Advanced Electrodynamics, Homework 3

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2D Green function (a) Derive the 2D Green function in polar coordinates.
Solution

(a) The 2D Green function is given by the solution of the two dimensional version of Helmholtz equation with an external source:

$$(\nabla^2 + k^2) G_0(\boldsymbol{r} - \boldsymbol{r}') = -\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}'). \tag{1}$$

The solution, in terms of Fourier transformation, is

$$G_0(\boldsymbol{R}) = -\int \frac{\mathrm{d}^2 \boldsymbol{p}}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{R}}}{k^2 - \boldsymbol{p}^2 + \mathrm{i}0^+}.$$

In polar coordinates where we consider the direction of $\boldsymbol{R}$ to be the $\theta = 0$ axis, we have

$$\begin{aligned}
G_0(\boldsymbol{R}) &= -\frac{1}{(2\pi)^2} \int_0^\infty p \,\mathrm{d}p \int_0^{2\pi} \mathrm{d}\theta \, \frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{R}|\cos\theta}}{k^2 - p^2 + \mathrm{i}0^+} \\
&= \frac{1}{(2\pi)^2} \frac{1}{2} \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}p \left(\frac{1}{p + k - \mathrm{i}0^+} + \frac{1}{p - k - \mathrm{i}0^+} \right) \mathrm{e}^{\mathrm{i}p|\boldsymbol{R}|\cos\theta} \\
&= \frac{1}{2(2\pi)^2} \left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \times 2\pi \mathrm{i} \mathrm{e}^{\mathrm{i}k|\boldsymbol{R}|\cos\theta} + \int_{\pi/2}^{3\pi/2} \mathrm{d}\theta \times 2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}k|\boldsymbol{R}|\cos\theta} \right) \\
&= \frac{\mathrm{i}}{4\pi} ((\pi J_0(k|\boldsymbol{R}|) + \mathrm{i}\pi \mathbf{H}(k|\boldsymbol{R}|)) + (\pi J_0(k|\boldsymbol{R}|) - \mathrm{i}\pi \mathbf{H}(k|\boldsymbol{R}|))) \\
&= \frac{\mathrm{i}}{4\pi} \times 2\pi J_0(k|\boldsymbol{R}|).
\end{aligned}$$

So we get

$$G_0(\boldsymbol{R}) = \frac{\mathrm{i}}{2} \tag{2}$$

(b)

Dyadic green function in Fourier space (a) Show that in vacuum the Maxwell equations can be rephrased into

$$\boldsymbol{M}^2 \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} c^2 \boldsymbol{k}\cdot\boldsymbol{k} - c^2 \boldsymbol{k}\boldsymbol{k} & 0 \\ 0 & c^2 \boldsymbol{k}\cdot\boldsymbol{k} - c^2 \boldsymbol{k}\boldsymbol{k} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \omega^2 \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix}. \tag{3}$$

(b) Find the eigenvalues and eigenvectors. (c) Derive the Green function in the Fourier space, and show why longitude modes are absent.
Solution

(a) In the Fourier space the Maxwell equations are

$$\begin{aligned}
\boldsymbol{k} \cdot \boldsymbol{E} &= 0, \\
\boldsymbol{k} \times \boldsymbol{E} &= \omega \boldsymbol{B}, \\
\boldsymbol{k} \cdot \boldsymbol{B} &= 0, \\
\boldsymbol{k} \times \boldsymbol{B} &= -\frac{1}{c^2} \omega \boldsymbol{E}.
\end{aligned}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are actually $\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega)$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{k},\omega)$, respectively. The first and the third equations have no ω dependence and therefore cannot be a part of the eigenvalue problem. From the second and the fourth equations we have

$$\begin{aligned}\omega\boldsymbol{k}\times\boldsymbol{B} &= \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E})\\ &= (\boldsymbol{k}\cdot\boldsymbol{E})\boldsymbol{k}-\boldsymbol{k}^2\boldsymbol{E}\\ &= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E},\end{aligned}$$

and therefore

$$-\frac{\omega^2}{c^2}\boldsymbol{E} = (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}. \tag{4}$$

Similarly we have

$$\begin{aligned}\frac{\omega}{c^2}\boldsymbol{k}\times\boldsymbol{E} &= -\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{B})\\ &= -(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{k}+\boldsymbol{k}^2\boldsymbol{B}\\ &= (-\boldsymbol{k}\boldsymbol{k}+\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B},\end{aligned}$$

and therefore

$$\frac{\omega^2}{c^2}\boldsymbol{B} = (-\boldsymbol{k}\boldsymbol{k}+\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B}. \tag{5}$$

From (4) and (5) we have

$$\begin{pmatrix}\boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}\boldsymbol{k} & \\ & \boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}\boldsymbol{k}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix} = \frac{\omega^2}{c^2}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix}. \tag{6}$$