Solution Manual For:
Introduction to Linear Optimization
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Introduction

Acknowledgements

Chapter 1 (Introduction)

Exercise 1.1

Since $f(\cdot)$ is convex we have that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1}$$

Since $f(\cdot)$ is concave we also have that

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y). \tag{2}$$

Combining these two expressions we have that f must satisfy each with equality or

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y). \tag{3}$$

This implies that f must be linear and the expression given in the book holds.

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Exercise 1.2

Part (a): We are told that f_i is convex so we have that

$$f_i(\lambda x + (1 - \lambda)y) \leq \lambda f_i(x) + (1 - \lambda)f_i(y), \tag{4}$$

for every i. For our function $f(\cdot)$ we have that

$$f(\lambda x + (1 - \lambda)y) = \sum_{i=1}^{m} f_i(\lambda x + (1 - \lambda)y) \tag{5}$$

$$\leq \sum_{i=1}^{m} \lambda f_i(x) + (1 - \lambda)f_i(y) \tag{6}$$

$$= \lambda \sum_{i=1}^{m} f_i(x) + (1 - \lambda)\sum_{i=1}^{m} f_i(y) \tag{7}$$

$$= \lambda f(x) + (1 - \lambda)f(y) \tag{8}$$

and thus $f(\cdot)$ is convex.

Part (b): The definition of a piecewise linear convex function f_i is that is has a representation given by

$$f_i(x) = \text{Max}_{j=1,2,\ldots,m}(c'_j x + d_j). \tag{9}$$

So our $f(\cdot)$ function is

$$f(x) = \sum_{i=1}^{n} \text{Max}_{j=1,2,\ldots,m}(c'_j x + d_j). \tag{10}$$

Now for each of the $f_i(x)$ piecewise linear convex functions $i \in 1, 2, 3, \ldots, n$ we are adding up in the definition of $f(\cdot)$ we will assume that function $f_i(x)$ has m_i affine/linear functions to maximize over. Now select a new set of affine values $(\tilde{c}_j, \tilde{d}_j)$ formed by summing elements from each of the $1, 2, 3, \ldots, n$ sets of coefficients from the individual f_i. Each pair of $(\tilde{c}_j, \tilde{d}_j)$ is obtained by summing one of the (c_j, d_j) pairs from each of the n sets. The number of such coefficients can be determined as follows. We have m_1 choices to make when selecting (c_j, d_j) from the first piecewise linear convex function, m_2 choices for the second piecewise linear convex function, and so on giving a total of $m_1 m_2 m_3 \cdots m_n$ total possible sums each producing a single pair $(\tilde{c}_j, \tilde{d}_j)$. Thus we can see that $f(\cdot)$ can be written as

$$f(x) = \text{Max}_{j=1,2,3,\ldots,\prod_{l=1}^{n} m_l} \tilde{c}'_j x + \tilde{d}_j, \tag{11}$$

since one of the $(\tilde{c}_j, \tilde{d}_j)$ will produce the global maximum. This shows that $f(\cdot)$ can be written as a piecewise linear convex function.

Exercise 1.3 (minimizing a linear plus linear convex constraint)

We desire to convert the problem $\min(c'x + f(x))$ subject to the linear constraint $Ax \geq b$, with $f(x)$ given as in the picture to the standard form for linear programming. The $f(\cdot)$

given in the picture can be represented as

$$f(\xi) = \begin{cases} -\xi + 1 & \xi < 1 \\ 0 & 1 < \xi < 2 \\ 2(\xi - 2) & \xi > 2, \end{cases} \tag{12}$$

but it is better to recognize this $f(\cdot)$ as a piecewise linear convex function given by the maximum of three individual linear functions as

$$f(\xi) = \max\left(-\xi + 1, 0, 2\xi - 4\right) \tag{13}$$

Defining $z \equiv \max\left(-\xi + 1, 0, 2\xi - 4\right)$ we see that or original problem of minimizing over the term $f(x)$ is equivalent to minimizing over z. This in tern is equivalent to requiring that z be the smallest value that satisfies

$$\begin{align} z &\geq -\xi + 1 \tag{14} \\ z &\geq 0 \tag{15} \\ z &\geq 2\xi - 4. \tag{16} \end{align}$$

With this definition, our original problem is equivalent to

$$\text{Minimize } (c'x + z) \tag{17}$$

subject to the following constraints

$$\begin{align} Ax &\geq b \tag{18} \\ z &\geq -d'x + 1 \tag{19} \\ z &\geq 0 \tag{20} \\ z &\geq 2d'x + 4 \tag{21} \end{align}$$

where the variables to minimize over are (x, z). Converting to standard form we have the problem

$$\text{Minimize}(c'x + z) \tag{22}$$

subject to

$$\begin{align} Ax &\geq b \tag{23} \\ d'x + z &\geq 1 \tag{24} \\ z &\geq 0 \tag{25} \\ -2d'x + z &\geq 4 \tag{26} \end{align}$$

Exercise 1.4

Our problem is

$$\text{Minimize}(2x_1 + 3|x_2 - 10|) \tag{27}$$

subject to

$$|x_1 + 2| + |x_2| \leq 5. \tag{28}$$

To convert this problem to standard form, first define $z_1 = |x_2 - 10|$ then the above problem is equivalent to

$$\text{Minimize}(2x_1 + 3z_1) \tag{29}$$

subject to

$$|x_1 + 2| + |x_2| \leq 5 \tag{30}$$
$$z_1 \geq x_2 - 10 \tag{31}$$
$$z_1 \geq -x_2 + 10 \,. \tag{32}$$

The variables to minimize over now include z i.e. are (x, z_1). To eliminate the other terms with an absolute value we define $z_2 = |x_2 + 2|$ and $z_3 = |x_2|$, which add the following inequalities to our constraints

$$z_2 \geq x_2 + 2 \tag{33}$$
$$z_2 \geq -x_2 - 2 \tag{34}$$
$$z_3 \geq x_2 \tag{35}$$
$$z_3 \geq -x_2 \tag{36}$$

In total then our linear programming problem has become is

$$\text{Minimize} \quad (2x_1 + 3z_1) \tag{37}$$

subject to

$$z_1 \geq x_2 - 10 \tag{38}$$
$$z_1 \geq -x_2 + 10 \tag{39}$$
$$z_2 + z_3 \leq 5 \tag{40}$$
$$z_3 \geq x_2 \tag{41}$$
$$z_3 \geq -x_2 \tag{42}$$
$$z_2 \geq x_2 - 10 \tag{43}$$
$$z_2 \geq -x_2 + 10 \,. \tag{44}$$

Where the variables to minimize over are $(x_1, x_2, z_1, z_2, z_3)$. Note the above can easily be converted to standard form, by introducing negative signs and moving all variables to the left hand side of the equation if required.

As a second formulation of this problem define the variables z_1 and z_2 as

$$z_1 = x_2 - 10 \tag{45}$$
$$z_2 = x_1 + 2 \tag{46}$$

Then our linear programming problem becomes

$$\text{Minimize} \, (2x_1 + 3|z_1|) \tag{47}$$

subject to

$$|z_2| + |x_2| \leq 5 \tag{48}$$
$$z_1 = x_2 - 10 \tag{49}$$
$$z_2 = x_1 + 2 \tag{50}$$

To eliminate the absolute values introduce $z_1^+, z_1^-, z_2^+, z_2^-, x_2^+, x_2^-$ such that

$$
\begin{aligned}
|z_1| &= z_1^+ + z_1^- && (51) \\
|z_2| &= z_2^+ + z_2^- && (52) \\
|x_2| &= x_2^+ + x_2^- && (53)
\end{aligned}
$$

and our linear program becomes

$$
\text{Minimize } (2x_1 + 3(z_1^+ + z_1^-)) \tag{54}
$$

subject to

$$
\begin{aligned}
z_2^+ + z_2^- + x_2^+ + x_2^- &\leq 5 && (55) \\
z_1^+ - z_1^- &= x_2^+ - x_2^- - 10 && (56) \\
z_2^+ - z_2^- &= x_1 + 2 && (57) \\
z_1^+, z_1^-, z_2^+, z_2^-, x_2^+, x_2^- &\geq 0 && (58)
\end{aligned}
$$

Since x_2 no longer explicitly appears we don't have to minimize over it and our variables are $x_1, z_1^+, z_1^-, z_2^+, z_2^-, x_2^+, x_2^-$.

Exercise 1.5

Part (a): The problem given in the text is

$$
\text{Minimize} \quad c'x + d'|x| \tag{59}
$$

subject to the constraints that

$$
Ax + B|x| \leq b \tag{60}
$$

Where the notation $|x|$ is used to represent component-wise absolute values of the vector x. Our unknown vector to minimize over is given by x.

In the first type of reformulation for problems with absolute values is to define an auxiliary variable equal to the absolute value of the variable under consideration. Thus for the problem given, y_i is already such an auxiliary variable. Our problem becomes

$$
\text{Minimize} \quad c'x + d'y \tag{61}
$$

subject to

$$
\begin{aligned}
Ax + By &\leq b && (62) \\
x_i &\leq y_i && (63) \\
-x_i &\leq y_i. && (64)
\end{aligned}
$$

In this case our unknown vector to minimize over, is given by all components of x with all components of y or the pair (x, y).

A second formulation replaces all absolute valued variables with two strictly positive variables with the substitution

$$|x| = x^+ + x^- , \tag{65}$$

in addition to a direct substitution of each variable by the difference of the same two positive variables as

$$x = x^+ - x^- . \tag{66}$$

With these substitutions or problem becomes

$$\text{Minimize} \quad c'(x^+ - x^-) + d'(x^+ + x^-) \tag{67}$$

subject to

$$A(x^+ - x^-) + B(x^+ + x^-) \leq b \tag{68}$$
$$x^+ \geq 0 \tag{69}$$
$$x^- \geq 0 \tag{70}$$

or equivalently

$$\text{Minimize} \quad (c + d)'x^+ + (-c + d)'x^- \tag{71}$$

subject to

$$(A + B)x^+ + (-A + B)x^- \leq b \tag{72}$$
$$x^+ \geq 0 \tag{73}$$
$$x^- \geq 0 \tag{74}$$

In this formulation, the variables to minimize over are given by x^+ and x^- or the pair (x^+, x^-).

Part (b): This follows from the one-to-one nature of the constructive solutions we have developed above. What is meant by this is that is any of the above formulations have a feasible solution vector then feasible solution vectors for the other two can be computed from the given feasible solution vector. This implies that if one of the problems was infeasible then the other ones would be also. It remains to be shown how to construct a feasible solution vector for each of the given problems. For ease of explanation we will call the *first* formulation (given in the book) F_I, the second formulation (given above) F_{II} and the third formulation (given above) F_{III}.

We first show that a feasible solution vector for F_I implies a feasible solution vector for F_{II}. If we have a feasible vector x^I, to get a feasible vector for F_{II} (which has two unknowns x^{II} and y^{II}) we construct our variables x^{II} and y^{II} as

$$x^{II} = x^I \tag{75}$$
$$y^{II} = |x^I| , \tag{76}$$

to be understood component-wise, and we will have a feasible vector for F_{II}.

We next show that a feasible solution vector for F_I implies a feasible solution vector for F_{III}. If we have a feasible vector x for F_I, to get a feasible vector for F_{III} (which has two

unknowns x^+ and x^-) we construct our variables x^+ and x^- to satisfy (component-wise)

$$|x| = x^+ + x^- \tag{77}$$
$$x = x^+ - x^- \tag{78}$$

Solving the above system for x^+ and x^- we have that

$$x^+ = \frac{1}{2}(|x| + x) \tag{79}$$

$$x^- = \frac{1}{2}(|x| - x). \tag{80}$$

By inspection these satisfy the requirements that $x^+ \geq 0$ and $x^- \geq 0$ giving a feasible vector for the third formulation.

Finally, given a feasible vector for the third formulation we have a feasible vector for the first formulation by defining

$$x = x^+ - x^- \tag{81}$$

With these three directions proven given any formulation of the problem we can derive the remaining equivalent formulations.

Part (c): Not included. Would anyone like to help provide a solution?

Exercise 1.6

If the objective function is $\sum |a_t|$ then our program is given by

$$\text{Minimize} \sum_{t=0}^{T-1} |a_t| \tag{82}$$

subject to

$$x_0 = 0 \tag{83}$$
$$v_0 = 0 \tag{84}$$
$$x_{t+1} = x_t + v_t \quad t = 0, 1, 2, \ldots, T-1 \tag{85}$$
$$v_{t+1} = v_t + a_t \quad t = 0, 1, 2, \ldots, T-1 \tag{86}$$
$$x_T = 1 \tag{87}$$
$$v_T = 0. \tag{88}$$

To remove the absolute values from the above define

$$\hat{a}_t = |a_t| \quad t = 0, 1, 2, \ldots, T-1 \tag{89}$$

and then our problem becomes

$$\text{Minimize} \sum_{t=0}^{T-1} \hat{a}_t \tag{90}$$

subject to

$$
\begin{array}{rcll}
a_t & \leq & \hat{a}_t & t = 0, 1, 2, \ldots, T-1 \qquad (91) \\
-a_t & \leq & \hat{a}_t & t = 0, 1, 2, \ldots, T-1 \qquad (92) \\
x_0 & = & 0 & (93) \\
v_0 & = & 0 & (94) \\
x_{t+1} & = & x_t + v_t & t = 0, 1, 2, \ldots, T-1 \qquad (95) \\
v_{t+1} & = & v_t + a_t & t = 0, 1, 2, \ldots, T-1 \qquad (96) \\
x_T & = & 1 & (97) \\
v_T & = & 0 & (98)
\end{array}
$$

The second formulation for this problem has as its objective function to minimize the following

$$
\text{Max}_{t=0,1,2,\ldots,T-1} |a_t| \qquad (99)
$$

subject to the same constraints as before. This can be formulated as a linear program by defining z to be

$$
z = \text{Max}_{t=0,1,2,\ldots,T-1} |a_t| \,. \qquad (100)
$$

With the minimizing program of

$$
\text{Minimize} \quad z \qquad (101)
$$

subject to

$$
\begin{array}{rcll}
z & \geq & a_t & t = 0, 1, 2, \ldots, T-1 \qquad (102) \\
z & \geq & -a_t & t = 0, 1, 2, \ldots, T-1 \qquad (103) \\
x_0 & = & 0 & (104) \\
v_0 & = & 0 & (105) \\
x_{t+1} & = & x_t + v_t & t = 0, 1, 2, \ldots, T-1 \qquad (106) \\
v_{t+1} & = & v_t + a_t & t = 0, 1, 2, \ldots, T-1 \qquad (107) \\
x_T & = & 1 & (108) \\
v_T & = & 0 & (109)
\end{array}
$$

Exercise 1.7 (The moment problem)

To find a lower bound on $E[Z^4] = \sum_{k=0}^{K} k^4 p_k$ consider the following linear program

$$
\text{Minimize} \sum_{k=0}^{K} k^4 p_k \qquad (110)
$$

subject to

$$\sum_{k=0}^{K} p_k = 1 \tag{111}$$

$$\sum_{k=0}^{K} k p_k = E[Z] \tag{112}$$

$$\sum_{k=0}^{K} k^2 p_k = E[Z^2] \tag{113}$$

$$p_k \geq 0 \tag{114}$$

Where the unknown variables to minimize over are p_k for $k = 1, 2, \ldots, K$. Equation 111 expresses that p_k is a probability mass function, Equation 112 expresses the known mean of the distribution, and Equation 113 expresses the known variance of the distribution. In terms of a standard form formulation of the problem we have

$$\text{Minimize} \sum_{k=0}^{K} k^4 p_k \tag{115}$$

subject to

$$\sum_{k=0}^{K} p_k \geq 1 \tag{116}$$

$$-\sum_{k=0}^{K} p_k \geq -1 \tag{117}$$

$$\sum_{k=0}^{K} k p_k \geq E[Z] \tag{118}$$

$$-\sum_{k=0}^{K} k p_k \geq -E[Z] \tag{119}$$

$$\sum_{k=0}^{K} k^2 p_k \geq E[Z^2] \tag{120}$$

$$-\sum_{k=0}^{K} k^2 p_k \geq E[Z^2] \tag{121}$$

$$p_k \geq 0 \tag{122}$$

The problem of maximizing the fourth moment is the same as the above but with the objective function replaced with

$$-\sum_{k=0}^{K} k^4 p_k \,. \tag{123}$$

Exercise 1.8 (Road lighting)

I'll formulate this problem so as to minimize the total cost in operating all of the lamps (per unit time). As such let c be the cost per unit power to operate a single lamp. Then a formulation where we desire to minimize the total cost of all lamps could have an objective function like

$$c \sum_{i=1}^{m} p_i. \tag{124}$$

In this problem we are constrained by the requirement that in each interval I_i we meet the required intensity or

$$I_i = \sum_{j=1}^{m} a_{ij} p_j \geq I_i^*, \tag{125}$$

combined with the additional requirement that the power in each lamp is nonnegative i.e. $p_j \geq 0$. In standard form the problem we have amounts to

$$\text{Minimize} \quad c \sum_{i=1}^{m} p_i \tag{126}$$

subject to the constraints that

$$\sum_{j=1}^{m} a_{ij} p_j \geq I_i^* \quad \text{for} \quad 1 \leq i \leq n \tag{127}$$

$$p_j \geq 0 \quad \text{for} \quad 1 \leq j \leq m \tag{128}$$

where the unknowns are the m values of the lamp's power p_j.

Exercise 1.9

The main constraints in this problem are that

- Each school can not exceed its capacity in each grade

- Each neighborhood must have all of its students enrolled in a school

To solve this problem we introduce the unknowns x_{ijg} represent the number of students from neighborhood i that go to school j, in grade g. Then to minimize the total distance traveled by all students our objective function is

$$c \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{g=1}^{G} x_{ijg} d_{ij}, \tag{129}$$

where d_{ij} is the distance between neighborhood i and school j. The constant c in the above is added for generality but could just as well be ignored. The above objective function is minimized subject to the following constraints

$$\sum_{i=1}^{I} x_{ijg} \leq C_{jg} \quad \text{for} \quad 1 \leq j \leq J \quad \text{and} \quad 1 \leq g \leq G \tag{130}$$

$$\sum_{j=1}^{J} x_{ijg} \geq S_{ig} \quad \text{for} \quad 1 \leq i \leq I \quad \text{and} \quad 1 \leq g \leq G \tag{131}$$

$$x_{ijg} \geq 0 \tag{132}$$

The first constraint is the class capacity constraint and the second constraint is that all the students from neighborhood i belonging to grade g go to some school.

Exercise 1.10 (Production and inventory planning)

In addition to the variables x_i as the number of units produced during month i, introduce s_i as the amount of inventory in storage at the *end* of month i. Then for during month, to meet the demand d_i either by monthly production x_i or from storage from a previous month s_{i-1} giving the following constraints

$$s_0 = 0 \tag{133}$$
$$x_i + s_{i-1} \geq d_i \quad \text{for} \quad i = 1, 2, \ldots, 12. \tag{134}$$

The amount in storage at the end of month i is the amount in storage at month $i-1$ plus the amount produced x_i minus the demand during month i giving

$$s_i = x_i + s_{i-1} - d_i \quad \text{for} \quad i = 1, 2, \ldots, 12. \tag{135}$$

We incur costs for either changing the production level or having product in storage. These constraints produce the following program.

$$\text{Minimize} \left(c_1 \sum_{i=1}^{11} s_i + c_2 \sum_{i=1}^{11} |x_{i+1} - x_i| \right) \tag{136}$$

subject to

$$s_0 = 0 \tag{137}$$
$$s_i = x_i + s_{i-1} - d_i \quad \text{for} \quad i = 1, 2, \ldots, 12 \tag{138}$$
$$x_i + s_{i-1} \geq d_i \quad \text{for} \quad i = 1, 2, \ldots, 12 \tag{139}$$
$$x_i \geq 0. \tag{140}$$

In the above, our unknowns to perform the minimization over are given by x_i and s_i for $i = 1, 2, \ldots, 12$ and $i = 1, 2, \ldots, 11$ respectively. Note that the summations in the above are from 1 to 11, reflecting the idea that there is no cost for storage of product for the 12th month.

To remove the absolute values in the above we can define $a_i = |x_{i+1} - x_i|$ and our program is then equivalent to

$$\text{Minimize} \left(c_1 \sum_{i=1}^{11} s_i + c_2 \sum_{i=1}^{11} a_i \right) \tag{141}$$

subject to

$$
\begin{align}
s_0 &= 0 \tag{142} \\
s_i &= x_i + s_{i-1} - d_i \quad \text{for} \quad i = 1, 2, \ldots, 12 \tag{143} \\
x_i + s_{i-1} &\geq d_i \quad \text{for} \quad i = 1, 2, \ldots, 12 \tag{144} \\
x_i &\geq 0 \tag{145} \\
a_i &\geq x_{i+1} - x_i \tag{146} \\
a_i &\geq -(x_{i+1} - x_i). \tag{147}
\end{align}
$$

Where our unknowns are now the 12 values of x_i, the 11 values of a_i, and the 11 values of s_i.

Exercise 1.11 (Optimal currency conversion)

We will assume that we can perform currency conversions at *discrete* times throughout our day. Take as our unknowns $c_{i,j,k}$ representing the amount of currency i converted to currency j during the kth time step. In addition, introduce an auxiliary variable $x_{i,k}$ to be the total amount of the ith currency at timestep k. Then after K timesteps we desire to maximize the amount of the Nth currency or $x_{N,K}$. Our linear program then becomes

$$\text{Maximize} \left(x_{N,K} \right) \tag{148}$$

subject to

$$
\begin{align}
c_{i,j,k} &\geq 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, K - 1 \tag{149} \\
c_{i,j,k} &\leq x_{i,k} \tag{150} \\
x_{i,k+1} &= x_{i,k} - \sum_{j=1}^{N} c_{i,j,k} \tag{151} \\
x_{j,k+1} &= x_{j,k} + \sum_{i=1}^{N} c_{i,j,k} r_{i,j} \tag{152} \\
\sum_{j=1}^{N} \sum_{k=1}^{K} c_{i,j,k} &\leq u_i \tag{153} \\
x_{1,0} &= B \tag{154} \\
x_{i,0} &= 0 \quad \text{for} \quad i = 2, 3, \ldots, K - 1 \tag{155}
\end{align}
$$

Each equation in the above has a specific meaning related to this problem. For instance equation 149 implies that the amount of any currency converted at each timestep is nonnegative, equation 150 enforces the constraint that we cannot convert more of the i currency at

timestep k than we currently have, equation 151 states that the amount of the i the currency left at the $k+1$st timestep is reduced by the current currency conversions, equation 152 states that the conversion of $c_{i,j,k}$ of currency i produces $c_{i,j,k}r_{i,j}$ amount of currency j, and finally equation 153 expresses the constraint that the maximal amount of currency i that can be converted in a single day is u_i. The total amount of i converted is computed by summing the amount of currency i converted at each timestep in the expression $\sum_{j=1}^{N} \sum_{k=1}^{K} c_{i,j,k}$.

Exercise 1.12 (Chebychev center)

Problem skipped ... would anyone like to make an attempt at solving this?

Exercise 1.13 (Linear fractional programming)

The problem we are asked to solve is given by

$$\text{Minimize} \quad \frac{c'x + d}{f'x + g} \tag{156}$$

subject to

$$Ax \leq b \tag{157}$$
$$f'x + g > 0. \tag{158}$$

Note that this is not strictly a linear programming problem. If we are given, a-priori, the fact that the optimal function value lies in the range $[K, L]$ then we can derive the optimal solution using linear programming and a bisection like method. As such, following the hint, define a subroutine that returns "yes" or "no" depending on if the minimal functional value is less than or equal to an input t. The functional less than or equal to t is equivalent to the following set of equivalent expressions

$$\frac{c'x + d}{f'x + g} \leq t \tag{159}$$
$$c'x + d \leq f'xt + tg \tag{160}$$
$$(c + ft)'x + (d - tg) \leq 0 \tag{161}$$

Which *is* a linear constraint when t is a given constant. Thus our subroutine should return "yes" if a feasible solution to the following linear program is found to exists.

$$\text{Minimize} \quad (c'x + d) \tag{162}$$

Subject to

$$Ax \leq b \tag{163}$$
$$c'x + d \leq f'xt + tg, \tag{164}$$

and return "no" if no feasible solution exists.

Using this subroutine we could iteratively search for the minimum of the linear fractional formulation in the following way. As a first step set t to be the midpoint of the range of our linear fractional functional i.e. $t = \frac{K+L}{2}$ and call our subroutine (defined above) looking for a yes/no answer to the existence of an optimal solution. If one does exist then this means that there exist feasible solutions to the linear fractional program that has functional values *less than* the midpoint given by $\frac{L+M}{2}$. More importantly the optimal linear fractional value is properly bracked by $[L, \frac{L+M}{2}]$. If a feasible solution does not exist then the linear fractional program must have functional values *greater than* the midpoint $\frac{L+M}{2}$ and its optimal solution is bracketed by $[\frac{L+M}{2}, M]$. In using a value of t corresponding to the midpoint we have been able to reduce our initial interval by 2. Depending on the result of the first experiment we can apply this routine again to producing an interval one quarter the size of the original. Continuing in this fashion we can bracket our optimal solution in as fine an interval as desired. I should note that in the formulation just given we have a "loop invariant" where by the left endpoint of our routine will have the value "no" when used as an argument to our subroutine and the right endpoint should have the value "yes" when used as an argument to our subroutine.

Exercise 1.14

Part (a): Let the variables p_1 and p_2 represent the amount of product 1 and 2 produced by our company respectively. Then we have that the amount of produce must be positive or $p_1 \geq 0$ and $p_2 \geq 0$. In addition, the machine hours constraint imposes

$$3p_1 + 4p_2 \leq 20000 . \tag{165}$$

The production cost constraint is a bit trickier. Imposing a constant constraint of 4000 on the production costs we would have

$$3p_1 + 2p_2 \leq 4000 . \tag{166}$$

But we are told that upon sale of product 1 45% is reinvested and provides production capital. In the same way, upon the sale of produce 2, 30% of the revenue becomes production capital. Since product 1 and 2 sell for $6 and $5.4 respectively, this modifies equation 166 to become

$$3p_1 + 2p_2 \leq 4000 + 0.45(6p_1) + 0.3(5.4p_2) . \tag{167}$$

This immediate reinvestment into production capital reduces the total profit (net income minus expenditures) to

$$0.65(6p_1) + 0.7(5.4p_2) = 3.9p_1 + 3.78p_2 . \tag{168}$$

So in total this gives the following linear program

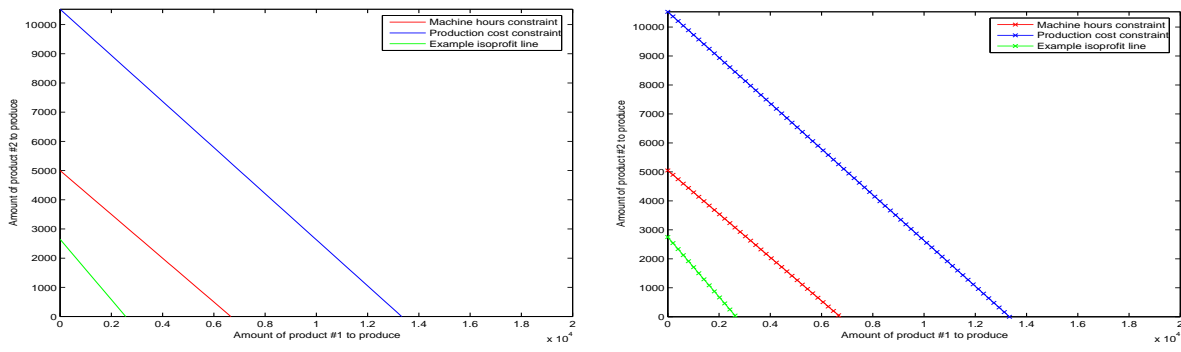$$\text{Maximize} \quad 3.9p_1 + 3.78p_2 \tag{169}$$

Figure 1: Constraint regions for Exercies 1.14. To the left is the constraint region for parts (a) and (b). To the right is the constraint region for part (c). Note in both cases that the production cost constraint is *not* active and thus the machine hours constraint is the bottleneck for further profit increase. Also note that the difference between the two constraint regions is very small.

subject to

$$p_1 \geq 0 \tag{170}$$
$$p_2 \geq 0 \tag{171}$$
$$3p_1 + 4p_2 \leq 20000 \tag{172}$$
$$0.3p_1 + 0.38p_2 \leq 4000 \tag{173}$$

Part (b): To solve this problem using graphical means we plot each constraint in the p_1 and p_2 plane. In figure 1 (left) the constraints above along with a single isoprofit line (in green) are plotted. From this plot we can see that the production cost constraint is never actually enforced since the machine hour constraint is more restrictive. Since we know that our minimizations should occur at a "corner" of the constrained set we can enumerate these as $(p_1, p_2) = (0, 0), (20000/3, 0), (0, 5000)$. Evaluating our profit function we find that the corresponding profits are $(0, 26000, 18900)$. Thus the largest profit occurs when $(p_1, p_2) = (20000/3, 0)$.

Part (c): If we increase the machine hours by 200 our constraint on the number of machine hours becomes

$$3p_1 + 4p_2 \leq 20200 \tag{174}$$

with our net profit now reduced, since we must pay for these repairs. As such our functional to maximize becomes

$$3.9p_1 + 3.78p_2 - 400 \,. \tag{175}$$

In figure 1 (right) we can see the results of a plot similar to that given i Part (b), but with the differing constraint. As in Part (b) the portion of the (p_1, p_2) plane to check for maximal profits are located at $(p_1, p_2) = (0, 0), (20200/3, 0), (0, 20200/4)$, which give profits of $(-400, 25860, 18689)$ respectively. We see that the addition of 200 hours does indeed modify the maximal profit values but not their locations. From the enumeration above the point of largest profit is given by $(20200/3, 0)$. That profit value is actually *less* than that given in Part (a) and thus the investment should *not* be made.

Exercise 1.15

Part (a): Introduce the variable p_1 and p_2 to be the amount of product 1 and product 2 respectively produced daily. By definition we must have

$$p_1 \geq 0 \tag{176}$$
$$p_2 \geq 0. \tag{177}$$

Using as a definition of profit the sale price minus the cost of materials we have

$$\text{Profit} = 9p_1 + 8p_2 - 1.2p_1 - 0.9p_2 = 7.8p_1 + 7.1p_2, \tag{178}$$

as the objective function to maximize. We must not exhaust our daily capacity for labor and testing giving constraints of

$$\frac{1}{4}p_1 + \frac{1}{3}p_2 \leq 90 \tag{179}$$
$$\frac{1}{8}p_1 + \frac{1}{3}p_2 \leq 80 \tag{180}$$

Part (b): To include modification i in our linear programming framework, we can introduce a variable n, as the number of overtime assembly labor to be scheduled. In this case our single hourly constraint expression equation 179 now becomes three constraints

$$\frac{1}{4}p_1 + \frac{1}{3}p_2 \leq 90 + n \tag{181}$$
$$n \geq 0 \tag{182}$$
$$n \leq 50. \tag{183}$$

With this much overtime scheduled the objective function now becomes

$$7.8p_1 + 7.1p_2 - 7n. \tag{184}$$

Our unknowns now include n and are (p_1, p_2, n).

The incorporation of modification ii (a possible discount on materials) we have that the raw materials would cost

$$r = 1.2p_1 + 0.9p_2 \tag{185}$$

if $r < 300$, otherwise

$$r = 0.9(1.2p_1 + 0.9p_2) \tag{186}$$

The functional to maximize is still given by

$$9p_1 + 8p_2 - r. \tag{187}$$

The constraints on the number of hours of labor and testing remain the same. I'm not sure how to exactly incorporate this problem modification into a linear programming formulation but what could be done, is to solve the problem *without* the discount (i.e. the original problem formulation) and the problem assuming that the discount was *always* true. With these optimal solutions in hand we can check which of them is consistent with the assumption of discount or not. For instance, in in the non discounted case if our formulation predicted a cost of materials greater than 300 we are not consistent with this situation. In that case the formulation including the discount in materials should yield a consistent and more optimal solution.

Exercise 1.16

Let x_1, x_2, x_3 be the number of times to "exercise" process 1, 2, and 3 respectively. As constraints on these variables we have the obvious nonnegative constraint $x_i \geq 0$ $i = 1, 2, 3$. In addition, we have the following set of constraints to not use up all of our crude oil

$$3x_1 + 1x_2 + 5x_3 \leq 8 \, 10^6 \tag{188}$$

$$5x_1 + 1x_2 + 3x_3 \leq 5 \, 10^6 . \tag{189}$$

Introducing auxiliary variables n_g and n_o for the number of barrels of gasoline and oil produced we see that in terms of our unknowns x_i we have

$$n_g = 4x_1 + 1x_2 + 3x_3 \tag{190}$$

$$n_0 = 3x_1 + 1x_2 + 4x_3 . \tag{191}$$

Our objective function to maximize is given by the profit minus expenditures giving

$$38n_g + 33n_o - 51x_1 - 11x_2 - 40x_3 . \tag{192}$$

Replacing n_g and n_o in terms of x_1, x_2, and x_3 we obtain the following linear program

$$\text{Maximize } (200x_1 + 60x_2 + 206x_3) \tag{193}$$

subject to

$$x_1 \geq 0 \tag{194}$$

$$x_2 \geq 0 \tag{195}$$

$$x_3 \geq 0 \tag{196}$$

$$3x_1 + 1x_2 + 5x_3 \leq 8 \, 10^6 \tag{197}$$

$$5x_1 + 1x_2 + 3x_3 \leq 5 \, 10^6 . \tag{198}$$

Exercise 1.17 (Investment under taxation)

To solve this problem, introduce the variables x_i to be the number of shares of stock i to be sold in raising the required K capital (we have s_i shares initially). With each transaction the capital gains tax is 30% of the profit made during the life time of the stocks and is given by

$$c_i = 0.3(q_i x_i - x_i p_i) = 0.3x_i(q_i - p_i) . \tag{199}$$

Where we have taken the difference between the current total stock value price and the originally purchased stock value. In a similar manner, the transaction fees on this sale is given by 1% of the total stock sale and is given by

$$t_i = 0.01q_i x_i . \tag{200}$$

In total, each transactions will produce an amount of capital equal to the difference between the current sale price and the sum of the capital gains tax and transaction fees given by

$$q_i x_i - c_i - t_i = q_i x_i - 0.3x_i(q_i - p_i) - 0.01q_i x_i = 0.69q_i x_i + 0.3x_i p_i . \tag{201}$$

Our constraint is to raise a total of K capital while maximizing our portfolio for the next year. The value of the portfolio next year is given by

$$\sum_{i=1}^{n}(s_i - x_i)r_i \,. \tag{202}$$

Our linear program is then given by

$$\text{Maximize} \quad \sum_{i=1}^{n}(s_i - x_i)r_i \tag{203}$$

subject to

$$s_i \;\geq\; x_i \tag{204}$$
$$x_i \;\geq\; 0 \tag{205}$$
$$\sum_{i=1}^{n}(0.69q_i + 0.3p_i)x_i \;\geq\; K \tag{206}$$

Here in addition to the captial requirement in equation 206, equation 204 represents the constraint that we can not sell more shares than we currently hold and equation 205 represents the constraint that we can only *sell* stock i (not buy).

Exercise 1.18

First assume that the set of vectors x_i *are* linearly independent. This means that no set of real coefficients a_i (not all of them zero) exist such that

$$\sum_{k=1}^{N} a_k x^k = 0 \,. \tag{207}$$

From this we see that a single vector could not be written as a linear combination of the others. For if it could we would have an expression like

$$x^l = \sum_{k=1;k\neq l}^{N} a_k x^k \tag{208}$$

which is equivalent to

$$\sum_{k=1}^{N} a_k x^k = 0 \,. \tag{209}$$

with $a_l = -1$, in contradiction to the linear independence assumption we started with.

In the other direction, if no vector can be expressed as a linear combination of the others then it is not possible to find a set of non-zero a_k such that

$$\sum_{k=1}^{N} a_k x^k = 0 \,. \tag{210}$$

For if we could then we could certainly express one of the vectors as a linear superposition of the others, in direct contradiction to the assumption that this is not possible.

Exercise 1.19

Assume that we are given a set of vectors x_i that form a basis in $\mathcal{R}^n$ and we wish to express an arbitrary vector y in terms of a superposition of x_i. This means that

$$y = \sum_{i=1}^{N} \alpha_i x_i .\tag{211}$$

We can recognize this summation to be equivalent to a matrix multiplication when we construct a matrix X with *columns* the vectors x_i. Then the above superposition is equivalent to

$$y = X\alpha \tag{212}$$

where α is a concatenation of the coefficients α_i. As such, we see that these coefficients can be obtained by matrix inversion or

$$\alpha = X^{-1}y \tag{213}$$

We know that X is invertible by Theorem 1.2 from the book.

Exercise 1.20

Not worked. Would anyone like to attempt this problem?

Chapter 2

Exercise 2.1

The definition of a polyhedron is a set that can be discribed in the form $\{x \in \mathbb{R}^n | Ax \geq b\}$ where A is an *mxn* matrix and b a *mx1* column vector

Part (a): Let $A(\theta)$ be defined as

$$A(\theta) = \begin{bmatrix} -\cos(\theta) & -\sin(\theta) \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{214}$$

Then this set is a polyhedron.

Part (b): No, this set is not a polyhedron because it cannot be written in the requied form.

Part (c): Pick A finite and b infinite, then no points will satistfy the inequality $Ax \geq b$ and this set is then the empty set, showing that the null set is a polyhedra.

Chapter 8: The ellipsoid method

Exercise 8.1

We can simplify notation if we define a vector v in terms of the vector u and e_1 as

$$v = u + ||u||e_1 \, .$$

Then the matrix R in terms of v is given by

$$R = 2\frac{vv'}{||v||^2} - I \, .$$

To show that $R'R = I$ we explicitly compute this product. We find (using the fact that $(v'v)' = v'v$) that

$$
\begin{aligned}
R'R &= \left(2\frac{vv'}{||v||^2} - I\right)\left(2\frac{vv'}{||v||^2} - I\right) \\
&= \frac{4}{||v||^4}(vv')(vv') - \frac{4}{||v||^2}vv' + I \, .
\end{aligned}
$$

Since matrix multiplication is associative we can simplify the first term in the above as

$$(vv')(vv') = v(v'v)v' = ||v||^2 vv' \, .$$

So that we find our product $R'R$ equal to

$$R'R = \frac{4}{||v||^4}||v||^2 vv' - \frac{4}{||v||^2}vv' + I = I \, ,$$

as expected.