Classification

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Binary Classification

First of all, let's consider a binary classification problem.

$$y \in 0, 1$$

Why not use least squares regression?

$$\min_{\theta} \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

Reasons

1. Least squares cost function does not work well, when your label is a binary number.
2. Least squares is highly sensitive to outliers.

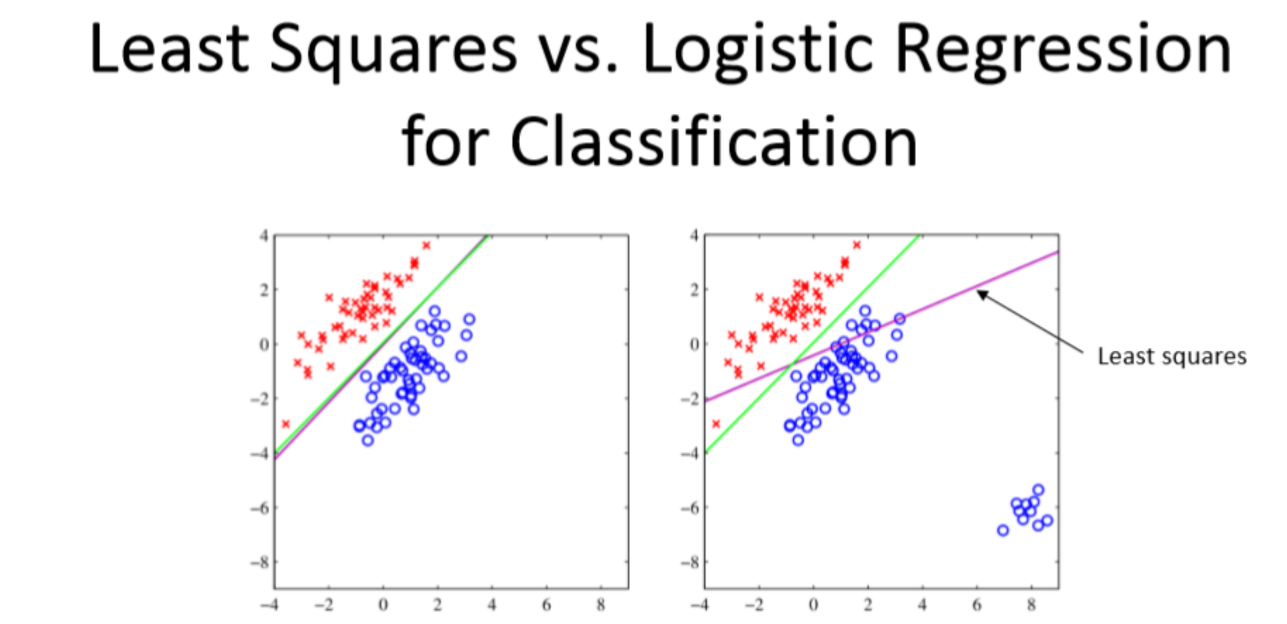


Figure 4.4 from Bishop. The left plot shows data from two classes, denoted by red crosses and blue circles, together with the decision boundary found by least squares (magenta) and also by the logistic regression model (green). The right-hand plot shows the corresponding results obtained when extra data points are added at the bottom left of the diagram, showing that least squares is highly sensitive to outliers, unlike logistic regression.

(see Bishop 4.1.3 for more details)

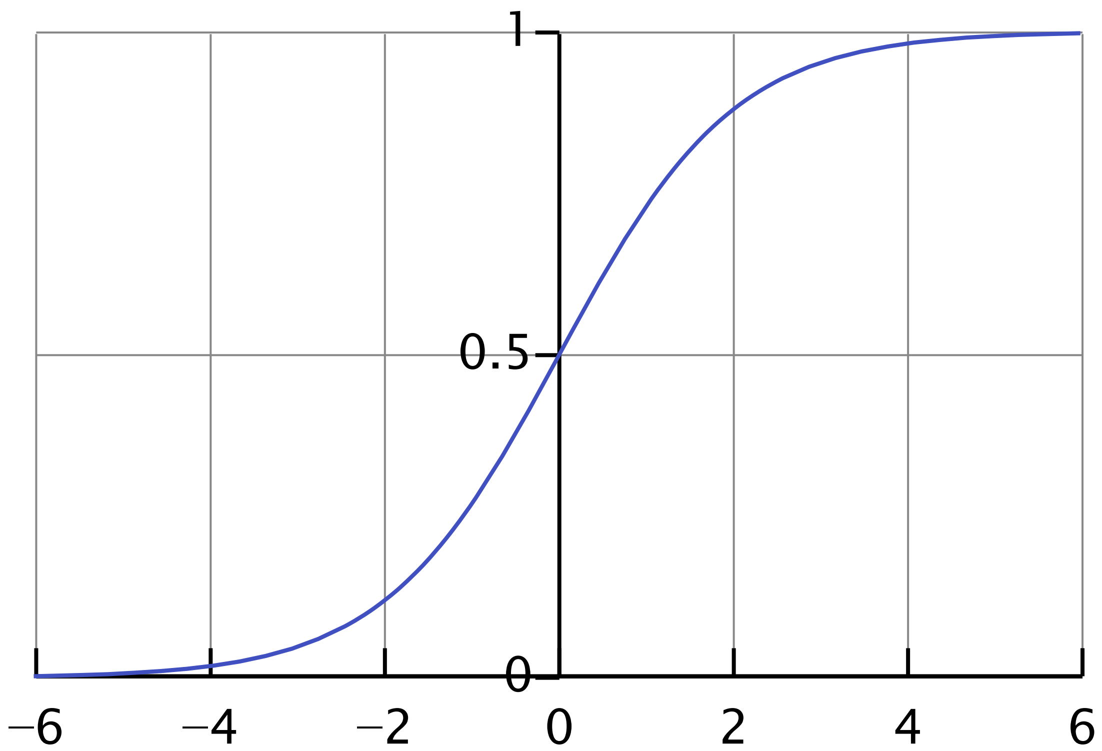
That's why we introduce Logistic Regression.

Logistic Regression

Sigmoid Function

We use sigmoid function to map $h_{\theta}(x)$ to [0, 1]:

$$\sigma(z) = \frac{1}{1+e^{-z}}$$



Derivative of Sigmoid Function

$$\frac{\partial \sigma(z)}{\partial z} = \frac{\partial}{\partial z}(\frac{1}{1+e^{-z}}) = (-1)(1+e^{-z})^{-2}(-e^{-z}) = \frac{e^{-z}}{(1+e^{-z})^2}$$

$$= \sigma(z)(1-\sigma(z))$$

$$= \frac{1}{1+e^{-z}}(1-\frac{1}{1+e^{-z}})$$

$$= \frac{(1+e^{-z})-1}{(1+e^{-z})^2}$$

Therefore, this is an interesting result.

$$\frac{\partial \sigma(z)}{\partial z} = \sigma(z)(1-\sigma(z))$$

Hypothesis

$$h_{\theta}(x) = \sigma(\theta^T x) = \frac{1}{1+e^{-\theta^T x}}$$

Predict "y=1", if $h_{\theta}(x) \geq 0.5$.
Predict "y=0", if $h_{\theta}(x) < 0.5$.

Logistic Regression Cost Function --- Cross Entropy

Information Entropy

衡量一条信息的量的多少。

$$H(X) = -\sum_{i=1}^{n} P(x_i) log_b P(x_i)$$

b通常取2, n是随机变量取值的n种可能性。

Cross Entropy

In information theory, the cross entropy between two probability distributions p and q over the **same underlying set of events measures** the average number of bits needed to identify an event drawn from the set if a coding scheme used for the set is optimized for an estimated probability distribution q, rather than the true distribution p.

$$H(p,q) = -\sum_{x \in X} p(x) \log q(x)$$

Cross Entropy Loss Function

Cross entropy can be used to define a loss function in machine learning and optimization.

Meanning of p and q

The true probability p_{i} is the true label, and the given distribution q_{i} is the predicted value of the current model.

More specifically, consider **logistic regression**, which (among other things) can be used to classify observations into two possible classes (often simply labelled 0 and 1).

The output of the model for a given observation, given a vector of input features x, **can be interpreted as a probability**, which serves as the basis for classifying the observation.

The probability is modeled using the logistic function

$$g(z) = 1/(1+e^{-z})$$

where z is some function of the input vector **x**, commonly just a linear function.

The probability of the output $y=1$ and $y=0$ is given by

$$q_{y=1} = \hat{y} = g(x\theta) = \frac{1}{1+e^{-x\theta}}$$

$$q_{y=0} = 1 - \hat{y}$$

Having set up our notation,

$$p \in y, 1-y, q \in \hat{y}, 1-\hat{y}$$

we can use cross entropy to get a measure of dissimilarity between p and q.

$$H(p,q) = -\sum_{i=1} p_i \log q_i = -y \log \hat{y} - (1-y)\log(1-\hat{y})$$

Why does it work so well?

Now we use different notations.

$$Cost(h_\theta(x), y) = -y \log h_\theta(x) - (1-y)\log(1-h_\theta(x))$$

Given $y=1$, we hope our prediction $h_{\theta}(x)$ gets close to 1.

Given $y=0$, we hope our prediction $h_{\theta}(x)$ gets close to 0.

Let's see what we've got when we plug in $y=0$ and $y=1$.

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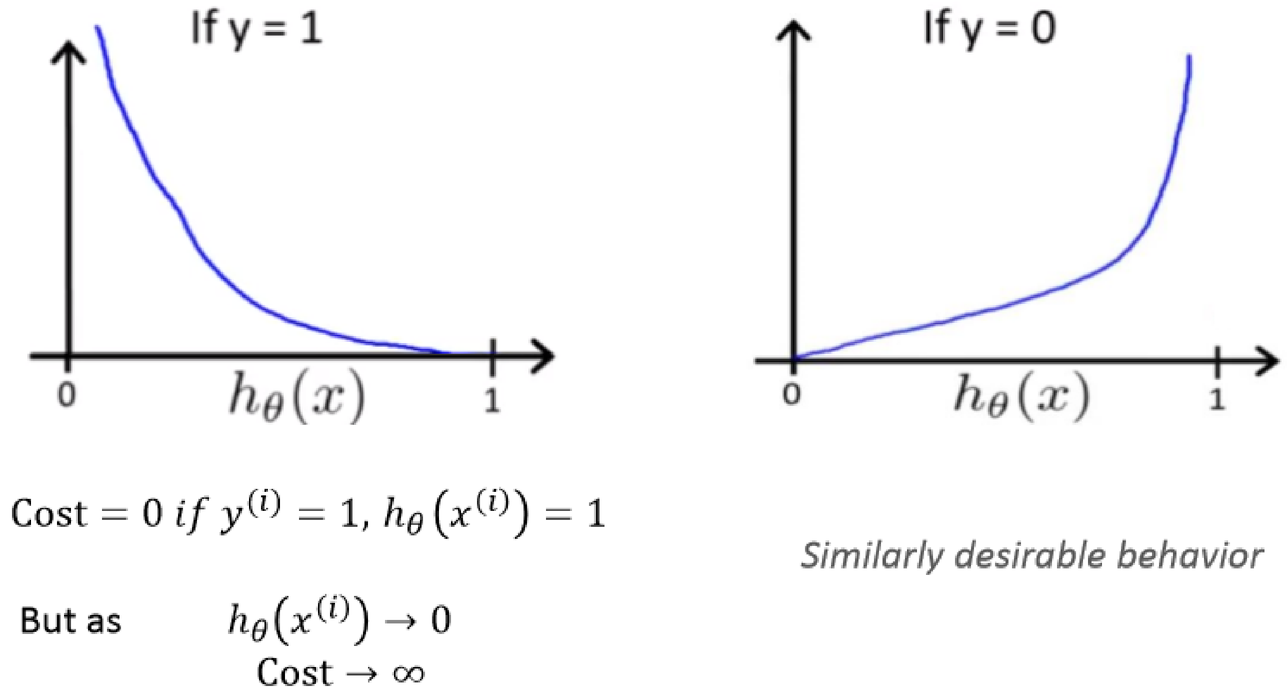


It obviously fulfills our expection of the cost function.

Derivative with respect to θ

Cost function and hypothesis.

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} Cost(h_\theta(x^{(i)}), y^{(i)})$$

$$= \frac{1}{m}\sum_{i=1}^{m} -y\log h_\theta(x^{(i)}) - (1-y^{(i)})\log(1-h_\theta(x^{(i)}))$$

where m is the number of samples.

$$h_\theta(x) = \sigma(\theta^T x) = \frac{1}{1+e^{-\theta^T x}}$$

Derivative:

$$\frac{\partial J(\theta)}{\partial \theta} = \frac{\partial J(\theta)}{\partial h_\theta(x)}\frac{\partial h_\theta(x)}{\partial \theta}$$

$$\frac{\partial J(\theta)}{\partial h_\theta(x)} = \frac{1}{m}\sum_{i=1}^{m} -y^{(i)}\frac{1}{h_\theta(x^{(i)})} - (1-y^{(i)})\frac{-1}{1-h_\theta(x^{(i)})}$$

$$= \frac{1}{m}\sum_{i=1}^{m}\frac{-y^{(i)}}{h_\theta(x^{(i)})} + \frac{1-y^{(i)}}{1-h_\theta(x^{(i)})}$$

$$= \frac{1}{m}\sum_{i=1}^{m}\frac{h_\theta(x^{(i)}) - y^{(i)}}{h_\theta(x^{(i)})(1-h_\theta(x^{(i)}))}$$

Focus on the demoninator $h_\theta(x)(1-h_\theta(x))$, it **seems like the derivative of a sigmoid function**.

$$\frac{\partial h_\theta(x)}{\partial \theta} = \frac{\partial}{\partial \theta}\sigma(x) = \frac{\partial}{\partial \theta}\frac{1}{1+e^{-\theta x}} = \frac{x\cdot e^{-\theta x}}{(1+e^{-\theta x})^2} = x\cdot\sigma(x)\cdot(1-\sigma(x))$$

Therefore,

$$\frac{\partial J(\theta)}{\partial \theta} = \frac{\partial J(\theta)}{\partial h_\theta(x)}\frac{\partial h_\theta(x)}{\partial \theta}$$

$$= \frac{1}{m}\sum_{i=1}^{m}\frac{h_\theta(x^{(i)}) - y^{(i)}}{h_\theta(x^{(i)})(1-h_\theta(x^{(i)}))}x^{(i)}\cdot\sigma(x^{(i)})\cdot(1-\sigma(x^{(i)}))$$

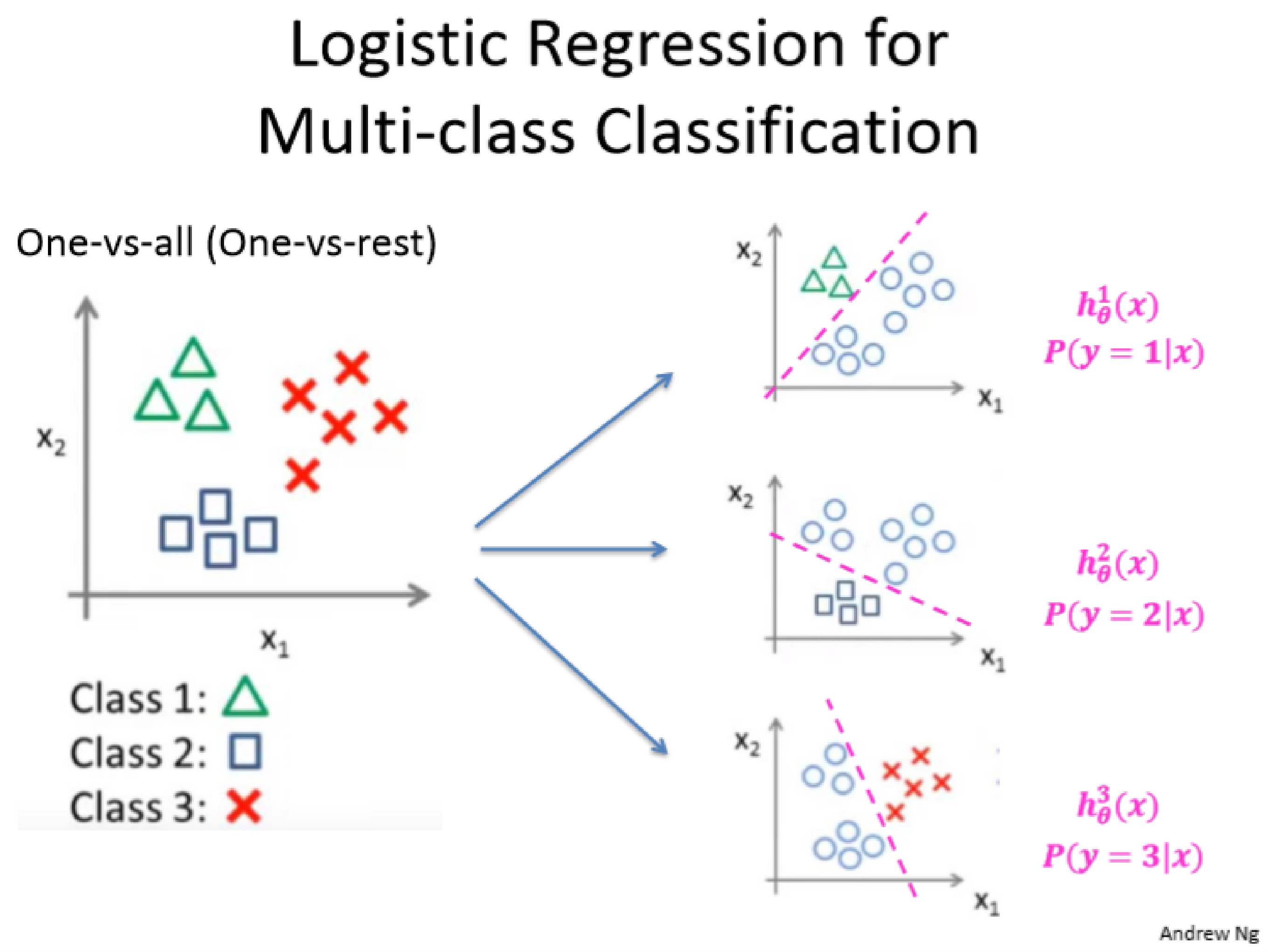
$$= \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

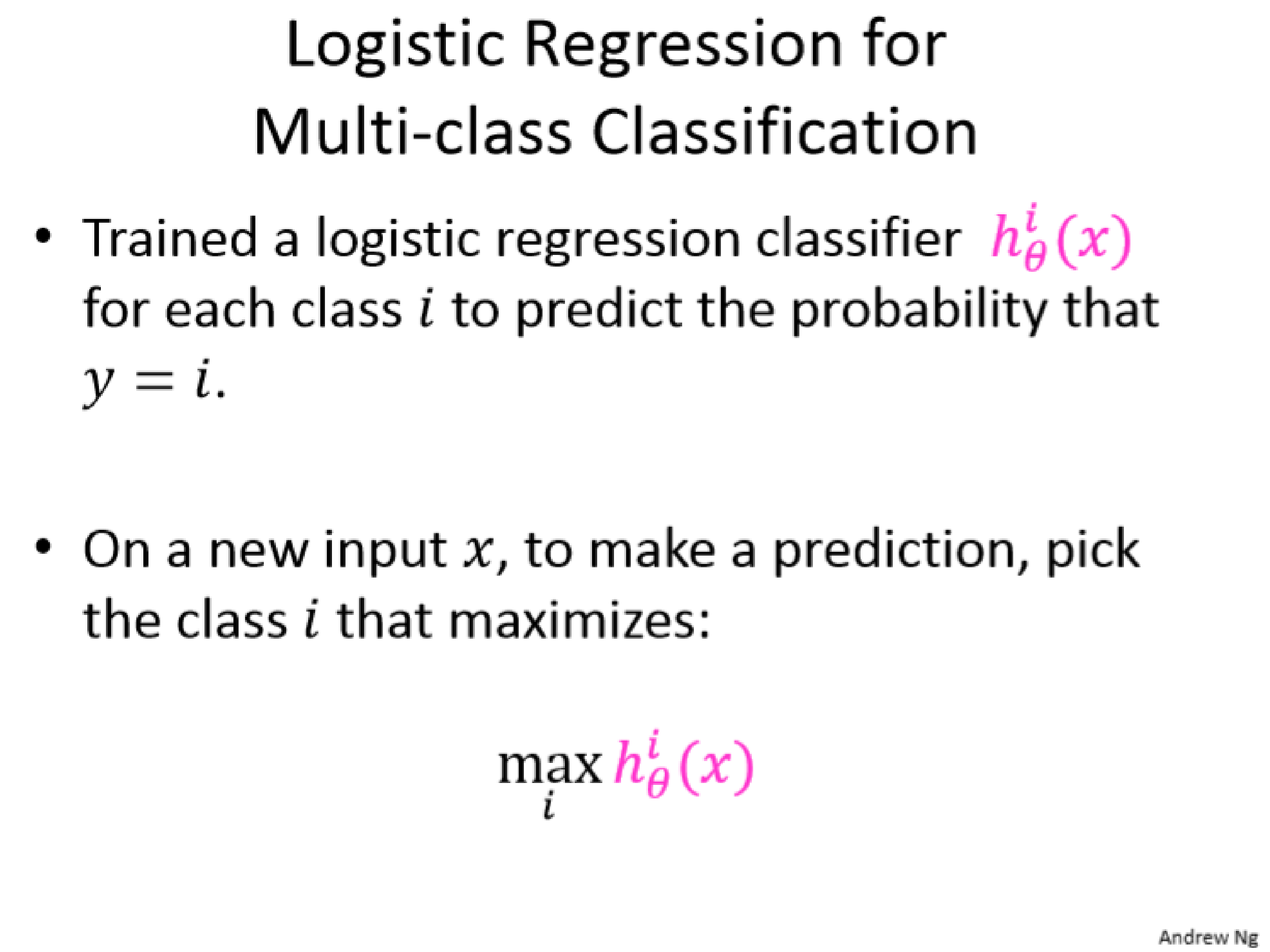
$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$$

Gradient descent for Logistic Regression

$$\theta_j = \theta_j - \alpha\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

Logistic Regression for Multi-class Classification


Logistic Regression for
Multi-class Classification
One-vs-all (One-vs-rest)
x_2
x_1
Class 1:
Class 2:
Class 3:
$h_\theta^1(x)$
$P(y=1|x)$
$h_\theta^2(x)$
$P(y=2|x)$
$h_\theta^3(x)$
$P(y=3|x)$
Andrew Ng



How to deal with non-linear features?

Transform features.

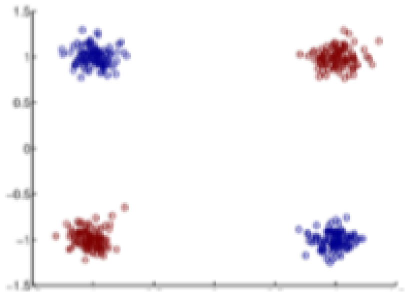
$$\phi(x) : x \in R^N \rightarrow z \in R^M$$

where M is the new dimensionality of the original feature/input x.

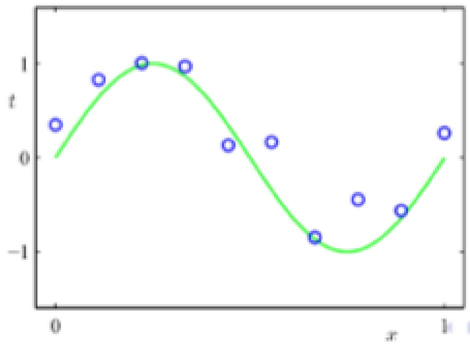
Note that M could be greater than N or less than, or the same.

What to do if data is nonlinear?

Example of nonlinear classification



Example of nonlinear regression

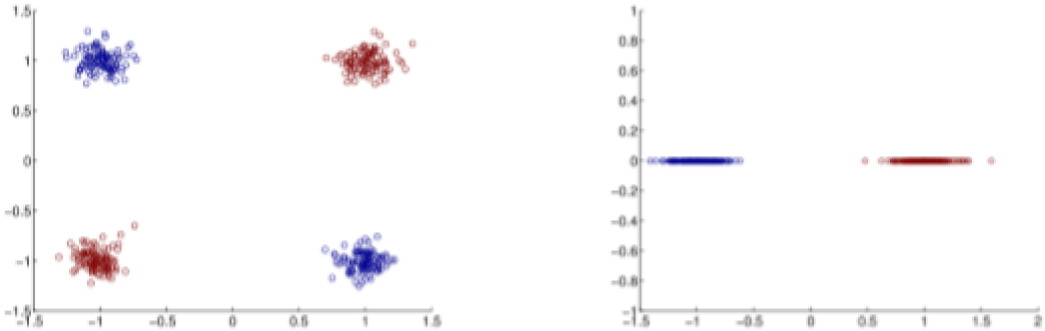


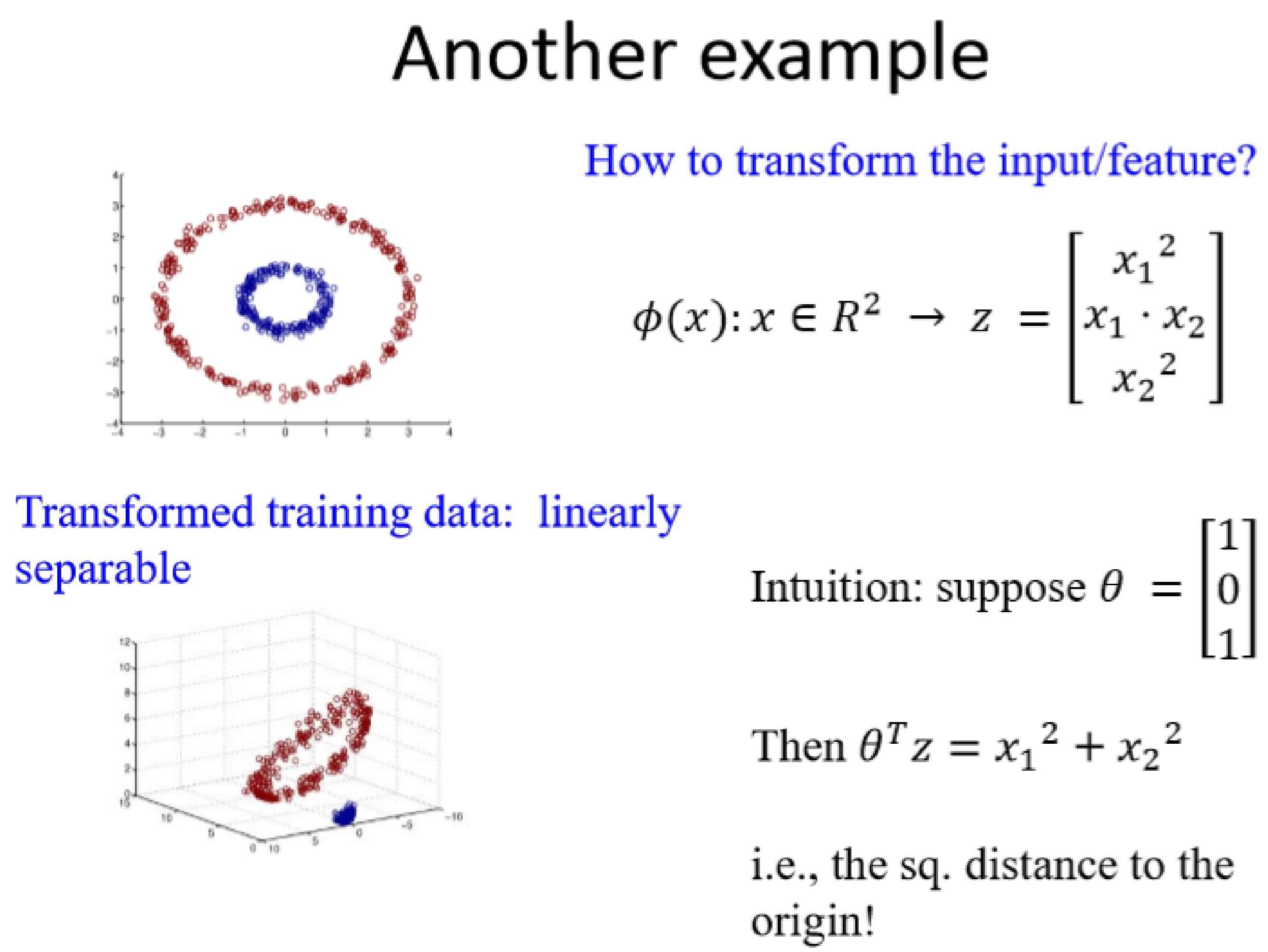
Nonlinear basis functions

Transform the input/feature

$$\phi(x) : x \in R^2 \rightarrow z = x_1 \cdot x_2$$

Transformed training data: linearly separable!





True or false question

Maximum likelihood can be used to derive a **closed-form solution** to logistic regression. **Answer: False, it can be used to derive cost, but no closed form solution exists.**

$$L(\theta) = \prod_{i=1}^{m} h(\theta, x_i)^{y_i}(1 - h(\theta, x_i))^{1-y_i}$$

解析解(closed-form solution)，又称为闭式解，是可以用解析表达式来表达的解。在数学上，如果一个方程或者方程组存在的某些解，是由有限次常见运算的组合给出的形式，则称该方程存在解析解。二次方程的根就是一个解析解的典型例子。在低年级数学的教学当中，解析解也被称为公式解。