A Brief Introduction to Percolation Theory

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Abstract

Consider a cube of water-permeable material. What is the probability that that if water is poured on top of the cube it may drain all the way through the cube and out the opposite face? Initially developed by Paul Flory and Walter Stockmayer in 1944, percolation theory attempts to answer such questions by rephrasing them in terms of vertices (sites) and edges (bonds) of graphs and examining the connectedness of such graphs. The connectedness of these graphs—in the infinite case—is determined by a threshold probability, p_c, describing whether the water may pass through each site or bond. This essay will introduce the ideas of site and bond percolation as well as the notion of clusters and critical (threshold) probabilities. We will also analyse the one dimensional case to garner a basic understanding before exploring higher dimensional cases. After discussing the concepts of percolation theory, we will move on and look at the many applications of the theory discussed in the earlier parts of the essay.

Contents

1 Introduction

1.1 The canonical example

Let us consider the example from the abstract of water filtering through a porus medium, but this time in two dimensions. How do we model this? One might imagine that the medium consists of many particles arranged (for simplicity) in a $n \times n$ square lattice and linked to each of their nearest neighbours. Clearly, this is the lattice on $\mathbb{Z}^2$. To setup the problem, each of the particles will be expressed as a vertex in a graph and each of the links will be an edge. In the context of percolation, a vertex is called a site and an edge is called a bond; these edges and bonds form a network. (We're sure the reader can visualise $\mathbb{Z}^2$, but just in case they can't, they may refer to Figure 1 beneath)

Figure 1: The lattice on $\mathbb{Z}^2$

So what does percolation actually mean? We shall consider two different types of percolation: bond and site percolation.

Definition 1.1. We say that we are considering **site percolation** if we let all of the sites in the network be open with probability $p \in [0, 1]$ (meaning that they allow the liquid through) and closed with probability $1 - p \in [0, 1]$ (meaning that they don't allow liquid through).

Definition 1.2. We say that we are considering **bond percolation** if we let all of the bonds in the network be open with probability $p \in [0, 1]$ (meaning that they allow the liquid through) and closed with probability $1 - p \in [0, 1]$ (meaning that they don't allow liquid through).

Now that we've defined site and bond percolation, what's the problem that we're trying to solve? In the case of water being poured on a porus medium, we would like to know whether there is a path from the one side of the network to the other.

2 The one dimensional case

3 Higher dimensional cases

4 Applications