where

$$\tan\theta(t) = -\frac{4at}{4a^2t^2-1}, \quad \cos\theta = \frac{4a^2t^2-1}{\sqrt{(4a^2t^2-1)^2+16a^2t^2}} = \frac{4a^2t^2-1}{4a^2t^2+1}$$

Therefore:

$$\dot{\theta}(t)\sec^2\theta(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{4at}{4a^2t^2-1} = \frac{4a(1+4a^2t^2)}{(1-4a^2t^2)^2}$$

Thus the motion on the unit circle is described by the angular velocity

$$\omega(t) = \dot{\theta}(t) = \cos^2\theta(t)\frac{4a(1+4a^2t^2)}{(1-4a^2t^2)^2} = \frac{4a}{1+4a^2t^2}$$

The relationship between angular velocity and time is plotted below:

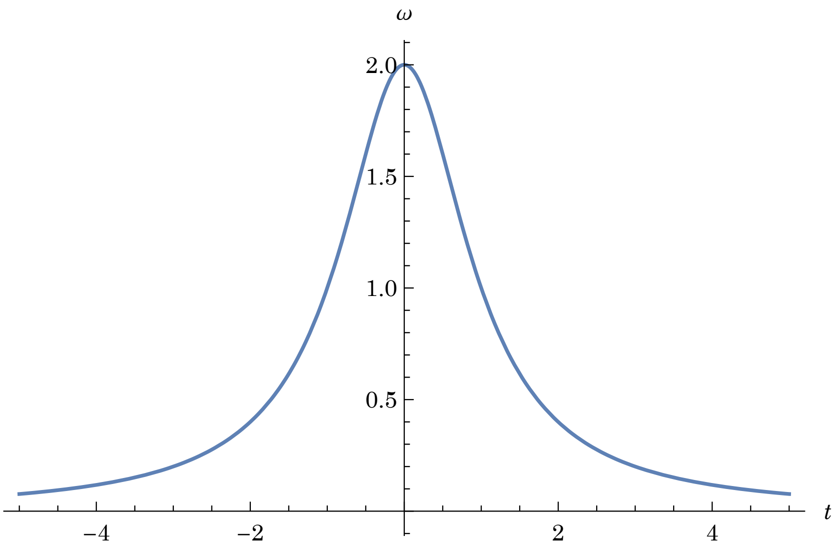


Figure 4: When the point $z$ is moving at a constant speed $|(d-b)/2|$ $(a=1/2)$ on the line $L$, the relationship between the angular velocity of its image $M(z)$ on the unit circle and time $t$.

We see that the angular velocity of $M(z)$ takes the maximum value $4a/(1+4a^2)$ when the point $z$ is passing through the point $(b+d)/2$. Noting that the integration:

$$\int_{-\infty}^{\infty}\omega(t)\mathrm{d}t = 4a\cdot\left.\frac{\arctan(2at)}{2a}\right|_{-\infty}^{\infty} = 2\pi$$

which is independent of the value of $a$ and is in agreement with the fact that $M(z)$ is a bijection. (This result also means that our result is correct.)

**Exercise 4.34**

Consider another special case where $M(z) = (z-a)/(a^*z-1)$. Show that

1. $M(M(z)) = z$;

2. The unit circle is invariant under $M(z)$;

3. If point $a$ lies in the unit disk, then $M(z)$ maps the unit disk to itself. *Hint: You perhaps need to show first that* $|a^*z-1|^2 - |z-a|^2 = (1-|a|)^2(1-|z|^2)$. What if $a$ doesn't lie in the unit disk?