

# A Brief Introduction to Percolation Theory

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## Abstract

Consider a cube of water-permeable material. What is the probability that if water is poured on top of the cube it may drain all the way through the cube and out the opposite face? Initially developed by Paul Flory and Walter Stockmayer in 1944, percolation theory attempts to answer such questions by rephrasing them in terms of vertices (sites) and edges (bonds) of graphs and examining the connectedness of such graphs. The connectedness of these graphs—in the infinite case—is determined by a threshold probability,  $p_c$ , describing whether the water may pass through each site or bond. This essay will introduce the ideas of site and bond percolation as well as the notion of clusters and critical (threshold) probabilities. We will also analyse the one dimensional case to garner a basic understanding before exploring higher dimensional cases. After discussing the concepts of percolation theory, we will move on and look at the many applications of the theory discussed in the earlier parts of the essay.

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# 1 Introduction

## 1.1 The canonical example

Let us consider the example from the abstract of water filtering through a porous medium, but this time in two dimensions. How do we model this? One might imagine that the medium consists of many particles arranged (for simplicity) in an  $n \times n$  square lattice and linked to each of their nearest neighbours. Clearly, this is the lattice on  $\mathbb{Z}^2$ . To set up the problem, each of the particles will be expressed as a vertex in a graph and each of the links will be an edge. In the context of percolation, a vertex is called a **site** and an edge is called a **bond**; these sites and bonds form a network.

**Definition 1.1.** A vertex in a graph is referred to as a **site**.

**Definition 1.2.** A edge in a graph is referred to as a **bond**.

**Definition 1.3.** A graph is referred to as a **network**.

So what does percolation actually mean? First we shall introduce the notion of open and closed sites and bonds and then we can discuss percolation.

**Definition 1.4.** A site or bond in a network is labeled **open** if it allows whatever we're considering to pass through.

**Definition 1.5.** A site or bond in a network is labeled **closed** if it doesn't allow whatever we're considering to pass through.

And now for the percolation definitions.

**Definition 1.6.** We say that we are considering **site percolation** if we let all of the sites in the network be open with probability  $p \in [0, 1]$  and closed with probability  $1 - p \in [0, 1]$ .

**Definition 1.7.** We say that we are considering **bond percolation** if we let all of the bonds in the network be open with probability  $p \in [0, 1]$  and closed with probability  $1 - p \in [0, 1]$ .

Now that we've defined site and bond percolation, what's the problem that we're trying to solve? In the case of water being poured on a porous medium, we would like to know whether there is an open path from the top of the network to the bottom. We shall model this using bond percolation.

**Definition 1.8.** We say that a path in a network is **open** if:

- when considering site percolation, every site in the path is open.
- when considering bond percolation, every bond in the path is open.

**Definition 1.9.** Let  $N = (V, E)$  be a network and let  $A, B \in V$ . The sites  $A, B$  are **openly connected** if there exists an open path connecting  $A$  and  $B$ .

**Definition 1.10.** Let  $N = (V, E)$  be a network and let  $A, B \in V$ . The sites  $A, B$  are **openly disconnected** if there does not exist an open path connecting  $A$  and  $B$ .<sup>1</sup>

The probability that an open path from the top of the network to the bottom exists depends on both our choices of both  $p$  and  $n$  from before. As a result of our context, our value for  $n$  should be large—this is the case with most percolation models—but we shall use small  $n$  for the sake of example and simplicity. Let us now fix  $n$  and see what happens as we vary  $p$ . Obviously we have two trivial cases,  $p = 0$  and  $p = 1$ , where the network is completely openly disconnected and completely openly connected respectively. What about when  $p \in (0, 1)$ ? Let's inspect three different values of  $p$  on our network:  $p = 0.25$ ,  $p = 0.5$  and  $p = 0.75$  as shown in figures 1a, 1b and 1c on page 5. As one might expect, as  $p$  increases, so does the "connectedness" of the network; i.e. the probability of having an open path from the top of the network to the bottom increases with  $p$ . Also, observe that as  $p$  increases we also get larger "clusters" of open connected sites or bonds.

**Definition 1.11.** Let  $N = (V, E)$  be a network. A **cluster** is a set of vertices,  $C \subset V$ , such that if  $v_1, v_2 \in C$  then  $v_1$  and  $v_2$  are openly connected.

**Definition 1.12.** Let  $N = (V, E)$  be a network and let  $C \subset V$  be a cluster. The **size** of  $C$ , denoted by  $|C|$ , is the number of sites in the cluster.

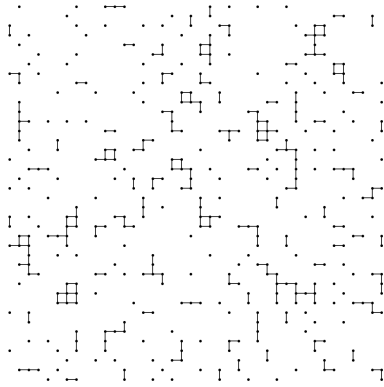
We will explore clusters in more detail in the next section, but for now let us introduce the idea of a critical probability. Although they don't exist in the finite cases, they do exist in the infinite cases.

**Definition 1.13.** Consider a network,  $N = (V, E)$ . A cluster,  $C \subseteq V$ , is considered **infinite** iff  $|C| = \infty$ .

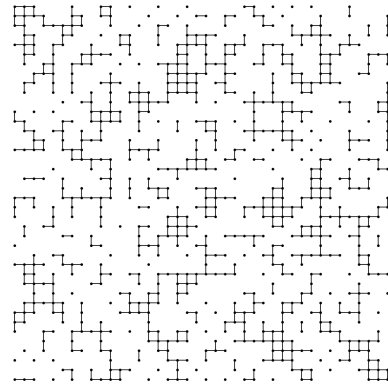
**Definition 1.14.** Consider an infinite network,  $N$ . The critical probability, denoted  $p_c$ , is the probability such that an infinite cluster exists. The value of  $p_c$  may not be equal for site and bond percolation.

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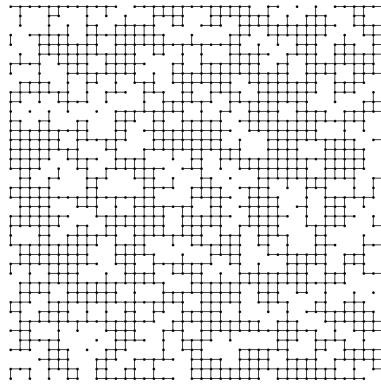
<sup>1</sup>Definitions 1.8, 1.9 and 1.10 are equivalent for both site and bond percolation.



(a)  $p = 0.25$



(b)  $p = 0.5$



(c)  $p = 0.75$

Figure 1: Examples of *bond* percolation for  $p \in (0, 1)$  on a  $40 \times 40$  network.

## 1.2 Other network configurations

We have already seen the lattice on  $\mathbb{Z}^2$  as an example, but there are many more. To remain within the scope of this essay, we shall only briefly mention some two and three dimensional examples and print their site and bond critical probabilities and a diagram.

[1, p. 11]

### Two dimensional network configurations

Clearly, one two dimensional network configuration is the lattice on  $\mathbb{Z}^2$ . Other regular two dimensional network configurations include, but are not limited to, the Bethe Lattice (Figure 2a), Honeycomb Lattice (Figure 2b), Kagome Lattice (Figure 2c) and the Triangular Lattice (Figure 2d). As one might imagine, each of these configurations has a different (but not necessarily distinct) critical probability. Below is a table showing the critical probabilities for each of the aforementioned network configurations. It should be noted that probabilities marked with a \* (star) are exact results. [1, p. 11]

Configuration	$p_c$ for site percolation	$p_c$ for bond percolation
$\mathbb{Z}^2$	$1/2$ *	0.5927
Honeycomb	$1 - 2\sin(\pi/18)^*$	0.6962
Bethe with $z = 3$	$1/2?$ *	$1/2?$ *
Kagome	0.522	0.652
Triangular	$2\sin(\pi/18)^*$	$1/2^*$

## 2 The one dimensional case

## 3 Higher dimensional cases

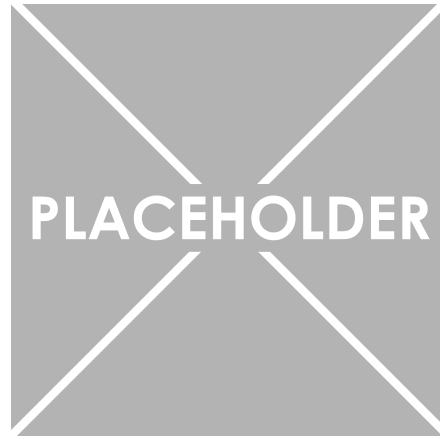
## 4 Applications

## References

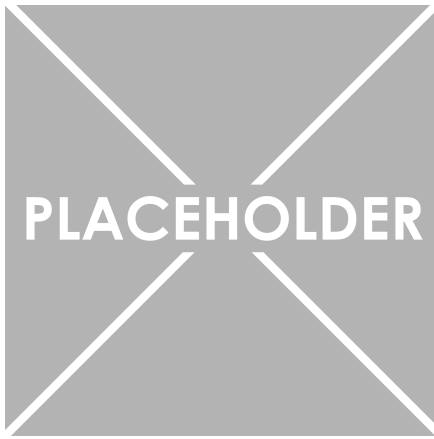
- [1] M. Sahimi. *Applications of Percolation Theory*. Taylor & Francis, University of Southern California, 1994.



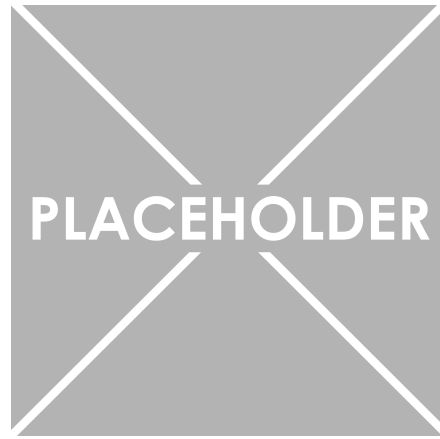
(a) Bethe Lattice



(b) Honeycomb Lattice



(c) Kagome Lattice



(d) Triangular Lattice

Figure 2: Two dimensional network configurations