Linear Regression

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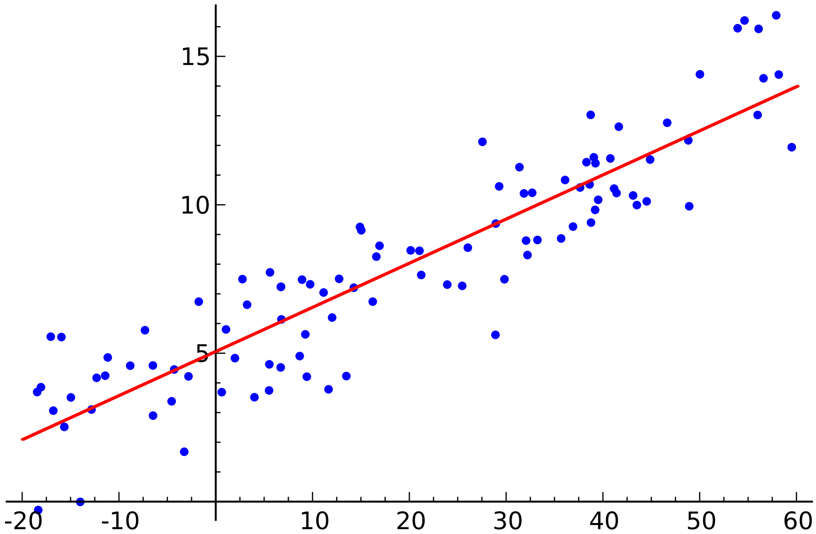
Definition

In statistics, linear regression is a linear approach to modeling the **relationship** between **a scalar response** (or dependent variable) and **one or more explanatory variables** (or independent variables).

Example:



Singlevariate Linear Regression

Hypothesis

$$h_\theta(x) = \theta_0 + \theta_1 x$$

where θ_{j} are paraters.

Cost Function

$$J(\theta_0, \theta1) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

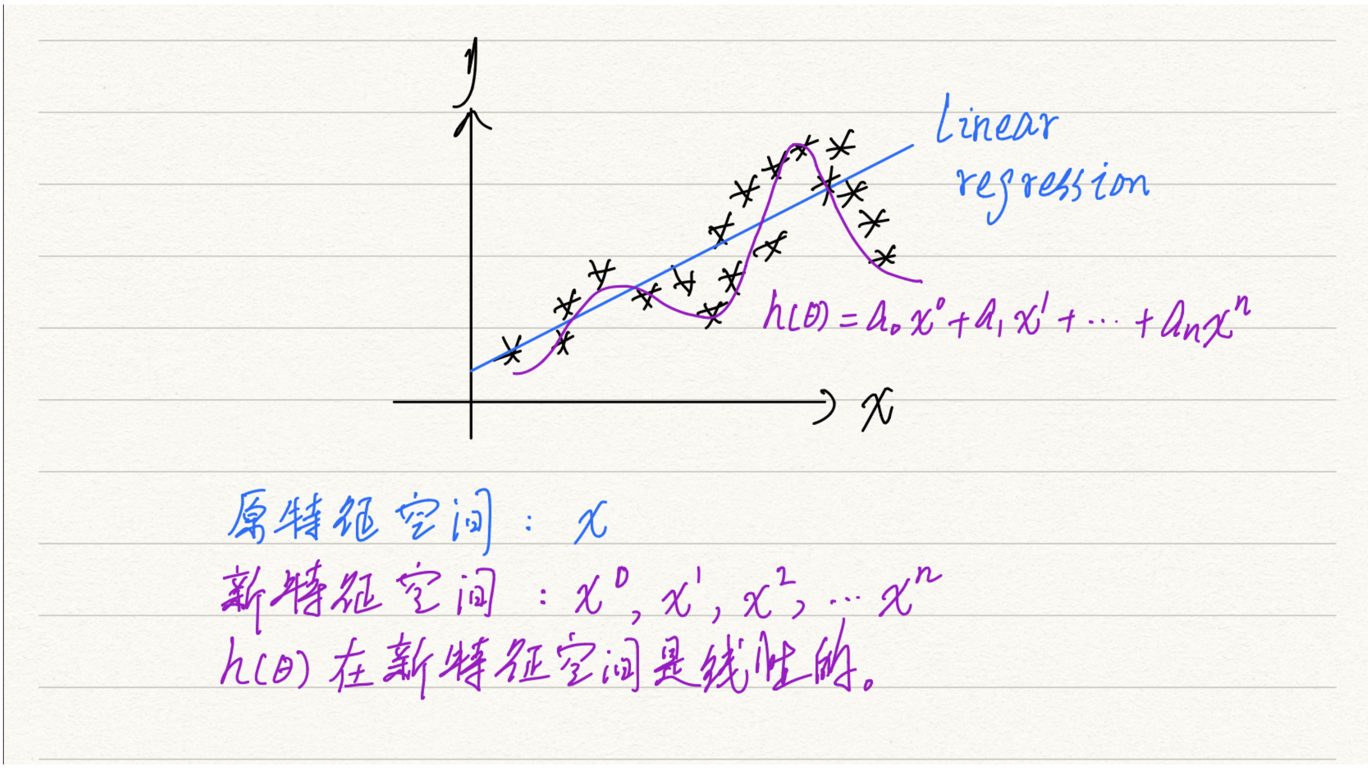
where m is the number of samples and we use superscript i here to notate the sample.

Cost function describes the distance between your prediction and the ground truth.

True of False Question

Question: Suppose we use polynomial features for linear regression, then the hypothesis is linear in the original features.

Answer: False, it is linear in the new polynomial features.



Multivariate Linear Regression

hypothesis

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

For convenience of notation and matrix multiplication, define $x_0 = 1$.

Cost Function

$$J(\theta_j) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

where $j=0,1,2, ... n$.

Minimize the cost function $J(\theta)$

Two potential solutions

1. Gradient descent.
   - Start with a guess for θ.
   - Change θ to decrese $J(\theta)$.
   - Until reach minimum.
2. Direct minimization.
   - Take derivative, set to zero.
   - Sufficient condition for minima.
   - Not possible for most "interesting" cost functions.

Gradient Descent Algorithm

Set θ = random value.

Repeat {

$$\theta_j = \theta_j - \alpha\frac{\partial}{\partial\theta_j}J(\theta)$$

} until convergence.

where α is the learning rate.

We take Least Squares Cost Function as an example.

SSD (Sum of square differences), also SSE (Sum of Square Errors).

$$J(\theta_j) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

where $j=0,1,2, … n$.

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

Calculation.

$$ \frac{\partial }{\partial \theta_j}J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\frac{\partial }{\partial \theta_j}(h_{\theta}(x^{(i)}) - y^{(i)})^{2} \ = \frac{1}{2m}\sum_{i=1}^{m} \cdot 2 \cdot (h_{\theta}(x^{(i)}) - y^{(i)}) \cdot \frac{\partial }{\partial \theta_j}(h_{\theta}(x^{(i)}) - y^{(i)}) \

\frac{1}{m}\sum_{i=1}^{m}(h_{\theta}(x^{(i)}) - y^{(i)})\frac{\partial }{\partial \theta_j}(\theta_{0} + \theta_{1}x_1^{(i)} + \theta_2 x_2^{(i)} + \cdot \cdot \cdot + \theta_n x_n^{(i)} - y^{(i)})\ = \frac{1}{m}\sum_{i=1}^{m}(h_{\theta}(x^{(i)}) - y^{(i)})x_j $$

Therefore, **loss gradient** is denoted as follows:

$$\frac{\partial}{\partial\theta_j}J(\theta) = (h_\theta(x^{(i)}) - y^{(i)})x_j$$

Gradient descent

Set θ = random value.

Repeat {

$$\theta_j = \theta_j - \alpha\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j$$

} until convergence.

where α is the learning rate.

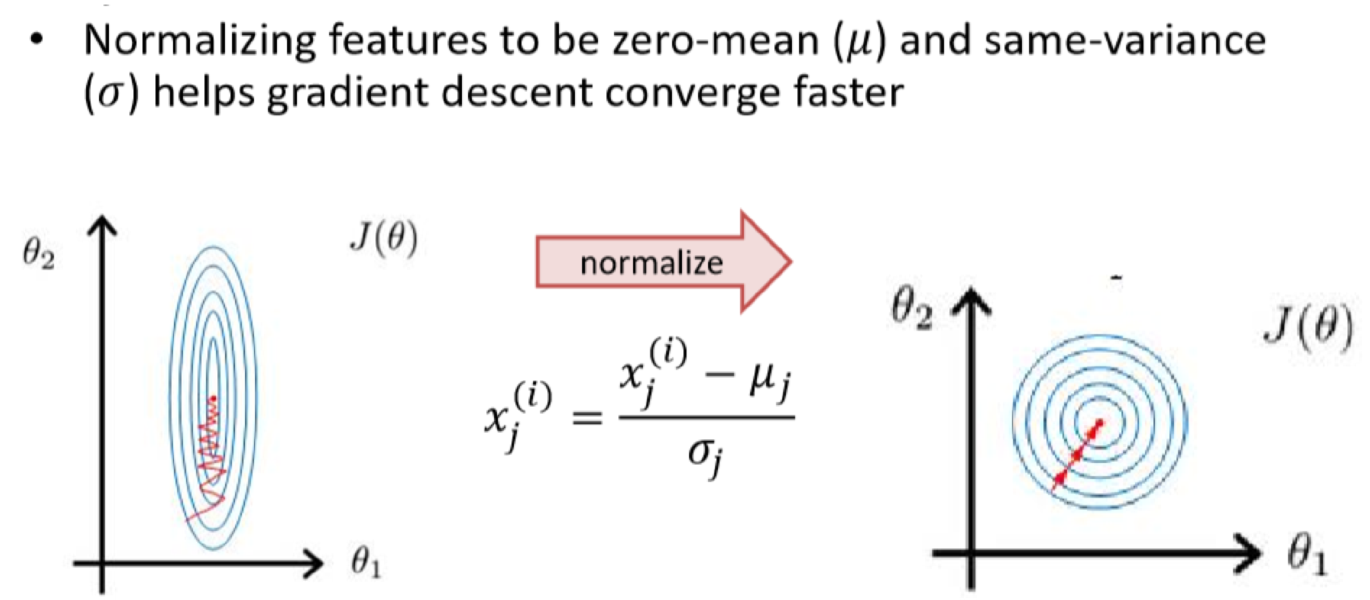
Feature Normalization (Feature Scaling)

Motivation

If features have very different scale, gradient descent can get "stuck" since x_j affects size of gradient in the direction of j^{th} dimension.

Definition

Feature scaling is a method used to **normalize the range** of independent variables or features of data. In data processing, it is also known as data normalization and is generally performed during the **data preprocessing** step.



Methods

Rescaling (min-max normalization)

$$x' = \frac{x - x_{min}}{x_{max} - x_{min}}$$

Mean normalization

$$x' = \frac{x - \bar{x}}{x_{max} - x_{min}}$$

where $\bar x$ is the average value.

Standardization (Z-score Normalization)

$$x' = \frac{x - \bar{x}}{\sigma}$$

where $\bar x$ is the average value and σ is standard deviation.

Scaling to unit length

To be continued...

Direction solution to minimize the cost function

Set

$$\frac{\partial J(\theta)}{\partial \theta_j} = 0$$

Matrix Notation

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{2m}(X\theta - y)^T(X\theta - y) = 0$$

where X is a $m \times n$ matrix,

θ is a $n \times 1$ matrix,

y is a $m \times 1$ matrix.

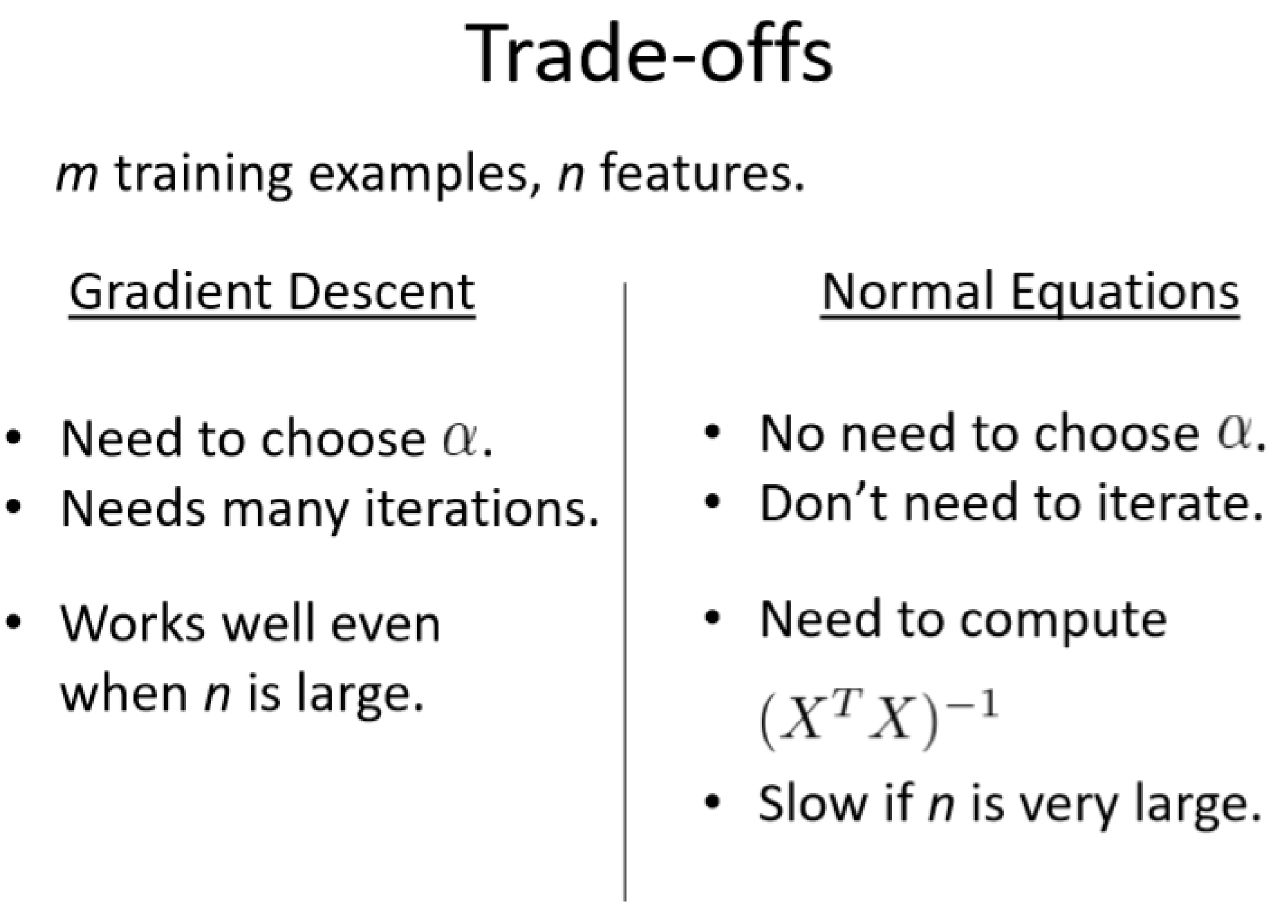
Calculation.

$$(X\theta - y)^T(X\theta - y) \ = (\theta^T X^T - y^T)(X\theta - y) \ = \theta^T X^T X\theta - y^T X\theta - \theta^T X^T y + y^T y \ = \theta^T X^T X\theta - 2y^T X\theta + y^T y$$

Ignore constant multiplier.

$$\frac{\partial J(\theta)}{\partial \theta_j} \propto X^T X\theta - X^T y = 0$$

$$\theta = (X^T X)^{-1} X^T y$$



Maximum Likelihood: Another view of cost function

New cost function

$$p((x^{(i)}, y^{(i)})|\theta)$$

maximize probability of data given model.

Maximum Likelihood: Example

- Intuitive example: Estimate a coin toss

  I have seen 3 flips of heads, 2 flips of tails, what is the chance of head (or tail) of my next flip?

- Model:

  Each flip is a Bernoulli random variable X

  X can take only two values: 1 (head), 0 (tail)

  $$p(X=1)=\theta, \quad p(X=0)=1-\theta$$

- θ is a parameter to be identified from data

Maximum Likelihood: Example

- 5 (independent) trials

$X_1=1$ $X_2=0$ $X_3=1$ $X_4=1$ $X_5=0$

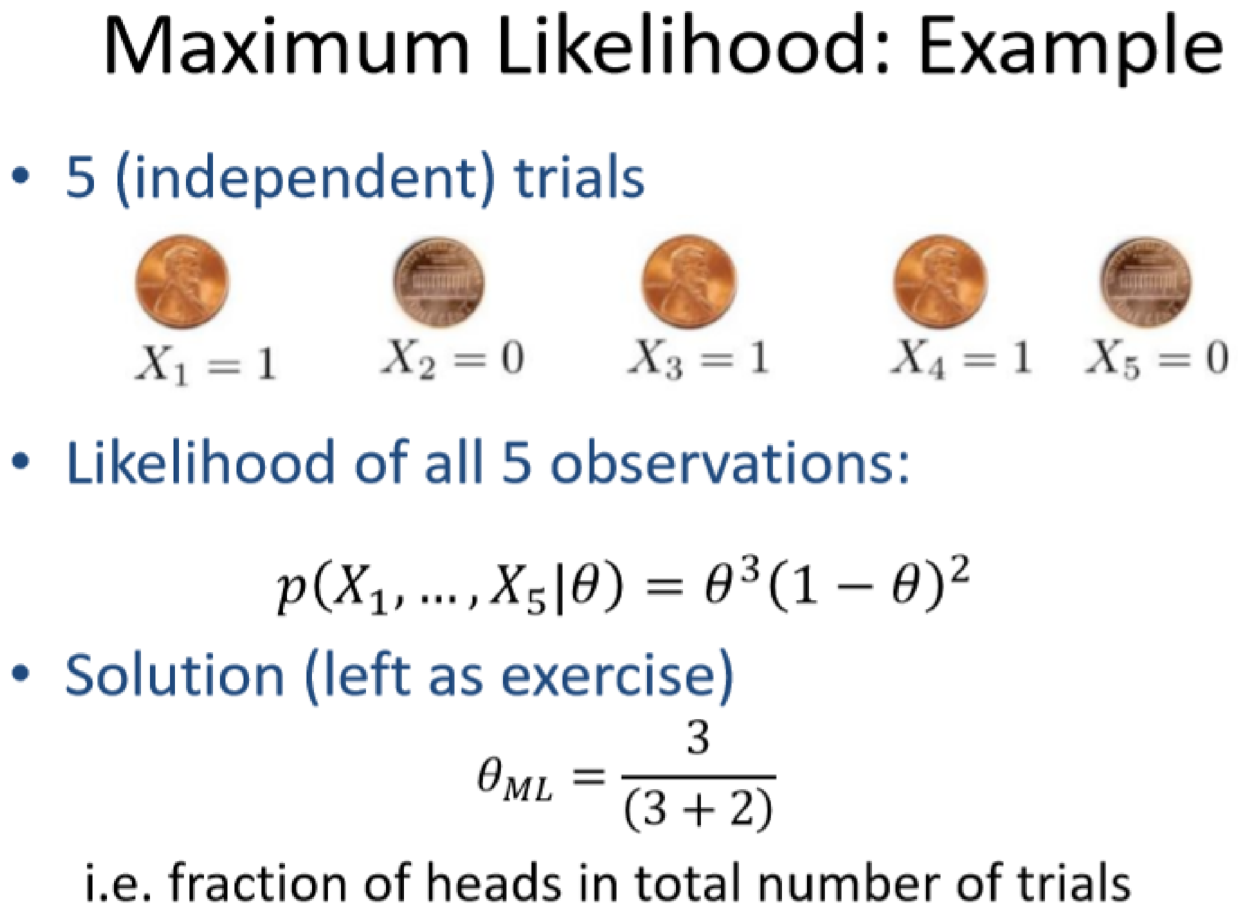
- Likelihood of all 5 observations:

$$p(X_1, \dots, X_5|\theta) = \theta^3(1-\theta)^2$$

- Intuition

  ML chooses θ such that likelihood is maximized

该方法的核心是条件概率，条件概率是给定参数θ时，最大化已知观测结果的概率，在这个例子里面，X就是五次观察结果(Observations)，θ是抛出硬币正面朝上的概率，我们在条件概率里面，认为这个参数是给定的，我们如何对这个参数取值，才能使得这个条件概率最大化。



Calculation:

$$\frac{d}{d\theta}(\theta^3(1-\theta)^2) = \theta^2(5\theta^2 - 8\theta + 3) = 0$$

Therefore, $\theta_{ML} = \frac{3}{5}.$