2025.11.18 HW7 习题四.

B12. (1) $E(\xi_2)=\frac{2a}{15}=\frac{4}{3}\Rightarrow a=10.$

(2) $E(\xi_9)=\frac{9a}{15}=6$

B15. 记X：最远的两个点的距离.

则 $E(X)=\int_0^1 A_n^2\ (1-x)\,x^{n-2}\cdot x\,dx$

$=n(n-1)\cdot\frac{1}{(n+1)\cdot n}=\frac{n-1}{n+1}$

B

18. 无放回：$P(\xi_2=0)=\frac{5}{15}\times\frac{4}{14}=\frac{2}{21}$

$P(\xi_2=1)=\frac{5}{15}\times\frac{10}{14}+\frac{10}{15}\times\frac{5}{14}=\frac{10}{21}$

$P(\xi_2=2)=\frac{10}{15}\times\frac{9}{14}=\frac{9}{21}$

$\therefore E(\xi_2)=\frac{2}{21}\times\frac{16}{9}+\frac{10}{21}\times\frac{1}{9}+\frac{9}{21}\times\frac{4}{9}$

$=\frac{78}{189}=\frac{26}{63}$

B22. (1) $P(X+Y\geq 1)=1-\frac{1}{2}\times\frac{1}{2}=\frac{3}{4}$

(2) $X\cdot(-1)^Y$ 的分布律为

记 $Z=X\cdot(-1)^Y$

Z	0	1	-1
$P(Z)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

故 $E(Z)=0$

$\mathrm{Var}(Z)=\frac{1}{4}\times 1+\frac{1}{4}\times 1=\frac{1}{2}$

B25.

(1) $\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)$

$=\int_{-1}^{1}\int_{-1}^{1}\frac{xy}{4}(1+xy)\,dx\,dy-\int_{-1}^{1}x\left(\int_{-1}^{1}\frac{1}{4}(1+xy)\,dy\right)dx\cdot$

$\int_{-1}^{1}y\left(\int_{-1}^{1}\frac{1}{4}(1+xy)\,dx\right)dy$

$=\frac{1}{9}-0\times 0=\frac{1}{9}.$

$D(X)=E(X^2)-E^2(X)=\int_{-1}^{1}x^2\left(\int_{-1}^{1}\frac{1}{4}(1+xy)\,dy\right)dx=\frac{1}{3}$.

同理 $D(Y)=\frac{1}{3}$.

$\therefore \rho_{XY}=\dfrac{\frac{1}{9}}{\sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{3}}}=\frac{1}{3}$. $\therefore$ 正相关. 不独立.

(2) $Cov(X^2,Y^2)=E(X^2Y^2)-E(X^2)E(Y^2)$

$=\frac{1}{9}-\frac{1}{3}\times\frac{1}{3}=0$.

$\therefore \rho_{XY}=0$. $\therefore$ 不相关.

又 $f(x^2,y^2)=f_X(x^2)\cdot f_Y(y^2)$. $\therefore$ 独立.

B26. $\begin{cases} P(X=k)=\left(\frac{1}{6}\right)^k\cdot\left(\frac{5}{6}\right)^{n-k}\cdot C_n^k \\ P(Y=k)=\left(\frac{1}{6}\right)^k\cdot\left(\frac{5}{6}\right)^{n-k}\cdot C_n^k. \end{cases}$

记 X: "1点朝上"的次数; Y: "6点朝上"的次数.

则 $Cov(X,Y)=E(XY)-E(X)E(Y)$

$=\sum_{i=0}^{n}\sum_{j=0}^{n-i} ij\,P(X=i,Y=j)-\frac{n}{6}\cdot\frac{n}{6}$.

$=\frac{n(n-1)}{36}-\frac{n^2}{36}=-\frac{n}{36}$　　$D(X)=n\cdot\frac{1}{6}\cdot\frac{5}{6}=\frac{5n}{36}$

$\therefore \rho_{XY}=\dfrac{-\frac{n}{36}}{\sqrt{\frac{5n}{36}}\cdot\sqrt{\frac{5n}{36}}}=-\frac{1}{5}$. 负相关

B29. (1) $P(\xi=k)=P(Y=1)\cdot P(X=\xi|Y=1)+$

$P(Y=-1)\cdot P(X=-\xi|Y=-1)$

$=p\cdot P(X=\xi)+(1-p)P(X=-\xi)$

$=P(X=\xi)$

$=P(X=k)$

$\therefore \xi\sim N(0,1)$

(2) $Cov(X,\xi)=E(X\xi)-E(X)E(\xi)$

$=E(X^2Y)$

$=E(X^2)\cdot E(Y)$

而 $E(X^2)=D(X)+E^2(X)=1$.

$E(Y)=p-(1-p)=2p-1$

$\therefore Cov(X,\xi)=2p-1$

$\Rightarrow \rho_{X\xi}=2p-1$

i) 当 $p=\frac{1}{2}$ 时, X 与 ξ 不相关;

ii) 当 $p>\frac{1}{2}$ 时, X 与 ξ 正相关;

iii) 当 $p<\frac{1}{2}$ 时, X 与 ξ 负相关. 固定t.

而 $P(\xi=t|X=p)$ 与 p 的取值有关

i) 若 $t=p$, 则 $P(\xi=t|X=p)=p$.

$t=-p$. $1-p$

$t\neq p$ 且 $t\neq -p$ 0

$\therefore \xi$ 与 p 不独立.

B32. (1) $\xi \sim N(-b, a^2+4b^2)$

$\eta \sim N(a, 4a^2+b^2)$

$\therefore$ 标准化变量 $\xi^* = \frac{\xi+b}{\sqrt{a^2+4b^2}}$

$$\eta^* = \frac{\eta-a}{\sqrt{4a^2+b^2}}.$$

$$Cov(\xi,\eta) = E(\xi\cdot\eta) - E(\xi)E(\eta)$$

$$= E(a^2XY - abX^2 - abY^2 + b^2XY) + ab$$

$$= -ab - 5ab + ab = -5ab$$

$$\therefore \rho_{\xi,\eta} = \frac{-5ab}{\sqrt{a^2+4b^2}\cdot\sqrt{4a^2+b^2}}$$

(2) $E(\xi) = -b$.

$$D(\xi) = a^2 Var(X) + b^2 Var(Y) - 2ab\cdot Cov(X,Y)$$

$$= a^2+4b^2 - 4ab\cdot(\tfrac{1}{2}) = a^2+4b^2-2ab$$

$\therefore \xi$ 的变异系数为 $\frac{\sqrt{a^2+4b^2-2ab}}{-b}$.

(3) $E(\eta) = a$. 由正态分布知中位数就是 a.

(4)

$\because Cov(X,Y) = -1 \cdot \sqrt{1} \cdot \sqrt{4} = -2$.

$E(XY) = Cov(X,Y) + E(X)E(Y) = -2$.

$\therefore E(\xi\eta) = -6ab - 2a^2 - 2b^2$

$\Rightarrow Cov(\xi,\eta) = -6ab - 2a^2 - 2b^2 + ab$

$= -(2a+b)(a+2b)$

在$b=-2a$或$a=-2b$时，ξ,η不相关，且相互独立

否则 $-\dfrac{(2a+b)(a+2b)}{b^2} = -\left(\dfrac{2a}{b}+1\right)\left(\dfrac{a}{b}+2\right)$

i) 若$-2 < \dfrac{a}{b} < -\dfrac{1}{2}$，则$\xi,\eta$正相关，且不独立.

ii) 否则，ξ,η负相关，且不独立.

B33. (1) $X_1 \sim N(0,1)$ $X_2 \sim N(0,16)$ $X_3 \sim N(1,4)$.

(2) $Cov(X_1,X_2)=2$，$Cov(X_1,X_3)=-1$，$Cov(X_2,X_3)=0$.

$\therefore X_1$与X_2正相关且不独立.

X_2与X_3不相关，且独立

X_1与X_3负相关，且不独立

$\therefore$进而，X_1, X_2, X_3不独立

(3) Y_1, Y_2 均服从正态分布.

$E(Y_1) = E(X_1) - E(X_2) = 0.$

$E(Y_2) = E(X_3) - E(X_1) = 1.$

$D(Y_1) = 1 + 16 - 2\times 2 = 13.$

$D(Y_2) = 4 + 1 - 2\times(-1) = 7$

又 $E(X_1^2) = 1$, $E(X_1X_2) = 2$, $E(X_1X_3) = -1.$

$E(X_2, X_3) = 0.$

$\therefore Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$

$= E(X_1X_3) - E(X_1^2) - E(X_2X_3) + E(X_1X_2) = 0$

$\therefore Y = (Y_1, Y_2)^T$ 的分布为 $N(a', B')$, 其中

$a' = (0, 1)^T$, $B' = \begin{pmatrix} 13 & 0 \\ 0 & 7 \end{pmatrix}$