ASTR6403P: Physical Cosmology
Homework Set 1

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Question 1

Convert the following quantities by inserting the appropriate factors of $c, \hbar$ and k_B:

i) $T_0 = 2.725\text{K} \to \text{eV}$

Solution:

$$\begin{aligned} T_0 = k_{\text{B}} T_0 &= 1.381 \times 10^{-23}\text{J} \cdot \text{K}^{-1} \times 2.725\text{K} \\ &= 3.762 \times 10^{-23}\text{J} \times 6.252 \times 10^{18}\text{eV/J} \\ &= 2.348 \times 10^{-5}\text{eV}. \end{aligned}$$

ii) $\rho_\gamma = \pi^2 T_0^4/15 \to \text{eV}^4$ and g cm^{-3}

Solution:

$$\begin{aligned} \rho_\gamma &= \frac{\pi^2 T_0^4}{15} = \frac{\pi^2}{15}(2.348 \times 10^{-5}\text{eV})^4 \\ &= 2.000 \times 10^{-14}\text{eV}^4 \\ &= 2.000 \times 10^{-14}\text{eV}^4 \times (5.068 \times 10^4\text{cm}^{-1} \cdot \text{eV}^{-1})^3 \times 1.783 \times 10^{-33}\text{g} \cdot \text{eV}^{-1} \\ &= 4.642 \times 10^{-33}\text{g} \cdot \text{cm}^{-3}. \end{aligned}$$

iii) $1/H_0 \to \text{cm}$

Solution:

[a]

$$\begin{aligned} \frac{1}{H_0} &= \frac{1}{67.4\text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}} \\ &= \frac{3 \times 10^8\text{m} \cdot \text{s}^{-1}}{67.4 \times 10^3\text{m} \cdot \text{sec}^{-1}} \times 3.086 \times 10^{24}\text{cm} = 1.374 \times 10^{29}\text{cm}. \end{aligned}$$

[a] $H_0 = 67.4 \pm 0.5\text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$ from Planck 2018 [1807.06209].

iv) $m_{Pl} = 1.2 \times 10^{19}\text{GeV} \to \text{K, cm}^{-1},\ \text{sec}^{-1}$

Solution:

$$\begin{aligned}
m_{Pl} = \frac{m_{Pl}}{k_B} &= \frac{1.2 \times 10^{19}\text{GeV}}{1.381 \times 10^{-23}\text{J} \cdot \text{K}^{-1}} \\
&= 8.689 \times 10^{50}\text{K} \\
&= 1.2 \times 10^{19}\text{GeV} \times 5.068 \times 10^{4}\text{cm}^{-1} \cdot \text{eV}^{-1} = 6.082 \times 10^{32}\text{cm}^{-1} \\
&= 6.082 \times 10^{32}\text{cm}^{-1} \times 3 \times 10^{10}\text{cm} \cdot \text{sec}^{-1} = 1.824 \times 10^{43}\text{sec}^{-1}.
\end{aligned}$$

Question 2

Calculate the Christoffel symbols for flat 2-dimensional space with polar coordinates, obtain the geodesic equation for a freely moving particle, and show the trajectory is as expected.

Solution:

$$\Gamma^{k}{}_{ij} = \frac{g^{kl}}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

The metric in 2-D polar coordinates (r, θ) is:

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{r\theta} = g_{\theta r} = 0.$$

and their inverses:

$$g^{rr} = 1, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad g^{r\theta} = g^{\theta r} = 0.$$

Then:

$$\Gamma^{r}{}_{rr} = \frac{g^{rr}}{2}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) = 0,$$

$$\Gamma^{r}{}_{r\theta} = \Gamma^{r}{}_{\theta r} = \frac{g^{rr}}{2}(\partial_\theta g_{rr}) = 0,$$

$$\Gamma^{r}{}_{\theta\theta} = \frac{g^{rr}}{2}(-\partial_r g_{\theta\theta}) = -r,$$

$$\Gamma^{\theta}{}_{rr} = \frac{g^{\theta\theta}}{2}(2\partial_r g_{\theta r} - \partial_\theta g_{rr}) = 0,$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{g^{\theta\theta}}{2}(\partial_r g_{\theta\theta}) = \frac{1}{r},$$

$$\Gamma^{\theta}{}_{\theta\theta} = \frac{g^{\theta\theta}}{2}(\partial_\theta g_{\theta\theta}) = 0.$$

The geodesic equations are:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^{i}{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = 0.$$

Consider $\lambda = t$, for $i = r$:

$$\ddot{r} - r\dot{\theta}^2 = 0,$$

for $i = \theta$:

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0.$$

These are exactly the equations of motion in polar coordinates in classical mechanics, as expected.

Question 3

In class, we have derived how the energy of a massless particle changes in a flat expanding Universe. Repeat the calculation for a massive non-relativistic particle and find how its energy changes.

Solution:

We consider a parameter λ which satisfies:

$$P^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda},$$

and $\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}x^0} = E\frac{\mathrm{d}}{\mathrm{d}t}$. Then in FLRW metric, with ${\Gamma^0}_{ij} = a\dot{a}\delta_{ij}$, we have:

$$E\frac{\mathrm{d}E}{\mathrm{d}t} = -{\Gamma^0}_{ij}P^iP^j = -a\dot{a}\delta_{ij}P^iP^j.$$

For a massive non-relativistic particle, we have mass-shell condition:

$$P^\mu P_\mu = -E^2 + a^2\delta_{ij}P^iP^j = -m^2,$$

then the evolution of energy is:

$$E\frac{\mathrm{d}E}{\mathrm{d}t} = -a\dot{a}\delta_{ij}P^iP^j = -\frac{\dot{a}}{a}(E^2 - m^2).$$

the solution is:

$$E^2 - m^2 \propto a^{-2}.$$

Question 4

Consider a galaxy of physical size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? redshift 1? Do the calculation in a flat Universe, first matter-dominated and then with 30% matter and 70% cosmological constant.

Solution:

The angle subtended in a flat Universe is:

$$\theta = \frac{l/a}{\chi(a)} = \frac{l(1+z)}{c\displaystyle\int_0^z \frac{\mathrm{d}z'}{H(z')}},$$

then use Friedmann Equation

$$H = \left[\frac{8\pi G}{3}\rho_{\mathrm{cr},0}\left(\Omega_{\mathrm{r}}(1+z)^4 + \Omega_{\mathrm{m}}(1+z)^3 + \Omega_{\mathrm{k}}(1+z)^2 + \Omega_\Lambda\right)\right]^{\frac{1}{2}}$$

Finally, we can compute the angle for different redshifts:

$$\begin{aligned}\theta(z) &= l(1+z)\left\{c\int_0^z \left[\frac{8\pi G}{3}\rho_{\mathrm{cr},0}\left(\Omega_{\mathrm{r}}(1+z)^4 + \Omega_{\mathrm{m}}(1+z)^3 + \Omega_{\mathrm{k}}(1+z)^2 + \Omega_\Lambda\right)\right]^{-\frac{1}{2}}\mathrm{d}z\right\}^{-1} \\ &= A(1+z)\left[\int_0^z \left(\Omega_{\mathrm{r}}(1+z)^4 + \Omega_{\mathrm{m}}(1+z)^3 + \Omega_{\mathrm{k}}(1+z)^2 + \Omega_\Lambda\right)^{-\frac{1}{2}}\mathrm{d}z\right]^{-1}.\end{aligned}$$

where $A = \frac{l}{c}\left(\frac{8\pi G}{3}\rho_{\mathrm{cr},0}\right)^{\frac{1}{2}}$.
Substitute $l = 5\mathrm{kpc} = 1.542\times10^{22}\mathrm{cm}$, $\rho_{\mathrm{cr},0} = 1.878\times10^{-29}h^2\mathrm{g}\cdot\mathrm{cm}^{-3}$ and $h = 0.674$[a], we have $A = 1.123\times10^{-6}$.
And use `Python` to calculate the angle for different cosmological parameters and redshifts (see the code in the Appendix):

- Matter-dominated Universe ($\Omega_{\mathrm{m}} = 1, \Omega_\Lambda = 0$): $\theta(0.1) = 2.73'', \theta(1) = 0.79''$.
- $\Omega_{\mathrm{m}} = 0.3, \Omega_\Lambda = 0.7$: $\theta(0.1) = 2.60'', \theta(1) = 0.60''$.

[a] $H_0 = 67.4\pm0.5\mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1}$ from Planck 2018 [1807.06209].

Question 5

Apply the Einstein equations to the case of non-flat Universe, with the space-time interval given by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

where r, θ, ϕ is the standard 3D spherical coordinates, and k is a constant that describes the Universe's curvature. Calculate the Christoffel symbols, Ricci tensor and Ricci scalar. Then derive the Friedmann equation for non-flat Universe.

Solution:

The FLRW metric of non-flat Universe is:

$$g_{tt} = -1, \quad g_{rr} = \frac{a^2}{1-kr^2}, \quad g_{\theta\theta} = a^2 r^2, \quad g_{\phi\phi} = a^2 r^2 \sin^2\theta,$$

with their inverses:

$$g^{tt} = -1, \quad g^{rr} = \frac{1-kr^2}{a^2}, \quad g^{\theta\theta} = \frac{1}{a^2 r^2}, \quad g^{\phi\phi} = \frac{1}{a^2 r^2 \sin^2\theta}.$$

and other components are zero. There're the non-zero components of Christoffel symbols. Ricci tensor and Ricci scalar (see Appendix for details)

Christoffel symbols:

$$\Gamma^t{}_{rr} = \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^t{}_{\theta\theta} = a\dot{a}r^2, \quad \Gamma^t{}_{\phi\phi} = a\dot{a}r^2\sin^2\theta,$$
$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \frac{\dot{a}}{a}, \quad \Gamma^r{}_{rr} = \frac{kr}{1-kr^2}, \quad \Gamma^r{}_{\theta\theta} = -r(1-kr^2), \quad \Gamma^r{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta,$$
$$\Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \frac{\dot{a}}{a}, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta,$$
$$\Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \frac{\dot{a}}{a}, \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}.$$

Ricci tensor:

$$\begin{aligned}
R_{tt} &= -3\frac{\ddot{a}}{a}, \\
R_{rr} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}, \\
R_{\theta\theta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k), \\
R_{\phi\phi} &= r^2\sin^2\theta(a\ddot{a} + 2\dot{a}^2 + 2k).
\end{aligned}$$

Ricci scalar:

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).$$

Then we can derive the Friedmann equation for non-flat Universe. The Einstein equations are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}.$$

For $\mu = \nu = t$, we have:

$$R_{tt} - \frac{1}{2}g_{tt}R = -3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho,$$

where we have used $T_{tt} = \rho$. Then take $H = \dfrac{\dot{a}}{a}$, we have the Friedmann equation for non-flat Universe:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}.$$

we can also suppose the "energy density of curvature" $\rho_k = -\dfrac{3k}{8\pi G a^2}$, then the Friedmann equation can be written as:

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_k).$$

A Code for Question 4

```python
import numpy as np
from scipy import integrate


def calculate_angle(z: float,  Omega_m: float, Omega_Lambda: float, Omega_r: float = 0, Omega_k: float = 0, l: float = 5, verbose: bool = False) -> float:
    """
    Calculate the angular size of a l kpc galaxy at redshift z

    Parameters:
    z : Redshift
    l : Physical size of the galaxy in kpc (default 5 kpc)
    Omega : Density fraction parameter

    Return:
    Angular size
    """

    l_cm = l * 3.086e21 # cm

    # Physical constants
    h = 0.674 # Planck 2018
    G = 6.674e-8  # cm^3 g^-1 s^-2
    C = 3e10  # cm/s
    rho_crit0 = 1.878e-29 * h**2  # g cm^-3


    A = l_cm * np.sqrt(8 * np.pi * G * rho_crit0 / 3) / C
    if verbose:
        print(f"A = {A:.3e} ")

    def integrand(z_prime):
        factor = (Omega_r * (1 + z_prime)**4 +
                  Omega_m * (1 + z_prime)**3 +
                  Omega_k * (1 + z_prime)**2 +
                  Omega_Lambda)
        return 1 / np.sqrt(factor)
    integral_result, _ = integrate.quad(integrand, 0, z)

    theta_rad = A * (1 + z) / integral_result

    return theta_rad
```

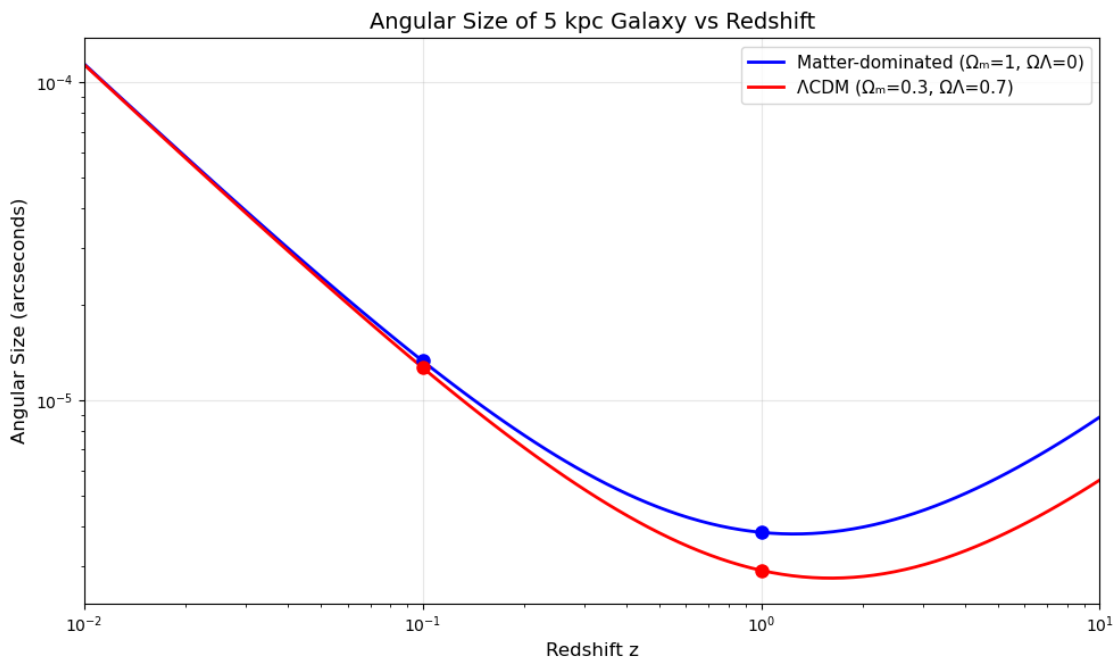

Figure 1: The angle subtended by a galaxy of physical size 5 kpc at different redshifts in two cosmological models: matter-dominated Universe ($\Omega_\mathrm{m} = 1, \Omega_\Lambda = 0$) and $\Omega_\mathrm{m} = 0.3, \Omega_\Lambda = 0.7$.

B Calculation of Question 5

Christoffel symbols:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{g^{\lambda\alpha}}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}).$$

For $\lambda = t$, we have:

$$\Gamma^t{}_{\mu\nu} = \frac{g^{t\alpha}}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) = \frac{g^{tt}}{2}(\partial_\mu g_{\nu t} + \partial_\nu g_{\mu t} - \partial_t g_{\mu\nu})$$

when $\mu = \nu = t$, $\Gamma^t{}_{tt} = 0$. When $\mu = \nu \neq t$, $\Gamma^t{}_{\mu\mu} = -\frac{g^{tt}}{2}\partial_t g_{\mu\mu}$, then $\Gamma^t{}_{rr} = \frac{a\dot{a}}{1-kr^2}, \Gamma^t{}_{\theta\theta} = a\dot{a}r^2, \Gamma^t{}_{\phi\phi} = a\dot{a}r^2\sin^2\theta$. Then consider $\mu = t, \nu \neq t$:

$$\Gamma^t{}_{t\nu} = \frac{g^{tt}}{2}(\partial_t g_{\nu t} + \partial_\nu g_{tt} - \partial_t g_{t\nu}) = 0.$$

For $\lambda = r$, we have:

$$\Gamma^r{}_{\mu\nu} = \frac{g^{r\alpha}}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) = \frac{g^{rr}}{2}(\partial_\mu g_{\nu r} + \partial_\nu g_{\mu r} - \partial_r g_{\mu\nu})$$

when $\mu = \nu = r$, $\Gamma^r{}_{rr} = \frac{g^{rr}}{2}\partial_r g_{rr} = \frac{kr}{1-kr^2}$. When $\mu = \nu \neq r$, $\Gamma^t{}_{\mu\mu} = -\frac{g^{rr}}{2}\partial_r g_{\mu\mu}$, then $\Gamma^r{}_{tt} = 0, \Gamma^r{}_{\theta\theta} = -r(1-kr^2), \Gamma^r{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta$. Then consider $\mu = r, \nu \neq r$:

$$\Gamma^r{}_{r\nu} = \frac{g^{rr}}{2}(\partial_r g_{\nu r} + \partial_\nu g_{rr} - \partial_r g_{r\nu}) = \frac{g^{rr}}{2}\partial_\nu g_{rr}.$$

then $\Gamma^r{}_{rt} = \dfrac{\dot{a}}{a}, \Gamma^r{}_{r\theta} = \Gamma^r{}_{r\phi} = 0$.

For $\lambda = \theta$, we have:

$$\Gamma^\theta{}_{\mu\nu} = \frac{g^{\theta\alpha}}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) = \frac{g^{\theta\theta}}{2}(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu})$$

when $\mu = \nu = \theta$, $\Gamma^\theta{}_{\theta\theta} = \dfrac{g^{\theta\theta}}{2}\partial_\theta g_{\theta\theta} = 0$. When $\mu = \nu \neq \theta$, $\Gamma^\theta{}_{\mu\mu} = -\dfrac{g^{\theta\theta}}{2}\partial_\theta g_{\mu\mu}$, then $\Gamma^\theta{}_{tt} = 0, \Gamma^\theta{}_{rr} = 0, \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$. Then consider $\mu = \theta, \nu \neq \theta$:

$$\Gamma^\theta{}_{\theta\nu} = \frac{g^{\theta\theta}}{2}(\partial_\theta g_{\nu\theta} + \partial_\nu g_{\theta\theta} - \partial_\theta g_{\theta\nu}) = \frac{g^{\theta\theta}}{2}\partial_\nu g_{\theta\theta},$$

then $\Gamma^\theta{}_{\theta t} = \dfrac{\dot{a}}{a}, \Gamma^\theta{}_{\theta r} = \dfrac{1}{r}, \Gamma^\theta{}_{\theta\phi} = 0$.

For $\lambda = \phi$, we have:

$$\Gamma^\phi{}_{\mu\nu} = \frac{g^{\phi\alpha}}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) = \frac{g^{\phi\phi}}{2}(\partial_\mu g_{\nu\phi} + \partial_\nu g_{\mu\phi} - \partial_\phi g_{\mu\nu})$$

when $\mu = \nu = \phi$, $\Gamma^\phi{}_{\phi\phi} = \dfrac{g^{\phi\phi}}{2}\partial_\phi g_{\phi\phi} = 0$. When $\mu = \nu \neq \phi$, $\Gamma^\phi{}_{\mu\mu} = -\dfrac{g^{\phi\phi}}{2}\partial_\phi g_{\mu\mu}$, then $\Gamma^\phi{}_{tt} = 0, \Gamma^\phi{}_{rr} = 0, \Gamma^\phi{}_{\theta\theta} = 0$. Then consider $\mu = \phi, \nu \neq \phi$:

$$\Gamma^\phi{}_{\phi\nu} = \frac{g^{\phi\phi}}{2}(\partial_\phi g_{\nu\phi} + \partial_\nu g_{\phi\phi} - \partial_\phi g_{\phi\nu}) = \frac{g^{\phi\phi}}{2}\partial_\nu g_{\phi\phi},$$

then $\Gamma^\phi{}_{\phi t} = \dfrac{\dot{a}}{a}, \Gamma^\phi{}_{\phi r} = \dfrac{1}{r}, \Gamma^\phi{}_{\phi\theta} = \dfrac{\cos\theta}{\sin\theta}$.

Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}.$$

For $\mu = \nu = t$, we have:

$$R_{tt} = \partial_\lambda \Gamma^\lambda{}_{tt} - \partial_t \Gamma^\lambda{}_{t\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{tt} - \Gamma^\lambda{}_{\sigma t}\Gamma^\sigma{}_{t\lambda}.$$

The first term is zero since $\Gamma^\lambda{}_{tt} = 0$. The second term is:

$$-\partial_t \Gamma^\lambda{}_{t\lambda} = -\partial_t(\Gamma^t{}_{tt} + \Gamma^r{}_{tr} + \Gamma^\theta{}_{t\theta} + \Gamma^\phi{}_{t\phi}) = -\partial_t\left(3\frac{\dot{a}}{a}\right) = -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2}.$$

The third term is zero since $\Gamma^\sigma{}_{tt} = 0$. The fourth term is:

$$-\Gamma^\lambda{}_{\sigma t}\Gamma^\sigma{}_{t\lambda} = -(\Gamma^t{}_{tt}\Gamma^t{}_{tt} + \Gamma^r{}_{rt}\Gamma^r{}_{tr} + \Gamma^\theta{}_{\theta t}\Gamma^\theta{}_{t\theta} + \Gamma^\phi{}_{\phi t}\Gamma^\phi{}_{t\phi}) = -3\frac{\dot{a}^2}{a^2}.$$

Then we have:

$$R_{tt} = -3\frac{\ddot{a}}{a}.$$

For $\mu = \nu = r$, we have:

$$R_{rr} = \partial_\lambda \Gamma^\lambda{}_{rr} - \partial_r \Gamma^\lambda{}_{r\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{rr} - \Gamma^\lambda{}_{\sigma r}\Gamma^\sigma{}_{r\lambda}.$$

The first term is:

$$\partial_\lambda \Gamma^\lambda{}_{rr} = \partial_t \Gamma^t{}_{rr} + \partial_r \Gamma^r{}_{rr} + \partial_\theta \Gamma^\theta{}_{rr} + \partial_\phi \Gamma^\phi{}_{rr} = \partial_t\left(\frac{a\dot{a}}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) = \frac{a\ddot{a}+\dot{a}^2}{1-kr^2} + \frac{k(1+kr^2)}{(1-kr^2)^2}.$$

The second term is:

$$-\partial_r \Gamma^\lambda{}_{r\lambda} = -\partial_r(\Gamma^t{}_{rt} + \Gamma^r{}_{rr} + \Gamma^\theta{}_{r\theta} + \Gamma^\phi{}_{r\phi}) = -\partial_r\left(\frac{\dot{a}}{a} + \frac{kr}{1-kr^2} + \frac{2}{r}\right) = -\frac{k(1+kr^2)}{(1-kr^2)^2} + \frac{2}{r^2}.$$

The third term is:

$$\Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{rr} = (\Gamma^t{}_{tt} + \Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\phi\phi})\Gamma^t{}_{rr} = 3\frac{\dot{a}}{a}\frac{a\dot{a}}{1-kr^2} = 3\frac{\dot{a}^2}{1-kr^2}.$$

The fourth term is:

$$-\Gamma^\lambda{}_{\sigma r}\Gamma^\sigma{}_{r\lambda} = -(\Gamma^t{}_{tr}\Gamma^t{}_{rt} + \Gamma^r{}_{rr}\Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta r}\Gamma^\theta{}_{r\theta} + \Gamma^\phi{}_{\phi r}\Gamma^\phi{}_{r\phi}) = -\left(\frac{\dot{a}^2}{a^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2}\right).$$

Then we have:

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}.$$

For $\mu = \nu = \theta$, we have:

$$R_{\theta\theta} = \partial_\lambda \Gamma^\lambda{}_{\theta\theta} - \partial_\theta \Gamma^\lambda{}_{\theta\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\theta\theta} - \Gamma^\lambda{}_{\sigma\theta}\Gamma^\sigma{}_{\theta\lambda}.$$

The first term is:

$$\partial_\lambda \Gamma^\lambda{}_{\theta\theta} = \partial_t \Gamma^t{}_{\theta\theta} + \partial_r \Gamma^r{}_{\theta\theta} + \partial_\theta \Gamma^\theta{}_{\theta\theta} + \partial_\phi \Gamma^\phi{}_{\theta\theta} = \partial_t(a\dot{a}r^2) + \partial_r(-r(1-kr^2)) = r^2(a\ddot{a}+\dot{a}^2) + (3kr^2-1).$$

The second term is:

$$-\partial_\theta \Gamma^\lambda{}_{\theta\lambda} = -\partial_\theta(\Gamma^t{}_{\theta t} + \Gamma^r{}_{\theta r} + \Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\theta\phi}) = -\partial_\theta\left(\frac{\dot{a}}{a} + \frac{1}{r} + \frac{\cos\theta}{\sin\theta}\right) = -\frac{1}{\sin^2\theta}.$$

The third term is:

$$\Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\theta\theta} = (\Gamma^t{}_{tt} + \Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\phi\phi})\Gamma^t{}_{\theta\theta} = 3\frac{\dot{a}}{a}a\dot{a}r^2 = 3\dot{a}^2r^2.$$

The fourth term is:

$$-\Gamma^\lambda{}_{\sigma\theta}\Gamma^\sigma{}_{\theta\lambda} = -(\Gamma^t{}_{t\theta}\Gamma^t{}_{\theta t} + \Gamma^r{}_{r\theta}\Gamma^r{}_{\theta r} + \Gamma^\theta{}_{\theta\theta}\Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\phi\theta}\Gamma^\phi{}_{\theta\phi}) = -\left(\frac{\dot{a}^2}{a^2} + \frac{1}{r^2} + \frac{\cos^2\theta}{\sin^2\theta}\right).$$

Then we have:

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k).$$

For $\mu = \nu = \phi$, we have:

$$R_{\phi\phi} = \partial_\lambda \Gamma^\lambda{}_{\phi\phi} - \partial_\phi \Gamma^\lambda{}_{\phi\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\phi\phi} - \Gamma^\lambda{}_{\sigma\phi}\Gamma^\sigma{}_{\phi\lambda}.$$

The first term is:

$$\begin{aligned}
\partial_\lambda \Gamma^\lambda{}_{\phi\phi} &= \partial_t \Gamma^t{}_{\phi\phi} + \partial_r \Gamma^r{}_{\phi\phi} + \partial_\theta \Gamma^\theta{}_{\phi\phi} + \partial_\phi \Gamma^\phi{}_{\phi\phi} \\
&= \partial_t (a\dot{a}r^2 \sin^2\theta) + \partial_r(-r(1-kr^2)\sin^2\theta) + \partial_\theta(-\sin\theta\cos\theta) \\
&= r^2 \sin^2\theta (a\ddot{a} + \dot{a}^2) + (3kr^2 - 1)\sin^2\theta.
\end{aligned}$$

The second term is:

$$\begin{aligned}
-\partial_\phi \Gamma^\lambda{}_{\phi\lambda} &= -\partial_\phi (\Gamma^t{}_{\phi t} + \Gamma^r{}_{\phi r} + \Gamma^\theta{}_{\phi\theta} + \Gamma^\phi{}_{\phi\phi}) \\
&= -\partial_\phi \left(\frac{\dot{a}}{a} + \frac{1}{r} + \frac{\cos\theta}{\sin\theta} \right) = 0.
\end{aligned}$$

The third term is:

$$\Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\phi\phi} = (\Gamma^t{}_{tt} + \Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\phi\phi})\Gamma^t{}_{\phi\phi} = 3\frac{\dot{a}}{a} a\dot{a}r^2 \sin^2\theta = 3\dot{a}^2 r^2 \sin^2\theta.$$

The fourth term is:

$$-\Gamma^\lambda{}_{\sigma\phi}\Gamma^\sigma{}_{\phi\lambda} = -(\Gamma^t{}_{t\phi}\Gamma^t{}_{\phi t} + \Gamma^r{}_{r\phi}\Gamma^r{}_{\phi r} + \Gamma^\theta{}_{\theta\phi}\Gamma^\theta{}_{\phi\theta} + \Gamma^\phi{}_{\phi\phi}\Gamma^\phi{}_{\phi\phi}) = -\left(\frac{\dot{a}^2}{a^2} + \frac{1}{r^2} + \frac{\cos^2\theta}{\sin^2\theta} \right).$$

Then we have:

$$R_{\phi\phi} = r^2 \sin^2\theta (a\ddot{a} + 2\dot{a}^2 + 2k).$$

Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}.$$

Substitute the components of metric and Ricci tensor, we have:

$$R = -3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} + \frac{1}{a^2 r^2} r^2 (a\ddot{a} + 2\dot{a}^2 + 2k) + \frac{1}{a^2 r^2 \sin^2\theta} r^2 \sin^2\theta (a\ddot{a} + 2\dot{a}^2 + 2k).$$

Then we have:

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right).$$