TQD Model by Prof. Wang

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1 The TQD model

The **twisted quantum double (TQD)** models are a type of models with intrinsic topological order in 2D. We know quantum double models are generalization of the toric code, and TQDs are generalization of quantum double models. Details may be found in PRB 87, 125114 (2013).

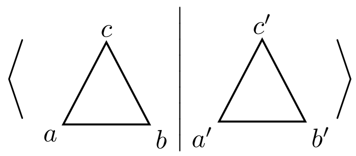
We consider a triangulated 2D plane. We index each sites with a natural number. In this way the sites are *ordered*. We place an arrow on each bond, the direction of which is from the site with a large index to the site with a small index. This lattice is the playground of the model. Note that if we view the arrows on the edges as oriented paths, then no closed loop will form on any triangle.

The input data of TQD on such a lattice Γ with E edges is as follows:

- A group G. We place an element g_i of G on each edge i. When $G = \mathbb{Z}_2$, for example, we have $g_i = \pm 1$. These group elements are all degrees of freedom in the model, and the Hilbert space is given.

  We use $[ab]$ to denote the group element on the edge between site a and site b, and usually we assume $a < b$. We further define
$$[ab] = [ba]^{-1}. \tag{1}$$

  We have the inner product relation



$$= \delta_{[ab][a'b']}\delta_{[bc][b'c']}\delta_{[ac][a'c']}, \tag{2}$$

where we assume the orders of a, b, c and $a,', b', c'$ are the same.

- A normalized 3-cocycle $\alpha \in H^3[G, U(1)]$. We require

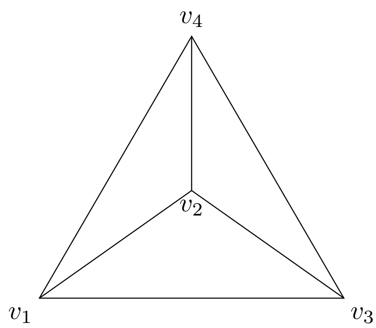
$$\delta\alpha\,(g_0, g_1 g_2, g_3) = \alpha(g_1, g_2, g_3)\alpha^{-1}(g_0 g_1, g_2, g_3)\alpha(g_0, g_1 g_2, g_3), \tag{3}$$

$$\alpha^{-1}(g_0, g_1, g_2 g_3)\alpha(g_0, g_1, g_2) = 1, \tag{4}$$

and the normalization condition is

$$\alpha(1, g, h) = \alpha(g, 1, h) = \alpha(g, h, 1) = 1. \tag{5}$$

These 3-cocycles can be placed on a subgraph of the lattice like the follows:



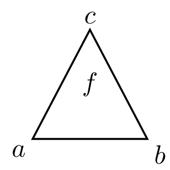
. (6)

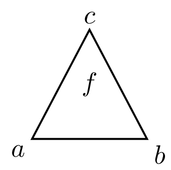
The subgraph is actually a flattened tetrahedron, and it can be oriented. We see the ascending order of v_1, v_2, v_3, v_4 as the orientation. For positive orientation we assign $\alpha([v_1 v_2], [v_2 v_3], [v_3 v_4])$ to the subgraph, and for negative orientation we assign $\alpha([v_1 v_2], [v_2 v_3], [v_3 v_4])^{-1}$. It can be seen that (6) is $\alpha([v_1 v_2], [v_2 v_3], [v_3 v_4])^{\epsilon(v_1, v_2, v_3, v_4)}$.

- Now the Hamiltonian can be defined. We define

$$H = -\sum_v A_v - \sum_f B_f, \tag{7}$$

where v runs over all vertices and f runs over all triangles. We have

$B_f \Bigg|$ 

$\Bigg\rangle = \delta_{[ab][bc][ca]} \Bigg|$ 

$\Bigg\rangle .$ (8)

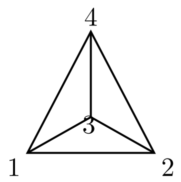
This quantity is unity when

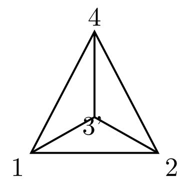
$$[ab][bc] = [ac]. \tag{9}$$

This is called the **flatness condition**, and B_f's presence in the Hamiltonian means it holds in the ground state. It means there is no flux in the ground state, if we regard G to be a gauge group and $\{g_i\}$ a gauge field. Sometimes (9) is also called the **chain rule**. The definition of A is

$$A_v = \frac{1}{|G|} \sum_{[v'v]=g\in G} A_v^g, \tag{10}$$

where

$A_3^g \Bigg|$ 

$\Bigg\rangle = \delta_{[3'3]}\alpha([12],[23'],[3'3])\alpha([23'],[3'3],[34])\alpha([13'],[3'3],[34])^{-1} \Bigg|$ 

$\Bigg\rangle .$ (11)

A_v induces time evolution. Under the assume that $1 < 2 < 3' < 3 < 4$, applying (9) to relevant triangles, we have

$$[13] = [13'][3'3], \quad [23] = [23'][3'3], \quad [3'4] = [3'3][34], \tag{12}$$

or in other words

$$[13'] = [13][33'], \quad [23'] = [23][33'], \quad [3'4] = [3'3][34]. \tag{13}$$

This can be seen as a gauge transformation: for a given vertex, applying gauge transformation g to it means to replace the group element g_0 on an incoming edge by gg_0 and to replace the group element g_0 on an outgoing edge by g_0g^{-1}. The topological order in (7) is actually the same as a Dijkgraaf-Witten TQFT. A TQFT does not have a Hamiltonian in the sense of Legendre transformation, and (7) can be seen as a Hamiltonian extension of the TQFT.

Note that for a given α we can find an equivalent 3-cocycle

$$\alpha' = \alpha \cdot \delta\beta\,, \tag{14}$$

and we have

$$\alpha'(g_0,g_1,g_2) = \underbrace{\frac{\beta(g_1,g_2)\beta(g_0,g_1g_2)}{\beta(g_0g_1,g_2)\beta(g_0,g_1)}}_{\text{3-coboundary}} \alpha(g_0,g_1,g_2),$$

and we will find all the additional β factors in (11) give rise to a equation in which each β factor corresponds to a triangle. So by redefinition states and absorb the β factors into the phase factors, (11) remains the same. So in the end, the input data include an equivalent class $[\alpha]$, a group G and a lattice.

2 The topological order in a TQD model

(7) is similar to the toric code, but since the operators are much more complicated and the algebra is not clear, we are unable to write down all ground states directly.

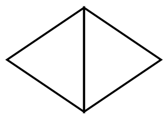
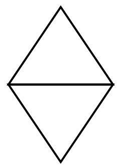
We can verify that again all B_f operators and A_v operators are commuting projectors. Hence projector to the ground state subspace is

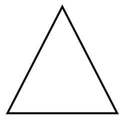
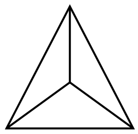
$$P_\Gamma^0 = \prod_f B_f \prod_v A_v. \tag{15}$$

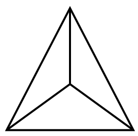
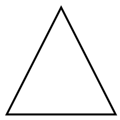
The ground state subspace $\mathcal{H}_\Gamma^0$ is invariant under P_Γ^0. The ground state degeneracy is

$$\mathrm{GSD} = \operatorname{tr} P_\Gamma^0. \tag{16}$$

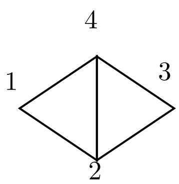
Since the lattice is complicated, there is no easy way to evaluate it directly from the Hamiltonian. Note, however, that we expect (7) to host a topological order, and if so, the GSD is independent of the details of triangulation. We consider the following **Pachner moves**:

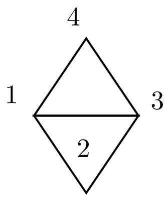
f_1 :

$\longrightarrow$

, (17)

f_2 :

$\longrightarrow$

, (18)

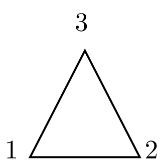
f_3 :

$\longrightarrow$

. (19)

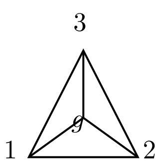
Note that we are working on a quantum theory and we need to find three operators on states to implement these moves. A natural choice is

$T_1 \Bigg|$


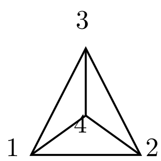
$\Bigg\rangle = \sum_{[13]\in G} \alpha([12],[23],[34]) \Bigg|$


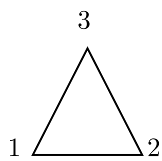
$\Bigg\rangle$ (20)

$T_2 \Bigg|$


$\Bigg\rangle = \sum_{[g1],[g2],[g3]\in G} \alpha([g1],[12],[23]) \Bigg|$


$\Bigg\rangle$, (21)

$T_3 \Bigg|$


$\Bigg\rangle = \alpha([12],[23],[34]) \Bigg|$


$\Bigg\rangle$. (22)

It can be proved that if $|\phi\rangle \in \mathcal{H}^0_\Gamma$, we also have $T\,|\phi\rangle \in \mathcal{H}^0_{\Gamma'}$. It can also be proved that T's are unitary.

Now we consider the example of a torus, which may be considered as a square with its left and right bonds identified, and its upper and lower bonds identified. We consider the following triangulation:

$$\begin{tikzcd} 2 \arrow[d, "h"'] & 4 \arrow[l, "g"'] \arrow[d, "h"] \\ 1 & 3 \arrow[l, "g"] \end{tikzcd},$$

where the boundary condition requires

$$[13] = [24] = g, \quad [12] = [34] = h.$$

The flatness conditions requires

$$[14] = gh = hg,$$

so in $\mathcal{H}^0_{\mathbb{T}}$ we have $gh = hg$.

Now we find (3) and (4) are actually two Pachner moves.

$$A^x\,|g,h\rangle = \tag{23}$$

$$\begin{aligned} \mathrm{GSD} &= \mathrm{tr}\left(\frac{1}{|G|}\sum_x A^x\right) \\ &= \frac{1}{|G|}\sum_{g,h}\sum_x \delta_{gh,hg}\,\langle gh|A^x|gh\rangle \\ &= \frac{1}{|G|} \end{aligned} \tag{24}$$