Generative model

Overview

In statistical classification, including machine learning, two main approaches are called the generative approach and the discriminative approach.

- Given an observable variable X and a target variable Y, a **generative model** is a statistical model of the joint probability distribution on $X × Y$, $P(X,Y)$;
- A **discriminative model** is a model of the conditional probability of the target Y, given an observation x, symbolically, $P(Y|X=x)$.

Examples

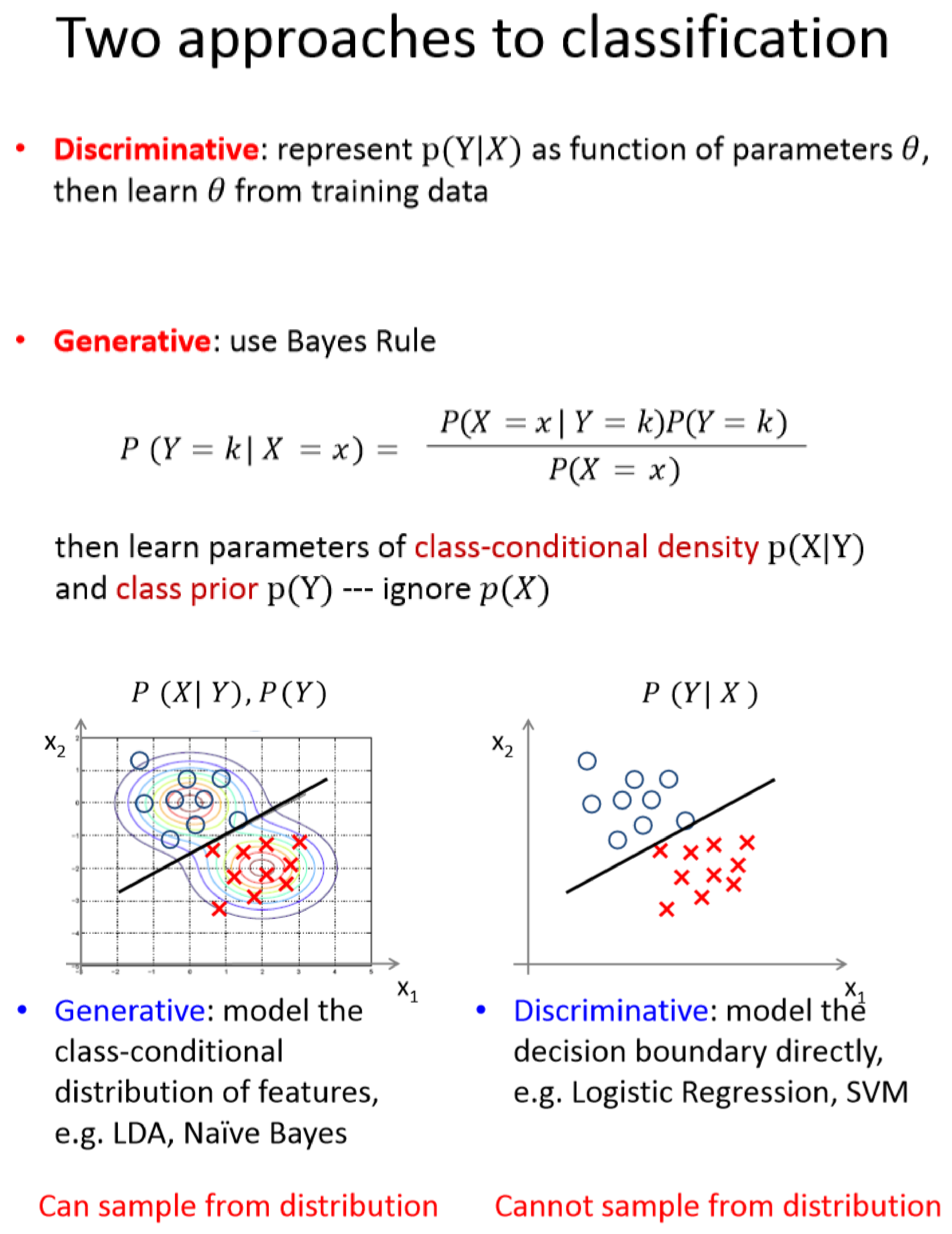
Standard examples of each, all of which are linear classifiers, are:

- generative classifiers:
  - naive Bayes classifier
  - linear discriminant analysis
- discriminative model:
  - logistic regression
  - support vector machine

Discriminative v.s. Generative

判别模型：不能产生新的数据样本； 生成模型，可以产生新的数据样本。

两者都能进行判别。



Linear Discriminant Analysis

线性判别分析 is a generative model

我们看看Webster怎么说这两个词有什么不同：

- discriminative: making distinctions.
- discriminant: a mathematical expression providing a criterion for the behavior of another more complicated expression, relation, or set of relations. If multiple antenna inputs are available, a very powerful **discriminant** is to see if all of the signals come from the same direction.

Generative Classifer

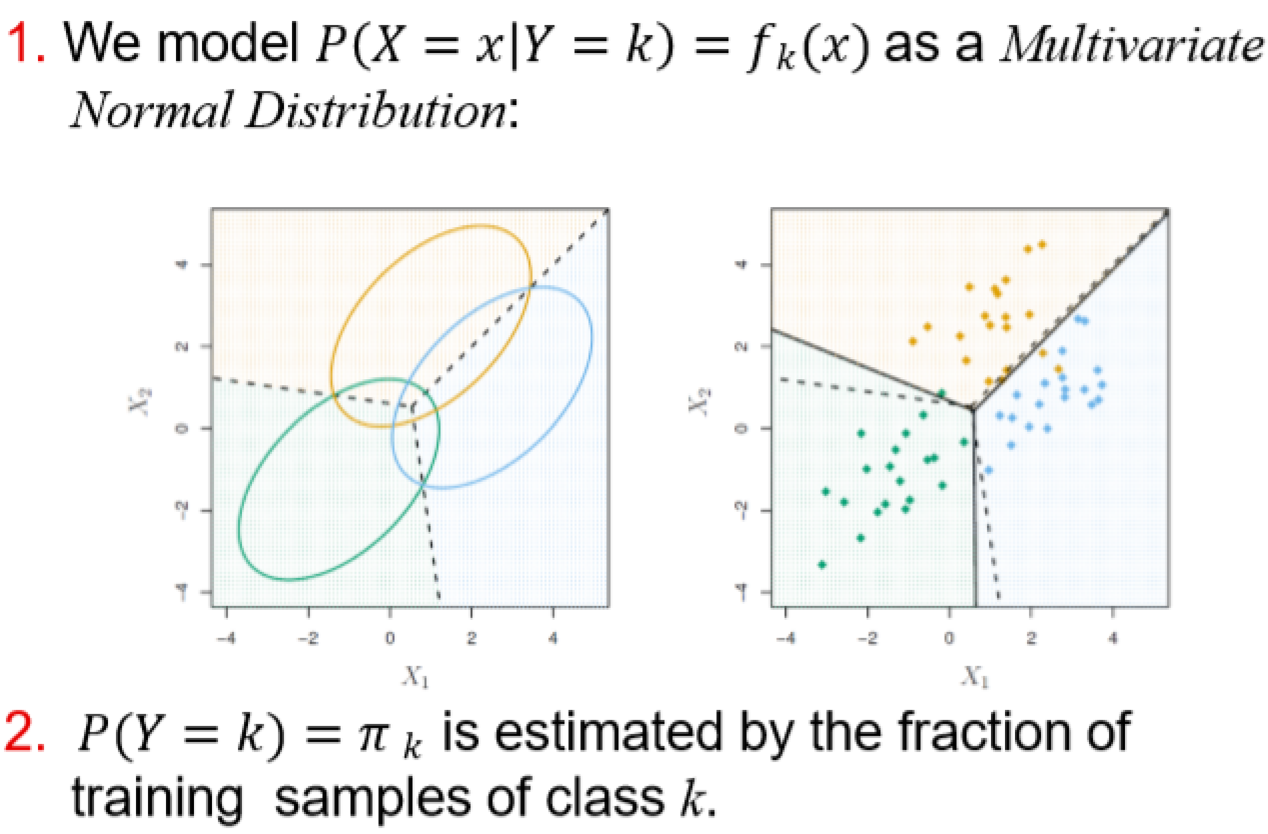
Instead of estimating $P(Y|X)$, we will estimate:

1. $P(X|Y)$: Given the category, what is the distribution of the inputs.
2. $P(Y)$: How likely are each of the categories.

Then, we use *Bayes rule* to obtain the estimate:

$$P(Y=k|X=x) = \frac{P(X=x|Y=k)P(Y=k)}{\sum_j P(X=x|Y=j)P(Y=j)}$$

Linear Discriminant Analysis (LDA)



LDA prior

假设： We know

$$P(Y=k)=\pi_k$$

exactly.

这个是Prior，是 class k 样本数量的占比。

LDA class-conditional density

$$P(X=x|Y=k)=f_k(x)=\frac{1}{(2\pi)^{p/2}\Sigma^{1/2}}\exp(-\frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k))$$

is Mutivariate Normal with density.

其中，p是变量维度。

Goal: maximize P(Y = k | X = x) over k

根据贝叶斯法则，得到：

$$P(Y=k|X=x)=\frac{f_k(x)\pi_k}{P(X=x)}$$

分母不取决于k，所以可以写成常数C.

$$P(Y=k|X=x)=C\times f_k(x)\pi_k$$

展开：

$$P(Y=k|X=x) = \frac{C\pi_k}{(2\pi)^{p/2}\Sigma^{1/2}}\exp(-\frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k))$$

把一切不取决于k的量，都吸收进C，得到C':

$$P(Y=k|X=x) = C'\pi_k\exp(-\frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k))$$

两边取对数：

$$\log P(Y=k|X=x) = \log C' + \log\pi_k - \frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k)$$

展开各项：

$$\log P(Y=k|X=x) = \log C' + \log\pi_k - \frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k)$$

注意，这里x和μ_k都是p by 1 matrix.

$$\log P(Y=k|X=x) = \log C' + \log\pi_k - \frac{1}{2}(x-\mu_k)^T(\Sigma^{-1}x - \Sigma^{-1}\mu_k)$$

$$= \log C' + \log\pi_k - \frac{1}{2}[x^T\Sigma^{-1}x - \mu_k^T\Sigma^{-1}x - x\Sigma^{-1}\mu_k + \mu_k^T\Sigma^{-1}\mu_k]$$

$$= \log C' + \log\pi_k - \frac{1}{2}[x^T\Sigma^{-1}x + \mu_k^T\Sigma^{-1}\mu_k] + x^T\Sigma^{-1}\mu_k$$

吸收与k无关的常量，得到：

$$\log P(Y=k|X=x) = C'' + \log\pi_k - \frac{1}{2}\mu_k^T\Sigma^{-1}\mu_k + x^T\Sigma^{-1}\mu_k$$

我们重新定义一下目标函数：

$$\delta_k(x) = \log\pi_k - \frac{1}{2}\mu_k^T\Sigma^{-1}\mu_k + x^T\Sigma^{-1}\mu_k$$

At an input x, we predict the output with the highest $\delta_k(x)$.

LDA Linear Decision Boundaries

如何确定分界线？ 当输入x恰好坐落在分界线时，例如 class k 和 class l 的分界线，那么他们的目标函数，判定这个输入x是 class k 和 class l的概率就会相等。

得到：

$$\delta_k(x) = \delta_l(x)$$

展开：

$$\log\pi_k - \frac{1}{2}\mu_k^T\Sigma^{-1}\mu_k + x^T\Sigma^{-1}\mu_k = \log\pi_l - \frac{1}{2}\mu_l^T\Sigma^{-1}\mu_l + x^T\Sigma^{-1}\mu_l$$

决定Decision Boundaries (超平面hyperplane)斜率的是：

$$\Sigma^{-1}(\mu_k - \mu_l)$$

决定位置的是：

$$\log \pi_k - \log \pi_l - \frac{1}{2}\mu_k^T \Sigma^{-1} \mu_k + -\frac{1}{2}\mu_l^T \Sigma^{-1} \mu_l$$

仅仅改变π不会改变超平面的方向，只会平移超平面。

Estimating π_k

You can't use 'macro parameter character #' in math mode

The fraction of training samples of class k.

Estimating the parameters of $f_k(x)$

- 均值

You can't use 'macro parameter character #' in math mode

- 协方差

单变量：

$$\sigma^2 = \frac{\sum_{k=1}^{K} \sum_{i; y_i = k} (x_i - \mu_k)^2}{n - K}$$

多变量： Compute the vectors of deviations

$$(x_1 - \mu_{y_1}), (x_2 - \mu_{y_2}), \ldots, (x_n - \mu_{y_n})$$

and use an estimate of its covariance matrix.

Quadratic Discriminant Analysis (QDA)

到这为止，我们认为每个class的协方差矩阵是一样的， 如果设置成不一样的，我们的目标函数就会变成：

In **quadratic discriminant analysis** we estimate a mean μ_k and a covariance matrix $\mathbf{\Sigma}_k$ for each class separately.

Given an input, it is easy to derive an objective function:

$$\delta_k(x) = \log \pi_k - \frac{1}{2}\mu_k^T \mathbf{\Sigma}_k^{-1} \mu_k + x^T \mathbf{\Sigma}_k^{-1} \mu_k - \frac{1}{2} x^T \mathbf{\Sigma}_k^{-1} x - \frac{1}{2} \log |\mathbf{\Sigma}_k|$$

This objective is now quadratic in x and so are the decision boundaries.

- Bayes boundary (– – –)
- LDA (· · · · · ·)
- QDA (——).

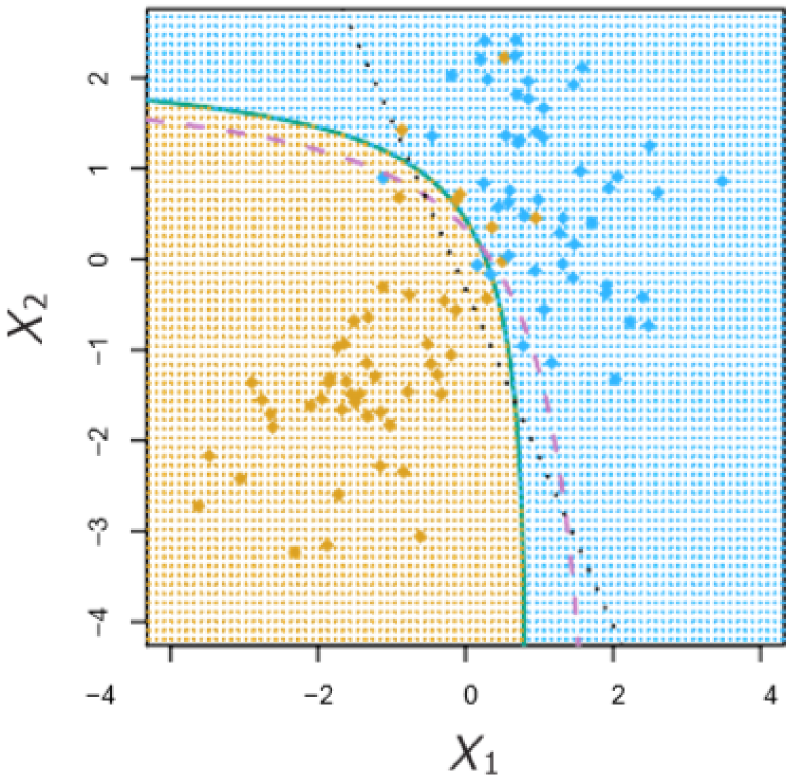
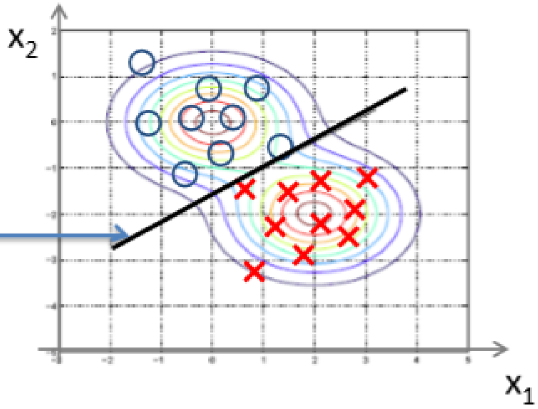


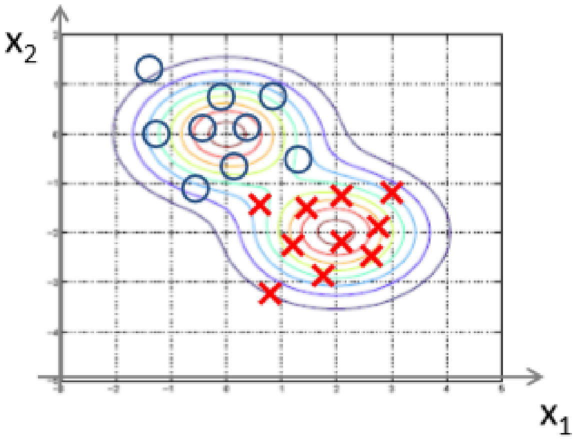
Illustration of Decision Boundary

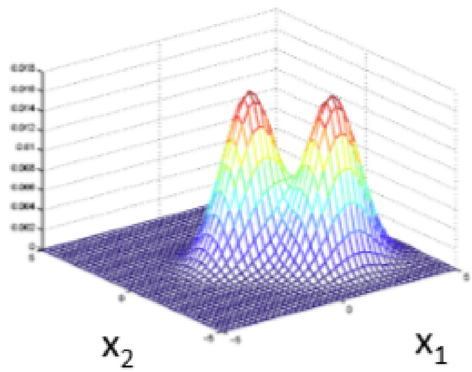
$$\underbrace{\log\frac{\pi_k}{\pi_l}}_{\text{class prior log-ratio}} - \underbrace{\frac{1}{2}(\mu_k+\mu_l)^T\Sigma^{-1}(\mu_k-\mu_l)}_{\text{constant}} + \underbrace{x^T}_{\text{input}}\underbrace{\Sigma^{-1}}_{\text{covariance}}\underbrace{(\mu_k-\mu_l)}_{\text{diff. in class means}} = 0$$



Can re-write as $\theta_0 + x^T\theta = 0$

Effect of Covariance Matrix





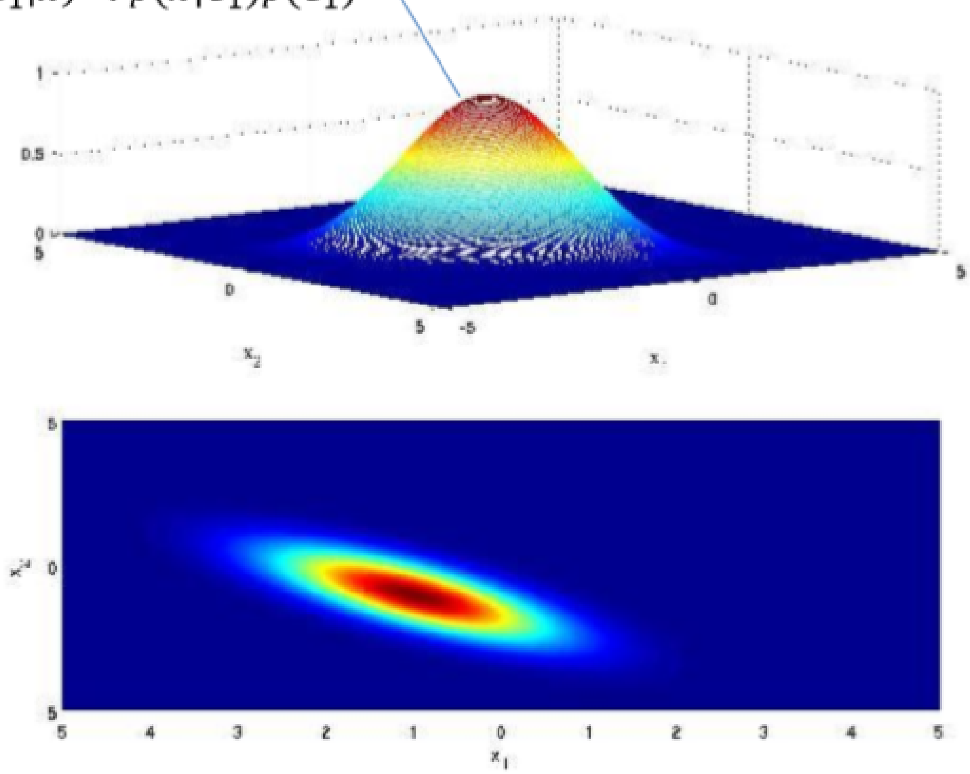
- covariance matrix determines the shape of the Gaussian density, so
- in LDA, the Gaussian densities for different classes have the same shape, but are shifted versions of each other (different mean vectors).

Effect of Class Prior

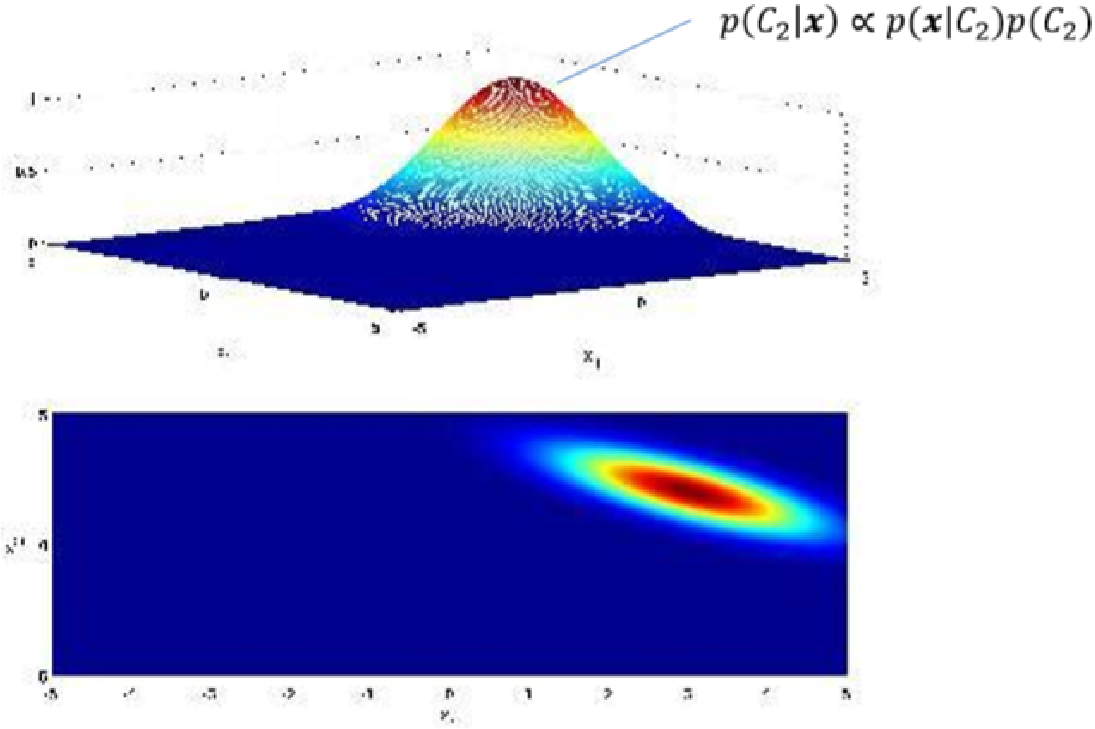
- What effect does the prior p(class), or π_k, have?
- Lets look at an example for 2 classes…

$$\underbrace{\boxed{\log \frac{\pi_k}{\pi_l}}}_{\text{class prior log-ratio}} - \boxed{\frac{1}{2}(\mu_k + \mu_l)^T \Sigma^{-1} (\mu_k - \mu_l)} + \boxed{x^T}\boxed{\Sigma^{-1}}\boxed{(\mu_k - \mu_l)} = 0$$

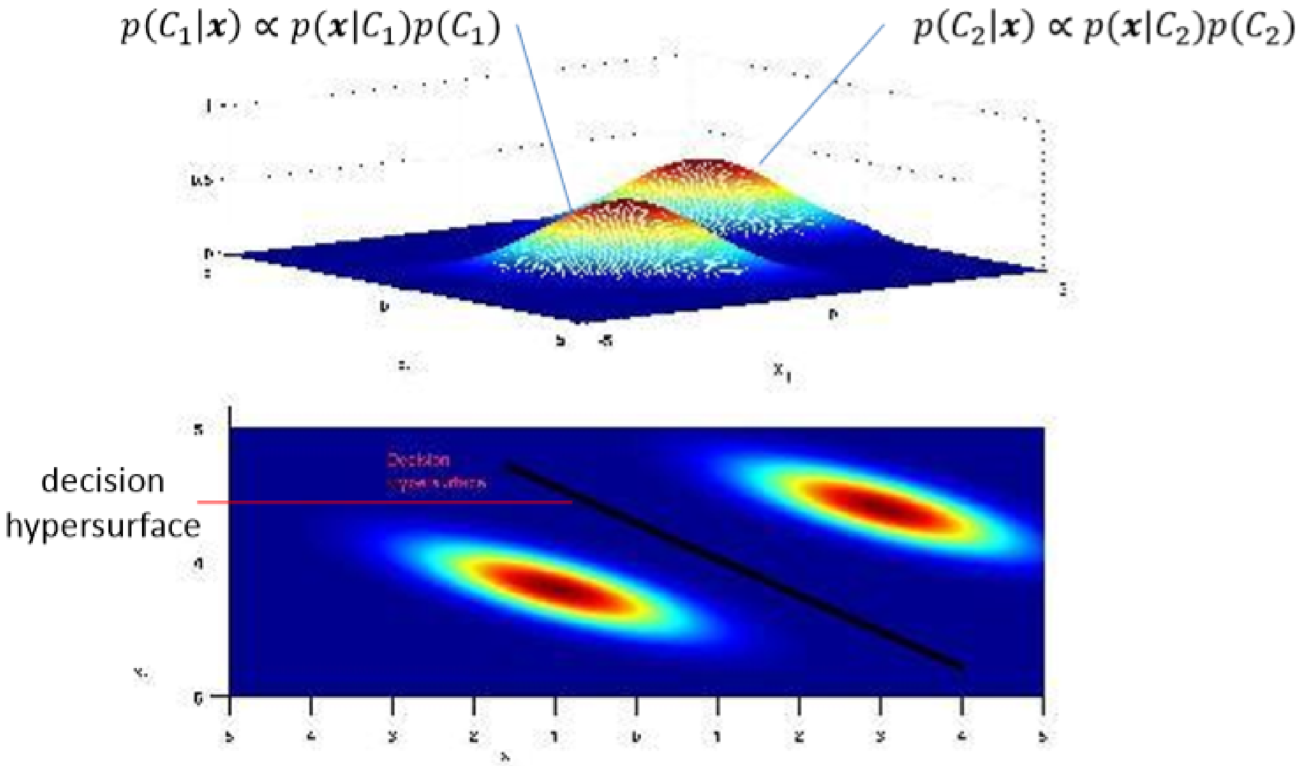
$p(C_1|\boldsymbol{x}) \propto p(\boldsymbol{x}|C_1)p(C_1)$



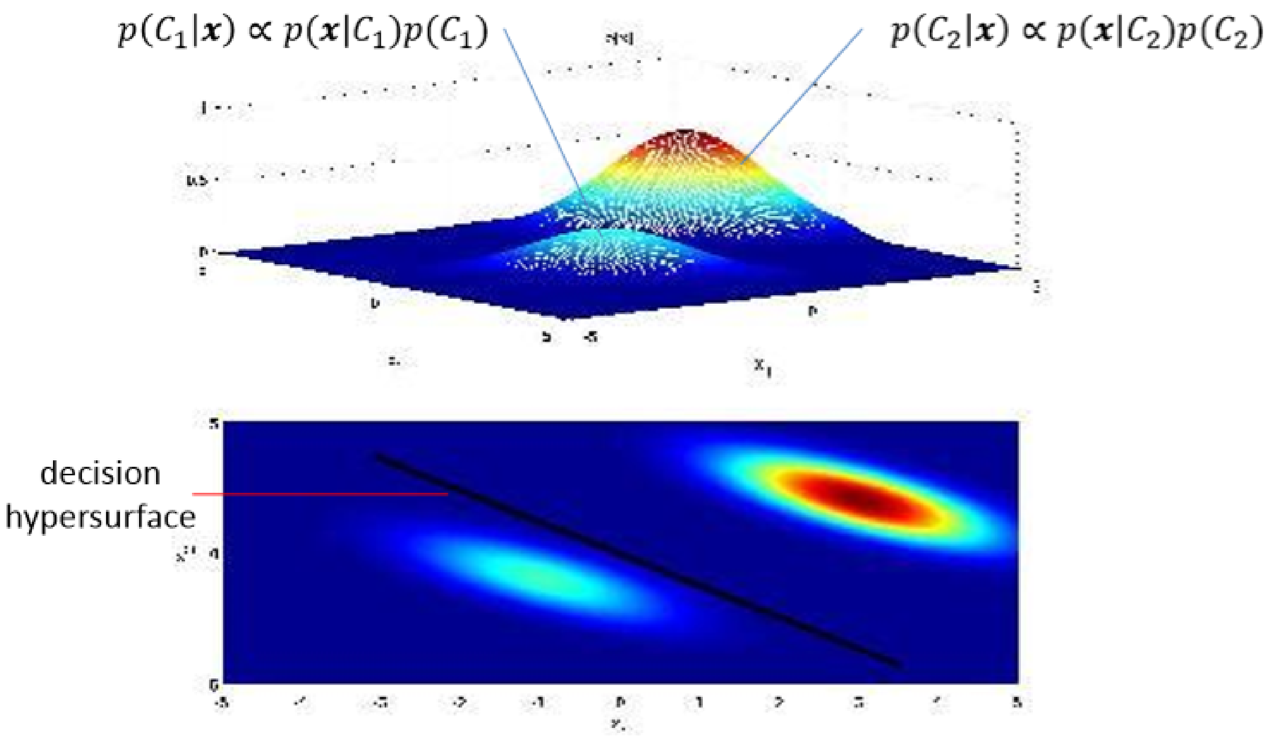
Model class-conditional probability of a 2D feature vector for class 1 as a multivariate Gaussian density.



Now consider class 2 with a similar Gaussian conditional density, which has the same covariance but a different mean



If the priors for each class are the same (i.e. 0.5), then the decision hypersurface cuts directly between the two means, with a direction parallel to the elliptical shape of the modes of the Gaussian densities shaped by their (identical) covariance matrices.



Now if the priors for each class are unequal, the decision hypersurface cuts between the two means with a direction as before, but now will be located further from the more likely class. This biases the predictor in favor of the more likely class.

More than two classes, unequal covariances

- more general case of unequal covariances (here shown for four classes)
- QDA
- the decision hypersurface is no longer a hyperplane, i.e. it is nonlinear.