概率论和数理统计

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Preface

- 预修课程：微积分一、二，要求会二重积分的计算;
- 课程教材：概率论与数理统计 (第二版), 高等教育出版社, 2023;
- 联系方式: hwmath2003@zju.edu.cn;
- 资源分享: 学在浙大、钉钉群
  (原始课件、作业本编号、作业题号、上课课件);
- 线上参考资源: 中国大学 MOOC(概率论与数理统计—浙大班 SPOC, 第十期);
  https://www.icourse163.org/spoc/learn/ZJU-1206351805?tid
  智慧树 (高等数理统计)
  （注：需待第三轮选课结束，确定教学班名单后方能进入学习，后续再通知）
- 成绩构成: 平时成绩 60%(包括作业及到课率等平时表现, 小测, 课程大作业, 等等)+ 期末考试成绩 40%

  总评成绩说明：期末考试成绩设置最低分, 低于期末最低分或总评低于 60 均为不及格.

Chapter 1 Introduction to Probability

关键词 (Keywords):

- 样本空间 (Sample space)
- 随机事件 (Events)
- 频率和概率 (Frequency and Probability)
- 等可能概型 (Classical Probability)
- 条件概率 (Conditional Probability)
- 独立事件 (Independent Events)

§1.1 样本空间和随机事件 (Sample Space and Events)

自然界与社会生活中的两类现象：确定性现象，不确定性现象.

♠ 确定性现象 (deterministic phenomenon)：结果确定；

♠ 不确定性现象 (indeterministic phenomenon)：结果不确定.

Example 1

- 向上抛出的物体会下落　　确定
- 抛一枚均匀的硬币，落地时正面朝上　　不确定
- 打靶，击中靶心　　不确定
- 买了彩票会中奖　　不确定

不确定性现象:

- 个别现象
- 随机现象 (random phenomenon): 在个别试验中其结果呈现出不确定性, 但在大量重复试验中其结果又具有统计规律性 (statistical regularity). ——研究对象

概率论与数理统计是研究随机现象数量规律性的学科.

对随机现象的观察、记录、实验统称为随机试验 (random experiment). 它具有以下特性:

- 可以在相同条件下重复进行;
- 每次试验的可能结果不止一个, 并且事先知道可能出现的所有结果;
- 完成试验之前不能确定哪一个结果会出现.

Example 2

- 一枚硬币抛一次, 观察正反面出现的情况;
- 对某路公交车某停靠站登记下车人数;
- 对某批电子产品测试其输入电压;
- 对听课人数进行登记.

(一) 样本空间 (Sample Space)

Definition 1 (样本空间)

随机试验 E 的所有结果构成的集合称为该随机试验 E 的样本空间 (the sample space of the experiment), 记为

$$S = \{s_1, s_2, \ldots, s_n\}.$$

有时记为 $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$. 称 S(或 Ω) 中的元素 s_i(或 ω_i) 为样本点 (sample point). 一个元素的单点集称为基本事件 (elementary event).

Definition 1 (Sample Space)

The collection of all possible outcomes of an experiment is called the sample space of the experiment.

The sample space of an experiment can be thought of as a set, or collection, of different possible outcomes; and each outcome can be thought of as a point, or an element, in the sample space. Similarly, events can be thought of as subsets of the sample space.

Example 3

- 一枚硬币抛一次，观察正反面出现的情况; $S = \{\text{正面}, \text{反面}\}$;
- 一颗骰子抛一次，观察其点数情况; $S = \{1, 2, 3, 4, 5, 6\}$;
- 记录一城市一日中发生交通事故次数; $S = \{0, 1, 2, \ldots\}$;
- 记录一批产品的寿命 x, $S = \{x | a \leq x \leq b\}$;
- 记录某地一昼夜最高温度 x，最低温度 y
  $S = \{(x, y) | T_0 \leq y \leq x \leq T_1\}$.

注: 样本空间的元素与试验目的有关.

样本空间在试验完成前是已知的.

随机事件 (Events)

Definition 2 (随机事件)

一般我们称 S 的子集 A 为随机试验 E 的随机事件 A，简称事件 (event) A. 当且仅当 A 所包含的一个样本点发生则称事件 A 发生.

随机事件有如下特征:

♠ 任意一事件 A 是相应的样本空间 S 的一个子集, 其关系可用维恩 (Venn) 图来表示;

♠ 事件 A 发生当且仅当 A 中的某一个样本点出现;

♠ 事件 A 的表示可用集合, 也可用语言来表示.

Example 4 (Rolling a Dice)

When a six-sided dice is rolled, the sample space can be regarded as containing the six numbers 1, 2, 3, 4, 5, 6, each representing a possible side of the dice that shows after the roll. Symbolically, we write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

One event A is that an even number is obtained, and it can be represented as the subset $A = \{2, 4, 6\}$. The event B that a number greater than 2 is obtained is defined by the subset $B = \{3, 4, 5, 6\}$.

Events are sets:

- Can describe in words
- Can describe in notation
- Can describe with Venn diagrams

Experiment: toss a coin 3 times.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Event:
You get 2 or more heads $= \{HHH, HHT, HTH, THH\}$.

Example 5

观察 89 路公交车浙大紫金港站候车人数.

$$S = \{0, 1, 2, \ldots\};$$

$$A = \{\text{至少有 10 人候车}\} = \{10, 11, 12, \ldots\}.$$

A 为随机事件, A 可能发生, 也可能不发生.

- 由一个样本点组成的单点集, 称为基本事件;
- 如果将 S 亦视作事件，则每次试验 S 总是发生，故又称 S 为必然事件;
- 为方便起见，记 $\varnothing$ 为不可能事件, $\varnothing$ 不包含任何样本点.

Some events are impossible. For example, when a dice is rolled, it is impossible to obtain a negative number. Hence, the event that a negative number will be obtained is defined by the subset of S that contains no outcomes.

Definition 3 (Empty Set)

The subset of S that contains no elements is called the empty set, or null set, and it is denoted by the symbol $\varnothing$.

In terms of events, the empty set is any event that cannot occur.

事件其本质即为集合.

Finite and Infinite Sets

Some sets contain only finitely many elements, while others have infinitely many elements. There are two sizes of infinite sets that we need to distinguish.

Definition 4 (Countable/Uncountable)

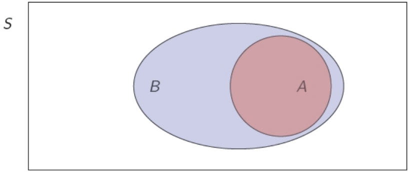
An infinite set A is countable if there is a one-to-one correspondence between the elements of A and the set of natural numbers 1, 2, 3, ... A set is uncountable if it is neither finite nor countable. If we say that a set has at most countably many elements, we mean that the set is either finite or countable.

事件的相互关系及运算 (Relations and Operations of Set Theory)

注：若无特殊说明，以下所提事件均在同一样本空间中.

♠ 事件的关系 (包含、相等)

- $A \subset B$: 事件 A 发生一定导致事件 B 发生. For events, to say that $A \subset B$ means that if A occurs then so does B.
  Venn diagram is often used to portray relationships between sets (events), e.g.,

  特别地, $\varnothing \subset A \subset S$.
- $A = B$: 若 $A \subset B$ 且 $B \subset A$.

Example 6

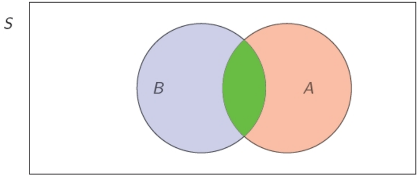
- 记 $A=\{$明天天晴$\}$, $B=\{$明天无雨$\}$ $A \subset B$
- 记 $A=\{$至少有 10 人候车$\}$, $B=\{$至少有 5 人候车$\}$ $A \subset B$
- 抛两颗均匀的骰子, 两颗骰子出现的点数分别记为 x, y. 记 $A=\{x+y$为奇数$\}$, $B=\{$两次的骰子点数奇偶性不同$\}$ $A=B$

♠ 事件的运算

- 事件 A 与事件 B 的和事件, 记为 $A \cup B$,

$$A \cup B = \{x | x \in A \ \text{ or } \ x \in B\}.$$

The union of A and B, denoted by $A \cup B$, is the set of all points in A or B or both (e.g., all colored area).

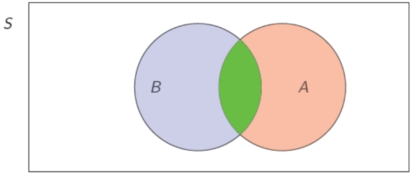


$A \cup B$ 发生, 即事件 A 与事件 B 至少有一个发生. 注意: 不要写成 $A + B$

- 事件 A 与事件 B 的积事件, 记为 $A\cap B$, $A\cdot B$ 或 AB

$$A\cap B=\{x|x\in A\ \text{ and }\ x\in B\}.$$

The intersection of A and B, denoted by $A\cap B$, is the set of all points both in A and B (e.g., green area).



$A\cap B$ 发生, 即事件 A 与事件 B 同时发生.

Definition 5 (Union of Many Sets)

The union of n sets $A_1, A_2, \ldots, A_n$ is defined to be the set that contains all outcomes that belong to at least one of these n sets. The notation for this union is either of the following:

$$A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{i=1}^{n} A_i.$$

Similarly, the union of an infinite sequence of sets $A_1, A_2, \ldots$ is the set that contains all outcomes that belong to at least one of the events in the sequence. The infinite union is denoted by $\bigcup_{i=1}^{\infty} A_i$.

In terms of events, the union of a collection of events is the event that at least one of the events in the collection occurs.

Definition 6 (Intersection of Many Sets)

The intersection of n sets $A_1, A_2, \ldots, A_n$ is defined to be the set that contains the elements that are common to all these n sets. The notation for this intersection is either of the following:

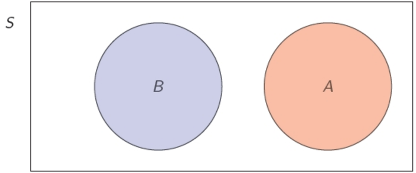
$$A_1 \cap A_2 \cap \cdots \cap A_n \quad \textit{or} \quad \bigcap_{i=1}^{n} A_i.$$

Similar notations are used for the intersection of an infinite sequence of sets or for the intersection of an arbitrary collection of sets. The infinite intersection is denoted by $\bigcap_{i=1}^{\infty} A_i$.

In terms of events, the intersection of a collection of events is the event that every event in the collection occurs.

对于“并”(或“交”)而言,都具有 Associative Property, 即运算与次序无关.

- 当 $A \cap B = \varnothing$ 时，称事件 A 和 B 互不相容或互斥.

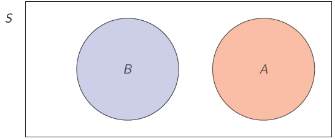


当事件 A 和 B 互不相容时，意味着两者不会同时发生.

例：基本事件是两两互不相容的.

Definition 7 (Disjoint/Mutually Exclusive)

It is said that two sets A and B are mutually exclusive, or disjoint, if A and B have no outcomes in common, that is, if $A \cap B = \varnothing$.

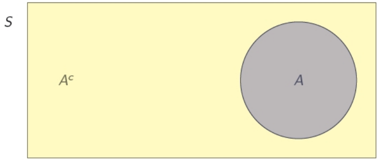


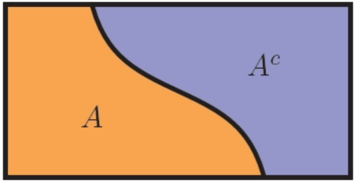
The sets $A_1, A_2, \ldots, A_n$ or the sets $A_1, A_2, \ldots$ are disjoint if for every $i \neq j$, we have that A_i and A_j are disjoint, that is, $A_i \cap A_j = \varnothing$ for all $i \neq j$. The events in an arbitrary collection are disjoint if no two events in the collection have any outcomes in common.

In terms of events, A and B are disjoint if they cannot both occur.

- A 的逆事件记为 $\overline{A}$ 或 A^c, 两者满足 $\begin{cases} A \cup \overline{A} = S, \\ A\overline{A} = \varnothing. \end{cases}$

若 $\begin{cases} A \cup B = S, \\ AB = \varnothing, \end{cases}$ 则称 A, B 互逆 (或互为对立事件).



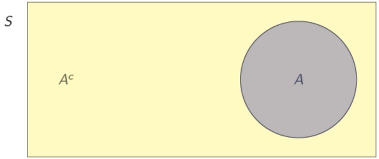


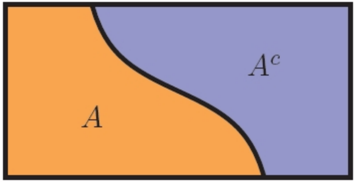
每次试验, A 与 $\overline{A}$ 必有一个发生, 且只有一个发生.

注意区分: 互斥 (互不相容) 与互逆 (互为对立) 事件.

Definition 8 (Complement)

The ***complement*** *of a set* A *is defined to be the set that contains all elements of the sample space* S *that* ***do not*** *belong to* A*. The notation for the complement of* A *is* A^c *or* $\overline{A}$*.*



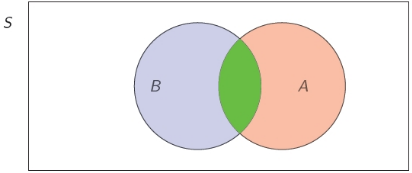


In terms of events, the event A^c is the event that A does not occur.

- 事件 A 对事件 B 的差事件, 记为 $A\overline{B}$ 或 $A-B$:

$$A\overline{B} = A - B = \{x|x \in A \ \text{ and } \ x \notin B\}.$$

$A-B$ is the intersection of A and $\overline{B}$, is the set of all points both in A and $\overline{B}$ (e.g., red area).



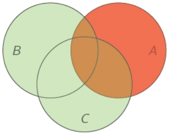
A 发生, 并且 B 不发生.

注意区分 $A-B$ 与 $B-A$

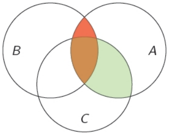
Laws of Set Operations

- 交换律 (Commutative Laws) $A \cup B = B \cup A$, $A \cap B = B \cap A$
- 结合律 (Associative Laws)
  $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$, $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$
- 分配律 (Distributive Laws)
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Brown area



Colored area



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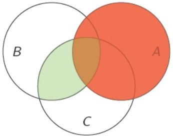
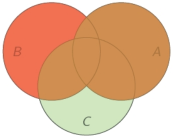


Brown area



Laws of Set Operations

- De Morgan's Laws(德摩根律)

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \ \overline{A \cap B} = \overline{A} \cup \overline{B};$$

$$\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}, \ \overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}.$$

Example 7

设事件 A 表示“学生甲来听课”，事件 B 表示“学生乙来听课”，则

- $A \cup B = \{$甲、乙至少有一人来听课$\}$
- $A \cap B = \{$甲、乙都来听课$\}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B} = \{$甲、乙都不来听课$\}$
- $\overline{A} \cup \overline{B} = \overline{A \cap B} = \{$甲、乙至少有一人不来听课$\}$

概率中有以下定义：由 n 个元件组成的系统，其中一个损坏，则系统就损坏，此时这一系统称为“串联系统”；若有一个不损坏，则系统不损坏，此时这一系统称为“并联系统”.

Example 8

由 n 个部件组成的系统，记

$$A_i = \{\text{第 } i \text{ 个部件没有损坏}\}, i = 1, 2, \cdots, n, \quad A = \{\text{系统没有损坏}\}.$$

串联系统: $A = \bigcap_{i=1}^{n} A_i$;

并联系统: $A = \bigcup_{i=1}^{n} A_i$.

§1.2 Frequency and Probability

Definition 9 (Frequency)

If event A occurs n_A times in n repeated experiments under a certain condition, then the frequency of A occurring in n experiments is defined as

$$f_n(A) = \frac{n_A}{n}.$$

Example 9

中国男子国家足球队,“冲出亚洲”共进行了 n 次,其中成功了一次,在这 n 次试验中“冲出亚洲”这事件发生的频率为 $1/n$;

Example 10

2000 年悉尼奥运会开幕前,气象学家对两个开幕候选日“9 月 10 日”和“9 月 15 日”的 100 年气象学资料分析发现,“9 月 10 日”的下雨天数为 86 天,“9 月 15 日”的下雨天数为 22 天. 即“9 月 10 日”和“9 月 15 日”的下雨频率分别为 86%和 22%,因此最后决定开幕日定为“9 月 15 日”.

频率 $f_n(A)$ 反映了事件 A 发生的频繁程度.

Properties of frequency

- $0 \leq f_n(A) \leq 1$;
- $f_n(S) = 1$;
- The sets $A_1, A_2, \ldots, A_k$ are disjoint if for every $i \neq j, i, j = 1, 2, \ldots, k$, then

$$f_n\Big(\bigcup_{j=1}^{k} A_j\Big) = \sum_{j=1}^{k} f_n(A_j).$$

Example 11 (Tossing a Coin)

Suppose that a coin is tossed once. Let $H = \{\text{head}\}$.

No.	n =5		n =50		n =500	
	n_H	$f_n(H)$	n_H	$f_n(H)$	n_H	$f_n(H)$
1	2	0.4	22	0.44	251	0.502
2	3	0.6	25	0.50	249	0.498
3	1	0.2	21	0.42	256	0.512
4	5	1.0	25	0.50	253	0.506
5	1	0.2	24	0.48	251	0.502
6	2	0.4	21	0.42	246	0.492
7	4	0.8	18	0.36	244	0.488
8	2	0.4	24	0.48	258	0.516
9	3	0.6	27	0.54	262	0.524
10	3	0.6	31	0.62	247	0.494

实验者	n	n_H	$f_n(H)$
德·摩根	2048	1061	0.5181
蒲 丰	4040	2048	0.5069
K·皮尔逊	12000	6019	0.5016
K·皮尔逊	24000	12012	0.5005

Properties of frequency (Cont.)

频率的重要性质：事件 A 的频率 $f_n(A)$ 随试验次数 n 的增大渐趋稳定.

When the times of experiments n is large enough, the frequency turns out to have a kind of stability, i.e. the value of $f_n(A)$ show fluctuations which become progressively weaker as n increase, until ultimately $f_n(A)$ stabilizes to a constant.

It is very natural to use this limit constant, called probability, to show the possibility that A occurs in a single experiment and it is commonly denoted by $\mathrm{P}(A)$ (or $\mathrm{Pr}(A)$). Such a definition of probability is said to be a statistical one.

Probability

Definition 10 (概率的统计性定义)

当试验次数足够大, 事件 A 的频率 $f_n(A)$ 的稳定值定义为 A 的概率 (probability), 记为 $P(A)$ (或 $Pr(A)$).

Definition 11 (概率的公理化定义)

1. *(非负性) For every event A, $P(A) \geq 0$;*
2. *(规范性) $P(S) = 1$;*
3. *(可列可加性) For every infinite sequence of disjoint events $A_1, A_2, \ldots,$*

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i).$$

A probability measure, or simply a probability, on a sample space S is a specification of numbers $P(A)$ for all events A that satisfy Axioms 1, 2, and 3.

Properties of probability

(1) $P(\varnothing) = 0$

Proof

Let $A_i = \varnothing, i = 1, 2, \ldots,$ *then*

$$\bigcup_{i=1}^{\infty} A_i = \varnothing, A_i A_j = \varnothing, i \neq j.$$

Hence

$$P(\varnothing) = P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\varnothing).$$

By noting $P(\varnothing) \geq 0$, *So*

$$P(\varnothing) = 0.$$

(2) (有限可加性) For every finite sequence of disjoint events $A_1, A_2, \ldots, A_n$, i.e., the sets $A_iA_j = \varnothing$ for every $i \neq j$, $i, j = 1, 2, \ldots, n$,

$$\mathrm{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mathrm{P}(A_i).$$

Proof

Let $A_{n+k} = \varnothing, k = 1, 2, \ldots,$ *then*

$$A_iA_j = \varnothing, \quad \textit{for every} \quad i \neq j, i, j = 1, 2, \ldots$$

Hence

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i).$$

(3) For every event A, $\mathrm{P}(\overline{A}) = 1 - \mathrm{P}(A)$

Proof

Since A *and* $\overline{A}$ *are disjoint events and* $A \cup \overline{A} = S$. *Together with* $P(S) = 1$, *we get* $P(\overline{A}) = 1 - P(A)$.

(4) If $A \subset B$, then $\mathrm{P}(B - A) = \mathrm{P}(B \cdot \overline{A}) = \mathrm{P}(B) - \mathrm{P}(A)$ and $\mathrm{P}(B) \geq \mathrm{P}(A)$.
$\mathrm{P}(A) \leq \mathrm{P}(S) = 1$ is followed.

Proof

Since $A \subset B$,

$$B = BS = B(A \cup \overline{A}) = BA \cup B\overline{A} = A \cup (B\overline{A}).$$

Noting that A *and* $(B\overline{A})$ *are disjoint, So* $P(B) = P(A \cup (B\overline{A})) = P(A) + P(\overline{A}B)$.
(Note. If $C = D$*, then* $P(C) = P(D)$*)*
Then

$$P(B) - P(A) = P(\overline{A}B) = P(B - A).$$

By noting $P(B - A) \geq 0$*, we get* $P(B) \geq P(A)$.

Question. For every two events A and B, $\mathrm{P}(B - A) = \mathrm{P}(B\overline{A}) =?\ \mathrm{P}(B) - \mathrm{P}(AB)$

(5) (概率的加法公式) For every two events A and B,

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(AB).$$

Proof

Since $A \cup B = A \cup (B\overline{A})$, *together with the disjoint of* A *and* $B\overline{A}$, *we have*

$$P(A \cup B) = P(A) + P(B\overline{A}) = P(A) + P(B) - P(AB).$$

(5') For every three events A, B and C,

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(AB) - \mathrm{P}(AC) - \mathrm{P}(BC) + \mathrm{P}(ABC).$$

(5'') For every n events $A_1, A_2, \ldots, A_n$,

$$\begin{aligned} \mathrm{P}\Big(\bigcup_{i=1}^{n} A_i\Big) &= \sum_{i=1}^{n} \mathrm{P}(A_i) - \sum_{1 \le i < j \le n} \mathrm{P}(A_i A_j) \\ &+ \sum_{1 \le i < j < k \le n} \mathrm{P}(A_i A_j A_k) + \ldots + (-1)^{n-1} \mathrm{P}(A_1 A_2 \ldots A_n). \end{aligned}$$

“多还少补”

Question

若 $\mathsf{P}(A) = 0$, 可否推出 $A = \varnothing$?
若 $\mathsf{P}(B) = 1$, 可否推出 $B = S$?

Answer

不能.
反例. 在 $[0,1]$ 中任意地取一数, 则 $S = [0,1]$. 记 $A = \{$取到的数是$0.3\} = \{0.3\}$, $B = \overline{A} = [0,0.3) \cup (0.3,1]$, 那么

$$\mathsf{P}(A) = 0, \text{ 但是 } A \neq \varnothing;$$

$$\mathsf{P}(B) = 1, \text{ 但是 } B \neq S = [0,1].$$

Example 12

设甲、乙两人向同一目标进行射击，已知甲击中目标的概率为 0.7，乙击中目标的概率为 0.6，两人同时击中目标的概率为 0.4. 求:

1. 目标不被击中的概率 $= \mathsf{P}(\overline{A \cup B}) = 1 - \mathsf{P}(A) - \mathsf{P}(B) + \mathsf{P}(AB) = 0.1.$
2. 甲击中目标而乙未击中目标的概率 $= \mathsf{P}(A \cap \overline{B}) = \mathsf{P}(A) - \mathsf{P}(AB) = 0.3.$

Solution

设 $A = \{$甲击中目标$\}$, $B = \{$乙击中目标$\}$, 则

$$\mathsf{P}(A) = 0.7, \quad \mathsf{P}(B) = 0.6, \quad \mathsf{P}(AB) = 0.4.$$

于是$\{$目标不被击中$\}$$=\overline{A} \cap \overline{B} = \overline{A \cup B}$, $\{$甲击中目标而乙未击中目标$\}$$=A \cap \overline{B}$

转到 Example 31.

Example 13

甲、乙、丙 3 人去参加某个集会的概率均为 0.4, 其中至少有两人参加的概率为 0.3, 三人都参加的概率为 0.05, 求 3 人中至少有 1 人参加的概率.

Solution

设 A, B, C 分别表示甲、乙、丙参加, 则

$$0.3 = \mathsf{P}(AB \cup AC \cup BC) = \mathsf{P}(AB) + \mathsf{P}(AC) + \mathsf{P}(BC) - 2\mathsf{P}(ABC)$$

可得

$$\mathsf{P}(AB) + \mathsf{P}(AC) + \mathsf{P}(BC) = 0.4.$$

因此

$$\begin{aligned} & \mathsf{P}(3\text{ 人中至少有 }1\text{ 人参加}) = \mathsf{P}(A \cup B \cup C) \\ = \; & \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) - \mathsf{P}(AB) - \mathsf{P}(AC) - \mathsf{P}(BC) + \mathsf{P}(ABC) = 0.85. \end{aligned}$$

Exercise

1. 已知 $\mathsf{P}(A \cup B) = 0.6$, $\mathsf{P}(A) = 0.3$, 则 $\mathsf{P}(\overline{A}B) = \underline{\ 0.3\ }$.
2. 已知 $\mathsf{P}(AB) = \mathsf{P}(\overline{A} \cdot \overline{B})$, $\mathsf{P}(A) = p$, 则 $\mathsf{P}(B) = \underline{1 - p}$.
3. 若已知 A, B 至少有一发生时 C 发生, 则下列说法正确的是 (B)
   A. $\mathsf{P}(A \cup B) = \mathsf{P}(C)$ B. $\mathsf{P}(A \cup B) \le \mathsf{P}(C)$
   C. $\mathsf{P}(AB) = \mathsf{P}(C)$ D. $\mathsf{P}(A \cup B) > \mathsf{P}(C)$

§1.3 Classical Probability (Simple Sample Space)

Definition 12 (古典概型)

若试验 E 满足:

- 样本空间 S 中样本点有限 (有限性);
- 出现每一样本点的概率相等 (等可能性),

则称这种试验为等可能概型 (或古典概型). 此时, 对于随机事件 A, 有

$$P(A) = \frac{A\text{所包含的样本点数}}{S\text{中的样本点数}}.$$

Definition 12 (Classical Probability)

A sample space S containing n outcomes $s_1, \ldots, s_n$ is called a simple sample space if the probability assigned to each of the outcomes $s_1, \ldots, s_n$ is $1/n$. If an event A in this simple sample space contains exactly m outcomes, then

$$\mathrm{P}(A) = \frac{m}{n}.$$

Definition 13 (排列)

从 n 个不同元素中，任取 k 个不同的元素按照一定的顺序排成一列，叫做从 n 个不同元素中取出 k 个元素的一个排列；从 n 个不同元素中取出 k 个元素的所有排列的个数，叫做从 n 个不同元素中取出 k 个元素的排列数，用符号 A_n^k (或 P_n^k, $P_{n,k}$) 表示.

Definition 13 (Permutations)

Suppose that a set has n elements. Suppose that an experiment consists of selecting k of the elements one at a time without replacement. Let each outcome consist of the k elements in the order selected. Each such outcome is called a permutation of n elements taken k at a time. We denote the number of distinct such permutations by the symbol A_n^k (or P_n^k, $\mathrm{P}_{n,k}$).

Definition 14 (组合)

从 n 个不同元素中，任取 k 个元素并成一组，叫做从 n 个不同元素中取出 k 个元素的一个组合；从 n 个不同元素中取出 k 个元素的所有组合的个数，叫做从 n 个不同元素中取出 k 个元素的组合数，用符号 C_n^k (或 $C_{n,k}$, $\binom{n}{k}$) 表示.

Definition 14 (Combinations)

Consider a set with n elements. Each subset of size k chosen from this set is called a combination of n elements taken k at a time. We denote the number of distinct such combinations by the symbol C_n^k (or $\mathrm{C}_{n,k}$, $\binom{n}{k}$).

Theorem 1 (Number of Permutations)

The number of permutations of n elements taken k at a time is

$$A_n^k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1).$$

Theorem 2 (Combinations)

The number of distinct subsets of size k that can be chosen from a set of size n is

$$C_n^k = \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!}.$$

Example 14

一袋中有 8 个球, 其中 3 个为红球, 5 个为黄球, 设摸到每一球的可能性相等.

1. 从袋中随机摸一球, 记 $A = \{$摸到红球$\}$, 求 $\mathsf{P}(A)$.
2. 从袋中不放回摸两球, 记 $B = \{$摸到两球颜色不同$\}$, 求 $\mathsf{P}(B)$.

Solution

1. 将球进行编号 1-8, 其中 1, 2, 3 为红球, 则 $S = \{1, 2, \ldots, 8\}$, $A = \{1, 2, 3\}$, 所以
$$\mathsf{P}(A) = \frac{\sharp A}{\sharp S} = \frac{3}{8}.$$
2. $\mathsf{P}(B) = \dfrac{\mathrm{C}_3^1 \mathrm{C}_5^1}{\mathrm{C}_8^2} = \dfrac{15}{28} \approx 53.6\%.$

Example 15

有 N 件产品，其中 D 件是次品，从中不放回地取 n 件，记 $A_k = \{$恰有 k 件次品$\}$，$k = 0, 1, \ldots, n$，求 $\mathsf{P}(A_k)$. $(D \le N, n \le N)$.

Solution

$$\mathsf{P}(A_k) = \frac{\mathsf{C}_D^k \mathsf{C}_{N-D}^{n-k}}{\mathsf{C}_N^n}, \quad k = 0, 1, \ldots, n$$

注：当 $L > m$ 或 $L < 0$ 时，规定 $\mathsf{C}_m^L = 0$.

Example 16

将 n 个不同的球, 投入 N 个不同的盒中 $(n \le N)$, 设每一球落入各盒的概率相同, 且各盒可放的球数不限, 记 $A = \{$恰有 n 个盒子各有一球$\}$, 求 $\mathsf{P}(A)$.

Solution

将 n 个球放入 N 个盒子中, 总样本点为N^n, 使 A 发生的样本点数为$\mathsf{C}_N^n \cdot n!$, 于是

$$\mathsf{P}(A) = \frac{\mathsf{C}_N^n \cdot n!}{N^n}.$$

Example 17 (生日问题)

在一个 $n(\leq 365)$ 人的班级里，至少有两人生日相同的概率是多少？

Solution

记 $B = \{至少两人生日相同\}$，则

$$\mathrm{P}(B) = 1 - \frac{\mathrm{C}_{365}^{n} \cdot n!}{365^n} = 1 - \frac{\mathrm{A}_{365}^{n}}{365^n}.$$

当 $n = 64$ 时，此概率 $p = 0.997$.

Example 18 (抽签问题)

一袋中有 a 个红球, b 个白球, 记 $a+b=n$. 有 n 个人一人一次不放回地进行摸球, 设每人每次摸到各球的概率相等, 求第 k 个人摸到红球的概率 $\mathsf{P}(A_k)$, 其中 $A_k=\{$第 k 个人摸到红球$\}$, $k=1,2,\ldots,n$.

Solution (方法 1)

可设想将 n 个球进行编号, 其中前 a 号球为红球, 将 n 个人也进行编号. 视人、球对应的任意一种情况, 即 $1\sim n$ 的任一排列, 为一个样本点, 则每点出现的概率相等. 因此, 第 k 个人摸到红球的概率

$$\mathsf{P}(A_k)=\frac{a(n-1)!}{n!}=\frac{a}{n}=\frac{a}{a+b},\quad k=1,2,\ldots,n.$$

注意到该结果与 k 无关.

Solution (方法 2)

视哪几人摸到红球为一样本点，则总样本点数为 C_n^a，每点出现的概率相等．而 A_k 中有 C_{n-1}^{a-1} 个样本点，所以

$$\mathrm{P}(A_k) = \frac{\mathrm{C}_{n-1}^{a-1}}{\mathrm{C}_n^a} = \frac{a}{n} = \frac{a}{a+b}.$$

Solution (方法 3)

将第 k 个人摸到的球号作为一个样本点，则 $S = \{1, 2, \ldots, n\}$，$A_k = \{1, 2, \ldots, a\}$，所以

$$\mathrm{P}(A_k) = \frac{a}{n} = \frac{a}{a+b}.$$

思考

记第 k 个人摸到的球的颜色为一样本点，则 $S = \{红色, 白色\}$，$A_k = \{红色\}$，所以

$$\mathrm{P}(A_k) = \frac{1}{2}.$$

此值不仅与 k 无关，而且与 a, b 都无关.

此法正确吗？为什么？

可以思考一种特殊情形，当 $a = 0$，$b = 2$.

此法有问题，当 $a \neq b$ 时，这样取不满足等可能概型.

Example 19 (配对问题)

一个小班有 n 个同学, 编号为 $1, 2, \ldots, n$ 号, $n \geq 1$. 中秋节前每人准备一件礼物, 相应编号为 $1, 2, \ldots, n$. 将所有礼物集中放在一起, 然后每个同学随机取一件, 求没有人拿到自己礼物的概率.

Solution

记 A_i 表示第 i 人拿到自己的礼物, $i = 1, 2, \ldots, n$, A 表示至少有一人拿到自己的礼物. 于是

$$\mathsf{P}(A) = \mathsf{P}(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathsf{P}(A_i) - \sum_{i<j} \mathsf{P}(A_i A_j) + \cdots + (-1)^{n-1} \mathsf{P}(A_1 \cdots A_n).$$

Solution (Cont.)

计算各项如下

$$
\begin{aligned}
\mathsf{P}(A_i) &= \frac{(n-1)!}{n!} = \frac{1}{n}, \quad \text{共 } n \text{ 项} \\
\mathsf{P}(A_iA_j) &= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}, i<j, \quad \text{共 } \mathsf{C}_n^2 \text{ 项} \\
\mathsf{P}(A_iA_jA_k) &= \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}, i<j<k, \quad \text{共 } \mathsf{C}_n^3 \text{ 项} \\
&\vdots \\
\mathsf{P}(A_1\cdots A_n) &= \frac{1}{n!}, \quad \text{共 } 1 \text{ 项}.
\end{aligned}
$$

Solution (Cont.)

因此

$$
\begin{aligned}
\mathsf{P}(\text{没有人取到自己礼物}) &= \mathsf{P}(\overline{A}) = 1 - \mathsf{P}(A_1 \cup \cdots \cup A_n) \\
&= 1 - \frac{\mathrm{C}_n^1}{n} + \frac{\mathrm{C}_n^2}{n(n-1)} - \frac{\mathrm{C}_n^3}{n(n-1)(n-2)} + \cdots + (-1)^n \frac{1}{n!} \\
&= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \\
&= \sum_{i=0}^{n} \frac{(-1)^i}{i!}.
\end{aligned}
$$

思考：当 $n \to \infty$ 时，这个概率趋向于$1/\mathrm{e}$.

抽签问题

一袋中有 a 个红球, b 个白球, 记 $a+b=n$. 有 n 个人一人一次不放回地进行摸球, 设每人每次摸到各球的概率相等, 求第 k 个人摸到红球的概率 $\mathsf{P}(A_k)$, 其中 $A_k=\{$第 k 个人摸到红球$\}$, $k=1,2,\dots,n$.

之前已解得 $\mathsf{P}(A_k)=\frac{a}{n}$, $k=1,2,\dots,n$.

请问 $\mathsf{P}(A_2|A_1)=?\quad \mathsf{P}(A_2|\overline{A_1})=?$

§1.4 Conditional Probability

A major use of probability in statistical inference is the updating of probabilities when certain events are observed.

The updated probability of event A after we learn that event B has occurred is the conditional probability of A given B,

$$\mathsf{P}(A|B).$$

Example 20 (Rolling a Dice)

When a six-sided dice is rolled, the sample space can be regarded as containing the six numbers 1, 2, 3, 4, 5, 6, each representing a possible side of the dice that shows after the roll. Symbolically, we write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

One event A is that a small number is obtained, and it can be represented as the subset $A = \{1, 2, 3\}$.The event B that an odd number is obtained is defined by the subset $B = \{1, 3, 5\}$. Please calculate the probability of the event A and the conditional probability of A given B.

Solution

易见 $\mathsf{P}(A)=\frac{\sharp A}{\sharp S}=\frac{3}{6}=\frac{1}{2}$. 若事件 B 已发生, 此时试验的所有可能结果只有三种: 1 或 3 或 5. 在这一前提下, 事件 A 发生只有两种情况: 1 或 3 出现, 所以在事件 B 发生的条件下, 事件 A 发生的条件概率为 $\frac{2}{3}$, 记为

$$\mathsf{P}(A|B)=\frac{2}{3}\neq\mathsf{P}(A).$$

其原因在于事件 B 的发生改变了样本空间, 使它由原来的 S 缩小为 $S_B=B$, 而 $\mathsf{P}(A|B)$ 是在新的样本空间 S_B 下, 对于事件 A 的概率度量, 其本质是在新的样本空间 S_B 下, 对于事件 AB 的概率度量.

分析:

若记 $\mathsf{P}(A|B) = x$, 则应有 $\mathsf{P}(AB) : \mathsf{P}(B) = x : 1$, 则有

$$x = \frac{\mathsf{P}(AB)}{\mathsf{P}(B)}.$$

条件概率 (Conditional Probability)

Definition 15 (条件概率)

如果$P(B) \neq 0$, 那么在事件 B 发生的条件下, 事件 A 发生的条件概率为

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

Definition 15 (Conditional Probability)

If $\mathrm{P}(B) \neq 0$, then the conditional probability that A occurs given that B occurs is defined to be

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(AB)}{\mathrm{P}(B)}.$$

注意区别于 $\mathrm{P}(AB)$

事实上, 条件概率 $\mathsf{P}(\cdot|B)$ 也是一种概率, 它满足概率的定义和性质, 如:

1. 非负性: $\mathsf{P}(A|B) \geq 0$;
2. 规范性: $\mathsf{P}(S|B) = 1$;
3. 可列可加性: 若 $A_1, A_2, \ldots, A_k, \ldots$ 两两互斥, 则 $\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} \mathsf{P}(A_i|B)$;
4. $\mathsf{P}(\overline{A}|B) = 1 - \mathsf{P}(A|B)$;
5. 加法公式: $\mathsf{P}(A \cup C|B) = \mathsf{P}(A|B) + \mathsf{P}(C|B) - \mathsf{P}(A \cap C|B)$;
6. 若 $A \supset C$, 则有 $\mathsf{P}(A|B) \geq \mathsf{P}(C|B)$.

注: 1. 当 $B \neq D$ 时, $\mathsf{P}(A|B) + \mathsf{P}(C|D)$ 无意义;
2. $A|B$, 不表示事件, 此记号无含义;
3. 由 $\mathsf{P}(A|B)$ 的含义, 可将 $\mathsf{P}(A)$ 记为 $\mathsf{P}(A|S)$, 只是 S 常常省略而已, 因此 $\mathsf{P}(A)$ 也可以视为条件概率.

Example 21

有一批产品，其合格率为 90%，合格品中有 95% 是优质品，从中任取一件，记

$$A=\{\text{取到一件合格品}\},\quad B=\{\text{取到一件优质品}\}.$$

则

$$\mathsf{P}(A)=90\%,\quad \mathsf{P}(B|A)=95\%,$$

且可得

$$\mathsf{P}(B)=\mathsf{P}(AB)=\mathsf{P}(A)\cdot\mathsf{P}(B|A)=85.5\%.$$

乘法公式 (Multiplication Formula)

当下面的条件概率都有意义时, 有:

- $\mathsf{P}(AB) = \mathsf{P}(A) \cdot \mathsf{P}(B|A) = \mathsf{P}(B) \cdot \mathsf{P}(A|B)$.
- $\mathsf{P}(ABC) = \mathsf{P}(A) \cdot \mathsf{P}(B|A) \cdot \mathsf{P}(C|AB)$.
- $\mathsf{P}(A_1A_2 \cdots A_n) = \mathsf{P}(A_1)\mathsf{P}(A_2|A_1)\mathsf{P}(A_3|A_1A_2) \cdots \mathsf{P}(A_n|A_1 \cdots A_{n-1}), n \geq 2$

Example 22

某厂生产的产品能直接出厂的概率为 70%, 余下 30% 的产品要调试后再定. 已知调试后有 80% 的产品可以出厂, 20% 的产品要报废. 求该厂产品的报废率.

Solution

设 $A = \{$生产的产品要报废$\}$, $B = \{$生产的产品要调试$\}$, 则由题意知, $\mathsf{P}(B) = 0.3$, $\mathsf{P}(A|B) = 0.2$. 由于 $A \subset B$, 所以 $A = AB$, 因此

$$\mathsf{P}(A) = \mathsf{P}(AB) = \mathsf{P}(B) \cdot \mathsf{P}(A|B) = 0.2 \times 0.3 = 0.06.$$

Example 23

某行业进行专业劳动技能考核，一个月安排一次，每人最多参加 3 次．某人第一次参加能通过的概率为 60%；如果第一次未通过就去参加第二次，这时能通过的概率为 80%；如果第二次再未通过，则去参加第三次，此时能通过的概率为 90%．求这人能通过考核的概率．

Solution

设 $A_i = \{这人第\ i\ 次通过考核\}, i = 1, 2, 3,\ A = \{这人通过考核\}$．则由题意知，
$P(A_1) = 0.6, P(A_2|\overline{A}_1) = 0.8, P(A_3|\overline{A}_1\overline{A}_2) = 0.9$．而 $A = A_1 \cup \overline{A}_1 A_2 \cup \overline{A}_1\overline{A}_2 A_3$．因此

$$\begin{aligned} P(A) &= P(A_1) + P(\overline{A}_1 A_2) + P(\overline{A}_1\overline{A}_2 A_3)(\text{利用有限可加性}) \\ &= P(A_1) + P(\overline{A}_1)P(A_2|\overline{A}_1) + P(\overline{A}_1)P(\overline{A}_2|\overline{A}_1)P(A_3|\overline{A}_1\overline{A}_2) \\ & \quad (\text{其中} P(\overline{A}_2|\overline{A}_1) = 1 - P(A_2|\overline{A}_1) = 1 - 0.8 = 0.2) \\ &= 0.6 + 0.4 \times 0.8 + 0.4 \times 0.2 \times 0.9 = 0.992. \end{aligned}$$

Solution (Cont.)

也可以通过考虑逆事件的概率来求解:

$$\begin{aligned} \mathsf{P}(A) &= 1-\mathsf{P}(\overline{A}) \\ &= 1-\mathsf{P}(\overline{A}_1\overline{A}_2\overline{A}_3) \\ &= 1-\mathsf{P}(\overline{A}_1)\mathsf{P}(\overline{A}_2|\overline{A}_1)\mathsf{P}(\overline{A}_3|\overline{A}_1\overline{A}_2) \\ &= 1-0.4\times 0.2\times 0.1 = 0.992. \end{aligned}$$

Example 24

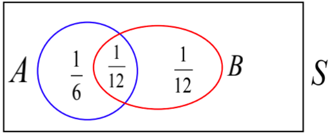
已知 $\mathsf{P}(A)=1/4$, $\mathsf{P}(B|A)=1/3$, $\mathsf{P}(A|B)=1/2$, 求 $\mathsf{P}(A\cup B)$, $\mathsf{P}(\overline{A}|A\cup B)$.

Solution

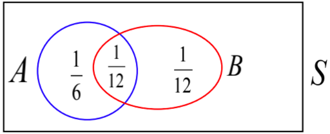
由乘法公式可知

$$\mathsf{P}(AB)=\mathsf{P}(A)\mathsf{P}(B|A)=1/12,\quad \mathsf{P}(B)=\mathsf{P}(AB)\big/\mathsf{P}(A|B)=1/6.$$

可得 Venn 图如下



Solution (Cont.)



所以

$$\mathrm{P}(A\cup B)=\frac{1}{6}+\frac{1}{12}+\frac{1}{12}=\frac{1}{3}.$$

$$\mathrm{P}(\overline{A}|A\cup B)=1-\mathrm{P}(A|A\cup B)=1-\frac{\mathrm{P}(A)}{\mathrm{P}(A\cup B)}=1-\frac{1/4}{1/3}=\frac{1}{4}.$$

或

$$\mathrm{P}(\overline{A}|A\cup B)=\frac{\mathrm{P}(\overline{A}B)}{\mathrm{P}(A\cup B)}=\frac{1/12}{1/3}=\frac{1}{4}.$$

全概率公式与 Bayes 公式

A set of events is jointly or collectively exhaustive if at least one of the events must occur. Another way to describe collectively exhaustive events is that their union must cover all the events within the entire sample space.

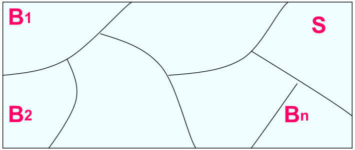
Definition 16 (完备事件组 (划分))

设 S 为试验 E 的样本空间, $B_1, B_2, \ldots, B_n$ 为 E 的一组事件, $n \geq 2$, 若满足

1. *(不漏)* $B_1 \cup B_2 \cup \cdots \cup B_n = S$;
2. *(不重)* $B_i \cap B_j = \varnothing,\ i \neq j,\ i, j = 1, 2, \ldots, n.$

则称 $B_1, B_2, \ldots, B_n$ 为 S 的一个划分 (partition), 或称为一组完备事件组 (collectively exhaustive events)

Each elementary event belongs to exactly one set in a partition of S.

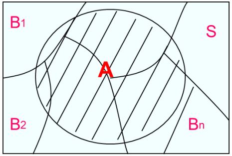


若 $B_1, B_2, \ldots, B_n$ 为 S 的一个划分, 则 $B_1, B_2, \ldots, B_n$ 至少有一个发生是必然的, 两两同时发生又是不可能的.

Theorem 3 (Law of Total Probability, 全概率公式)

Let $B_1, B_2, \ldots, B_n$ *be a partition of* S *such that* $P(B_i) > 0$ *for all* i. *Then for any events* A *in sample space* S,

$$P(A) = \sum_{j=1}^{n} P(B_j)P(A|B_j).$$



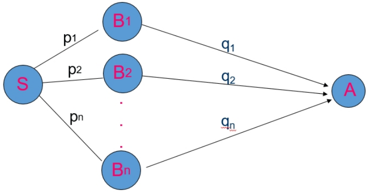
Proof

$A = AS = AB_1 \cup AB_2 \cup \cdots \cup AB_n$. This is a disjoint union, so

$$\mathrm{P}(A) = \sum_{j=1}^{n} \mathrm{P}(AB_j) = \sum_{j=1}^{n} \mathrm{P}(B_j)\mathrm{P}(A|B_j).$$

注：运用全概率公式的关键是构造一个合适的划分. 最简单的一个划分是 B 与 $\overline{B}$.

若记 $\mathrm{P}(B_j) = p_j, \mathrm{P}(A|B_j) = q_j$, 则全概率公式的示意图为:



Corollary 1 (Law of Total Probability for Conditional Probability, 条件概率的全概率公式)

Futhermore, let $B_1, B_2, \ldots, B_n$ be a partition of S such that $P(B_iC) > 0$ for all i, where C is a event in sample space S and $P(C) > 0$. Then for any events A in S,

$$P(A|C) = \sum_{j=1}^{n} P(AB_j|C) = \sum_{j=1}^{n} P(B_j|C) \cdot P(A|B_jC).$$

Theorem 4 (Bayes' Formula, 贝叶斯公式)

If $B_1, B_2, \cdots, B_n$ *is a partition of* S*, each* B_i *having positive probability, then for any events* A *such that* $P(A) > 0$*,*

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}.$$

Proof

robability and conditional probability,

$$P(B_i|A) = \frac{P(AB_i)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}.$$

Example 25

一单位有甲、乙两人，其中甲近期出差的概率为 70%. 已知若甲出差，则乙出差的概率为 10%; 若甲不出差，则乙出差的概率为 60%.

1. 求近期乙出差的概率 $= 0.7 \times 0.1 + 0.3 \times 0.6 = 0.25.$
2. 若已知乙近期出差在外，求甲出差的概率 $= \dfrac{0.7 \times 0.1}{0.25} = 0.28.$

Solution

设 $A = \{$甲出差$\}$, $B = \{$乙出差$\}$, 则 $\mathsf{P}(A) = 0.7$, $\mathsf{P}(B|A) = 0.1$, $\mathsf{P}(B|\overline{A}) = 0.6$. 于是

1. 由全概率公式 $\mathsf{P}(B) = \mathsf{P}(A)\mathsf{P}(B|A) + \mathsf{P}(\overline{A})\mathsf{P}(B|\overline{A}) = 0.25.$
2. 由 Bayes 公式 $\mathsf{P}(A|B) = \dfrac{\mathsf{P}(A)\mathsf{P}(B|A)}{\mathsf{P}(B)} = 0.28.$

Example 26

有三个箱子，第 1 箱装有 3 个白球和 5 个红球，第 2 箱装有 2 个白球和 2 个红球，第 3 箱装有 5 个白球和 2 个红球．现从 3 个箱子中任挑出一个，然后从该箱子中随机地取两次，每次取一球，作不放回抽样．

1. 求第一次取到的是白球的概率．
2. 已知第一次取到的是白球，求取到的是第 1 箱的概率．
3. 已知第一次取到的是白球，求第二次取到的还是白球的概率．

Solution

设 $A_i=\{$取到第 i 箱$\}$，$i=1,2,3$，则 $\mathsf{P}(A_i)=1/3$．设 $B_j=\{$第 j 次取到白球$\}$，$j=1,2$，则

$$\mathsf{P}(B_1|A_1)=\frac{3}{8},\quad \mathsf{P}(B_1|A_2)=\frac{1}{2},\quad \mathsf{P}(B_1|A_3)=\frac{5}{7}.$$

Solution (Cont.)

1. 由全概率公式

$$\mathsf{P}(B_1) = \sum_{i=1}^{3} \mathsf{P}(A_i)\mathsf{P}(B_1|A_i) = \frac{1}{3} \times \left(\frac{3}{8} + \frac{1}{2} + \frac{5}{7}\right) = \frac{89}{168}.$$

2. 由 Bayes 公式

$$\mathsf{P}(A_1|B_1) = \frac{\mathsf{P}(A_1)\mathsf{P}(B_1|A_1)}{\mathsf{P}(B_1)} = \frac{1/3 \times 3/8}{89/168} = \frac{21}{89}.$$

Solution (Cont.)

3 由于

$$\mathsf{P}(B_1B_2) = \sum_{i=1}^{3} \mathsf{P}(A_i)\mathsf{P}(B_1B_2|A_i) = \frac{1}{3} \times \left(\frac{\mathsf{C}_3^2}{\mathsf{C}_8^2} + \frac{\mathsf{C}_2^2}{\mathsf{C}_4^2} + \frac{\mathsf{C}_5^2}{\mathsf{C}_7^2}\right) = \frac{1}{4},$$

因此

$$\mathsf{P}(B_2|B_1) = \frac{\mathsf{P}(B_1B_2)}{\mathsf{P}(B_1)} = \frac{1/4}{89/168} = \frac{42}{89}.$$

Bayes 公式:

已知“结果”，求“原因”

Example 27

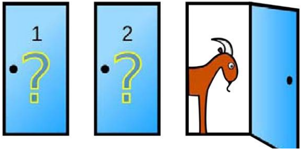
某人从甲地到乙地，乘飞机、火车、汽车迟到的概率分别为 0.1、0.2、0.3，他等可能地选择这三种交通工具. 若已知他最后迟到了，求他分别是乘飞机、火车、汽车的概率.

$$1/6, 2/6, 3/6$$

Bayes 公式: 在观察到事件 A 发生后, 对事件 B_i 概率 $\mathrm{P}(B_i)$(先验概率) 进行更新, 得到 $\mathrm{P}(B_i|A)$(后验概率).

Example 28 (Monty Hall Problem: Goats and Cars)

Suppose you're on a game show, and you're given the choice of three doors. One door hides a car, two hide goats. Suppose that you pick Door 1. The host, who knows what is behind the doors, opens a door (Door 3) with a goat behind it.



Now the host asks you: Do you want to switch your original choice?
What is the best strategy for winning a car?
(a) Switch (b) Don't switch (c) It doesn't matter

§1.5 Independence

Example 29

有 10 件产品, 其中 8 件为正品, 2 件为次品. 从中取 2 件, 每次取 1 件, 设 $A_i=\{$第 i 次取到正品$\}$, $i=1,2$.

- 不放回抽样时, $\mathsf{P}(A_2|A_1)=\dfrac{7}{9}\neq \mathsf{P}(A_2)=\dfrac{8}{10}$;
- 有放回抽样时, $\mathsf{P}(A_2|A_1)=\dfrac{8}{10}=\mathsf{P}(A_2|\overline{A}_1)=\mathsf{P}(A_2)=\dfrac{8}{10}$.

在有放回抽样时, A_1 的发生与否对 A_2 的概率没有不影响, 同样, A_2 的发生对 A_1 的概率也不影响. 此时有 $\mathsf{P}(A_1A_2)=\mathsf{P}(A_1)\cdot\mathsf{P}(A_2)$, $\mathsf{P}(\overline{A_1}A_2)=\mathsf{P}(\overline{A_1})\cdot\mathsf{P}(A_2)$.

Independent

In general, the occurrence of some event B changes the probability that another event A occurs, the original probability $\mathsf{P}(A)$ being replaced by $\mathsf{P}(A|B)$. If the probability remains unchanged, that is to say $\mathsf{P}(A|B) = \mathsf{P}(A)$, then we call A and B 'independent'. This is well defined only if $\mathsf{P}(B) > 0$. Definition of conditional probability leads us to the following.

Definition 17 (两事件相互独立)

设 A, B 为两随机事件, 如果 $P(AB) = P(A)P(B)$, 则称 A, B **相互独立**.

若 $0 < \mathsf{P}(A) < 1$, 则

$$\mathsf{P}(AB) = \mathsf{P}(A)\mathsf{P}(B) \Leftrightarrow \mathsf{P}(B) = \mathsf{P}(B|A) = \mathsf{P}(B|\overline{A}).$$

若 $\mathsf{P}(A) \neq 0, \mathsf{P}(B) \neq 0$, 则

$$\mathsf{P}(AB) = \mathsf{P}(A)\mathsf{P}(B) \Leftrightarrow \mathsf{P}(B|A) = \mathsf{P}(B) \Leftrightarrow \mathsf{P}(A|B) = \mathsf{P}(A).$$

Proposition 1

设 A, B 为两随机事件, 则

$$A, B \text{ 相互独立} \Leftrightarrow \overline{A}, B \text{ 相互独立} \Leftrightarrow A, \overline{B} \text{ 相互独立} \Leftrightarrow \overline{A}, \overline{B} \text{ 相互独立}.$$

Proof

Without the loss of generality, give a proof of the first equivalence below.

$$\begin{aligned}
A, B \text{ 相互独立} &\Leftrightarrow \mathsf{P}(AB) = \mathsf{P}(A)\mathsf{P}(B) \\
&\Leftrightarrow \mathsf{P}(B) - \mathsf{P}(AB) = \mathsf{P}(B) - \mathsf{P}(A)\mathsf{P}(B) \\
&\Leftrightarrow \mathsf{P}(B - AB) = \mathsf{P}(B)[1 - \mathsf{P}(A)] \\
&\Leftrightarrow \mathsf{P}(\overline{A}B) = \mathsf{P}(\overline{A})\mathsf{P}(B) \\
&\Leftrightarrow \overline{A}, B \text{ 相互独立}.
\end{aligned}$$

Notes

- 注意区分 “独立 (independent)” 和 “互斥 (exclusive)”.

  当 $\mathsf{P}(A)\mathsf{P}(B) \neq 0$, 即 $\mathsf{P}(A) > 0$ 且 $\mathsf{P}(B) > 0$ 时, “A, B 相互独立” 与 “A, B 互不相容”不会同时成立.

当 $\mathsf{P}(A)\mathsf{P}(B) \neq 0$ 时, 若 “A, B 相互独立”, 则有

$$\mathsf{P}(AB) = \mathsf{P}(A)\mathsf{P}(B) \neq 0,$$

故可知 $AB \neq \phi$.

另一方面, 当 $\mathsf{P}(A)\mathsf{P}(B) \neq 0$ 时, 若 “A, B 互不相容”, 则有

$$0 = \mathsf{P}(\phi) = \mathsf{P}(AB) \neq \mathsf{P}(A)\mathsf{P}(B),$$

因此 A, B 不相互独立.

Definition 18 (多事件相互独立)

设 $A_1, A_2, \ldots, A_n$ 为 n 个随机事件, $n \geq 2$. 若对 $2 \leq k \leq n$, 均有

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}),$$

则称 $A_1, A_2, \ldots, A_n$ **相互独立**.

Notes

- 对 $n \geq 3$ 个事件, 两两独立不蕴含 ($\not\Rightarrow$) 相互独立.
- 对 $n \geq 3$ 个事件, n 个事件相互独立可推出 ($\Rightarrow$) 两两独立.

Example 30

有一个正四面体，现在给一面漆上红色，一面漆上黄色，一面漆上蓝色，还有一面漆上红黄蓝三色．现在任取一面，考察该面的颜色情况．令 $A_1 =$ "这面含红色"，$A_2 =$ "这面含黄色"，$A_3 =$ "这面含蓝色"．问 A_1, A_2, A_3 三个事件是否相互独立？是否两两独立？

Solution

对这四面分别编号为 $1, 2, 3, 4$，现在任取一面，该面的颜色情况为一样本点，则
$S = \{1, 2, 3, 4\}$，$A_1 = \{1, 4\}$，$A_2 = \{2, 4\}$，$A_3 = \{3, 4\}$，
$A_1A_2 = A_1A_3 = A_2A_3 = A_1A_2A_3 = \{4\}$，因此 $P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$，
$P(A_1A_2) = P(A_1A_3) = P(A_2A_3) = P(A_1A_2A_3) = \frac{1}{4}$．注意到

$$P(A_iA_j) = P(A_i)P(A_j), \forall i \neq j, i, j = 1, 2, 3; \quad \text{但} P(A_1A_2A_3) \neq P(A_1)P(A_2)P(A_3),$$

所以 A_1, A_2, A_3 三个事件两两独立但不相互独立．

独立试验

试验结果互不影响的一系列试验.

重复试验

在相同条件下进行的试验.

实际问题中, 常常不是用定义去验证事件的独立性, 而是由实际情形来判断其独立性.

Example 31

甲、乙两人同时向一目标射击，甲击中率为 0.8，乙击中率为 0.7，求目标被击中的概率.

Solution

设 $A=\{\text{甲击中}\}$, $B=\{\text{乙击中}\}$, $C=\{\text{目标被击中}\}$, 则 $C=A\cup B$,

$$\mathsf{P}(C)=\mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(AB).$$

由于甲、乙同时射击，其结果互不影响，所以 A, B 相互独立，因此

$$\mathsf{P}(C)=0.8+0.7-0.8\times 0.7=0.94.$$

与 Example 12 做比较.

Example 32

甲、乙两人进行乒乓球比赛, 每局甲胜的概率为 p, 试问对甲而言, 采用三局二胜制有利, 还是采用五局三胜制有利? (设各局胜负相互独立.)

Solution

设 $A_i=\{$第 i 局甲胜$\}$, 则 $\mathsf{P}(A_i)=p$, $i=1,2,\ldots,5$. 再设 $A=\{$甲胜$\}$. 则

1. 若采用三局二胜制,

$$\mathsf{P}(A)=\mathsf{P}(A_1A_2\cup A_1\overline{A}_2A_3\cup\overline{A}_1A_2A_3)=p^2+2p^2(1-p)\xlongequal{\text{记为}}p_1.$$

2. 若采用五局三胜制,

$$\begin{aligned}\mathsf{P}(A)&=\mathsf{P}(A_1A_2A_3\cup(\text{前三次有一次输})A_4\cup(\text{前四次有两次输})A_5)\\&=p^3+\mathsf{C}_3^1(1-p)p^3+\mathsf{C}_4^2(1-p)^2p^3\xlongequal{\text{记为}}p_2.\end{aligned}$$

Solution (Cont.)

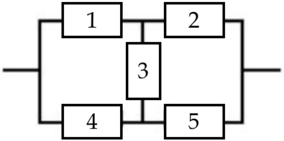
于是

$$
\begin{aligned}
p_2 - p_1 &= p^3[1+3(1-p)+6(1-p)^2] - p^2[1+2(1-p)] \\
&= 3p^2(p-1)^2(2p-1) \begin{cases} >0, & \text{当 } p>1/2, \\ =0, & \text{当 } p=1/2, \\ <0, & \text{当 } p<1/2. \end{cases}
\end{aligned}
$$

所以当 $p>1/2$ 时，五局三胜制对甲有利；当 $p<1/2$ 时，三局二胜制对甲有利；当 $p=1/2$ 时，两种赛制对甲利害情况相同.

Example 33

有 5 个独立元件构成的系统 (如图所示), 设每个元件能正常运行的概率为 p $(0 < p < 1)$, 求系统正常运行的概率.



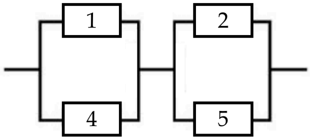
Solution

设 $A_i = \{$第 i 个元件正常运行$\}$, $i = 1, 2, 3, 4, 5$, 再设 $A = \{$系统正常运行$\}$, 则

$$A = AA_3 \cup A\overline{A}_3.$$

Solution (Cont.)

当元件 3 正常运行时，系统可以简化为如下图所示.

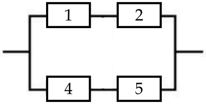


此时有

$$\begin{aligned} \mathsf{P}(A|A_3) &= \mathsf{P}((A_1 \cup A_4) \cap (A_2 \cup A_5)) \\ &= \mathsf{P}(A_1 \cup A_4) \cdot \mathsf{P}(A_2 \cup A_5)(\text{因为} A_1, A_2, \ldots, A_5 \text{相互独立}) \\ &= (2p - p^2)^2. \end{aligned}$$

Solution (Cont.)

当元件 3 不正常运行时，系统可以简化为如下图所示.



此时有

$$P(A|\overline{A}_3) = P(A_1A_2 \cup A_4A_5) = 2p^2 - p^4.$$

因此

$$\begin{aligned} P(A) &= P(AA_3 \cup A\overline{A}_3) = P(A_3)P(A|A_3) + P(\overline{A}_3)P(A|\overline{A}_3) \\ &= p(2p-p^2)^2 + (1-p)(2p^2-p^4) = 2p^2 + 2p^3 - 5p^4 + 2p^5. \end{aligned}$$

问：P$(A_1A_2A_3 \cup A_3A_4A_5) =?$

Example 34

一袋中有编号为 1,2,3,4 共 4 个球, 采用放回抽样, 每次取一球, 共取 2 次, 记录号码之和, 这样独立重复进行试验. 求 "和等于 3" 出现在 "和等于 5" 之前的概率.

Solution

设 A 表示 "'和等于 3' 出现在 '和等于 5' 之前", B 表示 "第一次号码之和为 3", C 表示 "第一次号码之和为 5", D 表示 "第一次号码之和既不为 3 也不为 5". 于是

$$\mathrm{P}(B) = \frac{2}{16}, \quad \mathrm{P}(C) = \frac{4}{16}, \quad \mathrm{P}(D) = \frac{10}{16}.$$

Solution (Cont.)

由全概率公式

$$
\begin{aligned}
\mathsf{P}(A) &= \mathsf{P}(B)\mathsf{P}(A|B) + \mathsf{P}(C)\mathsf{P}(A|C) + \mathsf{P}(D)\mathsf{P}(A|D) \\
&= \frac{2}{16} \times 1 + \frac{4}{16} \times 0 + \frac{10}{16} \times \mathsf{P}(A|D)
\end{aligned}
$$

注意到，在第一次和不等于 3 或 5 的情况下，求 A 的条件概率，相当于重新考虑 A 的概率，即 $\mathsf{P}(A|D) = \mathsf{P}(A)$. 因此可以得到

$$
\mathsf{P}(A) = \frac{1}{3}.
$$

Example 35

某技术工人长期进行某项技术操作，他经验丰富，因嫌按规定操作太过繁琐，就按照自己的方法进行，但这样做有可能发生事故．设他每次操作发生事故的概率为 p, $p>0$ 但很小很小．现设他独立重复进行了 n 次操作，求:

1. n 次都不发生事故的概率;
2. 至少有一次发生事故的概率.

Solution

设 $C_i=\{$第i次不发生事故$\}, i=1,\ldots,n$, 则 $C_1,\ldots,C_n$ 相互独立，且 $\mathsf{P}(C_i)=1-p$.

1. 设 $A=\{n$ 次都不发生事故$\}$，则 $\mathsf{P}(A)=\mathsf{P}(C_1\cap\cdots\cap C_n)=(1-p)^n$.
2. 设 $B=\{$至少有一次发生事故$\}$，则 $\mathsf{P}(B)=1-\mathsf{P}(A)=1-(1-p)^n$.

$\lim\limits_{n\to\infty}\mathsf{P}(B)=1$，故“小概率事件”在大量独立重复试验中“至少有一次发生”几乎是必然的.

条件独立 (Conditional Independent)

Example 36

一射击场有 10 支枪, 其中 7 支已校正, 3 支未校正. 某人用已校正的枪射击, 命中率为 0.8, 用未校正的枪射击, 命中率为 0.4. 若他随机地取一支枪, 用这支枪独立射击 2 次.

1. 求他第一枪命中的概率.
2. 求他两枪都命中的概率.
3. 若已知他第一枪命中, 求他取到的枪为已校正的枪的概率.

Solution

设 $A = \{$取到的枪为已校正的枪$\}$, $B_i = \{$第 i 枪命中$\}$, $i = 1, 2$.

Solution (Cont.)

1. 由全概率公式

$$P(B_1) = P(B_1|A)P(A) + P(B_1|\overline{A})P(\overline{A}) = 0.8 \times 0.7 + 0.4 \times 0.3 = 0.68.$$

2. 由全概率公式及条件独立性

$$P(B_1B_2) = P(B_1B_2|A)P(A) + P(B_1B_2|\overline{A})P(\overline{A}) = 0.8^2 \times 0.7 + 0.4^2 \times 0.3 = 0.496.$$

3. 由 *Bayes* 公式

$$P(A|B_1) = \frac{P(B_1|A)P(A)}{P(B_1)} = \frac{0.8 \times 0.7}{0.68} = \frac{14}{17}.$$

Example 37 (疾病诊断)

某种疾病的诊断试验有 5% 的假阳性和 4% 的假阴性，即令
$A=\{$试验反应是阳性$\}$，$B=\{$患有此种疾病$\}$，则 $\mathrm{P}(A|\overline{B})=0.05$，$\mathrm{P}(\overline{A}|B)=0.04$.
已知此病发病率是 0.01，即 $\mathrm{P}(B)=0.01$.

1. 当试验反应是阳性时，求此人患有此种疾病的概率.
2. 为提高准确率，通常会对第一次试验阳性的人再做一次独立的检查. 如果这两次都是阳性，求此人患有此种疾病的概率.

Solution

1. 由 Bayes 公式

$$\mathrm{P}(B|A)=\frac{\mathrm{P}(A|B)\mathrm{P}(B)}{\mathrm{P}(A|B)\mathrm{P}(B)+\mathrm{P}(A|\overline{B})\mathrm{P}(\overline{B})}=\frac{0.96\times 0.01}{0.96\times 0.01+0.05\times 0.99}=0.1624.$$

Solution

2 令 $A_i = \{$第 i 次试验阳性$\}$, $i = 1, 2$. 由 Bayes 公式

$$\begin{aligned} \mathsf{P}(B|A_1A_2) &= \frac{\mathsf{P}(A_1A_2|B)\mathsf{P}(B)}{\mathsf{P}(A_1A_2|B)\mathsf{P}(B) + \mathsf{P}(A_1A_2|\overline{B})\mathsf{P}(\overline{B})} \\ &= \frac{0.96^2 \times 0.01}{0.96^2 \times 0.01 + 0.05^2 \times 0.99} = 0.7883. \end{aligned}$$

谢谢!