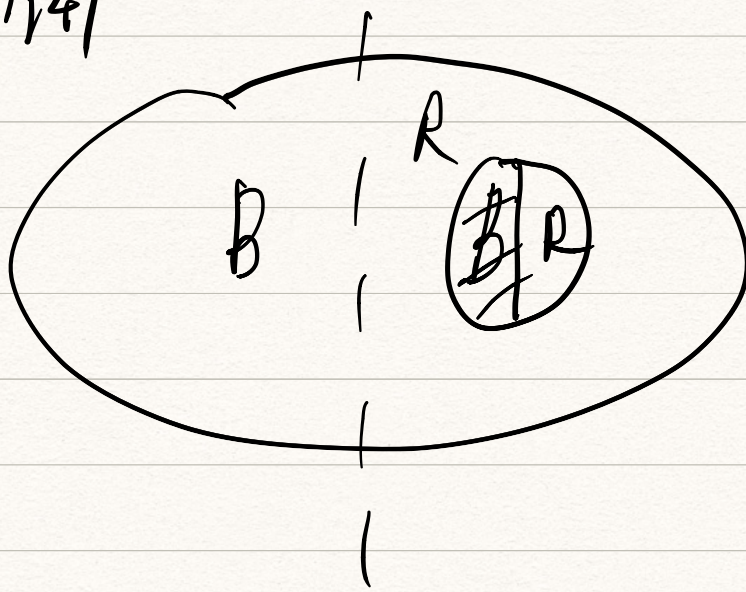
Chapter 29
Simplex Method

How to reduce a problem to linear problem

George Dantzig 1947



Max value can be found at one of the extreme points.

Simplex method:
Algorithm which visits the extreme points of feasible region and picks one extreme with maximum objective value.

GEORGE DANTZIG

In 1947 George Dantzig was a young mathematician looking for a method of solving very complicated optimization problems. The problems he considered were real-world problems originating in the military, business, and economics. He was employed by the military but it was his friends who were economists and mathematicians, not to mention the engineers who were working on building a new type of machine called a "computer," who led him on the path to success.

Many of the problems facing Dantzig involved systems of equations with 50 or even 100 equations and unknowns. The method he created, called the simplex method, became an instant success. One of the first problems that he applied the method to was a transportation problem involving the large steel industry. These companies had built a virtual monopoly by claiming that the transportation cost of steel was too complicated to compute, so all steel companies, large and small, had to use the same simplified formula that favored the large companies. The simplex method showed them how to compute the cost, thereby allowing smaller companies to use their proximity to the user to cut cost and remain competitive.

THE BERLIN AIRLIFT

One of the first concrete problems that George Dantzig's linear programming helped solve was the massive supply to West Berlin shortly after World War II.

Berlin surrendered to the Russian army on May 2, 1945. During the following weeks the Russians shipped most of the city's industrial goods to Russia. The American, British, and French troops did not arrive until July 1945. Even before the war ended the Allies decided to divide the city into four sectors, each country occupying one. Berlin lay deep in the Russian sector of the country, but the Western powers assumed that the Soviets would allow them free access to the city. However, on June 24, 1948 the Soviets blocked all land and water routes through East Germany to Berlin. They hoped to drive the Western powers from East Germany. The problem facing the Western Allies was how to keep West Berlin supplied during the Russian blockade.

The problem was turned over to the Planning Research Division of the U.S. Air Force. Their staff had been working on similar programming problems, most of which were theoretical. But now they were asked to apply their new methods to a very practical problem. Their solution helped shatter Soviet hopes of using the blockade to win total occupation of Berlin. A gigantic airlift was organized to supply more than 2 million people in West Berlin. It was a large-scale program that required intricate planning. To break the blockade, hundreds of American and British planes delivered massive quantities of food, clothing, coal, petroleum, and other supplies. At its peak a plane landed in West Berlin every 45 seconds.

The number of variables in the formulation of the problem exceeded 50. They included the number of planes, crew capacity, runways, supplies in Berlin, supplies in West Germany, and money.

Finite Mathematics with Applications, David Zitarelli and Raymond Coughlin, 2nd ed., Saunders College Publishing, Harcourt Brace Jovanovich, 1992

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Here is a *very* simplified version of the problem:

Maximize the cargo capacity of the planes subject to the restrictions:

1. The number of planes is limited to 44
2. The American planes are larger than the British planes and therefore need twice the number of personnel per flight, so if we let one "crew" represent the number of flight personnel required for a British plane, and thus two "crews" are required for an American flight, then the restriction is that the number of crews available is 64
3. The cost of an American flight is $9000 and the cost of a British flight is $5000 and the total cost per week is limited to $300,000.
4. The cargo capacity of an American plane is 30,000 cubic feet and the cargo capacity of a British plane is 20,000 cubic feet.

How many American and British planes should be used to maximize the cargo capacity?

Here is what you need to do:

Define your variables in English
Write the objective function and constraints
Solve the problem 2 ways:

1. **By graphing it and evaluating the objective function at the corner points of the feasible region.**
2. **By setting up the initial simplex tableau, finding the pivot point, clearing out that column, and continuing until the process is done. At each stopping point in the process, identify the "feasible" solution.**

Be sure to write your conclusion in clear English.

Finite Mathematics with Applications, David Zitarelli and Raymond Coughlin, 2nd ed., Saunders College Publishing, Harcourt Brace Jovanovich, 1992

standard form   slack form

Finally, we negate constraint (29.23). For consistency in variable names, we rename x'_2 to x_2 and x''_2 to x_3, obtaining the standard form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0\,.
\end{array}
\tag{29.24–29.28}
$$

(29.24) (29.25) (29.26) (29.27) (29.28)

By converting each constraint of a linear program in standard form, we obtain a linear program in a different form. For example, for the linear program described in (29.24)–(29.28), we introduce slack variables x_4, x_5, and x_6, obtaining

$$\text{maximize} \quad 2x_1 - 3x_2 + 3x_3 \tag{29.33}$$

subject to

basic variable

$$x_4 = 7 - x_1 - x_2 + x_3 \tag{29.34}$$

nonbasic

$$x_5 = -7 + x_1 + x_2 - x_3 \tag{29.35}$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3 \tag{29.36}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \; . \tag{29.37}$$

In this linear program, all the constraints except for the nonnegativity constraints are equalities, and each variable is subject to a nonnegativity constraint. We write each equality constraint with one of the variables on the left-hand side of the equality and all others on the right-hand side. Furthermore, each equation has the same set of variables on the right-hand side, and these variables are also the only ones that appear in the objective function. We call the variables on the left-hand side of the equalities ***basic variables*** and those on the right-hand side ***nonbasic variables***.

For linear programs that satisfy these conditions, we shall sometimes omit the words "maximize" and "subject to," as well as the explicit nonnegativity constraints. We shall also use the variable z to denote the value of the objective func-

tion. We call the resulting format ***slack form***. If we write the linear program given in (29.33)–(29.37) in slack form, we obtain

$$z = 2x_1 - 3x_2 + 3x_3 \tag{29.38}$$

$$x_4 = 7 - x_1 - x_2 + x_3 \tag{29.39}$$

$$x_5 = -7 + x_1 + x_2 - x_3 \tag{29.40}$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3 \; . \tag{29.41}$$

As with standard form, we find it convenient to have a more concise notation for describing a slack form. As we shall see in Section 29.3, the sets of basic and nonbasic variables will change as the simplex algorithm runs. We use N to denote the set of indices of the nonbasic variables and B to denote the set of indices of the basic variables. We always have that $|N| = n$, $|B| = m$, and $N \cup B = \{1, 2, \ldots, n + m\}$. The equations are indexed by the entries of B, and the variables

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The Simplex Algorithm - In brief.

1. Start with LP in slack form.

Maximize $13a + 23b$

subject to :

i. $5a + 15b + S_C = 480$
ii. $4a + 4b + S_H = 160$
iii. $35a + 20b + S_M = 1190$
$a, b, S_C, S_H, S_M \geq 0$.

2. FIND a feasible solution.
Set $a = b = 0$ (non-basic var.) $S_C = 480, S_H = 160, S_M = 1190$ (basic var.)

3. Pivot on non-basic var.
Choose b and first constraint (i), (by min ratio rule).
$\underline{15b}$
Solve for b:
$b = 32 - \frac{1}{3}a - \frac{1}{15}S_C$ $\frac{480}{15} = 32$ $\frac{160}{4} = 40$ $\frac{1190}{20} = 59.5$

Eliminate b from (ii), (iii) to obtain:

(i) $\frac{1}{3}a + b + \frac{1}{15}S_C = 32$ all vars ≥ 0.
(ii) $\frac{8}{3}a - \frac{4}{15}S_C + S_H = 32$
(iii) $\frac{85}{3}a - \frac{4}{3}S_C + S_M = 550$

OBJ. FUNCTION :
max $\frac{16}{3}a - \frac{23}{15}S_C$

②

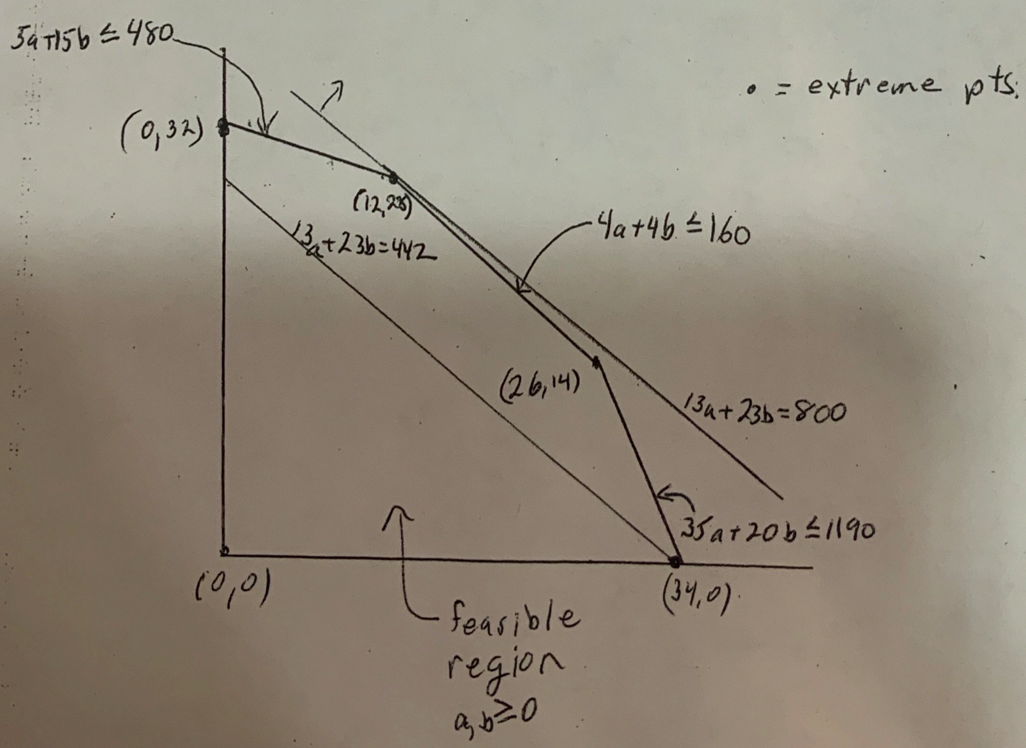
4. Pivot on non-basic variable
Choose a and constraint (ii), $\left(\frac{8}{3}a\right)$
(coeff $a>0$) (mrr)
Eliminate a from (i), (iii) to obtain.

$$
\begin{array}{llr}
i & b + \frac{1}{10} S_C + \frac{1}{8} S_H & = 28 \\
ii & a - \frac{1}{10} S_C + \frac{3}{8} S_H & = 12 \\
iii & - \frac{25}{6} S_C + \frac{85}{8} S_H + S_M & = 110
\end{array}
$$

$a, b, S_C, S_H, S_M \geq 0.$

Objective function:

$\max \; -S_C - 2S_H$



$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \, . \end{array}$$

Let $x_2 = x_2' - x_2''$

Next, we show how to convert a linear program in which some of the variables do not have nonnegativity constraints into one in which each variable has a nonnegativity constraint. Suppose that some variable x_j does not have a nonnegativity constraint. Then, we replace each occurrence of x_j by $x_j' - x_j''$, and add the nonnegativity constraints $x_j' \geq 0$ and $x_j'' \geq 0$. Thus, if the objective function has a term $c_j x_j$, we replace it by $c_j x_j' - c_j x_j''$, and if constraint i has a term $a_{ij} x_j$, we replace it by $a_{ij} x_j' - a_{ij} x_j''$. Any feasible solution $\hat{x}$ to the new linear program corresponds to a feasible solution $\bar{x}$ to the original linear program with $\bar{x}_j = \hat{x}_j' - \hat{x}_j''$ and with the same objective value. Also, any feasible solution $\bar{x}$ to the original linear program corresponds to a feasible solution $\hat{x}$ to the new linear program with $\hat{x}_j' = \bar{x}_j$ and $\hat{x}_j'' = 0$ if $\bar{x}_j \geq 0$, or with $\hat{x}_j'' = \bar{x}_j$ and $\hat{x}_j' = 0$ if $\bar{x}_j < 0$. The two linear programs have the same objective value regardless of the sign of $\bar{x}_j$. Thus, the two linear programs are equivalent. We apply this conversion scheme to each variable that does not have a nonnegativity constraint to yield an equivalent linear program in which all variables have nonnegativity constraints.

Continuing the example, we want to ensure that each variable has a corresponding nonnegativity constraint. Variable x_1 has such a constraint, but variable x_2 does not. Therefore, we replace x_2 by two variables x_2' and x_2'', and we modify the linear program to obtain

$$\begin{array}{lrcrcrcl} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & & & x_1, x_2', x_2'' & & & \geq & 0 \, . \end{array} \tag{29.22}$$

Next, we convert equality constraints into inequality constraints. Suppose that a linear program has an equality constraint $f(x_1, x_2, \ldots, x_n) = b$. Since $x = y$ if and only if both $x \geq y$ and $x \leq y$, we can replace this equality constraint by the pair of inequality constraints $f(x_1, x_2, \ldots, x_n) \leq b$ and $f(x_1, x_2, \ldots, x_n) \geq b$. Repeating this conversion for each equality constraint yields a linear program in which all constraints are inequalities.

Finally, we can convert the greater-than-or-equal-to constraints to less-than-or-equal-to constraints by multiplying these constraints through by -1. That is, any inequality of the form