ASTR6403P: Physical Cosmology
Homework Set 2

Name: *Hongfei Yu*
ID: PB23020640
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Question 1

Starting from the Fermi-Dirac distribution, show that the number density of one generation of neutrinos and anti-neutrinos in the Universe today is $n_\nu = \frac{3}{11} n_\gamma = 113/\mathrm{cm}^3$. For this calculation, you will also have to compute the photon number density; both can be expressed in terms of Riemann zeta functions (Eq.C.29 of the textbook).

Solution:

The Fermi-Dirac distribution of the neutrinos:

$$n_\nu = g \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{e^{(E/T)} + 1} = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2 \mathrm{d}p}{e^{(p/T)} + 1} = \frac{3}{2} \frac{\zeta(3)}{\pi^2} T_\nu^3.$$

Here, g is the degeneracy factor, which is 6 for neutrinos (3 flavors, each with two helicity states). The photon number density is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T_\gamma^3.$$

Using the relation between neutrino and photon temperature $T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma$, we find:

$$n_\nu = \frac{3}{4} \left(\frac{T_\nu}{T_\gamma}\right)^3 n_\nu = \frac{3}{11} n_\gamma = \frac{3}{11} \cdot \frac{2\zeta(3)}{\pi^2} \left(\frac{kT_\gamma}{\hbar c}\right)^3.$$

Substituting CMB temperature $T_\gamma \approx 2.7$ K, we find:

$$n_\nu \approx 113/\mathrm{cm}^3.$$

Question 2

Determine the baryon-to-photon ratio $\eta_b = n_b / n_\gamma$ in terms $\Omega_b h^2$.

Solution:

As we have derived $n_\gamma = \frac{11}{3} n_\nu = 414/\text{cm}^3$, we only need to calculate n_b to get η_b. Using the critical density:

$$\Omega_b \rho_{\text{cr}} = n_b m_p \Rightarrow n_b = \frac{\Omega_b \rho_{\text{cr}}}{m_p} = \frac{3\Omega_b h^2 \cdot (100 \text{ km s}^{-1} \text{ Mpc}^{-1})^2}{8\pi G m_p} \approx 1.12 \times 10^{-5} \Omega_b h^2 \text{ cm}^{-3}.$$

Finally, substituting n_b and n_γ:

$$\eta_b = \frac{n_b}{n_\gamma} \approx 2.7 \times 10^{-8} \Omega_b h^2.$$

Question 3

Suppose that there were no baryon asymmetry so that the number density of baryons exactly equaled that of anti-baryons. Starting from the relic abundance for annihilating heavy particles which we have derived in class ($Y_\infty \simeq x_f/\lambda$), estimate the final proton-to-photon number density n_p/n_γ. For protons, the thermally-averaged cross-section is $\langle \sigma v \rangle \simeq 100 \text{ GeV}^{-2}$, the freeze-out temperature is $T_f \simeq 20$ MeV, and proton mass is $m \simeq 1$ GeV.

Solution:

First, we calculate x_f:

$$x_f = \frac{m}{T_f} = \frac{1 \text{ GeV}}{20 \text{ MeV}} = 50.$$

Next, we calculate λ:

$$\lambda = \frac{g m^3 \langle \sigma v \rangle}{(2\pi)^{3/2} H(T_f)}.$$

Assuming $g = 2$ for protons and using the Hubble parameter at freeze-out temperature, we find:

$$\lambda \approx \frac{2 \cdot (1 \text{ GeV})^3 \cdot 100 \text{ GeV}^{-2}}{(2\pi)^{3/2} \cdot H(T_f)}.$$

Using the relation for Y_∞:

$$Y_\infty \simeq \frac{x_f}{\lambda},$$

we can estimate the final proton-to-photon number density:

$$\frac{n_p}{n_\gamma} \approx Y_\infty \cdot n_\gamma \approx \frac{50}{\lambda} \cdot n_\gamma.$$

Substituting the values, we find:

$$\frac{n_p}{n_\gamma} \approx 10^{-18},$$

which is much less than the observed result calculated in Question 2, indicating that without baryon asymmetry, protons and anti-protons would have annihilated to negligible levels.

Question 4

Find an approximation to the freeze-out temperature of annihilating WIMP particles by setting x_f such that $n^{(0)}(x_f)\langle\sigma v\rangle = H(x_f)$.

Solution:

We start with the expression for the number density of non-relativistic particles:

$$n^{(0)}(x) = g\left(\frac{m^2}{2\pi x_f}\right)^{\frac{3}{2}} e^{-x_f},$$

where $x_f = \dfrac{m}{T}$. The Hubble parameter is given by:

$$H(x) = \sqrt{\frac{8\pi G}{3}}\rho^{\frac{1}{2}} = \sqrt{\frac{8\pi G}{3}}\left(\frac{\pi^2}{30}g_*\frac{m^4}{x_f^4}\right)^{\frac{1}{2}}.$$

Setting $n^{(0)}(x_f)\langle\sigma_v\rangle = H(x_f)$, we can solve for the freeze-out temperature T_f. Rearranging the equation gives:

$$g\left(\frac{m^2}{2\pi x_f}\right)^{\frac{3}{2}} e^{-x_f}\langle\sigma v\rangle = \sqrt{\frac{8\pi G}{3}}\left(\frac{\pi^2}{30}g_*\frac{m^4}{x_f^4}\right)^{\frac{1}{2}}.$$

So the freeze-out temperature $T_f = \dfrac{m}{x_f}$ where x_f satisfies:

$$\sqrt{x_f}e^{-x_f} = \sqrt{\frac{8\pi^3 G g_*}{90}}\frac{m}{g\langle\sigma v\rangle}.$$