# 单纯形法及应用练习

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DOCTYPE - md & Katex. 使用基于Pandoc的渲染器输出PDF文档
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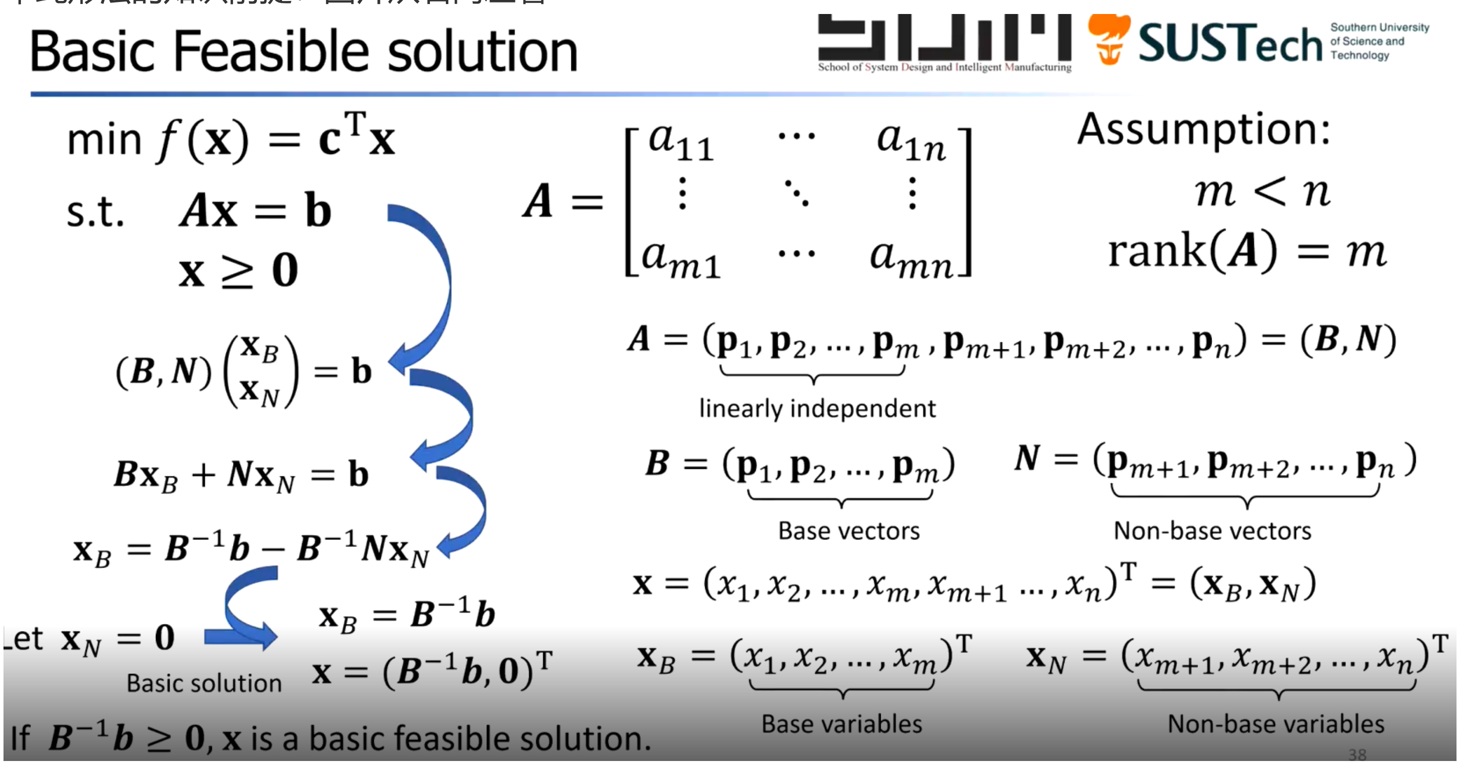
单纯形法的知识前提：图片从右向左看




当解出$\vec{x} = \left(\vec{B}^{\,-1}\,b, 0\right)^T$得到基本解(Basic solution).

当满足$\vec{B}^{\,-1}\,b \geq 0$，称$x$为一个基本可行解(Basic feasible solution).

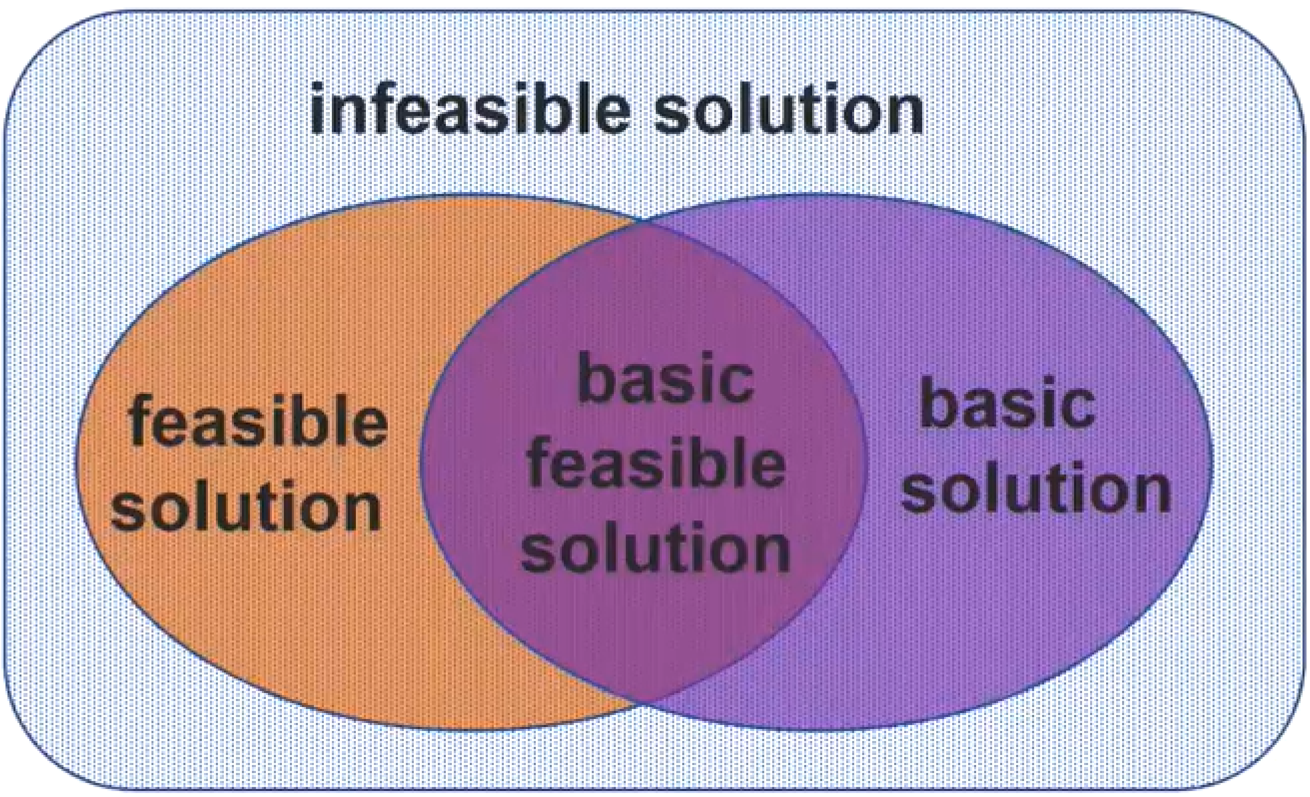
## Basic Feasible Solution

- Consider a LP problem in the standard form that contains *n* variables and *m* constraints (assume $n \geq m$)

- A basic solution is obtained by setting $n - m$ variables (NBV) equal to 0 and solving for the remaining *m* variables (BV)

$$\binom{n}{m} = \binom{n}{n-m} = \frac{n!}{m!\,(n-m)!}$$

- Any basic solution in which all variables are nonnegative is called a basic feasible solution (or BFS)




# 什么是“凸 Convex”和“凹 Concave”？

参考资料：

- 什么是凸组合(Convex combination)?
- Convex combination的一个特殊情况：两个向量的convex combination
- Operation Research Course 这个的课程蛮好，口音清晰，讲解清楚，比......

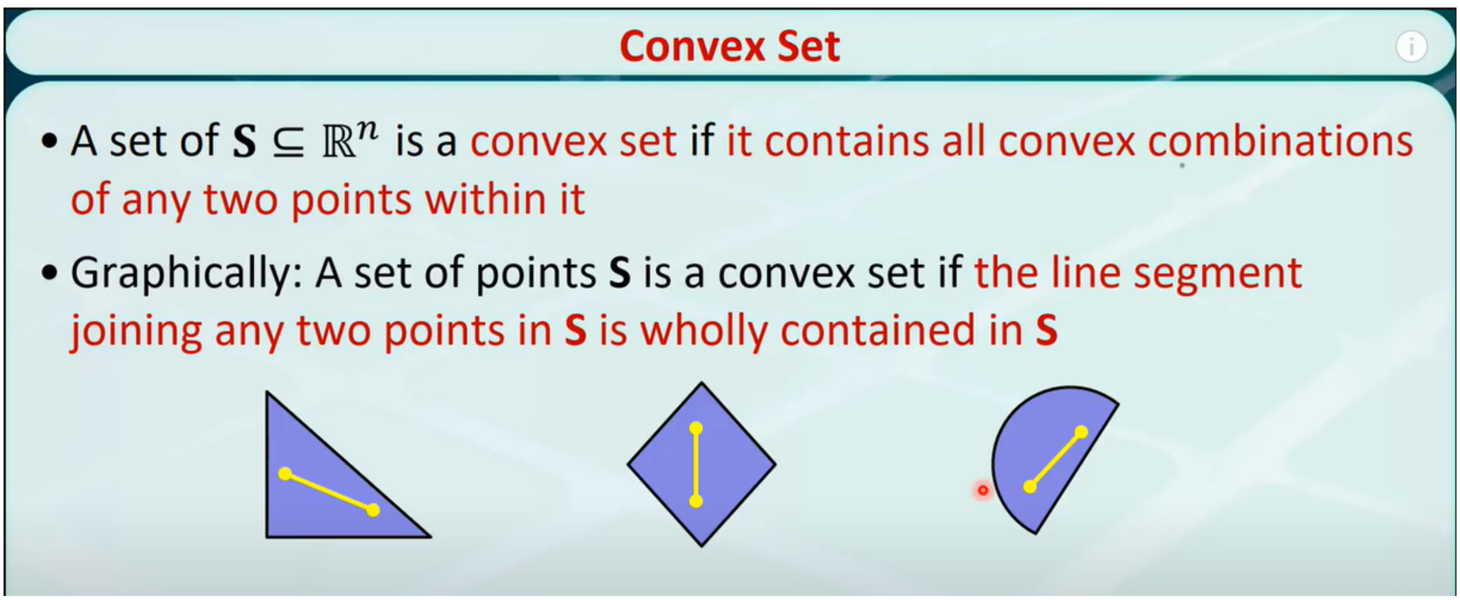
**凸组合 Convex Combination**的定义是：

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_k x_k \\ \lambda_1, \lambda_2, ...\lambda_k \in [0, 1] \\ \lambda_1 + \lambda_2 + ... + \lambda_k = 1 \quad or \quad \sum_{i=1}^{N} \lambda_i = 1 \end{cases}$$

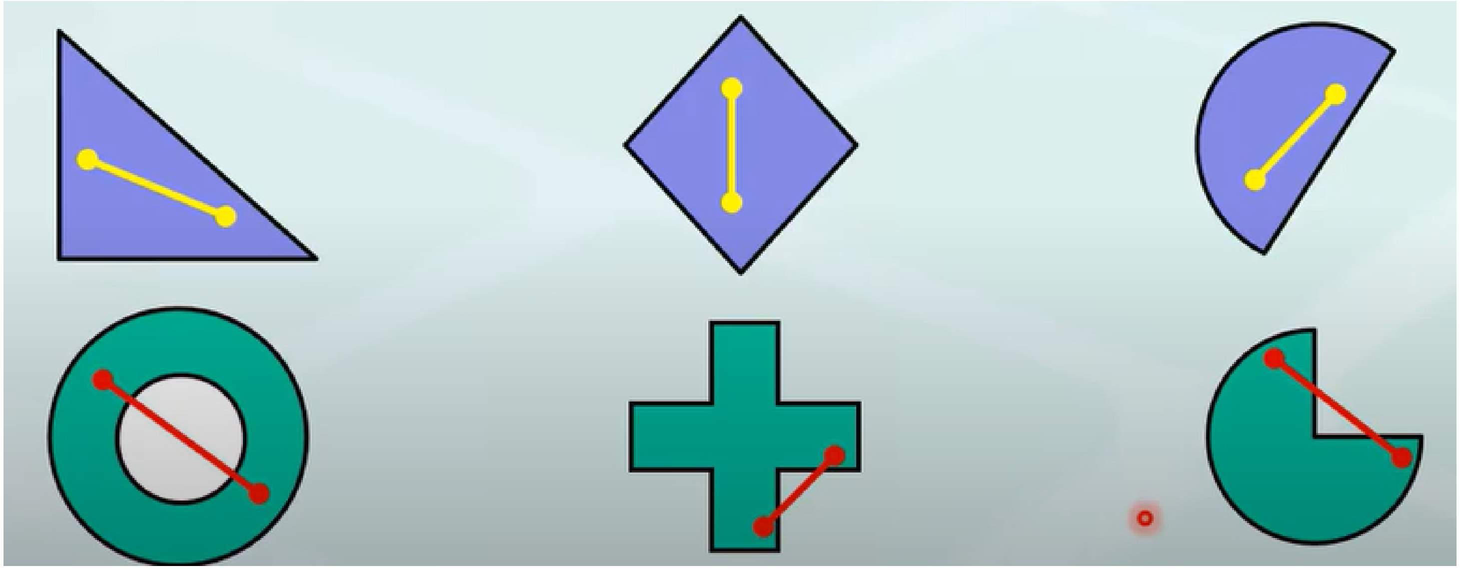
这里还有一个严格凸组合(Strict convex combination)的定义，仅仅需要将 Convex combination's definition约束到：

$$\lambda_1, \lambda_2, ...\lambda_k \in (0, 1)$$

**什么是凸集 Convex Set** -

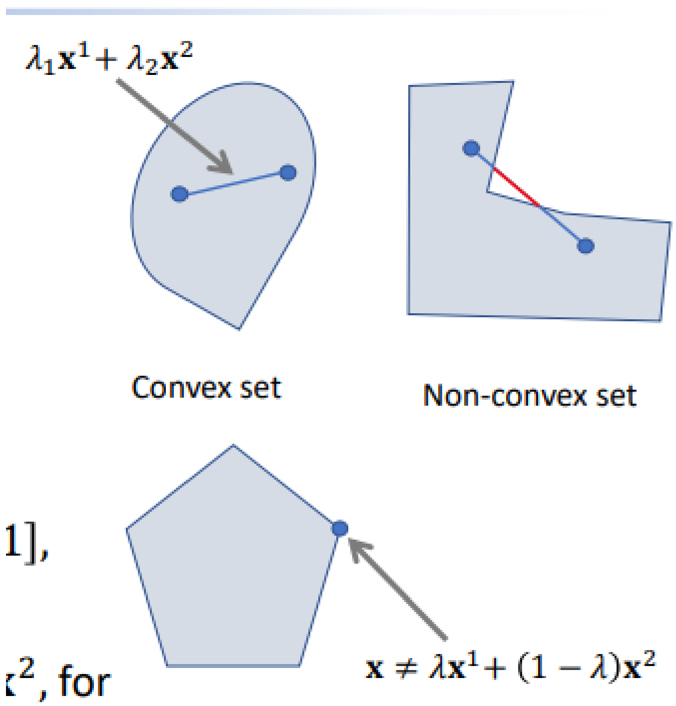



上排为Convex Set, 下排为 Non-Convex Set:



**补充** -

- Given a convex set $S$ and $x \in S, x_1 \in S, x_2 \in S, x_1 \neq x_2$, for any $\lambda \in (0,1)$, $x \neq \lambda x_1 + (1-\lambda) x_2$, **then $S$ is a vertex of $S$**



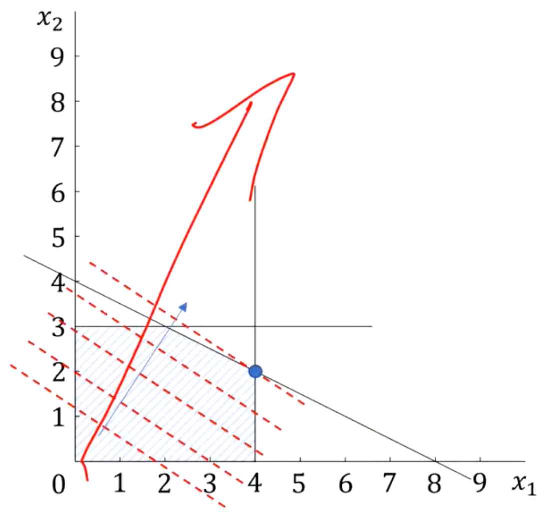

**Two special cases:**

- A empty set is a convex set.
- A single point set is a convex set.

# Basic Theories of LP

## Basic Theories of LP

- The feasible region of the LP problem is a convex set.
- One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem.
- The basic feasible solution of the linear programming problem lies at the vertex of its feasible region.




**Theorem - The feasible region of a LP problem is a convex set**

**证明**也很简单:

> Every linear constraint will split the space in half. In each half space, it is a convex set. It is easy to proof that the intersection of convex sets is still a convex set.

**Theorem - One of the vertices of the LP problem's feasible region must be the optimal solution of the LP problem**

**Theorem - The basic feasible solution of the linear programming problem lies at the vertex of its feasible region**

**证明**第三条theorem:

> 第三条Theorem的数学表达是 -
>
> $$For\ any\ \lambda \in (0,1), x \neq \lambda x_1 + (1-\lambda)x_2$$
>
> 那么要说明其不成立，可以利用反证法：假设$x$在$x_1, x_2 (x_1 \neq x_2)$的line segment上，那么有 $x = \lambda x_1 + (1-\lambda)x_2$.
> 存在如下关系：

$$
\begin{cases}
x_1 = \left(x_B^1, x_N^1\right)^T \\
x_2 = \left(x_B^2, x_N^2\right)^T \\
x_B = \lambda x_B^1 + (1-\lambda)\, x_B^2 \\
x_N = \lambda x_N^1 + (1-\lambda)\, x_N^2
\end{cases}
$$

由于我们知道，由于加入了slack varialbles，我们令$x_N = 0$且$\lambda \in (0,1)$，从而很显然地我们得到

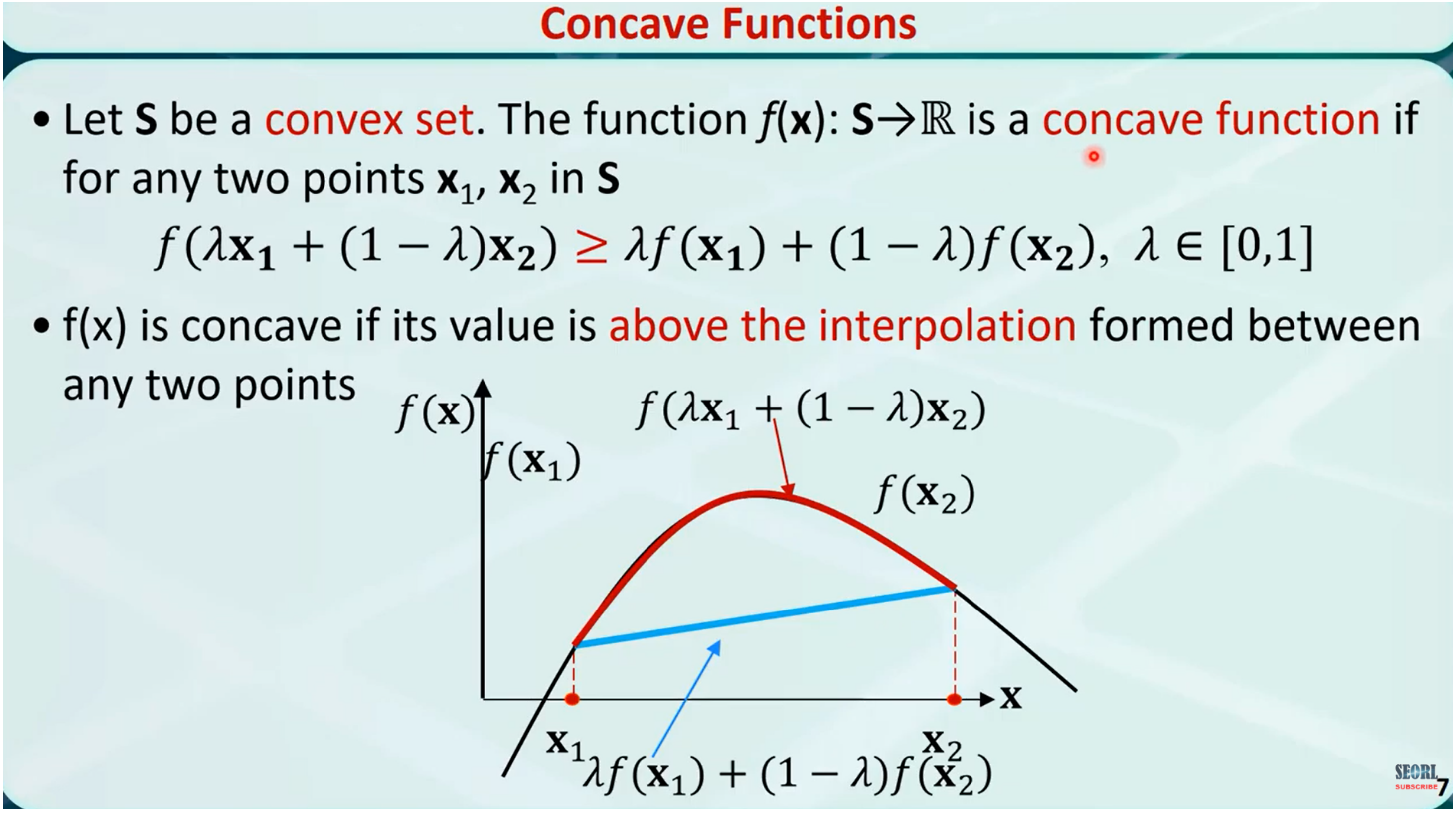
$$
\begin{cases}
x_N = x_N^1 = x_N^2 = 0 \\
x_1 = \left(x_B^1, 0\right)^T \\
x_2 = \left(x_B^2, 0\right)^T
\end{cases}
$$

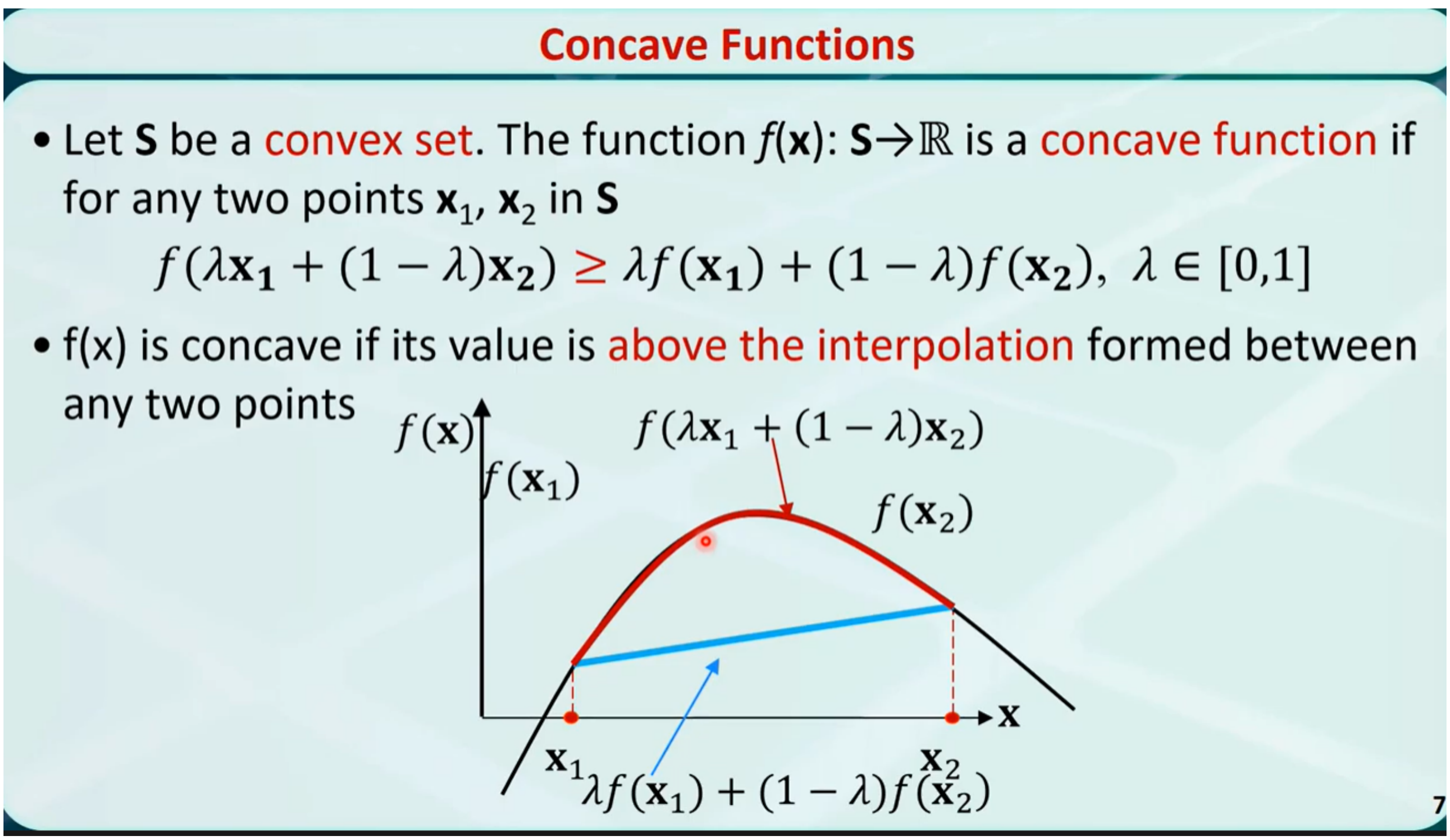
那么我们从最基础的条件出发$Ax = (B, N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = B$ 从而有

$$
Bx_B = Bx_B^1 = Bx_B^2 = b
$$

上面这个式子有同学问我为什么能够成立，其实很简单，因为我们的前提是$x_1$， $x_2$都是Feasible solutions.最后我们得到的结果就很容易了：由于$B$是一个空间的基，显然是一个非奇异矩阵(Non-Singular Matrix).因此我们得到了这样的结论——$x_B = x_1^B = x_2^B$, i.e. $x = x_1 = x_2$.

# 什么是 凸函数Convex Function 和 凹函数Concave Funtion




# 单纯形法 - 步骤与讲解

> 这一部分内容参考了textbook - Introduction to Operations Research (Frederick S. Hillier & Gerald J. Lieberman) 第十版（英文版）；

> 其他参考内容将在内容下注明.
> TODO: 期末复习单纯形法时回顾并把书上的笔记摘录
> TODO: 单纯形法优越性与成立的数学证明