Quantum Optics by Prof. Saijun Wu

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We consider the following non-Hermitian effective Hamiltonian:

$$\tilde{H}_{\text{eff}} = \left(H_{\text{SE}} - \sum_e \text{i}\hbar \frac{\Gamma_e}{2} |e\rangle\langle e| \right) |0\rangle\langle 0| + \hbar \sum_e g^*_{\text{eg,k}} \text{e}^{-\text{i}\Delta_k t} a^\dagger_k |g, 0\rangle\langle e, 0| . \tag{1}$$

We find

1 Random wave function approach to a three-level system

Now we consider three-level systems. We try to use the technique in Section 3 in the last lecture to describe these systems. It should be noted that in Section 3 in the last lecture we can guess the form of the collapse operator since there are only one damping channel, but in a three-level system, we need to determine the collapse operators according to how emitted photons are observed. For example, for a V-type system, two excited states may jump to one ground state. If we assume that observers in the environment can only detect the photon number, then there is only one collapse operator:

$$C = \sqrt{\Gamma_{eg}} |g\rangle\langle e| + \sqrt{\Gamma_{ag}} |g\rangle\langle a| . \tag{2}$$

If the environment can also see the color of the emitted photon, then we have two collapse operators

$$C_1 = \sqrt{\Gamma_{eg}} |g\rangle\langle e| , \quad C_2 = \sqrt{\Gamma_{ag}} |g\rangle\langle a| . \tag{3}$$

(2) and (3) give different damping Hamiltonian. (3) gives

$$H_{\text{damping}} = -\frac{\text{i}\hbar}{2} (\Gamma_{eg} |e\rangle\langle e| + \Gamma_{ag} |g\rangle\langle g|), \tag{4}$$

while (2) adds two cross terms.

A good demonstration of the physical consequence of this fact can be seen as follows. Suppose there is no quantum jump before $t = t_0$, and we have

$$|\psi_{\text{s}}(t)\rangle = \frac{1}{C} (\text{e}^{-\text{i}\omega_e t - \Gamma_e t/2} |e\rangle + \text{e}^{-\text{i}\omega_a t - \Gamma_a t/2} |a\rangle), \quad |C|^2 = \text{e}^{-\Gamma_e t} + \text{e}^{-\Gamma_a t}, \tag{5}$$

and the probability of quantum jump (i.e. spontaneous radiation) is

$$\gamma(t) = \frac{\text{d}P_{\text{emission}}}{\text{d}t} = \frac{1}{\text{e}^{-\Gamma_e t} + \text{e}^{-\Gamma_a t}} (\Gamma_e \text{e}^{-\Gamma_e t} + \Gamma_a \text{e}^{-\Gamma_a t} + 2\sqrt{\Gamma_e \Gamma_a} \text{e}^{-(\Gamma_e + \Gamma_a)t/2} \cos(\omega_e - \omega_a)t). \tag{6}$$

In the specific case of $\Gamma_a = \Gamma_e = \Gamma$, we have

$$\gamma(t) = 1 + \text{e}^{-\Gamma t} \cos(\Delta\omega t). \tag{7}$$

The oscillating form of γ is called **quantum beat**. It comes from the fact that the quantum jump $a \to g$ and $e \to g$ are indistinguishable.

We can also calculate the survival ratio until $t = t_0$, i.e.

$$\prod_{j=1}^N (1 - \delta P_{\text{j}}(t_j)) \to \text{e}^{-\int_0^t \gamma(\tau) \text{d}\tau}. \tag{8}$$

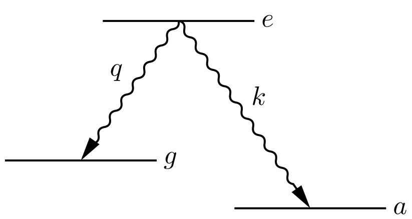

Figure 1: The energy level diagram of a Λ-type three-level system

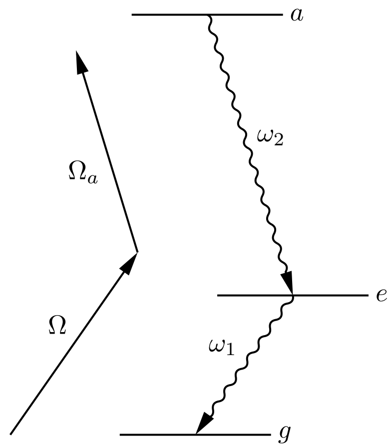

Figure 2: A ladder-type three-level system

2 Linear response in a two-level system

We consider a two level system with an external field, the Hamiltonian of which is

$$H = H_0 + \frac{\hbar}{2}\Omega \, |e\rangle\langle g| + \text{h.c.}, \tag{9}$$

and we take spontaneous radiation into account, so we work with

$$H_{\text{eff}} = H_0 - \frac{\mathrm{i}\Gamma_e}{2} \, |e\rangle\langle e| + \frac{\hbar\Omega}{2} \, |e\rangle\langle g| + \text{h.c.}. \tag{10}$$

We again use the random wave function approach, and we have

$$\mathrm{i}\dot{c}_e = \left(\Delta - \frac{\mathrm{i}\Gamma}{2}\right) c_e + \frac{\Omega}{2} c_g, \quad \mathrm{i}\dot{c}_g = \frac{\Omega^*}{2} c_e. \tag{11}$$

We find a stable solution

$$c_g \approx 1, \quad c_e = -\frac{\Omega/2}{\Delta - \mathrm{i}\Gamma/2}. \tag{12}$$

We therefore find the decay rate is

$$\gamma = \langle \psi_\mathrm{s} | C^\dagger C | \psi_\mathrm{s} \rangle = \frac{|\Omega|^2}{4\Delta^2 + \Gamma^2} \Gamma. \tag{13}$$

The response of the electric dipole can now be found to be

$$\langle \psi_\mathrm{s} | \boldsymbol{d} | \psi_\mathrm{s} \rangle = \frac{\boldsymbol{d}_{eg} E \mathrm{e}^{-\mathrm{i}\omega t}}{\hbar(\Delta - \mathrm{i}\Gamma/2)} + \text{h.c.}. \tag{14}$$

From the definition of

$$E_\text{ext} = E_\text{in} \mathrm{e}^{-\rho\sigma L/2}, \tag{15}$$

we have

$$\sigma = k\alpha = \frac{3\lambda^2}{2\pi} \frac{\Gamma/2}{\Delta - \mathrm{i}\Gamma/2}. \tag{16}$$

It can be found that $\operatorname{Im}\alpha$ is just the scattering rate, which is a result of the optical theorem.

3 Spontaneous radiation as energy correction

We already know that a random wave function theory is fully described by both the Hamiltonian and a set of collapse operators. Sometimes, for example (23) in the last lecture, wave function collapse caused by spontaneous radiation and the subsequent observation of emitted photon is weak, and the main effect of spontaneous radiation is a correction of energy levels.

In this section we consider a ladder-type three-level system shown in Figure 2 on page 2. The Hamiltonian is

$$\dot{\rho}(t) = \frac{1}{\mathrm{i}\hbar} H_{\mathrm{eff}}\rho - \frac{1}{\mathrm{i}\hbar}\rho H_{\mathrm{eff}}^{\dagger} + \sum_{j} C_j \rho C_j^{\dagger}. \tag{17}$$