Quantum Optics by Prof. Saijun Wu

Jinyuan Wu

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1 Review of linear quantum optics

Consider an optical circuit in Figure 1 on page 1. The linear transformation matrix of the beam splitter is

$$S = \begin{pmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}, \quad \varphi \ll 1. \tag{1}$$

It can be verified that the output port

The idea of *dark port* can be generalized to many interferometers.

Nonlinear interferometers can achieve much better precision. Consider a most simple optical circuit, which is just a beam splitter with joint measurement at the output ports. The output state is

$$\begin{aligned} |\psi\rangle &= a_1^\dagger a_2^\dagger |0\rangle \\ &= \left(\cos\frac{\varphi}{2} b_1^\dagger + \sin\frac{\varphi}{2} b_2^\dagger\right)\left(\cos\frac{\varphi}{2} b_2^\dagger - \sin\frac{\varphi}{2} b_1^\dagger\right)|0\rangle \\ &= \cos\varphi b_1^\dagger b_2^\dagger + \frac{1}{2}\sin\varphi((b_1^\dagger)^2 - (b_2^\dagger)^2)\,|0\rangle\,. \end{aligned} \tag{2}$$

This is a generalization of the Hong-Ou-Mandel effect, which occurs when $\varphi = \pi/2$. The two-point correlation function is

$$P^{(2)}(1,2) = \langle\psi|a_1^\dagger a_2^\dagger a_2 a_1|\psi\rangle = \cos^2\varphi, \tag{3}$$

and the standard variance is

$$\Delta P^{(2)}(1,2) = \sqrt{\langle\psi|:(n_1 n_2)^2:|\psi\rangle - P^{(2)}(1,2)^2} = \frac{1}{2}|\sin 2\varphi|\,. \tag{4}$$

It can be seen that nonlinear interferometers can achieve better precision. Nonetheless, the usage of nonlinear interferometers is still largely limited because currently, no detector is reliable enough to perform good joint measurement.

Quantum imaging is a possible approach to achieve super-resolution. Large biological molecules, like DNAs, cannot be seen clearly by optical microscopes because of the Abbe limit of resolution. When the wave length of the detecting radiation is small enough to see them clearly, they are likely to be destroyed devitalized by the radiation. Quantum approaches can be used We construct the so-called **noon state**

$$|N00N\rangle = \frac{1}{\sqrt{2}}(\,|N,0\rangle + |0,N\rangle). \tag{5}$$

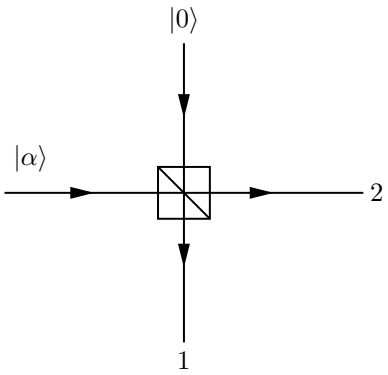

Figure 1: Vacuum displacement after going through a beam splitter

Consider a pulse

$$|\psi\rangle = D(\alpha)\,|0\rangle \tag{6}$$

that last for roughly T. It can also be considered as a

$$\langle n\rangle = \tag{7}$$

2 Semiclassical light-atom interaction

The Hamiltonian of an electron in a (non-relativistic) atom is

$$H_{\text{atom}} = \sum_i \frac{\boldsymbol{p}_i^2}{2m} - \sum_i \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r_i} + \sum_{i\neq j}\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}. \tag{8}$$

The dipole interaction Hamiltonian is

$$H_{\text{dipole}} = -\boldsymbol{d}\cdot\boldsymbol{E}, \quad \boldsymbol{d} = -e\boldsymbol{r}. \tag{9}$$

This is usually the only term worth considering (see Section 1.4 in the optics note for details). In principle we can just use (9) to couple an atom and the light field together. In practice, such an approach is far beyond existing computational resources.

In this section we discuss the case where the light field is almost always in a coherent state, and therefore can be described by classical electrodynamics. The approximation, therefore, is *not* quantum optics strictly speaking. It should be noted that semi-classical models are much easier to simulate and can reveal a lot of physics. The inverse semi-classical approximation, i.e. treating the atom as classical, can be found in here.

We consider the case with only one atom, which is place at $\boldsymbol{r} = 0$. We label the energy eigenstates as $\{|n\rangle\}$. The atom Hamiltonian is therefore

$$H_{\text{atom}} = \sum_n \hbar\omega_n\,|n\rangle\langle n|\,, \tag{10}$$

and the dipole operator is

$$\boldsymbol{d} = \sum_{m,n} \underbrace{\langle m|\boldsymbol{d}|n\rangle}_{\boldsymbol{d}_{mn}}\,|m\rangle\langle n| \tag{11}$$

For the sake of convenience we consider H_{atom} as the "free" Hamiltonian and switch to the interaction picture, and thus the dipole operator is[1]

$$\boldsymbol{d} = \sum_{m,n} \boldsymbol{d}_{mn}\mathrm{e}^{\mathrm{i}\omega_{mn}t}\,|m\rangle\langle n|\,, \quad \omega_{mn} = \omega_m - \omega_n, \tag{12}$$

and since the ω frequency component of the electric field is

$$\boldsymbol{E} = \boldsymbol{E}_0^+\mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{E}_0^-\mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{E}_0^+ = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\hat{\boldsymbol{\epsilon}}\mathrm{e}^{\mathrm{i}\varphi}a, \quad \boldsymbol{E}_0^- = (\boldsymbol{E}_0^+)^*, \tag{13}$$

the dipole Hamiltonian - now the interaction Hamiltonian - is

$$H_{\text{I}} = -(\boldsymbol{E}_0^+\mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.})\cdot\sum_{m,n}\boldsymbol{d}_{mn}\mathrm{e}^{\mathrm{i}\omega_{mn}t}\,|m\rangle\langle n|\,. \tag{14}$$

To make things easier for us, we can move the complex phase factor of $\boldsymbol{E}^+$ into the definition of $\hat{\boldsymbol{\epsilon}}$, and therefore

$$\boldsymbol{E}_0^+ = E_0^+\hat{\boldsymbol{\epsilon}}a, \quad \boldsymbol{E}_0^- = E_0^-\hat{\boldsymbol{\epsilon}}^* a^\dagger, \quad E_0^+ = E_0^- = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}, \tag{15}$$

and the expectation of $\boldsymbol{E}$ under a coherent state $|\alpha\rangle$ is proportion to $\cos(\omega t)$ and the amplitude is

$$E_0 = 2E_0^+. \tag{16}$$

[1] To switch to the interaction picture we replace all operators in the Hamiltonian with their version in the interaction picture. Note that if $\boldsymbol{d}$ evolves in accordance with H_{atom}, we must replace $|m\rangle$ with $|m\rangle\,\mathrm{e}^{\mathrm{i}\omega_m t}$ instead of $|m\rangle\,\mathrm{e}^{-\mathrm{i}\omega_m t}$.

A widely used model is the **two-level system**, where we only keep two atomic states $|g\rangle$ and $|e\rangle$ and higher excited states are integrated out. In a two-level system we have

$$H_{\mathrm{I}} = -\boldsymbol{E}_0^+ \cdot \boldsymbol{d}_{eg} \mathrm{e}^{-\mathrm{i}(\omega-\omega_{eg})t} |e\rangle\langle g| - \boldsymbol{E}_0^- \cdot \boldsymbol{d}_{eg}^* \mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})} |e\rangle\langle g| + \mathrm{h.c.}. \tag{17}$$

Note

If we view the electric field in (17) as a given external field, it is easy to find that there is no conservation of energy here. This is understandable because a time dependent external field destroys time translation symmetry. We can view the breaking of energy conservation from another point of view. In QFT, only asymptotic states can be viewed as energy eigenstates. A state where particles come closer is not an energy eigenstate and therefore does not have a definite energy, though the expectation of the energy is conserved. In the present case, the atom is always soaked in photons and they have strong coupling, so the atom does not have a definite energy, and since we ignore the energy of the photons, nor does the expectation value of the atom.

If we view the electric field in (17) as a quantum field with its own dynamics, we get some vertices where energy conservation of ω, ω_e and ω_g does not apply. The logic is the same as above. In the present case, the atom and the optical field is always interacting, and neither of them are in an energy eigenstate. The energies $\hbar\omega$, $\hbar\omega_g$ and $\hbar\omega_e$ are the energies before and after the interaction is turned on, and there is no reason to believe that we have an energy conservation equation involving these three quantities when the interaction is still going on and both the optical field and the atom do not have definite energies. Unlike the case in QFT, here the interaction is never turned off.

Nonetheless, certain degree of energy conservation still exists, as if a vertex violates "energy conservation of ω and ω_{eg}" too much, it introduces high frequency oscillation and disappears after coarse graining. This is exactly the underlying idea of RWA, which will be introduced below.

We can always redefine $|e\rangle$ and $|g\rangle$ to give $\boldsymbol{d}_{eg}$ a phase factor. We usually denote

$$\Omega = -\frac{2E_0^+ \hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{d}_{eg}}{\hbar} = -\frac{E_0 \hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{d}_{eg}}{\hbar} \tag{18}$$

as the **Rabi frequency**. Usually, we will adjust $|g\rangle$ and $|e\rangle$ to make $\Omega > 0$. Suppose the initial state is

$$|\psi(0)\rangle = c_g |g\rangle + c_e |e\rangle . \tag{19}$$

The first order perturbation is

$$c_e^{(1)}(g) = -\mathrm{i} \int_0^t \mathrm{d}\tau \, (\Omega \mathrm{e}^{-\mathrm{i}(\omega-\omega_{eg})t} + \Omega^* \mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})t}). \tag{20}$$

We immediately notice that the $\mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})t}$ term oscillates much faster and therefore after the integration has a $1/(\omega+\omega_{eg})$ factor. Therefore, it can be expected that when the time scale we are working on is significantly smaller than $1/(\omega+\omega_{eg})$, this term can be omitted, which is called the **rotational wave approximation**. The Hamiltonian corresponding to the rotational wave approximation is

$$H_{\mathrm{RWA}} = \frac{1}{2}\hbar\Omega \mathrm{e}^{-\mathrm{i}\Delta t} |e\rangle\langle g| + \mathrm{h.c.}, \quad \Delta = \omega - \omega_{eg}. \tag{21}$$

We usually call Δ the **detuning**.

It can then be noticed that (21), undergoing another picture transformation, is equivalent to

$$H = -\hbar\Delta |e\rangle\langle e| + \hbar\Omega |e\rangle\langle g| + \mathrm{h.c.}. \tag{22}$$

We do the following unitary transformation on wave functions:

$$U|e\rangle = \mathrm{e}^{\mathrm{i}\Delta t} |e\rangle , \quad U|g\rangle = |g\rangle , \tag{23}$$

and since the transformation is time-dependent, the new Hamiltonian acting on $U|\psi\rangle$ is

$$H' = UHU^\dagger + \mathrm{i}\hbar(\partial_t U)U^\dagger = \mathrm{e}^{\mathrm{i}\Delta t} \times \frac{1}{2}\hbar\Omega \mathrm{e}^{-\mathrm{i}\Delta t} |e\rangle\langle g| + \mathrm{h.c.} + \mathrm{i}\hbar(\mathrm{i}\Delta) |e\rangle\langle e| ,$$

which is just (22). The transformation is called the **rotational wave transformation (RWT)**.

Note

RWT is an *active* transformation, i.e. it acts on the vectors and keeps the basis unchanged. On the other hand, a *passive* transformation rotates the basis, and the vectors are unchanged (but the components change). The advantage of active transformations are that we do not need to always keep an eye on things like "whether the basis is before or after the transformation". The basis vectors $|e\rangle$ and $|g\rangle$ in (22) have exactly the *same* value as anywhere else.

Now since the Hamiltonian is much simpler we can then study it exactly instead of using perturbation theory, which is not that reliable with a strong external field or with a long time duration. The Schrödinger equation reads in terms of c_e and c_g

$$\mathrm{i}\dot{c}_g = \frac{1}{2}\Omega^* c_e, \quad \mathrm{i}\dot{c}_e = \Delta c_e + \frac{1}{2}\Omega c_g. \tag{24}$$

It can be seen that the excited component in the atomic wave function first goes up and then goes down, oscillating periodically, which is called the **Rabi oscillation**. In the language of phonons, the atom undergoes first absorption then stimulated emission.

A two-level system is essentially a qubit. The Hamiltonian can be rephrase into

$$H = \frac{\hbar\Delta}{2}\sigma^z + \frac{\hbar\Omega_r}{2}\sigma^x + \frac{\hbar\Omega_i}{2}\sigma^y \tag{25}$$

if we add a $\hbar\Delta(|g\rangle\langle g| + |e\rangle\langle e|)/2$ into it, and if we define

$$\boldsymbol{\Omega} = (\Omega_\mathrm{r}, \Omega_\mathrm{i}, \Delta), \tag{26}$$

where Ω_r and Ω_i are the real and the imaginary part of Ω, respectively, then the vector representing the atomic wave function on the Bloch sphere satisfies

$$\dot{\boldsymbol{n}} = \boldsymbol{\Omega} \times \boldsymbol{n}. \tag{27}$$

This equation is first written down for magnetic resonance imaging.

Finally we discuss when the two-level atom is a good approximation, and when RWA is a good approximation. It is easy to find that if we add an external driving field $\boldsymbol{E}^+\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.}$ to an atom with multiple energy levels, it can be seen that any energy level pair $|1\rangle, |2\rangle$ can be coupled by the external field as long as Ω_{12} is large enough. However, the selection rules and the frequency matching condition often mean only two energy levels are coupled strongly enough, where the two-level approximation works. In such a case, the detuning should not be too large (or otherwise the coupling is small), and therefore $1/(\omega - \omega_{eg})$ is indeed much larger than $1/(\omega + \omega_{eg})$. Therefore, we find that in a real atom system with an external driving field, when two energy levels are coupled overwhelmingly strong, i.e. the two energy level approximation works, then RWA also works. If RWA does not work, then it is impossible that only two energy levels are significantly coupled together.