Quantum Optics, Homework 3

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Stochastic wave function of a Λ system Figure 1 is a three-level Λ system. (a) Write down the effective Hamiltonian and quantum jump operators for Figure 1. (b) Suppose $|\psi_{\mathrm{s}}(t=0)\rangle = |\mathrm{g}\rangle$. Describe how the wave function evolves using pseudocode. (c) Consider a case in which there is no quantum jump in $0 < t < t_0$. Find the time evolution of the wave function and the scattering rate

$$\gamma_1 = \langle\psi_{\mathrm{s}}|C_1^\dagger C_1|\psi_{\mathrm{s}}\rangle\,, \quad \gamma_2 = \langle\psi_{\mathrm{s}}|C_2^\dagger C_2|\psi_{\mathrm{s}}\rangle\,. \tag{1}$$

Solution

(a) The effective Hamiltonian is

$$\begin{aligned} H_{\mathrm{eff}} &= -\hbar\Delta\,|\mathrm{e}\rangle\langle\mathrm{e}| + \left(\frac{1}{2}\hbar\Omega\,|\mathrm{e}\rangle\langle\mathrm{g}| + \mathrm{h.c.}\right) - \frac{\mathrm{i}\hbar}{2}(C_1^\dagger C_1 + C_2^\dagger C_2) \\ &= -\hbar(\Delta + \mathrm{i}\Gamma/2)\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hbar(\Omega\,|\mathrm{e}\rangle\langle\mathrm{g}| + \mathrm{h.c.})/2, \end{aligned} \tag{2}$$

where the quantum jump operators are

$$C_1 = \sqrt{\Gamma_1}\,|\mathrm{a}\rangle\langle\mathrm{e}|\,, \quad C_2 = \sqrt{\Gamma_2}\,|\mathrm{g}\rangle\langle\mathrm{e}|\,, \tag{3}$$

and

$$\Gamma = \Gamma_1 + \Gamma_2. \tag{4}$$

(b) The time evolution can be described using the following algorithm.

input : Time step Δt, maximal time t_0

Initialize an array $\{|\psi_{\mathrm{s}}(t)\rangle\}_{t=n\Delta t}$ of wave functions with $t_0/\Delta t$ elements

for $t \in 0 : \Delta t : t_0$

  Pick up a uniformly distributed random number x between 0 and 1

  $P_{\mathrm{g}} \leftarrow \Delta t\,\langle\psi_{\mathrm{s}}(t)|C_1^\dagger C_1|\psi_{\mathrm{s}}(t)\rangle$

  $P_{\mathrm{a}} \leftarrow \Delta t\,\langle\psi_{\mathrm{s}}(t)|C_2^\dagger C_2|\psi_{\mathrm{s}}(t)\rangle$

  `// jumping to` $|\mathrm{g}\rangle$

  if $0 < x < P_{\mathrm{g}}$

    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $C_1\,|\psi_{\mathrm{s}}(t)\rangle$

  `// jumping to` $|\mathrm{a}\rangle$

  elseif $P_{\mathrm{g}} < x < P_{\mathrm{g}} + P_{\mathrm{a}}$

    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $C_2\,|\psi_{\mathrm{s}}(t)\rangle$

  `// evolution according to the effective Hamiltonian`

  else

    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $|\psi_{\mathrm{s}}(t)\rangle + \frac{\Delta t}{\mathrm{i}\hbar}H_{\mathrm{eff}}\,|\psi_{\mathrm{s}}(t)\rangle$

  end

end

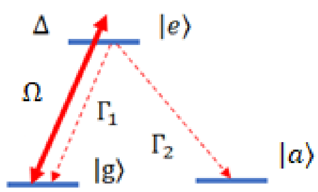

Figure 1: A three-level Λ system

(c) The wave function in this case evolves purely according to H_{eff}. Since Schrödinger equation is linear, we can leave the normalization to the end of our calculation. Note that (2) actually does not contain $|\mathrm{a}\rangle$ explicitly, nor does the initial state $|\mathrm{g}\rangle$. Therefore we can work in the two-level system spanned by $|\mathrm{e}\rangle$ and $|\mathrm{g}\rangle$. The effective Hamiltonian is

$$H_{\text{eff}} = \hbar \begin{pmatrix} 0 & \Omega^*/2 \\ \Omega/2 & -(\Delta + \mathrm{i}\Gamma/2) \end{pmatrix}, \tag{5}$$

where we let $|\mathrm{g}\rangle$ be the first component and $|\mathrm{e}\rangle$ the second. We have the decomposition

$$H_{\text{eff}} = -\frac{\hbar}{2}(\Delta + \mathrm{i}\Gamma) + \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\Omega} = (\Omega_{\mathrm{r}}, \Omega_{\mathrm{i}}, \Delta + \mathrm{i}\Gamma/2). \tag{6}$$

Note here we cannot "shift the energy zero point" to reshape the Hamiltonian into $\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$, because the value damping rate has physical meaning. Applying (6) on $|\mathrm{g}\rangle$, we have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}H_{\text{eff}}t/\hbar}|\mathrm{g}\rangle &= \mathrm{e}^{\mathrm{i}t(\Delta+\mathrm{i}\Gamma/2)/2}\mathrm{e}^{-\mathrm{i}t\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}/2}|\mathrm{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\sigma^0\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \frac{\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\mathrm{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2}|\mathrm{g}\rangle - \left(\frac{\Omega_{\mathrm{r}}}{|\boldsymbol{\Omega}|}|\mathrm{e}\rangle + \frac{\mathrm{i}\Omega_{\mathrm{i}}}{|\boldsymbol{\Omega}|}|\mathrm{e}\rangle + \frac{\Delta+\mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}|\mathrm{g}\rangle\right)\mathrm{i}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right),
\end{aligned}$$

where

$$|\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2 - \Gamma^2/4 + \mathrm{i}\Delta\Gamma}. \tag{7}$$

Note

Note that here $|\boldsymbol{n}|$ is defined as $\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}$ instead of $\sqrt{\boldsymbol{n}^*\cdot\boldsymbol{n}}$, because to make

$$\mathrm{e}^{\mathrm{i}\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}} = \sigma^0\cos\alpha + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\alpha$$

hold, it is required that

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \sigma^0,$$

which is equivalent to $\boldsymbol{n}\cdot\boldsymbol{n} = 1$, considering $\{\sigma^i, \sigma^j\} = 0$ when $i \neq j$. What is important here, therefore, is $\boldsymbol{n}\cdot\boldsymbol{n}$.

Therefore we have (we have omitted the complex factors, since they will be canceled by normalization anyway)

$$|\psi_{\mathrm{s}}(t)\rangle \propto \left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \mathrm{i}\frac{\Delta+\mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\mathrm{g}\rangle - \frac{\mathrm{i}\Omega}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}|\mathrm{e}\rangle, \tag{8}$$

and after normalization it is

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Cesium atom

EIT-assisted giant Kerr effect The "lambda"-system composed of $|\mathrm{a}\rangle, |\mathrm{e}\rangle, |\mathrm{b}\rangle$ is further coupled to excited state $|\mathrm{c}\rangle$, as in Fig. 1. We consider the situation of EIT-resonance: $\delta = 0$. We further consider atomic state to be initially in $|\psi(t=0)\rangle = |\mathrm{a}\rangle$, and weak-excitation limit is satisfied ($|\Omega_1|$ small "enough"). 3a) Write down the effective Hamiltonian for this problem for $\delta = 0$ 3b) Obtain the approximate stochastic wavefunction in its steady state $\left|\widetilde{\psi_{\mathrm{S}}}\right\rangle = |\mathrm{a}\rangle + c_{\mathrm{e}}|\mathrm{e}\rangle + c_{\mathrm{b}}|\mathrm{b}\rangle + c_{\mathrm{c}}|\mathrm{c}\rangle$ such that $\mathrm{H}_{\text{eff}}\,|\psi_{\mathrm{S}}\rangle \approx 0$

Solution

(a)