QFT I, Homework 4

Jinyuan Wu

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Scalar QED Consider the theory of a complex scalar field ϕ interacting with the electromagnetic field A^μ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(D_\mu \phi\right)^* D^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

where $D_\mu = \partial_\mu + \mathrm{i} e A_\mu$ is the usual gauge covaraint derivative.

(a) Show the Lagrangian is invariant under the gauge transformations

$$\phi(x) \to \mathrm{e}^{-\mathrm{i}\alpha(x)} \phi(x), \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x). \tag{2}$$

(b) Derive the Feynman rules for the interaction between photons and scalar particles.

(c) Draw all the leading-order Feynman diagrams and compute the amplitude for the process $\gamma\gamma \to \phi\phi^*$.

(d) Compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$. You can take an average over all initial state polarizations. For simplicity, you can restrict your calculation in the limit $m = 0$.

(e) Draw all leading order Feynman diagrams, that contribute to the Compton scattering process $\gamma\phi \to \gamma\phi$ and compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$ with $m = 0$.

Solution

(a) Under the gauge transformation (2), we have

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu \left(A_\nu + \frac{1}{e} \partial_\nu \alpha \right) - \partial_\nu \left(A_\mu + \frac{1}{e} \partial_\mu \alpha \right) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu},$$

so the first term in (1) remains the same. It is obvious that under (2)

$$\phi^* \phi \to \phi'^* \phi' = \mathrm{e}^{\mathrm{i}\alpha} \phi^* \mathrm{e}^{-\mathrm{i}\alpha} \phi = \phi^* \phi,$$

so the third term in (1) is also invariant. Also we have

$$\begin{aligned} D^\mu \phi \to (\partial^\mu + \mathrm{i} e A'^\mu) \phi' &= (\partial^\mu + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \mathrm{e}^{-\mathrm{i}\alpha} \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} (\partial^\mu - \mathrm{i} \partial^\mu \alpha + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} D^\mu \phi, \end{aligned}$$

and also

$$(D^\mu \phi)^* = \mathrm{e}^{\mathrm{i}\alpha} D^\mu \phi,$$

so $D^\mu \phi (D^\mu \phi)^*$ is also invariant. Therefore (1) is invariant under (2).

(b) We make the following expansion of Fourier transformation. For the complex scalar field we have

$$\phi(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i} p \cdot x} + b^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i} p \cdot x}). \tag{3}$$

which was proved in (10) in Homework 2. The vector field is expanded as

$$A_\mu(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \sum_{r=1}^{2} \epsilon^r_\mu(\boldsymbol{p}) \left(a^\dagger_{\boldsymbol{p},r} \mathrm{e}^{\mathrm{i} p \cdot x} + a_{\boldsymbol{p},r} \mathrm{e}^{-\mathrm{i} p \cdot x} \right). \tag{4}$$

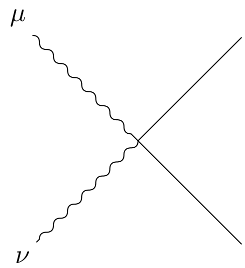
Expanding (2) we have

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{scalarQED}}, \tag{5}$$

where $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{vector}}$ are Lagrangians of free scalar field and free massless vector field, and

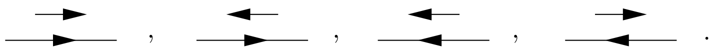
$$\begin{aligned}\mathcal{L}_{\text{scalarQED}} &= (D_\mu\phi)^* D^\mu\phi - (\partial_\mu\phi)^*\partial^\mu\phi \\ &= e^2\eta_{\mu\nu}A^\mu A^\nu\phi^*\phi - \mathrm{i}eA_\mu\phi^*\partial^\mu\phi + \mathrm{i}e\partial_\mu\phi^* A^\mu\phi.\end{aligned} \tag{6}$$

The first term has no derivatives. Therefore it gives the following (momentum space) vertex:



$$= 2\mathrm{i}e^2\eta_{\mu\nu}, \tag{7}$$

where the factor i comes from the time evolution operator and the factor 2 comes from the fact that there are two identical photon lines. The two ϕ lines can be any of the following four:



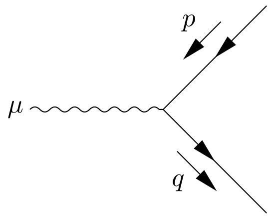
The second term gives

$$-\mathrm{i}eA_\mu\phi^*\partial^\mu\phi \sim -\mathrm{i}eA_\mu(a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x} + b_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x})(-\mathrm{i}(p'\cdot x)a_{\boldsymbol{p}'}\mathrm{e}^{-\mathrm{i}p'\cdot x} + \mathrm{i}(p'\cdot x)b^\dagger_{\boldsymbol{p}'}\mathrm{e}^{\mathrm{i}p'\cdot x}),$$

and the third term is its complex conjugate. Therefore, the $a^\dagger a$ term in the Lagrangian is

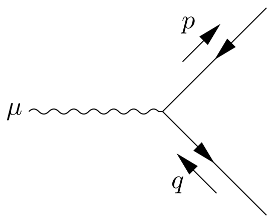
$$\sim -e(p_1+p_2)_\mu A^\mu a^\dagger_{\boldsymbol{p}_1}a_{\boldsymbol{p}_2},$$

so after adding the i factor from the time evolution operator we have



$$= -\mathrm{i}e(p_\mu + q_\mu), \tag{8}$$

and we can change the direction of a momentum line and a ϕ-particle line arbitrarily; if a momentum line goes in contrast to the corresponding particle line, then we need to add a minus sign to the corresponding momentum. For example we have



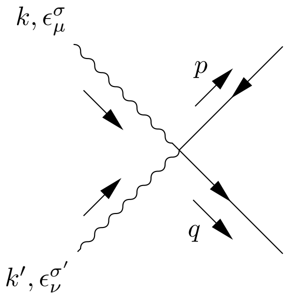
$$= \mathrm{i}e(p_\mu + q_\mu). \tag{9}$$

There are four vertices in this type in total.

Note

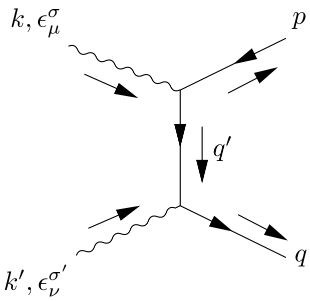
Here we follow the notation of Peskin, i.e. using the *momentum* arrow to denote whether this line represents creation or annihilation and using the arrow *on* a particle line to show whether this line represents a particle (if the direction of the particle line is parallel to the direction of the momentum line) or a antiparticle (otherwise). The real direction of a 4-momentum is *not* represented in any arrow.

(c) We enumerate over all possible diagrams. The vertex (7) itself is a diagram:



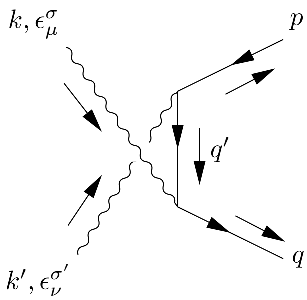
$$= (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \times 2\mathrm{i}e^2 \eta_{\mu\nu} =: \mathrm{i}\mathcal{M}_4. \tag{10}$$

Combining two (8)-type vertices we have a t-channel



$$\begin{aligned} &= \epsilon^\sigma_\mu e^{\sigma'}_\nu \times \mathrm{i}e(p - (k-p))^\mu \times \mathrm{i}e(-(k-p) - q)^\nu \times \frac{\mathrm{i}}{(k-p)^2 - m^2 + \mathrm{i}0^+} \\ &= -\mathrm{i}e^2 \epsilon^\sigma_\mu (2p - k)^\mu \epsilon^{\sigma'}_\nu (-k + p - q)^\nu \frac{1}{(k-p)^2 - m^2 + \mathrm{i}0^+} =: \mathrm{i}\mathcal{M}_t, \end{aligned} \tag{11}$$

and a u-channel



$$\begin{aligned} &= \epsilon^\sigma_\mu \epsilon^{\sigma'}_\nu \times \mathrm{i}e(p - (q-k))^\nu \times \mathrm{i}e(-q - (q-k))^\mu \times \frac{\mathrm{i}}{(q-k)^2 - m^2 + \mathrm{i}0^+} \\ &= \mathrm{i}e^2 \epsilon^\sigma_\mu (2q - k)^\mu \epsilon^{\sigma'}_\nu (p - q + k)^\nu \frac{1}{(q-k)^2 - m^2 + \mathrm{i}0^+} =: \mathrm{i}\mathcal{M}_u. \end{aligned} \tag{12}$$

Note

We *do not* need to distinguish the direction of the q' momentum line. This line can be either a particle line or an antiparticle line, but since the ordinary propagator $\mathrm{i}/(p^2 - m^2 + \mathrm{i}0^+)$ is obtained by summing up the two cases, when we write down this propagator, we have automatically considered both processes.

Summing everything up, we have

$$\begin{aligned} \mathrm{i}\mathcal{M}(\gamma\gamma \to \phi\phi^*) &= \mathrm{i}(\mathcal{M}_4 + \mathcal{M}_t + \mathcal{M}_u) \\ &= \mathrm{i}e^2 (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \left(2\eta_{\mu\nu} + \frac{(k-2p)_\mu (k'-2q)_\nu}{t - m^2} + \frac{(k-2q)_\mu (k'-2p)_\nu}{u - m^2} \right) \\ &=: \mathrm{i}(\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu e^2 \mathcal{M}_{\mu\nu}, \end{aligned} \tag{13}$$

where

$$t = (k-p)^2, \quad u = (q-k)^2. \tag{14}$$

(d) We work in the center-of-mass frame, and therefore we have $k = (|\boldsymbol{k}|, \boldsymbol{k})$, and $k' = (|\boldsymbol{k}|, -\boldsymbol{k})$. The massless limit can be calculated with Eq. (4.85) in Peskin, which is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\mathrm{CM}} = \sum_{\text{spins}} \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{CM}}^2}, \tag{15}$$

What we need is $|\mathcal{M}|^2$. We have

$$\begin{aligned}\sum_{\text{spins}}|\mathcal{M}|^2 &= e^4\sum_{\epsilon^\sigma,\epsilon^{\sigma'}}P(\epsilon^\sigma,\epsilon^{\sigma'})(\epsilon^\sigma)^\mu(\epsilon^{\sigma'})^\nu(\epsilon^\sigma)^{\rho*}(\epsilon^{\sigma'})^{\delta*}\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\delta}\\ &= e^4\sum_{\sigma=\pm1}(\epsilon^\sigma)^\mu(\epsilon^\sigma)^{\rho*}\sum_{\sigma'=\pm1}(\epsilon^{\sigma'})^\nu(\epsilon^{\sigma'})^{\delta*}\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\delta}.\end{aligned}$$

Using Ward identity (see Eq. (5.75) and relevant discussion about it in Peskin) we have

$$\begin{aligned}\sum_{\sigma=\pm1}(\epsilon^\sigma)^\mu(\epsilon^\sigma)^{\rho*}\sum_{\sigma'=\pm1}(\epsilon^{\sigma'})^\nu(\epsilon^{\sigma'})^{\delta*}\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\delta} &= (-g^{\mu\rho})(-g^{\nu\delta})\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\delta}\\ &= \mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^*.\end{aligned}$$

Substitute (13) into the above equation, we have

$$\begin{aligned}\mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^* = 4\eta_{\mu\nu}\eta^{\mu\nu} &+ \frac{4(k-2p)\cdot(k'-2q)}{t}+\frac{4(k-2q)\cdot(k'-2p)}{u}\\ &+2\frac{(k-2q)\cdot(k-2p)(k'-2p)\cdot(k'-2q)}{ut}\\ &+\frac{(k-2q)^2(k'-2p)^2}{u^2}+\frac{(k-2p)^2(k'-2q)^2}{t^2}.\end{aligned}$$

The equation above has been simplified using the fact that $k^2=k'^2=p^2=q^2=0$ and

$$\begin{aligned}t &= (k-p)^2=-2k\cdot p=-2k'\cdot q,\\ u &= (q-k)^2=-2q\cdot k=-2p\cdot k',\\ s &= (k+k')^2=2k\cdot k'=2p\cdot q,\end{aligned} \tag{16}$$

and we can evaluate terms in $\mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^*$. We have

$$4\eta_{\mu\nu}\eta^{\mu\nu}=16,$$

and

$$\begin{aligned}\frac{4(k-2p)\cdot(k'-2q)}{t} &= \frac{4(k\cdot k'-2p\cdot k'-2k\cdot q+4p\cdot q)}{t}\\ &= \frac{4(-s/2+4\times(-s/2)+u+u)}{t}=-\frac{10s}{t}+\frac{8u}{t},\end{aligned}$$

and similarly

$$\frac{4(k-2q)\cdot(k'-2p)}{u}=-\frac{10s}{u}+\frac{8t}{u}.$$

The fourth term is

$$\begin{aligned}&2\frac{(k-2q)\cdot(k-2p)(k'-2p)\cdot(k'-2q)}{ut}\\ &=2\frac{(-2(p+q)\cdot k+4p\cdot q)(-2(p+q)\cdot k'+4p\cdot q)}{(-2q\cdot k)(-2k\cdot p)}\\ &=2\frac{(-2(k+k')\cdot k+4p\cdot q)(-2(k+k')\cdot k'+4p\cdot q)}{(-2q\cdot k)(-2k\cdot p)}\\ &=2\frac{(-2k\cdot k'+4p\cdot q)^2}{4(q\cdot k)(p\cdot k)}\\ &=\frac{2s^2}{ut}.\end{aligned}$$

The fifth term is

$$4\frac{(k-2q)^2(k'-2p)^2}{u^2}=4\frac{(-2k\cdot q)(-2k'\cdot p)}{u^2}=4,$$

and so does the sixth term. So summing everything up, we have

$$\mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^*=24+\frac{2s^2}{ut}+\frac{8u}{t}+\frac{8t}{u}+\frac{10s}{ut}(t+u).$$

In the massless case we have

$$s + t + u = 0, \tag{17}$$

and therefore

$$\begin{aligned}
\mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^* &= 24 + \frac{2s^2}{ut} + \frac{8u}{t} + \frac{8t}{u} + \frac{10s}{ut}(-s) \\
&= 24 - \frac{8s^2}{ut} + \frac{8(u^2+t^2)}{ut} \\
&= 24 - \frac{8(u+t)^2}{ut} + \frac{8(u^2+t^2)}{ut} \\
&= 24 - 16 = 8,
\end{aligned}$$

so we have

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 8e^4. \tag{18}$$

Also, note that

$$E_{\text{CM}} = 2|\boldsymbol{k}|, \quad E_{\text{CM}}^2 = 4|\boldsymbol{k}|^2 = 2k \cdot k' = s,$$

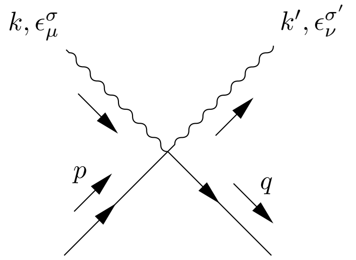
and therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \times 8e^4 \times \frac{1}{s} = \frac{e^4}{8\pi^2 s}, \tag{19}$$

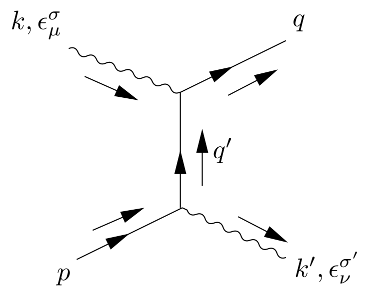
so we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = 2\pi \times \frac{e^4}{8\pi^2 s} = \frac{e^4}{4\pi s}. \tag{20}$$

(e) We enumerate $\gamma\phi \to \gamma\phi$ diagrams. We have

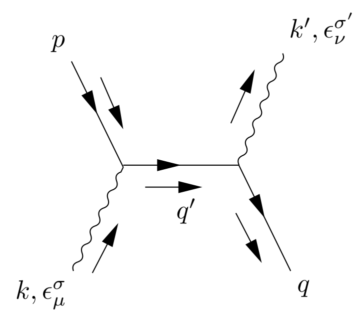


$$= (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \times 2\mathrm{i}e^2 \eta_{\mu\nu} =: \mathrm{i}\mathcal{M}_4, \tag{21}$$



$$\begin{aligned}
&= (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \times \mathrm{i}e(-q-(q-k))_\mu \\
&\quad \times \mathrm{i}e(-p-(q-k))_\nu \times \frac{\mathrm{i}}{(q-k)^2 - m^2 + \mathrm{i}0^+} \\
&= -\mathrm{i}e^2 (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu (k-2q)_\mu (k'-2p)_\nu \frac{1}{(q-k)^2 - m^2 + \mathrm{i}0^+} \\
&=: \mathrm{i}\mathcal{M}_t,
\end{aligned} \tag{22}$$

and



$$\begin{aligned}
&= (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \times \mathrm{i}e(-p-(k+p))_\mu \times \mathrm{i}e(-q-(k+p))_\nu \\
&\quad \times \frac{\mathrm{i}}{(p+k)^2 - m^2 + \mathrm{i}0^+} \\
&= -\mathrm{i}e^2 (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu (2p+k)_\mu (2q+k')_\nu \frac{1}{(k+p)^2 - m^2 + \mathrm{i}0^+} \\
&=: \mathrm{i}\mathcal{M}_s.
\end{aligned} \tag{23}$$

Summing everything up and take the $m \to 0$ limit, and we have

$$\begin{aligned} \mathrm{i}\mathcal{M} &= \mathrm{i}e^2(\epsilon^\sigma)^\mu(\epsilon^{\sigma'})^\nu\Big(2\eta_{\mu\nu} - (k-2q)_\mu(k'-2p)_\nu\frac{1}{(q-k)^2} - (2p+k)_\mu(2q+k')_\nu\frac{1}{(k+p)^2}\Big) \\ &=: \mathrm{i}e^2(\epsilon^\sigma)^\mu(\epsilon^{\sigma'})^\nu\mathcal{M}_{\mu\nu}. \end{aligned} \tag{24}$$

Now we can repeat the process in (d). We define

$$\begin{aligned} s &= (k+p)^2 = 2k\cdot p = 2k'\cdot q, \\ t &= (k-k')^2 = -2k\cdot k' = -2p\cdot q, \\ u &= (k-q)^2 = -2k\cdot q = -2k'\cdot p, \end{aligned} \tag{25}$$

and again we have

$$\begin{aligned} \sum_{\text{spins}}|\mathcal{M}|^2 &= e^4\sum_{\text{spins}}(\epsilon^\sigma)^\mu(\epsilon^{\sigma'})^\nu\mathcal{M}_{\mu\nu}(\epsilon^\sigma)^\rho(\epsilon^{\sigma'})^\delta(\mathcal{M}_{\rho\delta})^* \\ &= e^4(-g^{\mu\rho})(-g^{\nu\delta})\mathcal{M}_{\mu\nu}(\mathcal{M}_{\rho\delta})^* \\ &= e^4\mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^*, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^* = 16 &+ \frac{(k-2q)^2(k'-2p)^2}{(q-k)^4} + \frac{(k+2p)^2(k'+2q)^2}{(k+p)^4} - 4\frac{(k-2q)\cdot(k'-2p)}{(q-k)^2} \\ &- 4\frac{(k+2p)\cdot(k'+2q)}{(k+p)^2} + 2\frac{(k-2q)\cdot(k+2p)(k'-2p)\cdot(k'+2q)}{(k+p)^2(k-q)^2}. \end{aligned}$$

We have

$$\frac{(k-2q)^2(k'-2p)^2}{(q-k)^4} = \frac{(-4k\cdot q)(-4k'\cdot p)}{(-2q\cdot k)^2} = \frac{(-4k\cdot q)(-4k\cdot q)}{(-2q\cdot k)^2} = 4.$$

and the third term also evaluates to 4. The fourth term is

$$\begin{aligned} -4\frac{(k-2q)\cdot(k'-2p)}{(q-k)^2} &= -4\frac{k\cdot k' + 4q\cdot p - 2p\cdot k - 2q\cdot k'}{-2q\cdot k} \\ &= \frac{10t+8s}{u}, \end{aligned}$$

and similarly the fifth term is

$$-4\frac{(k+2p)\cdot(k'+2q)}{(k+p)^2} = \frac{10t+8u}{s}.$$

The last term is

$$\begin{aligned} 2\frac{(k-2q)\cdot(k+2p)(k'-2p)\cdot(k'+2q)}{(k+p)^2(k-q)^2} &= 2\frac{(2(p-q)\cdot k - 4p\cdot q)(-2(p-q)\cdot k' - 4p\cdot q)}{su} \\ &= 2\frac{(2(k'-k)\cdot k - 4p\cdot q)(-2(k'-k)\cdot k' - 4p\cdot q)}{su} \\ &= 2\frac{(2k\cdot k - 4p\cdot q)^2}{su} \\ &= 2\frac{t^2}{su}. \end{aligned}$$

Putting everything together, we have

$$\begin{aligned} \mathcal{M}_{\mu\nu}(\mathcal{M}^{\mu\nu})^* &= 24 + \frac{2t^2}{su} + \frac{10t+8s}{u} + \frac{10t+8u}{s} \\ &= 24 + \frac{2t^2}{su} + 8\frac{s^2+u^2}{su} + \frac{10t(u+s)}{su} \\ &= 24 + \frac{2t^2}{su} + 8\frac{s^2+u^2}{su} - \frac{10t^2}{su} \\ &= 24 + 8\frac{s^2+u^2}{su} - \frac{8(s+u)^2}{su} \\ &= 24 - 16 = 8, \end{aligned}$$

so we have

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 8e^4. \tag{26}$$

We also have

$$E_{\text{CM}} = 2|\boldsymbol{k}|\,, \quad E_{\text{CM}} = 4|\boldsymbol{k}|^2 = s,$$

so again we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \times 8e^4 \times \frac{1}{s} = \frac{e^4}{8\pi^2 s}, \tag{27}$$

so we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = 2\pi \times \frac{e^4}{8\pi^2 s} = \frac{e^4}{4\pi s}. \tag{28}$$