CS 530 - Fall 2019

Homework 5 (Last one)

Due: Tuesday, December 10 at 11 pm - to be submitted via Gradescope

Note: The last time to turn in any HW will be Wednesday, December 11 at midnight

Reading : Chapter 29, pages 843-885 and the example of the simplex algorithm in the extra reading section of the course home page.

1. Bipartite matching

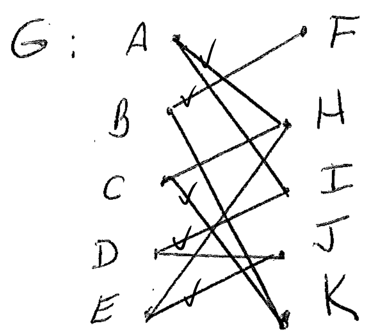
(i). Give an example of a connected bipartite graph G which has a maximal matching of some size t≥2 edges and a maximum matching of size greater than or equal to t+3.

(ii). How many perfect matchings does a complete bipartite graph of 2n nodes (n nodes on each side) have ? Give an exact value here (in terms of n). Briefly explain your reasoning. n

(iii). Let f(n) be your answer from part (ii). Show that the number of complete (perfect) matchings in a complete graph of 2n nodes is strictly bigger than O(f(n)). Here the graph is not bipartite. Briefly explain your answer.

每个点等级吗

2. Carry out the Hopcroft-Karp maximum matching algorithm on the bipartite graph G below. (You can find the algorithm on page 763, it is problem 26-6.)



Step 1 : A-H , B-F , C-K , D-I , E-J

Free vertices :

3. Consider the following linear programming problem;

maximize 3x+y

subject to

x+2y ≤ 12

(i) standard form

$x = x_1 - x_2 \quad y = y_1 - y_2$

maximize $3(x_1 - x_2) + (y_1 - y_2)$

s.t. $x_1 - x_2 + 2y_1 - 2y_2 \le 12$

$-2x_1 + 2x_2 + 4y_1 - 4y_2 \le 16$

$3x_1 - 3x_2 + 4y_1 - 4y_2 \le 12$

$x_1, x_2, y_1, y_2 \ge 0$

(ii) slack form

maximize $3x_1 - 3x_2 + y_1 - y_2$

s.t.

$z_1 = -x_1 + x_2 - 2y_1 + 2y_2 + 12$

$z_2 = 2x_1 - 2x_2 - 4y_1 + 4y_2 + 16$

$z_3 = -3x_1 + 3x_2 - 4y_1 + 4y_2 + 12$

$x_1, x_2, y_1, y_2, z_1, z_2, z_3 \ge 0$

(iii) maximize $3x_1 - 3x_2 + y_1 - y_2$

1 s.t. $-x_1 + x_2 - 2y_1 + 2y_2 - z_1 = -12$

$2x_1 - 2x_2 - 4y_1 + 4y_2 - z_2 = -16$

$-3x_1 + 3x_2 - 4y_1 + 4y_2 - z_3 = -12$ basic : z_1, z_2, z_3

for y_1 $r_1 = \frac{-12}{-2} = 6$ $r_2 = \frac{-16}{-4} = 4$ $r_3 = \frac{-12}{-4} = 3$

select r_3

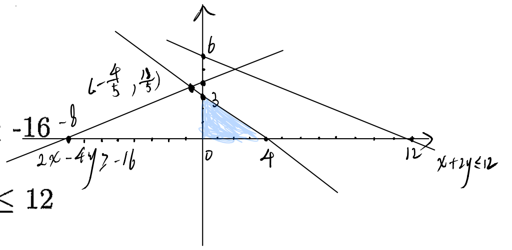
$y_1 = -\frac{3}{4}x_1 + \frac{3}{4}x_2 + y_2 - \frac{1}{4}z_3 + 3$

$3 - \frac{3}{4}$ $\quad -3 + \frac{3}{4}$

$\frac{12-3}{4}$ $\quad \frac{-12+3}{4}$

2x-4y $\geq$ -16 -8

3x+4y $\leq$ 12



$-4y_1 = 3x_1 - 3x_2 - 4y_2 + z_3 - 12$

$-2y_1 = \frac{3}{2}x_1 - \frac{3}{2}x_2 - 2y_2 + \frac{1}{2}z_3 - 6$

$$\begin{cases} \frac{1}{2}x_1 - \frac{1}{2}x_2 - z_1 + \frac{1}{2}z_3 = -6 \\ 5x_1 - 5x_2 - z_2 + z_3 = -4 \\ -3x_1 + 3x_2 - 4y_1 + 4y_2 - z_3 = -12 \end{cases}$$

basic: z_1, z_2, y_1
6 4 3

obj: $\frac{9}{4}x_1 - \frac{9}{4}x_2 - \frac{1}{4}z_3 + 3$

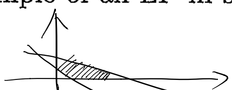
(i) Put the problem into standard LP form. (Don't forget the non-negativity requirements.)

(ii). Put your answer for part (i) into slack form, and draw the picture of the feasible space for this problem. Label your picture clearly indicating the feasible region, various constraints, and the extreme points.

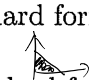
Say what the extreme points of your drawing are.

(iii). Now do one iteration (pivot) of the simplex algorithm on this LP and say which extreme point is the next basic feasible solution. $x_1 = x_2 = y_2 = z_3 = 0$ $z_1 = 6$ $z_2 = 4$ $y_1 = 3$

4. (i). Give an example of an LP in standard form which has feasible solutions but the point (0,0) is not feasible.



(ii). Give an example of an LP in standard form which has infinitely many optimal solutions.



(iii). Give an example of an LP in standard form with 2 variable in its objective function and which has an unbounded feasible region and has a unique optimal solution.

(iv). Give an example of an LP P in standard form with the property that both P and its dual have no feasible solutions.

5. Problem 29-3, parts a. and b. in the textbook, page 895.

6. Let M be an n by n matrix of 0's and 1's. We call M zeroable if there is a sequence of row switches and column switches of M which results in all 0's on the diagonal of M. (Any off-diagonal elements can be either 0 or 1.)

a. Give an example of an M with n at least 3 which is not zeroable but which has at least one 0 in every row and in every column.

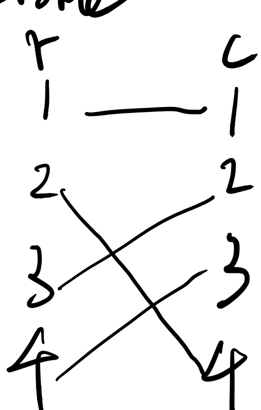
b. Give an efficient (polynomial number of steps) algorithm to decide if a matrix A is zeroable or not.

Write your algorithm using pseudocode, not as a full program in some programming language. Run your algorithm on your graph from part a. and explain how your algorithm works and why it works correctly..

Hint: One option is to use the max matching of a bipartite graph to solve this problem.

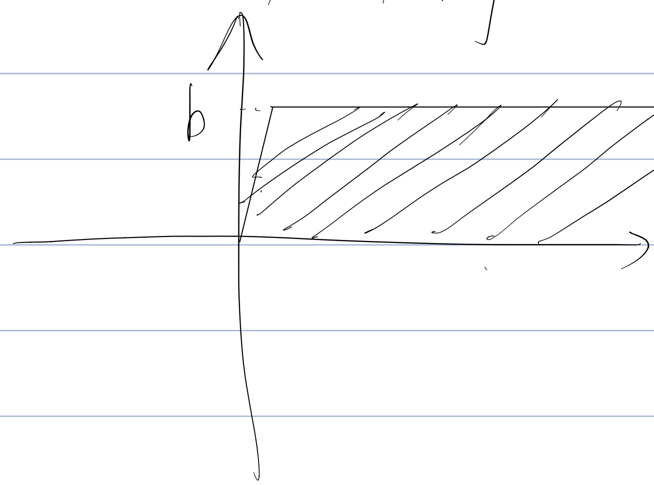
perfect matching $\Leftrightarrow$ zeroble

0 represents an edge



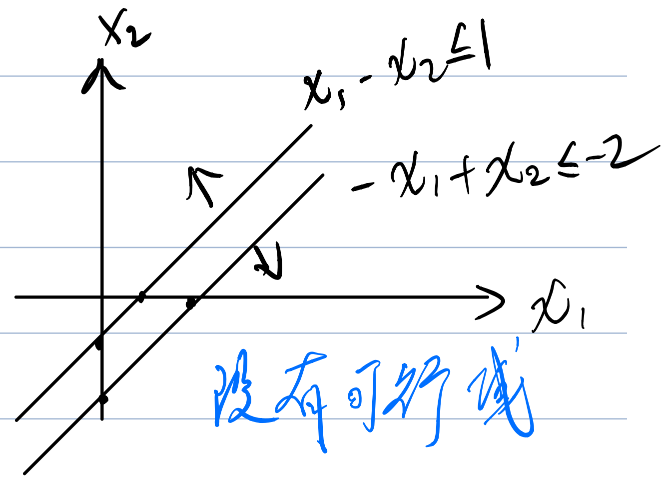
Use Edmond Matrix.
det(E)

4.(iii) $\max\ -x+y$ s.t. $y \le b$



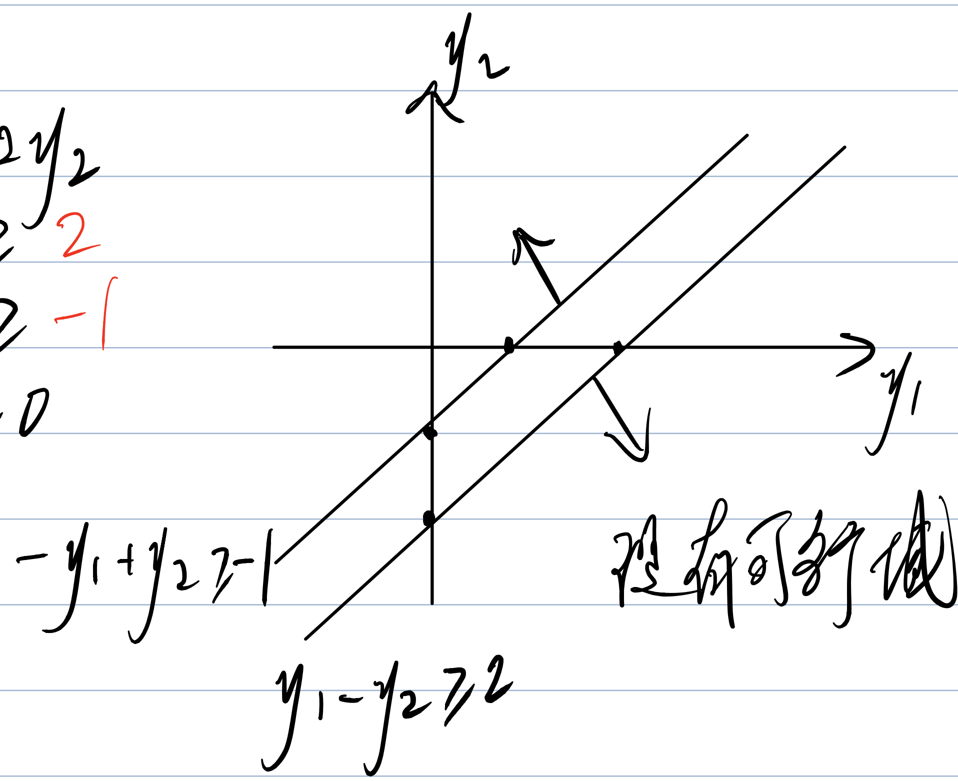
(iv) primal:

$$\begin{aligned} &\text{maximize } 2x_1 - x_2 \\ &\text{s.t. } x_1 - x_2 \le 1 \\ &\quad -x_1 + x_2 \le -2 \\ &\quad x_1, x_2 \ge 0 \end{aligned}$$



dual:

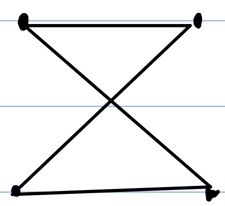
$$\begin{aligned} &\text{minimize } y_1 - 2y_2 \\ &\quad y_1 - y_2 \ge 2 \\ &\quad -y_1 + y_2 \ge -1 \\ &\quad y_1, y_2 \ge 0 \end{aligned}$$



1.(iii) prove that the perfecting matching in a complete graph with 2n nodes is strictly bigger than $O(f(n))$.

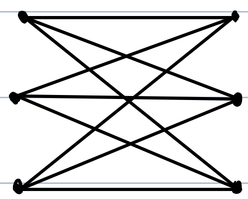
number of perfect matchings

$n=2$



2×1

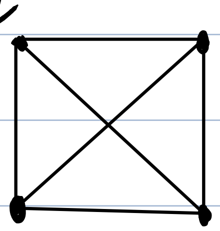
$n=3$



$3 \times 2 \times 1$

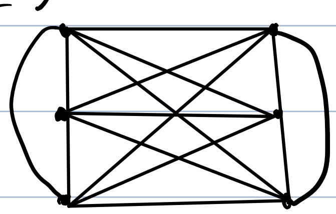
$f(n) = n!$

$n=2$



$$\underbrace{(2\times 2-1)}_{n}\underbrace{(2\times 1-1)}_{n-1}$$

$n=3$



$$\underbrace{(2\times 3-1)}_{n}\underbrace{(2\times 2-1)}_{n-1}\underbrace{(2\times 1-1)}_{n-2}$$

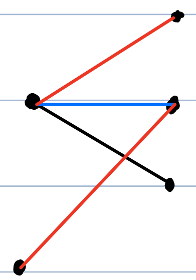
$$g(n) = (2n-1)(2(n-1)-1)(2(n-2)-1)\cdots(2\times 1-1)$$

$$= (2n-1)(2n-3)(2n-5)(2n-7)\cdots(2n-(2n-1))$$

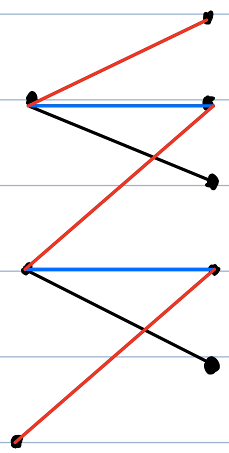
$> n(n-1)(n-2)\cdots 1$

(i) 二部图 maximal matching $= t \geq 2$ 　eg 2
maximum matching $\geq t+3$ 　5

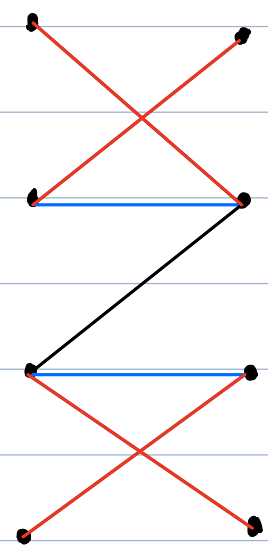
至少10个点



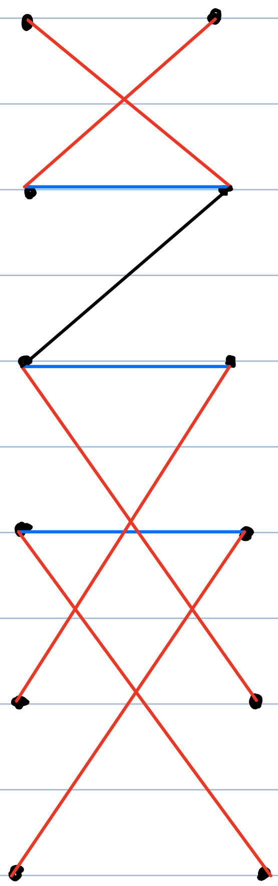
maximal : 1
maximum : 2



2
3



2
4



maximal matching : 3

maximum matching : 6

29-3 Integer linear programming

An ***integer linear-programming problem*** is a linear-programming problem with the additional constraint that the variables x must take on integral values. Exercise 34.5-3 shows that just determining whether an integer linear program has a feasible solution is NP-hard, which means that there is no known polynomial-time algorithm for this problem.

a. Show that weak duality (Lemma 29.8) holds for an integer linear program.

b. Show that duality (Theorem 29.10) does not always hold for an integer linear program.

c. Given a primal linear program in standard form, let us define P to be the optimal objective value for the primal linear program, D to be the optimal objective value for its dual, IP to be the optimal objective value for the integer version of the primal (that is, the primal with the added constraint that the variables take on integer values), and ID to be the optimal objective value for the integer version of the dual. Assuming that both the primal integer program and the dual integer program are feasible and bounded, show that

$$IP \le P = D \le ID .$$