CS 530 Advance Algorithm - Fall 2019
Homework #2

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1 LU Decomposition

Show the LUP decomposition of the 4 × 4 matrix M.

$$M = \begin{bmatrix} 3 & 2 & 1 & 9 \\ 6 & 4 & 9 & 12 \\ 9 & 0 & 6 & 3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \tag{1}$$

Answer:

(i) Initialize L and U.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} U = \begin{bmatrix} & & & \\ 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix} \tag{2}$$

(ii) Generate permutation matrix P.
Swap row 3 and row 1, because 9 is the biggest element in column 1.

$$PM = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 9 \\ 6 & 4 & 9 & 12 \\ 9 & 0 & 6 & 3 \\ 0 & 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 6 & 3 \\ 6 & 4 & 9 & 12 \\ 3 & 2 & 1 & 9 \\ 0 & 1 & 3 & 5 \end{bmatrix} \tag{3}$$

(iii)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & & 1 & 0 \\ 0 & & & 1 \end{bmatrix} U = \begin{bmatrix} 9 & 0 & 6 & 3 \\ 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix} \tag{4}$$

Schur complement:

$$\begin{bmatrix} 4 & 5 & 10 \\ 2 & -1 & 8 \\ 1 & 3 & 5 \end{bmatrix} \tag{5}$$

(iv)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{4} & & 1 \end{bmatrix} U = \begin{bmatrix} 9 & 0 & 6 & 3 \\ 0 & 4 & 5 & 10 \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix} \tag{6}$$

Schur complement:

$$\begin{bmatrix} -\frac{7}{2} & 3 \\ \frac{7}{4} & \frac{5}{2} \end{bmatrix} \tag{7}$$

(v)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & 1 \end{bmatrix} U = \begin{bmatrix} 9 & 0 & 6 & 3 \\ 0 & 4 & 5 & 10 \\ 0 & 0 & -\frac{7}{2} & 3 \\ 0 & 0 & 0 & \end{bmatrix} \tag{8}$$

Schur complement:

$$\begin{bmatrix} 4 \end{bmatrix} \tag{9}$$

(vi) LUP decomposition:

$$PM = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 9 \\ 6 & 4 & 9 & 12 \\ 9 & 0 & 6 & 3 \\ 0 & 1 & 3 & 5 \end{bmatrix} = \tag{10}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 6 & 3 \\ 0 & 4 & 5 & 10 \\ 0 & 0 & -\frac{7}{2} & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \tag{11}$$

2 Uniqueness of decomposition

In class we mentioned that if a non-singular matrix M has an LU decomposition, then the decomposition is unique. That is the there is only one pair (L,U) with M = LU.

(i) Is this same result true for every singular M which has an LU decomposition? Briefly explain why or why not.

Answer: If a singular M which has an LU decomposition, then the decomposition is non-unique.

Explanation: Consider the case where M is a n × n ($n > 2$) 0-matrix. Let U be a n × n ($n > 2$) 0-matrix. LU = M = 0 means that L can be any n × n unit lower triangular matrix. Therefore, there exisits different L which satisfy LU = M, meaning that if a singular M which has an LU decomposition, then the decomposition is non-unique.

(ii) LUP decompositions are not unique. Give an example of a non-singular 3 × 3 matrix A for which there are two different LUP decompositions. (No proof needed here, just write A and the L, U, P and L', U' and P' which show this.)

Answer:

$$P_1A = L_1U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 9 & 1 \\ 1 & 4 & 4 \\ 9 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{9} & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 9 & 9 & 1 \\ 0 & 3 & \frac{35}{9} \\ 0 & 0 & \frac{251}{27} \end{bmatrix} \tag{12}$$

$$P_2A = L_2U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 9 & 1 \\ 1 & 4 & 4 \\ 9 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{9} & -3 & 1 \end{bmatrix} \begin{bmatrix} 9 & 9 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & \frac{251}{9} \end{bmatrix} \tag{13}$$

3 Finding an inverse

Define

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix} \tag{14}$$

Answer:

(i) Find an LU decomposition for T.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{11} & 1 \end{bmatrix} U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & \frac{11}{3} & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \tag{15}$$

(ii) Use (i). to find the inverse of T.

We can reduce the problem finding an inverse to solving linear equations. Let T^{-1} be the inverse of T and then we have:

$$TT^{-1} = I = LUT^{-1} \tag{16}$$

where I is an 4 by 4 identity matrix.

Let m be a column vector.

$$T^{-1} = (m_1, m_2, m_3, m_4) \tag{17}$$
$$I = (e_1, e_2, e_3, e_4) \tag{18}$$

where $m_i (i = 1, 2, 3, 4)$ is the ith column vector of matrix T^{-1} and $e_i (i = 1, 2, 3, 4)$ is unit column vector with a 1 on ith row.

According to (16), we have:

$$LUm_i = e_i \tag{19}$$

We use two times of subsitution to get T^{-1}.

$$Ly = e_i \tag{20}$$
$$Um_i = y \tag{21}$$

We can get:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{11} & \frac{1}{11} & \frac{3}{11} & 1 \end{bmatrix} \tag{22}$$

$$T^{-1} = \begin{bmatrix} \frac{15}{11} & \frac{4}{11} & \frac{1}{11} & 0 \\ \frac{4}{11} & \frac{4}{11} & \frac{1}{11} & 0 \\ \frac{1}{11} & \frac{1}{11} & \frac{3}{11} & 0 \\ \frac{1}{44} & \frac{1}{44} & \frac{3}{44} & \frac{1}{4} \end{bmatrix} \tag{23}$$

4 Reducing general (non-square) matrix multiplication to matrix inversion

Let I be an algorithm which takes as input a matrix M and outputs its inverse M^{-1}. In class we showed how you can multiply two $n \times n$ matrices A and B by inverting a $3n \times 3n$ matrix C.

(i) Show how to do the same thing for non-square matrices A and B where A is $n \times r$ and B is $r \times s$ for any positive natural numbers n, r, s. (Hint: Use padding of matrices with 0's, sort of similar to what you may have done for Strassen's algorithm.)

Answer: Let integer m be max(n, r, s). Let A' be a m $\times$ m matrix where the elements in the first n rows and the first r columns are exactly the same as matrix A and the other elements are zeros. Let B' be a m $\times$ m matrix where the elements in the first r rows and the first s columns are exactly the same as matrix B and the other elements are zeros.

$$A' = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} B' = \begin{bmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

Then we can construct a 3m $\times$ 3m matrix D and its inverse D^{-1}:

$$DD^{-1} = \begin{bmatrix} I_m & A' & 0 \\ 0 & I_m & B' \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} I_m & -A' & A'B' \\ 0 & I_m & -B' \\ 0 & 0 & I_m \end{bmatrix} = I_{3m} \tag{25}$$

where I_m is m $\times$ m identity matrix and the elements in the first n rows and first s columns of $A'B'$ is the multiplication of A and B.

(ii) Give an example of how this would work when A is 4 $\times$ 2 and B is 2 $\times$ 3. Specifically, find the corresponding matrix C whose inverse would give you AB.

Answer: Here is an example.

$$A = \begin{bmatrix} 6 & 7 \\ 6 & 4 \\ 9 & 9 \\ 8 & 6 \end{bmatrix} B = \begin{bmatrix} 4 & 9 & 7 \\ 10 & 6 & 6 \end{bmatrix} \tag{26}$$

$$A' = \begin{bmatrix} 6 & 7 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 9 & 9 & 0 & 0 \\ 8 & 6 & 0 & 0 \end{bmatrix} B' = \begin{bmatrix} 4 & 9 & 7 & 0 \\ 10 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$C = \begin{bmatrix} I_4 & A' & 0 \\ 0 & I_4 & B' \\ 0 & 0 & I_4 \end{bmatrix} = \left[\begin{array}{cccc|cccc|cccc} 1 & 0 & 0 & 0 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 8 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 9 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 10 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{28}$$

$$C^{-1} = \begin{bmatrix} I_4 & -A' & A'B' \\ 0 & I_4 & -B' \\ 0 & 0 & I_4 \end{bmatrix} = \left[\begin{array}{cccc|cccc|cccc} 1 & 0 & 0 & 0 & -6 & -7 & 0 & 0 & 94 & 96 & 84 & 0 \\ 0 & 1 & 0 & 0 & -6 & -4 & 0 & 0 & 64 & 78 & 66 & 0 \\ 0 & 0 & 1 & 0 & -9 & -9 & 0 & 0 & 126 & 135 & 117 & 0 \\ 0 & 0 & 0 & 1 & -8 & -6 & 0 & 0 & 92 & 108 & 92 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & -9 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -10 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{29}$$

$$AB = \begin{bmatrix} 94 & 96 & 84 \\ 64 & 78 & 66 \\ 126 & 135 & 117 \\ 92 & 108 & 92 \end{bmatrix} \tag{30}$$

5 A problem about random permutations

Consider the following randomized algorithm R which produces a permutation of $1, 2, ..., n$ when run.
Algorithm R:
R starts with a vector V = (1,2,3,...,n) of the first n natural numbers specifying the identity permutation. You then "randomize" V by:

1. Independently at random pick n integers a1, a2, ..., an from the set 1,2,3,...,n. (This is done with replacement. That is, the integers you pick may be repeated in the list of a_i's.)
2. For each element ai from i=1 to n, switch i in V with a_i in V.
3. The result of the n switches in V is a permutation of the integers 1 through n which is the output of this randomized algorithm.

Call a permutation of 1,2,3,...,n random if the probablility that it is output by R is 1/n!.

Questions:

1. Show that any one of the n! different permutations of 1,2,3,...,n could be output by some choice of a1, a2, ..., an in step 1 of algorithm R.

   Answer: Algorithm R does not change the number of V but do some switch operations, which means any output will be a permutation of 1,2,3,...,n. For any possible permutation of 1,2,3,...,n, there always exists a way to get it by doing limitted times switch operations.

2. True or false: An output of step 3 of algorithm R is a random permutation of 1,2,3,...,n? Briefly explain your answer.

   Note: What you need to determine here is whether all of the n! permutations of V are equally likely to be output in step 3 of the algorithm.

   Answer: No! That's not a random permutation for $n > 2$.

   Explanation:
   Let V be the original vector $V = [1, 2, 3, ..., n]$.
   Let V' be the output of step 3 of algorithm R.
   When $n = 1$, V' will always be $V' = [1]$, thus which is a random permutation.
   When $n = 2$, the probability of V' being [1,2] is $\frac{1}{2}$ and the probability of V' being [2,1] is $\frac{1}{2}$:

   $$P(V' = [1, 2]) = P(V' = [2, 1]) = \frac{1}{2} \tag{31}$$

   Proof:
   For V'=[1,2]:

$$\text{Situation 1: } P(a_1 = 1) = \tfrac{1}{2},\ P(a_2 = 2) = \tfrac{1}{2}$$
$$P(Situation1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$\text{Situation 2: } P(a_1 = 2) = \tfrac{1}{2},\ P(a_2 = 1) = \tfrac{1}{2}$$
$$P(Situation2) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$P(V' = [1,2]) = P(Situation1) + P(Situation2) = \tfrac{1}{2}$$

For V'=[2,1]:

$$\text{Situation 1: } P(a_1 = 1) = \tfrac{1}{2},\ P(a_2 = 1) = \tfrac{1}{2}$$
$$P(Situation3) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$\text{Situation 2: } P(a_1 = 2) = \tfrac{1}{2},\ P(a_2 = 2) = \tfrac{1}{2}$$
$$P(Situation4) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$P(V' = [2,1]) = P(Situation3) + P(Situation4) = \tfrac{1}{2}$$

Therefore, $P(V' = [1,2]) = P(V' = [2,1]) = \frac{1}{2}$, which means algorithm R will generate a random permutaion.

However, for $n = 3$, algorithm R cannot generate a random permutation.
Use the same method that we use when $n = 2$,
we will have $3 \times 3 \times 3 = 27$ situations in all.
The number of permutations of $[1, 2, 3]$ is 6.
$\frac{27}{6} = 4.5$ is not an integer, which means every permutation of $[1, 2, 3]$ cannot be generated on equal probability.
Thus, algorithm R cannot generate a random permutaion.

Let's generalize this problem and **consider integer n**.
In step one, there will be n^n choices of $[a_1, a_2, ..., a_n]$.
For each choice of $[a_1, a_2, ..., a_n]$, step 3 will generate a permutaion of $[1, 2, 3, ...n]$.
If every permutation of $[1, 2, 3, ...n]$, the number of which is $n!$, is equally likely to be output, we will have:

$$\frac{n^n}{n!} \in \mathbf{Z}^+ \tag{32}$$

where $\mathbf{Z}^+$ is the set of positive integers.
Unfortunately, equation 33 is not possible. $\frac{n^n}{n!}$ is not an integer.
Therefore, algorithm R cannot generate a random permutation when $n > 2$.

3. Now change step 1 of algorithm R by not allowing repetitions of numbers in a1, a2, ..., an. That is we first choose a1 randomly from 1, 2,...n, but then choose a2 randomly from all the first n numbers except for a1, then choose a3 randomly from numbers 1,2,...n except for a1 and a2, etc.

   Steps 2 and 3 of the algorithm remain the same.

   Answer the same question 2 for this slightly changed version of R, and explain your answer.

Answer: Yes! That will be a random permutation.
Explanation:
In step one, there will be $n!$ choices of $[a_1, a_2, ..., a_n]$.
For differnent choices, step 3 will output different permutations of $[1, 2, 3, ...n]$.
Equation 33 will be reduced to:

$$\frac{n!}{n!} = 1 \in \mathbf{Z}^+ \tag{33}$$

Every permutation is equally to be output by algorithm R.