Notes on Spinors and Spacetime

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Contents

Chapter 1

The geometry of world-vectors and spin-vectors

1.1 MInkowski vector space

A Minkowske vector space is a 4-dim vector space $\mathbb{V}$ over $\mathbb{R}$, endowed with an orientation, a bilinear inner product of signature $(+,-,-,-)$, and a time-orientation. Set the basis: $\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{V}$, so any $\mathbf{U}\in\mathbb{V}$ can be uniquely expressed in:

$$\mathbf{U}=U^0\mathbf{t}+U^1\mathbf{x}+U^2\mathbf{y}+U^3\mathbf{z}$$

We call the basis of $\mathbb{V}$ a tetrad, denote $(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z})$ by $\mathbf{g_i}(i=0,1,2,3)$. So:

$$\mathbf{U}=\sum_i U^{\mathbf{i}}\mathbf{g_i}=U^{\mathbf{i}}\mathbf{g_i}$$

For another tetrad $(\mathbf{g}_{\hat{0}},\mathbf{g}_{\hat{1}},\mathbf{g}_{\hat{2}},\mathbf{g}_{\hat{3}})$, we have:

$$\mathbf{g_i}=g_{\mathbf{i}}^{\hat{\mathbf{j}}}\mathbf{g}_{\hat{\mathbf{j}}}$$

If $\det g_{\mathbf{i}}^{\hat{\mathbf{j}}}>0$, we say $\mathbf{g_i}$ and $\mathbf{g}_{\hat{\mathbf{i}}}$ have the same orientation. If $\det g_{\mathbf{i}}^{\hat{\mathbf{j}}}<0$, we say $\mathbf{g_i}$ and $\mathbf{g}_{\hat{\mathbf{i}}}$ have the opposite orientation. The orientation is a equivalence relations, call one class **proper** and another **improper**. The metric is then:

$$\eta_{\mathbf{ij}}=\eta^{\mathbf{ij}}=\mathbf{g_i}\cdot\mathbf{g_j}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

Then the inner product takes form:

$$\mathbf{U}\cdot\mathbf{V}=(U^{\mathbf{i}}\mathbf{g_i})\cdot(V^{\mathbf{i}}\mathbf{g_j})=U^{\mathbf{i}}V^{\mathbf{j}}\eta_{\mathbf{ij}}=U^0V^0-U^1V^1-U^2V^2-U^3V^3$$

Norm:

$$\|\mathbf{U}\|=\mathbf{U}\cdot\mathbf{U}=(U^0)^2-(U^1)^2-(U^2)^2-(U^3)^2$$

In terms of the norm:

$$\mathbf{U}\cdot\mathbf{V}=\frac{1}{2}\{\|\mathbf{U}+\mathbf{V}\|-\|\mathbf{V}\|-\|\mathbf{U}\|\}$$

We call $\mathbf{U}$:

$$\begin{cases}\left.\begin{array}{r}\text{timelike if } \|\mathbf{U}\|>0\\ \text{null if } \|\mathbf{U}\|=0\end{array}\right\}\text{causal}\\ \text{spacelike if } \|\mathbf{U}\|<0\end{cases}$$

Clearly, if $\mathbf{U}, \mathbf{V}$ are causal, then:

$$\begin{aligned}|U^0V^0| \geq &\{(U^1)^2 + (U^2)^2 + (U^3)^2\}^{\frac{1}{2}}\{(V^1)^2 + (V^2)^2 + (V^3)^2\}^{\frac{1}{2}} \\ \geq &U^1V^1 + U^2V^2 + U^3V^3\end{aligned}$$

Which means the sign of $\mathbf{U} \cdot \mathbf{V}$ is the same as the sign of U^0V^0, unless they are both null, proportional to one another, or one of them is 0. So all causal vectors falls into 2 disjoint classes, characterized by the sign of U^0, namely future-pointing or past-pointing. Call future-pointing timelike vector a **future-timelike** vector. If $\mathbf{t}$ is a future-timelike vector, then tetrad $(\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is called **orthochronous**. The triad $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is called **righted-handed** if tetrad $(\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is both proper and orthochronous, or neither, otherwise is **left-handed**. If a tetrad is both proper and orthochronous, it is called **restricted**.

1.1.1 Minkowski space-time

Minkowski vector space $\mathbb{V}$ can be regarded as the space of position vectors of the points which constitute Minkowski space-time $\mathbb{M}$, which is an affine space, id est invariant under translations, or has no preferred origin. Relation between $\mathbb{M}$ and $\mathbb{V}$ is:

$$\mathrm{vec} : \mathbb{M} \times \mathbb{M} \to \mathbb{V}$$

Where:

$$\mathrm{vec}(P, Q) + \mathrm{vec}(Q, R) = \mathrm{vec}(P, R)$$

So we set $\mathrm{vec}(P, Q) \equiv \overrightarrow{PQ} \in \mathbb{V}$. So $\mathbb{V}$ induces the norm Φ:

$$\Phi \equiv \| \mathrm{vec}(P, Q)\| = \left(Q^0 - P^0\right)^2 - \left(Q^1 - P^1\right)^2 - \left(Q^2 - P^2\right)^2 - \left(Q^3 - P^3\right)^2$$

A linear self-transformation of $\mathbb{V}$ preserves the Lorentz norm is called an (active) Lorentz transformation. If it preserves orientation and time-orientation, it is called a restricted Lorentz transformation. A self-transformation of $\mathbb{M}$ preserves the squared interval Φ is called an (active) Poincare transformation.

1.1.2 Coordinate change

Active transformation changes **coordinate**, while **passive** transformation changes **basis**. Consider $\mathbf{g_i} \mapsto \mathbf{g_{\hat{i}}} \in \mathbb{V}$, resulting:

$$G : \mathbf{U^i} \mapsto \mathbf{U^{\hat{i}}}$$

So:

$$\mathbf{U} = U^{\mathbf{\hat{i}}}\mathbf{g_{\hat{i}}} = U^{\mathbf{i}}\mathbf{g_i} = U^{\mathbf{i}}g_{\mathbf{i}}^{\mathbf{\hat{i}}}\mathbf{g_{\hat{i}}} \Rightarrow U^{\mathbf{\hat{i}}} = U^{\mathbf{i}}g_{\mathbf{i}}{}^{\mathbf{\hat{i}}}$$

For an active transformation $L : \mathbf{U} \mapsto \mathbf{V}$, we have:

$$V^{\mathbf{\hat{j}}} = U^{\mathbf{\hat{i}}}L_{\mathbf{\hat{i}}}^{\mathbf{j}}$$

Where

$$(L_{\mathbf{\hat{i}}}^{\mathbf{j}}) = (g_{\mathbf{j}}{}^{\mathbf{\hat{i}}})^{-1}$$

L preserve inner products:

$$\eta_{\mathbf{ij}}L_{\mathbf{k}}^{\mathbf{i}}L_{\mathbf{l}}^{\mathbf{j}} = \eta_{\mathbf{kl}}$$

If L is restricted, then:

$$\det\left(L_{\mathbf{i}}^{\mathbf{j}}\right) = 1, \quad L_0^0 > 0$$

The condition for passive restricted Lorentz transformation becomes:

$$\eta_{\mathbf{ij}}g_{\mathbf{\hat{i}}}{}^{\mathbf{i}}g_{\mathbf{\hat{j}}}{}^{\mathbf{j}} = \eta_{\mathbf{\hat{i}\hat{j}}}, \quad \det\left(g_{\mathbf{\hat{i}}}{}^{\mathbf{i}}\right) = 1, \quad g_{\hat{0}}{}^{0} > 0$$

1.2 Null directions and spin transformations

For a restricted Minkowski tetrad $(\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z})$, set:

$$\mathbf{U} = T\mathbf{t} + X\mathbf{x} + Y\mathbf{y} + Z\mathbf{z}$$

If it is null, then:

$$T^2 - X^2 - Y^2 - Z^2 = 0$$

Its image is call the null cone, as the figure1.1 shows:

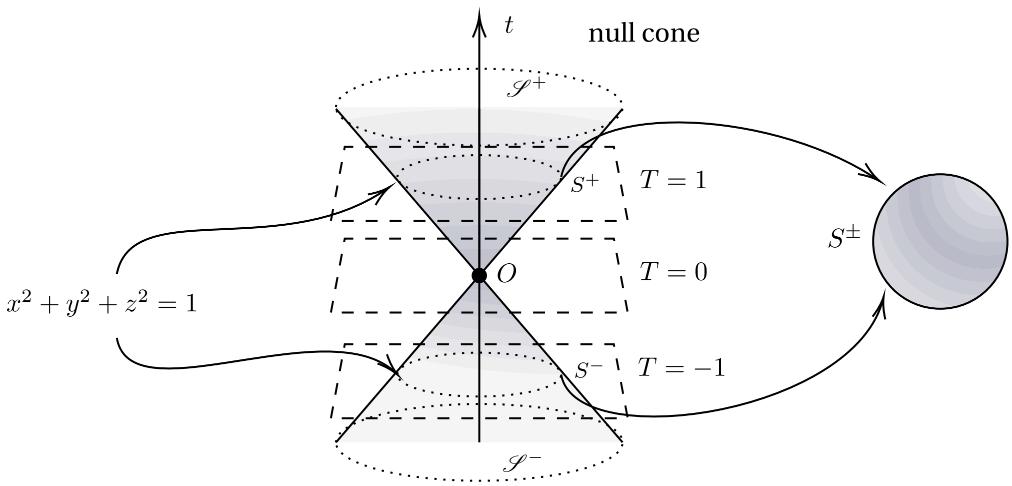

Figure 1.1: The null cone.

Call the abstract space whose elements are the future/past null directions $\mathscr{S}^+/\mathscr{S}^-$, and their projections to hyperplanes $T = 1/-1$ are called S^+/S^-, with equation:

$$x^2 + y^2 + z^2 = 1$$

The direction of $\mathbf{U}$ can be represented by the point:

$$\left(\frac{X}{|T|}, \frac{Y}{|T|}, \frac{Z}{|T|}\right)$$

The interior of S^+/S^- represents the set of future/past timelike directions. Their exteriors represent spacelike directions. We refer to $\mathscr{S}^-$ or S^- as the **celestial sphere** of O, because S^- is an accurate geometrical representation of the observer at O actually 'sees' provided he is stationary relative to the frame $(\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z})$. Call the mapping of the past null directions at O to the points of S^- the **sky mapping**, for S^+, call it **anti-sky mapping**.

Stereographic projection to Riemann sphere of S^+: Consider Euclidean 3-space $T = 1$. We projects S^+ to the Argand plane $z = 0$, as the figure 1.2 shows:

We can calculate that:

$$\zeta = \frac{x + y\mathtt{i}}{1 - z} \Rightarrow \zeta\bar{\zeta} = \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 + z}{1 - z}$$

In terms of ζ:

$$x = \frac{\zeta + \bar{\zeta}}{\zeta\bar{\zeta} + 1}, y = \frac{\zeta - \bar{\zeta}}{\mathtt{i}(\zeta\bar{\zeta} + 1)}, z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}$$

The north pole is projected to the point $\zeta = \infty$, so S^+ is a standard realization of the Argand plane of ζ.

In terms of spherical polar coordinates:

$$x = \sin\theta\cos\phi, y = \sin\theta\sin\phi, z = \cos\theta$$

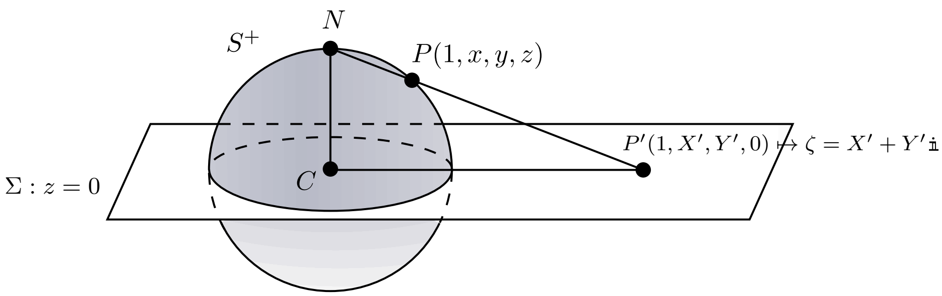

Figure 1.2: Stereographic projection of S^+ to the Argand plane

The relation between ζ and (θ, ϕ) is:

$$\zeta = \mathrm{e}^{\mathtt{i}\phi} \cot \frac{\theta}{2}$$

For S^-, just use the antipodal transformation: $(x, y, z) \mapsto -(x, y, z)$ or $(\theta, \phi) \mapsto (\pi - \theta, \pi + \phi)$, we can get:

$$\zeta = -\mathrm{e}^{\mathtt{i}\phi} \tan \frac{\theta}{2} \quad (\zeta \mapsto -\frac{1}{\bar{\zeta}})$$

We can also just use the hyperplane $\Pi : T - Z = 1$ and its intersection with $\Sigma : T = 1$ to stereographic project. See Fig.1-5 in the book.

1.2.1 Lorentz transformations and spin transformations

Instead of inhomogeneous coordinate ζ, we introduce homogeneous coordinate (ξ, η), where $(\lambda\xi, \lambda\eta)$ and (ξ, η) represent the same points on $S^+(\lambda \neq 0, \lambda \in \mathbb{C})$:

$$\zeta = \frac{\xi}{\eta}$$

In terms of homogeneous coordinate:

$$x = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\xi\bar{\xi} + \eta\bar{\eta}}, y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{\mathtt{i}(\xi\bar{\xi} + \eta\bar{\eta})}, z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}}$$

We can choose another point R on OP to represent the same null direction, multiplying the factor $(\xi\bar{\xi} + \eta\bar{\eta})/\sqrt{2}$ we can get $\mathbf{K} : \overrightarrow{OR}$:

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}(\xi\bar{\xi} + \eta\bar{\eta}), \qquad & X &= \frac{1}{\sqrt{2}}(\xi\bar{\eta} + \eta\bar{\xi}) \\ Y &= \frac{1}{\sqrt{2}}(\xi\bar{\eta} - \eta\bar{\xi}), \qquad & Z &= \frac{1}{\sqrt{2}}(\xi\bar{\xi} - \eta\bar{\eta}) \end{aligned} \tag{1.1}$$

R is independent of phase rescaling: $(\xi, \eta) \to (\mathrm{e}^{\mathtt{i}\theta}\xi, \mathrm{e}^{\mathtt{i}\theta}\eta)$, but not independent of the real scaling: $(\xi, \eta) \to (r\xi, r\eta), r \in \mathbb{R}$.

Consider a complex linear transformation of ξ and η:

$$\begin{aligned} \xi &\mapsto \xi = \alpha\xi + \beta\eta \\ \eta &\mapsto \tilde{\eta} = \gamma\xi + \delta\eta \end{aligned}$$

Then the unhomogeneous coordinate becomes:

$$\zeta \mapsto \tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

The normalization condition is the 'unimodular' condition:

$$\alpha\delta - \beta\gamma = 1$$

It is called **spin transformations**. We can also use **spin matrix** $\mathbf{A}$ to do the transformation:

$$\mathbf{A} := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det \mathbf{A} = 1$$

The transformation looks like:

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

The inverse transformation is of course:

$$\mathbf{A}^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

We can see the spin transformations form the group $SL(2, \mathbb{C})$.

But If $\mathbf{A}$ and $\mathbf{B}$ define the same transformation, then $\beta = \gamma = 0, \alpha = \delta = \pm 1$, which means $\mathbf{A} = \pm \mathbf{B}$. A spin transformation is therefore defined uniquely up to sign by its effect on the Riemann sphere of ζ.

Consider the effect of the spin transformation on the coordinate (T, X, Y, Z)(1.1), we find that it can be written as the form of projector:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = \begin{pmatrix} \xi\xi & \xi\bar{\eta} \\ \eta\xi & \eta\bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \xi & \bar{\eta} \end{pmatrix}$$

So it transforms like:

$$\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{T}+\tilde{Z} & \tilde{X}+\mathrm{i}\tilde{Y} \\ \tilde{X}-\mathrm{i}\tilde{Y} & \tilde{T}-\tilde{Z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \mathbf{A}^{\dagger} \tag{1.2}$$

Even if $\mathbf{U} = T\mathbf{t} + X\mathbf{x} + Y\mathbf{y} + Z\mathbf{z}$ is not null. On (T, X, Y, Z), the explicit form is:

$$\begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{T} \\ \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} = \frac{1}{2}\mathbf{B} \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}$$

Where

$$\mathbf{B} = \begin{pmatrix} \alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma}+\delta\delta & \alpha\bar{\beta}+\beta\bar{\alpha}+\gamma\delta+\delta\bar{\gamma} & \mathtt{i}(\alpha\beta-\beta\bar{\alpha}+\gamma\bar{\delta}-\delta\bar{\gamma}) & \alpha\bar{\alpha}-\beta\bar{\beta}+\gamma\bar{\gamma}-\delta\bar{\delta} \\ \alpha\bar{\gamma}+\gamma\bar{\alpha}+\beta\bar{\delta}+\delta\beta & \alpha\delta+\delta\bar{\alpha}+\beta\bar{\gamma}+\gamma\beta & \mathtt{i}(\alpha\delta-\delta\bar{\alpha}+\gamma\bar{\beta}-\beta\bar{\gamma}) & \alpha\bar{\gamma}+\gamma\bar{\alpha}-\beta\delta-\delta\beta \\ \mathtt{i}(\gamma\bar{\alpha}-\alpha\bar{\gamma}+\delta\bar{\beta}-\beta\bar{\delta}) & \mathtt{i}(\delta\bar{\alpha}-\alpha\delta+\gamma\beta-\beta\bar{\gamma}) & \alpha\delta+\delta\bar{\alpha}-\beta\bar{\gamma}-\gamma\bar{\beta} & \mathtt{i}(\gamma\bar{\alpha}-\alpha\bar{\gamma}+\beta\bar{\delta}-\delta\bar{\beta}) \\ \alpha\bar{\alpha}+\beta\bar{\beta}-\gamma\bar{\gamma}-\delta\delta & \alpha\beta+\beta\bar{\alpha}-\gamma\delta-\delta\bar{\gamma} & \mathtt{i}(\alpha\beta-\beta\bar{\alpha}+\delta\bar{\gamma}-\gamma\delta) & \alpha\bar{\alpha}-\beta\bar{\beta}-\gamma\bar{\gamma}+\delta\bar{\delta} \end{pmatrix}$$

This must be a restricted Lorentz transformation. Actually, we have proposition1.2.1

Prop. 1.2.1: Spin and Lorentz transformation

Every spin transformation corresponds to a unique restricted Lorentz transformation(1.2); conversely every restricted Lorentz transformation so corresponds to precisely two spin transformations, one being the negative of the other.

To prove the second part of the proposition, we can simply construct the spin-matrices corresponding to those Lorentz transformations who generates the group. There are space rotations and boosts, id est:

$$\begin{cases} \tilde{T} = \dfrac{T + vZ}{\sqrt{1 - v^2}} \\ \tilde{X} = x \\ \tilde{Y} = y \\ \tilde{Z} = \dfrac{Z + vT}{\sqrt{1 - v^2}} \end{cases}$$

[1]

For an arbitrary Lorentz transformation, just rotate it into **z** direction, boost in **z** direction, then rotate it back. For the rotation, we have the following proposition 1.2.2:

Prop. 1.2.2: Proper rotation and unitary spin transformation

Every unitary spin transformation corresponds to a unique proper rotation of S^+; conversely every proper rotation of S^+ corresponds to precisely two unitary spin transformations, one being the negative of the other.

Using this proposition, we just need to show that every z-boost can be obtained from a spin transformation. Just rewrite the boost:

$$\begin{cases} \tilde{T} + \tilde{Z} = w(T + Z) \\ \tilde{T} - \tilde{Z} = w^{-1}(T - Z) \\ \tilde{X} = X \\ \tilde{Y} = Y \end{cases} \quad \text{Where } w = \left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}$$ [2]

Thus we have:

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \pm \begin{pmatrix} w^{\frac{1}{2}} & 0 \\ 0 & w^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

In terms of inhomogeneous coordinate(the Argand plane), the expression is just

$$\tilde{\zeta} = w\zeta$$

So that Prop. 1.2.1 is finished. Then we prove Prop. 1.2.2. Noting that if the transformation is unitary, then according to the transformation rule 1.2, the trace $2T = 2(\xi\bar{\xi} + \eta\bar{\eta})$ is invariant under the transformation, so it is a proper rotation of S^+.

Conversely, consider a proper rotation on S^+, we can use Euler angles θ, ϕ, ψ. A rotation of S^+ about the z-axis is given by:

$$\tilde{\zeta} = \mathrm{e}^{\mathrm{i}\psi}\zeta$$

In spin transformations, that is:

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \pm \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

A rotation of S^+ through an angle θ about the y-axis is represented by:

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \pm \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

[1] Here v is the velocity parameter.
[2] Here w is called the relativistic Doppler factor and $\log w = \tanh^{-1} v$ is the rapidity.

As well as $\xi\tilde{\xi}+\eta\tilde{\eta}$, $\xi\tilde{\eta}-\eta\tilde{\xi}$ is also invariant, so it is a rotation of angle θ about y-axis. So Prop. 1.2.2 stands. Furthermore, the general rotation corresponds to:

$$\pm\begin{pmatrix}\cos\frac{\theta}{2}\mathrm{e}^{\mathbb{i}(\phi+\psi)/2} & -\sin\frac{\theta}{2}\mathrm{e}^{\mathbb{i}(\phi-\psi)/2}\\ \sin\frac{\theta}{2}\mathrm{e}^{-\mathbb{i}(\phi-\psi)/2} & \cos\frac{\theta}{2}\mathrm{e}^{-\mathbb{i}(\phi+\psi)/2}\end{pmatrix}$$

Its elements are called Cayley-Klein rotation parameters of mechanics.

1.2.2 Relation to quaternions

Obviously, the transformation rule 1.2 can be expressed in the quaternion form. Set

$$\mathbf{Q}=\begin{pmatrix}\mathbb{i}Z & \mathbb{i}X-Y\\ \mathbb{i}X+Y & -\mathbb{i}Z\end{pmatrix}=\mathbb{i}X+\mathbb{j}Y+\mathbb{k}Z$$

Then 1.2 reads:

$$\tilde{\mathbf{Q}}=\mathbf{A}\mathbf{Q}\mathbf{A}^{\dagger}$$

The most general unit quaternion can clearly be written in the form:

$$\begin{aligned}\mathbf{A}&=\mathbb{1}\cos\frac{\psi}{2}+(\mathbb{i}l+\mathbb{j}m+\mathbb{k}n)\sin\frac{\psi}{2}\\ &=\cos\frac{\psi}{2}+\mathbf{v}\sin\frac{\psi}{2}\end{aligned}$$

Where $\mathbf{v}=(l,m,n), l^2+m^2+n^2=1$. In the matrix notation:

$$\mathbf{A}=\begin{pmatrix}\cos\frac{\psi}{2}+\mathbb{i}n\sin\frac{\psi}{2} & (-m+\mathbb{i}l)\sin\frac{\psi}{2}\\ (m+\mathbb{i}l)\sin\frac{\psi}{2} & \cos\frac{\psi}{2}-\mathbb{i}n\sin\frac{\psi}{2}\end{pmatrix}$$