特征选择与变换习题

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1.

(1)

$\boldsymbol{H}\boldsymbol{e}=0$:

$$\boldsymbol{H}\boldsymbol{e}=(\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T)\boldsymbol{e}=\boldsymbol{I}\boldsymbol{e}-\boldsymbol{e}(\boldsymbol{e}^T\boldsymbol{e})=\boldsymbol{e}-\boldsymbol{e}=\boldsymbol{0}$$

$\boldsymbol{H}\boldsymbol{H}=\boldsymbol{H}$:

$$\boldsymbol{H}\boldsymbol{H}=(\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T)(\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T)=\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T-\boldsymbol{e}\boldsymbol{e}^T+\boldsymbol{e}(\boldsymbol{e}^T\boldsymbol{e})\boldsymbol{e}^T=\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T=\boldsymbol{H}$$

$(\boldsymbol{x}_1-\bar{\boldsymbol{x}},\ldots,\boldsymbol{x}_n-\bar{\boldsymbol{x}})^T=\boldsymbol{H}\boldsymbol{X}$:

$$(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}},\ldots,\bar{\boldsymbol{x}})^T=[\bar{\boldsymbol{x}}(1,1,\ldots,1)]^T=(\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}})^T=\boldsymbol{e}\boldsymbol{e}^T\boldsymbol{X}$$

$$(\boldsymbol{x}_1-\bar{\boldsymbol{x}},\ldots,\boldsymbol{x}_n-\bar{\boldsymbol{x}})^T=\boldsymbol{X}-\boldsymbol{e}\boldsymbol{e}^T\boldsymbol{X}=(\boldsymbol{I}-\boldsymbol{e}\boldsymbol{e}^T)\boldsymbol{X}=\boldsymbol{H}\boldsymbol{X}$$

$\boldsymbol{\Sigma}=\frac{1}{n}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}$:

$$\boldsymbol{\Sigma}=\frac{1}{n}(\boldsymbol{H}\boldsymbol{X})^T(\boldsymbol{H}\boldsymbol{X})=\frac{1}{n}\boldsymbol{X}^T(\boldsymbol{H}^T\boldsymbol{H})\boldsymbol{X}=\frac{1}{n}\boldsymbol{X}^T(\boldsymbol{H}\boldsymbol{H})\boldsymbol{X}=\frac{1}{n}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}$$

(2)

$\boldsymbol{v}_i$为$\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}$的特征值：

$$\begin{gathered}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i=\lambda_i\boldsymbol{v}_i\\ \boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i=\lambda_i\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i\\ \boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}(\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i)=\lambda_i(\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i)\end{gathered}$$

$\boldsymbol{u}_i$为$\boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}$的特征值：

$$
\begin{aligned}
\boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i &= l_i\boldsymbol{u}_i \\
\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i &= l_i\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i \\
\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i) &= l_i(\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i)
\end{aligned}
$$

又由于：

$$
\begin{aligned}
\boldsymbol{v}_i^T\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i = (\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i)^T(\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i) = \lambda_i, \qquad \boldsymbol{u}_i^T\boldsymbol{u}_i = 1 \\
\boldsymbol{u}_i^T\boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i = (\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i)^T(\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i) = l_i, \qquad \boldsymbol{v}_i^T\boldsymbol{v}_i = 1
\end{aligned}
$$

可以知道对应的比例为$\sqrt{\lambda_i}$和$\sqrt{l_i}$，即$\boldsymbol{v}_i = \frac{\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i}{\sqrt{l_i}}, \boldsymbol{u}_i = \frac{\boldsymbol{H}\boldsymbol{X}\boldsymbol{v}_i}{\sqrt{\lambda_i}}$。

2.

(1)

由于$\boldsymbol{u}$为$\boldsymbol{B}$的特征向量：

$$
\begin{aligned}
l_i\boldsymbol{u}_i = \boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i = \boldsymbol{H}\boldsymbol{H}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{u}_i = l_i\boldsymbol{H}\boldsymbol{u}_i \\
\Rightarrow \boldsymbol{u}_i = \boldsymbol{H}\boldsymbol{u}_i
\end{aligned}
$$

于是有：

$$
\boldsymbol{H}\hat{\boldsymbol{X}} = \boldsymbol{H}(\sqrt{l_1}\boldsymbol{u}_1, \ldots, \sqrt{l_n}\boldsymbol{u}_n) = (\sqrt{l_1}\boldsymbol{H}\boldsymbol{u}_1, \ldots, \sqrt{l_n}\boldsymbol{H}\boldsymbol{u}_n) = (\sqrt{l_1}\boldsymbol{u}_1, \ldots, \sqrt{l_n}\boldsymbol{u}_n) = \hat{\boldsymbol{X}}
$$

(2)

$$
\hat{\boldsymbol{B}} = \boldsymbol{H}\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}^T\boldsymbol{H} = \boldsymbol{H}\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}^T\boldsymbol{H}^T = \boldsymbol{H}\hat{\boldsymbol{X}}(\boldsymbol{H}\hat{\boldsymbol{X}})^T = \hat{\boldsymbol{X}}\hat{\boldsymbol{X}}^T = \boldsymbol{B}
$$

(3)

$$
\begin{aligned}
s_{ij} &= (\boldsymbol{x}_i - \boldsymbol{x}_j)^T(\boldsymbol{x}_i - \boldsymbol{x}_j) \\
&= ((\boldsymbol{x}_i - \bar{\boldsymbol{x}}) - (\boldsymbol{x}_j - \bar{\boldsymbol{x}}))^T((\boldsymbol{x}_i - \bar{\boldsymbol{x}}) - (\boldsymbol{x}_j - \bar{\boldsymbol{x}})) \\
&= (\boldsymbol{x}_i - \bar{\boldsymbol{x}})^T(\boldsymbol{x}_i - \bar{\boldsymbol{x}}) + (\boldsymbol{x}_j - \bar{\boldsymbol{x}})^T(\boldsymbol{x}_j - \bar{\boldsymbol{x}}) - 2(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^T(\boldsymbol{x}_j - \bar{\boldsymbol{x}}) \\
&= b_{ii} + b_{jj} - 2b_{ij}
\end{aligned}
$$

所以由于$\hat{\boldsymbol{B}} = \boldsymbol{B}$，即可以直接得到$\hat{\boldsymbol{S}} = \boldsymbol{S}$。

3.

$$\boldsymbol{\Sigma}_\varphi \boldsymbol{v} = \lambda \boldsymbol{v}$$
$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\varphi(\boldsymbol{x}_i)\varphi(\boldsymbol{x}_i)^T\boldsymbol{v} = \lambda\boldsymbol{v}$$

当$\lambda \neq 0$时：

$$\boldsymbol{v} = \frac{1}{n\lambda}\sum_{i=1}^{n}\varphi(\boldsymbol{x}_i)[\varphi(\boldsymbol{x}_i)^T\boldsymbol{v}] = \frac{1}{n\lambda}\sum_{i=1}^{n}[\varphi(\boldsymbol{x}_i)^T\boldsymbol{v}]\varphi(\boldsymbol{x}_i)$$

其中$[\varphi(\boldsymbol{x}_i)^T\boldsymbol{v}]$为标量，故可以表示为$\boldsymbol{v} = \sum_{i=1}^{n}\alpha_i\varphi(\boldsymbol{x}_i)$的形式。