar{a}_{ri} > 0, i = 1,2,..,m\}$

x_k is entry variable

x_r is exit variable

$\gamma_i \leq 0 \ (i = 1,2,..,n)$

$\mathbf{x} = \left(0,7,0,\frac{5}{12},1\right)^{\mathrm{T}}$ is the optimal, $f(\mathbf{x})$ = -70

Assignment

To learn:
- Duality Theory
- Dual Simplex Method

Reference

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