1 Euler-Lagrange Equations for UWT Fluid Dynamics Lagrangian

The Lagrangian for UWT's scalar-driven fluid dynamics is:

$$L = \frac{1}{2}\rho(u_r^2 + u_\theta^2 + u_z^2) + \frac{1}{2}(\partial_t\Phi_1)^2 + \frac{1}{2}(\partial_t\Phi_2)^2 \\ - \Big[\lambda(|\Phi_1\Phi_2| - v^2)^2 + k_U(2\Phi_1^2 + \Phi_1\Phi_2 + 2\Phi_2^2) + g_m|\Phi_1\Phi_2|\rho \\ + g_{\text{wave}}\varepsilon|\Phi_1\Phi_2|^2 R + k_{\text{damp}}(\Phi_1^2 + \Phi_2^2) + \nu|\nabla\mathbf{u}|^2 + \lambda_R(|\Phi_1 - \Phi_{1,\text{prev}}|^2 + |\Phi_2 - \Phi_{2,\text{prev}}|^2)\Big] \quad (1)$$

Parameters: $\rho = 1000\,\text{kg/m}^3$, $\lambda = 1$, $v = 0.226/6.242 \times 10^{18}\,\text{kg}$, $k_U = 2 \times 10^8\,\text{kg}^{-1}\text{m}^3\text{s}^{-2}$, $g_m = 0.01$, $g_{\text{wave}} = 19.5$, $\varepsilon = 10^{-30}\,\text{m}^2$, $k_{\text{damp}} = 0.001\,\text{s}^{-1}$, $\nu = 10^{-5}\,\text{m}^2/\text{s}$, $\lambda_R = 10^{-6}$, $R \approx 1/r^2\,\text{m}^{-2}$.

1.1 For Scalar Field Φ_1

$$\frac{\partial^2\Phi_1}{\partial t^2} = \lambda\left[2|\Phi_1\Phi_2|\left(\Phi_2\partial_t\Phi_1 + \Phi_1\partial_t\Phi_2\right) - 4v^2\Phi_1\Phi_2\right] + k_U(4\Phi_1 + \Phi_2) \\ + g_m\Phi_2\rho + g_{\text{wave}}\varepsilon 2|\Phi_1\Phi_2|\Phi_2 R - k_{\text{damp}}\Phi_1 + \nu\nabla^2\Phi_1 + 2\lambda_R(\Phi_1 - \Phi_{1,\text{prev}}) \quad (2)$$

1.2 For Scalar Field Φ_2

$$\frac{\partial^2\Phi_2}{\partial t^2} = \lambda\left[2|\Phi_1\Phi_2|\left(\Phi_2\partial_t\Phi_1 + \Phi_1\partial_t\Phi_2\right) - 4v^2\Phi_1\Phi_2\right] + k_U(\Phi_1 + 4\Phi_2) \\ + g_m\Phi_1\rho + g_{\text{wave}}\varepsilon 2|\Phi_1\Phi_2|\Phi_1 R - k_{\text{damp}}\Phi_2 + \nu\nabla^2\Phi_2 + 2\lambda_R(\Phi_2 - \Phi_{2,\text{prev}}) \quad (3)$$

1.3 For Velocity Field $\mathbf{u} = (u_r, u_\theta, u_z)$

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \nu\nabla^2\mathbf{u} - g_m|\Phi_1\Phi_2|\nabla\rho \quad (4)$$

Incompressibility constraint: $\nabla \cdot \mathbf{u} = 0$.