Quantum Optics by Prof. Saijun Wu

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1 Master equation

We consider a open quantum system, whose Hamiltonian is H, and at each time step, there is a probability of quantum jump to several given states. Suppose the probability of jumping to state $|i\rangle$ is Γ_i, we define

$$C_i = \sqrt{\Gamma_i}\,|i\rangle\langle i|\,, \tag{1}$$

and the system can be described using a stochastic wave function method shown in previous lectures. If we use a density matrix formalism, we find

$$\rho(t+\Delta t) = \sum_i \Gamma_i \Delta t\,|i\rangle\langle i| + (1-\sum_i \Gamma_i)\,|\psi_{\mathrm{s}}(t+\Delta t)\rangle\langle\psi_{\mathrm{s}}(t+\Delta t)|\,,$$

where $|\psi_{\mathrm{s}}\rangle$ evolves according to H. We therefore find

$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[H_{\mathrm{eff}}, \rho] + \sum_i C_i \rho C_i^\dagger, \tag{2}$$

where

$$H_{\mathrm{eff}} = H - \frac{\mathrm{i}\hbar}{2}\sum_i C_i^\dagger C_i. \tag{3}$$

(2) is called **the master equation in Lindblad form**.

We want to check the unitarity of (2). We have

$$\operatorname{tr}\dot{\rho} = -\sum_j \operatorname{tr}\left(\frac{1}{2}C_i^\dagger C_i \rho + \frac{1}{2}\rho C_i^\dagger C_i - C_i \rho C_i^\dagger\right). \tag{4}$$

The last term is called the **recycling term**, which makes the total probability increase, while the first two terms make the total probability decrease. With trace cyclic property, we find the total probability is conserved.

We consider a light-atom interacting system with RWA, where

$$H = \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}, \quad C = \sqrt{\Gamma}\,|g\rangle\langle e|\,, \tag{5}$$

and we have

$$\begin{aligned}
\dot{\rho}_{gg} &= \frac{\mathrm{i}\Omega}{2}\rho_{ge} - \frac{\mathrm{i}\Omega^*}{2}\rho_{eg} + \Gamma\rho_{ee}, \\
\dot{\rho}_{ee} &= -\dot{\rho}_{gg}, \\
\dot{\rho}_{ge} &= \left(-\frac{\Gamma}{2} + \mathrm{i}\Delta\right)\rho_{ge} - \frac{\mathrm{i}\Omega}{2}(\rho_{ee} - \rho_{gg}),
\end{aligned} \tag{6}$$

where

$$\rho_{ee} + \rho_{gg} = 1, \quad \rho_{eg} = \rho_{ge}^*. \tag{7}$$

We define

$$\langle\sigma^x\rangle = \operatorname{Re}\rho_{eg} =: u, \quad \langle\sigma^y\rangle = \operatorname{Im}\rho_{eg} =: v,\ \langle\sigma^z\rangle = \rho_{ee} - \rho_{gg} =: w, \tag{8}$$

and

$$\boldsymbol{n} = (\langle\sigma^x\rangle, \langle\sigma^y\rangle, \langle\sigma^z\rangle), \tag{9}$$

and we have

$$\dot{\boldsymbol{n}} = \boldsymbol{\Omega}\times\boldsymbol{n} - \begin{pmatrix} \gamma_{\mathrm{T}} u \\ \gamma_{\mathrm{T}} v \\ \gamma_{\mathrm{L}}(w+1) \end{pmatrix}, \tag{10}$$

where

$$\gamma_{\mathrm{T}} = \frac{\Gamma}{2} \tag{11}$$

is called the **transverse damping rate** and

$$\gamma_{\mathrm{L}} = \Gamma \tag{12}$$

is called the **longitude relaxing rate**. (10) is called the **optical Bloch equation**.

Now we try to find a stable solution of (10). It is

$$\rho_{ee}^{\text{stable}} = \frac{(\Omega/\Gamma)^2}{1 + 2(\Omega/\Gamma)^2 + 4(\Delta/\Gamma)^2}, \tag{13}$$

and

$$\rho_{ge}^{\text{stable}} = -\frac{\Omega/2}{\Delta - \mathrm{i}\Gamma/2}(2\rho_{ee} - 1). \tag{14}$$

We define

$$S = 2\left(\frac{\Omega}{\Gamma}\right)^2. \tag{15}$$

We can also evaluate the response of the electric dipole. We have

$$\langle d \rangle = \rho_{eg} d_{ge} + \text{h.c.} = \alpha E + \text{c.c.}, \quad \alpha = \frac{1}{1 + S + 4(\Delta/\Gamma)^2}\left(\frac{2\Delta}{\Gamma} + \mathrm{i}\right)\frac{3\lambda^3}{4\pi^2}\epsilon_0 =: \alpha(I). \tag{16}$$

saturated absorption, Saturated absorption spectroscopy **Lamb dip**

2 Rate equation

We choose a *adiabatic* basis, which are dressed states of H_{eff}. In this basis, assuming that the non-diagonal elements of the density matrix damp quickly enough, we have

$$\dot{\rho}_{nn} = -\gamma_n \rho_{nn} + \sum_{m \neq n} \gamma_{nm} \rho_{mm}, \quad \rho_{mn}|_{m \neq n} = 0, \tag{17}$$

which is called the **rate equation**.

Again for a two-level system where RWA works, we have

$$\gamma_{\tilde{g}} = \Gamma \sin^2\theta, \quad \gamma_{\tilde{e}} = \Gamma \cos^2\theta, \gamma_{\tilde{g}\tilde{e}} = \Gamma \sin^4\theta, \gamma_{\tilde{e}\tilde{g}} = \Gamma \cos^4\theta, \tag{18}$$

and the rate equation is

$$\dot{\rho}_{\tilde{g}\tilde{g}} = (-\sin^4\theta \rho_{\tilde{g}\tilde{g}} + \cos^4\theta \rho_{\tilde{e}\tilde{e}})\Gamma. \tag{19}$$

The stable solution is

$$\rho_{\tilde{g}\tilde{g}}^{\text{stable}} = \frac{\cos^4\theta}{\cos^4\theta + \sin^4\theta}, \quad \rho_{\tilde{e}\tilde{e}}^{\text{stable}} = \frac{\sin^4\theta}{\cos^4\theta + \sin^4\theta}. \tag{20}$$

3 How atoms

All previous discussions were based on the assumption that atoms are somehow "fixed" or "trapped" at a given point. This is of course possible (using laser trap or something), but a more interesting case is when atoms are not that constrained. In this case, we need to take the spacial motion of atoms into account.

Consider a stationary mode in a cavity:

$$E = E_0 \cos(kx) \mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}, \tag{21}$$

and at each point the energies of the ground state and the excited state of a two-level atom are different. Since

$$H = \frac{\hbar}{2} \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} \propto E, \tag{22}$$

we have

$$m\left\langle \ddot{\boldsymbol{r}} \right\rangle = -\boldsymbol{\nabla}\left\langle H \right\rangle \propto -\boldsymbol{\nabla} E. \tag{23}$$

Sisyphus cooling

Note: there are some subtleties here.

$$\langle F \rangle = \mathrm{Re}\,\frac{\boldsymbol{\nabla}\Omega}{\Omega}\alpha_r |E|^{\,2} + \mathrm{Im}\,\frac{\boldsymbol{\nabla}\Omega}{\Omega}\alpha_i |E|^{\,2} + \tag{24}$$

We find that the conservative force causes cooling, while the scattering force causes heating.

Doopler cooling is another cooling approach We will find

$$\Delta \to \Delta - kv, \tag{25}$$

$$F = -\beta v \tag{26}$$

What Doopler cooling fails to take into account is the Sisyphus cooling or heating mechanism above.

$$H = \hbar\Delta(|e+\rangle\langle e+| + |e-\rangle\langle e-|) + \frac{\hbar\Omega_{\mathrm{T}}}{2}(\,|e+\rangle\langle g+| + \,|e-\rangle\langle g-|) + \mathrm{h.c.} + \frac{\hbar\Omega_{\sigma-}}{2}(\,|e-\rangle\langle g+| + \mathrm{h.c.}), \tag{27}$$

and the dressed states are

$$|\tilde{g}+\rangle = |g+\rangle + \frac{\Omega_{\sigma-}/2}{\Delta}\,|e-\rangle\,, \quad |\tilde{g}-\rangle = |g-\rangle\,. \tag{28}$$