Advanced Electrodynamics by Prof. Kun Din

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Some contents of the first several lectures are covered in this optics note, but some technical details are worth a separate note.

1 Spherical functions

2 Effective continuous medium

Suppose $f(\boldsymbol{r})$ is a filter or "smoother". Suppose the electric charge is

$$\eta(\boldsymbol{r}) = \sum_n \sum_{i \in C_n} q_i \delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{1}$$

where n is the index of clusters of electrons, most frequently moleculars. The smoothened version is

$$\begin{aligned}
\langle \eta \rangle (\boldsymbol{r}) &= \int \mathrm{d}^3\boldsymbol{r}' \, \eta(\boldsymbol{r} - \boldsymbol{r}') f(\boldsymbol{r}') \\
&= \int \mathrm{d}^3\boldsymbol{r}' \sum_n \sum_{i \in C_n} q_i \delta(\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{r}_i) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \sum_{i \in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_n) + \cdots),
\end{aligned}$$

and if the perturbation of the charges is not strong, we can make a cutoff and get

$$\begin{aligned}
&\sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_n)) \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} (q_n - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{p}_n) \\
&= \sum_n \left(q_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \right) \\
&= \sum_n (q_n f(\boldsymbol{r} - \boldsymbol{r}_n) - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} f(\boldsymbol{r} - \boldsymbol{r}_n)) \\
&= \sum_n (q_n f(\boldsymbol{r} - \boldsymbol{r}_n) - \boldsymbol{\nabla} \cdot (\boldsymbol{p}_n f(\boldsymbol{r} - \boldsymbol{r}_n))).
\end{aligned}$$

Now we note that

$$\langle \delta(\boldsymbol{r} - \boldsymbol{r}_n) \rangle = f(\boldsymbol{r} - \boldsymbol{r}_n),$$

and we get

$$\langle \eta \rangle (\boldsymbol{r}) = \rho(\boldsymbol{r}) - \boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{r}), \tag{2}$$

where

$$\rho(\boldsymbol{r}) = \left\langle \sum_n q_n \delta(\boldsymbol{r} - \boldsymbol{r}_n) \right\rangle, \quad \boldsymbol{P}(\boldsymbol{r}) = \left\langle \sum_n \boldsymbol{p}_n \delta(\boldsymbol{r} - \boldsymbol{r}_n) \right\rangle. \tag{3}$$

TODO: magnetic average

3 Constitutive relations

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi_\mathrm{e}) \boldsymbol{E}, \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \tag{4}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho, \tag{5}$$

4 Frequency dispersion and the Clausius-Mossotti equation

Now we work on a frequent mechanism of frequency dispersion, i.e. the response of the material depending on ω. We consider a oscillator model, where the electrons can be seen as oscillators, the EOM of each of which is

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2 \boldsymbol{r} - m\gamma \boldsymbol{r} + e\boldsymbol{E}_\text{driving}. \tag{6}$$

Note that though the electrons has no interaction, but first, since the electrons are immersed in the radiation created by themselves, m, γ, and ω_0 will be modified, and second, an electron can feel the collective electric field of other electrons, and there must be a term for this in $\boldsymbol{E}_\text{drive}$. Since $\boldsymbol{E}_\text{other electrons} + \boldsymbol{E}_\text{this electron}$ has no long scale effects and only modifies m, γ, and ω_0 (because they it modifies the lattice potential and therefore change ω_0, and it also modifies the phonon spectrum, thus modifying γ, and since we know the radiation of a moving charge has a term proportion to $\ddot{\boldsymbol{d}}$, m is also modified), we can say that

$$\boldsymbol{E}_\text{driving} = \boldsymbol{E}_\text{external} - \boldsymbol{E}_\text{self}.$$

Here and after we write $\boldsymbol{E}_\text{external}$ as $\boldsymbol{E}$, and we have

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2 \boldsymbol{r} - m\gamma \boldsymbol{r} + e(\boldsymbol{E} - \boldsymbol{E}_\text{self}). \tag{7}$$

Note that this can also be seen as "using $\boldsymbol{E}_\text{self}$ to modify m, ω_0 and γ and subtracting $\boldsymbol{E}_\text{sef}$ from $\boldsymbol{E}$". A more generalized discussion of the modifications can be found in discussions around (16.12) in this optics note. What we are doing is actually a self-energy correction.

Suppose $\boldsymbol{r} \propto \mathrm{e}^{-\mathrm{i}\omega t}$, and we have

$$e\boldsymbol{r} =: \boldsymbol{d} = \alpha \boldsymbol{E}, \quad \alpha = \frac{e^2}{m(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega)}. \tag{8}$$

The long range radiation is summarized as

$$\boldsymbol{P} = N\boldsymbol{d} = N\alpha(\boldsymbol{E} - \boldsymbol{E}_\text{self}), \tag{9}$$

and if we approximate the self field as

$$\boldsymbol{E}_\text{sef} = -\frac{\boldsymbol{P}}{3\epsilon_0}, \tag{10}$$

which is the field strength in a empty cavity in a homogeneously polarized insulator, then we get

$$\boldsymbol{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}} \boldsymbol{E} =: \epsilon_0 \chi_\mathrm{e} \boldsymbol{E}, \tag{11}$$

$$\chi_\mathrm{e} = \frac{N\alpha/\epsilon_0}{1 - N\alpha/(3\epsilon_0)}, \quad \frac{\epsilon_\mathrm{r} - 1}{\epsilon_\mathrm{r} + 2} = \frac{N\alpha}{3\epsilon_0}. \tag{12}$$

This is the so-called **Clausius-Mossotti equations**.

Now we put (8) into (12), and get

$$\epsilon_\mathrm{r} = 1 + \chi_\mathrm{e} = 1 + \frac{1}{\left(\frac{Ne^2}{m\epsilon_0}\right)^{-1} (\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega) - \frac{1}{3}},$$

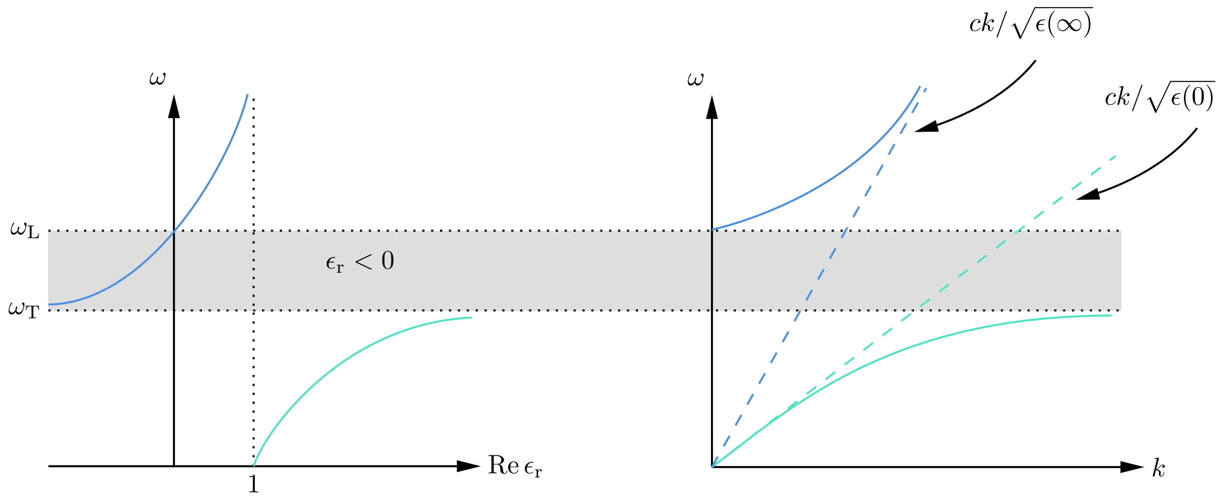

Figure 1: The photon band structure of (13)

and therefore after introducing the plasma frequency we get

$$\epsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}} = 1 + \frac{\omega_{\mathrm{p}}^2}{\tilde{\omega}_0^2 - \omega^2 - \mathrm{i}\gamma\omega}, \quad \tilde{\omega}_0^2 = \omega_0^2 - \frac{1}{3}\omega_{\mathrm{p}}^2, \quad \omega_{\mathrm{p}}^2 = \frac{Ne^2}{m\epsilon_0}. \tag{13}$$

This equation means a non-trivial photon band structure. Now we assume $\gamma = 0$, and because we have

$$k^2 = \left(\frac{\omega}{c}\right)^2 \epsilon_{\mathrm{r}}(\omega), \tag{14}$$

it is possible that $\epsilon_{\mathrm{r}} < 0$ in some frequencies. The photon band structure, therefore, has a forbidden band.