# Support Vector Machine

A supervised learning model.

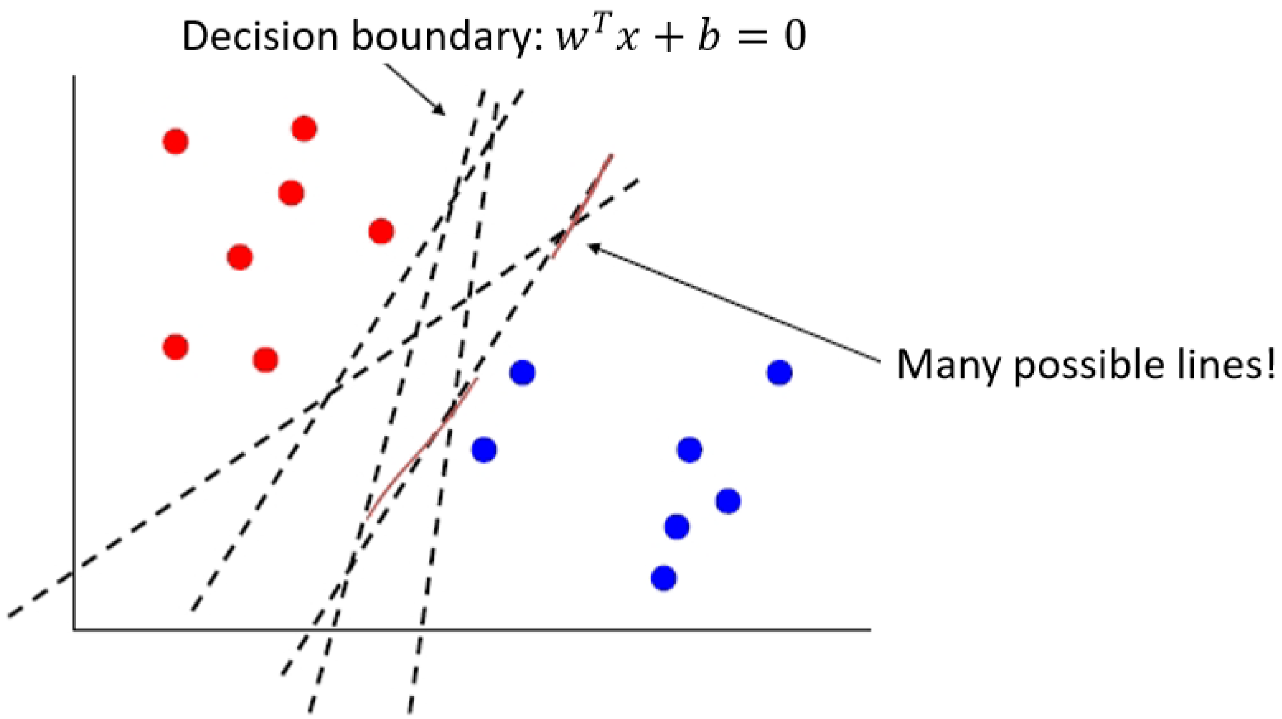
## Motivation and Logistics

- A **maximum margin method**, can be used for classification or regression.
- SVMs can efficiently perform a non-linear classification using what is called the **kernel trick**, implicitly mapping their inputs into high-dimensional feature spaces.
- First, we will derive **linear, hard-margin SVM** for linearly separable data, later for non-separable (soft margin SVM), and for nonlinear boundaries (kernel SVM).

## Maximum Margin

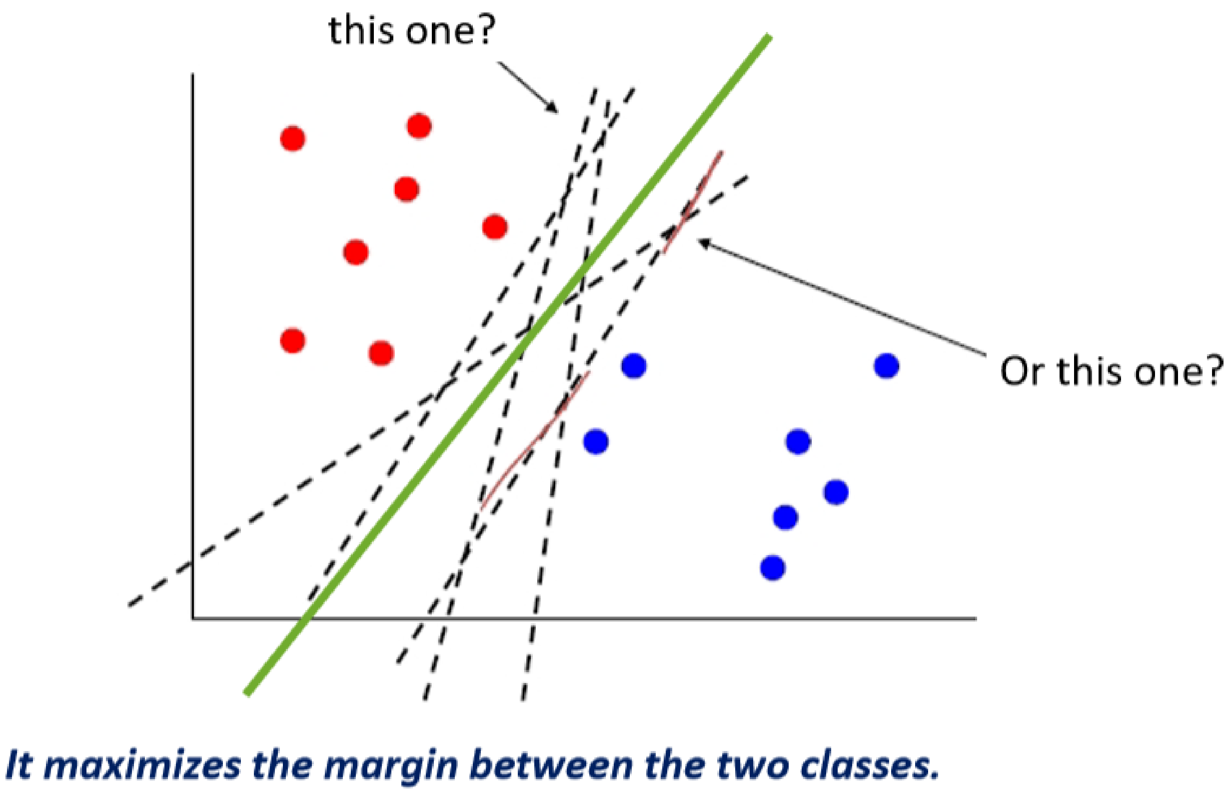
### Motivation

看一个二分类问题，现在可以有很多个Decision boundary，到底哪个才是最好的。



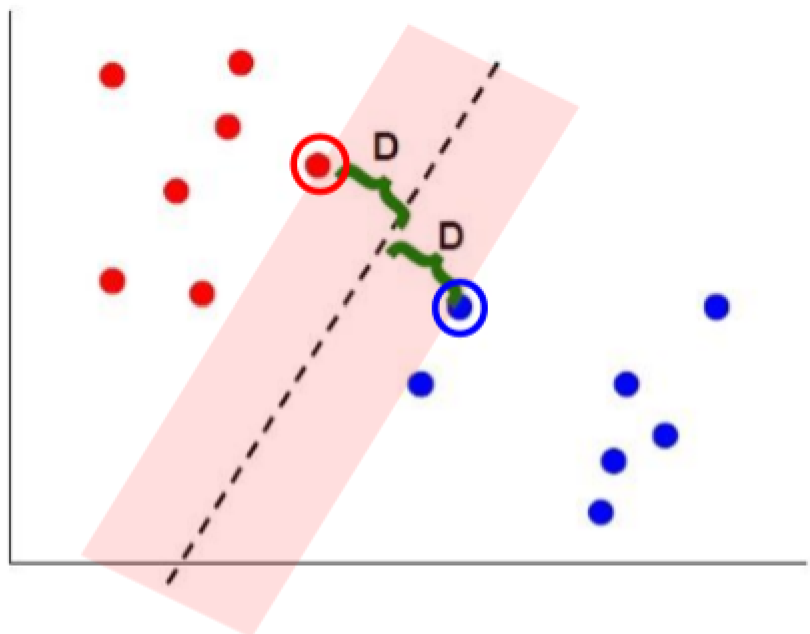

$$y = \begin{cases} +1 \text{ [red] if } \operatorname{sign}(\mathbf{w}^T \mathbf{x} + b \geq 0) \\ -1 \text{ [blue] if } \operatorname{sign}(\mathbf{w}^T \mathbf{x} + b < 0) \end{cases}$$

那当然是能远离所有点的那条线是最好的，如下图的绿线。




我们只需要关注boundary points.

我们的目标就是：learn a boundary that leads to the largest margin.

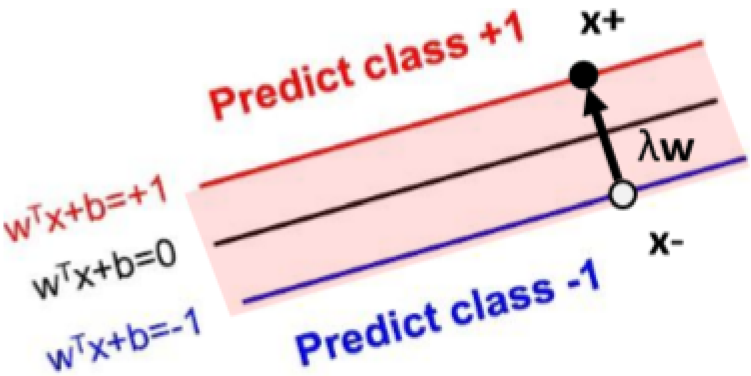



为什么叫支持向量？

Subset of vectors that support determine boundary are called the **support vectors (circled)**.

### Max Margin Classfier

我们对刚刚得到的decision boundary进行平移，得到一个margin.




我们看二分类问题，标签为1和-1，当然也可以标记为两个常数。

$$Class: +1 \quad if \; w^Tx+b \geq 1$$

$$Class: -1 \quad if \; w^Tx+b \leq 1$$

$$Class: Undefine \quad if -1 < w^Tx+b < 1$$

**我们怎么确定支持向量？**

假设现在如图这两个点$x^+$和$x^-$，是两个支持向量。

易知，

$$w^Tx^+ + b = 1$$

$$w^Tx^- + b = -1$$

则，

$$w^T(x^+ - x^-) = 2$$

我们需要maximize

$$||x^+ - x^-||$$

那我们如何用$w$来表示上式呢？

我们，需要引入一个中间变量$\lambda$. 令

$$x^+ - x^- = \lambda w$$

代入下式，

$$w^Tx^+ + b = 1$$

得到：

$$w^T(\lambda w + x^-) + b = 1$$

$$w^Tx^- + b + \lambda w^Tw = 1$$

$$-1 + \lambda w^T w = 1$$

$$\lambda = \frac{2}{w^T w}$$

现在我们可以求

$$||x^+ - x^-|| = ||\lambda w|| = \lambda \sqrt{w^T w} = \frac{2}{\sqrt{w^T w}}$$

Maximizing the margin is equivalent to regularization.

最大化这个margin相当于防止overfitting.

# Linear SVM

对于可以线性分类的问题。 我们这个模型要学习的是$w$和$b$这两个参数。

## Formulation

Objective function:

$$min \frac{1}{2} ||w||^2$$

$$s.t. (w^T x_i + b) y_i \geq 1, \forall i$$

如果样本标签为1，分类结果为-1或者0.5或者-0.5，都会破坏这个约束。

This is the primal formulation.

Apply Lagrange multipliers: formulate equivalent problem.

使用拉格朗日乘子，转化为没有约束的等价问题。

## Lagrange multipliers

Convert the primal constrained minimization to an unconstrained optimization problem: represent constraints as penalty terms:

$$\min_{w,b} \frac{1}{2} ||w||^2 + penalty$$

对每个样本${(x_i, y_i)}$的惩罚求和，

$$penalty = \sum_{i=1}^{n} \max_{\alpha_i \geq 0} \alpha_i [1 - (w^T x_i + b) y_i]$$

其中，n是样本数量，$\alpha_i$是拉格朗日乘子。

**max在这里到底是约束还是变量？**

如果约束满足：

$$1 - (w^T x_i + b) y_i \leq 0$$

如果约束不满足：

$$1-(w^Tx_i+b)y_i>0$$

大的拉格朗日乘子$\alpha_i$，确保惩罚足够大。

**提问，阿尔法怎么设？**

得到，

$$\min_{w,b}\frac{1}{2}||w||^2+\sum_{i=1}^{n}\max_{\alpha_i\geq 0}\alpha_i[1-(w^Tx_i+b)y_i]$$

代价函数：

$$\max_{\alpha_i\geq 0}\min_{w,b}J(w,b;\alpha)=\max_{\alpha_i\geq 0}\min_{w,b}\frac{1}{2}||w||^2+\sum_{i=1}^{n}\alpha_i[1-(w^Tx_i+b)y_i]$$

对w, b求偏导，

$$\frac{\partial J(w,b;\alpha)}{\partial w}=w-\sum_{i=1}^{n}\alpha_ix_iy_i=0$$

$$\frac{\partial J(w,b;\alpha)}{\partial b}=-\sum_{i=1}^{n}\alpha_iy_i=0$$

### Dual problem

重新代进原式，得到Dual problem，现在我们需要优化$\alpha$，训练的时候，我们

$$L=\max_{\alpha_i\geq 0}\sum_{i=1}^{n}\alpha_i-\frac{1}{2}\sum_{i,j=1}^{n}y_iy_j\alpha_i\alpha_j(x_i^Tx_j)$$

$$s.t.\alpha_i\geq 0;\sum_{i=1}^{n}\alpha_iy_i=0$$

Then use the obtained $\alpha_i$'s to solve for the weights and bias:

$$w=\sum_{i=1}^{n}\alpha_iy_ix_i$$

$$b=y_i-w^Tx_i$$

In practice, predict

$$y=sign[w^Tx+b]$$

### Primal v.s. Dual Problem

## Soft margin and slack variables

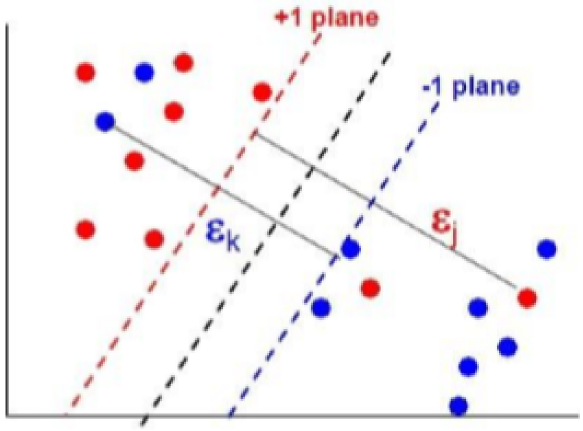
Introduce slack variables $\xi_i$

$$\min\left[\frac{1}{2}\|\mathbf{w}\|^2+\lambda\sum_{i=1}^{n}\xi_i\right]$$

subject to constraints (for all $i$):

$$y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\ge 1-\xi_i$$

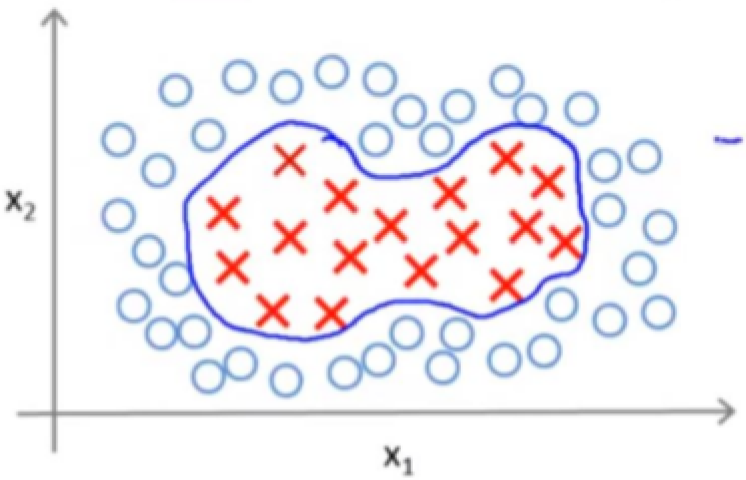
$$\xi_i\ge 0$$




Example lies on wrong side of hyperplane: $\xi_i > 1 \Rightarrow \sum_i \xi_i$ is upper bound on number of training errors

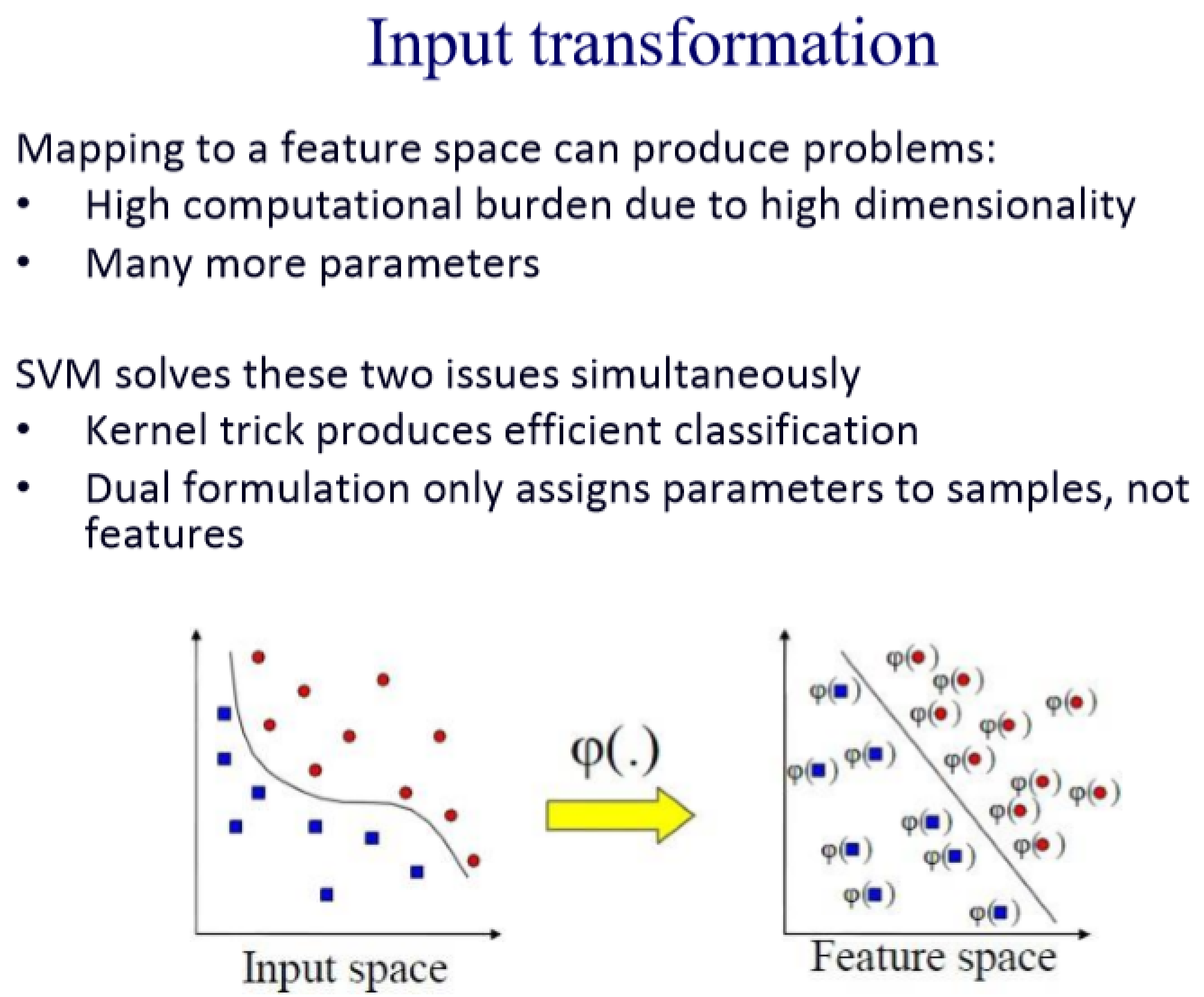
$\xi$的取值，如图所示，如果红点在+1平面的左方，$\xi$为0，如果在决策平面的左方，+1平面的右方，取值则为(0,1)，如果在决策平面另一侧，取值则大于1.

## Kernel trick for non-linear decision boundary




虽然可以用high order polynomial features，我们就希望用一个变换，把这些数据变得linear seperable,然后用linear SVM.

Input transformation




## Lankmark (Kernel Function)

Given data set

$$(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(m)}, y^{(m)})$$

Choose lankmarks

$$l^{(1)} = x^{(1)}, l^{(2)} = x^{(2)}, \ldots, l^{(m)} = x^{(m)}$$

原来的预测函数是这样： Predict "1" when:

$$\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n \geq 0$$

经过变换以后： Predict "1" when:

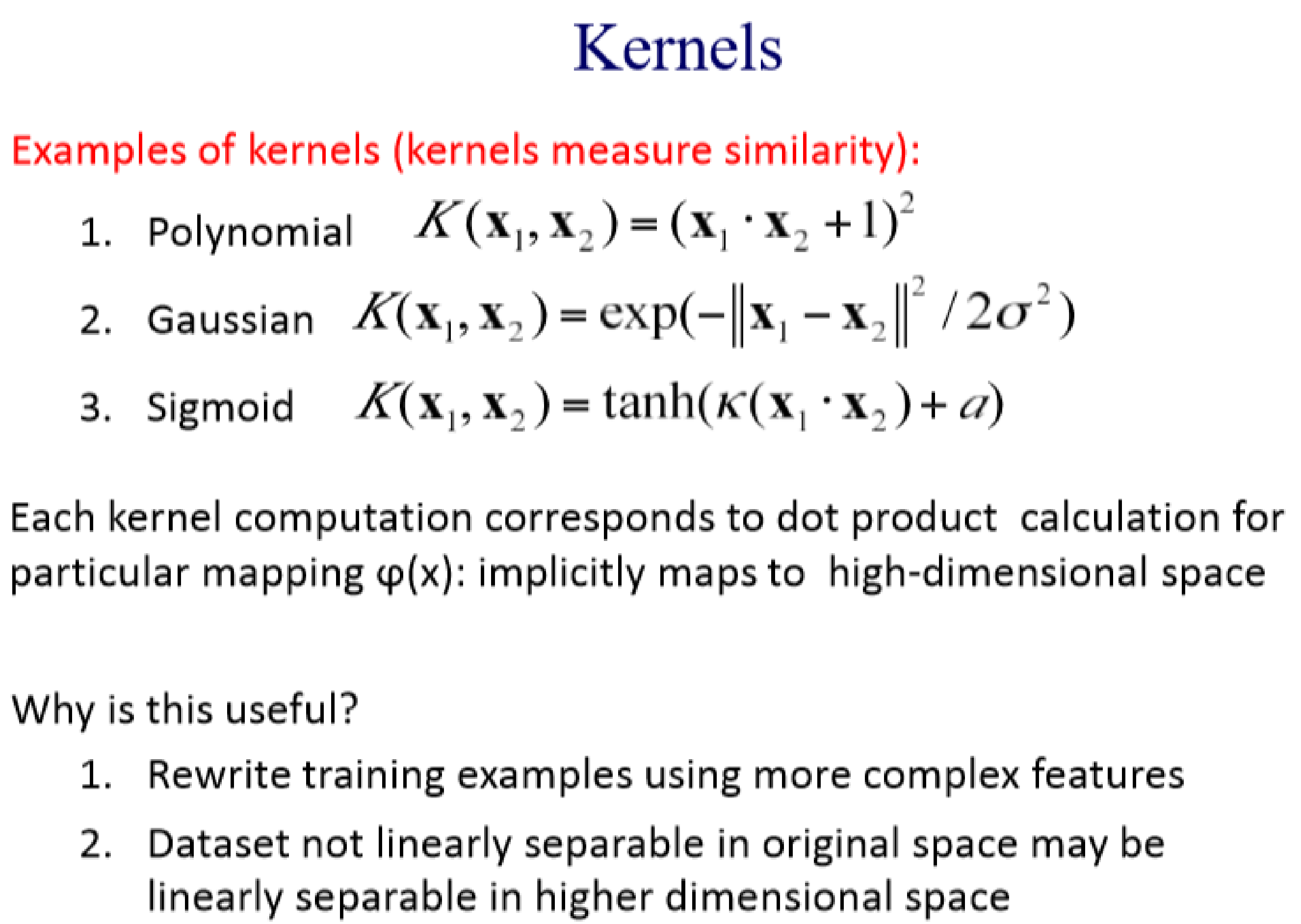
$$\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \ldots + \theta_n f_n \geq 0$$

其中，$f_i$就是Kernel Function.

Kernel Function需要自行选择，比如选的是高斯分布。

$$f_i = similarity(x, l^{(i)}) = \exp(-\frac{||x - l^{(i)}||^2}{2\sigma^2})$$

分布的参数需要自己设定，例如，我们这里设置均值为样本数据点本身，方差另外设定。



# Classification with non-linear SVMs

Non-linear SVM using kernel function *K()*:

$$L_K = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} y_i y_j \alpha_i \alpha_j K(\mathbf{x}_i, \mathbf{x}_j)$$

Maximize $L_K$ w.r.t. {α}, under constraints α≥0

Unlike linear SVM, cannot express w as linear combination of support vectors – now must retain the support vectors to classify new examples

Final decision function:

$$y = \mathrm{sign}[b - \sum_{i=1}^{n} y_i \alpha_i K(\mathbf{x}, \mathbf{x}_i)]$$

# Kernel SVM Summary

Advantages:

- Kernels allow very flexible hypotheses
- Poly-time exact optimization methods rather than approximate methods
- Soft-margin extension permits mis-classified examples
- Excellent results (1.1% error rate on handwritten digits vs. LeNet's 0.9%)

Disadvantages:

- Must choose kernel parameters
- Very large problems computationally intractable