Resource Management Techniques(RMT) UR_BBIT March-April 2025

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Terminology

- The British/Europeans refer to "**Resource Management Techniques**" or "Quantitative Methods" or "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use). Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS".

Why QM

- You have the opportunity to solve real-world problems
- You can use your analytical skills and your creativity.
- You become a better strategist
- You become an essential link between technology and organizational management.
- You're extremely relevant today

History of QM

- The development of **Quantitative methods** has taken a long time in history. Some of these methods were developed before during World War II. The methods were later improved. Their use expended into non-war activities. The work of Fisher and others have formed a today's very useful basis.

THE METHODOLOGY OF QM

Definition: Quantitative Methods (QM) is an interdisciplinary branch of applied mathematics and formal science that uses methods like mathematical modeling, statistics, and algorithms to arrive at optimal or near optimal solutions to complex problems.

The objective of operation research is to optimize the solution (minimize cost or maximize profit)

THE METHODOLOGY OF QM

- When QM is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. **Formulate the Problem**

OR analyst first defines the organization's problem.

Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Cont’d

- Step 2**. Observe the System**

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

- Step 3. **Formulate a Mathematical Model of the Problem**

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Cont’d

- Step 4. **Verify the Model and Use the Model for Prediction**

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Cont'd

Step 5**. Select a Suitable Alternative**

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Cont'd

- Step 6**. Present the Results and Conclusions of the Study**

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

Cont'd

- presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Cont’d

- Step 7. **Implement and Evaluate Recommendation**

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

BASIC OR CONCEPTS

- "OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing." We can also define a mathematical model as consisting of:

1. Decision variables, which are the unknowns to be determined by the solution to the model.
2. Constraints to represent the physical limitations of the system
3. An objective function

Cont'd

- **An optimal solution to the model** is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function. An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Example

- Two Mines Example

Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below

EXAMPLE

- Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below

Mine	Cost per day	Production		
		High	Medium	Low
A	180	6	3	4
B	160	1	1	6

- Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Solution

- What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an equivalent mathematical description. In dealing with problems of this kind we often do best to consider them in the order:
- Variables
- Constraints
- Objective

This process is often called formulating the problem (or more strictly formulating a mathematical representation of the problem).

Cont'd

- Variables represent the "decisions that have to be made" or the "unknowns". We have two decision variables in this problem: x = number of days per week mine A is operated y = number of days per week mine B is operated.

Note here that $x \geq 0$ and $y \geq 0$.

Cont'd

Constraints

- It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract Ore

- High $6x + 1y \geq 12$
- Medium $3x + 1y \geq 8$
- Low $4x + 6y \geq 24$

Cont'd

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

- $x \leq 5$
- $y \leq 5$

<u>Inequality constraints</u>

- Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

Cont'd

- For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over- constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

Cont'd

- The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).
- Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Cont'd

Objective Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$

Hence we have the complete mathematical representation of the problem:

- minimize $180x + 160y$

subject to
$$6x + y \geq 12$$
$$3x + y \geq 8$$
$$4x + 6y \geq 24$$
$$x \leq 5$$
$$y \leq 5$$
$$x, y \geq 0$$

Cont'd

Some notes: The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, 4x, 6y are linear terms but xy or x2 is a non-linear term)

Cont’d

- Any formulation which satisfies these three conditions is called a linear program (LP). We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under Integer Programming (IP).

LINEAR PROGRAMMING

It can be recalled from the two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

all variables continuous (i.e. can take fractional values)

a single objective (minimize or maximize)

the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

PROPERTIES OF LINEAR PROGRAMMING MODEL

Any linear programming model (problem) must have the following properties:

- The relationship between variables and constraints must be linear.
- The model must have an objective function.
- The model must have structural constraints.
- The model must have non-negativity constraint.

Cont’d

LP's are important - this is because:

many practical problems can be formulated as LP's

there exists an algorithm (called the simplex algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Cont’d

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems, Inventory model, Financial models, Work scheduling. Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

Cont'd

There are four basic assumptions in LP:

- Proportionality: The contribution to the objective function from each decision variable is proportional to the value of the decision variable
- Additivity: The contribution to the objective function for any decision variable is independent of the values of the other decision variables

Cont’d

- Divisibility : Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP
- Certainty : Each parameter is known with certainty

FORMULATING LP

1. Giapetto's wooden soldiers and trains.

Each soldier sells for \$27, uses \$10 of raw materials and takes \$14 of labor & overhead costs. Each train sells for \$21, uses \$9 of raw materials, and takes \$10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Cont'd

- **Answer**

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define: $x1$ = the number of soldiers produced per week $x2$ = the number of trains produced per week Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs).

Cont'd

- Giapetto can concentrate on maximizing the total weekly profit (z). Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto's objective function is: Maximize $z = 3x1 + 2x2$
- Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large.

Cont'd

- Here there are three constraints: Finishing time per week Carpentry time per week Weekly demand for soldiers Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto can not manufacture negative number of soldiers or trains!)

Cont'd

- All these characteristics explored above give the following Linear Programming (LP) model
- max $z = 3x1 + 2x2$ (The Objective function)
- s.t. $2x1 + x2 \leq 100$ (Finishing constraint)
- $x1 + x2 \leq 80$ (Carpentry constraint)
- $x1 \leq 40$ (Constraint on demand for soldiers)
- $x1, x2 \geq 0$ (Sign restrictions)
- A value of (x1, x2) is in the feasible region if it satisfies all the constraints and sign restrictions. Graphically and computationally we see the solution is $(x1, x2) = (20, 60)$ at which $z = 180$. (Optimal solution)

Report

- The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

Cont'd

- 2. Advertisement Example

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

XYZ Ltd makes furniture for computer users in categories of executive model desks for offices, general purpose desks for computer labs and single user desks for students. An executive model requires 2 hours in the cabinet section, 1 hour in the finishing department and 1 hour in the crating department. A general purpose desk requires 1 hour in the cabinet section, 2 hours in the finishing department and 1 hour in the crating department, while the a single user desk requires 1 hour in the cabinet section, 1 hour in the finishing department and ½ an hour in the crating department. On a daily basis the cabinet section has 16 hours available, the finishing department also has 16 hours available and the crating department has 10 hours. If the company realises a profit of Frw 150,000 on each executive desk, Frw 125,000 on general purpose and Frw 50,000 on each single user;

REQUIRED:

(i) Express the above information as a linear programming model. **(4 Marks)**

(ii) Using the simplex tableau method, find the number in each category of furniture that should be made and sold to maximise profits. **(7 Marks)**

Rukundo Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table below. At most 5 tables can be sold per week.

Resource	**Desk**	**Table**	**Chair**	**Max Arrival**
Lumber	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	0.5	8
Maximum demand	unlimited	5	unlimited	
Price	60	30	20	

Required

Maximize weekly revenue and recommend the appropriate results. **(10 marks)**

SOLVING LINEAR PROGRAMMING

- LP Solutions:

Four Cases When an LP is solved, one of the following four cases will occur:

- 1. The LP has a unique optimal solution.
- 2. The LP has alternative (multiple) optimal solutions. It has more than one (actually an infinite number of) optimal solutions

Cont'd

- 3. The LP is infeasible. It has no feasible solutions (The feasible region contains no points).
- 4. The LP is unbounded. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

Cont'd

- ***The Graphical Solution***

- Any LP with only two variables can be solved graphically

Example 1. Giapetto

- Since the Giapetto LP has two variables, it may be solved graphically.

Answer

- The feasible region is the set of all points satisfying the constraints.

max $z = 3x1 + 2x2$

s.t. $2x1 + x2 \leq 100$ (Finishing constraint)

$x1 + x2 \leq 80$ (Carpentry constraint)

$x1 \leq 40$ (Demand constraint)

$x1, x2 \geq 0$ (Sign restrictions)

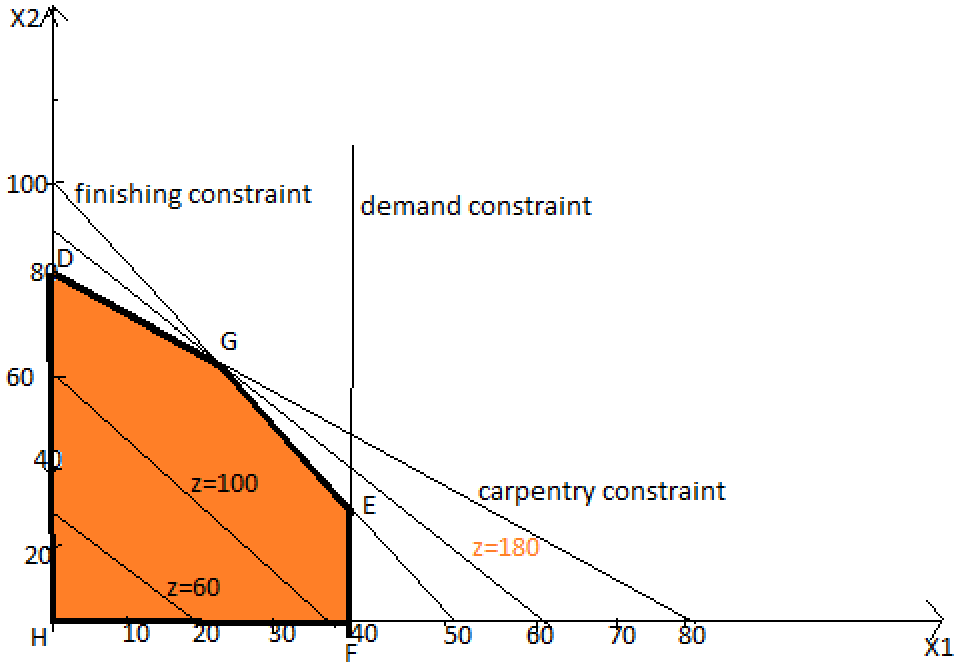
Cont'd

- The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point on or in the interior of this polygon (the shade area) is in the feasible region. Having identified the feasible region for the LP, a search can begin for the optimal solution which will be the point in the feasible region with the largest z-value (maximization problem).

Cont'd

- To find the optimal solution, a line on which the points have the same z-value is graphed. In a max problem, such a line is called an isoprofit line while in a min problem, this is called the isocost line. (The figure shows the isoprofit lines for $z = 60$, $z = 100$, and $z = 180$).

Cont'd



X2
100
80
60
40
20
D
G
E
F
H
finishing constraint
demand constraint
carpentry constraint
z=100
z=60
z=180
10
20
30
40
50
60
70
80
X1

Cont'd

- In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region. The optimal solution of this LP is point G where $(x1, x2) = (20, 60)$ giving $z = 180$.
- A constraint is binding (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint

Cont'd

- A constraint is nonbinding (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint. In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x1 < 40$ ($x1 = 20$).

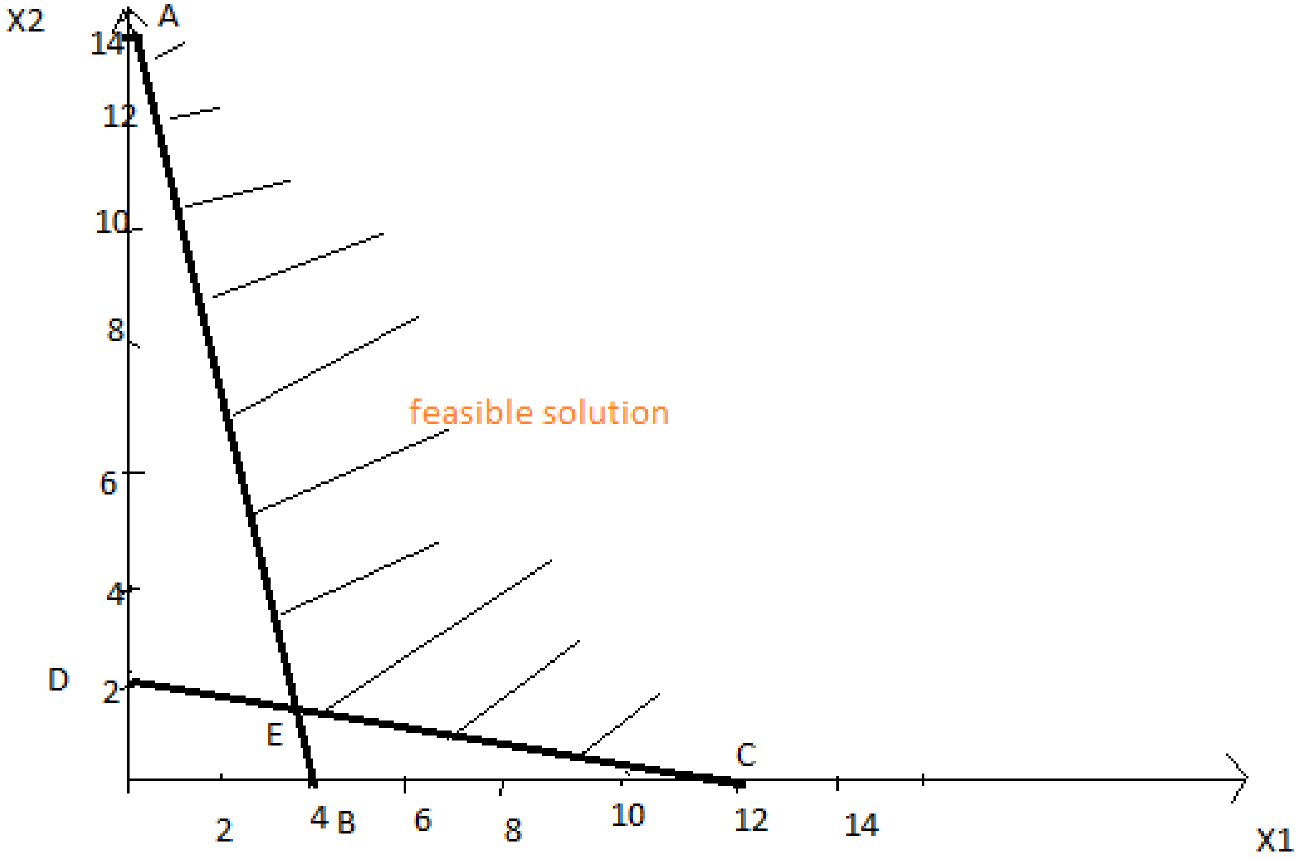
Cont'd

- Example 2. Advertisement
- Since the Advertisement LP has two variables, it may be solved graphically.
- Answer The feasible region is the set of all points satisfying the constraints.
- $\min z = 50x1 + 100x2$
- s.t. $7x1 + 2x2 \geq 28$ (high income women)

  $2x1 + 12x2 \geq 24$ (high income men)

  $x1, x2 \geq 0$

Cont'd



Cont'd

- Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value. An isocost line with the smallest z value passes through point E and is the optimal solution at $x1 = 3.6$ and $x2 = 1.4$ giving $z = 320$. Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Cont'd

- Example 3. Two Mines
- min $180x + 160y$
- st $6x + y \geq 12$

  $3x + y \geq 8$

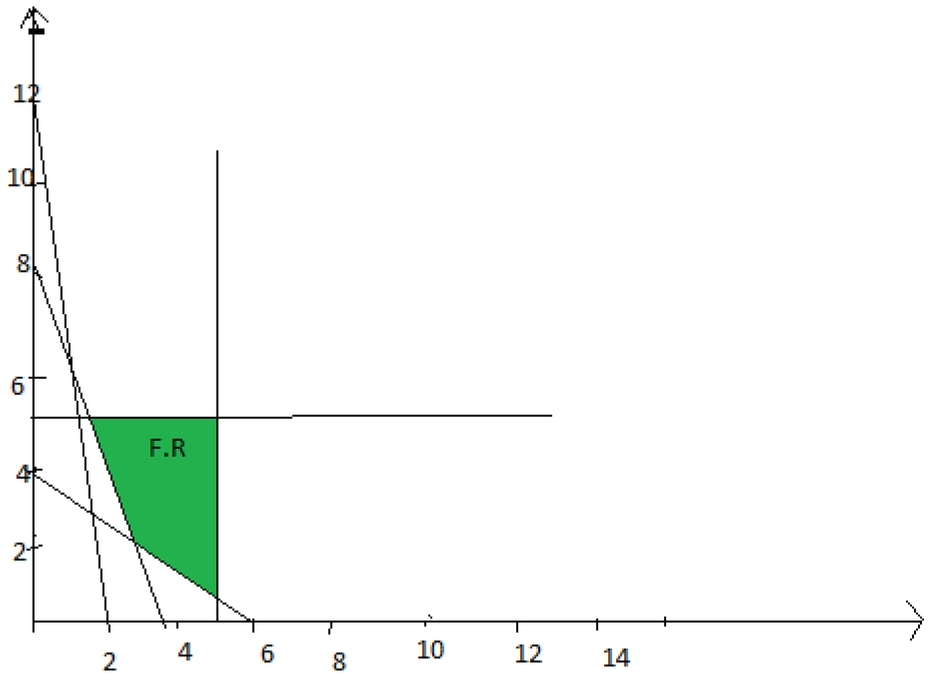
  $4x + 6y \geq 24$

  $x \leq 5$

  $y \leq 5$

  $x, y \geq 0$

Cont'd



Cont'd

- solution is 765.71.

i.e 1.71 days mine X and 2.86 days mine Y are operated.

Exercises

Rwamagana tailors group deals in making school uniforms, dresses and shorts. To make a dress requires 45 minutes cutting and 30 minutes of stitching. For a short requires 20 minutes of cutting and 40 minutes of stitching. For every 3 dresses produced at least 2 shorts are produced due to machine constraints. The profit on a dress is Fwr 930 and a short is Fwr 1150.The tailors work for 600 min per day.

Determine the:

(i) number of dresses and shorts to be produced to maximise profit using the graphical method. **(9 Marks)**

(ii) the maximum profit. **(2 Marks)**

The total revenue in "000" of Rwanda Francs earned by selling q hundred units of an item in Kigali in one day is described by the revenue function.

$R(q) = q^3(q-30)^2$.

Compute the marginal revenue function and hence determine the maximum value of R(q). **(5 Marks)**

A florist owns 100 acres of land and plans to plant three types of flowers. The seed for flower A, B and C cost Frw 40, 20 and 30 per acre respectively. A maximum of Frw 3,200 can be spent on flower seeds. Flower A, B and C require 1, 2,1 working days per acre respectively and there 160 work days. If the florist can make a profit of Frw 100 per acre of flower A, Frw 300 per acre from B and Frw 200 per acre from C. (All costs are in thousands Rwanda Francs.)

REQUIRED:

(i) Formulate a linear programming problem.

(3 Marks)

(ii) Using simplex tableau method, find the optimum acres that should be planted .Hence find the maximum profit.

(12 Marks)

The following are total revenue and total cost functions, $TR = 500q - 11q^2$and TC = $3q^3 - 2q^2 + 68q + 175$ expressed in terms of Frw '1000' from different departments of a firm dealing in oil products located in Kayonza district.

REQUIRED:

(i) Formulate a profit function for the firm.

(2 Marks)

(ii) Determine the value of q that yields maximum profit.

(5 Marks)

(iii) Determine the maximum profit attained.

(2 Marks)

Beds World Ltd (BWL) manufactures two types of beds: deluxe adjustable and deluxe ordinary. Each bed has to be processed by three machines and the time taken on each machine is indicated in the table below.

	Maximum time Available (hours)	Time (hours)	
		Deluxe adjustable	Deluxe ordinary
Machine 1	36	6	2
Machine 2	30	3	5
Machine 3	20	1	4

BWL realises a profit of Frw 250,000 on a deluxe adjustable bed and Frw 200,000 on a deluxe ordinary bed.

REQUIRED:

(i) Express the above information as linear programming model. **(2 Marks)**

(ii) Using the simplex tableau method, find the number of each type of bed BWL should produce in order to maximise profit. **(8 Marks)**

The Simplex Algorithm

- Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for any LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Cont’d

- Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

Cont'd

- Steps 1. Convert the LP to standard form
- 2. Obtain a basic feasible solution (bfs) from the standard form
- 3. Determine whether the current bfs is optimal. If it is optimal, stop.
- 4. If the current bfs is not optimal, determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable to find a new bfs with a better objective function value
- 5. Go back to Step 3.

Cont'd

Related concepts:

- Standard form: all constraints are equations and all variables are non negative
- bfs: any basic solution where all variables are nonnegative
- Non-basic variable: a chosen set of variables where variables equal to 0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

Example 1. Dakota Furniture

- Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	**Desk**	**Table**	**Chair**	**Max Arival**
Lumber	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	0.5	8
Maximum demand	unlimited	5	unlimited	
Price	60	30	20	

The Simplex Algorithm

- given

Solution with Simplex Algorithm

- First introduce slack variables and convert the LP to the standard form and write a canonical form

R0 $z - 60x1 - 30x2 - 20x3 = 0$

R1 $8x1 + 6x2 + x3 + s1 = 48$

R2 $4x1 + 2x2 + 1.5x3 + s2 = 20$

R3 $2x1 + 1.5x2 + .5x3 + s3 = 8$

R4 $x2 + s4 = 5$

$x1, x2, x3, s1, s2, s3, s4 \geq 0$

Solution with Simplex Algorithm

- Obtain a starting bfs. As $(x1, x2, x3) = 0$ is feasible for the original problem, the below given point where three of the variables equal 0 (the non-basic variables) and the four other variables (the basic variables) are determined by the four equalities is an obvious bfs: $x1 = x2 = x3 = 0$, $s1 = 48$, $s2 = 20$, $s3 = 8$, $s4 = 5$. . Determine whether the current bfs is optimal.

Con'd

- Determine whether there is any way that z can be increased by increasing some nonbasic variable. If each nonbasic variable has a nonnegative coefficient in the objective function row (row 0), current bfs is optimal. However, here all nonbasic variables have negative coefficients: It is not optimal.

Solution with Simplex Algorithm

- Find a new bfs

z increases most rapidly when x1 is made non-zero; i.e. x1 is the entering variable.

Examining R1, x1 can be increased only to 6. More than 6 makes s1 < 0. Similarly R2, R3, and R4, give limits of 5, 4, and no limit for x1 (ratio test).

Solution with Simplex Algorithm

- The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R3, x1 can only increase to x1 = 4 when s3 becomes 0. We say s3 is the leaving variable and R3 is the pivot equation.

Solution with Simplex Algorithm

- Determine whether the current bfs is optimal. Determine whether there is any way that z can be increased by increasing some nonbasic variable. If each nonbasic variable has a nonnegative coefficient in the objective function row (row 0), current bfs is optimal. However, here all nonbasic variables have negative coefficients: It is not optimal.

Solution with Simplex Algorithm

- Find a new bfs
- z increases most rapidly when x1 is made non-zero; i.e. x1 is the entering variable.
- Examining R1, x1 can be increased only to 6. More than 6 makes s1 < 0. Similarly R2, R3, and R4, give limits of 5, 4, and no limit for x1 (ratio test). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.

Solution with Simplex Algorithm

- Thus by R3, x1 can only increase to x1 = 4 when s3 becomes 0. We say s3 is the leaving variable and R3 is the pivot equation.
- Now we must rewrite the system so the values of the basic variables can be read off.

Solution with Simplex Algorithm

- The new pivot equation (R3/2) is
- R3': x1+.75x2+.25x3+0.5s3= 4
- Then use R3' to eliminate x1 in all the other rows.

  R0'=R0+60R3', R1'=R1-8R3', R2'=R2-4R3',

  R4'=R4 R0' z + 15x2 - 5x3 + 30s3 = 240 z = 240

R1' - x3 + s1 - 4s3 = 16 s1 = 16

R2' - x2 + .5x3 + s2 - 2s3 = 4 s2 = 4

R3' x1 + .75x2 + .25x3 + .5s3 = 4 x1 = 4

R4' x2 + s4 = 5 s4 = 5

Solution with Simplex Algorithm

- The new bfs is x2 = x3 = s3 = 0, x1 = 4, s1 = 16, s2 = 4, s4 = 5 making z = 240.

Check optimality of current bfs. Repeat steps until an optimal solution is reached

- We increase z fastest by making x3 non-zero (i.e. x3 enters).
- x3 can be increased to at most x3 = 8, when s2 = 0 (i.e. s2 leaves.)

Solution with Simplex Algorithm

- Rearranging the pivot equation gives R2” - 2x2 + x3 + 2s2 - 4s3 = 8 (R2’× 2). Row operations with R2” eliminate x3 to give the new system R0”= R0’ + 5R2”, R1” = R1’ + R2”, R3” = R3’ - .5R2”, R4” = R4’ The bfs is now x2 = s2 = s3 = 0, x1 = 2, x3 = 8, s1 = 24, s4 = 5 making z = 280.

Solution with Simplex Algorithm

Each nonbasic variable has a nonnegative coefficient in row 0 (5x2, 10s2, 10s3). THE CURRENT SOLUTION IS OPTIMAL

- Report: Dakota furniture's optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

Solution with Simplex Algorithm

- This was once written as a tableau. (Use tableau format for each operation in all HW and exams!!!)
- max z = 60x1 + 30x2 + 20x3

$$\begin{array}{lrrrl}
\text{s.t.} & 8x1 + & 6x2 + & x3 & \le 48 \\
& 4x1 + & 2x2 + & 1.5x3 & \le 20 \\
& 2x1 + & 1.5x2 + & .5x3 & \le 8 \\
& & x2 & & \le 5 \\
& & & x1, x2, x3 & > 0
\end{array}$$

Solution with Simplex Algorithm

z	x1	x2	x3	s1	s2	s3	s4	RHS	bv	ratio
1	-60	-30	-20	0	0	0	0	0	Z=0	
0	8	6	1	1	0	0	0	48	S1=48	
0	4	2	1.5	0	1	0	0	20	S2=20	
0	2	1.5	0.5	0	0	1	0	8	S3=8	
0	0	1	0	0	0	0	1	5	S4=5	

Solution with Simplex Algorithm

z	x1	x2	x3	s1	s2	s3	s4	RHS	BV	ratio
1	0	15	-5	0	0	30	0	20	240	
0	0	0	-1	1	0	-4	0	16	S1=16	
0	0	-1	0.5	0	1	-2	0	4	S2=4	8
0	1	0.75	0.25	0	0	0.5	0	4	X2=4	16
0	0	1	0	0	0	0	1	5	S4=5	

Solution with Simplex Algorithm

Z1	x1	x2	x3	s1	s2	s3	s4	RHS	BV
1	0	5	0	0	10	10	0	280	Z=280
0	0	-2	0	1	2	-8	0	24	S1=24
0	0	-2	1	0	2	-4	0	8	X3=8
0	1	1.25	0	0	-0.5	1.5	0	4	X1=2
0	0	1	0	0	0	0	1	5	S4=5

Solution with Simplex Algorithm

- Maxz=280
- X1=2
- X2=0
- X3=8

DUALITY

- **Definition:** The **Duality in Linear Programming** states that every linear programming problem has another linear programming problem related to it and thus can be derived from it. The original linear programming problem is called **"Primal,"** while the derived linear problem is called **"Dual."**

DUALITY

Finding the Dual of an LP

- The dual of a normal max problem is a normal min problem
- Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints. Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints. Similarly, the dual of a normal min problem is a normal max problem..

DUALITY

Finding the Dual of a Normal Max Problem

- PRIMAL max $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$$
\begin{aligned}
\text{s.t.}\quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& x_j \ge 0 \ (j = 1, 2, \ldots, n)
\end{aligned}
$$

DUALITY

DUAL $\min w = b1y1 + b2y2 + \ldots + bmym$

$$\text{s.t. } a11y1 + a21y2 + \ldots + am1ym \geq c1$$

$$a12y1 + a22y2 + \ldots + am2ym \geq c2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a1ny1 + a2ny2 + \ldots + amnym \geq cn$$

$$yi \geq 0 \ (i = 1, 2, \ldots, m)$$

DUALITY

- Finding the Dual of a Normal Min Problem

PRIMAL min $w = b_1y_1 + b_2y_2 + \ldots + b_my_m$

$$
\begin{aligned}
\text{s.t.}\quad & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1 \\
& a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2 \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \\
& a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n \\
& y_i \geq 0 \ (i = 1, 2, \ldots, m)
\end{aligned}
$$

DUALITY

- DUAL max $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$$
\begin{aligned}
\text{s.t.}\quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& x_j \ge 0 \ (j = 1, 2, \ldots, n)
\end{aligned}
$$

DUALITY

Finding the Dual of a Nonnormal

- Max Problem
- If the ith primal constraint is a $\geq$ constraint, the corresponding dual variable yi must satisfy yi ≤ 0
- If the ith primal constraint is an equality constraint, the dual variable yi is now unrestricted in sign (urs).
- If the ith primal variable is urs, the ith dual constraint will be an equality constraint

DUALITY

Finding the Dual of a Nonnormal Min Problem

- If the ith primal constraint is a ≤ constraint, the corresponding dual variable xi must satisfy xi ≤ 0
- If the ith primal constraint is an equality constraint, the dual variable xi is now urs.
- If the ith primal variable is urs, the ith dual constraint will be an equality constraint

DUALITY

- **In general,** the dual is found as follow:

Primal	**Dual**
(*a*) Maximize	Minimize
(*b*) Objective Function	Right hand side.
(*c*) Right hand side	Objective function.
(*d*) *i*th row of input-output coefficients.	*i*th column of input coefficients

DUALITY

- (*e*) *j*th column of input-output coefficients — *j*the row of input-output coefficients
- (*f*) *i*th relation of inequality (< =) — *i*th variable non-negative.

DUALITY

- **Duality theorem:** If an optimal solution exists to either the optimal or symmetric dual program, then the other also has an optimal solution and the two objective functions have the same optimal value. It may be computationally advantageous to solve a program's dual rather than the program itself.

DUALITY

- The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

EXAMPLE

- Min $w=0.12x1+0.15x2$

  s.t $60x1+60x2\geq 300$

  $12x1+6x2\geq 36$

  $10x1+30x2\geq 90$

  $x1, x2\geq 0$

$$\text{Dual: } \max z = 300y_1 + 36y_2 + 90y_3$$

$$\text{s.t } 60y_1 + 12y_2 + 10y_3 \le 0.12$$

$$60y_1 + 6y_2 + 30y_3 \le 0.15$$

$$y_1, y_2, y_3 \ge 0$$

Solving a min problem

- Max $z-300y1-36y2-90y3=0$

  s.t $60y1+12y2+10y3+s1=0.12$

  $60y1+6y2+30y3+s2=0.15$

  $y1,y2,y3 \geq 0$

Simplex tableau

z	y1	y2	y3	s1	s2	bi	bv	ratio
1	-300	-36	-90	0	0	0		
0	60	12	10	1	0	3/25	S1=3/25	1/500
0	60	6	30	0	1	3/20	S2=3/20	1/400

Solving a min problem

- Min(-300,-36,-90)=-300 we enter on column 2
- Min(1/500,1/400)=1/500 we enter on raw 2

z	y1	y2	y3	s1	s2	bi	bv	ratio
1	0	24	-40	5	0	3/5		
0	1	1/5	1/6	1/60	0	1/50 0	Y1=1 /500	
0	0	-6	20	-1	1	3/10 0	S2=3 /100	

Solving a min problem

- Min(24,-40)=-40 we enter on column 4
- Min(1/500,3/20)=1/500 we enter on raw 2

z	y1	y2	y3	s1	s2	bi	bv
1	0	12	0	3	2	33/50	
0	1	1/4	0	1/120	1/40	7/400 0	Y1=7 /4000
0	0	-3/10	1	1/20	-1/20	3/200 0	3/200 0

Solving a min problem

- Z=w=33/50
- X1=3, x2=2

TRANSPORTATION PROBLEMS

FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

- A set of m supply points from which a good/service is shipped. Supply point i can supply at most si units.
- A set of n demand points to which the good/service is shipped. Demand point j must receive at least dj units.
- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of cij.

TRANSPORTATION PROBLEMS

- **Some Definitions:**
- The following terms are to be defined with reference to the transportation problems:
- **Feasible Solution (F.S.)**
- A set of non-negative allocations x_{ij} which satisfies the row and column restrictions is known as feasible solution.
- **Basic Feasible Solution (B.F.S.)**
- A feasible solution to a m-origin and n-destination problem is said to be basic feasible solution if the number of positive allocations are $(m+n-1)$.

TRANSPORTATION PROBLEMS

- If the number of allocations in a basic feasible solutions are less than (m+n–1), it is called degenerate basic feasible solution (DBFS) (otherwise non-degenerate).
- **Optimal Solution**
- A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation

FORMULATION OF T.P.

- A transportation problem basically deals with the problem, which aims to find the best way to fulfill the demand of n demand points using the capacities of m supply points. While trying to find the best way, generally a variable cost of shipping the product from one supply point to a demand point or a similar constraint should be taken into consideration.

FORMULATION OF T.P.

- Example 1: REG Ltd has three electric power plants that supply the electric needs of four cities. The associated supply of each plant and demand of each city is given in the table below. The cost of sending 1 million kwh of electricity from a plant to a city depends on the distance the electricity must travel.

	city1	city2	city3	city4	Total supply
plant1	8	6	10	9	35
plant2	9	12	13	7	50
plant3	14	9	16	5	40
Total demand	45	20	30	30	

FORMULATION OF T.P.

Decision Variable:

- Since we have to determine how much electricity is sent from each plant to each city;
- X_{ij}= Amount of electricity produced at plant i and sent to city j
- X_{14} = Amount of electricity produced at plant 1 and sent to city 4

Assignment 1

Amahoro Industries produces two products (A and B) that are sold as raw materials to companies that manufacture sweets. Based on analysis of current inventory levels and potential demand for the coming month, the production manager of Amahoro has specified that the total production for products A and B combined must at least be 350 kgs. Also, a major customer's order for 125 kgs of product A must be satisfied. Product A requires 120 minutes of processing time per kg and product B requires 60 minutes of processing time per kg for the coming month, a total of 600 hours are available for producing the two products. Production costs are Frw 2 per kg of product A and Frw 3 per kg of product B.

a) Determine the production quantities that will satisfy the requirements at minimum costs. **(5 marks)**

b) What is the minimum production cost? **(2 marks)**

c) Identify the products produced in excess of the above conditions. **(3 marks)**

Assignment 2

- A factory manufactures two products A and B on three machines X, Y, and Z. Product A requires 10 hours of machine X and 5 hours of machine Y a one hour of machine Z. The requirement of product B is 6 hours, 10 hours and 2 hours of machine X, Y and Z respectively. The profit contribution of products A and B are \$ 23 per unit and \$ 32 per unit respectively. In the coming planning period the available capacity of machines X, Y and Z are 2500 hours, 2000 hours, and 500 hours respectively.

i) Formulate a mathematical model to explain the problem. **(5 Marks)**

ii) Find the optimal product mix for maximizing the profit. **(5 Marks)**

FORMULATION OF T.P.

Objective function

- Since we want to minimize the total cost of shipping from plants to cities;
- Minimize $Z = 8X_{11}+6X_{12}+10X_{13}+9X_{14}$
- $+9X_{21}+12X_{22}+13X_{23}+7X_{24}$
- $+14X_{31}+9X_{32}+16X_{33}+5X_{34}$

FORMULATION OF T.P.

Supply constraints

- Since each supply point has a limited production capacity;
- $X_{11}+X_{12}+X_{13}+X_{14} <= 35$
- $X_{21}+X_{22}+X_{23}+X_{24} <= 50$
- $X_{31}+X_{32}+X_{33}+X_{34} <= 40$
- **Demand constraints**
- Since each supply point has a limited production capacity;
- $X_{11}+X_{21}+X_{31} >= 45$
- $X_{12}+X_{22}+X_{32} >= 20$
- $X_{13}+X_{23}+X_{33} >= 30$
- $X_{14}+X_{24}+X_{34} >= 30$

FORMULATION OF T.P.

- **Sign constraints**
- Since a negative amount of electricity cannot be shipped all Xij's must be non-negative;
- $Xij >= 0$ (i= 1,2,3; j= 1,2,3,4).

FORMULATION OF T.P.

- LP Formulation of Powerco's Problem

$MinZ=8X_{11}+6X_{12}+10X_{13}+9X_{14}+9X_{21}+12X_{22}+13X_{23}+7X_{24}+$
$14X_{31}+9X_{32}+16X_{33}+5X_{34}$

S.T.: $X_{11}+X_{12}+X_{13}+X_{14} <= 35$ (Supply Constraints)

$X_{21}+X_{22}+X_{23}+X_{24} <= 50$

$X_{31}+X_{32}+X_{33}+X_{34} <= 40$

$X_{11}+X_{21}+X_{31} >= 45$ (Demand Constraints)

$X_{12}+X_{22}+X_{32} >= 20$

$X_{13}+X_{23}+X_{33} >= 30$

$X_{14}+X_{24}+X_{34} >= 30$

Xij >= 0 (i= 1,2,3; j= 1,2,3,4)

SOLVING A T.P.

- A transportation problem will have feasible solutions if and only if $S_1 + S_2 + \ldots + S_m = d_1 + d_2 + \ldots + d_n$

- How to deal with cases when the equation doesn't hold?

SOLVING A T.P.

- With the simplex method the initial solution is predetermined by the constraint structure. The initial set of basic variables will always consist of the slack and artificial variable in the problem.
- With transportation models the stepping stone algorithm (or the MODI method) will accept any feasible solution as a starting point.

SOLVING A T.P.

- Consequently, various approaches to finding a good starting solution have been proposed. These include the northwest corner method, the least cost method, and Vogel's approximation method. The transportation simplex starts with an initial basic feasible solution (as does regular simplex).

SOLVING A T.P.

- **How to obtain the initial basic feasible solution these three different methods:**
- (i) North-West corner rule or Northwest Corner Method
- (ii) Lowest cost entry method or Minimum Cost Method
- (iii) Vogel's approximation method.

Northwest Corner Method

- x The Northwest Corner Method does not consider the shipping cost in its iterations.
- **Step 1** Begin in the upper left corner of the transportation tableau (**the northwest corner).**
- **Step 2** Set **x11** as large as possible. **x11 = min** {**s1, d1**}

Northwest Corner Method

- **Step 3** If $\mathbf{x_{11} = s_1}$**,** cross out the first row of the tableau. This indicates that no more basic variables will come from this row. Also change $\mathbf{d_1}$ **to** $\mathbf{d_1 - s_1}$**. If** $\mathbf{x_{11} = d_1}$, cross out the first column of the tableau. This indicates that no more basic variables will come from this column. Also change $\mathbf{s_1}$ **to** $\mathbf{s_1 - d_1}$**. If** $\mathbf{x_{11} = s_1 = d_1}$ cross out either row 1 or column 1, but not both. If you cross out row 1 change d_1 to 0. If you cross out column 1, change s_1 to 0.

Northwest Corner Method

- **Step 4** Continue to apply this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.
- **Step 5** When you come to the point that only one cell can be assigned a value, assign this cell a value equal to its row or column demand, and cross out both the cell's row and column. This is a bfs.

EXAMPLE

	city1		city2		city3		city4		si	
plant1	8	35	6		10		9		35	15
plant2	9	10	12	20	13	20	7		50 40	20
plant3	14		9		16	10	5	30	40	10
di	45	10 0	20		30	10	30	0		

EXAMPLE

- $X11=35$; $x21=10$; $x22=20$; $x23=20$; $x33=10$; $x43=30$

- Min $z=8(35)+12(20+9(10)+13(20)+16(10)+5(30)=1180$

Minimum Cost Method

- The Minimum Cost Method uses shipping costs in an effort to produce a bfs which does not have a high total cost. This may help to reduce the number of pivots required from the bfs to the optimal solution.
- **Step 1** Find the variable with the smallest shipping cost. Call this x_{ij}. (**Break ties arbitrarily**.)
- **Step 2** Set x_{ij} as large as possible (its largest possible value). $x_{ij} = \min \{s_i, d_j\}$

Minimum Cost Method

- **Step 3** If $x_{ij} = s_i$, cross out the ith row of the tableau. This indicates that no more basic variables will come from this row. Also change d_j to $d_j - s_i$. If $x_{ij} = d_j$, cross out the jth column of the tableau. This indicates that no more basic variables will come from this column. Also change s_i to $s_i - d_j$. If $x_{ij} = s_i = d_j$, cross out either row i or column j, but not both. If you cross out row i, change d_j to 0. If you cross out column j, change s_i to 0.

Minimum Cost Method

- **Step 4** Continue to apply this procedure to the cell with the lowest shipping cost that does not lie in a crossed out row or column.
- **Step 5** When you come to the point that only one cell can be assigned a value, assign this cell a value equal to its row or column demand, and cross out both the cell's row and column. This is a bfs.

EXAMPLE : **Minimum Cost Method**

										si
	8	15	6	20	10		9		35	15
	9	30	12		13	20	7		50	20
	14		9		16	10	5	30	40	10
di	45	30	20		30	10	30	0		

EXAMPLE: Minimum Cost Method

- X11=15; x12=20; x21=30; x23=20; x33=10; x430

- Min z= 8(15)+6(20)+9(30)+13(20)+16(10)+5(30)=1080

Vogel's Method

- Vogel's Method, like the Minimum Cost Method considers the shipping cost in an effort to produce a bfs which does not have a high total cost. Associated with Vogel's Method is a penalty cost. The **penalty cost** is calculated for each row and column. It is equal to the difference between the two smallest costs in the row or column.
- **Step 1** Calculate the penalty cost for each row and column.
- **Step 2** Find the row or column which has the highest penalty cost. (Break ties arbitrarily.)

Vogel's Method

- **Step 3** Choose as the basic variable the variable in this row or column that has the smallest shipping cost. Call this xij. (Break ties arbitrarily.)
- **Step 4** If xij =si, cross out the ith row of the tableau. This indicates that no more basic variables will come from this row. Also change dj to dj - si If xij=dj, cross out the jth column of the tableau. This indicates that no more basic variables will come from this column. Also change si to si – dj. If xij = si = dj, cross out either row i or column j, but not both. If you cross out row i, change dj to 0. If you cross out column j, change si to 0.

Vogel's Method

- **Step 5** Repeat this procedure by re-computing new penalties (using only cells that do not lie in a crossed-out row or column).
- **Step 6** When you come to the point that only one cell can be assigned a value, assign this cell a value equal to its row or column demand, and cross out both the cell's row and column. This is a bfs.

Quiz/10 Marks

- A company has three factories located in three different districts. The factory can supply their products in four supermarkets. The production capacity of the factories x, y, and z are 1000, 700, and 900 kilograms per week, respectively. The total demand of the supermarkets A, B and C are 900, 800, 500 and 400, respectively. The profit per kilogram is given in the table below:

Factory		A	B	C	D
	X	6	6	6	4
	Y	4	2	4	5
	Z	5	6	7	8

i) formulate a mathematical model that summarize the above information

ii) determine a suitable allocation to maximize the total profit, using all Methods with comment on the results.

UNBALANCED T.P

Excess Supply

- If total supply exceeds total demand, we can balance a transportation problem by creating a dummy demand point that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity. Unmet Demand

UNBALANCED T.P

- If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with unmet demand. This means that a dummy supply point should be introduced.

UNBALANCED T.P

- Example: Modified REG Ltd for Excess Supply Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

ASSIGNMENT PROBLEM

Introduction

- In earlier discussion in previous chapter, we have dealt with two types of linear programming problems, *i.e.* Resource allocation method and Transportation model. We have seen that though we can use simplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them

ASSIGNMENT PROBLEM

- Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given the name **assignment model**.

Description

- Assignment Problems are a particular class of balanced transportation problems.
- **all supplies equal 1**
- **all demands equal 1**

ASSIGNMENT PROBLEM

- Hence an assignment problem must have all the properties of linear programming model. That is it must have:
- (*i*) an objective function,
- (*ii*) it must have structural constraints,
- (*iii*) It must have non-negativity constraint and
- (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

ASSIGNMENT PROBLEM

- Min $\sum\sum cijxij$
- S.t $\sum xij=1$ supply constraint
- $\sum xij=1$ demand constraint

$xij=0$ or $xij=1$

SOLVING AN A. P.

- **Minimization in an Assignment Problem**
- Some assignment problems are related to minimization of cost, time, and distance etc. In this type of assignment problem, our objective is to minimize total cost, total time, total distance etc
- **Step 1:** Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows(Row reduced matrix).
- **Step 2:** Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus first modified matrix is obtained.

SOLVING AN A. P.

- **Step 3:** Draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities
- If N=n, the number of rows (columns) of the given matrix then an optimal assignment can be made. So make the zero assignment to get the required solution.
- If N <n then proceed to step4

Cont'd

- **Step 4:** Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus these cond modified matrix is obtained.
- **Step 5:** Repeat step3 and step4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N =n.

Cont'd

- **Step 6:** To make zero assignment – examine the rows successively until a row – wise exactly single zero found; mark this zero by ' 'to make the assignment. Then, mark a 'X' over all zeros if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Cont'd

Step 7: Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If the relies more than one of the unmarked zeroes in any column or row, then mark ' 'one of the unmarked zeroes arbitrarily and mark across in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Cont'd

- **Step 8:** Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment

Cont'd

Example 1:

- A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

Cont’d

Task	subcordinate	I	II	III	IV
A		8	26	17	11
B		13	28	4	26
C		38	19	18	15
D		19	26	24	20

- Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table. Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.

	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Cont'd

- Four positions (I, II, III, IV) are assigned to four workers, James, Jane, Job and Joy in a way that they will make maximum output

	position			
worker	I	II	III	IV
James	90	35	40	30
Jane	35	70	40	30
Job	75	45	35	60
Joy	90	40	35	40

Maximization in an Assignment Problem

- Basically assignment model is a minimization model.
- If we want to maximize the objective function, it is easy to obtain an equivalent minimization problem by converting all numbers in the table to opportunity costs by subtracting every number in the original payoff table from the largest single number in that table.
- It turns out that the minimizing opportunity cost produces the same assignment as the original maximization problem.

Exercises

- Four companies W, X, Y and Z supply the requirements of three warehouses A, B and C respectively. The companies' availability, warehouses requirements and the unit cost of transportation are given in the following table.
-

	A	B	C	Supply
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35
Requirement	25	26	49	

- Find an initial basic feasible solution using
- a) North West Corner Method
- b) Least Cost Method c.
- c) Vogel Approximation Method (VAM

(ii) How many tonnes of 'Soap' to transport from cities of Kigali, Huye and Ruzizi to the cities of Muhanga, Rwamangana and Musanze on monthly basis. Table 1 indicate the supply and demand and Table 2 indicate the costs.

Table 1

Supply			Demand		
S1	Kigalii	150	A	Muhanga,	200
S2	Huye	175	B	Rwamangana	100
S3	Ruzizi	275	C	Musanze	300

Table 2

	A	B	C
S1	\$6	\$8	\$10
S2	\$7	\$11	\$11
S3	\$4	\$5	\$12

1 Find the transportation tableau **[2 Marks]**

2 Find the initial feasible solution by using NWCM and LCM. **[4 Marks]**

3 Test the IFS from LCM is optimal. **[4 Marks]**

NETWORK ANALYSIS

- Network analysis looks at the way large projects such as construction projects, computerization projects or managerial projects are planned and executed. The objective of network analysis is to set out a framework or plan such that the project is carried out and completed efficiently

Cont'd

- So network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects. A **network** is a set of points, called **nodes, event or vertices,** and a set of curves, called **branches (**or **arcs** or **links),**that connect certain pairs of nodes (or represent a direction of motion that may occur between vertices)**. Event or node** is either starting of an activity or ending of an activity

NETWORK ANALYSIS

- Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a ***network***, which is the basic tool in network-based management.

Cont’d

- An event represents a point in time signifying the completion of some activities and the beginning of new ones.
- This is usually represented by a circle in a net work which is also called a node or connector. The events are classified into three categories:
  - **Merge event**

When more than one activity comes and joins an event such an event is known as **merge event**

Cont’d

- **Burst event**

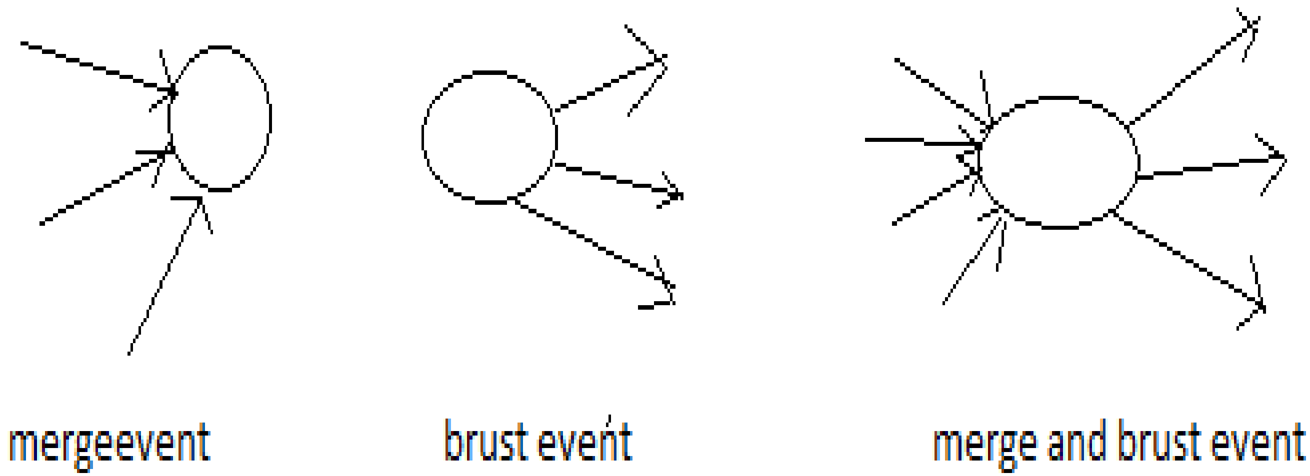
When more than one activity leaves an event such an event is known as burst event.

- **Merge and Burst events**

An activity may be a merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event

Cont’d

- 3 types



Cont'd

- **Some of the points to be remembered while drawing the network are**

(*a*) There must be only one beginning and one end for the network

(*b*) Event number should be written inside the circle or node (or triangle/square/rectangle etc).

- Activity name should be capital alphabetical letters and would be written above the arrow.
- The time required for the activity should be written below the arrow

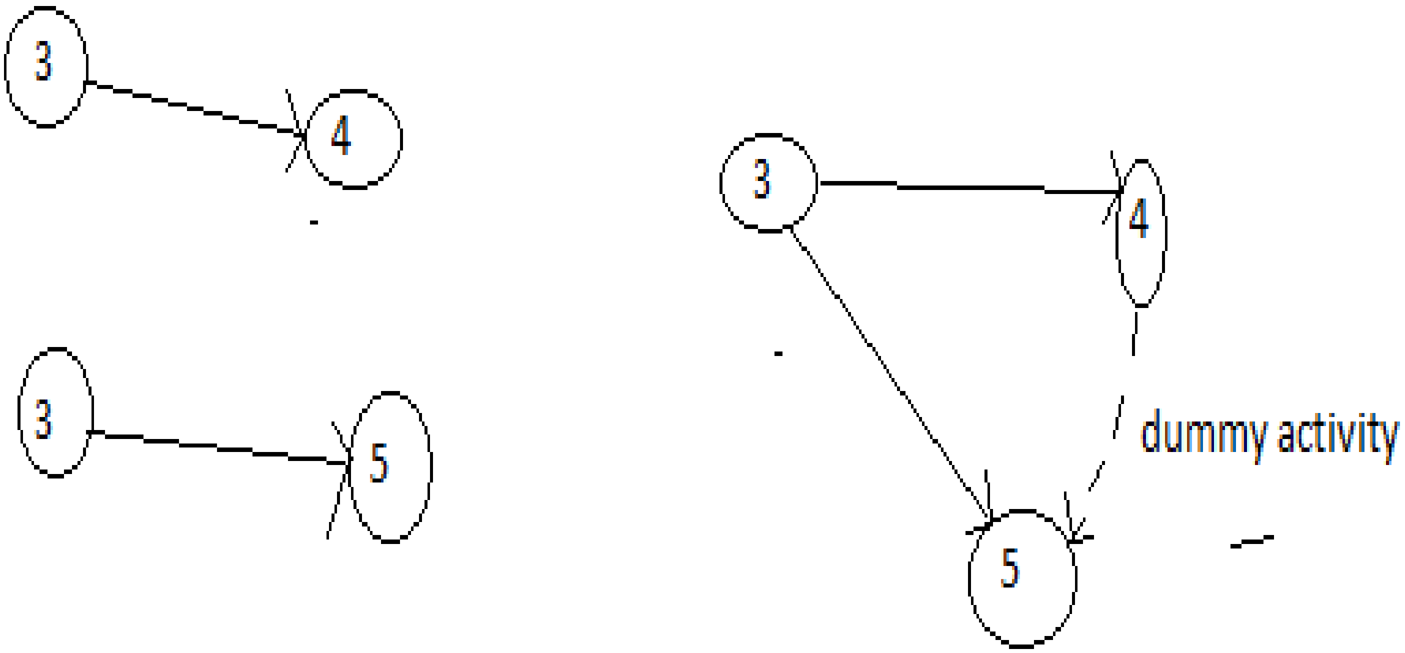
Cont'd

(*c*) While writing network, looping should be avoided. This is to say that the network arrows should move in one direction, *i.e.* starting from the beginning should move towards the end.

(*d*) When two activities start at the same event and end at different event, they should be shown by means of a **dummy activity**.

- Dummy activity is an activity, which simply shows the logical relationship and does not consume any resource. But, It should be represented by a dotted line as shown.
- In the figure, activities *C* and *D* start at the event 3 and end at event 4.
- *C* and *D* are shown in full lines, whereas the dummy activity is shown in dotted line

Cont'd



NETWORK ANALYSIS

- (*e*) When the event is written at the tail end of an arrow, it is known as **tail event**. If event is written on the head side of the arrow it is known as **head event**. A tail event may have any number of arrows (activities) emerging from it. This is to say that an event may be a tail event to any number of activities. Similarly, a head event may be a head event for any number of activities. This is to say that many activities may conclude at one event.

RULES FOR DRAWING NETWORK DIAGRAM

- **Rule 1**.Each activity is represented by one & only one, arrow.
- **Rule 2**.Each activity must be identified by two distinct events **&**No two or more activities can have the same tail and head events. Following figures shows how a dummy activity can be used to represent two concurrent activities, A&B. By definition, a dummy activity, which normally is depicted by a dashed arrow, consumes no time or resources

Cont'd

- **Rule 3**. To maintain correct precedence relationship, the following questions must be answered as each activity is added to the network:
- What activities must be immediately preceding the current activity?
- (b) What activities must follow the current activity?
- (c) What activities must occur concurrently with the current activity?

The answers to these questions may require the use of dummy activities to ensure correct precedence's among the activities

Cont'd

In case of large network, it is essential that certain good habits be practiced to draw an easy to follow network.

- Try to avoid arrows which cross each other
- Use straight arrows
- Do not attempt to represent duration of activity by its arrow length
- Use arrows from left or right. Avoid mixing two directions, vertical and standing arrows may be used if necessary.
- Use dummies freely in rough draft but final network should not have any redundant dummies.
- The network has only one entry point called start event and one point of emergence called the end event.

DRAWING A NETWORK DIAGRAM

- Supposing a project is to build a house. Then in network analysis each stage of the process would be itemized and the time taken to do each part estimated. A network or diagram of the events would then be drawn up and the critical path identified.

The critical path

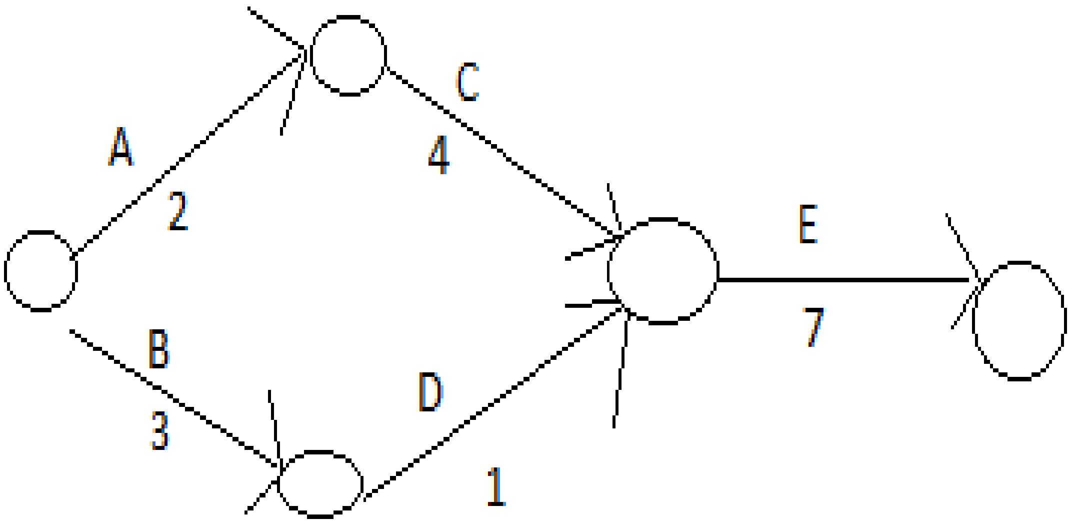
- If all the various ways (paths) through a network are identified and timed, the path with the longest time is known as the critical path. If an activity on the critical path is delayed, then the entire project will be delayed. Activities on the critical path must finish on time if the project is to finish on time.

EXAMPLE

- Draw the following network and find out the critical path

activity	Preceding activity	Time for each activity
A	-	2
B	-	3
C	A	4
D	B	1
E	C D	7

solution



Cont'd

- The critical path is A---C---E=13

. Network models

1.Minimum- Span problems

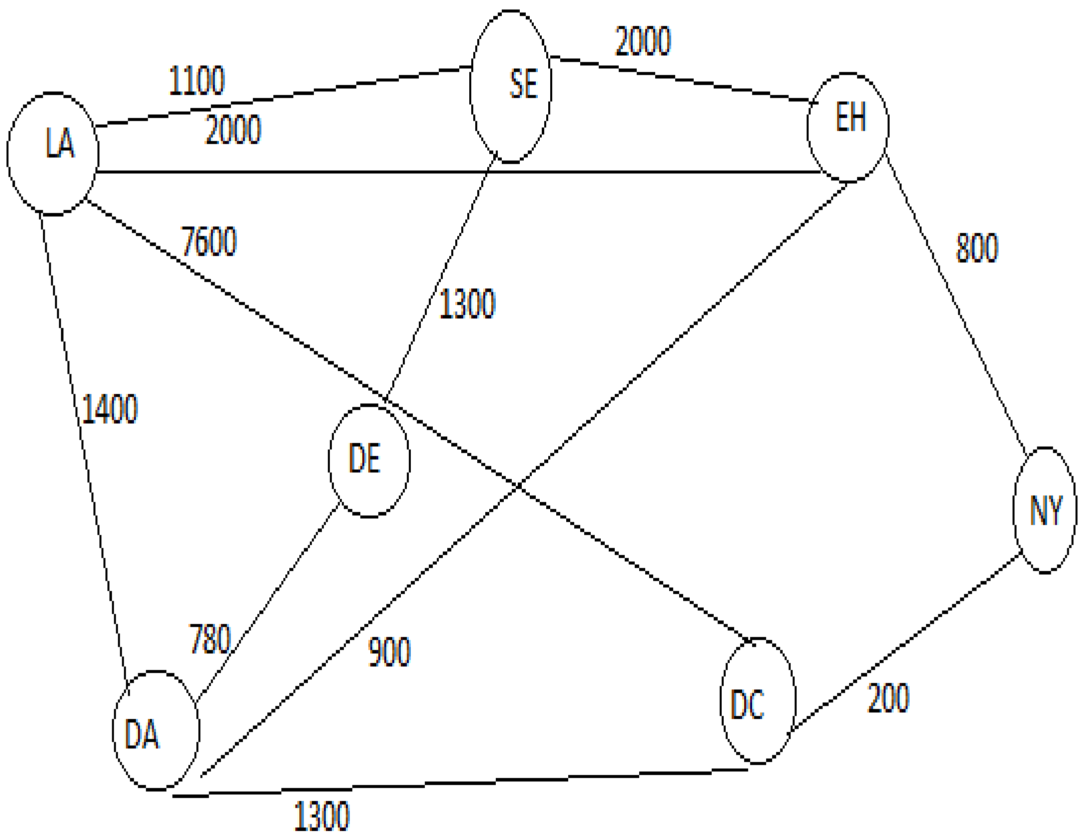
- The minimum spanning tree algorithm deals with linking the nodes of a network, directly or indirectly, using the shortest total length of connecting branches.
- The objective is to satisfy this requirement in a way that minimizes the total length of links inserted into the network.
- For a network with n nodes, a spanning tree is a group of **n-1** arcs that connects all nodes of the network and contains no loops.

Cont'd

- **Some Applications**
- Design of telecommunication networks (fiber-optic networks, computer networks, leased-line telephone networks, cable television networks, etc.)
- Design of lightly used transportation network to minimize the total cost of providing the links (rail lines, roads, etc.)
- Design of a network of high-voltage electrical transmission lines
- Design of a network of wiring on electrical equipment (e.g., a digital computer system) to minimize the total length of the wire

Cont'd

- **Example 1:** Suppose it is desired to establish a cable communication network that links major cities, which is shown in the figure beside. Determine how the cities are connected such that the total cable mile age is minimized



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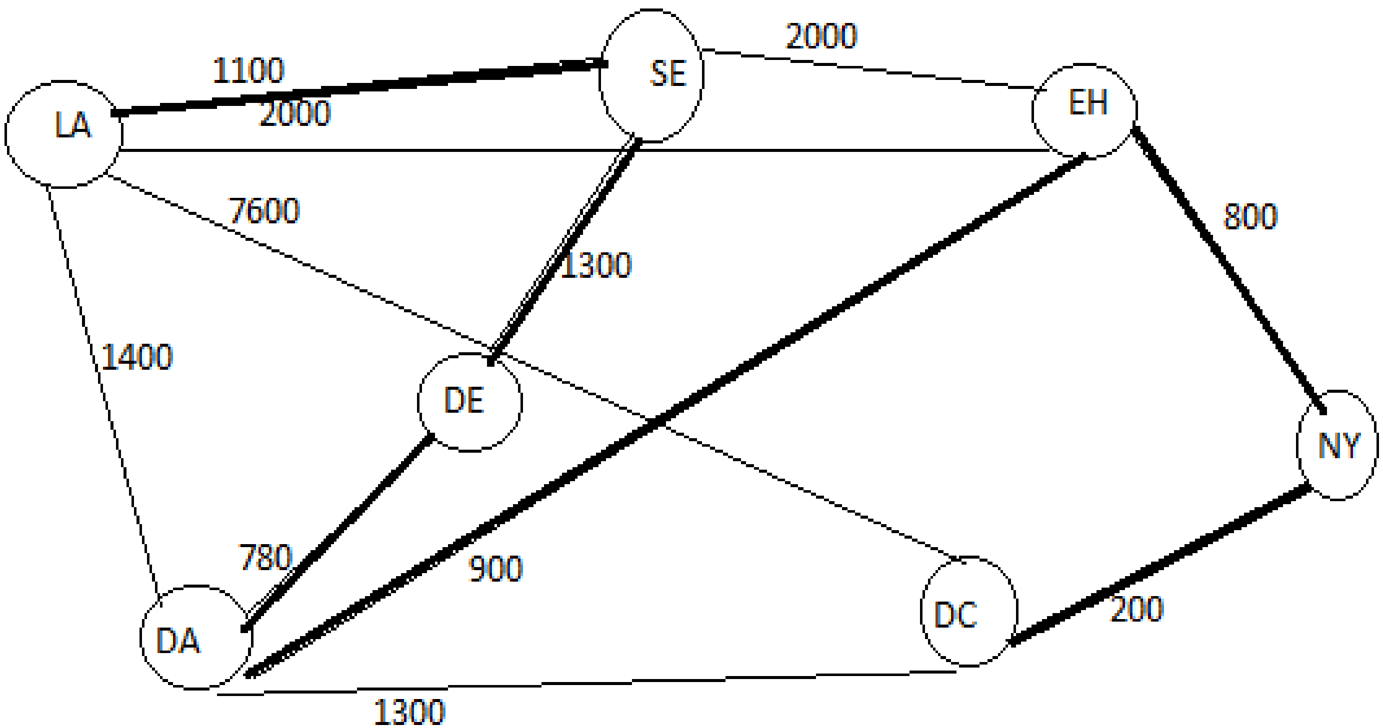


1100
2000
2000
SE
EH
LA
7600
1300
800
1400
DE
NY
780
900
DC
200
DA
1300

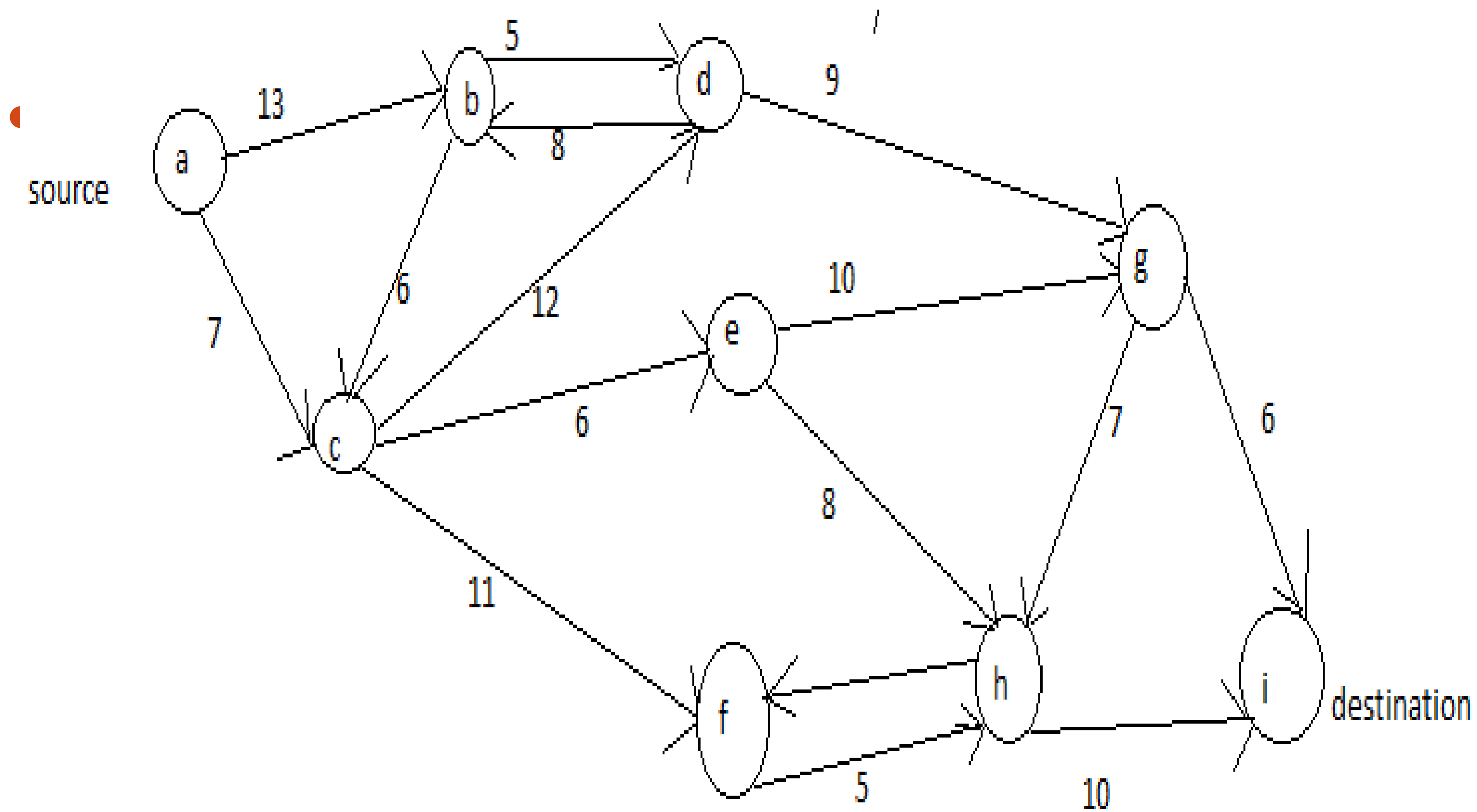
Cont'd

- Thus the total cable mile age is 1100+1300+780+900+800+200=5080

Cont'd

- **2. Shortest-Route problems**
- A shortest route-problem involves a connected network having a nonnegative cost associated with each branch. The shortest-route problem determines the shortest route between a source and destination in a transportation network. The objective is to determine a path joining the source and the sink such that the sum of the costs associated with the branches in the path is minimum. The criterion of this method is to find the shortest distance between two nodes with minimal cost.
- **Example1: Find the shortest path**

Cont'd



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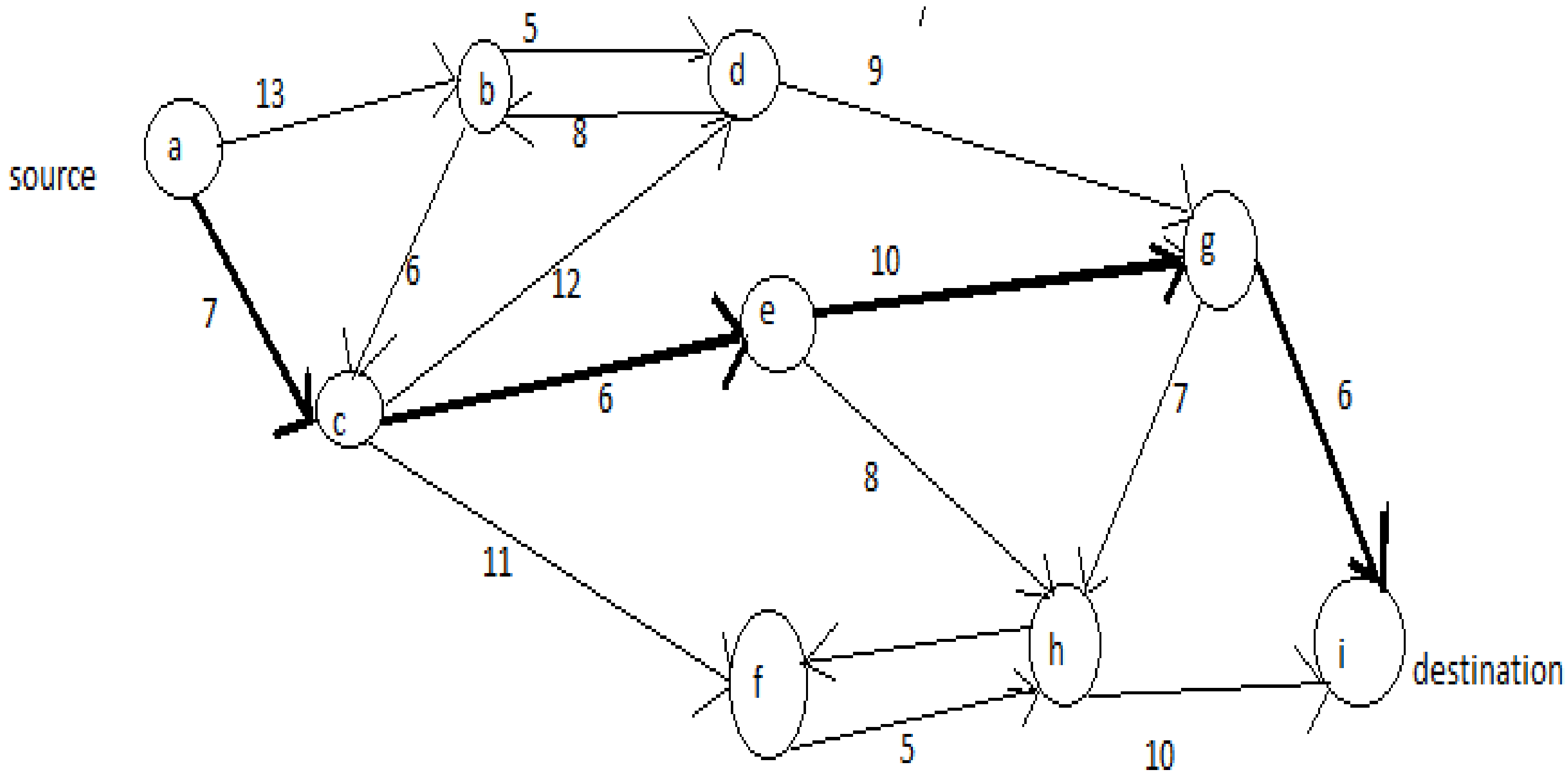


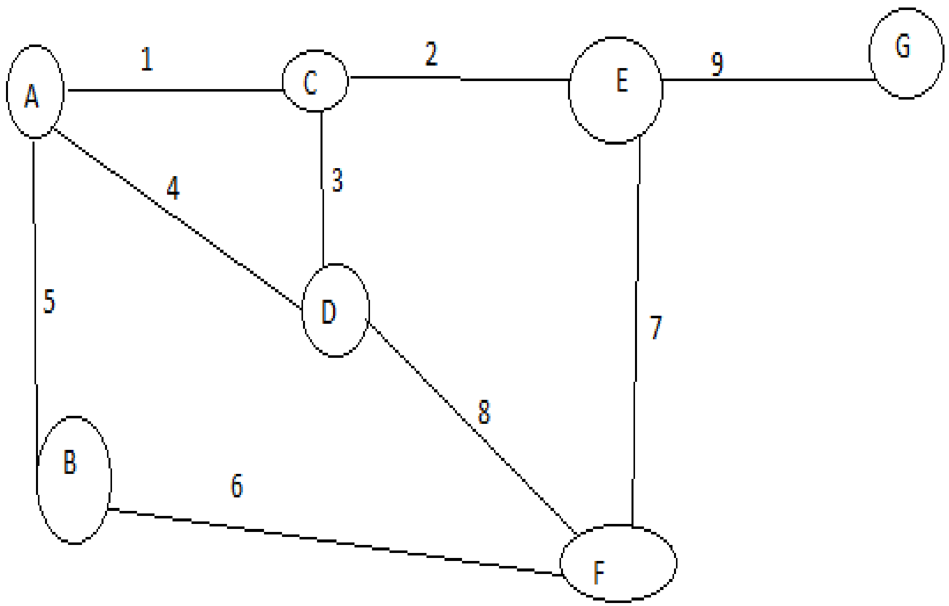
Cont'd

- The shortest path from a to I is a→c→e →g→i; Distance=7+6+10 +6=29units

Cont'd

- **Exercises**

- Find the minimum spanning tree for connecting the nodes



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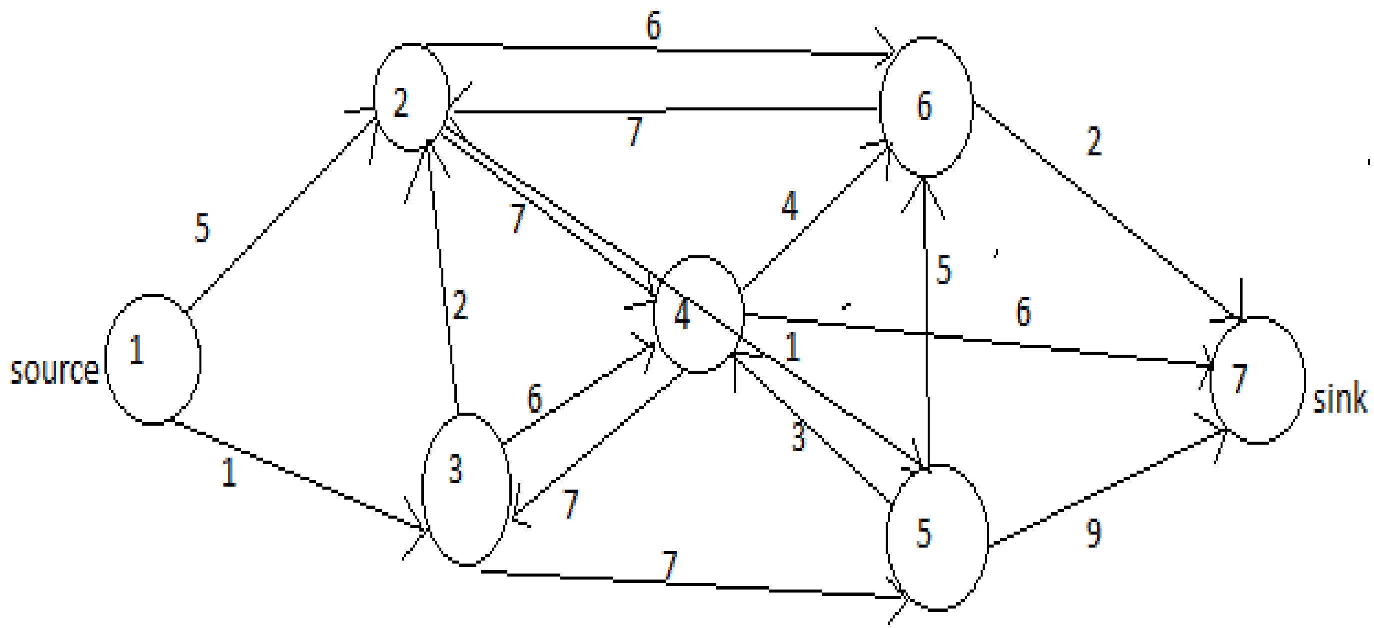
- Find the shortest path from 1 to 7



PROJECT MANAGEMENT WITH PERT/CPM

- This chapter introduces the **critical path method (CPM)** and the **program evaluation review technique (PERT),** two management science techniques developed in the late1950s to plan, schedule, and control large, complex projects with many activities.

Cont'd

- These approaches differ primarily on how the duration and the cost of activities are processed .In the case of CPM, it is assumed that details about these inputs are known with certainty, whereas for PERT, these details are not known with certainty. Both approaches use a network representation to display the relationships between project activities and to help managers to address questions such as:

Cont’d

- 1. What is the total time required to complete the project (the expected total time for PERT)?
- **2.** What are the start and the completion times for individual activities?
- **3.** Which critical activities must be completed as scheduled to meet the estimated project completion time?
- **4.** How much delay can be tolerated for non-critical activities without incurring a delay in the estimated project completion time?

Cont'd

- **5.** What is the least expensive way to speed up a project to meet a targeted completion time? PERT also provides answers to the following additional questions:
- **6.** What is the probability of completing a project within a given time frame?
- **7.** What is the variability in the project completion time?

Cont'd

- **PERT**: It is generally used for those projects where time required to complete various activities are not known as *a priori*. It is probabilistic model & is primarily concerned for evaluation of time. It is event oriented.
- **Critical Path Analysis–** It is a commonly used for those projects which are repetitive in nature &where one has prior experience of handling similar projects. It is a deterministic model & places emphasis on time &cost for activities of a project

Cont'd

- **Definitions Used in PERT and CPM**

In order to explain the purpose, structure and operation of PERT and CPM, it is helpful to define the following terms:

- A **project** is a collection of activities and events with a definable beginning and a definable end (the goal). For example: getting a college degree, patenting an invention, building a bridge, or installing new machinery.
- **Activity:** An activity is an effort(or an attempt or a try) that requires resources and takes a certain amount of time for completion. Examples of activities are: studying for an examination, designing a part, or training an employee.

Cont'd

- **Dummy activity**– An activity which does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.
- The dummy activity is inserted in the network to clarify the activity pattern in the following two situations
- To make activities with common starting and finishing points distinguishable
- To identify and maintain the proper precedence relationship between activities that is not connected by events.

Cont’d

Path: A path is a series of adjacent (or closest) activities leading from one event to another.

Critical path: A critical path is the sequence of critical activities that forms a continuous path between the start of a project and its completion. A path is a sequence of linked tasks going from beginning to end Critical path is the longest path in the project network.

Critical activity: A critical activity is an activity that, if even slightly delayed, will hold up the scheduled completion date of the entire project.Thisactivitymustbecompletedasscheduledtopreventdelayingtheproject completion.

Cont'd

- **Basic Steps in PERT/ CPM**

Projects scheduling by PERT/CPM consists of four mains steps

- **Planning**
- The planning phase is started by splitting the total project into small projects. These smaller projects in turn are divided into activities and are analyzed by the department or section.
- The relationship of each activity with respect to other activities are defined and established and the corresponding responsibilities and the authority are also stated.
- Thus the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

Cont'd

- **Scheduling**
- The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project.
- More over the schedule must pinpoint (identify) the critical path activities which require special attention if the project is to be completed in time.
- For non-critical activities, the schedule must show the amount of slack or float times which can be used advantageously when such activities are delayed or when limited
- Resources are to be utilized effectively.

Cont’d

- **Allocation of resources**
- Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labor, finance, equipment and space which will impose a limitation on time for the project.
- When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential.
- Resource allocation usually in curs a compromise and the choice of this compromise depends on the judgment of managers.

Cont'd

- **Controlling**

The final phase in project management is controlling.

- By having progress reports from time to time and up dating the network continuously, a better financial as well as technical control over the project is exercised.
- Arrow diagrams and time charts are used for making periodic progress reports. If required, a new course of action is determined for the remaining portion of the project.

Cont'd

- In the critical path method, the time estimates are assumed to be known with certainty. In certain projects like research and development, new product introductions, it is difficult to estimate the time of various activities. Hence PERT is used in such projects with a probabilistic method using three time estimates for an activity, rather than a single estimate,

Cont'd

Scheduling of Activities: Earliest Time and Latest Time

- Before the critical path in a network is determined, it is necessary to find the earliest and latest time of each event to know the earliest expected time (TE) at which the activities originating from the event can be started and to know the latest allowable time (TL) at which activities terminating at the event can be completed.

Cont'd

Forward Pass Computations (to calculate Earliest Time TE)

Procedure

- Step 1: Begin from the start event and move towards the end event.
- Step 2: Put TE = 0 for the start event.
- Step 3: Go to the next event (i.e node 2) if there is an incoming activity for event 2, add the calculated TE of previous event (i.e event 1) and activity time. Note: If there are more than one incoming activities, calculate TE for all incoming activities and take the maximum value. This value is the TE for event 2.
- Step 4: Repeat the same procedure from step 3 till the end event.

Cont'd

Backward Pass Computations (to calculate Latest Time T L)

Procedure

- Step 1: Begin from end event and move towards the start event. Assume that the direction of arrows is reversed.
- Step 2: Latest Time TL for the last event is the earliest time. TE of the last event.
- Step 3: Go to the next event, if there is an incoming activity, subtract the value of TL of previous event from the activity duration time. The arrived value is TL for that event. If there are more than one incoming activities, take the minimum TE value. Step 4: Repeat the same procedure from step 2 till the start event.

Cont'd

- DETERMINATION OF FLOAT AND SLACK TIMES
- As discussed earlier, the non – critical activities have some slack or float. The float of an activity is the amount of time available by which it is possible to delay its completion time without extending the overall project completion time.
- For an activity i = j, let tij = duration of activity
- TE = earliest expected time
- TL = latest allowable time
- ESij = earliest start time of the activity
- EFij = earliest finish time of the activity
- LSij = latest start time of the activity
- LFij = latest finish time of the activity

Cont’d

- TFij: The total float of an activity is the difference between the latest start time and the earliest start time of that activity.
- TFij = LS ij – ESij
- or TFij = (TL – TE) – tij
- Free Float FFij: The time by which the completion of an activity can be delayed from its earliest finish time without affecting the earliest start time of the succeeding activity is called free float. FF ij = (Ej – Ei) – tij

Cont’d

- FF_{ij} = Total float – Head event slack Independent Float IF_{ij}: The amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.
- $IF_{ij} = (E_j - L_i) - t_{ij}$
- IF_{ij} = Free float – Tail event slack Where tail event slack = $L_i - E_i$ The negative value of independent float is considered to be zero.

Cont'd

- Critical Path: After determining the earliest and the latest scheduled times for various activities, the minimum time required to complete the project is calculated. In a network, among various paths, the longest path which determines the total time duration of the project is called the critical path. The following conditions must be satisfied in locating the critical path of a network.
- An activity is said to be critical only if both the conditions are satisfied.

1. $TL - TE = 0$
2. $TLj - tij - TEj = 0$

Cont'd

- Example: A project schedule has the following characteristics as shown in Table below:

activity	name	time	activity	name	time
1-2	A	4	5-6	G	4
1-3	B	1	5-7	H	8
2-4	C	1	6-8	I	1
3-4	D	1	7-8	J	2
3-5	E	6	8-10	K	5
4-9	F	5	9-10	L	7

Cont'd

i. Construct PERT network.

- ii. Compute TE and TL for each activity.
- iii. Find the critical path.

Cont'd

- In PERT calculation, all values are used to obtain the percent expected value

For each activity, three time estimates are taken

- The Most Optimistic (hopeful) time
- The Most Likely time
- The Most Pessimistic time

The Duration of an activity is calculated using the following formula:

- Where $\mathbf{t_e}$ is the Expected time, t_o is the Optimistic time, t_m is the most probable activity time and t_p is the Pessimistic time.

Cont'd

$$t_e = \frac{t_0 + 4t_m + t_p}{6}$$

Cont'd

- The Standard Deviation, which is a good measure of the variability of each activity is calculated by the rather simplified formula: (The Variance is the Square of the Standard Deviation.)

$$s_1 = \frac{t_p - t_o}{6}$$

Cont'd

Probability for Project Duration

- The probability of completing the project within the scheduled time D (Ts) or contracted time may be obtained by using the standard normal deviate where Te is the expected time of project completion.

$$Z = \frac{D - t_e}{\sqrt{\sum \sigma^2 (incriticalpath)}}$$

Quiz

The following table shows precedence activities in a fast simple project of preparing tea at break time at a Kindergarten.

Activity	Activity description	Preceding activity	Duration (seconds)
A	Fill electric kettle with water	-	8
B	Boil the water	A	60
C	Find and clean teapot and cups	-	25
D	Put tea leaves in teapot	C	6
E	Pour water in the teapot	B, D	10
F	Allow tea to brew	E	45
G	Put milk in the cups	C	10
H	Put tea into the cups	F, G	20

REQUIRED

(i) Draw an activity network for the above project. **(3 Marks)**

(ii) Identify the earliest start time (EST) for each activity. **(3 Marks)**

(iii) Determine the critical path and duration of the project. **(2 Marks)**

Beds World Ltd (BWL) manufactures two types of beds: deluxe adjustable and deluxe ordinary. Each bed has to be processed by three machines and the time taken on each machine is indicated in the table below.

	Maximum time Available (hours)	Time (hours)	
		Deluxe adjustable	Deluxe ordinary
Machine 1	36	6	2
Machine 2	30	3	5
Machine 3	20	1	4

BWL realises a profit of Frw 250,000 on a deluxe adjustable bed and Frw 200,000 on a deluxe ordinary bed.

REQUIRED:

(i) Express the above information as linear programming model. **(2 Marks)**

(ii) Using the simplex tableau method, find the number of each type of bed BWL should produce in order to maximise profit. **(8 Marks)**

A job placement agency works on a contract basis with employers. A computer manufacturer is opening a new plant and has contracted with the placement agency to process job applications for prospective employees. Because of the uneven demands in workload at the agency, it often uses part-time personnel for the purpose of processing applications. For this particular contract, five placement analysts must be hired. Each analyst has provided an estimate of the maximum number of job applications he or she can evaluate during the coming month. Analysts are compensated on a piecework basis, with the rate determined by the type of application evaluated and the experience of the analyst. If x_{ij} equals the number of job applications of type j assigned to analyst i, the problem can be formulated as shown in the model below. Notice that the total supply (the maximum number of applications which can be processed by five analysts) exceeds total demand (expected number of applications). As a result, constraints (1) to (5) cannot be stated as equalities. According to assumption 2, total supply and demand must be brought into balance, artificially, before solving the problem.

Maximize $z = 15x_{11} + 10x_{12} + 8x_{13} + 7x_{14} + \ldots 6x_{54}$

subject to

$x_{11} + x_{12} + x_{13} + x_{14} \leq 90$ (1)

$x_{21} + x_{22} + x_{23} + x_{24} \leq 120$ (2)

$x_{31} + x_{32} + x_{33} + x_{34} \leq 140$ (3)

$x_{41} + x_{42} + x_{43} + x_{44} \leq 100$ (4)

$x_{51} + x_{52} + x_{53} + x_{54} \leq 110$ (5)

$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 100$ (6)

$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 150$ (7)

$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 175$ (8)

$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} = 125$ (9)

$x_{ij} \geq 0$ for all i and j

Let:

- x_{ij}: The number of job applications of type j assigned to analyst i, where:
  - $i = 1, 2, 3, 4, 5$ (the five analysts),
  - $j = 1, 2, \ldots, n$ (the types of job applications).

2. Objective Function

The objective of the agency is to minimize or maximize total compensation costs based on the number of applications assigned to each analyst. Since compensation is determined by a piecework rate c_{ij} (the rate analyst i is paid per application of type j), we want to minimize the total cost of assigning all the applications.

Thus, the **objective function** is:

$$\text{Minimize } Z = \sum_{i=1}^{5} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

Where:

- c_{ij} is the compensation rate for assigning one job application of type j to analyst i,
- x_{ij} is the number of job applications of type j assigned to analyst i.

3. Constraints

a. **Demand Constraints**:

Each type of job application has a certain demand (total number of applications that need to be processed). Let d_j be the total demand for job applications of type j. The sum of the applications assigned to all analysts must meet this demand.

For each type j:

$$\sum_{i=1}^{5} x_{ij} = d_j \quad \forall j$$

b. **Supply Constraints**:

Each analyst i has a maximum capacity (or supply) s_i, which is the total number of job applications the analyst can process. The total number of job applications assigned to each analyst cannot exceed their processing capacity.

For each analyst i:

$$\sum_{j=1}^{n} x_{ij} \leq s_i \quad \forall i$$

c. **Non-Negativity Constraints**:

The number of job applications assigned to any analyst must be non-negative.

$$x_{ij} \geq 0 \quad \forall i, j$$

Mathematical model

Objective Function:

$$\text{Minimize } Z = \sum_{i=1}^{5} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

Subject to:

$$\sum_{i=1}^{5} x_{ij} = d_j \quad \forall j \quad \text{(Demand constraints)}$$

$$\sum_{j=1}^{n} x_{ij} \leq s_i \quad \forall i \quad \text{(Supply constraints)}$$

$$x_{ij} \geq 0 \quad \forall i, j \quad \text{(Non-negativity constraints)}$$

Cont'd

- Consider the table below:

activity	Activity name	T0	tm	tp
1-2	A	4	6	8
1-3	B	2	3	10
1-4	C	6	8	16
2-4	D	1	2	3
3-4	E	6	7	8
3-5	F	6	7	14
4-6	G	3	5	7
4-7	H	4	11	12
5-7	I	2	4	6
6-7	J	2	9	10

Cont'd

a. Draw the project network.

b. Find the critical path.

c. Find the probability that the project is completed in 19 days.

d. If the probability is less that 20%, find the probability of completing it in 24 days.

- A project of constructing a new Library building in University, has the following characteristics

Activity	Most optimistic time(o)	Most pessimistic time(p)	Most likely time(m)
(1 – 2)	1	5	1.5
(2 – 3)	1	3	2
(2 – 4)	1	5	3
(3 – 5)	3	5	4
(4 – 5)	2	4	3
(4 – 6)	3	7	5
(5 – 7)	4	6	5
(6 - 7)	6	8	7
(7 - 8)	2	6	4
(7 - 9)	5	8	6
(8 - 10)	1	3	2
(9 - 10)	3	7	5

i) Construct a PERT network

ii) Find the critical path and variance for each event

iii) Determine the probability that the project will be completed within 25 days

CRASHING OF A PROJECT

- The process of shortening the time to complete a project is called crashing and is usually achieved by putting into service additional labor or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.

Cont'd

- STEPS IN PROJECT CRASHING:
- Assumption: It is assumed that there is a linear relationship between time and cost.   Let us consider project crashing by the critical path method. The following four-step procedure is adopted.
- Step 1: Find the critical path with the normal times and normal costs for the activities and identify the critical activities.
- Step 2: Find out the crash cost per unit time for each activity in the network. This is calculated by means of the following formula.

Cont'd

$$\frac{crash\cos t}{timeperiod} = \frac{crash\cos t - normal\cos t}{normaltime - crashtime}$$

Cont'd

- Step 3: Select an activity for crashing. The criteria for the selection is as follows: Select the activity on the critical path with the smallest crash cost per unit time. Crash this activity to the maximum units of time as may be permissible by the given data. Crashing an activity requires extra amount to be spent. However, even if the company is prepared to spend extra money, the activity time cannot be reduced beyond a certain limit in view of several other factors.

Cont'd

- In step 1, we have to note that reducing the time of on activity along the critical path alone will reduce the completion time of a project. Because of this reason, we select an activity along the critical path for crashing.
- In step 3, we have to consider the following question: If we want to reduce the project completion time by one unit, which critical activity will involve the least additional cost? On the basis of the least additional cost, a critical activity is chosen for crashing. If there is a tie between two critical activities, the tie can be resolved arbitrarily.

Cont'd

- Step 4: After crashing an activity, find out which is the critical path with the changed conditions. Sometimes, a reduction in the time of an activity in the critical path may cause a non-critical path to become critical. If the critical path with which we started is still the longest path, then go to Step 3. Otherwise, determine the new critical path and then go to Step 3.

Example

A project has activities with the following normal and crash times and cost

Activity	PHredecess or activity	Normal time(week)	Crash time	Normal cost	Crash cost
A	-	4	3	8000	9000
B	A	5	3	16000	20000
C	A	4	3	12000	13000
D	B	6	5	34000	35000
E	C	6	4	42000	44000
F	D	5	4	16000	16500
G	E	7	4	66000	72000
H	G	4	3	2000	5000

Determine a crashing scheme for the above project so that the total project time is reduced by 3 weeks.

QUEUING THEORY

- In this lesson we are going to talk about the queuing theory which is also known as waiting line. These queuing theory will facilitate in solving the queue related problem of the industry. The most important point will be taken into consideration in the designing queue system which should balance service to customers. Queuing theory deals with problems that involve waiting (or queuing). It is quite common that instances of queue occurs everyday in our daily life.

Cont'd

- Examples of queues or long waiting lines might be
- Waiting for service in banks and at reservation counters.
- Waiting for a train or a bus.
- Waiting for checking out at the Supermarket.

Whenever a customer arrives at a service facility, some of them usually have to wait before they receive the desired service. This forms a queue or waiting line and customers feel discomfort either mentally or physically because of long waiting queue.

Cont'd

Attitude of Customers

- Patient Customer: Customer arrives at the service system, stays in the queue until served, no matter how much he has to wait for service.
- Impatient Customer: Customer arrives at the service system, waits for a certain time in the queue and leaves the system without getting service due to some reasons like long queue before him.
- Balking: Customer decides not to join the queue by seeing the number of customers already in service system.

Cont'd

- Reneging: Customer after joining the queue, waits for some time and leaves the service system due to delay in service.
- Jockeying: Customer moves from one queue to another thinking that he will get served faster by doing so.

Cont’d

Queuing systems

A queuing system comprises with two components, the queue and the service facility

Cont’d

Characteristics of queuing system

In designing a good queuing system, it is necessary to have a good information about the model. The characteristics listed below would provide sufficient information.

- 1. The arrival pattern.
- 2. The service mechanism.
- 3. The queue discipline.
- 4. The number of customers allowed in the system.
- 5. The number of service channels.

Cont'd

- The Arrival Pattern

The arrival pattern describes how a customer may become a part of the queuing system. The arrival time for any customer is unpredictable. Therefore, the arrival time and the number of customers arriving at any specified time intervals are usually random variables.

- The Service Mechanism

The service mechanism is a description of resources required for service. If there are infinite number of servers, then there will be no queue. If the number of servers is finite, then the customers are served according to a specific order.

Cont'd

- The Queue Discipline

The most common queue discipline is the "**First Come First Served**" (FCFS) or "**First-in, First-out**" (FIFO). Other disciplines include "**Last In First Out**" (LIFO) where last customer is serviced first, "**Service In Random Order**" (SIRO) in which the customers are serviced randomly irrespective of their arrivals. "**Priority service**" is when the customers are grouped in priority classes based on urgency. "**Preemptive Priority**" is the highest priority given to the customer who enters into the service, immediately, even if a customer with lower priority is in service. "**Non-preemptive priority**" is where the customer goes ahead in the queue, but will be served only after the completion of the current service.

Cont'd

- The Number of Customers allowed in the System

Some of the queuing processes allow the limitation to the capacity or size of the waiting room, so that the waiting line reaches a certain length, no additional customers is allowed to enter until space becomes available by a service completion. This type of situation means that there is a finite limit to the maximum queue size.

- The Number of Service Channels

The more the number of service channels in the service facility, the greater the overall service rate of the facility.

Cont'd

To analyze queuing situations, the questions of interest that are typically concerned with measures of queuing system performance include,

- What will be the waiting time for a customer before service is complete?
- What will be the average length of the queue?
- What will be the probability that the queue length exceeds a certain length?
- How can a system be designed at minimum total cost?
- How many servers should be employed?
- Should priorities of the customers be considered?
- Is there sufficient waiting area for the customers?

Cont'd

- SYMBOLS AND NOTATIONS
- The symbols and notations used in queuing system are as follows:
- n = Number of customers in the system (both waiting and in service).
- λ= Average number of customers arriving per unit of time.
- μ = Average number of customers being served per unit of time.
- λ / μ = P, traffic intensity.
- Ls = Average or expected number of customers in the system (both waiting and in service).

Cont'd

- Lq = Average or expected number of customers in the queue.
- Ws = Average waiting time in the system (both waiting and in service).
- Wq = Average waiting time of a customer in the queue.
- Pn = Time independent probability that there are n customers in the system (both waiting and in service).
- $Pn(t)$ = Probability that there are n customers in the system at any time t (both waiting and in service).

Cont'd

SINGLE SERVER QUEUING MODEL

- Model 1: (MM1) : (FIFO)

This model is based on the following assumptions:

- (i) The arrivals follow Poisson distribution, with a mean arrival rate lamda.
- (ii) The service time has exponential distribution, average service rate miu.
- (iii) Arrivals are infinite population
- (iv) Customers are served on a First-in, First-out basis (FIFO).
- (v) There is only a single server.

Cont'd

- System of Steady-state Equations
- In this method, the question arises whether the service can meet the customer demand. This depends on the values of λ and μ.
- If $\lambda \geq \mu$ i.e., if arrival rate is greater than or equal to the service rate, the waiting line would increase without limit. Therefore for a system to work, it is necessary that $\lambda < \mu$.
- As indicated earlier, traffic intensity $\rho = \lambda / \mu$. This refers to the probability of time. The service station is busy. We can say that, the probability that the system is idle or there are no customers in the system, P0 = $1 - \rho$.

Cont'd

- The expected number of customers in the queue is given by,

$$l_n = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

- The expected number of customers in the system is given by,

$$l_s = \frac{\lambda}{\mu-\lambda}$$

Cont'd

- Queuing Equations

The evaluation of Model I is listed below:

- 1. Expected number of customers in the system,

$$l_s = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

2. Expected number of customers in the queue,

$$l_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$

Cont'd

- Average waiting time in the system,

$$w_s = \frac{1}{\mu - \lambda}$$

- Average waiting time in the queue,

$$w_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

Cont'd

- Average waiting time for a customer,

$$w = \frac{1}{\mu - \lambda}$$

- Probability that there is nobody in the system,

$$p_0 = 1 - \frac{\lambda}{\mu}$$

Cont'd

- Probability that there is at least one customer or queue is busy,

$$p_b = 1 - p_0$$

- Traffic intensity,

- Probability that there is n customer on queue is

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$$

Queuing theory

- Queue- Aggregation of items awaiting a service. It can be people, cars in a service station, components awaiting machining, telephone calls, aeroplanes etc.
- Queuing models deals with construction of mathematical models to enable predictions about how the system can work more efficiently to deal with demand for service.
- Queues form when rate of arrival of items requiring service is greater than the service rate.

Elements in a queue

- Queuing system is composed of:

- Arrivals → queue → service → outlet
- Arrivals- We only consider arrivals of discrete items.
- Queue – what happens between arrival and service. How are served, it can be first come first served, last in last out, or urgency method such as in hospitals
- Service – time taken to be served. It can be a constant time or it can vary with service. The service time can be reduced with better training, more service providers, more and equipments etc and all this at extra cost.
- The organisation should consider the cost of reducing the service time with the benefits to be realised
- Outlet- How do the items leave after being served. Are there any delays as they exit that can cause delay in service to others.
- The queuing system – the entire process from when a person/item arrives until served.

Types of queues

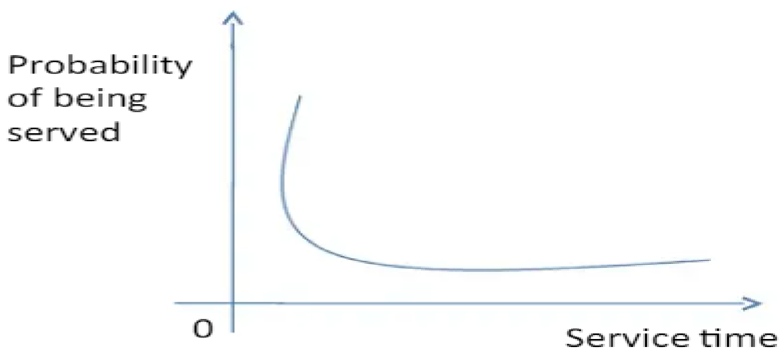
- Multiple queues with multiple service points (ATMs)
- Single queues with multiple service points (Bank halls).
- Multiple queues with single service points (server with many PCs.)

Arrival rate

- Arrivals are assumed to be random and to follow an Poisson distribution (a type of a discrete binomial distribution). i.e. arrivals are discrete such as0, 1, 2, 3, 4, ; they can not be continuous.

Service rate

- The time taken to serve can be constant but can also vary as per the service. For example, consultation time can vary as per illness.
- It follows a negative exponential distribution as shown below:



- The probability for short service time is high while the probability of higher service time is low.

- The arrival rate is represented by the greek letter λ. For example if customers arrive at the rate of 50 customers per hour then $\lambda = 50$
- The service rate is represented by the greek letter μ. For example an ATM can service 5 customers for every 20 minutes. Service rate per hour $\mu = 5/20*60 = 15$

Assumptions made when dealing with Simple queues

- It is a simple queue with 1 service time.
- The items are served as per first come first served.
- Queue has infinite capacity – it can be as big as possible
- Arrivals are random and follows a poisson distribution
- No simultaneous arrivals
- Service time follows a negative exponential distribution.
- There are discrete items from an infinite population of potential items
- There is service discipline and queues are followed.
- Traffic intensity is less than 1. Traffic intensity is the average arrival rate

  Average Service rate and it is represented by the greek letter $\rho = \lambda/\mu$.

Formulas

- Traffic intensity (average utilisation rate) = λ / μ
- Average number of items in queue when there is a queue (length of queue):

  $L_q = \lambda^2 / \mu(\mu - \lambda)$.
- Average number of items in system:

  $L_s = \lambda/(\mu - \lambda)$

Average waiting time in the queue

$W_q = \lambda/ \mu(\mu - \lambda)$

Average waiting time in the system =

$W_s = 1/(\mu - \lambda)$

Probabilities

- Probability of queuing on arrival (probability that the server is busy) $= \lambda/\mu$
- Probability of not queuing on arrival (probability that the server is idle)

$= (1 - \lambda/\mu)$

- Probability that there are n objects in the queue = $(1 - \lambda/\mu)(\lambda/\mu)^{n+1}$
- Probability that there are n objects in the system = $(1 - 1/\mu)(1/\mu)^{n}$

Example 1

- Customers arrive at a sales counter manned by 1 person at the rate of 20 per hour. The time required to serve a customer is has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.

Solution

- Arrival rate = λ= 20 customers per hour
- Service rate= μ= 3600/100=36 customers per hour.
- Average waiting time= W_q= $\lambda/\ \mu(\mu - \lambda)$= 20/(36(36-20)= 125 seconds
- Average waiting time in system=Ws=1/($\mu - \lambda$)=1/(36-20)=225seconds

Example 2

- An airlines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservations and flight timings. Assume that the number of customers arriving during any given time period is poison distributed with an arrival rate of eight per hour and that the reservations clerk can serve a customer in six minutes on an average which is exponentially distributed.

i) What is the probability that the clerk is idle?

ii) What is the average time a customer spends in the system?

iii) What is the average length of the queue?

Solution

Service rate = 60/6=10

i) Probability the system is idle= $(1 – \lambda/\mu)$

= (1-8/10)=0.2

ii) Average time a customer spends in the system = W_s= $1/(\mu – \lambda)$= 1/(10-8)=30 minutes

iii) Average length of the queue = L_q= $\lambda^2/ \mu(\mu - \lambda)$

= 8*8/(10(10-8))= 64/20= 3.2 customers

Example 3

- Arrivals at a telephone booth are considered to follow a poisson distribution with an average time of 10 minutes between one arrival and the next. The length of a phone call is exponentially distributed with a mean of 6 minutes.

i) What is the probability that a person arriving at the booth will have to wait?

ii) What is the average length of the queues that form.

iii) Calculate the probability of having 2 people in the queue

Practice questions

- Self service in a cafeteria takes an average of 8 minutes per customer and is slower than the attendant rate which takes 6 minutes per customer. The manager of the cafeteria wants to calculate the average number of customers in the cafeteria, the average time that each customer spends waiting for service. Assuming that customers arrive at the rate of 5 customers per hour, determine if there is need to change the system to the attendant type if waiting time for the customer in the queue should not be more than 13 minutes.

- A fertiliser company distributes its products by trucks loaded at its only loading station. Both company trucks and contractors trucks are used for this purpose. It was found out that on average every 5 minutes one truck arrived and the average loading time was 3 minutes. 40% of the trucks belong to the contractors. Making suitable assumptions, determine:

- i) the probability that a truck has to wait
- ii) the waiting time of a truck that waits
- iii) The expected waiting time of the contractors trucks before being loaded per day.

Cont'd

- **Example1:** Trucks at a single platform weigh-bridge arrive according to Poisson probability distribution. The time required to weigh the truck follows an exponential probability distribution. The mean arrival rate is 12 trucks per day, and the mean service rate is 18 trucks per day. Determine the following:
- (a) What is the probability that no trucks are in the system?
- (b) What is the average number of trucks waiting for service?
- (c) What is the average time a truck waits for weighing service to begin?
- (d) What is the probability that an arriving truck will have to wait for service?

Response

For this system, we again have an **M/M/1 queue** where:

- The arrival rate (λ) is 12 trucks per day.
- The service rate (μ) is 18 trucks per day.

Utilization Factor (ρ):

The system utilization is given by: $\rho = \dfrac{\lambda}{\mu}$

(a) Probability that no trucks are in the system (P_0):

The probability that no trucks are in the system is given by: $P_0 = 1 - \rho$

(b) Average number of trucks waiting for service (L_q):

The average number of trucks waiting in the queue is given by: $L_q = \dfrac{\rho^2}{1 - \rho}$

(c) Average waiting time for weighing service (W_q):

This is the average time a truck waits in the queue before service starts: $W_q = \dfrac{\rho}{\mu(1 - \rho)}$

(d) Probability that an arriving truck will have to wait for service:

This is simply the system utilization, which is the probability that the server is busy, so: $P_{\text{wait}} = \rho$

the performance measures for the trucks at the weigh-bridge:

(a) The probability that no trucks are in the system (P_0) is approximately **0.33** (or 33.33%).

(b) The average number of trucks waiting for service (L_q) is **1.33 trucks**.

(c) The average time a truck waits for weighing service to begin (W_q) is approximately **0.111 hours** (or about 6.67 minutes).

(d) The probability that an arriving truck will have to wait for service is approximately **0.67** (o 66.67%).

codes

- # Given data for the weigh-bridge system
- lambda_rate_truck = 12 # trucks per day (arrival rate)
- mu_rate_truck = 18   # trucks per day (service rate)

- # Utilization (rho) for this system
- rho_truck = lambda_rate_truck / mu_rate_truck

- # Probability that no trucks are in the system (P0)
- P_0_truck = 1 - rho_truck

- # Average number of trucks waiting for service (Lq)
- L_q_truck = (rho_truck ** 2) / (1 - rho_truck)

- # Average waiting time for weighing service (Wq)
- W_q_truck = rho_truck / (mu_rate_truck * (1 - rho_truck))

- # Probability that an arriving truck will have to wait for service (P_wait)
- P_wait_truck = rho_truck

- P_0_truck, L_q_truck, W_q_truck, P_wait_truck

```
Result
(0.33333333333333337,
 1.3333333333333333,
 0.11111111111111109,
 0.6666666666666666)
```

Cont'd

- Example 2: Auto car service provides a single channel water wash service. The incoming arrivals occur at the rate of 4 cars per hour and the mean service rate is 8 cars per hour. Assume that arrivals follow a Poisson distribution and the service rate follows an exponential probability distribution. Determine the following measures of performance:
- (a) What is the average time that a car waits for water wash to begin?
- (b) What is the average time a car spends in the system?
- (c) What is the average number of cars in the system?

Response

In this system, we again have an **M/M/1 queue**, where:

- The arrival rate (λ) is 4 cars per hour.
- The service rate (μ) is 8 cars per hour.

Utilization Factor (ρ):

The system utilization is given by: $\rho = \dfrac{\lambda}{\mu}$

(a) Average waiting time in the queue (W_q):

This is the average time a car waits in the queue before the service starts. It's given by: $W_q = \dfrac{\rho}{\mu(1-\rho)}$

(b) Average time a car spends in the system (W):

This is the total time a car spends in the system, both waiting in the queue and being served. It's given by: $W = \dfrac{1}{\mu(1-\rho)}$

(c) Average number of cars in the system (L):

This is the average number of cars either waiting in the queue or being served. It's given by: $L = \dfrac{\rho}{1-\rho}$

the performance measures for the auto car service:

(a) The average time that a car waits for the water wash to begin (W_q) is **0.125 hours** (or 7.5 minutes).

(b) The average time a car spends in the system (W) is **0.25 hours** (or 15 minutes).

(c) The average number of cars in the system (L) is **1 car.** [>-]

Codes from python

- # Given data for the auto car service system
- lambda_rate_auto = 4 # cars per hour (arrival rate)
- mu_rate_auto = 8   # cars per hour (service rate)

- # Utilization (rho) for this system
- rho_auto = lambda_rate_auto / mu_rate_auto

- # Average waiting time in the queue (Wq)
- W_q_auto = rho_auto / (mu_rate_auto * (1 - rho_auto))

- # Average time spent in the system (W)
- W_auto = 1 / (mu_rate_auto * (1 - rho_auto))

- # Average number of cars in the system (L)
- L_auto = rho_auto / (1 - rho_auto)

- W_q_auto, W_auto, L_auto

```
Result
(0.125, 0.25, 1.0)
```

Cont'd

- **Example3:** In a single pump petrol station, vehicles arrive at the rate of 20 customers per hour and petrol filling takes 2 minutes on an average. Assume the arrival rate is Poisson probability distribution and service rate is exponentially distributed, determine
- (a) What is the probability that no vehicles are in the petrol station?
- (b) What is the probability that 1 customer is filling and no one is waiting in the queue?
- (c) What is the probability that 1 customer is filling and 2 customers are waiting in the queue?
- (d) What is the probability that more than 2 customers are waiting?

Response

This problem describes a **queueing system** and can be modeled using the **M/M/1 queue** (which means the arrival process follows a Poisson distribution, the service times are exponentially distributed, and there's a single server). The following parameters are given:

- Arrival rate (λ): 20 customers per hour = $\frac{20}{60} = \frac{1}{3}$ customers per minute.
- Service rate (μ): 1 customer is served in 2 minutes = $\frac{1}{2}$ customers per minute.

For an M/M/1 queue, the system utilization (ρ) is given by:

$$\rho = \frac{\lambda}{\mu}$$

(a) Probability that no vehicles are in the petrol station (P_0):

This is the probability that the system is empty, i.e., no customers are being served or waiting. It is given by:

$$P_0 = 1 - \rho$$

(b) Probability that 1 customer is filling and no one is waiting in the queue (P_1):

This is the probability that exactly 1 customer is in the system, i.e., being served, and no one is in the queue. For an M/M/1 system, this is given by:

$$P_1 = (1 - \rho)\rho$$

(c) Probability that 1 customer is filling and 2 customers are waiting in the queue (P_3):

This is the probability that there are exactly 3 customers in the system, i.e., 1 being served and 2 waiting. For an M/M/1 queue:

$$P_3 = (1 - \rho)\rho^3$$

(d) Probability that more than 2 customers are waiting ($P(\text{queue length} > 2)$):

This is the probability that more than 3 customers are in the system, i.e., more than 2 customers are waiting (since 1 customer is always being served). This is the sum of probabilities for 4 or more customers in the system. The probability that there are more than 3 customers in the system is given by:

$$P(\text{queue length} > 2) = \sum_{n=4}^{\infty} P_n = \rho^4$$

calculate these probabilities using the values for λ and μ.

(a) The probability that no vehicles are in the petrol station (P_0) is approximately **0.33** (or 33.33%).

(b) The probability that 1 customer is filling and no one is waiting in the queue (P_1) is approximately **0.22** (or 22.22%).

(c) The probability that 1 customer is filling and 2 customers are waiting in the queue (P_3) is approximately **0.099** (or 9.88%).

(d) The probability that more than 2 customers are waiting in the queue is approximately **0.198** (or 19.75%). [>-]

The following are the codes used.

- # Given data
- lambda_rate = 1 / 3 # customers per minute (arrival rate)
- mu_rate = 1 / 2   # customers per minute (service rate)
- # Utilization (rho)
- rho = lambda_rate / mu_rate

- # Calculating probabilities
- P_0 = 1 - rho # Probability that no vehicles are in the petrol station
- P_1 = (1 - rho) * rho # Probability that 1 customer is filling and no one is waiting
- P_3 = (1 - rho) * (rho ** 3) # Probability that 1 customer is filling and 2 customers are waiting
- P_queue_more_than_2 = rho ** 4 # Probability that more than 2 customers are waiting

- P_0, P_1, P_3, P_queue_more_than_2

Thank You