Diffraction and Scattering in Electrodynamics by Prof. Kun Din

Jinyuan Wu

November 24, 2021

In previous lectures we only discussed boundaries at infinity. The details of boundaries do not affect the electromagnetic waves. Their role - if any - is to “guide” the electromagnetic waves, as is the case in spherical optical cavities.

In this lecture we are going to discuss the physics of boundaries in electrodynamics. Suppose κ to be the magnitude of the curvature of a boundary, which estimates its geometric details. When $\kappa\lambda \ll 1$, the boundary is almost flat in the eye of the light, and the theory on what happens around the boundary is *geometrical optics*. On the other hand, when $\kappa\lambda \gg 1$, the details of the wave fronts of electromagnetic waves are not clear in the eye of the boundary, and this case is called **Rayleigh scattering**. Rayleigh scattering usually happens when small particles are placed in light, and what the particles feel is almost a homogeneous variating potential.

Between the two limits are **scattering** and **diffraction**. Scattering is defined as the phenomenon that the electrodynamic modes couple with inner modes of matter strongly, and the momenta of both change significantly. Diffraction, on the other hand, means that light beams are split by the boundary, and there is no strong change on momentum, but the distribution in the momentum space changes (for example, if a plane wave is injected into a system, the direction of the output light beam is still the same, but the intensity now has spatial distribution, or in other words we can see a smooth distribution in the momentum space).

1 The general theory of scattering

Consider a optical system is placed in vacuum, and we investigate its behavior when light is shed on it. Energy conservation gives

$$-\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{\nabla}\cdot\boldsymbol{S} - \frac{\partial u}{\partial t} = \int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{j}\cdot\boldsymbol{E}, \tag{1}$$

where the volume is the volume of the system and the boundary is the boundary of the system. We make the decomposition

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{in}} + \boldsymbol{E}_{\mathrm{sca}}, \quad \boldsymbol{H} = \boldsymbol{H}_{\mathrm{in}} + \boldsymbol{H}_{\mathrm{sca}}, \tag{2}$$

and from now on we work with monochromic light. In this way,

$$\langle\boldsymbol{S}\rangle = \frac{1}{2}\operatorname{Re}\boldsymbol{E}\times\boldsymbol{H}^* = \frac{1}{2}\operatorname{Re}(\boldsymbol{E}_{\mathrm{in}} + \boldsymbol{E}_{\mathrm{sca}})\times(\boldsymbol{H}^*_{\mathrm{in}} + \boldsymbol{H}^*_{\mathrm{sca}}),$$

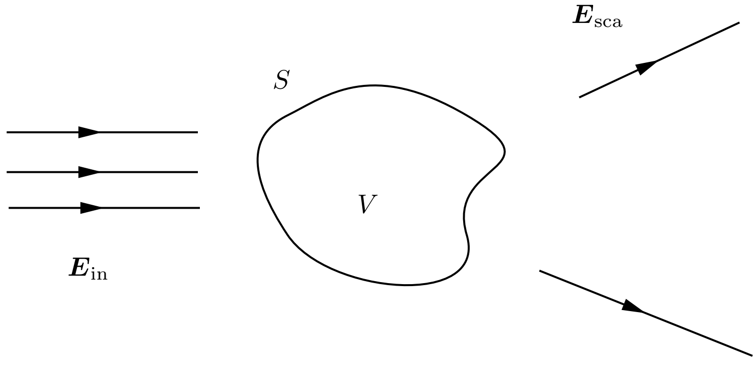

Figure 1: Scattering system

and we define

$$\langle \boldsymbol{S}_{\text{in}} \rangle = \frac{1}{2} \operatorname{Re} \boldsymbol{E}_{\text{in}} \times \boldsymbol{H}_{\text{in}}^*, \tag{3}$$

$$\langle \boldsymbol{S}_{\text{sca}} \rangle = \frac{1}{2} \operatorname{Re} \boldsymbol{E}_{\text{sca}} \times \boldsymbol{H}_{\text{sca}}^*, \tag{4}$$

and

$$\langle \boldsymbol{S}_{\text{ext}} \rangle = \frac{1}{2} \operatorname{Re}(\boldsymbol{E}_{\text{sca}} \times \boldsymbol{H}_{\text{in}}^* + \boldsymbol{E}_{\text{in}} \times \boldsymbol{H}_{\text{sca}}^*). \tag{5}$$

The last term is called **extinction**, because it depicts how scattering interrupts the energy flow of the input light. We use W to denote the power of energy flow $\boldsymbol{S}$, i.e. the flux of the flow on a certain surface.

We often define

$$C = \frac{W}{I} \tag{6}$$

as the **cross section**. By energy conservation we have

$$C_{\text{ext}} = C_{\text{sca}} + C_{\text{abs}}, \tag{7}$$

so the extinction cross section is also the **total cross section**, including both scattering and absorption effects. In geometrical optics, the total cross section cannot be larger than the geometric cross section of the absorber, but when the wave nature of light is taken into account, the total cross section can be larger - the diameter of the total cross section may be several wavelengths larger than the geometric diameter. This is often quite hard in engineering. The scattering cross section, even in geometrical optics, can be as large as twice of the geometric cross section. Actually in geometrical optics, the total cross section is always twice of the geometrical cross section.

When $\boldsymbol{E}_{\text{in}}$ is a plane wave, the asymptotic behavior of $\boldsymbol{E}$ as $|\boldsymbol{r}| \to \infty$ is

$$\boldsymbol{E} = E_0 \hat{\boldsymbol{e}}_0 \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}} + E_0 \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{f}(\boldsymbol{k}). \tag{8}$$

We are sure that the spacial dependence is in the form of $\mathrm{e}^{\mathrm{i}kr}/r$ because when $|\boldsymbol{r}| \to 0$ the electromagnetic waves can be seen as spherical waves (the center of the sphere being the optical system where scattering happens). Information about the optical system, i.e. the angular distribution of scattered electromagnetic waves, the amplitude and the polarization is stored in $\boldsymbol{f}(\boldsymbol{r})$.

Since outside the optical system there is no current, we have

$$\boldsymbol{H} = \frac{\hat{\boldsymbol{k}} \times \boldsymbol{E}}{Z_0}, \quad Z_0 = \sqrt{\frac{\mu}{\epsilon}}. \tag{9}$$

Note that if the optical system in question is an array of smaller optical components, then Z_0 *cannot* be obtained from the photon band structure of the array - we can only read the refractive index from the band structure. We have to evaluate E/H in that array to decide Z_0. Then we find that

$$\begin{aligned} W_{\text{ext}} &= -\frac{1}{2} \oint \mathrm{d}S \, \hat{\boldsymbol{k}} \cdot \operatorname{Re}(\boldsymbol{E}_{\text{in}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}_{\text{sca}}^*) + \boldsymbol{E}_{\text{sca}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}_{\text{in}}^*)) \\ &= -\frac{E_0^2}{2Z_0} \oint \frac{1}{r} \operatorname{Re}((\hat{\boldsymbol{e}}_0^* \cdot \boldsymbol{f}^*) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}} \mathrm{e}^{-\mathrm{i}kr} + (\hat{\boldsymbol{e}}_0 \cdot \boldsymbol{f})(\hat{\boldsymbol{k}}_0 \cdot \hat{\boldsymbol{k}}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}} \mathrm{e}^{\mathrm{i}kr} \\ &\quad - (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{e}}_0)(\hat{\boldsymbol{k}} \cdot \boldsymbol{f}^*) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}} \mathrm{e}^{-\mathrm{i}kr} - (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{e}}_0^*)(\hat{\boldsymbol{k}} \cdot \boldsymbol{f}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}} \mathrm{e}^{\mathrm{i}kr}) \, \mathrm{d}S \, . \end{aligned} \tag{10}$$

Note that the factor $E_0^2/(2Z_0)$ is just the intensity of the input light, i.e.

$$I_{\text{in}} = \frac{1}{2} \epsilon_0 E_0^2 c, \tag{11}$$

and $\mathrm{d}S$ contains r^2 and a solid angle element $\mathrm{d}\Omega$, so we can evaluate

$$\int_{-\pi}^{\pi} \cos\theta \, \mathrm{d}\theta \, \mathrm{e}^{\pm \mathrm{i} \cos\theta kr}$$

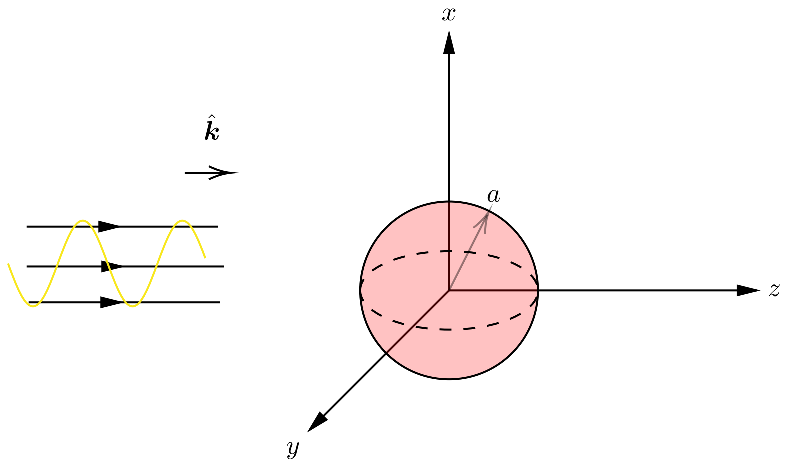

Figure 2: Mie scattering

factors in (10). Note that (10) is only correct when we are far from the optical system, so we just need to take the The final result is

$$C_{\text{in}} = \frac{W_{\text{ext}}}{I_{\text{in}}} = \frac{4\pi}{k} \operatorname{Im} \boldsymbol{e}_0^* \cdot \boldsymbol{f}(\hat{\boldsymbol{k}}_0). \tag{12}$$

This formula is called the **optical theorem**. This is a result of energy conservation, or more generally, a result of *unitarity*. (12) is actually a specific version of the optical theorem for plane waves, and it breaks for a highly convergent light beam. The generalized version of the optical theorem can be found in Born's famous textbook.

Similar to (10), we have

$$C_{\text{sca}} = \frac{1}{2Z_0 I_{\text{in}}} \oint \mathrm{d}S\, \hat{\boldsymbol{k}} \cdot \operatorname{Re} \boldsymbol{E}_{\text{sca}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}_{\text{sca}}) = \int_{4\pi} |\boldsymbol{f}(\boldsymbol{k})|^2 \, \mathrm{d}\Omega\,, \tag{13}$$

2 Mie scattering

Mie scattering is among few examples that can be solved exactly. It studies a sphere made of dielectric. The scattered fields are expanded using spherical functions:

$$\boldsymbol{E}_{\text{sca}} = \sum_{n=1}^{\infty} E_n (\mathrm{i} a_n \boldsymbol{N}_{e1n}^{(3)} - b_n \boldsymbol{M}_{o1n}^{(3)}), \quad \boldsymbol{H}_{\text{sca}} = \frac{k}{\omega\mu} \sum_{n=1}^{\infty} E_n (\mathrm{i} b_n \boldsymbol{N}_{o1n}^{(3)} + a_n \boldsymbol{M}_{e1n}^{(3)}), \tag{14}$$