Dirac Theory

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1 The Dirac equation

We have found spinors are representations of Lorentz group in this article.

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{1}$$

2 Free-particle solutions of the Dirac equation

Now we solve (1). We work under the chiral basis. We search for a plane wave solution

Peskin Section 3.3.

$$\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}, \quad p^2 = m^2. \tag{2}$$

In the rest frame, $p = p_0 = (m, 0)$, and (1) becomes

$$(m\gamma^0 - m)u(p_0) = m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0,$$

and the solutions are

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \tag{3}$$

for *any* Weyl spinor ξ. The normalization condition is $\xi^\dagger \xi = 1$, and

$$\xi_{S_z=\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{S_z=\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4}$$

We can find the solution for an arbitrary momentum by boosting. From (6) and (12) in this article, we have

$$(\mathcal{J}^{03})^\alpha{}_\beta = \mathrm{i}(\eta^{0\alpha}\delta^3{}_\beta - \delta^0{}_\beta \eta^{3\alpha}) = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$S^{03} = -\frac{\mathrm{i}}{2}\begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix},$$

and therefore these two transformations

$$\exp\left(\eta\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right), \quad \exp\left(\frac{-\eta}{2}\begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}\right)$$

are the same Lorentz group element acting on different objects. To boost $(m, 0)$ into (E, p^3), one need an *eta* defined by

Peskin (3.48)

$$\begin{aligned} \begin{pmatrix} E \\ p^3 \end{pmatrix} &= \exp\left[\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \left[\cosh\eta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}, \end{aligned} \tag{5}$$

and therefore $u(p)$ is obtained by a boost on the z direction with the same η, which is Peskin (3.49)

$$\begin{aligned}
u(p) &= \exp\left[-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \left[\cosh\left(\frac{1}{2}\eta\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\left(\frac{1}{2}\eta\right)\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right)+e^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right) & 0\\ 0 & e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right)+e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right)\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \qquad (6)\\
&= \begin{pmatrix}\left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right)+\sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi\\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right)+\sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi\end{pmatrix},
\end{aligned}$$

where the last line comes from the fact that since (5), we have

$$E+p^3 = m(\cosh\eta+\sinh\eta) = me^{\eta}, \quad E-p^3 = m(\cosh\eta-\sinh\eta) = me^{-\eta}.$$

The last line gives Peskin (3.50)

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix}, \qquad (7)$$

where $\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma})$. To prove (7), we note that Peskin (3.41)

$$\left(\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right)+\sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right)^2 = \begin{pmatrix}E-p^3 & \\ & E+p^3\end{pmatrix} = p^0\sigma^0 - p^3\sigma^3 = p\cdot\sigma,$$

and the same thing hold for $\sqrt{p\cdot\bar{\sigma}}$.

3 Symmetries of the Dirac theory

This section is based on Peskin Section 3.6.