Quantum Optics by Prof. Saijun Wu

Jinyuan Wu

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In previous lectures we are discussing atoms in a given optical field. Now we turn to discuss the optical field itself in a cavity.

We consider such a system: We have a cavity, in which there is an atom (or a small sphere, etc.), and only one optical mode we are interested in is coupled to the atom. There are other optical modes coupled to the atom, but we are not that interested in them, and therefore they effectively create a damping channel for the mode we are interested in. A beam of light is shed on the atom.

The Hamiltonian of photons can be obtained by solving the optical modes of the cavity, and the electric field component of mode c - the mode we are interested in - is

$$\boldsymbol{E}_c(\boldsymbol{r}, t) = \sqrt{\frac{\hbar\omega_c}{2\epsilon_0 V}} f(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\omega_c t} + \text{c.c.} \tag{1}$$

For example, for a Fabry-Perot cavity, we have

$$\boldsymbol{E}_c(\boldsymbol{r}, t) = \sqrt{\frac{\hbar\omega_c}{2\epsilon_0 V}} f(x, y) \cos(k_z z) \mathrm{e}^{-\mathrm{i}\omega_c t} + \text{c.c.} \tag{2}$$

We take a inverse approach of what has been used in previous lectures, trying not to explicitly include the atom. Note, however, that the atom is not a many-body system and the inner states of it cannot be simply ignored, and therefore the atom is not a fixed external deriving field. We assume a large detuning, and the atom just "move according to the external field", and in this way

$$\langle \boldsymbol{d} \rangle \approx \alpha \boldsymbol{E}_k. \tag{3}$$

In this way we do not need to rigorously integrate out the atom. Therefore the interaction Hamiltonian is effectively

$$V = \sum_k \alpha \boldsymbol{\mathcal{E}}(\boldsymbol{r}_0)^* \boldsymbol{\mathcal{E}}_k(\boldsymbol{r}_\mathbf{0}) a_k a^\dagger + \text{h.c.}, \tag{4}$$

where a_k is the annihilation operator of a mode we are not interested in, and we have assume only *one* mode in the cavity is strongly coupled to the atom. After RWA, the interaction Hamiltonian is

$$H_{\text{int}} = \sum_k g_k \mathrm{e}^{-\mathrm{i}\Delta_k} a^\dagger a_k + \text{h.c.}, \quad \Delta_k = \omega - \omega_k, \quad g_k = \frac{\alpha}{\hbar} \boldsymbol{\mathcal{E}}^*(\boldsymbol{r}_0) \boldsymbol{\mathcal{E}}_k(\boldsymbol{r}_0). \tag{5}$$

Now we solve the Hamiltonian. We work in the subspace where the photon number change is limited to 1, i.e.

$$|\psi(t)\rangle = c_n |n, 0\rangle + c_k |n-1, 1_k\rangle. \tag{6}$$

Unlike the case in the two-level atom system, this approximation may fail when the time is sufficiently long, but we can assume that the environment is noisy enough so that an observation has already taken place before a second photon transition occurs. The time evolution equation turns into

$$\mathrm{i}\dot{c}_n = \sqrt{n} \sum_k g_k \mathrm{e}^{\mathrm{i}\Delta_k t} c_k, \quad \mathrm{i}\dot{c}_k = \sqrt{n} g_k^* \mathrm{e}^{-\mathrm{i}\Delta_k t} c_n. \tag{7}$$

Again we can make a Markovian approximation. We have

$$\begin{aligned}
\mathrm{i}\dot{c}_n &= \int_0^t \mathrm{d}\tau \, n \sum_k |g_k|^2 \mathrm{e}^{\mathrm{i}\Delta_k (t-\tau)} c_n(\tau) \\
&=: n \int_0^t \mathrm{d}\tau \, K(t-\tau) c(\tau),
\end{aligned}$$

and by the Markovian approximation, we have

$$\dot{c}_n(t) = n(\delta_c - \mathrm{i}\kappa/2)c_n(t), \tag{8}$$

where

$$\kappa = 2\pi \sum_k |g_k|^2 \delta(\omega - \omega_k) \sim \frac{\alpha^2 k_c^3 |\boldsymbol{\mathcal{E}}_c|}{\hbar \epsilon_0}, \tag{9}$$

and

$$\delta_c \sim \mathrm{P} \int_{-\infty}^{\infty} \rho(\omega) |g_k|^2 \frac{1}{\Delta_k} \, \mathrm{d}k \,. \tag{10}$$

Therefore we get an effective damping Hamiltonian

$$H_{\text{eff}} = -\frac{\mathrm{i}n\kappa}{2} |n\rangle\langle n| \,, \quad C = \sqrt{n\kappa} \, |n-1\rangle\langle n| \,. \tag{11}$$

Now we generalize to the second quantized formulation without explicit dependence on the photon number, and we have

$$H_{\text{eff}} = -\frac{\mathrm{i}\kappa}{2} a^\dagger a, \quad C = \sqrt{\kappa} a. \tag{12}$$

The probability of a quantum jump is

$$\delta P_{\text{jump}} = \langle \psi_{\mathrm{s}}(t) | C^\dagger C | \psi_{\mathrm{s}}(t) \rangle \, \delta t = \langle n \rangle \, \kappa \, \delta t =: \gamma \, \delta t \,. \tag{13}$$

The master equation is therefore

$$\dot{\rho} = -\frac{\kappa}{2} \left(a^\dagger a \rho + \rho a^\dagger a - 2 a \rho a^\dagger \right) . \tag{14}$$

If we turn the external driving field on, the Hamiltonian is now

$$H_{\text{eff}} = -\frac{\mathrm{i}\kappa}{2} a^\dagger a + \hbar g \alpha \mathrm{e}^{\mathrm{i}\Delta_c t} + \text{h.c.}. \tag{15}$$

Plotting the time evolution according to (15) gives us a quite intuitive picture. We see the last two terms correspond to a displacement operator, and the first term drags a coherent state to the vacuum. If the input state is highly non-classical, the damping term also destroys quantum interference. For example, for an input cat state

$$|\psi(0)\rangle \approx \frac{1}{\sqrt{2}} (\, |\alpha \mathrm{e}^{-\kappa t/2}\rangle + |-\alpha \mathrm{e}^{-\kappa t/2}\rangle),$$

and after a quantum jump we have

$$|\psi(t)\rangle \approx \frac{1}{\sqrt{2}} (\, |\alpha \mathrm{e}^{-\kappa t/2}\rangle - |-\alpha \mathrm{e}^{-\kappa t/2}\rangle).$$

After several quantum jumps, the pure input state becomes a mixed state where the non-trivial relative phase factor between the two components of the wave functions vanish, and a cat state damps into a classical 1/2 - 1/2 probability distribution.