ASTR6403P: Physical Cosmology
Homework Set 4

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Question 1

Use the ${}^1{}_2$ component of the Einstein equation to show that $h_\times$ obeys the same equation as does h_+.

Solution:

The ${}^1{}_2$ component of the Einstein equation in the context of gravitational waves can be expressed as:

$$\delta G^1{}_2 = \delta^{1k}\left[\frac{3}{2}Hh^{\mathrm{TT}}_{k2,0} + \frac{h^{\mathrm{TT}}_{k2,00}}{2} + \frac{k^2}{2a^2}h^{\mathrm{TT}}_{k2}.\right]$$

also like what we did for h_{11}, we consider $h^{\mathrm{TT}}_{12} = h^{\mathrm{TT}}_{21} = h_\times$, then we have

$$\delta G^1{}_2 + \delta G^2{}_1 = 3Hh_{\times,0} + h_{\times,00} + \frac{k^2}{a^2}h_\times = 0.$$

Similarly, we change to conformal time so that $h_{\times,0} = \frac{1}{a}h'_\times$ and $h_{\times,00} = \frac{1}{a^2}h''_\times - \frac{a'}{a^3}h'_\times$, then we have

$$h''_\times + 2\frac{a'}{a}h'_\times + k^2 h_\times = 0,$$

which is exactly the same as the equation for h_+.

Question 2

In class, we have derived how the perturbation to the time-time component of the metric tensor A changes under a general coordinate transformation. In this exercise, complete the derivation for the other three scalar perturbations to the metric, i.e., Ψ, B, E. Show that Φ_{A} and Φ_{H} as identified by Bardeen (1980) indeed do not change under a general coordinate transformation.

Solution:

Consider the general coordinate transformation

$$\tilde{x}^0 = x^0 + \zeta, \quad \tilde{x}^i = x^i + \xi^{,i},$$

For Ψ:

$$\tilde{g}_{ij} = a^2 \left[(1-2\tilde{\Psi})\delta_{ij} + 2\partial_i\partial_j\tilde{E}\right] = \frac{\partial\tilde{x}^\alpha}{\partial x^i}\frac{\partial\tilde{x}^\beta}{\partial x^j}g_{\alpha\beta}.$$

Considering only first-order terms, we have

$$-2a^2\tilde{\Psi}\delta_{ij} = -2a^2\Psi\delta_{ij} - 2a^2\frac{a'}{a}\delta_{ij}\zeta,$$

so that

$$\tilde{\Psi} = \Psi + \frac{a'}{a}\zeta.$$

For B:

$$\tilde{g}_{0i} = -a^2\partial_i\tilde{B} = \frac{\partial\tilde{x}^\alpha}{\partial x^0}\frac{\partial\tilde{x}^\beta}{\partial x^i}g_{\alpha\beta}.$$

Considering only first-order terms, we have

$$-a^2\partial_i\tilde{B} = -a^2\partial_i B + a^2\partial_i\zeta - a^2\frac{a'}{a}\partial_i\xi,$$

so that

$$\tilde{B} = B - \zeta + \frac{a'}{a}\xi.$$

For E:

$$\tilde{g}_{ij} = a^2 \left[(1-2\tilde{\Psi})\delta_{ij} + 2\partial_i\partial_j\tilde{E}\right] = \frac{\partial\tilde{x}^\alpha}{\partial x^i}\frac{\partial\tilde{x}^\beta}{\partial x^j}g_{\alpha\beta}.$$

Considering only first-order terms, we have

$$2a^2\partial_i\partial_j\tilde{E} = 2a^2\partial_i\partial_j E - 2a^2\partial_i\partial_j\xi,$$

so that

$$\tilde{E} = E - \xi.$$

Now we can check the gauge invariance of Φ_A and Φ_H:

$$\begin{aligned}
\tilde{\Phi}_\mathrm{A} &= \tilde{A} - \frac{1}{a}\left[a(\tilde{B}-\tilde{E}')\right] \\
&= A - \zeta' - \frac{a'}{a}\zeta - \frac{1}{a}\left[a\left(B - \zeta + \frac{a'}{a}\xi - (E'-\xi')\right)\right]' \\
&= A - \frac{1}{a}\left[a(B-E')\right] = \Phi_\mathrm{A},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Phi}_\mathrm{H} &= \tilde{\Psi} + \frac{a'}{a}(\tilde{B}-\tilde{E}') \\
&= \Psi + \frac{a'}{a}\zeta + \frac{a'}{a}\left(B - \zeta + \frac{a'}{a}\xi - (E'-\xi')\right) \\
&= \Psi + \frac{a'}{a}(B-E') = \Phi_\mathrm{H}.
\end{aligned}$$

Question 3

Inflation also solves the flatness problem, in addition to the horizon problem and generation of initial conditions for structure formation, which we will discuss in this exercise.

(a) Suppose that $\Omega(t) \equiv \dfrac{8\pi G\rho(t)}{3H^2(t)}$ is equal to 0.3 today, where ρ counts the energy density in matter and radiation (ignore the cosmological constant). From the Friedmann equation for non-flat Universe that you have derived in previous exercises, plot $\Omega(t) - 1$ as a function of the scale factor. How close to 1 would $\Omega(t)$ have been back at the Planck epoch (assuming no inflation took place so that the scale factor at the Planck epoch was of order 10^{-32})? This fine-tuning of the initial conditions is the flatness problem.

(b) Now show that inflation solves the flatness problem. Extrapolate $\Omega(t) - 1$ back to the end of inflation, and then through 60 e-folds of inflation. What is $\Omega(t) - 1$ right before these 60 e-folds of inflation? This shows how inflation flattens space.

Solution:

(a) The Friedmann equation for a non-flat Universe is given by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

so that we have

$$\Omega(t) - 1 = \frac{k}{a^2 H^2}.$$

Consider $H^2 = H_0^2\left(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{k,0}a^{-2}\right)$, where $\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0}$, we have

$$\Omega(t) - 1 = \frac{\Omega_{k,0}}{\Omega_{m,0}a^{-1} + \Omega_{r,0}a^{-2} + \Omega_{k,0}}.$$

And we can see that at the Planck epoch ($a \sim 10^{-32}$), $\Omega(t) - 1 \sim -10^{-60}$, which means $\Omega(t)$ must be extremely close to 1 at that time.

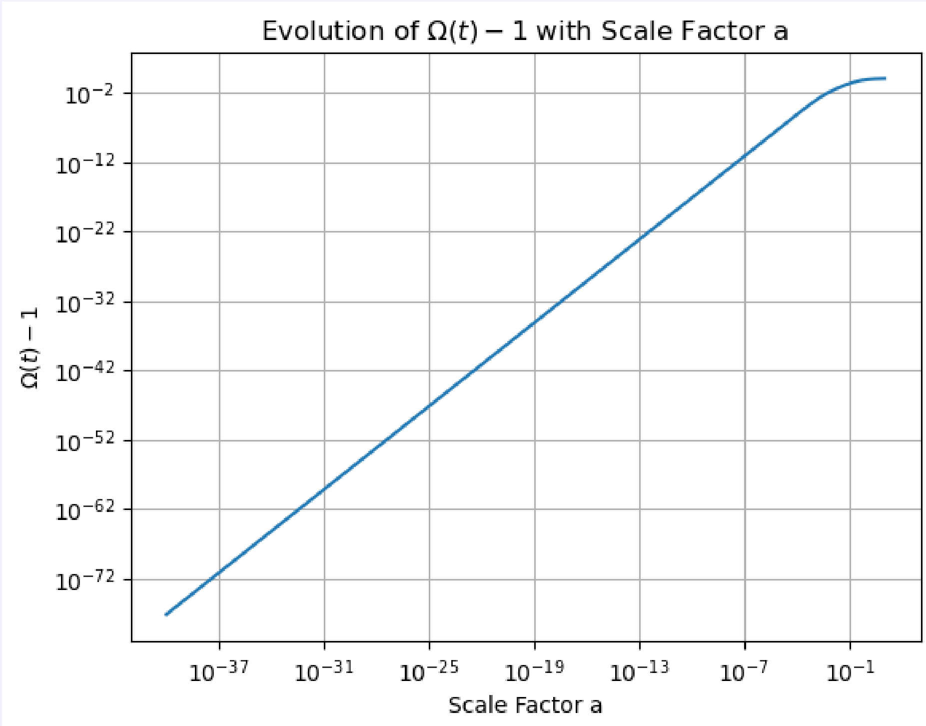
(b) During the inflation, we think $H = const.$, so that

$$\Omega(t) - 1 = \frac{k}{a^2 H^2} \propto a^{-2}.$$

Therefore, after 60 e-folds of inflation, we have

$$\Omega(t_{\text{start}}) - 1 = (\Omega(t_{\text{end}}) - 1)e^{120} \simeq 10^{-8}.$$

So that before the 60 e-folds of inflation, we have $\Omega(t) - 1$ change from 10^{-8} to 10^{-60} after inflation, which means inflation can effectively flatten the space.



Evolution of Ω(t) − 1 with Scale Factor a
Ω(t) − 1
10^−2
10^−12
10^−22
10^−32
10^−42
10^−52
10^−62
10^−72
10^−37
10^−31
10^−25
10^−19
10^−13
10^−7
10^−1
Scale Factor a