Theory

While g (acceleation of gravity) is often considered as a constant, it changes with altitude. This is because the force of gravity that acts upon any two objects is given by Newton's law of universal gravitation:

$$F = G\frac{m_1 m_2}{r} \tag{1}$$

Where G is the gravitational constant, r is the distance between center of mass of both objects, $m1$ is the mass of object 1 and $m2$ is the mass of object 2

However, because earth is not a perfect sphere, r varies depending on altitude. This means that g as mesaured in J-18 will be slightly different from the given value of $9.8N/kg$. Since measuring the distance between the center of mass and the earth precisely is not something that can be easily done, another method has to be used.

Since the period of a pendulum T only depend on its length L and g, it can be used to find g at any location.

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{2}$$

Since a massless rope does not exist in real life, it is not possible really make a simple pendulum, and thus one can not simply use Eq (2). Using a Kater's pendulum because it allows us to calculate gravity without finding the center of mass and center of oscillation. It is a pendulum with two masses that can be shifted along the length of the pendulum and two fixed pivot points. Finding the equation for the angular displacement, solving it to find the period andeliminating variables with clever algebra yields a simple formula.

When oscillating around Point A, the torque that pushes the pendulum back to its original position is given by:

$$\tau = |\vec{R}x\vec{F}| = Mgasin\,(\theta) \tag{3}$$

Since clockwise rotation is a negative torque by convention, the restoring force is a negative toque.

$$\tau = -Mgasin\,(\theta) \tag{4}$$

Defining τ in function of inertia I_a, time t and the angle θ using a different equation.

$$\tau = I\alpha$$
$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$
$$\tau = I_a \frac{d^2\theta}{dt^2}$$

Using the parallel axis theorem to define I_a in terms of the moment of inertia about the center of mass I_cm.

$$\begin{aligned} I_a &= I_cm + Ma^2 \\ I_a &= Mk^2 + Ma^2 \\ I_a &= M(k^2 + a^2) \end{aligned} \tag{5}$$

Where $I_cm = Mk^2$,k defines the radius of gyration about the center of mass, which is the distance from the axis of rotation the mass of the body would be if it was concentrated in a single point while keeping the same moment of inertia. Since Eq. () and Eq. (5) both represent torque, they are equal. Thus:

$$I_a \frac{d^2\theta}{dt^2} = -Mgasin\left(\theta\right) \tag{6}$$

Solving Eq. (6) for T (in this case the period of oscillation with pivot point a) give us:

$$\begin{aligned} T_a &= 2\pi\sqrt{\frac{I_a}{Mga}} \\ T_a &= 2\pi\sqrt{\frac{M(k^2 + a^2)}{Mga}} \\ T_a &= 2\pi\sqrt{\frac{k^2 + a^2}{ga}} \end{aligned} \tag{7}$$

Doing the same with pivot point B gives us:

$$T_b = 2\pi\sqrt{\frac{k^2 + b^2}{gb}} \tag{8}$$

When $T_a = T_b$:

$$
\begin{aligned}
2\pi\sqrt{\frac{k^2+a^2}{ga}} &= 2\pi\sqrt{\frac{k^2+b^2}{gb}} \\
\frac{k^2+a^2}{ga} &= \frac{k^2+b^2}{ga} \\
\frac{k^2+a^2}{ga} &= \frac{k^2+b^2}{gb} \\
\frac{k^2+a^2}{a} &= \frac{k^2+b^2}{b} \\
(k^2+a^2)b &= (k^2+b^2)a \\
(k^2+a^2)b-(k^2+b^2)a &= 0 \\
a^2b-(k^2+b^2)a+k2b &= 0
\end{aligned}
$$

The above is an equation of the form ax^2+bx+c, thus, apply the quadratic formula

$$
a = \frac{\left(k^2+b^2\right) \pm \sqrt{\left(-\left(k^2+b^2\right)\right)^2-4bk^2b}}{2b}
$$

$$
a = \frac{\left(k^2+b^2\right) \pm (k+b)(k-b)}{2b}
$$

$$
\begin{aligned}
a_1 &= \frac{(k^2+b^2)+(k+b)(k-b)}{2b} & a_2 &= \frac{(k^2+b^2)-(k+b)(k-b)}{2b} \\
a_1 &= \frac{k^2+b^2+k2+kb-kb-b^2}{2b} & a_2 &= \frac{k^2+b^2-k2-kb+kb+b^2}{2b} \\
a_1 &= \frac{2k^2}{2b} & a_2 &= \frac{2b^2}{2b} \\
a_1 &= \frac{k^2}{b} & a_2 &= \frac{b^2}{b} \\
ab &= k^2 & a &= b
\end{aligned}
\tag{9}
$$

By using $ab = k^2$ obtained in Eq. (9) to replace k^2 in 7 and 8

$$
\begin{aligned}
T_a &= T_b \\
2\pi\sqrt{\frac{k^2+a^2}{ga}} &= 2\pi\sqrt{\frac{k^2+b^2}{gb}} \\
2\pi\sqrt{\frac{ab+a^2}{ga}} &= 2\pi\sqrt{\frac{ab+b^2}{gb}} \\
2\pi\sqrt{\frac{a\left(a+b\right)}{ga}} &= 2\pi\sqrt{\frac{b\left(a+b\right)}{gb}} \\
2\pi\sqrt{\frac{\left(a+b\right)}{g}} &= 2\pi\sqrt{\frac{\left(a+b\right)}{g}} \\
T &= 2\pi\sqrt{\frac{a+b}{g}}
\end{aligned}
\tag{10}
$$

We then solve the Eq. (10) for g

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{a+b}{g}} \\
T^2 &= 4\pi^2\frac{a+b}{g} \\
g &= 4\pi^2\frac{a+b}{T^2}
\end{aligned}
\tag{11}
$$

$a+b$ being the distance between the fixed pivot points, which is given in this case as being $a+b=100.06cm$