QFT I, Homework 1

Jinyuan Wu

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1. Lorentz invariance This is problem 2.6 on p. 28 of Schwartz. (a) Show that

$$\int_{-\infty}^{\infty} \mathrm{d}k^0 \delta\left(k^2 - m^2\right) \theta\left(k^0\right) = \frac{1}{2\omega_k}. \tag{1}$$

where $\omega_{\boldsymbol{k}} = \sqrt{|\boldsymbol{k}|^2 + m^2}$ (b) Show the integration measure $\mathrm{d}^4 k$ is Lorentz invariant. (c) Show that

$$\int \frac{\mathrm{d}^3 \boldsymbol{k}}{2\omega_k} \tag{2}$$

is Lorentz invariant.

Solution

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}k^0 \delta\left(k^2 - m^2\right) \theta\left(k^0\right) &= \int_{0}^{\infty} \mathrm{d}k^0 \delta\left((k^0)^2 - \omega_k^2\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}((k^0)^2 - \omega_k^2) \, \frac{\mathrm{d}k^0}{\mathrm{d}((k^0)^2 - \omega_k^2)} \delta((k^0)^2 - \omega_k^2) \\
&= \int_{-\infty}^{\infty} \mathrm{d}((k^0)^2 - \omega_k^2) \, \frac{1}{2k^0} \delta((k^0)^2 - \omega_k^2) \\
&= \int_{-\omega_k^2}^{\infty} \mathrm{d}x \, \frac{1}{2\sqrt{x + \omega_k^2}} \delta(x) \\
&= \frac{1}{2\omega_{\boldsymbol{k}}},
\end{aligned}$$

which is exactly (1).

(b) A Lorentz transformation changes the four-momentum in this way:

$$k^\mu \longrightarrow k'^\mu = \Lambda^\mu{}_\nu k^\mu.$$

Under such a transformation, a differential form changes according to the Jacobian determinant in the way of

$$\mathrm{d}^4 k \longrightarrow \mathrm{d}^4 k' = \left| \frac{\partial k'^\mu}{\partial k^\nu} \right| \mathrm{d}^4 k = \left| \Lambda^\mu{}_\nu \right| \, \mathrm{d}^4 k \,.$$

We know the determinant of any Lorentz transformation matrix is 1, and thus $\mathrm{d}^4 k = \mathrm{d}^4 k'$, so it is a Lorentz invariant.

(c) Since $\mathrm{d}^4 k$ is Lorentz invariant, as well as $\delta(k^2 - m^2)$, if $\theta(k^0)$ is also Lorentz invariant then due to

$$\int_{k^0 = -\infty}^{\infty} \mathrm{d}^4 k \, \delta(k^2 - m^2) \theta(k^0) = \int_{k^0 = -\infty}^{\infty} \mathrm{d}k^0 \, \delta(k^2 - m^2) \theta(k^0) \, \mathrm{d}^3 \boldsymbol{k} = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{2\omega_{\boldsymbol{k}}},$$

we find that $\int \mathrm{d}^3 \boldsymbol{k} \,/ 2\omega_{\boldsymbol{k}}$ is also Lorentz invariant. So the key point is to verify the invariance of $\theta(k^0)$. A rotational transformation definitely does not change the sign of k^0. After a boost the time component of the four-momentum is

$$k'^0 = \gamma(k^0 - \boldsymbol{v} \cdot \boldsymbol{k}), \tag{3}$$

where c is set to 1 and a timelike boost satisfies $|\boldsymbol{v}| < 1$. A four-momentum that makes physical sense must be timelike, and therefore we have $k^0 > |\boldsymbol{k}|$ when $k^0 > 0$, and the Cauchy inequality tells us that

$$|\boldsymbol{k}| > |\boldsymbol{v}||\boldsymbol{k}| \geq \boldsymbol{v} \cdot \boldsymbol{k},$$

and therefore we have

$$k^0 - \boldsymbol{v}\cdot\boldsymbol{k} > 0$$

when $k^0 > 0$.When $k^0 < 0$, the timelike condition for the four-momentum is $k^0 < -|\boldsymbol{k}|$, and again by the Cauchy inequality we have

$$-|\boldsymbol{k}| < -|\boldsymbol{k}||\boldsymbol{v}| \le \boldsymbol{k}\cdot\boldsymbol{v},$$

so

$$k^0 - \boldsymbol{v}\cdot\boldsymbol{k} < 0.$$

Therefore we find that the sign of k'^0 always agrees with k^0, and therefore $\theta(k^0) = \theta(k'^0)$, so under a *timelike* Lorentz invariant $\theta(k^0)$ is indeed invariant, completing the proof of the invariance of $\int \mathrm{d}^3\boldsymbol{k}\,/2\omega_{\boldsymbol{k}}$.

2. Yukawa potential This is problem 3.6 on p. 43 of Schwartz. (a) Calculate the equation of motion for a massive vector A_μ from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J^\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Assuming $\partial_\mu J^\mu = 0$, use the equations to find a constraint on A_μ. (b) For J_μ the current of a point charge, show that the equations of motion for A_0 reduces to

$$A_0(r) = \frac{e}{4\pi^2 \mathrm{i} r}\int_{-\infty}^{\infty} \frac{k\mathrm{d}k}{k^2+m^2}\mathrm{e}^{\mathrm{i}k\cdot r}. \tag{4}$$

(c) Evaluate this integral with contour integration to get an explicit form for $A_0(r)$ (d) Show that as $m \to 0$ you reproduce the Coulomb potential.

Solution

(a) By the Euler-Lagrangian equation we have

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu} = 0.$$

The first term is

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = m^2 A^\mu - J^\mu.$$

As for the second term, we have

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu} &= -\frac{1}{2}F^{\rho\sigma}\frac{\partial(\partial_\rho F_\sigma - \partial_\sigma F_\rho)}{\partial\partial_\nu A_\mu} \\
&= -\frac{1}{2}F^{\rho\sigma}(\delta_{\rho\nu}\delta_{\sigma\mu} - \delta_{\sigma\nu}\delta_{\rho\mu}) \\
&= -\frac{1}{2}(F^{\nu\mu} - F^{\mu\nu}) \\
&= F^{\mu\nu},
\end{aligned}$$

and therefore

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu} = \partial_\nu F^{\mu\nu} = -\partial_\nu\partial^\nu A^\mu + \partial_\nu\partial^\mu A^\nu.$$

So the equation of motion is

$$m^2 A^\mu - J^\mu + \partial_\nu\partial^\nu A^\mu - \partial^\mu\partial_\nu A^\nu = 0,$$

or

$$\partial_\nu\partial^\nu A^\mu - \partial^\mu\partial_\nu A^\nu + m^2 A^\mu = J^\mu. \tag{5}$$

Under the assumption that $\partial_\mu J^\mu = 0$, we have

$$\partial_\mu\partial_\nu\partial^\nu A^\mu - \partial_\mu\partial^\mu\partial_\nu A^\nu + m^2\partial_\mu A^\mu = \partial_\mu J^\mu = 0,$$

where the first two terms cancels, and we arrive at a constraint on A^μ that

$$\partial_\mu A^\mu = 0. \tag{6}$$

This, in turn, simplifies the equation of motion into

$$\partial_\nu\partial^\nu A^\mu = J^\mu. \tag{7}$$

(b) Note that vector components in (7) are decoupled and for A^0 we have

$$\partial_\nu \partial^\nu A^0 = J^0. \tag{8}$$

The current of a point charge without dynamics is

$$J^0 = e\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_0), \quad J^i = 0,$$

and under such a current A^μ has no time evolution, and therefore (8) can be further simplified into

$$(-\nabla^2 + m^2)A^0 = e\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_0). \tag{9}$$

Denoting $\boldsymbol{x} - \boldsymbol{x}_0$ as $\boldsymbol{r}$, by the Fourier transformation we have

$$\begin{aligned}
A^0 &= \frac{1}{-\nabla^2 + m^2} e \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= e \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{|\boldsymbol{k}|^2 + m^2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= e \frac{1}{(2\pi)^3} \times (2\pi) \times \int_0^\pi \sin\theta \, \mathrm{d}\theta \int_0^\infty k^2 \, \mathrm{d}k \, \frac{1}{k^2 + m^2} \mathrm{e}^{\mathrm{i}kr\cos\theta} \\
&= e \frac{1}{(2\pi)^3} \times (2\pi) \times \int_0^\infty k^2 \, \mathrm{d}k \, \frac{1}{k^2 + m^2} \frac{1}{\mathrm{i}kr} \mathrm{e}^{\mathrm{i}kr} \\
&= \frac{e}{(2\pi)^2} \int_{-\infty}^\infty k^2 \, \mathrm{d}k \, \frac{1}{k^2 + m^2} \frac{1}{\mathrm{i}kr} \mathrm{e}^{\mathrm{i}kr} \\
&= \frac{e}{4\pi^2 \mathrm{i}r} \int_0^\infty k \, \mathrm{d}k \, \frac{1}{k^2 + m^2} \mathrm{e}^{\mathrm{i}kr},
\end{aligned}$$

which is exactly (4).

(c) There are two poles on the imaginary axis, i.e. $k = \pm\mathrm{i}m$. Since $\mathrm{e}^{\mathrm{i}kr}$ decreases sufficiently fast only on the upper plane, we will use a contour plotted as Figure 1 on page 4. That gives

$$\begin{aligned}
A^0 &= \frac{e}{4\pi^2 \mathrm{i}r} \times 2\pi\mathrm{i} \lim_{k \to \mathrm{i}m} k \frac{1}{k^2 + m^2} \mathrm{e}^{\mathrm{i}kr} \times (k - \mathrm{i}m) \\
&= \frac{e}{4\pi^2 \mathrm{i}r} \times 2\pi\mathrm{i} \times \frac{\mathrm{i}m}{2\mathrm{i}m} \mathrm{e}^{-mr} \\
&= \frac{e}{4\pi r} \mathrm{e}^{-mr},
\end{aligned}$$

so we get the potential

$$A^0 = \frac{e}{4\pi r} \mathrm{e}^{-mr}. \tag{10}$$

(d) Taking the limit $m \to 0$ in (10) we immediately obtain

$$A^0 = \frac{e}{4\pi r} \sim \frac{e}{r}, \tag{11}$$

which is a Coulomb potential.

3. Lorentz currents This is problem 3.2 on p. 42 of Schwartz. (a) Calculate the conserved currents associated with Lorentz transformations. Express the currents in terms of the energy momentum tensor. (b) Evaluate the currents for $\mathcal{L} = \frac{1}{2}\phi\left(\partial^2 + m^2\right)\phi$.

Solution A Lorentz transformation can be expressed in the most general form

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{12}$$

where Λ preserves the metric. Suppose J is a generator, then the metric-preserving requirement is just

$$J\eta + \eta J^\top = 0, \tag{13}$$

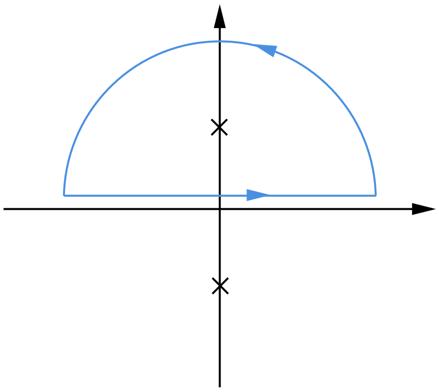

Figure 1: The integral contour

which contains 10 independent equations, leaving $16 - 10 = 6$ degrees of freedom. We can, therefore, encode the 6-element parameter of a Lorentz transformation into an antisymmetric tensor $\omega^{\mu\nu}$, and an arbitrary Lorentz transformation reads

$$\Lambda^{\mu}{}_{\nu} = \mathrm{e}^{\mathrm{i}J^{\mu}{}_{\nu\alpha\beta}\,\omega^{\alpha\beta}}. \tag{14}$$

It can be verified that

$$J^{\mu\nu}{}_{\alpha\beta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha} \tag{15}$$

is a good choice, and parameters ω^{ij} are parameters for rotation and ω^{0i}'s are parameters for boost.

(a) We are going to investigate a theory $\mathcal{L}$ invariant under this transformation. The Noether current can be labeled as $K^{\alpha\mu\nu}$, and $\partial_\alpha K^{\alpha\mu\nu} = 0$. Under a Lorentz transformation there is no substantial change on the field ϕ, i.e.

$$\phi(x) \longrightarrow \phi'(x') = \phi(x),$$

which means

$$\delta\phi + \partial_\mu\phi\,\delta x^\mu = 0,$$

where

$$\delta x^\mu = \mathrm{i}J^{\mu}{}_{\nu\alpha\beta}\,\delta\omega^{\alpha\beta}\,x^\nu.$$

By Noether's theorem we have

$$\begin{aligned}
0 &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\,\delta\phi + \mathcal{L}\,\delta x^\mu\right)\\
&= \partial_\mu\left(\left(-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi + \mathcal{L}\delta^\mu_\nu\right)\delta x^\nu\right)\\
&= -\partial_\mu\left(T^{\mu}{}_{\nu}\,\delta x^\nu\right) = -\mathrm{i}\partial_\mu\left(T^{\mu}{}_{\nu}\,J^{\nu}{}_{\sigma\alpha\beta}\,\delta\omega^{\alpha\beta}\,x^\sigma\right)\\
&= -\mathrm{i}\partial_\mu\left(T^{\mu}{}_{\nu}\left(\delta^\nu_\alpha\eta_{\sigma\beta} - \delta^\nu_\beta\eta_{\sigma\alpha}\right)\delta\omega^{\alpha\beta}\,x^\sigma\right)\\
&= -\mathrm{i}\partial_\mu(T^{\mu}{}_{\alpha}\,x^\beta - T^{\mu}{}_{\beta}\,x^\alpha)\,\delta\omega^{\alpha\beta}\,.
\end{aligned}$$

Note that $\delta\omega^{\alpha\beta}$ is antisymmetric, so only six Noether current can be obtained, each of which has the form of

$$\partial_\mu(T^{\mu}{}_{\alpha}\,x^\beta - T^{\mu}{}_{\beta}\,x^\alpha)\,\delta\omega^{\alpha\beta} + \partial_\mu(T^{\mu}{}_{\beta}\,x^\alpha - T^{\mu}{}_{\alpha}\,x^\beta)\,\delta\omega^{\beta\alpha} = 0,$$

where $\alpha \neq \beta$. Due to the antisymmetric natural of $T^{\mu}{}_{\alpha}\,x^\beta - T^{\mu}{}_{\beta}\,x^\alpha$, we obtain

$$\partial_\mu(T^{\mu}{}_{\alpha}\,x^\beta - T^{\mu}{}_{\beta}\,x^\alpha) = 0.$$

Therefore we have

$$K^{\alpha\mu\nu} = T^{\alpha\mu}x^\nu - T^{\alpha\nu}x^\mu, \tag{16}$$

which are the currents of Lorentz transformations in terms of the energy-momentum tensor. Note that the derivation presented here only works for scalar fields. For a field with non-zero spin, as the coordinates change under the Lorentz transformation, the components of the field will also mix together, creating more terms in $\delta\phi$.

(b) An equivalent Lagrangian containing only first order derivatives is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2, \tag{17}$$

from which we have

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi - \mathcal{L}\delta^\mu_\nu \\ &= -\partial^\mu\phi\partial_\nu\phi + \frac{1}{2}\delta^\mu_\nu\partial_\sigma\phi\partial^\sigma\phi - \frac{1}{2}\delta^\mu_\nu m^2\phi^2. \end{aligned}$$

Therefore

$$\begin{aligned} K^{\alpha\mu\nu} &= (x^\mu\partial^\nu\phi - x^\nu\partial^\mu\phi)\partial^\alpha\phi + \frac{1}{2}(\eta^{\alpha\mu}x^\nu - \eta^{\alpha\nu}x^\mu)(\partial_\sigma\phi)^2 + \frac{1}{2}(x^\mu\eta^{\alpha\nu} - x^\nu\eta^{\alpha\mu})m^2\phi^2 \\ &= (x^\mu\partial^\nu\phi - x^\nu\partial^\mu\phi)\partial^\alpha\phi + \frac{1}{2}(\eta^{\alpha\mu}x^\nu - \eta^{\alpha\nu}x^\mu)((\partial_\sigma\phi)^2 - m^2\phi^2). \end{aligned} \tag{18}$$

4. Classical electromagnetism This is problem 2.1 on p. 33 of Peskin. Action:

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{19}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{20}$$

(a) Construct the energy-momentum tensor for this theory. (b) The usual procedure does not result in a symmetric tensor. To remedy that, one can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{21}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu \tag{22}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right), \quad \boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{B}, \tag{23}$$

where $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

Solution

(a) The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{24}$$

and therefore the energy-momentum tensor is

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial\partial_\mu A^\rho}\partial_\nu A^\rho - \mathcal{L}\delta^\mu_\nu \\ &= -\frac{1}{2}F_{\sigma\delta}\frac{\partial F^{\sigma\delta}}{\partial\partial_\mu A^\rho}\partial_\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^\mu_\nu \\ &= -\frac{1}{2}F_{\sigma\delta}(\eta^{\sigma\mu}\delta^\delta_\rho - \eta^{\mu\delta}\delta^\sigma_\rho)\partial_\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^\mu_\nu \\ &= -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial_\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^\mu_\nu, \end{aligned}$$

so now we have defined an energy-momentum tensor

$$T^{\mu}{}_{\nu} = -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial_\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^\mu_\nu \tag{25}$$

or in other words,

$$T^{\mu\nu} = -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\eta^{\mu\nu}. \tag{26}$$

(b) With (21) and (22) we have

$$\hat{T}^{\mu\nu} = -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\nu A^\rho + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\eta^{\mu\nu} + F^{\mu\lambda}\partial_\lambda A^\nu + A^\nu\partial_\lambda F^{\mu\lambda},$$

where

$$\begin{aligned}
&-(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\nu A^\rho + F^{\mu\lambda}\partial_\lambda A^\nu \\
&= -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\nu A^\rho + (\partial^\mu A^\lambda - \partial^\lambda A^\mu)\partial_\lambda A^\nu \\
&= -(\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\nu A^\rho + (\partial^\mu A_\rho - \partial_\rho A^\mu)\partial^\rho A^\nu \\
&= (\partial^\mu A_\rho - \partial_\rho A^\mu)(\partial^\rho A^\nu - \partial^\nu A^\rho) \\
&= F^{\mu}{}_{\rho}F^{\rho\nu},
\end{aligned}$$

and since the equation of motion is

$$\begin{aligned}
0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu A_\nu} - \frac{\partial \mathcal{L}}{\partial A_\nu} \\
&= -\frac{1}{2}\partial_\mu(F^{\rho\sigma}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\sigma_\nu)) \\
&= -\frac{1}{2}\partial_\mu(F^{\mu\nu} - F^{\nu\mu}) \\
&= -\partial_\mu F^{\mu\nu},
\end{aligned}$$

we have $A^\nu\partial_\lambda F^{\mu\lambda} = 0$. So finally

$$\hat{T}^{\mu\nu} = F^{\mu}{}_{\rho}F^{\rho\nu} + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\eta^{\mu\nu}. \tag{27}$$

The second term is obviously symmetric, and for the first term we have

$$F^{\nu}{}_{\rho}F^{\rho\mu} = F^{\nu\rho}F_{\rho}{}^{\mu} = (-F^{\rho\nu})(-F^{\mu}{}_{\rho}) = F^{\mu}{}_{\rho}F^{\rho\nu},$$

so it is also symmetric. Therefore $\hat{T}^{\mu\nu}$ is a symmetric energy-momentum tensor.

Now we evaluate $\hat{T}^{\mu\nu}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. We have

$$\begin{aligned}
\mathcal{E} = \hat{T}^{00} &= F^{0\sigma}\eta_{\sigma\rho}F^{\rho 0} + \frac{1}{4}\eta_{\mu\sigma}\eta_{\nu\rho}F^{\mu\nu}F^{\sigma\rho} \\
&= -F^{0i}F^{i0} + \frac{1}{4}(\eta_{0\sigma}\eta_{\nu\rho}F^{0\nu}F^{\sigma\rho} + \eta_{i\sigma}\eta_{\nu\rho}F^{i\nu}F^{\sigma\rho}) \\
&= \boldsymbol{E}^2 + \frac{1}{4}(\eta_{\nu\rho}F^{0\nu}F^{0\rho} - \eta_{\nu\rho}F^{i\nu}F^{i\rho}) \\
&= \boldsymbol{E}^2 + \frac{1}{4}(-F^{0i}F^{0i} - F^{i0}F^{i0} + F^{ij}F^{ij}) \\
&= \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{4}F^{ij}F^{ij} = \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{4}\epsilon^{ijk}B^k\epsilon^{ijl}B^l \\
&= \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{4}(\delta^{jj}\delta^{lk} - \delta^{jk}\delta^{lj})B^kB^l \\
&= \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{4}\times 2\delta^{kl}B^kB^l = \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2,
\end{aligned}$$

which is the first equation in (23). Also,

$$\begin{aligned}
S^i = \hat{T}^{i0} &= F^{i\sigma}\eta_{\sigma\rho}F^{\rho 0} \\
&= F^{i\sigma}\eta_{\sigma j}F^{j0} = -F^{ij}F^{j0} \\
&= -(-\epsilon^{ijk}B^k)E^j \\
&= \epsilon_{ijk}E^jB^k = (\boldsymbol{E}\times\boldsymbol{B})^i,
\end{aligned}$$

which is the second equation in (23). So now (23) has been proved.