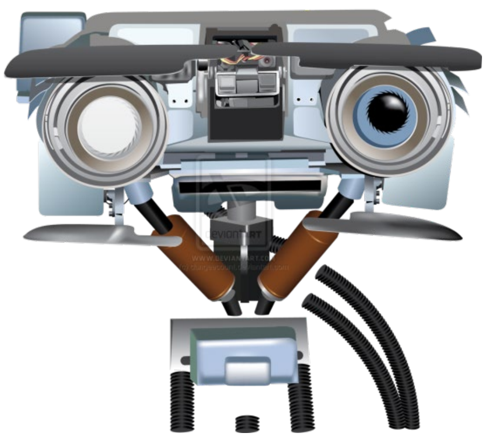


Preliminaries

Who should take this class?

- This is a difficult, math- and programming-intensive class geared primarily towards graduate students

- Historically, much fewer undergraduates manage an A than graduate students

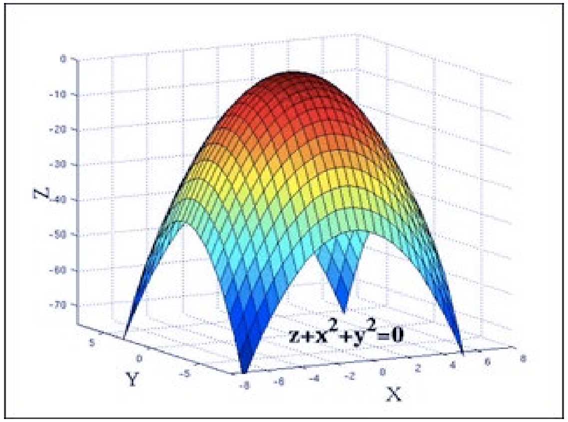
Course Prerequisites

- Linear algebra
- Multivariate Calculus, including partial derivatives
- Probability
- Comfort with programming in Python

Intro to Optimization (CAS CS 507) is not a formal prerequisite, but is highly recommended before taking this class

Course Prerequisites

- Multivariate Calculus

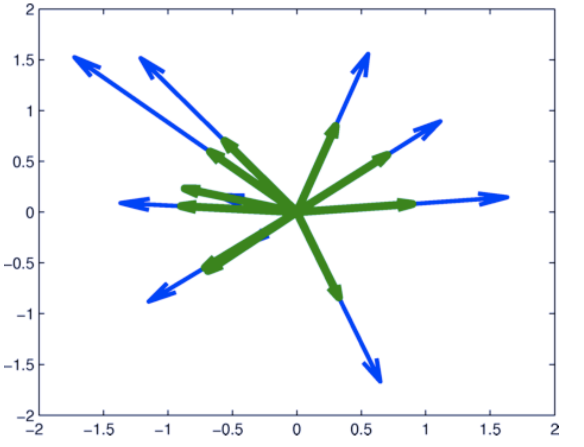


  - Vectors; dot product
  - Determinants; cross product
  - Matrices; inverse matrices
  - Square systems; equations of planes
  - Parametric equations for lines and curves
  - Max-min problems; least squares
  - Second derivative test; boundaries and infinity
  - Level curves; partial derivatives; tangent plane approximation
  - Differentials; chain rule
  - Gradient; directional derivative; tangent plane
  - Lagrange multipliers
  - Non-independent variables
  - Double integrals
  - Change of variables

- and other Calculus concepts such as convexity, etc.

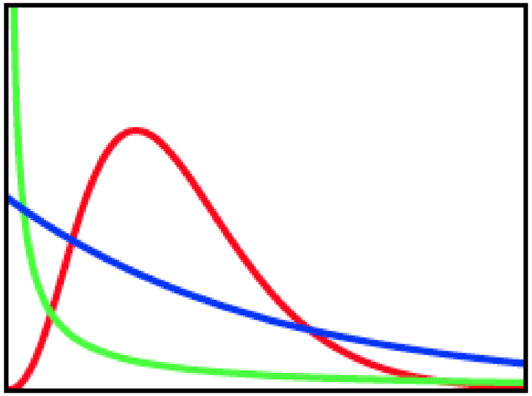
Course Prerequisites

- Linear algebra
  - Vectors and matrices
    - Basic Matrix Operations
    - Determinants, norms, trace
    - Special Matrices
  - Matrix inverse
  - Matrix rank
  - Eigenvalues and Eigenvectors
  - Matrix Calculus



Course Prerequisites

- Probability
  - Rules of probability, conditional probability and independence, Bayes rule



  - Random variables (expected value, variance, their properties); discrete and continuous variables, density functions, vector random variables, covariance, joint distributions
  - Common distributions: Normal, Bernoulli, Binomial, Multinomial, Uniform, etc.

A review: http://cs229.stanford.edu/section/cs229-prob.pdf

Course Prerequisites

python™

NumPy

"..but I really want to take this course!"

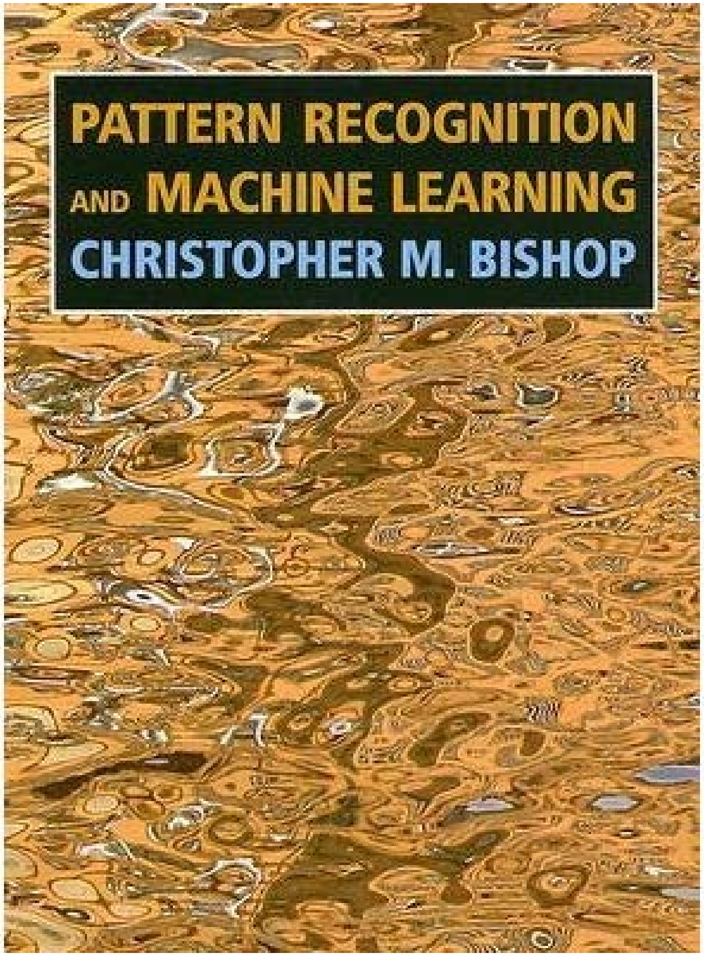
- If you lack any of these prerequisites, you SHOULD NOT take this class
- we cannot teach you the class material and also the prerequisite material
- we are not miracle workers!

- instead, please consider these alternative courses:
  - *EC 414 Introduction to Machine Learning*
  - *CS 506 Computational Tools for Data*
  - *CS 504 Data Mechanics*

Sufficient background

Insufficient background

Read the book



Matrix Algebra Review

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
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Matrix Algebra Review

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Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

T denotes the transpose operation

Vector

- We'll default to column vectors in this class

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Matrix

- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size by , i.e. m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

Basic Matrix Operations

- What you should know:
  – Addition
  – Scaling
  – Dot product
  – Multiplication
  – Transpose
  – Inverse / pseudoinverse
  – Determinant / trace

Vectors

- **Norm** $$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$
- More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties:

- **Non-negativity:** For all $x \in \mathbb{R}^n,\ f(x) \geq 0$
- **Definiteness**: f(x) = 0 if and only if x = 0.
- **Homogeneity**: For all $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ f(tx) = |t| f(x)$
- **Triangle inequality**: For all

$$x, y \in \mathbb{R}^n,\ f(x + y) \leq f(x) + f(y)$$

Matrix Operations

- **Example Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_i |x_i|$$

- General ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

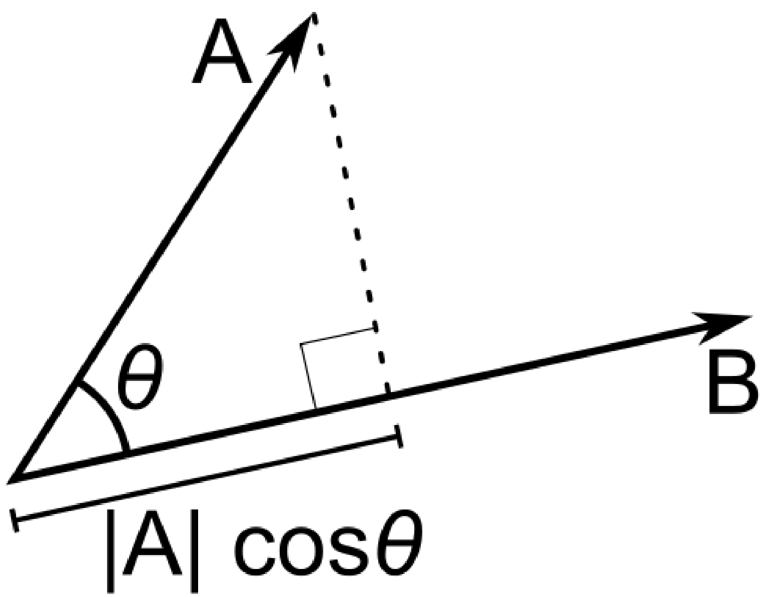
Matrix Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - x·y is also |x||y|Cos (*the angle between x and y*)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

Matrix Operations

- Inner product (dot product) of vectors
  - If B is a unit vector, then A·B gives the length of A which lies in the direction of B



Matrix Operations

- The product of two matrices

Matrix multiplication is associative: $(AB)C = A(BC)$.

Matrix multiplication is distributive: $A(B + C) = AB + AC$.

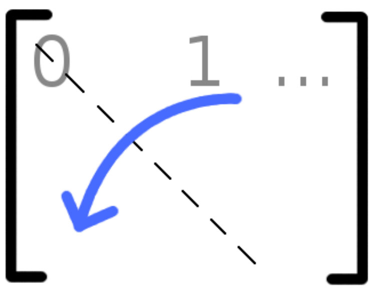
Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$. (For example, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$, the matrix product BA does not even exist if m and q are not equal!)

Matrix Operations

- Powers
  - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
  - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

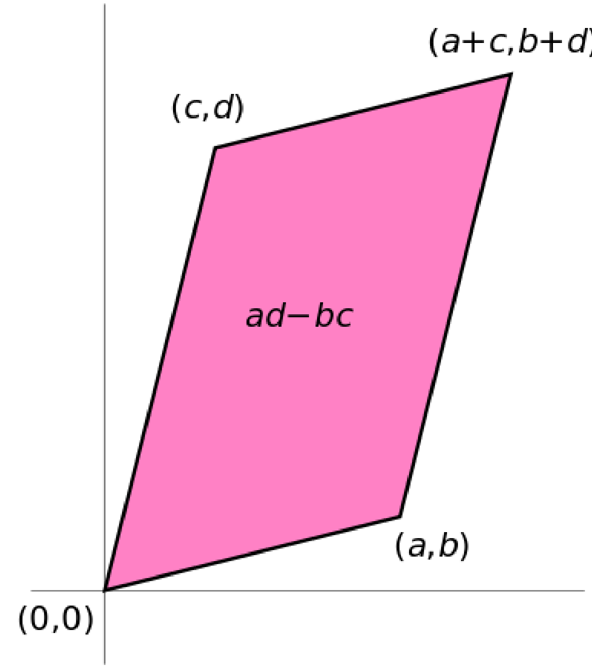
- Determinant
  - $\det(\mathbf{A})$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$
  - Properties:

$$\det(\mathbf{AB}) = \det(\mathbf{BA})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^{T}) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$



Matrix Operations

- Trace

$$\mathrm{tr}(\mathbf{A}) = \text{sum of diagonal elements}$$

$$\mathbf{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  – Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely in this class though.)

  – Properties:

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$
$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

Matrix Operations

- **Vector Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- Matrix norms: Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

Special Matrices

- Symmetric matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

$$\mathbf{A}^T = \mathbf{A}$$

- Skew-symmetric matrix

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

$$\mathbf{A}^T = -\mathbf{A}$$

- Identity matrix **`I`**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Matrix Algebra Review

- Vectors and matrices
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- Matrix inverse
- Matrix rank
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- Matrix Calculate

Inverse

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non-singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$$

Matrix Operations

- Pseudoinverse
  - Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X

Matrix Operations

- Pseudoinverse
  - Say you have the matrix equation $AX=B$, where A and B are known, and you want to solve for X
  - You could calculate the inverse and pre-multiply by it: $A^{-1}AX=A^{-1}B \rightarrow X=A^{-1}B$

Matrix Operations

- Pseudoinverse
  - Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X
  - You could calculate the inverse and pre-multiply by it: $A^{-1}AX=A^{-1}B \rightarrow X=A^{-1}B$
  - Python command would be **np.linalg.inv(A)*B**
  - But calculating the inverse for large matrices often brings problems with computer floating-point resolution (because it involves working with very small and very large numbers together).
  - Or, your matrix might not even have an inverse.

Matrix Operations

- Pseudoinverse
  - Fortunately, there are workarounds to solve AX=B in these situations. And python can do them!
  - Instead of taking an inverse, directly ask python to solve for X in AX=B, by typing **np.linalg.solve(A, B)**
  - Python will try several appropriate numerical methods (including the pseudoinverse if the inverse doesn't exist)
  - Python will return the value of X which solves the equation
    - If there is no exact solution, it will return the closest one
    - If there are many solutions, it will return the smallest one

Matrix Operations

- Python example:

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>> import numpy as np
>> x = np.linalg.solve(A,B)
x =
    1.0000
   -0.5000
```

Matrix Algebra Review

- Vectors and matrices
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Linear independence

- Suppose we have a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$
- If we can express $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2 \ldots \mathbf{v}_n$, then $\mathbf{v}_1$ is linearly *dependent* on the other vectors.
  – The direction $\mathbf{v}_1$ can be expressed as a combination of the directions $\mathbf{v}_2 \ldots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7\,\mathbf{v}_2 - .7\,\mathbf{v}_4$)

Linear independence

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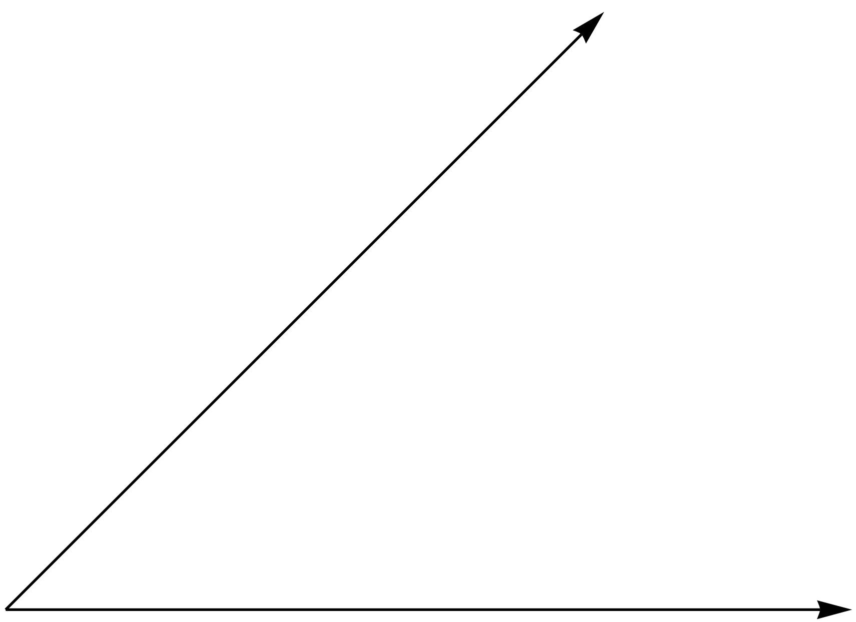
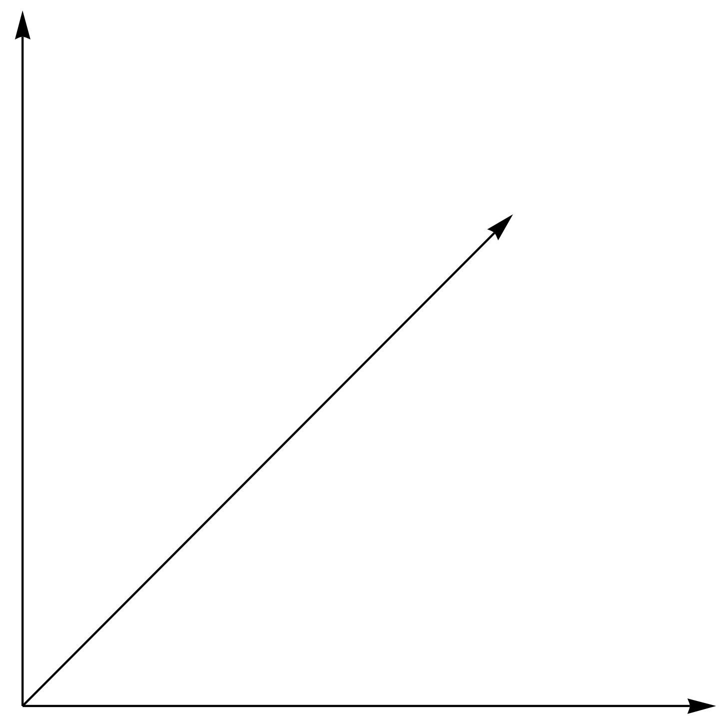
  - The direction $\mathbf{v}_1$ can be expressed as a combination of the directions $\mathbf{v}_2 \ldots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7\ \mathbf{v}_2 - .7\ \mathbf{v}_4$)
- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero)

Linear independence

Linearly independent set



Not linearly independent



Matrix rank

- Column/row rank

$$\text{col-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent column vectors of } \mathbf{A}$$
$$\text{row-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent row vectors of } \mathbf{A}$$

  - Column rank always equals row rank

- Matrix rank

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$

Matrix rank

- For transformation matrices, the rank tells you the dimensions of the output
- E.g. if rank of **A** is 1, then the transformation

p'=Ap

maps points onto a line.

- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+2y \end{bmatrix}$$

All points get mapped to the line y=2x

Matrix rank

- If an m x m matrix is rank m, we say it's "full rank"
  - Maps an m x 1 vector uniquely to another m x 1 vector
  - An inverse matrix can be found
- If rank < m, we say it's "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also doesn't exist for non-square matrices

Matrix Algebra Review

- Vectors and matrices
  - Basic Matrix Operations
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- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors(SVD)
- Matrix Calculus

Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.

$$Ax = \lambda x, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.
- Applying *A* to the eigenvector only scales the eigenvector by the scalar value λ, called an eigenvalue.

$$Ax = \lambda x, \quad x \neq 0.$$

Properties of eigenvalues

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i.$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i.$$

- The rank of A is equal to the number of non-zero eigenvalues of A.
- The eigenvalues of a diagonal matrix D = diag(d1, . . . dn) are just the diagonal entries d1, . . . dn

Diagonalization

- Eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

  - Where D is a diagonal matrix of the eigenvalues

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Diagonalization

- Eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

- Assuming all λ_i's are unique:

$$A = VDV^T$$

  - Remember that the inverse of an orthogonal matrix is just its transpose and the eigenvectors are orthogonal

Symmetric matrices

- Properties:
  - For a symmetric matrix A, all the eigenvalues are real.
  - The eigenvectors of A are orthonormal.

$$A = VDV^T$$

Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y \;\; = \sum_{i=1}^{n} \lambda_i y_i^2$$

  – where $y = V^T x$

- So, if we wanted to find the vector x that:

$$\max_{x \in \mathbb{R}^n} \;\; x^T A x \qquad \text{subject to } \|x\|_2^2 = 1$$

Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y \;\; = \sum_{i=1}^{n} \lambda_i y_i^2$$

  – where $y = V^T x$

- So, if we wanted to find the vector x that:

$$\max_{x \in \mathbb{R}^n} \;\; x^T A x \qquad \text{subject to } \|x\|_2^2 = 1$$

  – Is the same as finding the eigenvector that corresponds to the largest eigenvalue.

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Matrix Calculus – The Gradient

- Let a function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ take as input a matrix A of size m × n and returns a real value.
- Then the **gradient** of **f:**

$$\nabla_A f(A) \in \mathbb{R}^{m \times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

Matrix Calculus – The Gradient

- Every entry in the matrix is: $(\nabla_A f(A))_{ij} = \frac{\partial f(A)}{\partial A_{ij}}.$
- the size of $\nabla_A f(A)$ is always the same as the size of A. So if A is just a vector x:

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Find: $\dfrac{\partial f(x)}{\partial x_k} = ?$

$$\nabla_x f(x) = ?$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \sum_{i=1}^{n} b_i x_i$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} b_i x_i = b_k.$$

- From this we can conclude that: $\nabla_x b^T x = b.$

Matrix Calculus – The Gradient

- Properties
  - $\nabla_x(f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x)$.
  - For $t \in \mathbb{R}$, $\nabla_x(t\, f(x)) = t\nabla_x f(x)$.

Matrix Calculus – The Jacobian

- if you have a vector valued function $\boldsymbol{y} = f(\boldsymbol{x})$ e.g., $\mathrm{x}, y \in \mathbb{R}^n$, then the gradient of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ is a Jacobian matrix:

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

Matrix Calculus – The Hessian

- The Hessian matrix with respect to x, written $\nabla_x^2 f(x)$ or simply as H is the n × n matrix of partial derivatives

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$

- Exercise: Why is the Hessian always symmetric?

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$

- The Hessian is always symmetric, because

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}.$$

- This is known as Schwarz's theorem: The order of partial derivatives don't matter as long as the second derivative exists and is continuous.

Matrix Calculus – The Hessian

- Note that the hessian is not the gradient of whole gradient of a vector (this is not defined). It is actually the gradient of **every entry** of the gradient of the vector.

$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Eg, the first column is the gradient of $\frac{\partial f(x)}{\partial x_1}$

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \boxed{\begin{matrix} \frac{\partial^2 f(x)}{\partial x_1^2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} \end{matrix}} & \begin{matrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2^2} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \begin{matrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n^2} \end{matrix} \end{bmatrix}$$

Common vector derivatives

Scalar derivative			Vector derivative		
$f(x)$	$\rightarrow$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	$f(\mathbf{x})$	$\rightarrow$	$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{B}$	$\rightarrow$	$\mathbf{B}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{b}$	$\rightarrow$	$\mathbf{b}$
x^2	$\rightarrow$	$2x$	$\mathbf{x}^T\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
bx^2	$\rightarrow$	$2bx$	$\mathbf{x}^T\mathbf{B}\mathbf{x}$	$\rightarrow$	$2\mathbf{B}\mathbf{x}$