ASTR6403P: Physical Cosmology
Homework Set 5

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Question 1

Express the slow-roll parameters ϵ and η in terms of the potential V and its derivatives with respect to ϕ. By utilizing the Friedmann equation and the equation of motion for the scalar field, show that, to lowest order, $\epsilon \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$ and $\eta \equiv \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$, where primes denote derivatives with respect to ϕ evaluated at $\phi^{(0)}$.

Solution:

The slow-roll parameters ϵ and η are defined in terms of the Hubble parameter H and the scalar field ϕ as follows:

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = \frac{\ddot{\phi}}{H\dot{\phi}}.$$

Using the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

and the equation of motion for the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,$$

we can express ϵ and η in terms of the potential V and its derivatives. First, we compute $\dot{H}$:

$$2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right).$$

Substituting the equation of motion for $\ddot{\phi}$, we get

$$2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\phi}(-3H\dot{\phi} - V'(\phi)) + V'(\phi)\dot{\phi}\right) = -\frac{8\pi G}{3}3H\dot{\phi}^2.$$

Thus,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G\dot{\phi}^2}{H^2}.$$

Next, we express $\dot{\phi}$ in terms of V using the slow-roll approximation, which gives

$$3H\dot{\phi} \approx -V'(\phi) \implies \dot{\phi} \approx -\frac{V'(\phi)}{3H}.$$

Substituting this into the expression for ϵ, we have

$$\epsilon = \frac{4\pi G}{H^2}\left(-\frac{V'(\phi)}{3H}\right)^2 = \frac{4\pi G}{9H^4}V'^2(\phi).$$

Using the Friedmann equation to express H^2 in terms of V, we find

$$H^2 \approx \frac{8\pi G}{3}V(\phi),$$

which leads to

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2.$$

For η, we have

$$\eta = \frac{\ddot{\phi}}{H\dot{\phi}}.$$

Using the equation of motion for $\ddot{\phi}$, we get

$$\eta = \frac{-3H\dot{\phi} - V'(\phi)}{H\dot{\phi}} = -3 - \frac{V'(\phi)}{H\dot{\phi}}.$$

Substituting the slow-roll approximation for $\dot{\phi}$, we find

$$\eta = -3 + \frac{V'(\phi)}{H\left(-\frac{V'(\phi)}{3H}\right)} = -3 + \frac{3H^2}{V'(\phi)}.$$

Using the Friedmann equation again, we can express this as

$$\eta = -3 + \frac{3\left(\frac{8\pi G}{3}V(\phi)\right)}{V'(\phi)} = -3 + \frac{8\pi G V(\phi)}{V'(\phi)}.$$

Finally, we can express η in terms of $V''(\phi)$:

$$\eta = \epsilon - \frac{1}{8\pi G}\frac{V''(\phi)}{V(\phi)}.$$

Question 2

Show that the curvature in conformal Newtonian gauge is equal to $4k^2\Phi/a^2$. To do this, compute the three-dimensional Ricci scalar arising from the spatial part of the metric $g_{ij} = \delta_{ij}a^2(1+2\Phi)$.

Solution:

The spatial part of the metric in conformal Newtonian gauge is given by

$$g_{ij} = a^2(1+2\Phi)\delta_{ij}.$$

To compute the three-dimensional Ricci scalar $R^{(3)}$, we first need to calculate the Christoffel symbols for the spatial metric. The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

Calculating the derivatives, we find

$$\partial_i g_{jl} = 2a^2\delta_{jl}\partial_i\Phi,$$

and similarly for the other terms. Substituting these into the expression for the Christoffel symbols, we get

$$\Gamma^k_{ij} = \delta^{kl}\left(\delta_{jl}\partial_i\Phi + \delta_{il}\partial_j\Phi - \delta_{ij}\partial_l\Phi\right).$$

Next, we compute the Ricci tensor R_{ij} using the formula

$$R_{ij} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^l_{ik}\Gamma^k_{jl} = -2a^2\delta_{ij}\nabla^2\Phi.$$

Finally, the Ricci scalar $R^{(3)}$ is obtained by contracting the Ricci tensor with the inverse metric:

$$R^{(3)} = g^{ij}R_{ij} = \frac{1}{a^2(1+2\Phi)}(-2a^2\nabla^2\Phi) = -\frac{2\nabla^2\Phi}{1+2\Phi} \approx -2\nabla^2\Phi.$$

In Fourier space, we find

$$R^{(3)} \approx \frac{2k^2\Phi}{a^2}.$$

Thus, the curvature in conformal Newtonian gauge is given by

$$\kappa = \frac{4k^2\Phi}{a^2}.$$

Question 3

One way to characterize the amplitude of matter fluctuations on a particular scale is to compute the expected root mean square (RMS) overdensity in a sphere of comoving radius R, $\sigma_R^2 \equiv \langle\delta^2_{m,R}(\vec{x})\rangle$. Here

$$\delta_{m,R}(\vec{x}) = \int d^3x'\delta_m(\vec{x}')W_R(|\vec{x}-\vec{x}'|),$$

where $W_R(x)$ is the tophat window function, equal to $3/(4\pi R^3)$ for $x < R$ and 0 otherwise; the angular brackets denote the ensemble average.

(a) By Fourier transforming, express σ_R^2 in terms of an integral over the power spectrum.

(b) Choose the integration variable as $\ln k$, and plot the corresponding integrand for σ_8 ($R = 8\ h^{-1}$ Mpc) at redshift 0 in a standard CDM model ($\Omega_m = 1$, with other parameters set at $h = 0.7, n_s = 1, A_s = 2.1 \times 10^{-9}$). For simplicity, use the BBKS transfer function here.

Solution:

(a) The Fourier transform of the smoothed overdensity $\delta_{m,R}(\vec{x})$ is given by

$$\delta_{m,R}(\vec{k}) = \delta_m(\vec{k}) W_R(k),$$

where $W_R(k)$ is the Fourier transform of the tophat window function. The variance σ_R^2 can then be expressed as

$$\begin{aligned}
\sigma_R^2 := \langle \delta_{m,R}^2(\vec{x}) \rangle &= \int_{\vec{k}_1} \int_{\vec{k}_2} \langle \delta_{m,R}(\vec{k}_1) \delta_{m,R}(\vec{k}_2) \rangle \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \\
&= \int_{\vec{k}_1} \int_{\vec{k}_2} \langle \delta(\vec{k}_1) \delta(\vec{k}_2) \rangle W_R(\vec{k}_1) W_R(\vec{k}_2) \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \\
&= \int_{\vec{k}_1} \int_{\vec{k}_2} (2\pi)^3 P(k_1) \delta^D(\vec{k}_1 + \vec{k}_2) W_R(\vec{k}_1) W_R(\vec{k}_2) \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P(k) W_R^2(k) \\
&= \int \frac{\mathrm{d}k}{k} \Delta^2(k) W_R^2(k)
\end{aligned}$$

where $P(k)$ is the matter power spectrum.

(b)

$$\sigma_8^2 = \int \mathrm{d}\ln k \Delta^2(k) \left(\frac{3 j_1(8k)}{8k} \right)^2,$$

here j_1 is the spherical Bessel function of the first kind. The primordial power spectrum is given by

$$P_0 = A_s \left(\frac{k}{k_*} \right)^{n_s - 1} = A_s = 2.1 \times 10^{-9}.$$

The BBKS transfer function is given by

$$T(q) = \frac{\ln(1 + 2.34q)}{2.34q} \left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4 \right]^{-1/4},$$

where $q = k/h^2$ for $\Omega_\mathrm{m} = 1$. The matter power spectrum is then

$$P(k) = \frac{8\pi^2}{25}\frac{k}{H_0^4\Omega_m^2}P_0T^2(k)D_+^2(z),$$

where $D_+(z) = 1/(1+z)$ when $\Omega_\mathrm{m} = 1$. Finally, $\Delta(k)\Big|_{z=0}$ is given by

$$\Delta(k) = \frac{k^3P(k)}{2\pi^2} = \frac{4k^4}{25H_0^4}A_sT^2(k).$$