A Brief Introduction to Percolation Theory

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Abstract

Consider a cube of water-permeable material. What is the probability that that if water is poured on top of the cube it may drain all the way through the cube and out the opposite face? Initially developed by Paul Flory and Walter Stockmayer in 1944, percolation theory attempts to answer such questions by rephrasing them in terms of vertices (sites) and edges (bonds) of graphs and examining the connectedness of such graphs. The connectedness of these graphs—in the infinite case—is determined by a threshold probability, p_c, describing whether the water may pass through each site or bond. This essay will introduce the ideas of site and bond percolation as well as the notion of clusters and critical (threshold) probabilities. We will also analyse the one dimensional case to garner a basic understanding before exploring higher dimensional cases. After discussing the concepts of percolation theory, we will move on and look at the many applications of the theory discussed in the earlier parts of the essay.

Contents

1 Introduction

1.1 The canonical example

Let us consider the example from the abstract of water filtering through a porus medium, but this time in two dimensions. How do we model this? One might imagine that the medium consists of many particles arranged (for simplicity) in an $n \times n$ square lattice and linked to each of their nearest neighbours. Clearly, this is the lattice on $\mathbb{Z}^2$. To setup the problem, each of the particles will be expressed as a vertex in a graph and each of the links will be an edge. In the context of percolation, a vertex is called a site and an edge is called a bond; these sites and bonds form a network. (We're sure the reader can visualise $\mathbb{Z}^2$, but just in case they can't, they may refer to figure 1 beneath)

Figure 1: The lattice on $\mathbb{Z}^2$

So what does percolation actually mean? First we shall introduce the notion of open and closed sites and bonds and then we can discuss percolation.

Definition 1.1. A site or bond in a network is labeled **open** if it allows whatever we're considering to pass through.

Definition 1.2. A site or bond in a network is labeled **closed** if it doesn't allow whatever we're considering to pass through.

And now for the percolation definitions.

Definition 1.3. We say that we are considering **site percolation** if we let all of the sites in the network be open with probability $p \in [0, 1]$ and closed with probability $1 - p \in [0, 1]$.

Definition 1.4. We say that we are considering **bond percolation** if we let all of the bonds in the network be open with probability $p \in [0, 1]$ and closed with probability $1 - p \in [0, 1]$.

Now that we've defined site and bond percolation, what's the problem that we're trying to solve? In the case of water being poured on a porus medium, we would like to know whether there is an open path from the top of the network to the bottom.

Definition 1.5. We say that a path in a network is **open** if:

- when considering site perolation, every site in the path is open.
- when considering bond percolation, every bond in the path is open.

Definition 1.6. Let $N = (V, E)$ be a network and let $A, B \in V$. The sites A, B are **openly connected** if $\exists$ an open path connecting A and B.

Definition 1.7. Let $N = (V, E)$ be a network and let $A, B \in V$. The sites A, B are **openly disconnected** if $\nexists$ an open path connecting A and B.

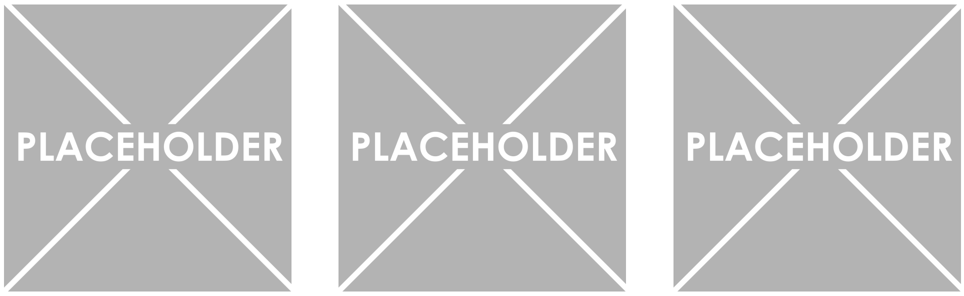
The probability that an open path from the top of the network to the bottom exists depends on both our choices of both p and n. As a result of our context, our value for n should be large—this is the case with most percolation models—but we shall use small n for the sake of example and simplicity. Let us now fix n and see what happens as we vary p. Obviously we have two trivial cases, $p = 0$ and $p = 1$, where the network is completely openly disconnected and completely openly connected respectively (see figures 2a and 2b below).

(a) $p = 0$

(b) $p = 1$

Figure 2: Examples of percolation for $p \in \{0, 1\}$.

What about when $p \in (0, 1)$? Let's inspect three different values of p on our network: $p = 0.25$, $p = 0.5$ and $p = 0.75$ as shown in figures 3a, 3b and 3c below.



(a) $p = 0.25$ (b) $p = 0.5$ (c) $p = 0.75$

Figure 3: Examples of percolation for $p \in (0, 1)$.

As one might expect, as p increases, so does the ”connectedness” of the network; i.e. the probability of having an open path from the top of the network to the bottom increases with p. Also observe that, as p increases we also get larger ”clusters” of open connected sites or bonds.

2 The one dimensional case

3 Higher dimensional cases

4 Applications