Free Fermion Theories by Tian Yuan

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This presentation is mostly based on Commun.Math.Phys. 257, 725-771. This article is about what constraints and symmetry can be added to the Hilbert space of a free fermion system, and we can get the space of all possible Hamiltonians.

From now on we use $\overline{(\cdot)}$ to denote complex conjugation and use $(\cdot)^*$ to denote adjoint.

We define a **Nambu space** to be a Hilbert space $W = V \oplus V^*$, where V is also a Hilbert space, and V^* is the dual space of V and an inner product can be induced. An inner product can be defined on W if we assign the inner product of V^* to be

$$\langle f, f' \rangle = \overline{\langle Cf, Cf' \rangle_V}, \quad f, f' \in V^*, \tag{1}$$

where $C : W \to W$ is defined as

$$C : V \mapsto \langle V, - \rangle . \tag{2}$$

The complex conjugate in (1) comes from the fact that C is anti-linear and the requirement that the first element in an inner product is anti-linear. We also assume there is a canonical bilinear form

$$b : W \otimes W \to \mathbb{C}, \quad b(v + f, v' + f') = f(v') + f'(v). \tag{3}$$

We show the motivation to define Nambu spaces. The Hilbert space V is actually the linear space spanned by $\{c_i^\dagger\}$, and V^* is the linear space spanned by $\{c_i\}$. The many-body Fock space is

$$\wedge V = \mathbb{C} \oplus V \oplus \wedge^2 V \oplus \cdots \tag{4}$$

and using this construction we can verify that V^* is indeed the dual space of V. C is similar to a particle-hole transformation, though a physical particle-hole transformation may also contain some unitary transformations. Now we see W is the space of all field operators, and b actually imposed the anti-commutative relation. Therefore the definition of Nambu spaces summarizes the properties of Fermi operators.

It can be verified that the $b(\cdot, \cdot)$ structure and the $\langle \cdot, \cdot \rangle$ structure are not independent. We make the decomposition

$$w = v + f, \quad v \in V, f \in V^*, \tag{5}$$

and we have

$$\langle w, w' \rangle = \langle v, v' \rangle + \langle Cf, Cf' \rangle , \tag{6}$$

A generalized quadratic Hamiltonian can be written as

$$H = \sum_{i,j} A_{ij} c_i^\dagger c_j + \frac{1}{2} \sum_{i,j} (B_{ij} c_i^\dagger c_j^\dagger + C_{ij} c_j c_i). \tag{7}$$

We require $H = H^*$, and this in turn requires

$$A_{ij} = \overline{A_{ji}}, \quad C_{ij} = \overline{B_{ij}}, \tag{8}$$

or in other words

$$\mathbf{A} = \mathbf{A}^*, \quad \mathbf{B}^\top = -\mathbf{B}. \tag{9}$$

Suppose

$$\psi = \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ f_1 \\ \vdots \\ f_n \end{pmatrix}, \tag{10}$$

the EOM

$$\mathrm{i}\hbar\dot{\psi} = [\psi, H]$$

now becomes

$$\dot{\psi} = \frac{\mathrm{i}}{\hbar}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & -\mathbf{A}^\top \end{pmatrix}\psi. \tag{11}$$

The linear space of all possible Hamiltonians is actually an algebra on W, which we denote as $C(W)$. We have

$$C(W) = C^0(W) \oplus C^1(W) \oplus C^2(W) \oplus \cdots, \tag{12}$$

where $C^n(W)$ is all possible Hamiltonians containing only n-operator terms. Note that the product structure on $C(W)$ is ordinary operator product, but $C^n(W)$ is not closed under ordinary operator product, so $C^n(W)$ is actually a Lie algebra.

Now we are able to write down a theorem: *We have the isomorphism* $C^2(W) \simeq \mathfrak{so}(W, b)$.

Now we can define symmetry on a Nambu space. An arbitrary unitary element g and an anti-unitary element h in symmetric group G satisfied the conditions that

$$\langle\psi, \psi'\rangle = \langle g\psi, g'\psi\rangle = \overline{\langle h\psi, h'\psi\rangle}, \tag{13}$$

and

$$b(\psi, \psi') = b(g\psi, g'\psi) = \overline{b(h\psi, h'\psi)}, \tag{14}$$