ω-spectrum and SPT by Tian Yuan

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Suppose $X \in \mathsf{Top}$. We define the **cone** of X as

$$\mathrm{C}\, X \coloneqq X \times I / X \times \{1\}. \tag{1}$$

The meaning of "cone" is quite clear: we first make a cylinder with cross section X and then make it into a cone. Similarly we define the **suspension** of X as

$$\mathrm{S}\, X = X \times I / (X \times \{1\} \cup X \times \{0\}). \tag{2}$$

For pointed space $X \in \mathsf{Top}_*$ we have

$$\Sigma\, X = \mathrm{S}\, X / x_0 \times I. \tag{3}$$

Now suppose we have two topological spaces A and X, and A is included in X. We have the following sequence

$$A \hookrightarrow X \hookrightarrow \underbrace{X \cup \mathrm{C}\, A}_{\simeq X/A} \hookrightarrow \underbrace{(X \cup \mathrm{C}\, A) \cup \mathrm{C}\, X}_{\simeq \mathrm{S}\, A} \hookrightarrow \underbrace{((X \cup \mathrm{C}\, A) \cup \mathrm{C}\, X) \cup \mathrm{C}(X \cup \mathrm{C}\, A)}_{\simeq \mathrm{S}\, X} \hookrightarrow \cdots. \tag{4}$$

Here the union operation is the "geometric" one and uses the natural identifications like the one of A and $A \times \{0\}$. Continuing this sequence we have

$$\mathrm{S}\, A \hookrightarrow \mathrm{S}\, X \hookrightarrow \underbrace{\mathrm{S}\, X / \mathrm{S}\, A}_{\mathrm{S}(X/A)} \hookrightarrow \mathrm{S}^2\, A \hookrightarrow \mathrm{S}^2\, X \hookrightarrow \cdots. \tag{5}$$

We also have a version of (5) for pointed spaces, which can be obtained by replacing S with Σ.

(5) and its pointed space version give us a strong sense of cohomology. Here we define the **generalized cohomology**. This is defined by a functor $h^n : (\mathsf{hCW}_*)^{\mathrm{op}} \to \mathsf{Ab}$ where hSW_* is the category of CW complexes in which we have modded out homotopy equivalence, and the following conditions (called the **axioms of generalized cohomology**) hold:

1. We have the following exact sequence

$$\cdots \to h^n(X/A) \xrightarrow{q^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} h^{n+1}(X/A) \to \cdots, \tag{6}$$

   where δ is a boundary operator (the choice of which is not specified by the axioms of generalized cohomology), and we have the commutative diagram

$$\begin{array}{ccc} h^n(A) & \xrightarrow{\delta} & h^{n+1}(X/A) \\ \downarrow & & \downarrow \\ h^n(B) & \xrightarrow{\delta} & h^{n+1}(Y/B). \end{array} \tag{7}$$

2. we have

$$h^n(V_\alpha X_\alpha) \xrightarrow{\prod_\alpha i_\alpha^*} \prod_\alpha h^n(X_\alpha). \tag{8}$$

Condition 1 is equivalent to the condition that $h^n(X)$ is naturally equivalent to $h^{n+1}(\Sigma\, X)$ and we have a short exact sequence

$$h^n(X/A) \to h^n(X) \to h^n(A). \tag{9}$$

Define the **loop space** $\Omega\, X$ of X as the set of all possible $I \to X$ with good enough properties. It can be verified that $\Omega\, X$ has a topological structure. Define $\langle A, B\rangle$ to be homotopy classes of (basepoint preserving?) maps from A to B. We have group structure on $\langle A, B\rangle$. We have

$$\langle \Sigma\, X, Y\rangle \simeq \langle X, \Omega\, Y\rangle, \tag{10}$$

and we know that x_0 in ΣX is mapped to the trivial loop because it is contractable. Note that from (10) we have

$$\pi_{n+1}(Y) \simeq \pi_n(\Omega Y). \tag{11}$$

It can also be proved that $\langle \cdot, \Omega^2 \cdot \rangle$ is always an Abelian group.

The definition of **Ω-spectrum** can now be given. $\{K_n\}_{n\in\mathbb{Z}}$ is an Ω-spectrum if

$$\Omega\, K_{n+1} \simeq K_n. \tag{12}$$

From this definition we find a realization of generalized cohomology. The functor is

$$h^n(\cdot) = \langle \cdot, K_n \rangle. \tag{13}$$

Since $\langle \cdot, \Omega^2 \cdot \rangle$ is always an Abelian group, we find

$$\langle X, K_n \rangle \simeq \langle X, \Omega\, K_{n+1} \rangle \simeq \langle X, \Omega^2\, K_{n+2} \rangle \in \mathsf{Ab},$$

so we see (13) is definitely to Ab.

The proof of the exact sequence condition 1 can be finished with the help of (5).

We finish this introduction with the **Brown representation**, which says that every generalized cohomology can be implemented by an Ω-spectrum in that we can always find a series $\{K_n\}_{n\in\mathbb{Z}}$ such that

$$\langle X, K_n \rangle \simeq h^n(X). \tag{14}$$