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GRASP with path relinking for commercial districting

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ABSTRACT

The problem of grouping basic units into larger geographic territories subject to dispersion, connectivity, and balance requirements is addressed. The problem is motivated by a real-world application from the bottled beverage distribution industry. Typically, a dispersion function is minimized as compact territories are sought. Existing literature reveals that practically all the works on commercial districting use center-based dispersion functions. These center-based functions yield mixed-integer programming models with some nice properties; however, they have the disadvantage of being very costly to be properly evaluated when used within heuristic frameworks. This is due to the center updating operations frequently needed through the heuristic search. In this work, a more robust dispersion measure based on the diameter of the formed territories is studied. This allows a more efficient heuristic search computation. For solving this particular territory design problem, a greedy randomized adaptive search procedure (GRASP) that incorporates a novel construction procedure where territories are formed simultaneously in two main stages using different criteria is proposed. This also differs from previous literature where GRASP was used to build one territory at a time. The GRASP is further enhanced with two variants of forward-backward path relinking, namely static and dynamic. Path relinking is a sophisticated and very successful search mechanism. This idea is novel in any districting or territory design application to the best of our knowledge. The proposed algorithm and its components have been extensively evaluated over a wide set of data instances. Experimental results reveal that the construction mechanism produces feasible solutions of acceptable quality, which are improved by an effective local search procedure. In addition, empirical evidence indicate that the two path relinking strategies have a significant impact on solution quality when incorporated within the GRASP framework. The ideas and components of the developed method can be further extended to other districting problems under balancing and connectivity constraints.

1. Introduction

The territory design problem (TDP) may be viewed as the problem of grouping small geographic basic units (BUs) into larger geographic clusters, called territories, in a way that the territories are
Q2 acceptable (or optimal) according to relevant planning criteria. Territory design or districting has a broad range of applications such as political districting, sales territory design, school districting, power districting, and public services, to name a few. The reader can find in the works of Kalcsics, Nickel, and Schröder (2005) and Duque, Ramos, and Suriñach (2007) state of the art surveys on models, algorithms, and applications to districting problems.

The problem addressed in this paper is a commercial territory design problem (CTDP) motivated by a real-world application from the bottled beverage distribution industry. The problem, introduced by Ríos-Mercado and Fernández (2009), considers finding a design of *p* territories with minimum dispersion subject to planning requirements such as exclusive BU-to-territory assignment, territory connectivity, and territory balancing with respect to three BU attributes: number of customers, product demand, and workload.

An important criterion in territory design problems is compactness. Typically this is achieved by minimizing a dispersion function. In commercial territory design, several models based on dispersion functions from the well-known *p*-center and *p*-median location problems have been studied in the past. These are center-based dispersion functions, that is, the dispersion is measured with respect to a centroid of a territory. However, there are other non-center-based measures of dispersion that can be used. Center-based functions rely heavily on the location of the centers; if the centers are "badly" located, the resulting design may cause a serious deterioration in objective function. In addition, in location problems, the centers represent a physical entity or facility that provides some service; however,

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in CTDPs the centers are artificially located as no facility is actually placed there, it is just a reference for the dispersion measure. These limitations motivate the study on other ways of measuring dispersion. For instance, a measure such as the diameter, which measures the longest distance between any two basic units in a territory, is a more robust function since it does not depend on a center location, providing more flexibility. Even from the algorithmic perspective, heuristic methods for tackling TDPs under center-based dispersion functions need to constantly update and recompute as centers keep moving along every time the territory suffers a change. This time-consuming task can be avoided if other measures such as the diameter are used.

In this work, we focus our study in a commercial territory design problem that seeks to minimize territory dispersion based on a diameter dispersion measure. To the best of our knowledge, this type of problem has not been addressed before in the territory design literature. Since the aim is to target large instances, we present a greedy randomized adaptive search procedure (GRASP) with path relinking for this NP-hard CTDP. The algorithm is denoted as GPR_CTDP. In our proposed GRASP we develop a procedure that builds exactly p territories at once simultaneously, that is, we start with p node seeds and start associating nodes to the seeds until all of them are assigned. By growing the territories simultaneously rather than one at a time one expects that the violation of the balancing constraints be considerably lower. In addition, we develop two path relinking (PR) strategies, one dynamic and one static, motivated by the work of Resende, Martí, Gallego, and Duarte (2010a), who successfully applied it to the max–min diversity problem. In our work, these PR strategies rely on finding a "path" between two different territory designs. To this end, an associated assignment subproblem for finding the best match between territory centers is solved. The solution to this problem provides a very nice way of generating the trajectory between two given designs. This idea is novel in any districting or territory design application to the best of our knowledge.

To assess its efficiency, the proposed GPR_CTDP with many of its components and strategies has been extensively evaluated over a wide set of data instances. We have found, for instance, that building territories simultaneously results in feasible solutions of acceptable quality. The two PR variants implemented in GPR_CTDP allowed us to obtain better solutions than those obtained when using straight local search; although, the static strategy resulted more helpful. The main algorithmic ideas incorporated in the developed algorithm can be extended so as to handle other districting problems with similar structure.

The paper is organized as follows. In Section 2 we describe the problem in detail and present a combinatorial optimization model. Section 3 gives an overview of relevant previous related work. Section 4 describes in detail the components of the proposed heuristic, and Section 5 presents the empirical evaluation of the method. We end the paper in Section 6, with some conclusions and final remarks.

2. Problem description

Let $G = (V, E)$ denote a graph where V is the set of city blocks or basic units (BUs), and E is the set of edges representing adjacency between blocks, that is, $(i, j) \in E$ if and only if BUs i and j are adjacent blocks. Let d_{ij} denote the Euclidean distance between BUs i and j, with $i, j \in V$. For each BU $i \in V$ there are three associated parameters. Let w_i^a be the value of activity a at node i, where $a = 1$ (number of customers), $a = 2$ (product demand), and $a = 3$ (workload). The number of territories is given by the parameter p. A p-partition of V is denoted by $X = (X_1, \ldots, X_p)$, where $X_k \subset V$ is called a territory of V. Let $w^a(X_k) = \sum_{i \in X_k} w_i^a$ denote the size of territory X_k with respect to activity $a \in A = \{1, 2, 3\}$ and $k \in K = \{1, \ldots, p\}$. The balancing planning requirements are modeled by introducing a user-specified tolerance parameter τ^a that measures the allowable relative deviation from the target average size μ^a, given by $\mu^a = w^a(V)/p$, for each activity $a \in A$. Another planning requirement is that all of the nodes assigned to each territory are connected by a path contained totally within the territory. In other words, each of the territories X_k must induce a connected subgraph of G. Finally, we seek to maximize territory compactness or, equivalently, minimize territory dispersion, where dispersion is given by the largest diameter over all territories, that is $\max_{k=1,\ldots,p} \max_{i,j \in X_k} \{d_{ij}\}$.

Let Π be the collection of all p-partitions of V. The combinatorial optimization model is given as follows.

Model (CTDP)

$$\min_{X \in \Pi} \quad f(X) = \max_{k \in K} \max_{i,j \in X_k} \{d_{ij}\} \tag{1}$$

$$\text{subject to} \quad \frac{w^a(X_k)}{\mu^a} \in [1 - \tau^a, 1 + \tau^a] \quad k \in K,\ a \in A \tag{2}$$

$$G_k = G(V_k, E(V_k)) \text{ is connected} \quad k \in K \tag{3}$$

Objective (1) measures territory dispersion. Constraints (2) represent the territory balance with respect to each activity measure as it establishes that the size of each territory must lie within a range (measured by tolerance parameter τ^a) around its average size. Constraints (3) guarantee the connectivity of the territories, where G_k is the graph induced in G by the set of nodes X_k. Note that there is an exponential number of such constraints.

The model can be viewed as partitioning G (the contiguity graph representing the BUs) into p connected components (contiguous districts) under the additional side constraints on balancing product demand, number of customers, and workload of each territory, and minimizing a dispersion measure of the BUs in a territory. The basic contiguity graph model for the representation of a territory divided into elementary units has been adopted in political districting (Ricca & Simeone, 2008).

3. Related work

Territory design or districting has a broad range of applications such as political districting (Bozkaya, Erkut, & Laporte, 2003; Browdy, 1990; Forman & Yue, 2003; Mehrotra, Johnson, & Nemhauser, 1998; Pukelsheim, Ricca, Simeone, Scozzari, & Serafini, 2012; Ricca & Simeone, 2008), sales territory design (Drexl & Haase, 1999; Zoltners & Sinha, 1983; 2005), school districting (Caro, Shirabe, Guignard, & Weintraub, 2004), power districting (de Assis, Franca, & Usberti, 2014; Bergey, Ragsdale, & Hoskote, 2003), and public services (Blais, Lapierre, & Laporte, 2003; D'Amico, Wang, Batta, & Rump, 2002; Muyldermans, Cattryse, Oudheusden, & Lotan, 2002), to name a few. The reader can find in the works of Kalcsics et al. (2005) and Duque et al. (2007) state of the art surveys on models, algorithms, and applications to districting problems. Zoltners and Sinha (2005) present a survey focusing on sales districting and Ricca, Scozzari, and Simeone (2013) present a survey on political districting.

Here we discuss the related work on commercial territory design. Ríos-Mercado and Fernández (2009) introduced the commercial TDP by incorporating a territory compactness criterion and a fixed number of territories p. They seek to maximize this compactness criterion subject to planning requirements such as exclusive BU-to-territory assignment, territory connectivity, and territory balancing with respect to three BU attributes: number of customers, product demand, and workload. In their work, the authors consider as a minimization function a dispersion function based on the objective function of the well-known p-Center Problem. After establishing the NP-completeness of the problem, the authors propose a Reactive GRASP for obtaining high-quality solutions to this problem. The core of their GRASP is a three-phase iterative procedure composed by a construction phase, an adjustment phase, and a local search phase. In

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the construction phase a solution with q territories, where q is usually larger than p, satisfying the connectivity constraints is built. Then an adjustment phase based on a pairwise merging mechanism is applied to obtain a solution with p territories. Afterwards, a local search phase attempting both to eliminate the infeasibility with respect to the balancing requirements and to improve the dispersion objective function is applied. One interesting observation is that the construction and adjustment phases produce solutions with very high degree of infeasibility. This is very nicely repaired by the local search, at a very high computational cost though. The reason for this is that attempting to merge two territories into one in the adjustment phase may result in a high violation of the upper bound of the balancing constraints.

Salazar-Aguilar, Ríos-Mercado, and Cabrera-Ríos (2011a) present an exact optimization framework for tackling relatively small instances of several CDTP models. They studied two linear models that differ in the way they measure dispersion, one model uses a dispersion function based on the objective of the p-Median Problem (MPTDP) and the other is based on the p-Center Problem (CPTDP). They can successfully solve instances of up to 100 BUs for the CPTDP and up to 150 BUs for the MPTDP. This concludes that p-center-based dispersion measures yield more difficult models as they have weaker LP relaxations than the median-based models.

Ríos-Mercado and Salazar-Acosta (2011) present a heuristic based on GRASP and adaptive memory programming for a CTDP that considers the minimization of a p-Center Problem function subject to additional budget routing constraints.

López-Pérez and Ríos-Mercado (2013) and Ríos-Mercado and López-Pérez (2013) extend the CTDP model by incorporating additional planning criteria such as joint and disjoint assignment requirements and similarity with the existing plan. Joint (disjoint) assignment means that a given set of customers must be assigned to the same (different) territory. Similarity with existing plan means that the new plan must be similar to the previous plan by allowing only a small portion of the basic units to be assigned to different territories. In this work, the authors use a p-Median Problem objective function for measuring dispersion. The authors develop a mathematical programming approach for dealing with the customer allocation level with relatively success by solving a surrogate mixed-integer programming model.

One of the most popular methods for addressing districting problems is the location-allocation technique (Kalcsics et al., 2005). However, this technique is not applicable to our problem mainly because the nature of the dispersion objective function is different. As it has been show, the location-allocation method seems to work well when a p-Median Problem-based objective function is used. From a theoretical perspective, Elizondo-Amaya, Ríos-Mercado, and Díaz (2014) develop a lower bounding scheme for the CDTP based on Lagrangian relaxation that considers a p-center based objective function.

CTDP has also been addressed from a multiobjective optimization perspective. Salazar-Aguilar, Ríos-Mercado, and González-Velarde (2011b) present an exact optimization method for obtaining Pareto fronts for relatively small instances for the problem where both territory compactness and balance are simultaneously optimized. Salazar-Aguilar, Ríos-Mercado, González-Velarde, and Molina (2012) and Salazar-Aguilar, Ríos-Mercado, and González-Velarde (2013) develop heuristic methods based on scatter search and GRASP, respectively, for addressing larger instances. Their heuristics find good quality approximations to the Pareto fronts; however, in each of these multi-objective optimization approaches center-based functions are used for measuring territory dispersion.

Summarizing these most relevant works on commercial TDP, all of them address dispersion functions based on territory centers. One of the reasons is that center-based functions yield well-structured mixed-integer programming models which in turn can lead to relatively good optimization algorithms. However, this relative advantage is somewhat lost when addressing a problem from the heuristic perspective. For instance, every time a territory changes, one must check and recompute if necessary a new center which involves computation between all pairs of basic units. In the past, one way authors have addressed this issue is by choosing not to update the centers every time but periodically. This has the negative consequence of not having the correct and precise value of the dispersion objective function all the time. As stated previously, CTDPs with diameter-based dispersion functions have not been studied in the past. To the best of our knowledge, our work is the first to introduce a non-center based measure of dispersion in the CDTP context. In fact, using non-center-based functions such as the diameter may be more convenient since no time consuming center updating operations are needed. It is a more robust measure in this sense.

4. Proposed heuristic

It is important to note that we are introducing a new model that has not been studied before to the best of our knowledge. As stated in the previous section, all existing methods developed for commercial districting are not applicable in this case given the different nature of the objective function being optimized. In the same vein, existing clustering software is not tailored for handling highly constrained problems such as the one being addressed.

This section introduces the proposed GRASP heuristic with path relinking for the commercial territory design problem (GPR_CTDP). GRASP is a well known meta-heuristic based on greedy search and random construction mechanisms Feo and Resende (1995) that has been successfully used for many combinatorial optimization problems, including CTDP Ríos-Mercado and Fernández (2009). We propose a GRASP improved with path relinking (PR). One important feature about GRASP when compared with other methods such as population-based heuristics (genetic algorithms, particle swarm optimization, etc.) is that one can design the construction mechanism in such a way to guarantee that the difficult constraints (such as connectivity) are met, something very difficult to achieve by other methods. Naturally, the incorporation of a sophisticated search mechanism such as PR is expected to render solutions of much better quality than those obtained by simple local search. The heuristic comprises a new construction procedure and a very effective PR mechanism. The construction procedure intelligently handles a strategy for building territories simultaneously, while the PR formulation allows us to obtain better solutions than those obtained when using straight local search, see Section 5. The rest of this section describes in detail the components of the GPR_CTDP approach, which receives as input an instance of the CTDP and a set of parameters as described below.

4.1. GRASP

A GRASP is an iterative process in which each major iteration consists of two phases: construction and local search (Feo & Resende, 1995). The construction phase attempts to build a feasible solution and the local search phase attempts to improve it. This process is repeated for a fixed number of iterations and the best overall solution is returned as the result. GRASP incorporates greedy search and randomization mechanisms that allow it to obtain high quality solutions to combinatorial problems in acceptable times. Despite the simplicity of this multi-start heuristic it has proved to be very effective in a wide range of problems and applications (see Resende and Ribeiro, 2010, chap. 10). Previous work on GRASP for the CTDP is presented in Section 3. In this paper we propose procedure GPR_CTDP, which is in essence a GRASP augmented with PR mechanisms, accordingly, in this section we describe the particular construction and local search procedures of the GRASP and the next subsection presents the PR strategies.

4.1.1. Construction phase

At a given iteration, the construction phase consists of building p territories, $X_1, \ldots, X_p$, simultaneously in such a way that connectivity is always satisfied while infeasibility in terms of dispersion and balance is allowed to some extent. Each territory X_k is formed by a subset of BUs or nodes such that $\cup_{k=1,\ldots,p} X_k = V$ and $X_k \cap X_l = \emptyset$, for all $k \neq l$. Under the proposed procedure each territory X_k is associated to a center, $c(k)$. This is not a requirement of the problem but a feature of the proposed formulation that was adopted for convenience when measuring dispersion of territories.

Procedure 1 grasp_construction(δ, L, α).

```
Input:  δ: fraction of nodes assigned by the distance criteria;
  L: interval for updating centers;
  α: RCL quality parameter;
Output:  X: A p-partition of V;
  (c(1), ..., c(p)) ← max_disp( p );{Compute p initial centers}
  i ← 0;   V̄ ← V;
  while ( n − |V̄| ≤ δn ) do
    for all ( k ∈ {1, ..., p} ) do
      N_q(X_k) ← q nearest (unassigned) neighbors of X_k;
      X_k ← X_k ∪ N_q(X_k);   V̄ ← V̄ \ N_q(X_k);
    end for
    i ← i + 1;
    if ( i module L = 0 ) then
      c(k) ← min ( max d_{v,w}), ∀v, w ∈ X_k, k = 1, ..., p; {Update centers}
    end if
  end while
  open(k) ← TRUE, k = 1, ..., p;
  while ( |V̄| > 0 and ∃k such that open(k) == TRUE ) do
    for all ( k = 1, ..., p ) do
      if ( open(k) = TRUE ) then
        Compute φ_k(v) in Eq. (4), ∀ v ∈ N(X_k);
        Φ_min ← min{φ_k(v)}; Φ_max ← max{φ_k(v)};
        RCL ← {h ∈ N(X_k) : φ_k(h) ≤ Φ_min + α(Φ_max − Φ_min)};
        Choose v ∈ RCL randomly; X_k ← X_k ∪ {v}; V̄ ← V̄ \ {v};
        if ( N(X_k) = ∅ or w^a(X_k) > (1 + τ^a) for any a ) then
          open(k) ← FALSE; {Close this territory}
        end if
      end if
    end for
  end while
  if (|V̄| > 0) then
    for all ( v ∈ V̄ ) do
      X_v ← Nearest territory to node v;
      X_v ← X_v ∪ {v}; V̄ ← V̄ \ {v};
    end for
  end if
  return X = {X_1, ..., X_p};
```

Procedure 1 presents the construction phase of the proposed GPR_CTDP. $\bar{V}$ denotes the set of nodes that have not been assigned to any territory and $n = |V|$ the number of BUs. The process starts by selecting p seeds or centers, $\{c(1), \ldots, c(p)\}$, which are the first nodes assigned to each territory; that is, $c(k) \in X_k$, $k \in \{1, \ldots, p\}$. Territories are then built iteratively in two main stages followed by a postprocessing stage. In the first stage q BUs are iteratively assigned to each territory X_k. For each territory X_k, we iteratively assign the q (unassigned) nearest neighboring nodes of that territory, $v \in N_q(X_k)$. The BUs in $N_q(X_k)$ that are assigned to X_k must be connected by an edge to a BU already assigned to X_k. The latter process is iterated until a fraction δ of the total of BUs have been assigned to one of the p territories (i.e. $\lfloor \delta n \rfloor$ BUs have been assigned), where the centers $c(1), \ldots, c(p)$ are updated every L iterations. One should note that the notion of centers is only used for this very-first phase of the construction procedure and it is not used elsewhere.

Fig. 1 shows the BUs assigned after stage one of the construction phase for an instance of the CTDP considered for experimentation. From this stage the p territories have been simultaneously built by using a neighborhood criteria completely ignoring the balance constraints. The rationale behind this is that nodes that belong to the same territory must be close to each other, hence a portion of nodes

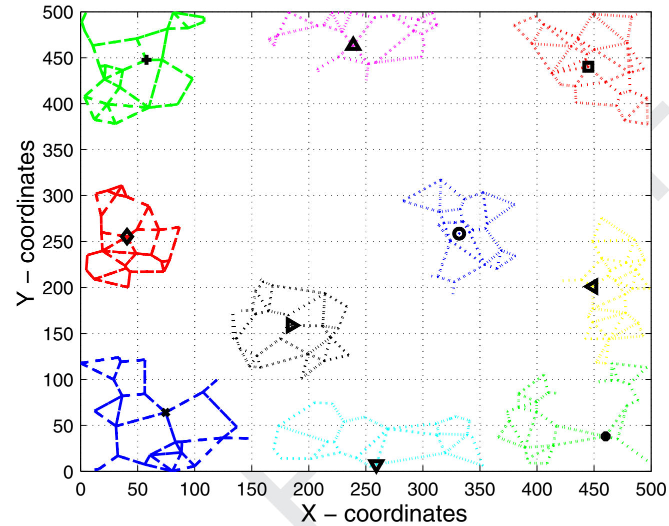

Fig. 1. First stage of the proposed construction procedure for an instance of the CTDP.

can be assigned with a closeness criterion. The remaining nodes will lie at boundaries among territories; therefore, balance and dispersion information is taken into account for assigning those nodes.

An important aspect of stage one is that of selecting seed centers. Clearly, randomness must be considered for this process as we want to generate fairly different centers at each iteration of the GPR_CTDP approach. To this end, we view the problem of choosing an appropriate set of p initials seeds as a p-Dispersion Problem (Erkut, Ülkücal, & Yeniçerioğlu, 1994), which is a combinatorial optimization problem that places p points in the plane as far way of each other as possible by using an appropriate measure for maximizing dispersion. In our procedure, we used an approach that selects centers randomly with a maximum dispersion criteria. The particular strategy starts with a randomly selected node as the center for the first territory and the rest of centers are obtained by using a greedy heuristic for the p-dispersion problem (Erkut et al., 1994).

The second stage of the construction phase consists of assigning the remaining $n - \lfloor \delta n \rfloor$ nodes that were not assigned in stage one. For this stage BUs are assigned to territories using a greedy randomized adaptive procedure that takes into account both balance and dispersion constraints. For each territory X_k, the cost of assigning every neighboring node $v \in N(X_k)$ to X_k is evaluated according to Eq. (4). Then a restricted candidate list (RCL) is formed, from which a single BU is randomly selected and assigned to the current territory X_k. This RCL is restricted by a quality parameter α, that is, RCL is formed by those BUs whose greedy function evaluation falls within α percent from the best evaluation. Eq. (4) determines the cost incurred when assigning node v to a territory X_k. This cost is determined by a linear combination of the weights assigned to nodes in territory $X_k \cup \{v\}$, as determined by the term $G_k(v)$, and the dispersion of those nodes, as estimated by the term $F_k(v)$, with $G_k(v)$ and $F_k(v)$ defined in Eqs. (5)(6), respectively

$$\phi_k(v) = \lambda F_k(v) + (1 - \lambda) G_k(v), \tag{4}$$

$$G_k(v) = \sum_{a \in A} g_k^a(v), \tag{5}$$

$$F_k(v) = \left(\frac{1}{d_{max}}\right) f(X_k \cup \{v\}) = \left(\frac{1}{d_{max}}\right) \max\{f(X_k), \max_{i \in X_k, v}\{d_{iv}\}\}, \tag{6}$$

where $f(X_k) = \max_{k \in K} \max_{i,j \in X_k}\{d_{ij}\}$ is the dispersion measure (as dictated by the objective function) and $g_k^a(v) = \frac{1}{\mu^a} \max\{w^a(X_k \cup \{v\}) - (1 + \tau^a)\mu^a, 0\}$ accounts for the sum of relative infeasibilities

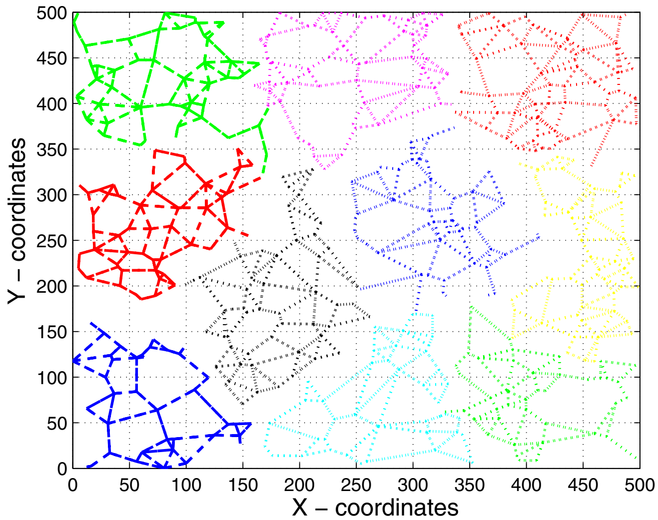

Fig. 2. Second and third stages of the proposed construction procedure for an instance of the CTDP.

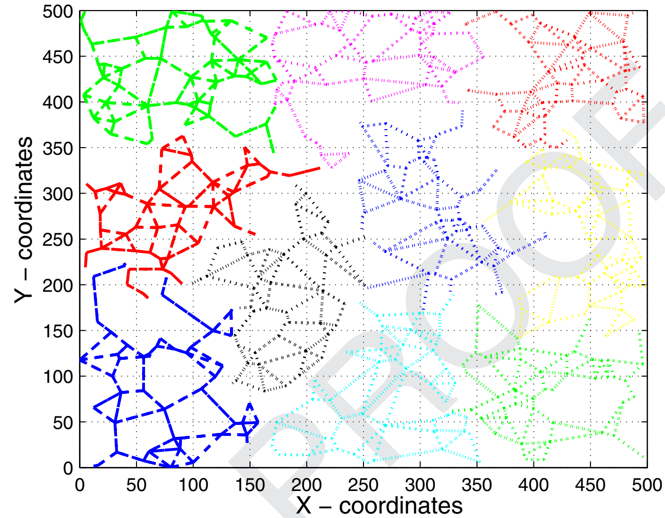

Fig. 3. Solution found after applying the local search procedure for an instance of the CTDP.

for the balancing constraints. Here $d_{max} = \max_{i,j \in V}\{d_{ij}\}$, the maximum distance between any pair of nodes, is used for normalizing the objective function. One should note that $g_k^a(v)$ represents the infeasibility with respect to the upper bound of the balance constraint for activity a. Both factors dispersion and balancing are weighted by a parameter λ in expression (4). The process is repeated for every territory k. If a territory exceeds the expected average weight for a territory it is considered *closed* (i.e., $open(j) = false$) and no further node can be assigned to it. The latter process iterates until either every node has been assigned to a territory or every territory is considered closed. Since stage two of this construction phase does not guarantee that all nodes will be assigned to a territory, a third stage is applied in which each unassigned node gets assigned to its nearest territory. Fig. 2 shows the distribution of BUs for an instance of the CTDP after stages two and three of the construction procedure.

4.1.2. *Local search*

After a solution is build a postprocessing phase consisting of local search is performed. The goal in this phase is to improve the objective function value and recovering feasibility (if violated) in the constructed solution, X. In this local search, a merit function that weights both the infeasibility with respect to balancing constraints and the objective function value is used. This function is indeed similar to the greedy function used in the construction phase with the exception that now the sum of relative infeasibilities take into consideration lower and upper bound violation of the balancing constraints. Specifically, the merit function for a given territory design $X = \{X_1, \dots, X_p\}$ is given by

$$\psi(X) = \lambda F(X) + (1 - \lambda) G(X) \tag{7}$$

where

$$F(X) = \left(\frac{1}{d_{max}}\right) \max_{k \in K} \max_{i,j \in X_k}\{d_{ij}\} \tag{8}$$

and

$$G(X) = \sum_{k=1}^{p} \sum_{a \in A} g^a(X_k), \tag{9}$$

with $g^a(X_k) = \frac{1}{\mu^a} \max\{w^a(X_k) - (1 + \tau^a)\mu^a, (1 - \tau^a)\mu^a - w^a(X_k), 0\}$ being the sum of the relative infeasibilities of the balancing constraints. The quality of solutions is then determined by expression (7), we now describe the mechanism for exploring solutions around the constructed territory design. Let $t(i)$ denote the territory node i belongs to, $i = 1, \dots, n$. A move $move(i, j)$ is defined as moving a node i from its current territory to a territory $t(j)$, where $t(j) \neq t(i)$. Only moves $move(i, j)$ where $(i, j) \in E$ and $t(i) \neq t(j)$ are allowed. Thus, $move(i, j)$ transforms a solution $X = (X_1, \dots, X_{t(i)}, \dots, X_{t(j)}, \dots, X_p)$ into $X^T = (X_1, \dots, X_{t(i)} \setminus \{i\}, \dots, X_{t(j)} \cup \{i\}, \dots, X_p)$. If connectivity must be kept, only moves where $X_{t(i)} \backslash \{i\}$ remains connected are allowed. Note that in general $move(i, j)$ is asymmetric.

The basic idea of the local search is to start the search with a given territory, say territory k, and then consider first the moves emanating from territory k, that is, if we let $N(X_k)$ denote the feasible moves $move(i, j)$ with $t(i) = k$ evaluate first all the moves in $N(X_k)$, and take the best that improves the current solution, if any. If none found, proceed with territory $(k + 1) \mod p$. As soon as a better move is found, perform the move, and restart the search from this new solution X^T but setting $k + 1$ as the starting territory, where k was the last territory examined, that is, in a new move the starting territory is $k + 1$ and the final territory to be examined is k. By using this cyclic strategy for starting territory we avoid performing many unnecessary move evaluations. A move is performed using a different territory each time until no improvements can be found. In practice an additional stopping criterion: the maximum number of allowed evaluations of the fitness function (*limit_evals*), is added to avoid performing an extensive search for long periods of time. Therefore, the postprocessing step stops when either a local optima is found or the number of moves exceeds *limit_evals*. The postprocessing phase is described in Procedure 2. Fig. 3 shows a solution obtained after applying the local search procedure.

4.2. *Path relinking*

Path relinking (PR) was originally proposed by Glover and colleagues as a way of incorporating intensification and diversification strategies in tabu search (Glover, 1996, chap. 1). PR consists of exploring the path of intermediate solutions between two selected solutions called starting (X^S) and target (X^T) with the hypothesis that some of the intermediate solutions can be either better than X^S and X^T (intensification) or comparable but different enough from X^S and X^T (diversification). Intermediate solutions are generated by performing moves from the starting solution in such a way that these moves introduce attributes that are present in the target solution. Successful applications of PR in the context of Tabu and Scatter Search are reported in Resende, Ribeiro, Glover, and Martí (2010b, chap. 4).

Procedure 2 local_search (X).

Input: X: A solution to the CTDP;
Output: X: Improved solution to the CTDP;
  $nmoves \leftarrow 0$; $local\_optima \leftarrow$ FALSE;
  $k \leftarrow 1$; {starting territory}
  while ($nmoves \leq limit\_evals$ AND $\neg local\_optima$) **do**
    $improvement \leftarrow$ FALSE;
    while ($|N(X_k)| > 0$ and $\neg improvement$) **do**
      $move(i, j) \leftarrow$ Choose valid move from $N(X_k)$;
      $N(X_k) \leftarrow N(X_k) \setminus \{(i, j)\}$;
      Evaluate $\psi(X^T)$ using Eq. (7);
      if ($\psi(X^T) < \psi(X)$) **then**
        $X \leftarrow X^T$; {perform move}
        $nmoves \leftarrow nmoves + 1$;
        $improvement \leftarrow$ TRUE;
        $kend \leftarrow k$;
        $k \leftarrow (k+1) \mod p$;
      end if
    end while
    if ($\neg improvement$) **then**
      $k \leftarrow (k+1) \mod p$;
    end if
    if ($k = kend$) **then**
      $local\_optima \leftarrow$ TRUE;
    end if
  end while
  return X

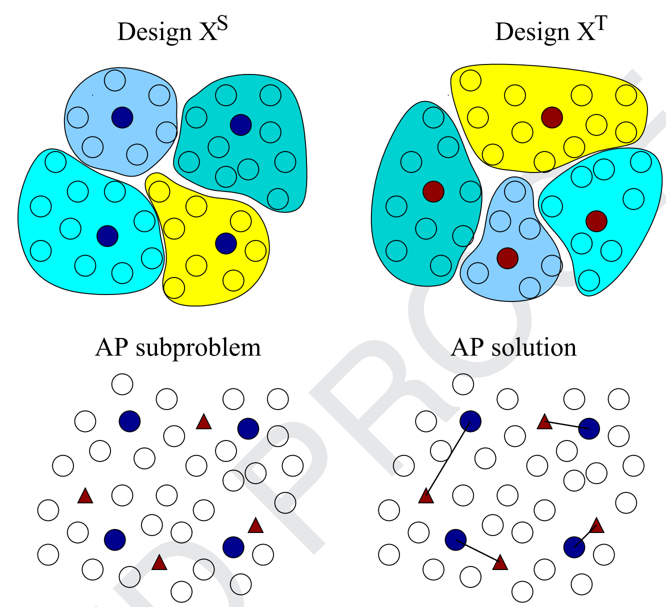

Fig. 4. Illustration of how to set up a search trajectory from two given designs (top) by solving an associated Assignment Problem (bottom).

Despite the fact that PR was originally proposed for Tabu and Scatter search, it has been successfully used with GRASP as well (Resende & Ribeiro, 2010, chap. 4; Resende et al., 2010b, chap. 10). In the context of GRASP, PR can be considered as a way of introducing memory into the search process. To the best of our knowledge PR has not been used in the context of territory design, although it has been recently applied to the related problem of capacitated clustering Deng and Bard (2011). Whereas both problems are related, the proposed formulations differ significantly. For example, Deng and Bard did not consider centers in their PR approach and they proposed a single PR variant (at a cluster-level basis). Deng and Bard report experiments with less than 90 nodes and 5 clusters, while in Section 5 we report instances of up to 500 nodes and 10 territories.

Different PR variants have been proposed so far each having benefits and limitations in terms of efficiency and efficacy. In this work we consider two variants of forward–backward PR, namely static and dynamic, that have proven very effective in related problems (Resende et al., 2010a). For excellent surveys on applications of GRASP with PR we refer the reader to the work of Resende and Ribeiro (2010, chap. 10).

The so called, forward–backward PR strategies explore the paths between X^S and X^T in two different ways (i.e., from X^S to X^T and viceversa) (Resende and Ribeiro, 2010, chap. 10). The main benefit of these strategies is that more and different solutions can be generated, although it has been found that there is little gain over one-way strategies (Ribeiro, Uchoa, & Werneck, 2002). This can be due to the greediness of usual PR methods, which evaluate every possible solution that can be generated by making a move from an initial solution and choose the move that results in the best intermediate solution (Resende et al., 2010a; Ribeiro et al., 2002). Thus, these methods explore a large number of solutions and, therefore, forward–backward PR does not help to improve the quality of final solutions. In this work we select moves in such a way that a single move is evaluated for generating intermediate solutions. This form of PR is more efficient at the expense of scarifying the benefit of greedy strategies. Nevertheless, we believe that in the considered setting the use of a forward–backward PR strategy is advantageous.

Besides the direction of the search, there are other aspects that make PR strategies different (Resende and Ribeiro, 2010; Resende et al., 2010a, chap. 10). For example, greedy-randomized PR methods form a RCL with candidate moves and select a move randomly as in GRASP (Faria, Binato, Resende, & Falcao, 2005). Truncated PR techniques explore partially the trajectory between X^S and X^T. Evolutionary PR consists of evolving a reference set of solutions in a similar way as the reference set is evolved in scatter search (Resende & Werneck, 2004). In this work we developed static and dynamic PR strategies that resulted very effective for the CTDP. Both strategies have been successfully used in other combinatorial optimization problems (Resende et al., 2010a). The rest of this section describes the PR strategies incorporated in GPR_CTDP.

Recall each solution of the CTDP is an assignment of every node $i \in V$ to one of p territories $X_1, \ldots, X_p$. Let $t(X, i) \in \{1, \ldots, p\}$ denote the index of the territory to which node i is assigned according to solution X. Given two particular solutions X^S and X^T, PR aims at generating intermediate solutions or p-partitions in the path starting at X^S and finishing at X^T. In GPR_CTDP intermediate solutions are created by changing $t(X^S, i)$, the territory to which node i is assigned in solution X^S into the corresponding territory $t(X^T, i)$. Because both X^S and X^T solutions are created independently, and the territory ordering may be arbitrary, it is not clear what territory in X^S corresponds to what territory in X^T. Hence, a correspondence between territories must be obtained before starting the search process. The problem of finding the best match between territories can be set as an Assignment Problem (AP) by considering the territory centers only. Let $C(X)$ be the set of p node centers corresponding to solution X. Then a complete bipartite graph is formed with sets $C(X^S)$ and $C(X^T)$, where the cost between node $i \in C(X^S)$ and $j \in C(X^T)$ is given by d_{ij}. The AP can be solved in polynomial time. We use one of the most recent implementations of the Hungarian algorithm (Burkard, Dell'Amico, & Martello, 2009). A solution to the AP represents a minimum cost assignment between territory centers, and therefore a match between territories. Let M be the solution to AP given by $M = \{(i_1, j_1), \ldots, (i_p, j_p)\}$. The idea of the PR is then to "transform" each territory $X_{t(i_k)}$ to territory $X_{t(j_k)}$ for each $(i_k, j_k) \in M$. The rationale for this matching stems from the fact that it is expected that relatively close territories (from different designs) will have many BUs in common. This scheme is illustrated in Fig. 4. One should note that the notion of centers is adopted at this stage for convenience, as centers allow us to establish a correspondence between territories in an efficient way.

Once that correspondence between territories has been established it is possible to perform moves from one solution X^S to another

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X^T. As a consequence, in order to arrive at solution X^T starting from X^S, every node in X^S such that $t(X^S, i)) \neq t(X^T, i)$ must be moved to its associated territory in X^T. We define a PR move, $move_{PR}(X^S, X^T, i)$, as a function that moves or reassigns a node i from territory $t(X^S, i)$ to territory $t(X^T, i)$. The move is valid as long as $t(X^S, i) \neq t(X^T, i)$ and the resulting p-partition remains connected, that is, if and only if $X_{t(X^T,i)} \cup \{i\}$ is connected and $X_{t(X^S,i)} \backslash \{i\}$ remains connected. One should note that moves are always made between boundary nodes as it is not possible to exchange a non-boundary node from one territory to another territory in a single move because loss of connectivity.

Intermediate solutions between X^S and X^T are generated by making moves from X^S to X^T and updating the solution X^S accordingly. Clearly, the order in which nodes i are selected may give rise to different trajectories between X^S and X^T. In this work we chose nodes i in lexicographical order, we also tried a random node selection approach although no difference in performance was obtained. After an intermediate solution is created it is evaluated using formula (8). The generation-evaluating process is repeated for every node with $t(X^S, i) \neq t(X^T, i)$ and the process stops when $t(X^S, i) = t(X^T, i)$ for all $i \in V$. Thus, the PR procedure receives as input a pair of solutions X^S and X^T, generates and evaluates all of the intermediate solutions from X^S to X^T and the best intermediate solution X^R is returned as output. In the following we denote with $PR(X^S, X^T)$ the application of PR starting at solution X^S and finishing at solution X^T.

Procedure 3 grasp_pr_static (i_{max}).

Input: i_{max}: number of global iterations;
Output: X^{best}: A p-partition of V;
for all ($i \in \{1, \dots, b\}$) **do**
  $X \leftarrow$ **grasp_construction**();
  $B_i \leftarrow$ **local_search**(X);
end for
Sort B from best to worst;
for all ($iter = 1, \dots, i_{max}$) **do**
  $X^S \leftarrow$ **grasp_construction**();
  $X^S \leftarrow$ **local_search**(X^S);
  if ($(\psi(X^S) < \psi(B_1))$ or $(\psi(X^S) < \psi(B_b)$ and $d_\mu^{sol}(X^S, B) > \theta)$) **then**
    $E_j \leftarrow$ closest solution to X^S in B with $\psi(X^S) < \psi(B_j)$;
    $E_j \leftarrow X^S$;
    Update B;
  end if
end for
$X^{best} \leftarrow B_1$;
for all ($i \in \{1, \dots, b-1\}$) **do**
  for all ($j \in \{i+1, \dots, b\}$) **do**
    Apply $PR(B_i, B_j)$ and $PR(B_j, B_i)$ and let $X^S \leftarrow$ best solution found;
    $X^S \leftarrow$ **local_search**(X^S);
    if ($\psi(X^S) < \psi(X^{best})$) **then**
      $X^{best} \leftarrow X^S$;
    end if
  end for
end for
return X^{best};

Procedures 3 and 4 present the static and dynamic variants of PR implemented in GPR_CTDP, respectively. Both static and dynamic variants maintain a set of b elite solutions $B = \{B_1, \dots, B_b\}$. B is initialized by running the construction and local search procedures for b times. Solutions in B are always kept sorted in ascending order of their objective function value estimated with Eq. (8).

4.2.1. Static GPR_CTDP

In the static variant, PR is performed at the end of i_{max} iterations of a typical GRASP. In each iteration of the GRASP a solution is constructed and improved with local search, X^S. This solution is compared with the solutions in B. If X^S is better than the best solution in B (i.e., B_1) or if X^S is better than the worst solution in B (i.e., B_b) and is at a distance larger than a given threshold θ from solutions in B, then the most similar solution to X^S in B is replaced by X^S. Solutions in B are then sorted from best to worst. After i_{max} iterations the static PR starts. Every path between solutions in B is evaluated and the best solution is returned. The distance between X^S and solutions in B is estimated as $d_\mu^{sol}(X^S, B) = \frac{1}{b}\sum_{i=1}^{b} g(X^S, B_i)$, where $g(X^S, B_i)$ is the fraction of nodes in X^S and B_i that are assigned to different territories; that is, $d_\mu^{sol}(X^S, B)$ is the average number of nodes assigned to different territories in X^S and B_i. Alternative measures of similarity/distance between territory designs have been described before, see for example the work by Tavares Pereira, Figueira, Mousseau, and Roy (2009). However, such measures do not take advantage of the information we have available when solving the AP. That is, those measures do not know the correspondence between territories beforehand. Besides, distance measures described in Tavares Pereira et al. (2009) are defined in terms of a single attribute and it is not clear how to extend the similarity measure to incorporate information of more than one attribute (e.g., the three activities considered in this work). For that reasons we adopted a simple, yet very informative, measure for computing the distance between territory designs. The pseudocode of the static variant of PR is shown in Procedure 3. $\theta \in [0, 1]$ is a scalar that is set empirically.

4.2.2. Dynamic GPR_CTDP

The dynamic PR variant differs from the static one in that in each iteration of the GRASP the solution X^S is compared to a randomly selected solution from B, say B'. The intermediate solutions between X^S and B' are evaluated, and the best solution found in the path is denoted X^R. Then if X^R is better than B_1 or if X^R is better than B_b and it is at a distance of at most θ from the solutions in B, then the closest solution in B to X^R is replaced with X^R. Then solutions in B are sorted from best to worst. After i_{max} iterations the best solution, namely B_1, is returned. The pseudocode is shown in Procedure 4.

Procedure 4 grasp_pr_dynamic (i_{max}).

Input: i_{max} : number of global iterations;
Output: X^{best}; A p-partition of V;
for all ($i = \{1, \dots, b\}$) **do**
  $X^S \leftarrow$ **grasp_construction**();
  $B_i \leftarrow$ **local_search**(X^S);
end for
Sort B in ascending order;
for all ($iter = 1, \dots, i_{max}$) **do**
  $X^S \leftarrow$ **grasp_construction**();
  $X^S \leftarrow$ **local_search**(X^S);
  Randomly select B' from B;
  Apply $PR(X^S, B')$ and $PR(B', X^S)$ and let $X^R \leftarrow$ best solution found;
  if ($(\psi(X^R) < \psi(B_1))$ or $(\psi(X^R) < \psi(B_b)$ and $d_\mu^{sol}(X^R, B) > \theta)$) **then**
    $B_j \leftarrow$ closest solution to X^R in B with $\psi(X^R) < \psi(B_j)$;
    $B_j \leftarrow X^R$;
    Update B;
  end if
end for
return $X^{best} \leftarrow B_1$;

A number of parameters are associated with GPR_CTDP in both variants, namely δ the fraction of nodes assigned with a distance criterion, k the number of neighbors that are considered for building a territory, λ the tradeoff parameter of the objective function, α the GRASP quality parameter for the RCL, *limit_evals* the maximum number of evaluations for the local search, b the number of solutions in the elite set B and θ the distance threshold in PR. In this work we have fixed all of these parameters based on preliminary experimentation. The next section reports experimental results with the proposed GPR_CTDP.

5. Computational experiments

This section reports experimental results obtained with GPR_CTDP. The proposed method was implemented in Matlab^R. The code and data sets are publicly available for research purposes

Table 1
Summary of values used for the algorithmic parameters of GPR_CTDP.

Parameter	Value	Description
δ	0.5	Fraction of nodes assigned with a distance criterion
k	3	Number of neighbors that are considered for growing a territory
λ	0.7	Weight parameter in the meritfunction
α	0.3	RCL quality parameter
limit_evals	1000	The maximum number of fitness function evaluations in the local search
b	20	The number of solutions in the elite set E
θ	0.6	The distance threshold in PR
i_{max}	500	Number of global iterations for GPR_CTDP

from the authors upon request. All of the experiments were run in a 64-bit workstation with a Corei7 processor at 3.4 GHz and 8 GB in RAM.

5.1. Experimental setting

For the experiments we used the data base from Ríos-Mercado and Fernández (2009). These are randomly generated instances based on real-world data. Data sets DS and DT are considered for experimentation. The former generate the BU weights from a uniform distribution and the latter uses a triangular distribution. Data set DT more closely resembles real-world instances. These data sets are fully described in Ríos-Mercado and Fernández (2009). For each of DS and DT data sets there are 20 different instances of size $n = 500$ and $p = 10$.

For all of the instances in both DS and DT data sets we use a tolerance level $\tau^a = 0.05,\ a \in A$. Recall that τ^a measures the allowable relative deviation from the target average size μ^a for activity a. Hence, a value of $\tau^a = 0.05$ implies that instances are tightly constrained in all activities and therefore the problem is more difficult to solve than instances that use a larger value of τ^a. In previous work (Ríos-Mercado & Fernández, 2009), experiments have been reported with other values for $\tau^a \in [0.05, 0.30]$. Here we focus on the most difficult instances.

Throughout the evaluation, the GRASP is run with $i_{max} = 500$. Based on preliminary experimentation for fine-tuning the algorithmic parameters for GPR_CTDP, we will use the values reported in Table 1. Showing the fine-tuning of these parameters is out of the scope of this paper.

In the following sections we report the obtained experimental results. We have divided experimental results in three sections that aim at assessing different aspects of the GPR_CTDP.

5.2. Assessing the construction and local search procedures within a GRASP framework

This section describes results of experiments designed to evaluate the effect of the proposed construction and local search procedures. To this end we apply the new construction phase within a GRASP framework, that is, no PR phase is applied in this experiment. First, we apply the GRASP with construction phase only and then we apply the complete GRASP with both construction and local search phases. For each of these, we tested the two different data sets. Figs. 5 and 6 show the performance of the construction and local search procedures for DT and DS data sets, respectively. In each figure we plot the values of the objective, $F(S)$, and infeasibility, $G(S)$, for each instance and for each mechanism. As expected, from these figures we can see that local search improves significantly the construction procedure, in terms of both infeasibility and dispersion. For both data sets, local search (triangle marker) obtains feasible solutions (i.e., $G(S) = 0$) for most of the instances starting from the highly infeasible solutions generated by the construction mechanism (diamond marker). Besides, there are considerable improvements in terms of $F(S)$ for all of the instances in the DT data set, see Fig. 5. Lower improvements

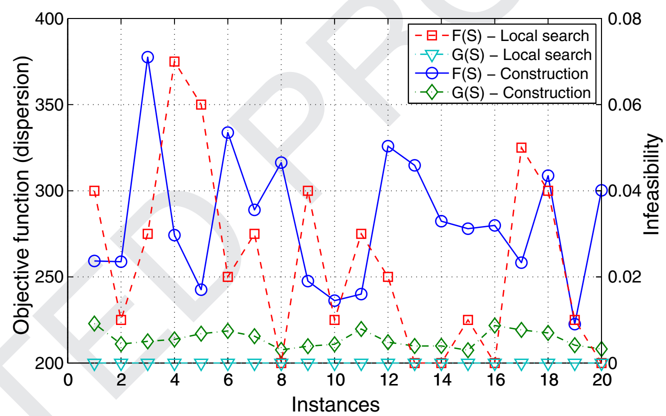

Fig. 5. Performance of the construction and local search mechanisms for instances in the DT data set. We show the values of $F(S)$ (left y-axis) and $G(S)$ (right y-axis).

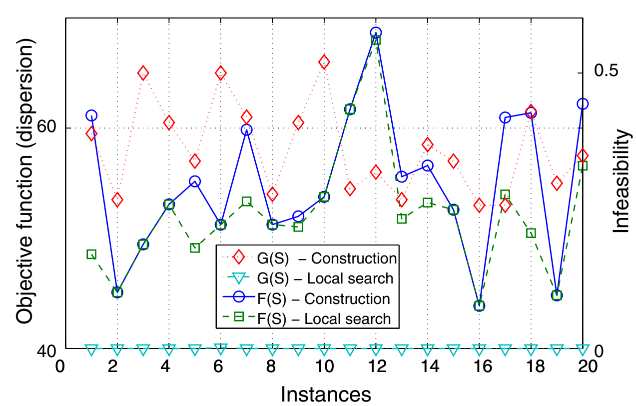

Fig. 6. Performance of the construction and local search mechanisms for instances in the DS data set. We show the values of $F(S)$ (left y-axis) and $G(S)$ (right y-axis).

in terms of dispersion are observed for the DS data set, see Fig. 6, although local search always obtained returned better solutions. It is expected that the PR mechanisms further improve the dispersion of solutions obtained with my local search.

Fig. 7 shows the search profile for a particular instance of the DT data set, that is, we show how the quality of solutions, as measured by the weighted merit function $\psi(X)$, improves as a function of time. We plot the average value, across 500 iterations of GRASP, of Expression (7) during the local search process for a specific instance in DT. As before, It can be seen that the local search procedure improves considerably the quality of solutions generated with the construction procedure, where the most important improvements are obtained at the first stages of the local search. Please note that we stop local search when either no further improvement is possible or the maximum number of evaluations is performed (in our case 1000 evaluations); in practice, the maximum number of evaluations was barely used as stopping criterion for the local search procedure.

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Table 2
Evaluation of the construction and local search procedures of GPR_CTDP.

Data set		DT		DS	
Measure/mechanism		Construction	Local search	Construction	Local search
RDB	Best	5.81%	0.00%	0.00%	0.00%
	Average	34.12%	1.51%	20.91%	14.51%
	Worst	81.45%	6.04%	58.28	56.76%
$G(S)$	Best	$0.00E+00$	$0.00E+00$	$2.60E-01$	$0.00E+00$
	Average	$2.37E-02$	$0.00E+00$	$3.61E-01$	$3.01E-04$
	Worst	$7.06E-02$	$0.00E+00$	$5.24E-01$	$3.55E-03$

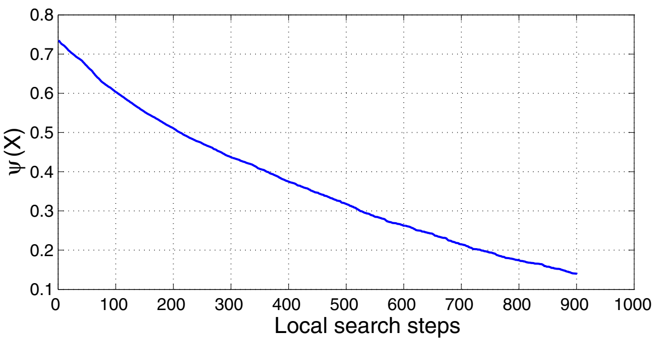

Fig. 7. Quality of solutions as a function of local search steps (time) for a selected instance.

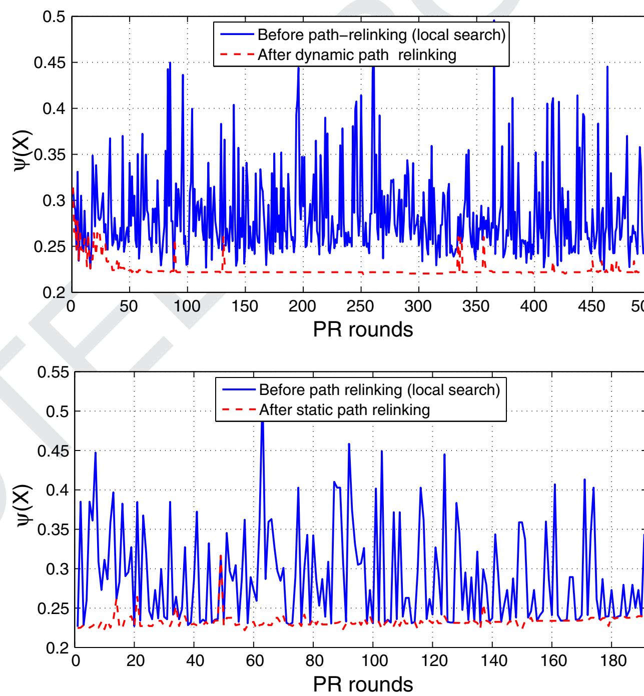

Fig. 8. Quality of solutions before and after applying PR for a selected instance. Top: dynamic PR. Bottom: static PR.

Table 2 summarizes the performance of the construction and local search procedures across all instances of both DT and DS data sets. For the dispersion term $F(S)$, we show the relative deviation between the solution obtained with each procedure and the best known solution for each instance RDB $= (F(S) - F(S_{best})/F(S_{best})$. The column labeled "local search" indicates that both construction and local search phases are applied. From this table we can see that the average of the sum of relative infeasibilities is maintained low in the construction procedure for both data sets. This result shows that the proposed procedure is able to obtain acceptable solutions in terms of the degree of satisfaction of the balance constraints despite the fact part of the construction procedure is based on a purely distance-based criterion.

After applying local search to the constructed solutions, the dispersion measure $F(S)$ is improved as it shows a reduction in the relative deviation with respect to the best dispersion value. In the case of the DT data set the solutions obtained with local search are very close to the best ones in terms of dispersion (average deviation of 1.51%), while for DU there is much more room for improvement (average deviation of 14.51%). For the DT data set the objective function is improved in average by 32.61%, while for the DS data set the improvement is of 6.4%. These are rather important differences that evidence the effectiveness of the proposed local search mechanism. It is very important to emphasize that dispersion is improved by also considerably reducing $G(S)$.

5.3. GRASP vs. GPR_CTDP

This section reports experimental results on the improvements of the PR strategies over the straight GRASP implementation described in Section 4.1. Table 3 shows the performance of GPR_CTDP under both static (GPR-ST column) and dynamic (GPR-DY column) PR strategies for DT and DS data sets. In the table, we compare the performance of GPR_CTDP when using PR and when only GRASP without PR is adopted. We show the relative deviation between the best solution obtained with each method and the best known solution for each instance.

As we can see, for the DT data set, the improvements obtained with PR over local search are small yet non-negligible. We believe this result can be due to the fact that we are approaching to the global optimum for this data set and since the local search procedure provides very competitive solutions by itself the improvements due to PR are rather small. However, it is important to emphasize that all of the solutions found with GRASP and GPR_CTDP are feasible for this data set. For this data set the static PR strategy outperformed the dynamic one by less than 1% in terms of the objective function. For the DS data set the improvements due to PR are larger. GPR_CTDP with static PR outperforms the results of local search by an average of ≈ 13% in terms of the objective function, whereas the dynamic strategy outperforms local search by less than 1%. The static variant of PR achieve important improvements in terms of the dispersion objective ($F(S)$), while also reducing the infeasibility term.

Fig. 8 shows the difference in performance gained by applying PR after local search for a specific instance from the DT data set (for which smaller differences were obtained). Each point in the x-axis corresponds to one round of PR. For dynamic PR there are 500 rounds of PR because it is applied every iteration, while for static PR there are fewer rounds because it is applied only for the elite set. It can be seen from these plots that even though differences reported in Table 3 are small, PR improves the local search solution in every case.

Finally, it is important to point out that even in the case when GRASP is allowed to run by itself for an amount of time equal to the total amount of time employed by GPR_CTDP, the results reported by

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Table 3
Evaluation of GPR_CTDP with static and dynamic PR.

Data set		DT			DS		
Measure		GRASP	GPR-ST	GPR-DY	GRASP	GPR-ST	GPR-DY
RDB	Best	0%	0%	0%	0%	0%	0%
	Average	1.51%	0.51%	1.27%	14.51%	0.76%	13.92%
	Worst	6.04%	3.09%	3.91%	56.76%	11.44%	56.76%
$G(S)$	Best	$0.00E+00$	$0.00E+00$	$0.00E+00$	$0.00E+00$	$0.00E+00$	$0.00E+00$
	Average	$0.00E+00$	$0.00E+00$	$0.00E+00$	$3.01E-04$	$2.53E-04$	$2.84E-04$
	Worst	$0.00E+00$	$0.00E+00$	$0.00E+00$	$3.55E-03$	$5.07E-04$	$3.55E-03$

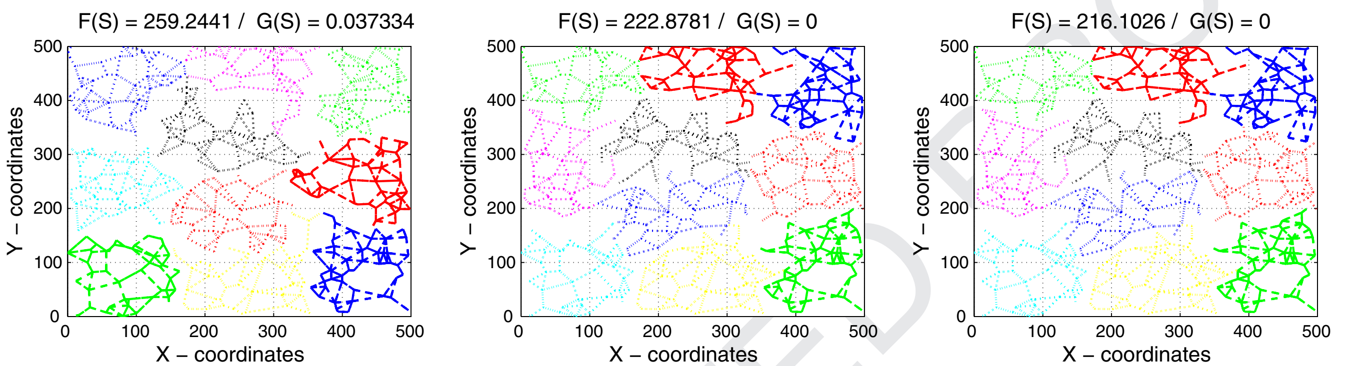

Fig. 9. Solutions obtained by the construction, local search and static PR procedures for a particular instance of the DT data set.

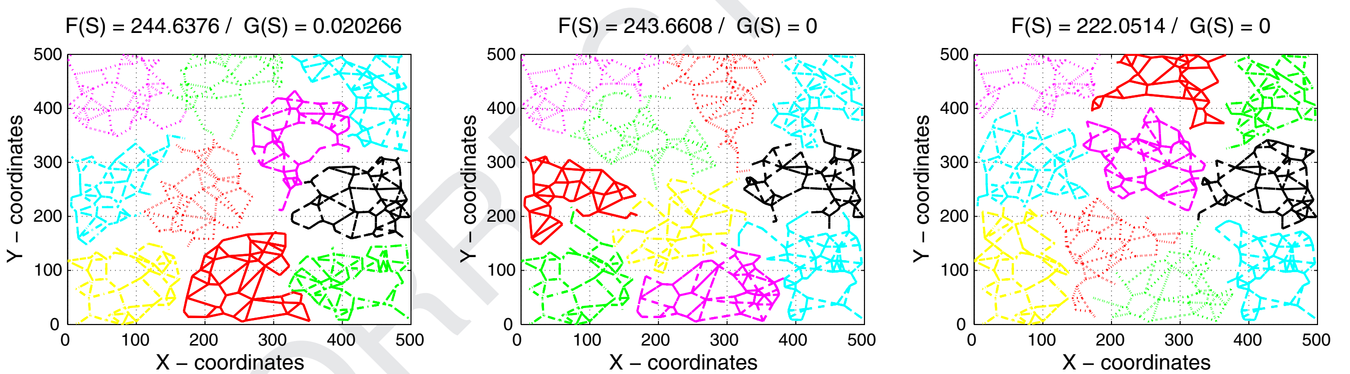

Fig. 10. Solutions obtained by the construction, local search and dynamic PR procedures for a particular instance of the DT data set.

the later are still better. This is due to the fact that the GRASP seems to converge within the first iterations, thus a better solution is hardly found by GRASP afterwards.

Figs. 9 and 10 show the territories obtained with the construction, local search, and PR GPR_CTDP procedures for a particular instance of the DT data set. Fig. 9 shows the solution from a run of the static PR GPR_CTDP and Fig. 9 shows the corresponding solution for the dynamic PR GPR_CTDP. These figures illustrate the advantages of GPR_CTDP over the construction and local search mechanisms. Territories generated after the construction procedure present infeasibilities. The local search process eliminates infeasibilities and reduces the dispersion objective. However, the dispersion is further minimized with both PR variants. Visually, it can be seen that territories generated with local search (center plots) are more disperse than those generated with GPR_CTDP (rightmost plots). For this particular instance, a better solution was obtained with the static version of PR, which agrees with results presented in this section.

5.4. Static vs. dynamic path relinking

This section elaborates on the difference in performance between the static and dynamic PR variants of GPR_CTDP. From Table 3 we can see that the improvements of static and dynamic GPR_CTDP over local search are of 1% and 0.24% for the DT data set and of 13.75% and 0.59% for the DS data set (in terms of the objective function). Thus, despite the fact both strategies resulted effective, the use of the static one is advantageous. We think this can be due to the fact that static GPR_CTDP explores all of the paths between elite solutions at the end of the search process. Hence a global picture of the search process is considered during the execution of static GPR_CTDP. Dynamic GPR_CTDP on the other hand, explores the paths between every solution processed by local search and a random solution from the elite set. Since it is not guaranteed that PR is performed over two competitive solutions, it is less likely that an effective solution can be found after exploring the paths.

Fig. 11 shows the relative deviation of the solutions found with each tested method and the best known solution for each instance for DT data set. This figure give us more insight into the performance of the different methods across the instances, it is rather clear that the static PR strategy obtained the best solutions for most of the instances (those instances for which the relative deviation is zero), followed by the dynamic PR approach.

Table 4 reports the processing time for each variant of GPR_CTDP and for each data set. In general terms a new territory design can

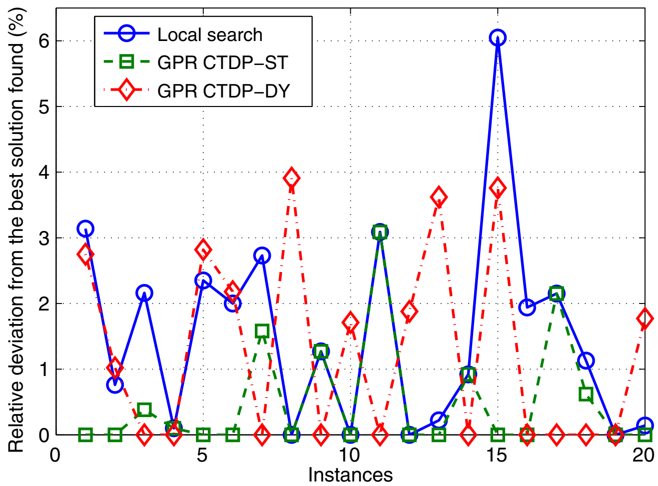

Fig. 11. A comparison among the methods in terms of relative deviation from best objective function value for the DT data set on each individual instance.

Table 4
CPU time (min) comparison for static and dynamic GPR_CTDP.

	DT		DS	
	GPR-ST	GPR-DY	GPR-ST	GPR-DY
Best	124.84	119.93	179.74	179.37
Average	136.25	133.70	204.42	200.24
Worst	152.96	150.34	240.30	227.46

be obtained with either variant of GPR_CTDP in a few hours. For instance, in average, it took about 2.2 h for DT and 3.4 h for DU to obtain a solution; whereas, decisions involving territory designs (e.g., re-assignments or modifications) are taken at periods spaced by no less than a month (3–4 months in average). Therefore, the proposed solution and implementation satisfies loosely the demands of industry.

One final comment, it was observed that, in the GPR_CTDP method, around 80% of the time is spend in the GRASP and 20% doing the path relinking. Therefore, we have empirically observed that this additional amount of effort pays off significantly.

6. Conclusions

We have introduced a new model in commercial territory design. The new model makes use of a diameter-based dispersion function instead of the traditional center-based functions.

We have described a GRASP with path relinking (GPR_CTDP) for this CTDP. The problem, motivated by a real-world application, consists of grouping commercial units into geographic territories subject to dispersion, connectivity and balance constraints. A novel construction procedure was developed and two variants of PR were explored in GPR_CTDP, namely, static and dynamic PR. The components of GPR_CTDP were evaluated and compared extensively in instances that are known to be very challenging from previous work.

Experimental results show that the proposed construction procedure is able to construct very competitive solutions, mainly in terms of the dispersion criterion. The local search of the GPR_CTDP improves solutions in terms of both dispersion and balance requirements. Both versions of PR improve the performance of the application of the construction and local search mechanisms, confirming previous work on the combination of GRASP and PR. In particular we found that with the static PR variant better solutions can be obtained for the TDP. This can be due to the fact that the PR process is applied over elite instances, which increases the chances of finding a better solution. In general terms the processing time of both PR variants lies in reasonable ranges for the application.

We have identified several future work directions in the context of GPR_CTDP. In particular we would like to explore other variants of PR that are known to be very effective, for example, evolutionary PR. Further, we are interested in the development of an adaptive filtering step that allows us to identify pairs of solutions that can be potentially improved by applying PR. This is in addition to the rules used for updating the set of elite solutions. We think that such a filtering strategy will have a very positive impact in the efficiency of GPR_CTDP. Since we found evidence that maintaining a set of elite solutions can be beneficial for TDP, we would like to explore the use of other "population-based" metaheuristics such as scatter search. Another promising and direct future work direction is that of improving the processing time of our implementation. Specifically, we would like to explore distributed and parallel implementations of GPR_CTDP.

It is important to note that the method developed in this work can also be extended and applied to other districting problems under balancing and connectivity constraints. The presence of the connectivity constraints make the path relinking process more challenging. For instance, path relinking has been applied in a different manner in related partitioning problems such as capacitated clustering (Deng & Bard, 2011). In this particular work, we have successfully exploited the problem structure by solving an associated Assignment Problem whose solution will guide the relinking process in a more intelligent fashion. To the best of our knowledge this PR idea is novel and worthwhile for further exploration in other districting or clustering problems under connectivity constraints.

An idea worthwhile exploring could be the definition of different neighborhood topologies. In the present work, and most of the literature, the typical move of reassigning a BU to a different territory has been considered. However, a swap neighborhood where two BUs from different territories are swapped defines an entire different topology. Furthermore, advanced search procedures such as iterated greedy local search (IGLS) could also deliver solutions of better quality. IGLS is a local search mechanism that iteratively destroys and reconstructs solutions generating an entirely different search path. For instance, in our particular problem, it is clear that the territory having the worst objective function value (diameter) is a bottleneck in the sense that the overall objective function cannot be improved unless the diameter of this territory is improved. Therefore by carefully selecting this territory and adequate neighbor territories, one can unassign all basic units associated to these territories and make a better reconstruction that would give a lower diameter value.

Another idea is as follows. During the search process, our heuristic keeps feasibility once it is attained. However, there are also some state-of-the-art metaheuristic components such as strategic oscillation that may be worthy of future investigation. Strategic oscillation is a concept introduced by Glover and Jao (2011) that allows the search process to examine infeasible solutions hoping that would yield solutions of better quality once feasibility is recovered. This concept has proven very successful in other combinatorial optimization problems.

A natural area for future work is the development of lower bounding schemes. While it is clear that the inherent problem complexity make it very difficult to find optimal solutions, one can attempt to find at least lower bounds on the objective function value that would allow to give an estimate on the quality of the solutions delivered by the heuristics. Finding good lower bounds is a very challenging area of research.

Finally, a natural extension of this work is to consider stochastic optimization models. That is, in most of these models in the districting literature, deterministic models are often tackled. However, if one assumes that some parameters such as product demand, for instance, may be a random variable, the resulting model is an integer stochastic optimization problem which is of course harder to solve. However,

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the ideas developed in this work can be useful to generate insight and help in the development of efficient solution algorithms to such hard models.

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