Quantum Optics by Prof. Saijun Wu

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1 Measurement of parameters in linear optics systems and the standard quantum error

Suppose we have a linear optics system with a few parameters, and we want to measure them according to the measurement results of photon numbers - which is almost the only observable that is really observable use present tools. Let $\bar{n}_l$ be the average observed photon number at detector l. If we do enough times of measurement, we have

$$\langle n_l \rangle = \bar{n}_l. \tag{1}$$

Let P be the collection of parameters, and $\bar{n}_l$ is a function of P. If we have an explicit expression $\bar{n}_l = \bar{n}_l(P)$, we must already have a computationally efficient model of the linear optics system, which is often hard to obtain. Anyway, if there are enough detectors, we have

$$P = n^{-1}(\bar{n}), \tag{2}$$

where $\bar{n}$ is the collection of photon number operators at all detectors. The error can then be estimated using standard error propagation techniques.

Consider a device like Figure 1 on page 1, where we use two detectors to measure the reflective coefficient of a beam splitter. Suppose we give an input consisting a few pulses, and the wave function of the system, under Heisenberg picture, is

$$|\psi\rangle = \prod_{j=1}^{N} a_j^\dagger |0\rangle = \prod_{j=1}^{N} \left(\cos\frac{\theta}{2} b_{j1}^\dagger + \sin\frac{\theta}{2} b_{j2}^\dagger \right). \tag{3}$$

We have

$$\langle n_1 \rangle = \sum_j \langle b_{j1}^\dagger b_{j1} \rangle = N \cos^2\frac{\theta}{2}, \quad \langle n_2 \rangle = \sum_j \langle b_{j2}^\dagger b_{j2} \rangle = N \sin^2\frac{\theta}{2}. \tag{4}$$

In principle we know everything about the incident light; in practice we do not, so from the equations above we have

$$\tan^2\frac{\theta}{2} = \frac{\langle n_1 \rangle}{\langle n_2 \rangle}, \tag{5}$$

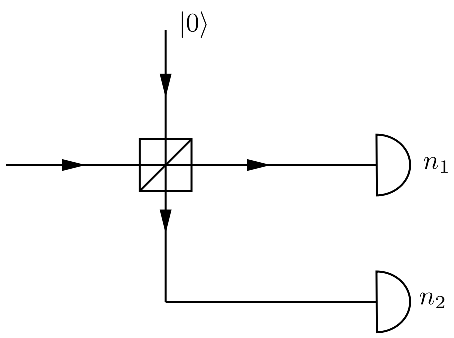

Figure 1: Measuring the reflective coefficient of a beam splitter

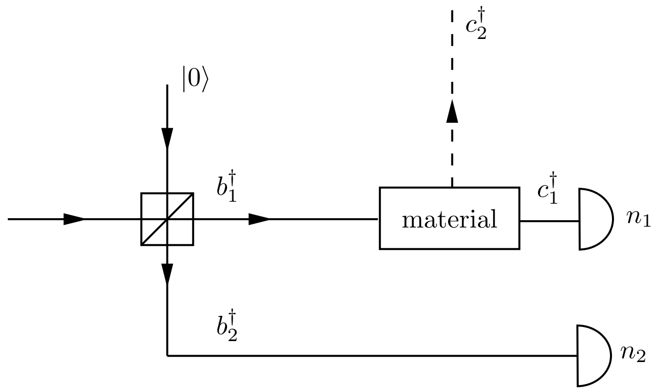

Figure 2: Measuring light absorption

where parameters that we may not know are not present. We can consider this as a trick to get rid of classical errors, like the (non-coherent and usually thermal) fluctuation of N. We can then evaluate the standard error of the two quantities, which are

$$\delta n_1 = \delta n_2 = \frac{1}{2}\sqrt{N}\sin\theta. \tag{6}$$

Thus

$$\delta\theta = \frac{\delta\bar{n}_1}{\partial\bar{n}_1/\partial\theta} = \frac{1}{2\sqrt{N}}. \tag{7}$$

We can also use a coherent light input, and the wave function of the whole system, again in the Heisenberg picture, is

$$|\psi\rangle = \prod_{i=1}^{N} \mathrm{e}^{\alpha_i a_i^\dagger - \alpha_i^* a_i} |0\rangle\,, \tag{8}$$

and we have

$$\langle n_1\rangle = N|\alpha|^2\cos^2\frac{\theta}{2}, \quad \langle n_2\rangle = N|\alpha|^2\sin^2\frac{\theta}{2}, \tag{9}$$

and again, we have

$$\tan^2\frac{\theta}{2}\frac{\langle n_1\rangle}{\langle n_2\rangle}. \tag{10}$$

The error of θ can be estimated as

$$\delta\theta = \frac{\delta\bar{n}_1}{\partial\bar{n}_1/\partial\theta} = \frac{1}{2\sqrt{N}|\alpha|\,\cos\frac{\theta}{2}}. \tag{11}$$

The errors in two cases, (7) and (11), have a shared factor: $1/\sqrt{N}$, or in other words usually we have $\delta n \sim \sqrt{N}$. This is called the **shot-noise error**. It arises from the quantum natural of the optical field, rather than technical issues (as we see, the discuss above involves nothing about "technical" errors). Since the final state of the optical field is usually not an eigenstate of the particle number operator, there is absolutely a fluctuation of photon number measurement, no matter high precise the device is.

The idea in Figure 1 on page 1 can be generated to some more realistic measurements. Consider we are analyzing light absorption in a certain kind of material, and we have an optical circuit depicted in Figure 2 on page 2. To make things easier we use a 50-50 beam splitter. Though the light absorption is a non-unitary process, we can view it as another beam splitter, where a part of the incident light is scattered to somewhere else. We can therefore derive

$$\langle n\rangle_1 = \frac{N|\alpha|^2}{2}\cos^2\frac{\theta}{2}, \quad \langle n\rangle_2 = \frac{1}{2}N|\alpha|^2. \tag{12}$$

Since

$$I = I_0\mathrm{e}^{-\rho\sigma L} =: \mathrm{e}^{-\mathrm{OD}}, \tag{13}$$

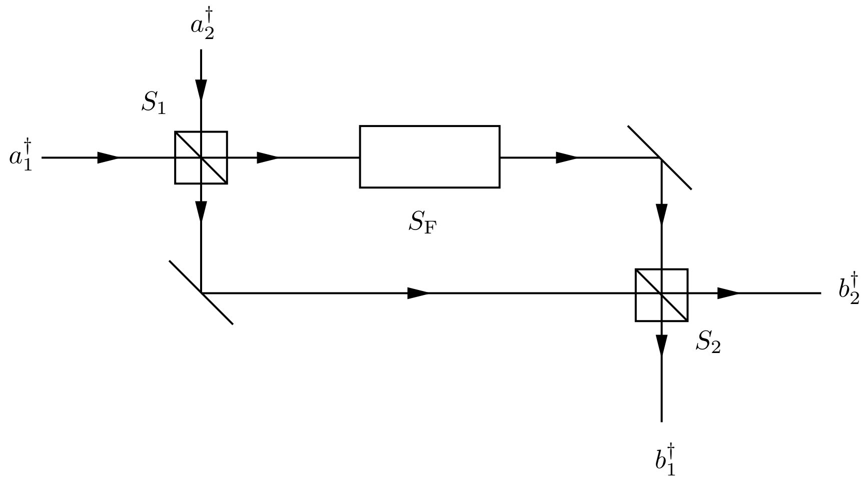

Figure 3: A Mach–Zehnder interferometer used to measure absorption

we have

$$\sigma = -\frac{1}{\rho L} \ln \frac{\langle n_1 \rangle}{\langle n_2 \rangle}. \tag{14}$$

It should be noted that large absorption cannot be measured in an easy way. We have

$$n = \frac{P\tau}{\hbar\omega},$$

and we have

$$\langle c_1^\dagger c_1 \rangle \simeq \frac{P\tau}{\hbar\omega} \mathrm{e}^{-\mathrm{OD}}. \tag{15}$$

When the absorption rate is large, $\langle c_1^\dagger c_1 \rangle$ can be quite small compared to its shot-noise error, and the measurement can be extremely unreliable. That is why absorption experiments are usually carried out with dilute solutions. But on the other hand, OD must be large to make sure

$$\frac{\delta\sigma}{\sigma} = \frac{\delta\mathrm{OD}}{\mathrm{OD}} \tag{16}$$

is small enough, or otherwise technical errors - like some small vibration of the device - will add some fluctuation to σ and OD and the result is not reliable again.

So now we are in a dilemma: we need a large OD for a small technical error, but on the other hand we need a small OD for a small shot-noise error. So in the end, the precision of the device cannot be improved unboundedly. The lower limit of the error is called the **standard quantum error**.

2 Measuring the position of a nanosphere with light interferometry

One way to eschew this problem is to measure the *phase*. We construct a Mach-Zehnder interferometer shown in Figure 3 on page 3. The two beam splitters are 50-50 ones, i.e. their transition matrix being

$$S_1 = S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{17}$$

The spacial propagating process can be described by

$$S_\mathrm{F} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}kL_1 - \varphi_1} & \\ & \mathrm{e}^{\mathrm{i}kL_2 + \varphi_2} \end{pmatrix} =: \text{unitary const} \times \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} & \\ & \mathrm{e}^{-\mathrm{i}\varphi/2} \end{pmatrix}. \tag{18}$$

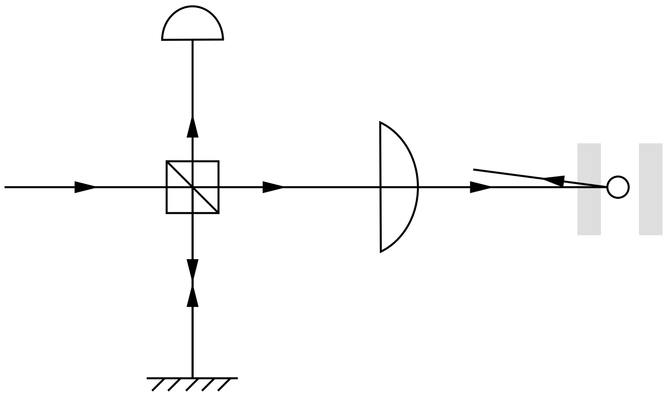

Figure 4: Measuring the position of a nanosphere with phase measurement

Ignoring unimportant factors, we have

$$\begin{aligned} n_1 &= \cos^2\frac{\varphi}{2}a_1^\dagger a_1 + \sin^2\frac{\varphi}{2}a_2^\dagger a_2 + \frac{1}{2}\sin\varphi(a_1^\dagger a_2 + a_2^\dagger a_1), \\ n_2 &= \cos^2\frac{\varphi}{2}a_2^\dagger a_2 + \sin^2\frac{\varphi}{2}a_1^\dagger a_1 - \frac{1}{2}\sin\varphi(a_1^\dagger a_2 + a_2^\dagger a_1). \end{aligned} \tag{19}$$

By measuring n_1 and n_2, we are able to measure φ, and hence the absorption.

The phase measure technique can also be used to measure an unknown position in the light circuit. For example, consider Figure 4 on page 4, where a nanosphere is contained in a light trap and we want to use an light beam to determine where exactly it is. The equation of motion is

$$m\ddot{x} = -\omega_0^2 x_1 + F, \tag{20}$$

where teh radiational force is roughly

$$F = \frac{P}{c} \sim \frac{SA}{c} \sim a^\dagger a|\mathcal{E}_l|^2 A, \tag{21}$$

where P is the light power, S the Poynting vector, and A is some "effective area" of the nanosphere. To reflect the fact that light is made of photons, we replace the F in (20) with its expectation plus a classical fluctuation, i.e.

$$F(t) = \langle F\rangle + \delta F = \frac{|\mathcal{E}_l|^2|\alpha|^2 A}{c} + \delta F\,, \tag{22}$$

where

$$\langle \delta F(t)\,\delta F(t')\rangle = \hbar k\,\langle F\rangle\,\delta(t-t') =: D\delta(t-t'). \tag{23}$$

The errors are

$$(\delta x)_{\text{shot noise}} = \frac{1}{k}\,\delta\varphi = \frac{1}{k}\frac{1}{2\sqrt{\frac{P\tau}{\hbar\omega}}}, \quad (\delta x)_{\text{radiation force}} \sim \sqrt{\hbar k\frac{P\tau}{c}\frac{\tau}{m}}. \tag{24}$$

The total error is

$$\delta x = \sqrt{(\delta x)^2_{\text{shot noise}} + (\delta x)^2_{\text{radiation force}}} \tag{25}$$

and we may find its minimum is

$$\delta x_{\min} \sim \sqrt{\frac{\hbar\tau}{m}}. \tag{26}$$

So this is a quantitative estimation of the standard quantum error.

Of course, the length measurement technique developed for nanospheres is definitely not limited to this usage, and actually this idea is key to *gravitational wave measurement*. When gravitational waves in certain modes go by the interferometer, we have

$$x_1 = x_1(0) + h(t)L, \quad x_2 = x_2(0) - h(t)L. \tag{27}$$

The measurement of the position of a nanosphere and the gravitational wave measurement both reveal a fact that simply increase the power cannot lift the precision unboundedly. Some part of the measurement error arises from the quantum nature of light.

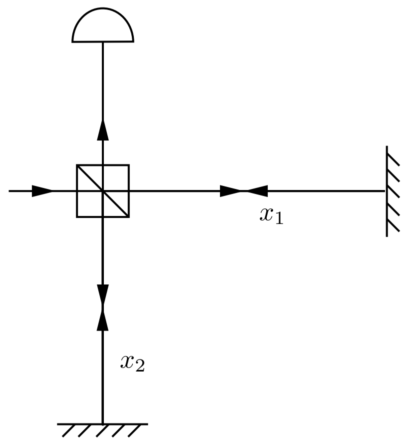

Figure 5: Gravitational wave measurement

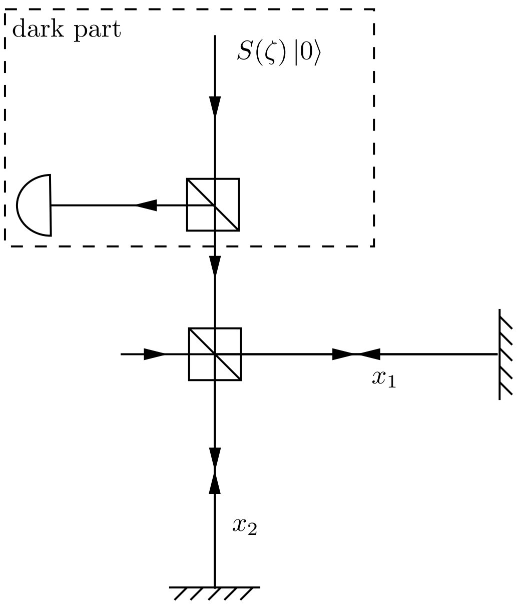

Figure 6: One solution beyond the standard quantum limit

3 Beyond the standard quantum limit

To go beyond the standard quantum limit we can make use of the other input port, i.e. *the dark port* of the beam splitter, for example we can use a device shown in Figure 6 on page 5, where

$$S(\zeta) = \mathrm{e}^{|\zeta|(a^\dagger a^\dagger - aa)} \tag{28}$$

is the squeezing operator. We define

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad P = \frac{1}{\sqrt{2}}(a - a^\dagger), \tag{29}$$

and we find that

$$S^\dagger(\zeta) X S(\zeta) = X(\zeta), \quad S^\dagger(\zeta) P S = P(\zeta), \tag{30}$$

and

$$\frac{\mathrm{d}X(\zeta)}{\mathrm{d}\zeta} = -X. \tag{31}$$

Measuring n_1, we have

$$\delta n_1 = |\alpha|\, \mathrm{e}^{-\zeta}, \tag{32}$$

so we find tha the noise is *squeezed* by the injection of $S(\zeta)\,|0\rangle$.

Nonlinear interferometry and entangled state.