Burning Time and Rendezvous

Kerbal Space Program Mathematics

July 24, 2020

1 The Situation

Bob is on board off a CraftA, doing some science stuff. He's orbiting Kerbin, his orbit is circular, altitude of 100,000 m (At Sea Level - ASL), inclination of 0°.
Valentina is on board off a CraftB, doing some pilot stuff. She's orbiting Kerbin, her orbit is circular, altitude of 300,000 m (ASL), inclination of 0°.

They have decided to sea a human movie together (The Martian).

2 First formulas

A craft is on an circular orbit. Let a be the semi-major axis (m), V the velocity of the craft ($\mathrm{m} \cdot \mathrm{s}^{-1}$) and P the period of the orbit (s). To calculate a, we need Ap and Pe, respectively the Apoapsis and the Periapsis of an orbit ASL (in the same way that KSP defines Ap and Pe) and R, the radius of the body (m). I note $R_A = Ap + R$ and $R_P = Pe + R$. In this case,

$$a = \frac{R_A + R_P}{2} \quad ; \quad V = \sqrt{2\mu\left(\frac{1}{r} - \frac{1}{R_A + R_P}\right)} \quad ; \quad P = 2\pi\sqrt{\frac{a^3}{\mu}}.$$

μ is the gravitational parameter of Kerbin: $\mu = 3.5316 \times 10^{12}\,\mathrm{m}^3 \cdot \mathrm{s}^{-2}$.

In case of a circular orbit, $a = R_A = R_p$ and $V = \sqrt{\dfrac{\mu}{R_A}} = \sqrt{\dfrac{\mu}{R_P}}$.

Bob's craft:
$a = 700{,}000\,\mathrm{m}$, $V = 2246.140\,\mathrm{m} \cdot \mathrm{s}^{-1}$, $P = 1958.128\,\mathrm{s}$

Valentina's craft:
$a = 900{,}000\,\mathrm{m}$, $V = 1980.909\,\mathrm{m} \cdot \mathrm{s}^{-1}$, $P = 2854.683\,\mathrm{s}$

3 Engine characteristics

Bob and Valentina are both using one 48-7S "Spark" liquid fuel engine (in vacuum).

Thrust: $F = 20\,\mathrm{kN} = 20{,}000\,\mathrm{N}$.

Specific Impulse: $I_{sp} = 320\,\mathrm{s}$.

Effective exhaust gas velocity: $V_e = I_{sp} \times g_0 = 320 \times 9.81 = 3139.2\,\mathrm{m \cdot s^{-1}}$.

Rate of the ejected mass flow: $q = \dfrac{F}{V_e} = \dfrac{20{,}000}{3139.2} = 6.371\,\mathrm{kg \cdot s^{-1}}$.

4 Bob doesn't move, Valentina is going to him

4.1 Hohmann Transfer

Valentina's craft as apoapsis and periapsis at 300,000 m. Valentina first needs to lower her periapsis to 100,000 m to join Bob. Her new orbit is not circular anymore so **the velocity is not constant**. The new data are:

Semi-major axis: $\dfrac{900{,}000 + 700{,}000}{2} = 800{,}000\,\mathrm{m}$.

Velocity at Apoapsis: $\sqrt{2\mu\left(\dfrac{1}{R_A} - \dfrac{1}{R_A + R_P}\right)} = \sqrt{2\mu\left(\dfrac{R_P}{R_A(R_A + R_P)}\right)} = 1852.971\,\mathrm{m \cdot s^{-1}}$.

Delta-v: $\Delta v = 1852.971 - 1980.909 = -127.938\,\mathrm{m \cdot s^{-1}}$.

Period: $2\pi\sqrt{\dfrac{a^3}{\mu}} = 2392.374\,\mathrm{s}$.

Possible Procedure : At any moment of her circular orbit, Valentina has to burn retrograde, lowering down her periapsis to 100,000 m, using $127.938\,\mathrm{m \cdot s^{-1}}$ of $\Delta - v$. Doing that, the probability that Valentina meets Bob is very small. The best is to wait the good moment.

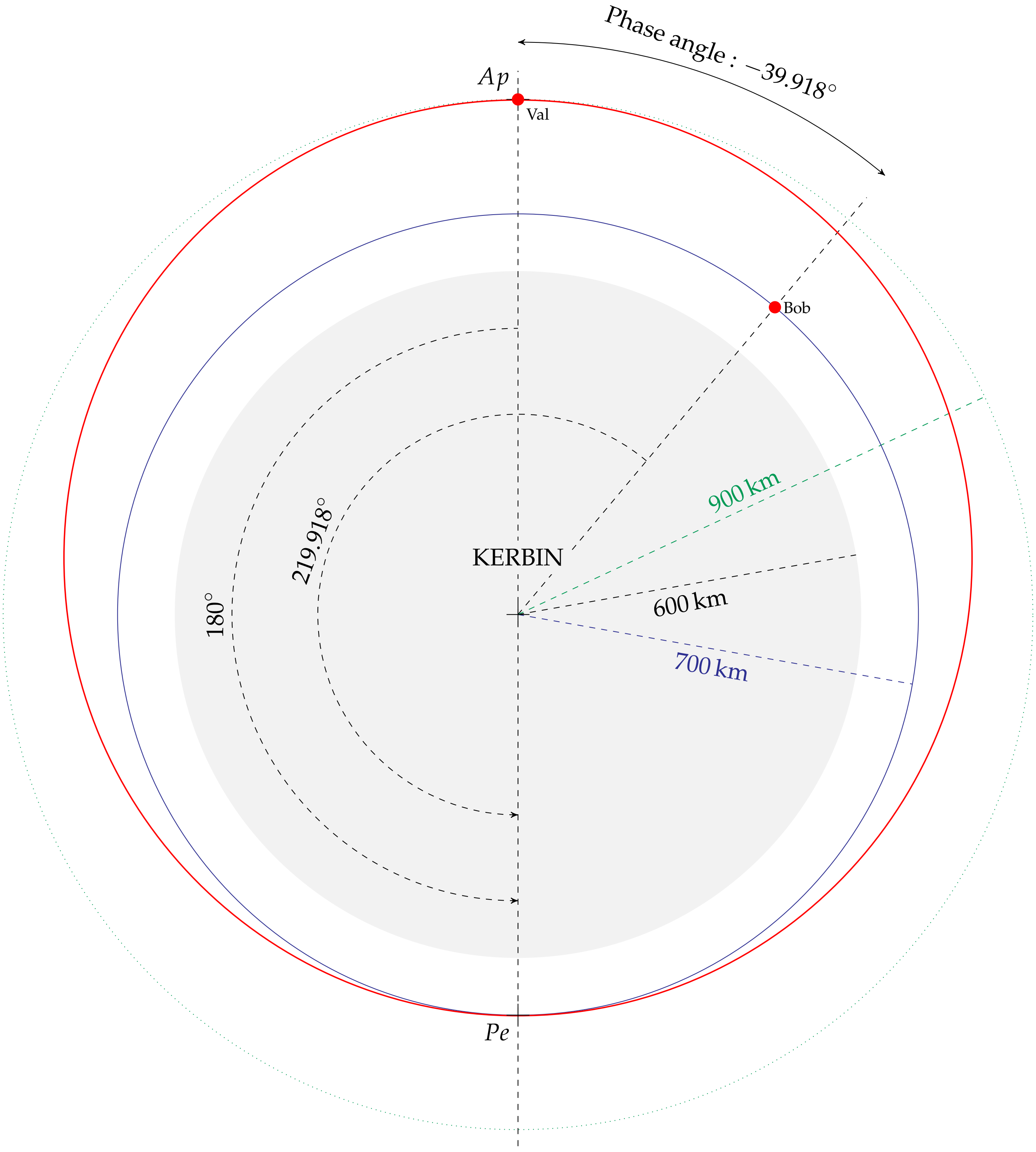
4.2 Angle Phase

Valentina (still in the first circular orbit) is reaching her apoapsis, pointing retrograde, ready to burn. Her new periapsis will be a point on Bob's orbit. She executes her burn. Going from apoapsis to periapsis corresponds to the half of her total new orbit. We know that her new orbit's period is $P = 2392.374\,\mathrm{s}$. So, Valentina needs to wait $\dfrac{2392.374}{2} = 1196.187\,\mathrm{s}$ to go from her apoapsis to her new periapsis.
Reaching her periapsis, she wants to see Bob's craft. So, Bob and Valentina needs to reach Valentina's periapsis at the same time. What we need to know is where was Bob 1196.187 s earlier.

Bob completes his orbit in 1958.128 s corresponding of a 360°-turn. Bob's orbit is circular so angles and times are proportional.

$$\frac{1196.187 \times 360}{1958.128} = 219.918^\circ.$$

The following figure shows that Valentina must burn when Bob is -39.918° behind her. (and more math after the figure)


Phase angle : −39.918°
Ap
Val
Bob
900 km
KERBIN
219.918°
180°
600 km
700 km
Pe

4.3 Meeting Point and Last Burning Time

Valentina and Bob are both reaching Valentina's periapsis. Bob's velocity is contant:

$$V_B = 2246.140\,\mathrm{m}\cdot\mathrm{s}^{-1}.$$

Valentina's Velocity is given by the formula

$$V = \sqrt{2\mu\left(\frac{1}{R_P} - \frac{1}{R_A + R_P}\right)} = \sqrt{2\mu\left(\frac{R_A}{R_P(R_A + R_P)}\right)} = 2382.391\,\mathrm{m}\cdot\mathrm{s}^{-1}.$$

At the meeting point, the relative velocity of the two crafts is:

$$V_{rel} = 2382.391 - 2246.140 = 136.251\,\mathrm{m}\cdot\mathrm{s}^{-1}.$$

Finally, Valentina needs to burn retrograde (relatively to her target) to reduce the V_{rel} to 0. How much time does she need? The following formula gives the answer:

$$\Delta t = \frac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right) \tag{1}$$

where M is the mass of the craft just before the burn (the unit of M is kg because the unit of q is $\mathrm{kg}\cdot\mathrm{s}^{-1}$).
Suppose that $M = 2.120\,\mathrm{t} = 2120\,\mathrm{kg}$ (information given by the map view). After substituting all the numbers, we find $t = 14.134\,\mathrm{s}$. Valentina needs $t = 14.134\,\mathrm{s}$ to reduce the relative velocity (**with those specific mass and specific engine!!**).

$\rightarrowtail$ *Does it mean that Valentina needs to wait her target to be at 14s to start her burn?* No, she can wait a little more. During the burn, the mass and the relative velocity are both decreasing, meaning that the burning time is not proportional to the distance to the target.

4.4 When Will Valentina Start Her Burn?

Valentina has to start her burn when her target is at distance d (unit is m) where:

$$D = \frac{F}{q^2}\left[(q\Delta t - M)\ln\left(\frac{M}{M - q\Delta t}\right) + q\Delta t\right]. \tag{2}$$

We know all the values we need and a calculator gives: $D = 955.944\,\mathrm{m}$.

Because in KSP it's a little tricky to read the distance to a target (in particular is the relative velocity is big), I suggest to look at the HUD in the bottom left corner of the screen and note the time remaining until the intersection point with the target.
With a relative velocity of $136.251\,\mathrm{m}\cdot\mathrm{s}^{-1}$ and a remaining distance of $955.944\,\mathrm{m}$, Valentina must burn (full throttle) $\dfrac{955.944}{136.251} = 7.0\,\mathrm{s}$ before the intersection point (but the burn will still last $14.134\,\mathrm{s}$!!).
In fact, we don't need the burning time to calculate the distance we are looking for. Indeed, we also have this formula:

$$D = \frac{M(V_e)^2}{F}\left[1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\left(\frac{\Delta v}{V_e} + 1\right)\right]. \tag{3}$$

Finally, the total Δv budget is: $127.938 + 136.251 = 264.189\,\mathrm{m}\cdot\mathrm{s}^{-1}$.

5 Some Mathematical Demonstrations

5.1 Burning Time

In the previous section, we use that formula (1):

$$\Delta t = \frac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right).$$

Where does it come from?

The Tsiolkovsky's rocket equation gives $\Delta v = V_e \ln\left(\frac{m_0}{m_f}\right)$ where m_0 is the initial mass and m_f the final mass. In our example, $m_0 = M$ and $m_f = M - q\Delta t$ because q is the rate of ejected mass flow. After a time of Δt, the craft is lighter because it consumed fuel.

The rocket equation becomes:

$$\begin{array}{rrcl}
 & \Delta v & = & V_e \ln\left(\dfrac{M}{M - q\Delta t}\right) \\
\Leftrightarrow & \dfrac{\Delta v}{V_e} & = & \ln\left(\dfrac{M}{M - q\Delta t}\right) \\
\Leftrightarrow & \mathrm{e}^{\frac{\Delta v}{V_e}} & = & \dfrac{M}{M - q\Delta t} \\
\Leftrightarrow & M - q\Delta t & = & \dfrac{M}{\mathrm{e}^{\frac{\Delta v}{V_e}}} \\
\Leftrightarrow & q\Delta t & = & M - \dfrac{M}{\mathrm{e}^{\frac{\Delta v}{V_e}}} \\
\Leftrightarrow & q\Delta t & = & M\left(1 - \dfrac{1}{\mathrm{e}^{\frac{\Delta v}{V_e}}}\right) \\
\Leftrightarrow & \Delta t & = & \dfrac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right) \qquad QED
\end{array}$$

5.2 Distance

After calculating the burning time, we used formula 2 to calculate the distance from Valentina to Bob just before Valentina executes her burn:

$$D = \frac{F}{q^2}\left[(q\Delta t - M)\ln\left(\frac{M}{M - q\Delta t}\right) + q\Delta t\right].$$

Where does it come from?

According to Newton's second law of motion, $F = ma$ where F is the force of the engine, m the total mass of the craft and a the total force of acceleration. Valentina is in vacuum so the total acceleration is the acceleration of the engine. Both m and a change with respect to time. We can note $a(t)$ and $m(t)$ and then, $a(t) = \dfrac{F}{m(t)}$.

We know that $m(t) = M - qt$ so $a(t) = \dfrac{F}{M - qt}$.

Moreover, $a(t) = \frac{d}{dt}v(t)$, which implies $v(t) = \int a(t)\mathrm{d}t = \int \frac{F}{M-qt}\mathrm{d}t = \frac{F}{q}\ln\left(\frac{M}{M-qt}\right) + C$ where C is a constant number and v is the relative velocity.
Let Δt be the time of the burn. At the end of the burn, Valentina and Bob want $v = 0$, as we do. $v(\Delta t) = 0 \Rightarrow C = 0$ and, finally $v(t) = \frac{F}{q}\ln\left(\frac{M}{M-qt}\right)$.

We are looking for the distance $D(t)$ (the distance also changes with respect to t). We know that $v(t) = \frac{d}{dt}D(t)$ which implies $d(t) = \int v(t)\mathrm{d}t$. Integrating from $t = 0$ to $t = \Delta t$, we obtain:

$$\begin{aligned}
D(t) &= \int_0^{\Delta t} v(t)\mathrm{d}t \\
&= \int_0^{\Delta t} \frac{F}{q}\ln\left(\frac{M}{M-qt}\right)\mathrm{d}t \\
&= \frac{F}{q}\int_0^{\Delta t} \ln\left(\frac{M}{M-qt}\right)\mathrm{d}t \\
&= \frac{F}{q^2}\left[(qt-M)\ln\left(\frac{M}{M-qt}\right) + qt\right]_0^{\Delta t} \qquad (\star) \\
&= \frac{F}{q^2}\left[(q\Delta t-M)\ln\left(\frac{M}{M-q\Delta t}\right) + q\Delta t\right] \qquad \mathit{QED}
\end{aligned}$$

($\star$) Let f be the function $f(t) = (qt-M)\ln\left(\frac{M}{M-qt}\right) + qt$
Then,

$$\begin{aligned}
\frac{d}{dt}f(t) &= \frac{d}{dt}(qt-M)\ln\left(\frac{M}{M-qt}\right) + (qt-M)\times\frac{d}{dt}\ln\left(\frac{M}{M-qt}\right) + \frac{d}{dt}qt \\
&= q\ln\left(\frac{M}{M-qt}\right) + (qt-M)\frac{\frac{Mq}{(M-qt)^2}}{\frac{M}{M-qt}} + q \\
&= q\ln\left(\frac{M}{M-qt}\right) + (qt-M)\frac{q}{M-qt} + q \\
&= q\ln\left(\frac{M}{M-qt}\right) + (-q) + q \\
&= q\ln\left(\frac{M}{M-qt}\right) \\
\text{so} \quad \frac{1}{q}\frac{d}{dt}f(t) &= \ln\left(\frac{M}{M-qt}\right).
\end{aligned}$$

That is why $\frac{F}{q}\int_0^{\Delta t}\ln\left(\frac{M}{M-qt}\right)\mathrm{d}t = \frac{F}{q^2}\left[(qt-M)\ln\left(\frac{M}{M-qt}\right) + qt\right]_0^{\Delta t}$.

5.3 Distance Without Time

We know that $D = \frac{F}{q^2}\left[(q\Delta t-M)\ln\left(\frac{M}{M-q\Delta t}\right) + q\Delta t\right]$ and $\Delta t = \frac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right)$. So, each time there is Δt is the formula of the distance, we can substitute it by its own formula to express D with as less variables as possible.

Let's do this step by step.

1. Because $q = \frac{F}{V_e}$ then $\frac{F}{q^2} = \frac{F}{\left(\frac{F}{V_e}\right)^2} = \frac{(V_e)^2}{F}$.

2. $q\Delta t - M = q \times \frac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right) - M = M\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right) - M = M - M\mathrm{e}^{\frac{-\Delta v}{V_e}} - M = -M\mathrm{e}^{\frac{-\Delta v}{V_e}}$.

   Note that $M - q\Delta t = -(q\Delta t - M) = M\mathrm{e}^{\frac{-\Delta v}{V_e}}$.

3. $\ln\left(\frac{M}{M - q\Delta t}\right) = \ln\left(\frac{M}{M\mathrm{e}^{\frac{-\Delta v}{V_e}}}\right) = \ln\left(\frac{1}{\mathrm{e}^{\frac{-\Delta v}{V_e}}}\right) = -\ln\left(\mathrm{e}^{\frac{-\Delta v}{V_e}}\right) = \frac{\Delta v}{V_e}$.

4. $q\Delta t = q \times \frac{M}{q}\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right) = M\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right)$.

5. Coming back to D:

$$\begin{aligned}
D &= \frac{F}{q^2}\left[(q\Delta t - M)\ln\left(\frac{M}{M - q\Delta t}\right) + q\Delta t\right] \\
&= \frac{(V_e)^2}{F}\left[-M\mathrm{e}^{\frac{-\Delta v}{V_e}} \times \frac{\Delta v}{V_e} + M\left(1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right)\right] \\
&= \frac{M(V_e)^2}{F}\left[-\mathrm{e}^{\frac{-\Delta v}{V_e}} \times \frac{\Delta v}{V_e} + 1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\right] \\
&= \frac{M(V_e)^2}{F}\left[1 - \mathrm{e}^{\frac{-\Delta v}{V_e}}\left(\frac{\Delta v}{V_e} + 1\right)\right] \qquad QED
\end{aligned}$$

6 TODO List

1. Calculate the angle phase of a rendezvous when Bob (the target) is orbiting Kerbin while Valentina is on the launch pad (harder because depends on gravity, atmosphere, gravity turn... but finding at least an estimation $\pm 3°$ could be great.)

2. Find the suicide burn time when landing on the Mun or Minmus. It's the same of having $V_{rel} = 0$ when reaching the ground but with gravity.