Bayesian Method

先看贝叶斯公式和条件概率的相关知识。

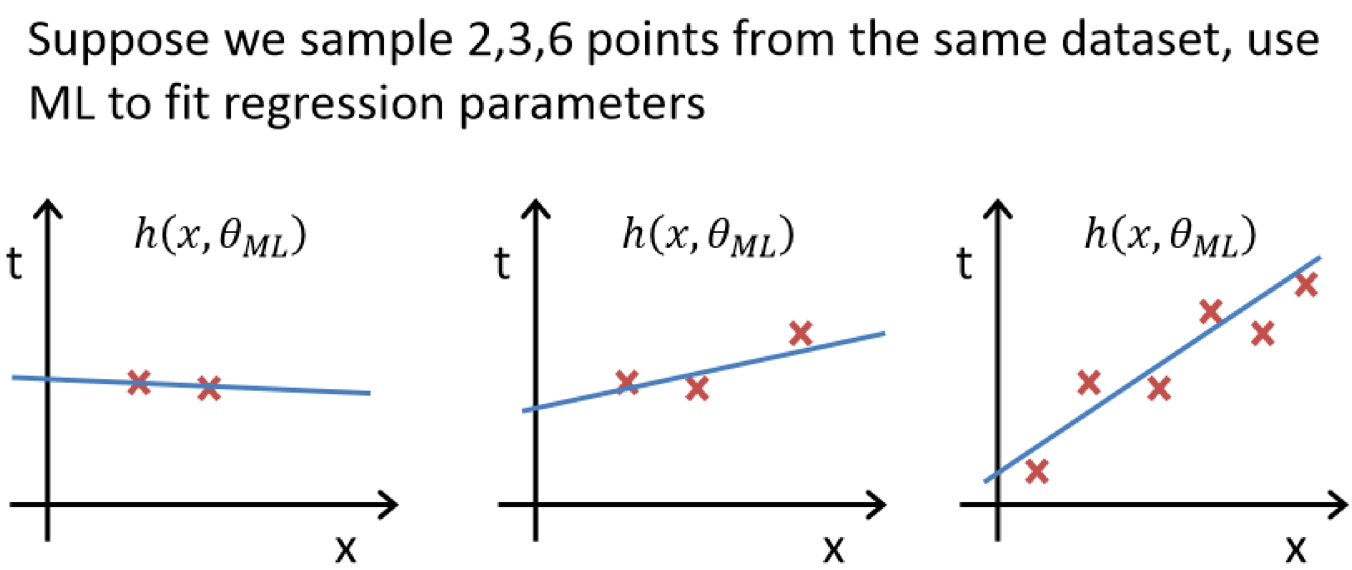
Frequentist vs. Bayesian

- Frequentists: Related to the frequencies of related events.
- Bayesians: Related to our own certainty/uncertainty of events.
- Frequentists: Variation of data in terms of fixed model parameters.
- Bayesians: Variation of beliefs about parameters in terms of fixed observed data.

Problem with Maximum Likelihood

Bias

当数据量很少的时候，Maximum Likelihood会有bias,受数据量影响大。



Overfitting

maximum Likelihood cannot be used to choose complexity of model.

E.g. suppose we want to estimate the number of the basis functions（特征变换函数）.

- Choose K = 1?
- Or K=15 例如，我们用多项式特征去拟合，K就是多项式的最高次项。

Maximum Likelihood will always choose K that best fits training data (in this case, K=15).

Solution: Bayesian method

Define a prior distribution over the parameters (**results in regularization)**.

Frequentist vs. Bayesian

Frequentist -- maimize data likelihood

$$p(D|model) = p(D|\theta)$$

Bayesian -- treat θ as random variable, maximize posterior

Bayes' Rule

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

- $p(D|\theta)$ is the data likelihood which is the same as before in Maximum Likelihood.
- $p(\theta)$ is the prior over the model parameters, which is a new **distribution** we model; specifies which parameters are more likely a priori, **before seeing any data**.
- $p(D)$ does not depend on θ, constant when choosing θ with highest posterior probability.

Prior over model parameters -- Intuition

Prior Distribution $p(\theta)$

Prior distributions $p(\theta)$ are probability distributions of model parameters **based on some a priori knowledge about the parameters**.

换句话说，这个先验分布，与你观测到的实验结果是相互独立的。

Prior distributions are independent of the observed data.

Example: toss a coin

What is the probability of heads (θ)?

在这之前，先看一个beta分布。为什么我们选择这个分布，后续会讲。

Beta Distribution

$$Be(\alpha, \beta)$$

Probability density function

$$p(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

where

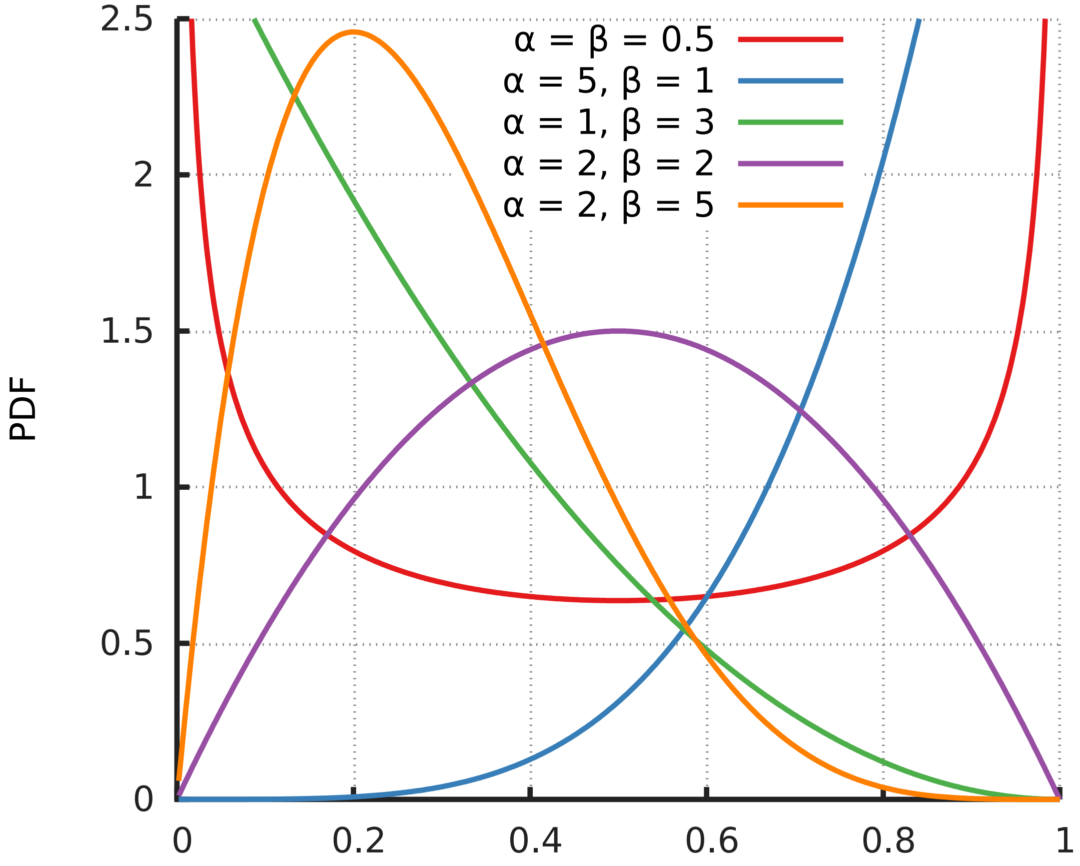
$$\alpha > 0, \beta > 0$$

$$\Gamma(z) = \int_0^\infty \frac{t^{z-1}}{e^t} dt$$

if z is a complex number with a positive real part.

$$\Gamma(z) = (z-1)!$$

if z is a positive integer.



回顾一下贝叶斯方法：

Bayes' Rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

$p(\theta)$是我们刚刚选择的beta分布。

Likelihood function

我们看看已知的实验结果：

- n = 10 coin tosses
- y = 4 number of heads

$p(y|\theta)$ 是Likelihood function for the Data.

独立重复实验，服从伯努利分布。

Typesetting math: 100%

$$p(y|\theta) = Binomial(n, \theta) = C_n^y \theta^y (1-\theta)^{n-y}$$

看一下刚刚的beta分布，统计意义上是我们对参数的估计。

Prior Distribution

记住，这个Prior是我们开始实验之前，就对这个分布有所了解。

Uninformative Prior

如果我们事先对这个分布没有任何了解，那我们的Prior就是uninformative.

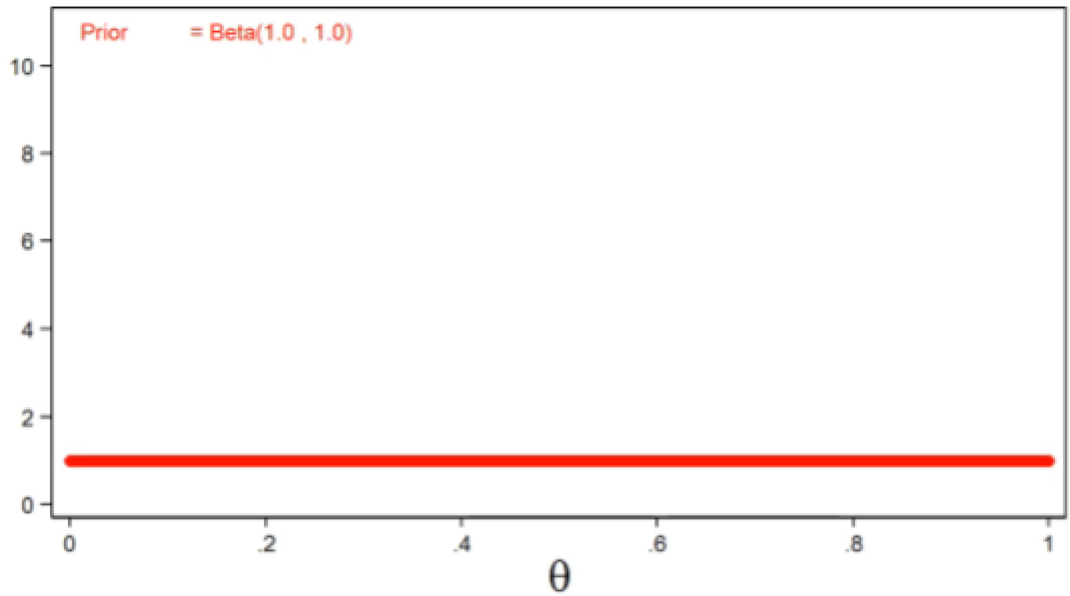
参数设置为：

$$\alpha = \beta = 1$$

beta分布就会变成：

$$p(\theta) = 1$$

意义就是无论你抛硬币多少次，θ正面朝上的次数，总是均匀的。



Informative Prior

我们事先就知道，正面朝上的概率大概为0.5.

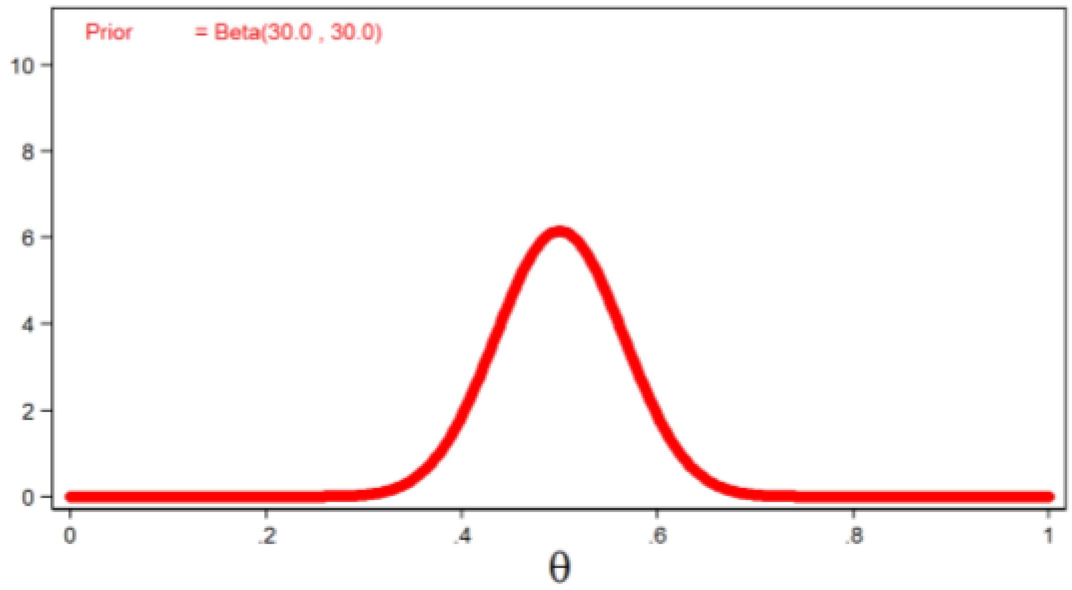
参数设置为：

$$\alpha = \beta = 30$$

beta分布就会变成：

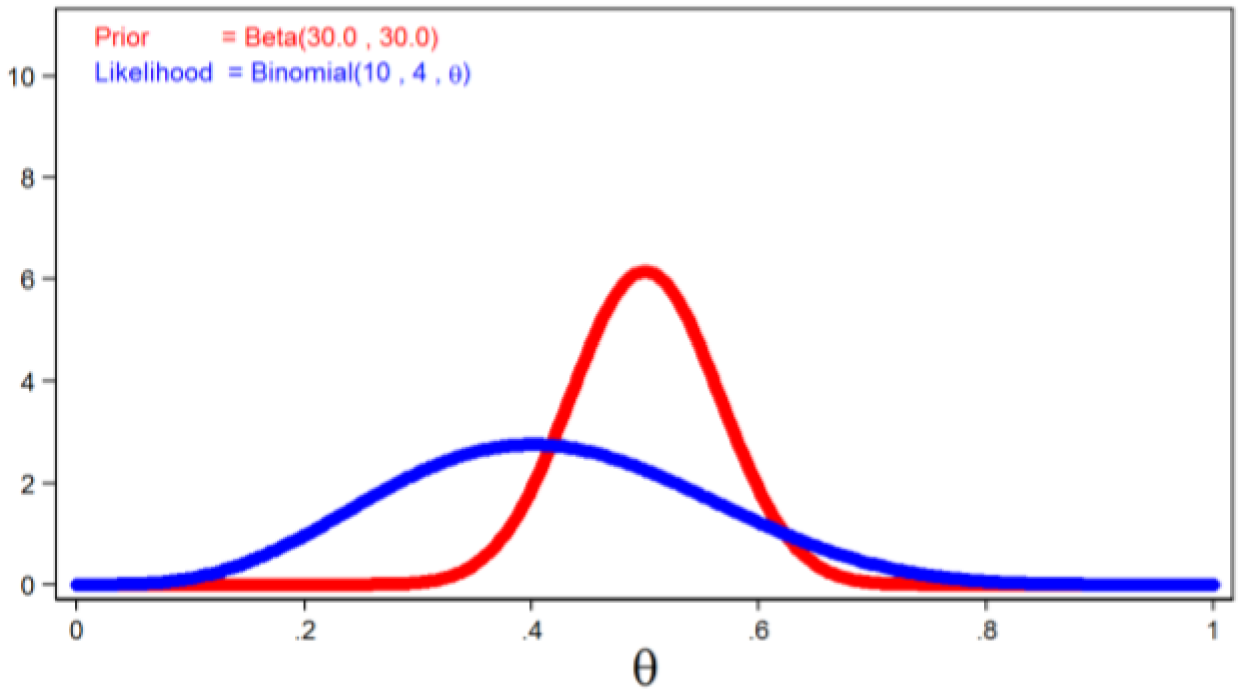
$$p(\theta) = \frac{60!}{30! \times 30!} \theta^{29} (1-\theta)^{29}$$

Typesetting math: 100%



Prior and Likelihood PDF

看一下Prior和Likelihood的概率密度分布。



如果我们只看Likelihood，它是有bias的，因为我们只抛了10次硬币，正面朝上只有4次。

Posterior Distribution

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

Typesetting math: 100%

$$Posterior = Prior \times Likelihood$$

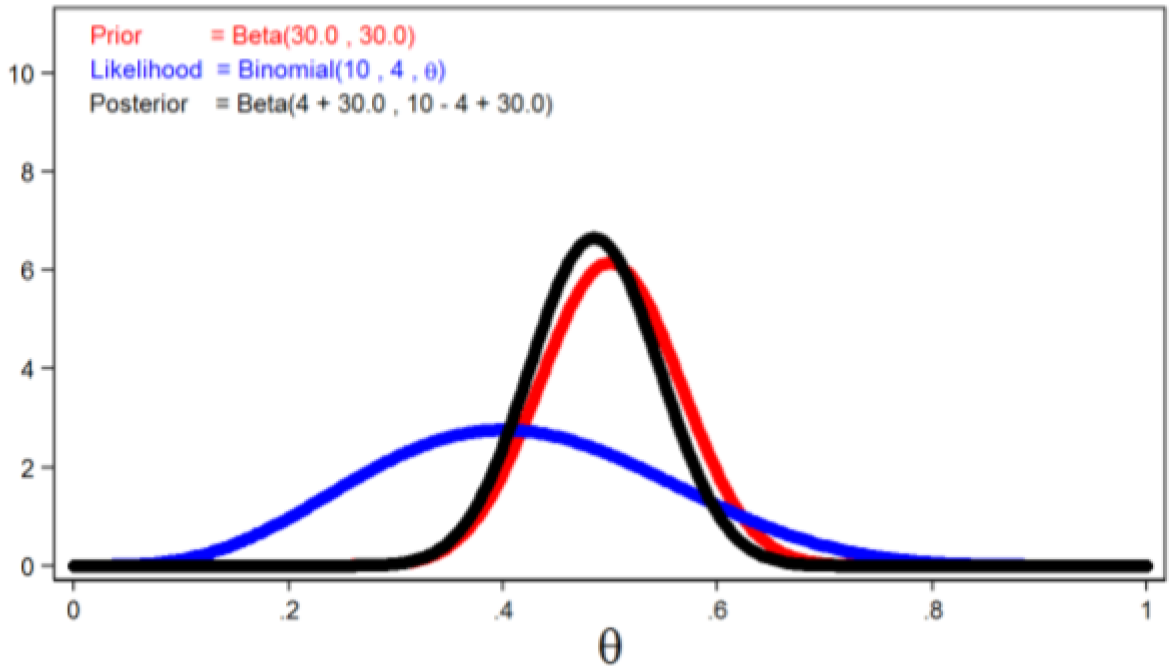
$$P(\theta|y) = P(\theta)P(y|\theta)$$

$$P(\theta|y) = Beta(\alpha, \beta) \times Binomial(n, \theta)$$

$$= Beta(y + \alpha, n - y + \beta)$$

这里选择beta分布，是为了数学上计算的方便。

看我们Posterior的概率密度分布：



Likelihood被Prior修正了。

Bayesian Linear Regression

直观上讲，我们用贝叶斯来修正Linear Regression，比如说，加上高斯噪声。

Reference

https://towardsdatascience.com/introduction-to-bayesian-linear-regression-e66e60791ea7

从频率主义者的角度，就是比较常规的Gradient descent梯度下降法来进行训练。最终会得到一个solution:

$$y = \theta^T X$$

[illegible]，是y服从一个概率分布，比如：

Typesetting math: 100%

$$N(\theta^T X, \sigma^2 I)$$

把原来频率主义者得到的y当作是均值，再叠加上高斯噪声。

The posterior probability of the model parameters is conditional upon the training inputs and outputs:

$$p(\theta|y, X) = \frac{p(y|\theta, X)p(\theta|X)}{p(y|X)}$$

$$Posterior = \frac{Likelihood \times Prior}{Normalization}$$

Typesetting math: 100%