Phenomena That Can Be Explained Solely by Band Theory

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This article is a reading note of Xiaogang Wen's Quantum Field Theories of Many-body Systems, Chapter 4.

1 The shape of the Fermi surface and algebraic long-range orders

In this section we explicitly evaluate the equal-time Green function. An important fact is that it is highly affected by the shape of the Fermi surface. When $T = 0$, we have (when not explicitly mentioned, when there is no spin polarization mentioned, we are working with only one spin polarization) Sec. 4.2.4

$$
\begin{aligned}
\mathrm{i}G(-0^+, \boldsymbol{x}) &= \mathcal{T} \left\langle c(\boldsymbol{x}, -0^+) c^\dagger(0,0) \right\rangle = - \left\langle c^\dagger(0,0) c(\boldsymbol{x}, 0) \right\rangle \\
&= - \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} n_\mathrm{F}(\xi_{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = - \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \Theta(-\xi_{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.
\end{aligned} \tag{1}
$$

We define

$$
\begin{aligned}
\tilde{N}(k, \hat{\boldsymbol{x}}) &:= \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \Theta(-\xi_{\boldsymbol{k}}) \delta(k - \boldsymbol{k}\cdot\hat{\boldsymbol{x}}) \\
&= \frac{1}{(2\pi)^d} \times \text{intersection area between the Fermi sea and the } k = \boldsymbol{k}\cdot\hat{\boldsymbol{x}} \text{ plane},
\end{aligned} \tag{2}
$$

where the second line is since the δ-function is non-zero on the plane $\boldsymbol{k}\cdot\hat{\boldsymbol{x}} = k$ in the momentum space, and we have

$$
\int \mathrm{d}^d \boldsymbol{k}\, \delta(k - \boldsymbol{k}\cdot\hat{\boldsymbol{x}}) \cdots = \int \mathrm{d}^{d-1} S \, \frac{1}{|\boldsymbol{\nabla}_{\boldsymbol{k}}(\boldsymbol{k}\cdot\hat{\boldsymbol{x}})|} \cdots = \int \mathrm{d}^{d-1} S \cdots .
$$

Since when $k = \boldsymbol{k}\cdot\hat{\boldsymbol{x}}$, we have $k|\boldsymbol{x}| = \boldsymbol{k}\cdot\boldsymbol{x}$, we have

$$
\mathrm{i}G(-0^+, \boldsymbol{x}) = - \int_{-\infty}^{\infty} \mathrm{d}k \, \tilde{N}(k, \hat{\boldsymbol{x}}) \mathrm{e}^{\mathrm{i}k|\boldsymbol{x}|}. \tag{3}
$$

Now the most important task is to evaluate (2). Since it strongly depends on the shape of the Fermi surface, we are not going to give a generalized form of (2). What we want to focus on is the fact that Fermionic systems usually have *algebraic long-range orders*. The Fourier transformation relation in the first equation of (2) means that a long-range component in $G(-0^+, \boldsymbol{x})$ is caused by a highly localized feature in $\tilde{N}(k, \hat{\boldsymbol{x}})$, and smooth, continuous components in $N(k, \hat{\boldsymbol{x}})$ with regard to k contribute to details with small characteristic length scales in $G(-0^+, \boldsymbol{x})$. Therefore, to investigate possible long-range orders in $G(-0^+, \boldsymbol{x})$, we need to look for something that is kind of *singular* in $\tilde{N}(k, \hat{\boldsymbol{x}})$.

From Figure 1 find that when $\hat{\boldsymbol{x}}$ is given, $\tilde{N}(k, \hat{\boldsymbol{x}})$ is only non-zero in the interval

$$
\boldsymbol{k}_\mathrm{F}(-\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} < k < \boldsymbol{k}_\mathrm{F}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}}, \tag{4}
$$

where we define $\boldsymbol{k}_\mathrm{F}(\hat{\boldsymbol{x}})$ to be the point of tangency of a tangent plane of the Fermi surface that is perpendicular to $\boldsymbol{x}$. Therefore, there are two singularities in the derivative of $\tilde{N}(k, \hat{\boldsymbol{x}})$ with respect to k, which are the upper and lower limits of the interval. Therefore, the terms that contribute to a long-range order are

$$
\begin{aligned}
\tilde{N}(k, \hat{\boldsymbol{x}}) \sim\, & c_+ \Theta\left(\boldsymbol{k}_F(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k\right) \left|\boldsymbol{k}_F(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k\right|^{(d-1)/2} \\
&+ c_- \Theta\left(-\boldsymbol{k}_F(-\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} + k\right) \left|-\boldsymbol{k}_F(-\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} + k\right|^{(d-1)/2},
\end{aligned} \tag{5}
$$

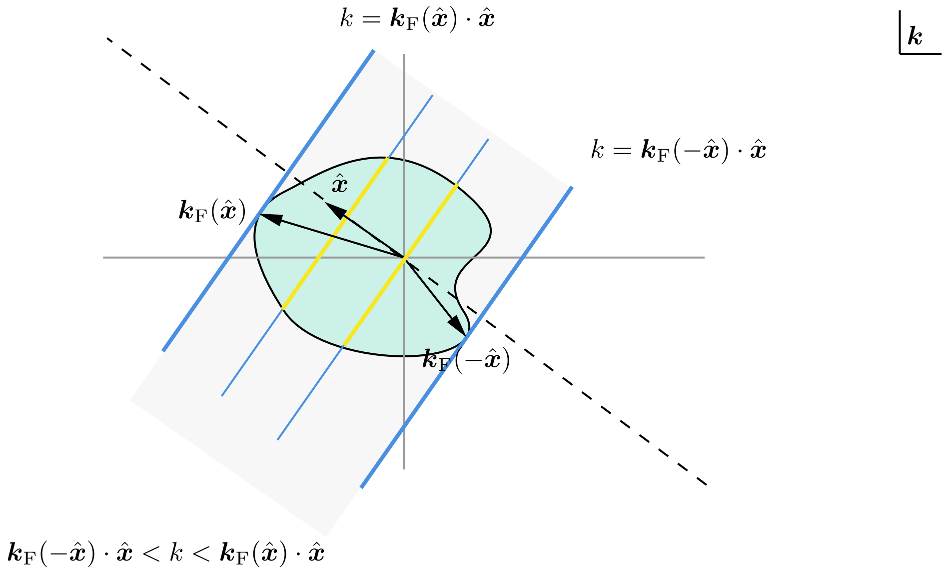

Figure 1: The shape of the Fermi surface and (2). The green shadow is the Fermi sea. The slim blue lines are $k = \boldsymbol{k} \cdot \hat{\boldsymbol{x}}$ planes. The yellow lines represents the intersection between the Fermi sea and $k = \boldsymbol{k} \cdot \hat{\boldsymbol{x}}$ planes. The two thick blue lines are the $\boldsymbol{k} \cdot \hat{\boldsymbol{x}} = \boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{x}}$ plane and the $\boldsymbol{k} \cdot \hat{\boldsymbol{x}} = \boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{x}}$ plane. $\tilde{N}$ is only non-zero in the grey area where $\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{x}} < k < \boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{x}}$.

where the exponent $(d-1)/2$ can be explained by Figure 2(a). Now we can evaluate (3) and get

$$\mathrm{i}G\left(-0^{+}, \boldsymbol{x}\right)\big|_{\boldsymbol{x}\to\infty} \sim \text{const} \times \left(c_{+}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\boldsymbol{x}}\mathrm{c}^{-\mathrm{i}\frac{\pi(d+1)}{4}} + c_{-}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}})\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\frac{\pi(d+1)}{4}}\right)\frac{1}{|x|^{(d+1)/2}}. \tag{6}$$

The first term comes from

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k|\boldsymbol{x}|}\,\mathrm{d}k\,\Theta(\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k)|\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k|^{(d-1)/2} \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k')|\boldsymbol{x}|}\Theta(k')|k'|^{(d-1)/2} \\
&= \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\boldsymbol{x}} \int_{0}^{\infty} k'^{(d-1)/2}\mathrm{e}^{-\mathrm{i}k'x}\,\mathrm{d}k',
\end{aligned}$$

and we have

$$\int_0^\infty k^D \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = (\mathrm{i}x)^{-1-D}\Gamma(1+D), \quad \operatorname{Re} D > -1,\ \operatorname{Im} x > 0.$$

The second term can be obtained in a similar manner. Note that (5) is not zero outside (4), and when integrating over k, we need to set a cutoff for both terms in (5), but this does not change the form of (6), since we just have

$$\int_0^\Lambda k^D \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = (\mathrm{i}x)^{-1-D}(\Gamma(1+D) - \Gamma(1+D, \mathrm{i}x\Lambda)).$$

Actually, the form of (6) can be observed by a dimensional analysis and some common sense about the step function – the dimensional analysis of $k^{(d-1)/2}\,\mathrm{d}k$ means we have a $|\boldsymbol{x}|^{-((d-1)/2+1)}$ factor in $(-0^{+}, \boldsymbol{x})$, and the step function brings in the oscillation. We see that (6) oscillates as $|\boldsymbol{x}| \to \infty$ and damps in an algebraic way. The algebraic damping eventually comes from the mere existence of a sharp boundary of $n_{\mathrm{F}}(\xi_{\boldsymbol{k}})$, or in other words, comes from the existence of the Fermi surface.

Our derivation above does not include the case in which there are uncountably infinite $\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})$'s. This happens when a part of the Fermi surface is flat. This means we should not approximate the behavior of $\tilde{N}(k, \hat{\boldsymbol{x}})$ near the boundary points of (4) with Figure 1(b). In this case, we have

$$\tilde{N}(k, \hat{\boldsymbol{x}}) \sim c_{+}\Theta\left(\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} - k\right) + c_{-}\Theta\left(-\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} + k\right)\left|-\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{x}} + k\right|^{(d-1)/2}. \tag{7}$$

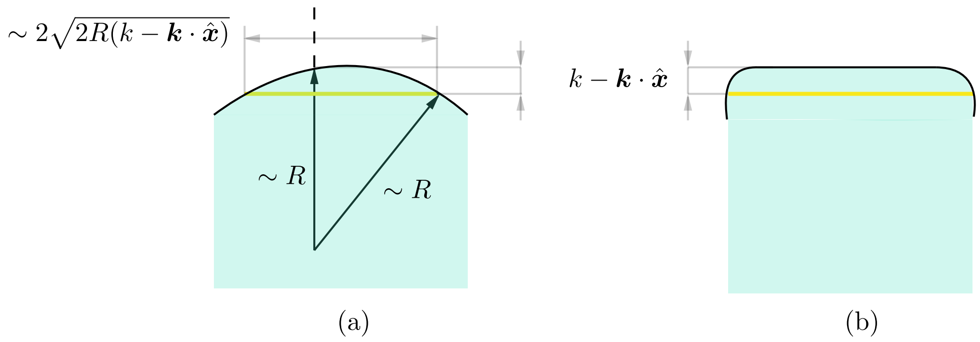

Figure 2: Estimation of $\tilde{N}(k,\hat{\boldsymbol{x}})$. (a) The case when the Fermi surface is not flat. The intersection in (2) is a $d-1$ hypersurface, the "radius" of which is about $2\sqrt{2R(k-\boldsymbol{k}\cdot\hat{\boldsymbol{x}})} \sim \sqrt{k-\boldsymbol{k}\cdot\hat{\boldsymbol{x}}}$, and therefore the "area" of the intersection is proportion to $(k-\boldsymbol{k}\cdot\hat{\boldsymbol{x}})^{(d-1)/2}$. (b) The Fermi surface is flat on some directions. In this case there is no well-defined R, and the "area" of the intersection is almost constant when k enters (4).

Again we evaluate (3) and get

$$\mathrm{i}G(-0^+,\boldsymbol{x})\big|_{\boldsymbol{x}\to\infty} \sim -\mathrm{i}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})}\frac{1}{|\boldsymbol{x}|}. \tag{8}$$

Note that here we throw away the term contributed by the second term of (7) because it decays more quickly ($\sim |\boldsymbol{x}|^{-(d+1)}$) than the contribution of the first term ($\sim |\boldsymbol{x}|$). In other words, a flat Fermi surface induces an algebraic long-range order that decays slower. If the Fermi surface is flat on both side – i.e. $\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})$ and $\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}})$ – then we have

$$\mathrm{i}G(-0^+,\boldsymbol{x})\big|_{\boldsymbol{x}\to\infty} \sim -\mathrm{i}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(\hat{\boldsymbol{x}})}\frac{1}{|\boldsymbol{x}|} + \mathrm{i}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{F}}(-\hat{\boldsymbol{x}})\cdot\boldsymbol{x}}\frac{1}{|\boldsymbol{x}|}. \tag{9}$$

2 Density-density correlation function

Now we discuss the simplest two-body correlation function. This topic is important for two reasons. First, it gives the electrostatics of a band material. Second, if a **charge density wave (CDW)** order forms, we can see a periodic pattern in the real space density-density correlation function, and if we are able to find something like the algebraic long-range order discussed in the past section, it is a good hint of what kind of Fermi surface tends to induce a CDW. Note that in the jellium model (see here), the only strong correlation effect is the Wigner crystal. We can conclude that the rich strong correlation effects in condensed matter physics are mostly controlled by the *lattice*, and the lattice shapes the bands and thus the Fermi surface. Sec. 4.3.1

3 Linear response and effective theory

Chern-Simons