Quantum Optics, Homework 3

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Interference between Gaussian pulses Consider two Gaussian pulses with wave vectors $\boldsymbol{k}_{1,2} = k(\pm\sin\theta, 0, \cos\theta)$, respectively. They are incident to a plane detector on the surface $z = 0$. The intensity distributions of the two beams are all

$$|\mathcal{E}|^2 \propto e^{-\left(x^2+y^2\right)/\sigma^2}, \tag{1}$$

with $\sigma \gg \lambda$. The pulses arrive at the detector simultaneously. The detector absorbs the pulses completely and there is no reflection. Calculate $P^{(1)}(\boldsymbol{r})$ and $P^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for the following states of the optical field:

(a) $|\psi\rangle = \frac{1}{\sqrt{2^N N!}} \left(a_1^\dagger + a_2^\dagger\right)^N |V\rangle$.

(b) $|\psi\rangle = \frac{1}{N!} \left(a_1^\dagger a_2^\dagger\right)^N |V\rangle$.

(c) $|\psi\rangle = \frac{1}{\sqrt{2N!}} \left(\left(a_1^\dagger\right)^N + \left(a_2^\dagger\right)^N\right) |V\rangle$.

(d) $|\psi\rangle = D_1(\alpha) D_2(\alpha) |V\rangle, \quad D_j(\alpha) \equiv e^{\alpha a_j^\dagger - \alpha^* a_j}$.

(e) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(D_1(\alpha) + D_2(\alpha)\right) |V\rangle$.

Solution The electric field operator is

$$\boldsymbol{E} = \sum_{i=1,2} \boldsymbol{\mathcal{E}}_i \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r} - \mathrm{i}\omega t} a_i + \text{h.c.}. \tag{2}$$

(a) We define

$$b^\dagger = \frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger),$$

and now the wave function is

$$|\psi\rangle = \frac{1}{\sqrt{N!}} (b^\dagger)^N |0\rangle .$$

We have

$$P^{(1)}(\boldsymbol{r}) = \frac{1}{N!} |\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0| b^N (a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2 - \boldsymbol{k}_1)\cdot\boldsymbol{r}} a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{r}} a_2^\dagger a_1) (b^\dagger)^N |0\rangle .$$

Evaluating the terms in the RHS above, we have

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_1 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

and similarly

$$\langle 0| b^N a_2^\dagger a_2 (b^\dagger)^N |0\rangle = \frac{1}{2} N^2 (N-1)!,$$

and

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_2 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

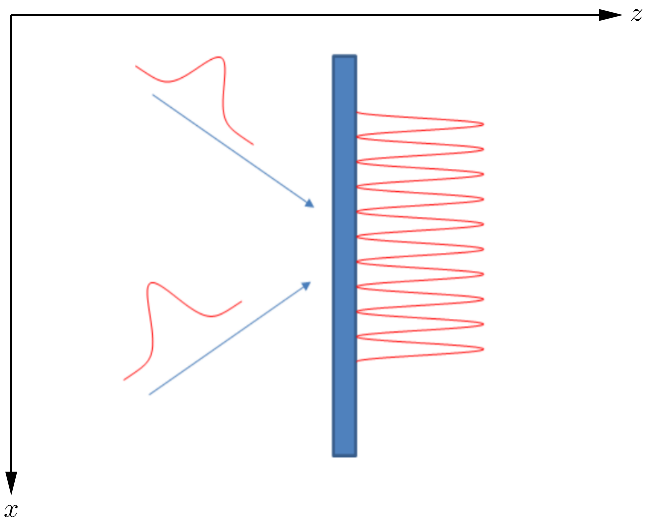

Figure 1: The two Gaussian beams incident to a detector

and similarly

$$\langle 0|b^N a_1^\dagger a_2 (b^\dagger)^N|0\rangle = \frac{1}{2}N^2(N-1)!.$$

Putting everything together we have

$$\begin{aligned}
P^{(1)}(\boldsymbol{r}) &= \eta \frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times \frac{1}{2}N^2(N-1)! \times (2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2))\cdot\boldsymbol{r}}) \\
&= \eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}) \\
&= \eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(2k\sin\theta x)) \\
&= 2\eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta),
\end{aligned}$$

so finally

$$P^{(1)}(\boldsymbol{r}) = 2N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta) \propto 2N\mathrm{e}^{-(x^2+y^2)/\sigma^2}\cos^2(kx\sin\theta). \tag{3}$$

The two-photon joint probability is

$$\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,b^N(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2}) \\
&\quad\times(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(b^\dagger)^N\,|0\rangle \\
&= \eta^2\frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\times N\,\langle 0|b(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})|0\rangle \\
&\quad\times(N-1)\,\langle 0|b(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})|0\rangle \\
&\quad\times N\,\langle 0|(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})b^\dagger|0\rangle \\
&\quad\times(N-1)\,\langle 0|(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})b^\dagger|0\rangle \\
&\quad\times\text{contraction between } N\ b\text{'s and } b^\dagger\text{'s} \\
&= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\frac{1}{N!}\times\frac{N}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})\times\frac{N-1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2}) \\
&\quad\times\frac{N-1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})\times\frac{N}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})\times(N-2)! \\
&= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2N(N-1)(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}_1)(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}_2),
\end{aligned}$$

so

$$\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= 4\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2N(N-1)\cos^2(kx_1\sin\theta)\cos^2(kx_2\sin\theta) \\
&\propto 4\eta^2\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}N(N-1)\cos^2(kx_1\sin\theta)\cos^2(kx_2\sin\theta).
\end{aligned} \tag{4}$$

(b) We have

$$P^{(1)}(\boldsymbol{r}) = \frac{\eta}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0|(a_2a_1)^N(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(a_1^\dagger a_2^\dagger)^N|0\rangle$$
$$= \frac{\eta}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0|a_2^N a_1^N(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle\,.$$

Evaluating the terms in the RHS, we have

$$\begin{aligned}\langle 0|a_1^N a_2^N a_1^\dagger a_1 (a_1^\dagger)^N(a_2^\dagger)^N|0\rangle &= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_1a_1^\dagger|0\rangle\\ &\quad\times \text{contraction of } (N-1)\ a_1\text{'s and } (N-1)\ a_1^\dagger\text{'s}\\ &\quad\times \text{contraction of } N\ a_2\text{'s and } N\ a_2^\dagger\text{'s}\\ &= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_1a_1^\dagger|0\rangle \times (N-1)!\,\langle 0|a_1a_1^\dagger|0\rangle \times N!\,\langle 0|a_2a_2^\dagger|0\rangle\\ &= N^2N!(N-1)!,\end{aligned}$$

and similarly we have

$$\langle 0|a_1^N a_2^N a_2^\dagger a_2 (a_1^\dagger)^N(a_2^\dagger)^N|0\rangle = N^2N!(N-1)!.$$

Also we have

$$\begin{aligned}\langle 0|a_2^N a_1^N a_1^\dagger a_2 (a_1^\dagger)^N(a_2^\dagger)^N|0\rangle &= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_2a_2^\dagger|0\rangle\\ &\quad\times \text{contraction of } N\ a_2\text{'s, } (N-1)\ a_1\text{'s, } N\ a_1^\dagger\text{'s and } (N-1)\ a_2^\dagger\text{'s}\\ &= 0,\end{aligned}$$

so it vanishes, and so does $\langle 0|a_2^N a_1^N a_2^\dagger a_1(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle$. Putting everything together we have

$$P^{(1)}(\boldsymbol{r}) = \eta\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times 2\times N^2N!(N-1)! = 2N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2,$$

so the single photon probability is

$$P^{(1)}(\boldsymbol{r}) = 2\eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \propto 2\eta N\mathrm{e}^{-(x^2+y^2)/\sigma^2}. \tag{5}$$

The two-photon joint probability is

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\langle 0|\,a_1^Na_2^N(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})\\ &\quad\times(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(a_1^\dagger)^N(a_2^\dagger)^N\,|0\rangle\\ &= \eta^2\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\langle 0|\,a_1^Na_2^N(a_1^\dagger a_1^\dagger a_1a_1 + a_2^\dagger a_2^\dagger a_2a_2\\ &\quad+ a_1^\dagger a_2^\dagger a_2a_1\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}_2+\boldsymbol{k}_2\cdot\boldsymbol{r}_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}+\text{h.c.})(a_1^\dagger)^N(a_2^\dagger)^N\,|0\rangle\\ &= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\frac{1}{(N!)^2}(N^2(N-1)^2(N-2)!N!\times 2\\ &\quad+N^4(N-1)!(N-1)!(\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}+\text{h.c.})),\end{aligned}$$

where the second equation uses the conservation of particle numbers. So we have

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= 2\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2(N(N-1)+N^2\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2))\\ &= 2\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2(N(N-1)+N^2\cos(2k(x_1-x_2)\sin\theta))\\ &\propto 2\eta^2\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}(N(N-1)+N^2\cos(2k(x_1-x_2)\sin\theta)).\end{aligned}\tag{6}$$

(c) The single photon probability is now

$$\begin{aligned}P^{(1)} &= \eta\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\langle 0|\,(a_1^N+a_2^N)(a_1^\dagger a_1+a_2^\dagger a_2+\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2+\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)\\ &\quad\times((a_1^\dagger)^N+(a_2^\dagger)^N)\,|0\rangle\,.\end{aligned}$$

Evaluating the terms on the RHS, we have

$$\begin{aligned}\langle 0|(a_1^N + a_2^N)a_1^\dagger a_1((a_1^\dagger)^N + (a_2^\dagger)^N)|0\rangle &= \langle 0|a_1^N a_1^\dagger a_1(a_1^\dagger)^N|0\rangle + \langle 0|a_2^N a_1^\dagger a_1(a_2^\dagger)^N|0\rangle \\ &= N\cdot N\cdot(N-1)! + N!,\end{aligned}$$

as well as

$$\langle 0|(a_1^N + a_2^N)a_2^\dagger a_2((a_1^\dagger)^N + (a_2^\dagger)^N)|0\rangle = N\cdot N\cdot(N-1)! + N!.$$

The third term and fourth term vanish because the photon numbers in the bra and the ket is not the same. So we have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \eta\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times 2\times(N^2(N-1)! + N!) \\ &= \eta(N+1)|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \propto (N+1)\mathrm{e}^{-(x^2+y^2)/\sigma^2}.\end{aligned} \tag{7}$$

The two-photon joint probability is

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,(a_1^N + a_2^N)(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1}) \\ &\quad\times(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1}) \\ &\quad\times(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})((a_1^\dagger)^N + (a_2^\dagger)^N)\,|0\rangle \\ &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,(a_1^N + a_2^N)(a_1^\dagger a_1^\dagger a_1 a_1 + a_2^\dagger a_2^\dagger a_2 a_2)((a_1^\dagger)^N + (a_2^\dagger)^N)\,|0\rangle \\ &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2 \times N^2(N-1)!\times 2,\end{aligned}$$

where the second equation uses conservation of particle numbers. So we have

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2 N(N-1)|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2 \\ &\propto \eta^2 N(N-1)\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}.\end{aligned} \tag{8}$$

(d) We have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\,\langle\alpha,\alpha|(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)|\alpha,\alpha\rangle \\ &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(2 + 2\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}) \\ &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta) \propto \eta\mathrm{e}^{-(x^2+y^2)/\sigma^2}\cos^2(kx\sin\theta).\end{aligned} \tag{9}$$

Similarly by the definition of coherent states we have

$$P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) = P^{(1)}(\boldsymbol{r}_1)P^{(1)}(\boldsymbol{r}_2). \tag{10}$$

(e) We have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \frac{1}{2}\eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(\,\langle\alpha,0| + \langle 0,\alpha|)(a_1^\dagger a_1 + a_2^\dagger a_2 \\ &\qquad + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(\,|\alpha,0\rangle + |0,\alpha\rangle) \\ &= \frac{1}{2}\eta(|\alpha|^2 + |\alpha|^2),\end{aligned}$$

so

$$P^{(1)}(\boldsymbol{r}) = \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2|\alpha|^2 \propto \eta\mathrm{e}^{-(x^2+y^2)/\sigma^2}|\alpha|^2. \tag{11}$$

Similarly, all terms involving both a_1 and a_2 vanish because either of them gives 0 when acting on $|\alpha,0\rangle$ or $|0,\alpha\rangle$, and we just have (10) as well.

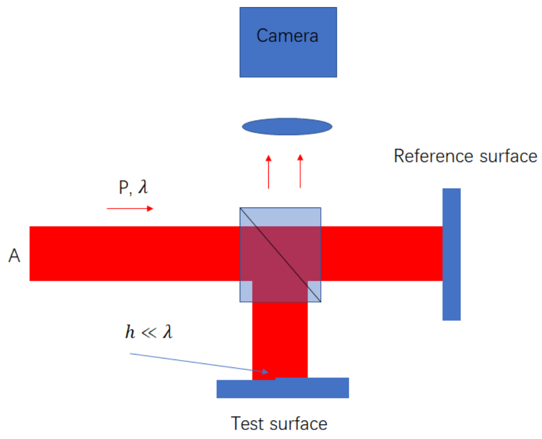

Figure 2: Surface profile measuring using lasers

Discussion Actually by calculating commutators the problems can be solved much easier. For example, to evaluate $\langle 0|b_1^N E^- E^+ (b_1^\dagger)^N|0\rangle$ is just to evaluate

$$\langle 0|b_1^N E^- E^+ (b_1^\dagger)^N|0\rangle = \langle 0|[b_1^N, E^-]\,[E^+, (b_1^\dagger)^N]\,|0\rangle\,, \tag{12}$$

and then we can invoke the formula that connect commutators to derivatives.

(b) It can be found that (5) is homogeneous in space. It should be noted that a **single event** in which we detect the photon number at each point has interference stripes. The fact that (5) is homogeneous in space is a consequence of the fact that the position of the interference stripes can vary freely, and after calculating the average the stripes vanish. This fact that a single event has interference stripes can also be seen from the fact that (6) has interference stripes.

(c) The state we are discussing is called the **NOON** state as it is actually

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\,|N,0\rangle + |0,N\rangle). \tag{13}$$

This is a highly entangled state, which can be used to achieve super-resolution. For example, in this problem, we have

$$P^{(N)}(\boldsymbol{r},\boldsymbol{r},\ldots,\boldsymbol{r}) = \eta^N \left|\frac{1}{\sqrt{2}}(\mathcal{E}_1(\boldsymbol{r})^N + \mathcal{E}_2(\boldsymbol{r})^N)\right|^2, \tag{14}$$

which has a $\cos N(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{r}$ term. We can see that if the space resolution of the camera is l, then by using $\mathcal{E}(\boldsymbol{r})$ to construct a NOON state and then measuring it, we can determine the details of $\mathcal{E}(\boldsymbol{r})$ with a space resolution with the magnitude of Nl.

NOON states are difficult to prepare. There are already successful examples of NOON states obtained by post-selection experiments.

Laser surface measurement

(a) Consider the Michaelson interferometer in Figure 2. Suppose that there is a step on the test surface with height $h \ll \lambda$, and that the step has no scattering effects and there is no interference between the left and the right reflected light beam. Describe the output, and estimate the necessary power P to achieve $\delta h/h = 0.1$ within time duration T.

(b) Replace the laser by a series of single photon pulses.

(c) Replace the laser by a thermal light source where

$$\bar{n} = \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1} \gg 1. \tag{15}$$

Discuss the relation between this case and the case of coherent light.

Solution Consider Figure 3. The transformation matrix of the light propagating in the space is

$$\begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix},$$

where

$$\varphi = k(x_1 - x_2) =: kx \ll 1. \tag{16}$$

The transformation matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -i\sin\varphi/2 \\ -i\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -i\sin\varphi/2 \\ -i\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}. \tag{17}$$

By detecting $\langle b_2^\dagger b_2\rangle$ i.e.

$$\begin{aligned} \langle n_2\rangle = \langle b_2^\dagger b_2\rangle &= \langle(-i\sin\varphi/2 a_1^\dagger + \cos\varphi/2 a_2^\dagger)(i\sin\varphi/2 a_1 + \cos\varphi/2 a_2)\rangle \\ &= \sin^2\varphi/2\,\langle a_1^\dagger a_1\rangle \end{aligned} \tag{18}$$

we can measure x. Also we have

$$\begin{aligned} \langle n_2^2\rangle &= \langle((-i\sin\varphi/2 a_1^\dagger + \cos\varphi/2 a_2^\dagger)(i\sin\varphi/2 a_1 + \cos\varphi/2 a_2))^2\rangle \\ &= \langle(\sin^2\varphi/2 a_1^\dagger a_1 - i\sin\varphi/2\cos\varphi/2 a_1^\dagger a_2)(\sin^2\varphi/2 a_1^\dagger a_1 + i\sin\varphi/2\cos\varphi/2 a_2^\dagger a_1)\rangle \\ &= \sin^4\varphi/2\,\langle a_1^\dagger a_1 a_1^\dagger a_1\rangle + \sin^2\varphi/2\cos^2\varphi/2\,\langle a_1^\dagger a_2 a_2^\dagger a_1\rangle \\ &= \sin^4\varphi/2\,\langle(a_1^\dagger)^2 a_1^2\rangle + \sin^4\varphi/2\,\langle a_1^\dagger a_1\rangle + \sin^2\varphi\cos^2\varphi/2\,\langle a_1^\dagger a_1\rangle \\ &= \sin^4\varphi/2\,\langle(a_1^\dagger)^2 a_1^2\rangle + \sin^2\varphi/2\,\langle a_1^\dagger a_1\rangle, \end{aligned}$$

and the error is

$$\delta n_2 = \sqrt{\sin^4\varphi/2\,\langle(a_1^\dagger)^2 a_1^2\rangle + \sin^2\varphi/2\,\langle a_1^\dagger a_1\rangle - \sin^4\varphi/2\,\langle a_1^\dagger a_1\rangle^2}. \tag{19}$$

(a) For a coherent input on $a_1^\dagger$ mode, the measurement result is

$$\langle n_2\rangle = |\alpha|^2\sin^2\varphi/2. \tag{20}$$

By (19), the fluctuation of $\langle n_2\rangle$ is

$$\delta n_2 = |\alpha|\,\sin\varphi/2,$$

and since φ is small, we have $\langle n_2\rangle \propto \varphi^2$, and therefore

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2}\frac{\delta n_2}{\langle n_2\rangle} = \frac{1}{2|\alpha|\,\sin\varphi/2}. \tag{21}$$

Note that

$$|\alpha|^2 = \langle\text{numbers output photons}\rangle = \frac{PT}{\hbar\omega},$$

and again by using the fact that φ is small, we obtain

$$\delta\varphi \approx \sqrt{\frac{\hbar\omega}{PT}} = \sqrt{\frac{2\pi\hbar c}{PT\lambda}}. \tag{22}$$

Now we want to measure h, which is

$$h = \frac{\varphi_L - \varphi_R}{k}, \tag{23}$$

so we have

$$\begin{aligned} \frac{\delta h}{h} &= \frac{\sqrt{\delta\varphi_L{}^2 + \delta\varphi_R{}^2}}{kh} = \frac{\sqrt{2}\,\delta\varphi}{kh} \\ &= \frac{\lambda}{2\pi h}\sqrt{\frac{4\pi\hbar c}{PT\lambda}}, \end{aligned} \tag{24}$$

where $\varphi = \varphi_L \approx \varphi_R$. The condition $\delta h/h < 0.1$ is equivalent to

$$P > \frac{100\hbar\lambda c}{\pi T h^2}. \tag{25}$$

(b) This time the input state is

$$|\psi\rangle = a_1^\dagger |0\rangle\,, \tag{26}$$

so

$$\langle n_2\rangle = \sin^2\varphi/2, \tag{27}$$

and

$$\delta n_2 = \sin\varphi/2. \tag{28}$$

Therefore, after N pulses being measured we have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{\sqrt{N}}\frac{1}{2\sin\varphi/2}. \tag{29}$$

It can be seen that the form of the equation is the same as (21). Since the derivation in (a) after (21) has nothing to do with the exact meaning of $|\alpha|$, everything should be same for (21) and (29), and we have (25)

$$P > \frac{100\hbar\lambda c}{\pi T h^2} \tag{30}$$

again.

(c) For a thermal optical field, we have

$$\langle n_2\rangle = \bar{n}\sin^2\varphi/2, \tag{31}$$

and

$$\begin{aligned}
\langle (a_1^\dagger)^2 a_1^2\rangle &= \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{1+n}}\langle n|(a_1^\dagger)^2 a_1^2|n\rangle \\
&= \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{1+n}} n(n-1) \\
&= \frac{\alpha^2}{1+\bar{n}}\sum_{n\geq 2}\alpha^{n-2}n(n-1) \qquad (\alpha := \frac{\bar{n}}{1+\bar{n}}) \\
&= \frac{\alpha^2}{1+\bar{n}}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\sum_{n\geq 0}\alpha^n \\
&= \frac{\alpha^2}{1+\bar{n}}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\frac{1}{1-\alpha} = \frac{\alpha^2}{1+\bar{n}}\frac{2}{(1-\alpha)^3} \\
&= 2\bar{n}^2,
\end{aligned}$$

so

$$\begin{aligned}
\delta n_2 &= \sqrt{\sin^4\varphi/2\cdot 2\bar{n}^2 + \sin^2\varphi/2\bar{n} - \sin^4\varphi/2\bar{n}^2} \\
&= \sin\varphi/2\sqrt{\bar{n}^2\sin^2\varphi/2 + \bar{n}}.
\end{aligned} \tag{32}$$

From (31) and the fact that φ is small we also have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2}\frac{\delta n_2}{\langle n_2\rangle} = \frac{1}{2\sin\varphi/2}\sqrt{\sin^2\varphi/2 + \frac{1}{\bar{n}}}.$$

Now we need to be smart to choose φ. If φ is not very small, then since $\bar{n}$ is large, we have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2}. \tag{33}$$

(33) is a unique character of a thermal optical field that the fluctuation of the photon number is of the same order of the photon number itself. It can be seen that the precision of a single measurement cannot be improved unboundedly even when considering solely the shot noise. Using a thermal light source, in this way, is not an efficient idea. If, however, φ is small enough, then we have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2\sqrt{\bar{n}}\sin\varphi/2}. \tag{34}$$

Note that when it comes to the energy flow, the position of $\sqrt{\bar{n}}$ is the same of the ones of $|\alpha|$ and $\sqrt{N}$, so again we have (25).

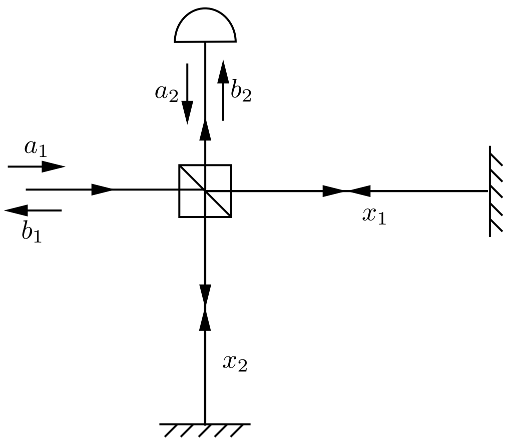

Figure 3: A standard Michaelson interferometer

Discussion Figure 2 is a simplified version of real-world surface profile measurement, where every point on the surface and every point on the detector forms a interferometer. If, for example, we have 100 pixels on the camera, than we have 100^2 interferometers.

When the quantum nature of the optical field is important, we usually measure the position of *dark* stripes, because when φ changes the positions of bright stripes move, but when we see the intensity on a certain position increases, it may also be a result of the shot noise.

We can see that all the three light sources have the same shot noise error.

Michaelson light clock The Michaelson interferometer (see again Figure 3) can also be used to measure photon frequency when $\Delta x = x_1 - x_2$ is already known. Derive its precision and compare the result with the Ramsey atomic clock.

Solution We can reuse results in the last problem. We have

$$\omega = \frac{c}{\Delta x}\varphi, \tag{35}$$

and φ is measured from $\langle n_2 \rangle$. If the input light is laser, we have (21), and from (35) we have

$$\frac{\delta\omega}{\omega} = \frac{\delta\varphi}{\varphi} = \frac{1}{2|\alpha|\,\sin\left(\frac{\omega\Delta x}{2c}\right)}, \tag{36}$$

or

$$\delta\omega \approx \frac{\omega}{2|\alpha|\,\frac{\omega\Delta x}{2c}} = \frac{1}{|\alpha|\,\Delta x/c}. \tag{37}$$

This is similar to the case in the Ramsey atomic clock, which is

$$\delta\omega = \frac{1}{\sqrt{N}T}, \quad T = \Delta x/v, \tag{38}$$

but here v is the speed of atoms instead of light. Therefore using Michaelson interferometer as a clock is not a good idea since the precision is poor compared to a Ramsey atomic clock.

Discussion We can see that in the Michaelson interferometer, photons play the role of atoms in atomic blocks. Since light moves so fast, atoms are usually used to define a standard time unit - that is exactly how "one second" is defined nowadays.

It does not mean that light frequency cannot be measured efficiently using an interferometer. Now people can construct a cavity made of single crystal silicon, and light can live in it for seconds. Such a device can be used to determine the frequency of light with $\sim 1\,\mathrm{Hz}$ uncertainty.