Quantum Optics, Homework 4

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Random wave function Numerically simulate a two-level atom under an external laser using the random wave function method.

Solution The code can be found with this document. The results are Figure 1 on page 2. The "mean" curves are the average value of 1000 runs.

Stochastic wave function of a Λ system Figure 2 on page 2 is a three-level Λ system. (a) Write down the effective Hamiltonian and quantum jump operators for Figure 2 on page 2. (b) Suppose $|\psi_{\mathrm{s}}(t=0)\rangle = |\mathrm{g}\rangle$. Describe how the wave function evolves using pseudocode. (c) Consider a case in which there is no quantum jump in $0 < t < t_0$. Find the time evolution of the wave function and the scattering rate

$$\gamma_1 = \langle \psi_{\mathrm{s}} | C_1^\dagger C_1 | \psi_{\mathrm{s}} \rangle , \quad \gamma_2 = \langle \psi_{\mathrm{s}} | C_2^\dagger C_2 | \psi_{\mathrm{s}} \rangle . \tag{1}$$

(d) Plot the time evolution of γ_1 and γ_2 under the circumstance of (i) $\Delta = 0, \Omega \gg \Gamma_1 \gg \Gamma_2$; (ii) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$; (iii) $\Delta = 0, \Omega \ll \Gamma_1, \Gamma_2$.

Solution

(a) The effective Hamiltonian is

$$\begin{aligned} H_{\mathrm{eff}} &= -\hbar\Delta |\mathrm{e}\rangle\langle\mathrm{e}| + \left(\frac{1}{2}\hbar\Omega |\mathrm{e}\rangle\langle\mathrm{g}| + \text{h.c.}\right) - \frac{\mathrm{i}\hbar}{2}(C_1^\dagger C_1 + C_2^\dagger C_2) \\ &= -\hbar(\Delta + \mathrm{i}\Gamma/2) |\mathrm{e}\rangle\langle\mathrm{e}| + \hbar(\Omega |\mathrm{e}\rangle\langle\mathrm{g}| + \text{h.c.})/2, \end{aligned} \tag{2}$$

where the quantum jump operators are

$$C_1 = \sqrt{\Gamma_1} |\mathrm{a}\rangle\langle\mathrm{e}| , \quad C_2 = \sqrt{\Gamma_2} |\mathrm{g}\rangle\langle\mathrm{e}| , \tag{3}$$

and

$$\Gamma = \Gamma_1 + \Gamma_2. \tag{4}$$

(b) The time evolution can be described using the following algorithm.

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input : Time step Δt, maximal time t_0

Initialize an array $\{ |\psi_{\mathrm{s}}(t)\rangle \}_{t=n\Delta t}$ of wave functions with $t_0/\Delta t$ elements

for $t \in 0 : \Delta t : t_0$

  Pick up a uniformly distributed random number x between 0 and 1

  $P_{\mathrm{g}} \leftarrow \Delta t \langle \psi_{\mathrm{s}}(t) | C_1^\dagger C_1 | \psi_{\mathrm{s}}(t) \rangle$

  $P_{\mathrm{a}} \leftarrow \Delta t \langle \psi_{\mathrm{s}}(t) | C_2^\dagger C_2 | \psi_{\mathrm{s}}(t) \rangle$

  `// jumping to` $|\mathrm{g}\rangle$

  if $0 < x < P_{\mathrm{g}}$

    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $C_1 |\psi_{\mathrm{s}}(t)\rangle$

  `// jumping to` $|\mathrm{a}\rangle$

  elseif $P_{\mathrm{g}} < x < P_{\mathrm{g}} + P_{\mathrm{a}}$

    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $C_2 |\psi_{\mathrm{s}}(t)\rangle$

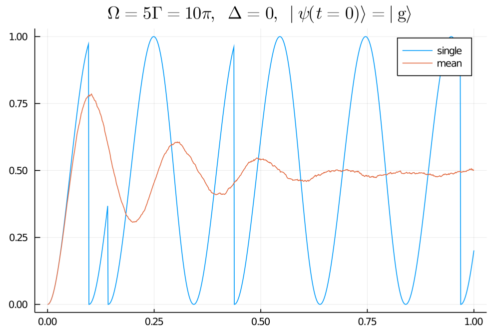
  `// evolution according to the effective Hamiltonian`

  else

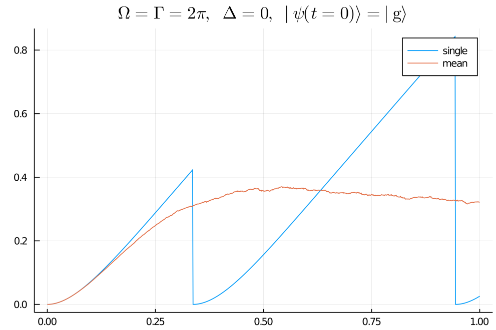
    $|\psi_{\mathrm{s}}(t+\Delta t)\rangle \leftarrow$ normalized $|\psi_{\mathrm{s}}(t)\rangle + \frac{\Delta t}{\mathrm{i}\hbar} H_{\mathrm{eff}} |\psi_{\mathrm{s}}(t)\rangle$

  end

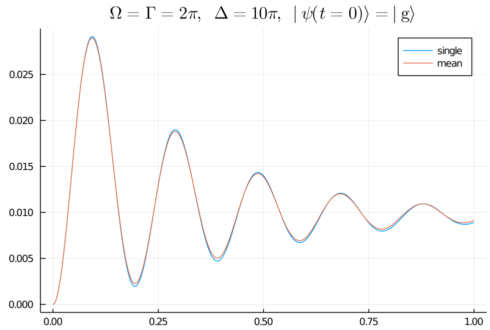
end



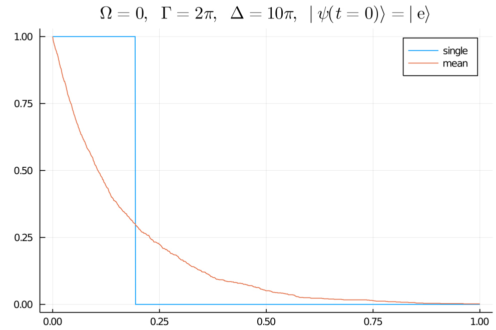
(a)



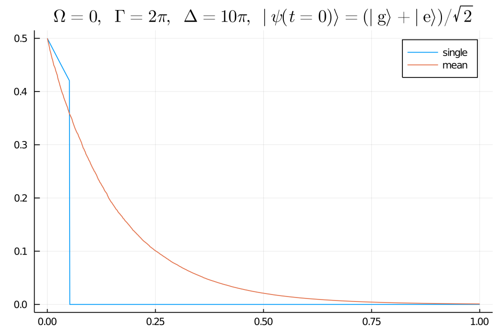
(b)



(c)



(d)



(e)

Figure 1: Numerical simulated P_{e}.

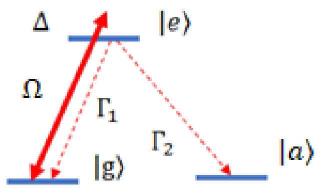


Figure 2: A three-level Λ system

(c) The wave function in this case evolves purely according to H_{eff}. Since Schrödinger equation is linear, we can leave the normalization to the end of our calculation. Note that (2) actually does not contain $|\text{a}\rangle$ explicitly, nor does the initial state $|\text{g}\rangle$. Therefore we can work in the two-level system spanned by $|\text{e}\rangle$ and $|\text{g}\rangle$. The effective Hamiltonian is

$$H_{\text{eff}} = \hbar \begin{pmatrix} 0 & \Omega^*/2 \\ \Omega/2 & -(\Delta + \mathrm{i}\Gamma/2) \end{pmatrix}, \tag{5}$$

where we let $|\text{g}\rangle$ be the first component and $|\text{e}\rangle$ the second. We have the decomposition

$$H_{\text{eff}} = -\frac{\hbar}{2}(\Delta + \mathrm{i}\Gamma) + \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\Omega} = (\Omega_{\text{r}}, \Omega_{\text{i}}, \Delta + \mathrm{i}\Gamma/2). \tag{6}$$

Note here we cannot "shift the energy zero point" to reshape the Hamiltonian into $\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$, because the value damping rate has physical meaning. Applying (6) on $|\text{g}\rangle$, we have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}H_{\text{eff}}t/\hbar}|\text{g}\rangle &= \mathrm{e}^{\mathrm{i}t(\Delta+\mathrm{i}\Gamma/2)/2}\mathrm{e}^{-\mathrm{i}t\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}/2}|\text{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\sigma^0 \cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \frac{\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\text{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2}|\text{g}\rangle - \left(\frac{\Omega_{\text{r}}}{|\boldsymbol{\Omega}|}|\text{e}\rangle + \frac{\mathrm{i}\Omega_{\text{i}}}{|\boldsymbol{\Omega}|}|\text{e}\rangle + \frac{\Delta + \mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}|\text{g}\rangle\right)\mathrm{i}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right),
\end{aligned}$$

where

$$|\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2 - \Gamma^2/4 + \mathrm{i}\Delta\Gamma}. \tag{7}$$

Note

Note that here $|\boldsymbol{n}|$ is defined as $\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}$ instead of $\sqrt{\boldsymbol{n}^*\cdot\boldsymbol{n}}$, because to make

$$\mathrm{e}^{\mathrm{i}\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}} = \sigma^0\cos\alpha + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\alpha$$

hold, it is required that

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \sigma^0,$$

which is equivalent to $\boldsymbol{n}\cdot\boldsymbol{n} = 1$, considering $\{\sigma^i, \sigma^j\} = 0$ when $i \neq j$. What is important here, therefore, is $\boldsymbol{n}\cdot\boldsymbol{n}$.

Therefore we have (we have omitted the complex factors, since they will be canceled by normalization anyway)

$$|\psi_{\text{s}}(t)\rangle = \frac{1}{C}\left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \mathrm{i}\frac{\Delta + \mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\text{g}\rangle - \frac{\mathrm{i}\Omega}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}|\text{e}\rangle, \tag{8}$$

the normalization constant being

$$C = \sqrt{\left|\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \mathrm{i}\frac{\Delta + \mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right|^2 + \frac{|\Omega|^2}{|\boldsymbol{\Omega}|^2}\sin^2\frac{|\boldsymbol{\Omega}|\,t}{2}}. \tag{9}$$

Since

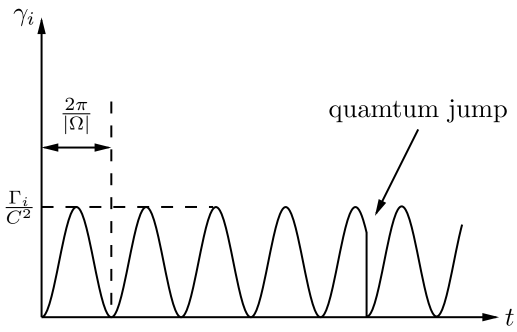
$$C_1^\dagger C_1 = \Gamma_1 |\text{e}\rangle\langle\text{e}|, \quad C_2^\dagger C_2 = \Gamma_2 |\text{e}\rangle\langle\text{e}|,$$

it is then straightforward that

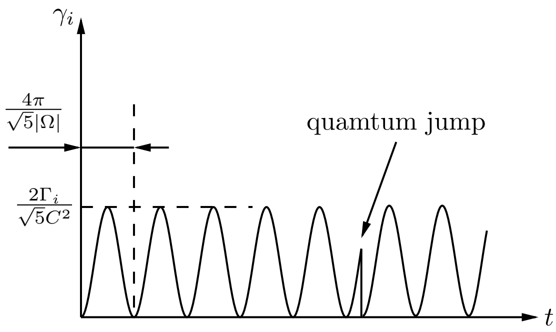
$$\gamma_1 = \frac{\Gamma_1}{C^2}\frac{\Omega^2}{|\boldsymbol{\Omega}|^2}\left|\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right|^2, \quad \gamma_2 = \frac{\Gamma_2}{C^2}\frac{\Omega^2}{|\boldsymbol{\Omega}|^2}\left|\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right|^2. \tag{10}$$

Note that in these equations $|\boldsymbol{\Omega}|^2$ means the norm, i.e. $|\boldsymbol{\Omega}|\cdot|\boldsymbol{\Omega}|^*$.

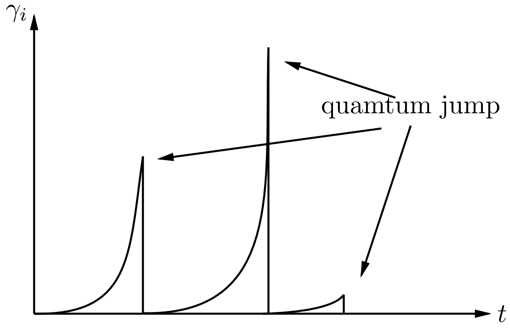
The scattering rates are calculated under the assumption that no quantum jump happened before, so what they mean are actually the probabilities that "no quantum jump happened before, and a quantum jump will happen in the next second".



(a) $\Delta = 0, \Omega \gg \Gamma_1 \gg \Gamma_2$



(b) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$



(c) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$

Figure 3: Time evolution of γ_1 and γ_2

(d) Note that γ_1 and γ_2 only differ with a factor, so their plots only differ in scaling.

(i) In this case $|\boldsymbol{\Omega}| \approx |\Omega|$, so γ_1 and γ_2 oscillate as $A \sin^2 |\Omega| t/2$ until a quantum jump happens, and both of them drop to zero, and then another oscillation starts. See Figure 3a on page 4.

(ii) In this case

$$|\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2 - \Gamma^2/4 + \mathrm{i}\Delta\Gamma} \approx \sqrt{|\Omega|^2 + \frac{1}{4}|\Omega|^2} = \frac{\sqrt{5}}{2}|\Omega|,$$

so the curve of γ is similar to (i), but the oscillating period is now $4\pi/\sqrt{5}|\Omega|$. See Figure 3b on page 4.

(iii) In this case

$$|\boldsymbol{\Omega}| \approx \sqrt{-\Gamma^2/4} = \frac{\mathrm{i}}{2}\Gamma,$$

and we have

$$\begin{aligned} \gamma_i &= \frac{\Gamma_i}{C^2}\frac{\Omega^2}{\Gamma^2/4}\left|\sin\frac{\mathrm{i}\Gamma}{2}t\right|^2 \\ &= \frac{\Gamma_i}{C^2}\frac{\Omega^2}{\Gamma^2}(\mathrm{e}^{-\Gamma t/4} - \mathrm{e}^{\Gamma t/4})^2. \end{aligned}$$

Its prefactor is small considering that Ω/Γ is small, but it increases exponentially. Before it grows too large, a quantum jump will happen. See Figure 3c on page 4.

Cesium atom Consider a simplified cesium level diagram in Figure 4 on page 5 (We ignore Zeeman and hyperfine structure). The laser beams at 895 nm, 761 nm and 794 nm induce coupling between the ground state 6 $S_{1/2} - 6P_{1/2}$ (with Rabi freq Ω_1), $6P_{1/2} - 8$ $S_{1/2}$ (with Rabi freq Ω_2), and 8 $S_{1/2} - 6P_{3/2}$ (with Rabi freq Ω_3) respectively. The 1-photon detuning Δ_a, 2−photon detuning Δ_b, and 3-photon detuning Δ_c are sketched as in the diagram. We consider the case where all these detunings are at GHz level or smaller, which are tiny comparing with the optical frequencies of the lasers.

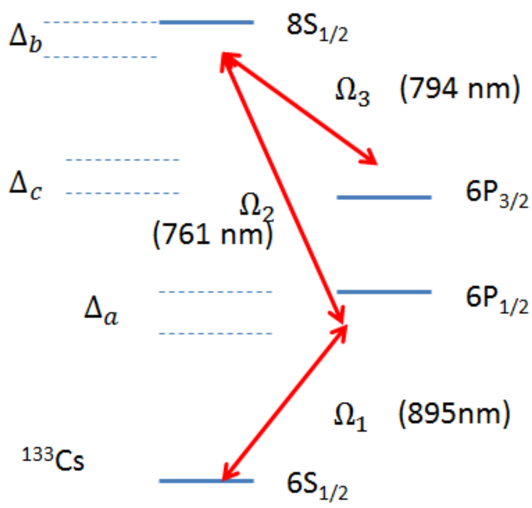


Figure 4: Cesium atom subjected to three laser fields for nearly resonant excitations

Task 1: With $|g\rangle = |6S_{1/2}\rangle, |a\rangle = |6P_{1/2}\rangle; |b\rangle = |8S_{1/2}\rangle; |c\rangle = |6P_{3/2}\rangle$, invent your own additional notations to write down the time-dependent Hamiltonian for the cesium atom in the laser fields, that explicitly have the optical frequencies of the lasers.

Task 2: Write down a time-independent Hamiltonian.

Task 3: Consider radiative life time for $6P_{1/2}, 8S_{1/2}$ and $6P_{3/2}$ are given by $\tau_a = 1/\Gamma_a, \tau_b = 1/\Gamma_b$, and $\tau_c = 1/\Gamma_c$. Write down the effective non-Hermitian Hamiltonian H_{eff} that includes an anti-Hermitian part to account for the spontaneous emissions.

Task 4: Consider the internal state of the cesium has a wavefunction $|\psi_S(t)\rangle = c_g|g\rangle + c_a|a\rangle + c_b|b\rangle + c_c|c\rangle$. Consider the weak perturbation limit (ie laser intensities are small enough that atoms are barely excited), write down the differential equations for the coefficients and approximately solve for $|\psi_{\text{S}}(t)\rangle \approx |\tilde{\psi}_{\text{S}}\rangle$ with $\text{H}_{\text{eff}}|\tilde{\psi}_S\rangle = 0$.

Task 5: Perturbative calculate the 3-photon scattering rate given by $\gamma_3 = |c_c|^2 \Gamma$, by assuming $|\psi(t)\rangle$ quickly relax to eigenstate of the effective Hamiltonian with $c_g \approx 1$.

Task 6: Consider strong 761 nm and 795 nm laser, calculate the atomic polaraizability $\alpha(\Omega_2, \Omega_3)$ for the 852 nm laser excitation by evaluating $\langle \text{d} \rangle = \left\langle \widetilde{\Psi}_S \left| d_{ag} \right| a \right\rangle \left\langle g \mid \widetilde{\psi}_S \right\rangle +$ c.c., and express it as $\alpha \text{E}_1 +$ c.c..

Task 7: Discuss the validity of your method of calculating γ_3 and $\alpha(\Omega_2, \Omega_3)$ using an ill-defined wavefunction (since its norm cannot be unity) and an effective nonHermitian Hamitonian, instead of using a density matrix and master equations (or the full Monte-Carlo wavefunction method). Your discussion may involve Ω, Δ, Γ and the total time of observation T.

Solution

(1) The time-dependent Hamiltonian is

$$H = H_0 + H_{\text{dipole}}, \tag{11}$$

where

$$H_0 = \hbar\omega_{\text{g}} |\text{g}\rangle\langle\text{g}| + \hbar\omega_{\text{a}} |\text{a}\rangle\langle\text{a}| + \hbar\omega_{\text{b}} |\text{b}\rangle\langle\text{b}| + \hbar\omega_{\text{c}} |\text{c}\rangle\langle\text{c}|, \tag{12}$$

and

$$\begin{aligned} H_{\text{dipole}} = &- \boldsymbol{d}_{\text{ag}} \cdot (\boldsymbol{E}_1 \text{e}^{-\text{i}\omega_1 t} + \boldsymbol{E}_1^* \text{e}^{\text{i}\omega_1 t}) |\text{a}\rangle\langle\text{g}| - \boldsymbol{d}_{\text{ba}} \cdot (\boldsymbol{E}_2 \text{e}^{-\text{i}\omega_2 t} + \boldsymbol{E}_2^* \text{e}^{\text{i}\omega_2 t}) |\text{b}\rangle\langle\text{a}| \\ &- \boldsymbol{d}_{\text{bc}} \cdot (\boldsymbol{E}_3 \text{e}^{-\text{i}\omega_3 t} + \boldsymbol{E}_3^* \text{e}^{\text{i}\omega_3 t}) |\text{b}\rangle\langle\text{c}| + \text{h.c.}, \end{aligned} \tag{13}$$

where we denote the external electric fields as

$$\boldsymbol{E}_i = \boldsymbol{E}_{i0} \text{e}^{\text{i}\omega_i t} + \boldsymbol{E}_{i0}^* \text{e}^{-\text{i}\omega_i t}, \quad i = 1, 2, 3. \tag{14}$$

The laser frequencies satisfy

$$\omega_1 + \Delta_{\text{a}} = \omega_{\text{a}} - \omega_{\text{g}}, \quad \omega_2 + \Delta_{\text{b}} - \Delta_{\text{a}} = \omega_{\text{b}} - \omega_{\text{a}}, \quad \omega_3 + \Delta_{\text{b}} + \Delta_{\text{c}} = \omega_{\text{b}} - \omega_{\text{c}}, \tag{15}$$

from which we find

$$\omega_1 = \omega_{\mathrm{a}} - \omega_{\mathrm{g}} - \Delta_{\mathrm{a}}, \quad \omega_2 = \omega_{\mathrm{b}} - \omega_{\mathrm{a}} + \Delta_{\mathrm{a}} - \Delta_{\mathrm{b}}, \quad \omega_3 = \omega_{\mathrm{b}} - \omega_{\mathrm{c}} - \Delta_{\mathrm{b}} - \Delta_{\mathrm{c}}. \tag{16}$$

(2) We switch to the interaction picture, using H_0 as the free Hamiltonian, and we have

$$\begin{aligned} H = H_{\text{dipole, int}} = &- \boldsymbol{d}_{\mathrm{ag}} \cdot (\boldsymbol{E}_1 \mathrm{e}^{-\mathrm{i}\omega_1 t} + \boldsymbol{E}_1^* \mathrm{e}^{\mathrm{i}\omega_1 t}) \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{a}} - \omega_{\mathrm{g}})t} \\ &- \boldsymbol{d}_{\mathrm{ba}} \cdot (\boldsymbol{E}_2 \mathrm{e}^{-\mathrm{i}\omega_2 t} + \boldsymbol{E}_2^* \mathrm{e}^{\mathrm{i}\omega_2 t}) \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{a}})t} \\ &- \boldsymbol{d}_{\mathrm{bc}} \cdot (\boldsymbol{E}_3 \mathrm{e}^{-\mathrm{i}\omega_3 t} + \boldsymbol{E}_3^* \mathrm{e}^{\mathrm{i}\omega_3 t}) \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{c}})t} + \text{h.c.}. \end{aligned} \tag{17}$$

We make the rotating wave approximation, i.e. omitting all terms that vibrate much faster than all detunings, and get

$$\begin{aligned} H = &\frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{a}} - \omega_{\mathrm{g}} - \omega_1)t} + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{a}} - \omega_2)t} \\ &+ \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{c}} - \omega_3)t} + \text{h.c.}, \end{aligned} \tag{18}$$

where we define

$$\frac{1}{2}\hbar\Omega_1 = -\boldsymbol{d}_{\mathrm{ag}} \cdot \boldsymbol{E}_1, \quad \frac{1}{2}\hbar\Omega_2 = -\boldsymbol{d}_{\mathrm{ba}} \cdot \boldsymbol{E}_2, \quad \frac{1}{2}\hbar\Omega_3 = -\boldsymbol{d}_{\mathrm{bc}} \cdot \boldsymbol{E}_3. \tag{19}$$

Warning

If we define the electric field as

$$\boldsymbol{E}_i = \frac{1}{2}(\boldsymbol{E}_{i0} \mathrm{e}^{\mathrm{i}\omega_i t} + \boldsymbol{E}_{i0}^* \mathrm{e}^{-\mathrm{i}\omega_i t}) = |\boldsymbol{E}_i| \cos(\omega_i t + \varphi_i), \quad i = 1, 2, 3, \tag{20}$$

then there will be no 1/2 factor in the definition of Ω_i's. However, later we will evaluate $\langle \boldsymbol{d} \rangle$, which has the form of something$^{-\mathrm{i}\omega t}$ + h.c., and if we insist on (20), an additional and easy-to-forget factor 2 must be added when we evaluate $\alpha = d/E$.

There are three phase factors, and we have four states, so it is possible to use a rotating wave transformation to eliminate them all. By

$$\begin{aligned} U \left|\mathrm{a}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{a}} - \omega_{\mathrm{g}} - \omega_1)t} \left|\mathrm{a}\right\rangle = \mathrm{e}^{-\mathrm{i}\Delta_{\mathrm{a}} t} \left|\mathrm{a}\right\rangle, \\ U \left|\mathrm{b}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{a}} - \omega_2)t} \mathrm{e}^{-\mathrm{i}\Delta_{\mathrm{a}} t} \left|\mathrm{b}\right\rangle = \mathrm{e}^{-\mathrm{i}\Delta_{\mathrm{b}} t} \left|\mathrm{b}\right\rangle, \\ U \left|\mathrm{c}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{b}} - \omega_{\mathrm{c}} - \omega_3)t} \mathrm{e}^{\mathrm{i}\Delta_{\mathrm{b}} t} \left|\mathrm{c}\right\rangle = \mathrm{e}^{\mathrm{i}\Delta_{\mathrm{c}} t}, \end{aligned} \tag{21}$$

we have

$$\begin{aligned} H \to H' &= U H U^\dagger - \mathrm{i}\hbar U \partial_t U^\dagger \\ &= \frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| + \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| + \text{h.c.} + \hbar\Delta_{\mathrm{a}} \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{a}\right| + \hbar\Delta_{\mathrm{b}} \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{b}\right| - \hbar\Delta_{\mathrm{c}} \left|\mathrm{c}\right\rangle\!\left\langle\mathrm{c}\right|. \end{aligned} \tag{22}$$

This is the time-independent Hamiltonian we want.

(3) The effective Hamiltonian is

$$\begin{aligned} H_{\mathrm{eff}} = &\frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| + \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| + \text{h.c.} + \hbar \underbrace{\left(\Delta_{\mathrm{a}} - \frac{\mathrm{i}\Gamma_{\mathrm{a}}}{2}\right)}_{\tilde{\Delta}_{\mathrm{a}}} \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{a}\right| \\ &+ \hbar \underbrace{\left(\Delta_{\mathrm{b}} - \frac{\mathrm{i}\Gamma_{\mathrm{b}}}{2}\right)}_{\tilde{\Delta}_{\mathrm{b}}} \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{b}\right| - \hbar \underbrace{\left(\Delta_{\mathrm{c}} + \frac{\mathrm{i}\Gamma_{\mathrm{c}}}{2}\right)}_{\tilde{\Delta}_{\mathrm{c}}} \left|\mathrm{c}\right\rangle\!\left\langle\mathrm{c}\right|. \end{aligned} \tag{23}$$

(4) In the weak perturbation limit, $c_{\mathrm{g}} \approx 1$, and the equation $H_{\mathrm{eff}} \left|\psi_{\mathrm{s}}\right\rangle = 0$ is equivalent to

$$
\begin{aligned}
&-\frac{1}{2}\hbar\Omega_1^* c_{\mathrm{a}} = 0, \\
&\frac{1}{2}\hbar\Omega_1 + \hbar\left(\Delta_{\mathrm{a}} - \frac{\mathrm{i}\Gamma_{\mathrm{a}}}{2}\right) c_{\mathrm{a}} + \frac{1}{2}\hbar\Omega_2^* c_{\mathrm{b}} = 0, \\
&\frac{1}{2}\hbar\Omega_2 c_{\mathrm{a}} + \hbar\left(\Delta_{\mathrm{b}} - \frac{\mathrm{i}\Gamma_{\mathrm{b}}}{2}\right) c_{\mathrm{b}} + \frac{1}{2}\hbar\Omega_3 c_{\mathrm{c}} = 0, \\
&\frac{1}{2}\hbar\Omega_3^* - \hbar\left(\Delta_{\mathrm{c}} + \frac{\mathrm{i}\Gamma_{\mathrm{c}}}{2}\right) c_{\mathrm{c}} = 0.
\end{aligned}
$$

The first equation can be throw away because it merely means c_{a} is small. The solutions are therefore

$$
\begin{aligned}
c_{\mathrm{a}} &= -\frac{2\Omega_1\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + \left|\Omega_3\right|^2\Omega_1/2}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}}, \\
c_{\mathrm{b}} &= \frac{\Omega_1\Omega_2\tilde{\Delta}_{\mathrm{c}}}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}}, \\
c_{\mathrm{c}} &= \frac{\Omega_1\Omega_2\Omega_3^*/2}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}}.
\end{aligned} \tag{24}
$$

(5) We have

$$
\gamma_3 = \Gamma_3\left|c_{\mathrm{c}}\right|^2 = \frac{\left|\Omega_1\right|^2\left|\Omega_2\right|^2\left|\Omega_3\right|^2/4}{\left|\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}\right|^2}\Gamma_3. \tag{25}
$$

(6) The total unitary transformation from the original picture to the current pictures is partly given by

$$
U_{\mathrm{total}}\left|\mathrm{g}\right\rangle = \mathrm{e}^{\mathrm{i}\omega_{\mathrm{g}}t}\left|\mathrm{g}\right\rangle, \quad U_{\mathrm{total}}\left|\mathrm{a}\right\rangle = \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{a}}-\omega_{\mathrm{g}}-\omega_1)t}\mathrm{e}^{\mathrm{i}\omega_{\mathrm{a}}t}\left|\mathrm{a}\right\rangle,
$$

so in the current picture, we have

$$
\boldsymbol{d} = U_{\mathrm{total}}\boldsymbol{d}_{\mathrm{original}}U_{\mathrm{total}}^{-1},
$$

and therefore

$$
\begin{aligned}
\langle\boldsymbol{d}\rangle &= \left\langle\psi_{\mathrm{s}}|\mathrm{a}\right\rangle \mathrm{e}^{\mathrm{i}(\omega_1+\omega_{\mathrm{g}})t}\boldsymbol{d}_{\mathrm{ag}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{g}}t}\left\langle\mathrm{g}|\psi_{\mathrm{s}}\right\rangle + \text{h.c.} = \mathrm{e}^{-\mathrm{i}\omega_1 t}\boldsymbol{d}_{\mathrm{ga}}\left\langle\psi_{\mathrm{s}}|\mathrm{g}\right\rangle\left\langle\mathrm{a}|\psi_{\mathrm{s}}\right\rangle + \text{h.c.} \\
&= -\mathrm{e}^{-\mathrm{i}\omega_1 t}\boldsymbol{d}_{\mathrm{ga}}\frac{-2\boldsymbol{d}_{\mathrm{ag}}\cdot\boldsymbol{E}_1}{\hbar}\frac{2\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + \left|\Omega_3\right|^2/2}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}} + \text{h.c.},
\end{aligned} \tag{26}
$$

so we have

$$
\overset{\leftrightarrow}{\boldsymbol{\alpha}} = \frac{\boldsymbol{d}_{\mathrm{ga}}\boldsymbol{d}_{\mathrm{ag}}}{\hbar}\frac{4\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + \left|\Omega_3\right|^2}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}}. \tag{27}
$$

When the other two laser beams are very, very strong, we have approximately

$$
\overset{\leftrightarrow}{\boldsymbol{\alpha}} = \frac{\boldsymbol{d}_{\mathrm{ga}}\boldsymbol{d}_{\mathrm{ag}}}{\hbar}\frac{\left|\Omega_3\right|^2}{\left|\Omega_3\right|^2\tilde{\Delta}_{\mathrm{a}} - \left|\Omega_2\right|^2\tilde{\Delta}_{\mathrm{c}}}. \tag{28}
$$

(7) What we are doing is to assume (a) that quantum jump is highly impossible on the time scale that we are interested, so the main effect of spontaneous radiation is to renormalize parameters in a pure state problem, and (b) that all perturbations, be it from Ω or Γ, are small enough.

The first condition is equivalent to $\gamma T \ll 1$, where T is the time scale we are interested in, where

$$
\Gamma \sim \Gamma_{\mathrm{a}}\left|c_{\mathrm{a}}\right|^2 + \Gamma_{\mathrm{b}}\left|c_{\mathrm{b}}\right|^2 + \Gamma_{\mathrm{c}}\left|c_{\mathrm{c}}\right|^2. \tag{29}
$$

The second condition is equivalent to $\left|c_{\mathrm{a}}\right|^2 + \left|c_{\mathrm{b}}\right|^2 + \left|c_{\mathrm{c}}\right|^2 \ll 1$.

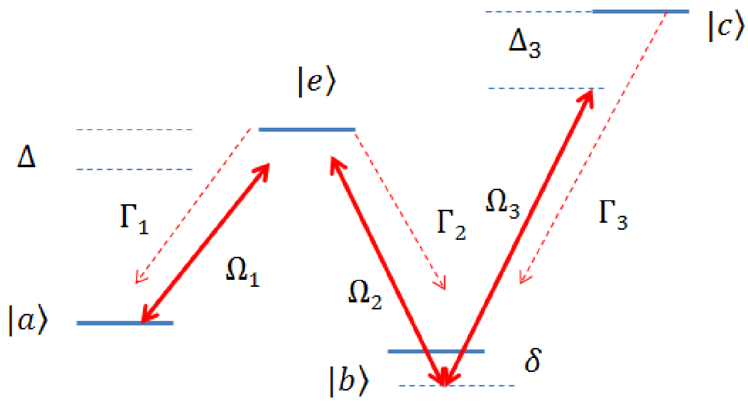


Figure 5: A three-level system coupled to an additional energy level

EIT-assisted giant Kerr effect The "lambda"-system composed of $|\mathrm{a}\rangle, |\mathrm{e}\rangle, |\mathrm{b}\rangle$ is further coupled to excited state $|\mathrm{c}\rangle$, as in Figure 5 on page 8. We consider the situation of EIT-resonance: $\delta = 0$. We further consider atomic state to be initially in $|\psi(t=0)\rangle = |\mathrm{a}\rangle$, and weak-excitation limit is satisfied ($|\Omega_1|$ small "enough"). (a) Write down the effective Hamiltonian for this problem for $\delta = 0$. (b) Obtain the approximate stochastic wavefunction in its steady state $|\tilde{\psi}_{\mathrm{S}}\rangle = |\mathrm{a}\rangle + c_{\mathrm{e}}|\mathrm{e}\rangle + c_{\mathrm{b}}|\mathrm{b}\rangle + c_{\mathrm{c}}|\mathrm{c}\rangle$ such that $H_{\mathrm{eff}}\,|\psi_{\mathrm{S}}\rangle \approx 0$. (c) Approximately evaluate the atomic dipole moment $\langle d\rangle = \langle \psi_S\,| d_{ae}|\, a\rangle \,\langle e\mid \psi_S\rangle + \text{c.c.}$ oscillating at the E_1 frequency.

Solution

(a) Repeating procedures in Task 1 of the previous problem, after RWA, we have

$$H = H_0 + \frac{1}{2}\hbar\Omega_1\,|\mathrm{e}\rangle\langle\mathrm{a}|\,\mathrm{e}^{-\mathrm{i}\omega_1 t} + \frac{1}{2}\hbar\Omega_2\,|\mathrm{e}\rangle\langle\mathrm{b}|\,\mathrm{e}^{-\mathrm{i}\omega_2 t} + \frac{1}{2}\hbar\Omega_3\,|\mathrm{c}\rangle\langle\mathrm{b}|\,\mathrm{e}^{-\mathrm{i}\omega_3 t} + \text{h.c.}, \tag{30}$$

where

$$H_0 = \hbar\omega_{\mathrm{a}}\,|\mathrm{a}\rangle\langle\mathrm{a}| + \hbar\omega_{\mathrm{e}}\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\omega_{\mathrm{b}}\,|\mathrm{b}\rangle\langle\mathrm{b}| + \hbar\omega_{\mathrm{c}}\,|\mathrm{c}\rangle\langle\mathrm{c}|\,. \tag{31}$$

Switching to the interaction picture, the Hamiltonian becomes

$$H = \frac{1}{2}\hbar\Omega_1\,|\mathrm{e}\rangle\langle\mathrm{a}|\,\mathrm{e}^{\mathrm{i}(\omega_{\mathrm{e}}-\omega_{\mathrm{a}}-\omega_1)t} + \frac{1}{2}\hbar\Omega_2\,|\mathrm{e}\rangle\langle\mathrm{b}|\,\mathrm{e}^{\mathrm{i}(\omega_{\mathrm{e}}-\omega_{\mathrm{b}}-\omega_2)t} + \frac{1}{2}\hbar\Omega_3\,|\mathrm{c}\rangle\langle\mathrm{b}|\,\mathrm{e}^{\mathrm{i}(\omega_{\mathrm{c}}-\omega_{\mathrm{b}}-\omega_3)t} + \text{h.c.}. \tag{32}$$

The detunings are

$$\Delta + \omega_1 = \omega_{\mathrm{e}} - \omega_{\mathrm{a}}, \quad \omega_{\mathrm{e}} - \omega_{\mathrm{b}} = \omega_2, \quad \Delta_3 + \omega_3 = \omega_{\mathrm{c}} - \omega_{\mathrm{b}}, \tag{33}$$

so (20) is

$$H = \frac{1}{2}\hbar\Omega_1\,|\mathrm{e}\rangle\langle\mathrm{a}|\,\mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2\,|\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3\,|\mathrm{c}\rangle\langle\mathrm{b}|\,\mathrm{e}^{\mathrm{i}\Delta_3 t} + \text{h.c.}. \tag{34}$$

Now we do rotating wave transformation

$$U\,|\mathrm{e}\rangle = \mathrm{e}^{-\mathrm{i}\Delta t}\,|\mathrm{e}\rangle\,, \quad U\,|\mathrm{c}\rangle = \mathrm{e}^{-\mathrm{i}\Delta_3 t}\,|\mathrm{c}\rangle\,, \quad U\,|\mathrm{a}\rangle = |\mathrm{a}\rangle\,, \quad U\,|\mathrm{b}\rangle = |\mathrm{b}\rangle\,, \tag{35}$$

we have

$$\begin{aligned} H \to H' &= UHU^\dagger - \mathrm{i}\hbar U\partial_t U^\dagger \\ &= \frac{1}{2}\hbar\Omega_1\,|\mathrm{e}\rangle\langle\mathrm{a}|\,\mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2\,|\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3\,|\mathrm{c}\rangle\langle\mathrm{b}|\,\mathrm{e}^{\mathrm{i}\Delta_3 t} + \text{h.c.} + \hbar\Delta\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\Delta_3\,|\mathrm{c}\rangle\langle\mathrm{c}|\,. \end{aligned} \tag{36}$$

The effective Hamiltonian can be obtained by adding the damping terms, i.e.

$$\begin{aligned} H_{\mathrm{eff}} = \frac{1}{2}\hbar\Omega_1\,|\mathrm{e}\rangle\langle\mathrm{a}|\,\mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2\,|\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3\,|\mathrm{c}\rangle\langle\mathrm{b}|\,\mathrm{e}^{\mathrm{i}\Delta_3 t} + \text{h.c.} \\ + \hbar\Delta\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\Delta_3\,|\mathrm{c}\rangle\langle\mathrm{c}| - \frac{\mathrm{i}\hbar(\Gamma_1+\Gamma_2)}{2}\,|\mathrm{e}\rangle\langle\mathrm{e}| - \frac{\mathrm{i}\hbar\Gamma_3}{2}\,|\mathrm{c}\rangle\langle\mathrm{c}|\,. \end{aligned} \tag{37}$$

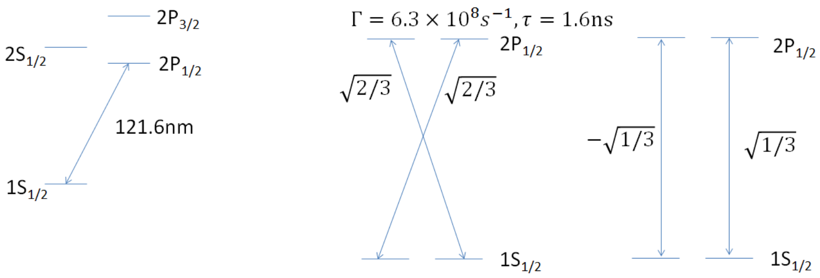


Figure 6: Left: Simplified Hydrogen levels and excitation of 1 S atom with 121.6 nm light resonant to 1 S$-$2P$_{1/2}$ transition. Right: The Clebsch-Gordon coefficients for the 1 S$_{1/2}-$2P$_{1/2}$ dipole transitions.

(b) The equation $H_{\text{eff}}\,|\tilde{\psi}_{\text{s}}\rangle \approx 0$ is equivalent to

$$\begin{aligned}
&\frac{1}{2}\hbar\Omega_1^* c_{\text{e}} = 0,\\
&\frac{1}{2}\hbar\Omega_1 + \frac{1}{2}\hbar\Omega_2 c_{\text{b}} + \hbar\left(\Delta - \frac{\mathrm{i}(\Gamma_1+\Gamma_2)}{2}\right)c_{\text{e}} = 0,\\
&\frac{1}{2}\hbar\Omega_2^* c_{\text{e}} + \frac{1}{2}\hbar\Omega_3^* c_{\text{c}} = 0,\\
&\frac{1}{2}\hbar\Omega_3 c_{\text{b}} + \hbar\left(\Delta_3 - \frac{\mathrm{i}\Gamma_3}{2}\right)c_{\text{c}} = 0.
\end{aligned}$$

We throw away the first equation because it merely requires that c_{c} is small, and solving the rest of the equations gives

$$\begin{aligned}
c_{\text{b}} &= -\frac{\Omega_1\Omega_2^*\tilde{\Delta}_3}{|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta}},\\
c_{\text{c}} &= \frac{\Omega_1\Omega_2^*\Omega_3}{2(|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta})},\\
c_{\text{e}} &= -\frac{\Omega_1|\Omega_3|^2}{2(|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta})},
\end{aligned} \tag{38}$$

where we define

$$\tilde{\Delta} = \Delta - \frac{\mathrm{i}(\Gamma_1+\Gamma_2)}{2}, \quad \tilde{\Delta}_3 = \Delta_3 - \frac{\mathrm{i}\Gamma_3}{2}. \tag{39}$$

(c) The total picture transformation from the original picture to the picture we are using is

$$U_{\text{total}}\,|\text{a}\rangle = \mathrm{e}^{\mathrm{i}\omega_{\text{a}}t}\,|\text{a}\rangle\,, \quad U_{\text{total}}\,|\text{e}\rangle = \mathrm{e}^{-\mathrm{i}\Delta t}\mathrm{e}^{\mathrm{i}\omega_{\text{e}}t}\,|\text{e}\rangle = \mathrm{e}^{\mathrm{i}(\omega_1+\omega_{\text{a}})t}, \tag{40}$$

and therefore in the current picture, we have

$$\langle\text{e}|\boldsymbol{d}|\text{a}\rangle = \langle\text{e}|U_{\text{total}}\boldsymbol{d}_{\text{original}}U_{\text{total}}^\dagger|\text{a}\rangle = \boldsymbol{d}_{\text{ea}}\mathrm{e}^{\mathrm{i}\omega_1 t}, \quad \langle\text{a}|\boldsymbol{d}|\text{e}\rangle = \boldsymbol{d}_{\text{ae}}\mathrm{e}^{-\mathrm{i}\omega_1 t},$$

so

$$\begin{aligned}
\langle\boldsymbol{d}\rangle &= \boldsymbol{d}_{\text{ae}}\mathrm{e}^{-\mathrm{i}\omega_1 t}\,\langle\psi_{\text{s}}|\text{a}\rangle\,\langle\text{e}|\psi_{\text{s}}\rangle + \text{h.c.}\\
&= \frac{\boldsymbol{d}_{\text{ae}}\boldsymbol{d}_{\text{ea}}\cdot\boldsymbol{E}_{10}\mathrm{e}^{-\mathrm{i}\omega_1 t}}{\hbar}\,\frac{|\Omega_3|^2}{|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta}} + \text{h.c.}.
\end{aligned} \tag{41}$$

Effective Hamiltonian and Master Equation for the Hydrogen "D1" manifold This problem exercises on setting up effective Hamiltonian and master equation for multi-level system including radiative decays. We choose hydrogen atom as an example, and for simplicity we ignore the hyperfine structure.

We consider a hydrogen atom subjected to 121.6 nm laser radiation. The laser is linearly polarized, and its frequency is exactly resonant to the 2 S$_{1/2}$ $-$ 2P$_{1/2}$ transition. The Rabi frequency is defined as $\Omega = \frac{E\langle 1S_{1/2},m|d_z|2P_{1/2},m\rangle}{\hbar}$, and we choose the polarization direction of light to be along z, which is also chosen to be the quantization axis.

(a) Write down the non-Hermitian effective Hamiltonian matrix that includes both light-atom interaction and the radiative decay from $2P_{1/2}$, in the rotating frame with no explicit time-dependence. (b) Write down the Master equation for the density matrix.

Solution

(a) After coupling the orbital angular momentum and the spin angular momentum together, for S we get $j = 1/2, m_j = \pm 1/2$, and for P we have $j = 1/2, m_j = \pm 1/2$, and $j = 3/2, m_j = \pm 1/2, \pm 3/2$. These are shown in Figure 6 on page 9, and the external laser connects $1S_{1/2}$ and $2P_{1/2}$. Since the direction of $\boldsymbol{E}$ is z direction, the atom-light coupling Hamiltonian is proportion to z, and since $[z, L_z] = 0$, and spins are not directly coupled to $\boldsymbol{E}$, we find $J_z = L_z + S_z$ is conserved. Therefore, there are four states that are involved in the presence of the 121.6 nm laser radiation: $1S_{1/2}, m = \pm 1/2$, $2P_{1/2}, m = \pm 1/2$, and only states with the same m can be coupled.

To be concise we consider $1S_{1/2}$ to be g and $2P_{1/2}$ to be e. So the Hamiltonian after RWA is

$$H = -\sum_{m=\pm 1/2} \hbar\Omega\, |\mathrm{e}, m\rangle\langle \mathrm{g}, m|\, \mathrm{e}^{\mathrm{i}(\omega_\mathrm{e} - \omega_\mathrm{g} - \omega)} + \text{h.c.}. \tag{42}$$

Since the detuning

$$\Delta = \omega_\mathrm{e} - \omega_\mathrm{g} - \omega \tag{43}$$

is zero, the RWA Hamiltonian is

$$H = -\sum_{m=\pm 1/2} \hbar\Omega\, |\mathrm{e}, m\rangle\langle \mathrm{g}, m| + \text{h.c.}. \tag{44}$$

Now we include the damping terms. Note that the polarization of spontaneous radiation is not necessarily along the z axis. Emission of a σ_+ photon introduces a $1/2 \to -1/2$ decay channel. Emission of a σ_- photon introduces a $-1/2 \to 1/2$ decay channel. Emission of a π photon introduces $1/2 \to 1/2$, $-1/2 \to -1/2$ channels. Therefore

$$\begin{aligned} C_{\sigma_+} &= \sqrt{\frac{2}{3}\Gamma_\mathrm{ge}}\, |\mathrm{g}, -1/2\rangle\langle \mathrm{e}, 1/2|\,, \quad C_{\sigma_-} = \sqrt{\frac{2}{3}\Gamma_\mathrm{ge}}\, |\mathrm{g}, 1/2\rangle\langle \mathrm{e}, -1/2|\,, \\ C_\pi &= -\sqrt{\frac{1}{3}\Gamma_\mathrm{ge}}\, |\mathrm{g}, -1/2\rangle\langle \mathrm{e}, -1/2| + \sqrt{\frac{1}{3}\Gamma_\mathrm{ge}}\, |\mathrm{g}, 1/2\rangle\langle \mathrm{e}, 1/2|\,, \end{aligned} \tag{45}$$

and

$$\begin{aligned} H_\mathrm{eff} &= -\sum_{m=\pm 1/2} \hbar\Omega\, |\mathrm{e}, m\rangle\langle \mathrm{g}, m| + \text{h.c.} - \frac{\mathrm{i}\hbar}{2} \sum_{p=\sigma_+,\sigma_-,\pi} C_p^\dagger C_p \\ &= -\sum_{m=\pm 1/2} \hbar\Omega\, |\mathrm{e}, m\rangle\langle \mathrm{g}, m| + \text{h.c.} - \frac{\mathrm{i}\hbar}{2}\Gamma_\mathrm{ge}(|\mathrm{e}, -1/2\rangle\langle \mathrm{e}, -1/2| + |\mathrm{e}, 1/2\rangle\langle \mathrm{e}, 1/2|). \end{aligned} \tag{46}$$

(b) The master equation is just

$$\dot\rho = \frac{1}{\mathrm{i}\hbar}[H_\mathrm{eff}, \rho] + \sum_{p=\sigma_+,\sigma_-,\pi} C_p^\dagger \rho C_p. \tag{47}$$