Advanced Electrodynamics by Prof. Kun Din

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Some contents of the first several lectures are covered in this optics note, but some technical details are worth a separate note.

1 Spherical functions

2 Effective continuous medium

Suppose $f(\boldsymbol{r})$ is a filter or "smoother". Suppose the electric charge is

$$\eta(\boldsymbol{r}) = \sum_n \sum_{i\in C_n} q_i \delta(\boldsymbol{r}-\boldsymbol{r}_i), \tag{1}$$

where n is the index of clusters of electrons, most frequently moleculars. The smoothened version is

$$\begin{aligned}
\langle \eta \rangle (\boldsymbol{r}) &= \int \mathrm{d}^3\boldsymbol{r}'\, \eta(\boldsymbol{r}-\boldsymbol{r}') f(\boldsymbol{r}') \\
&= \int \mathrm{d}^3\boldsymbol{r}' \sum_n \sum_{i\in C_n} q_i \delta(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{r}_i) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \sum_{i\in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i\in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i\in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_n) + \cdots),
\end{aligned}$$

and if the perturbation of the charges is not strong, we can make a cutoff and get

$$\begin{aligned}
&\sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i\in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_n)) \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} (q_n - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{p}_n) \\
&= \sum_n \left(q_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \right) \\
&= \sum_n (q_n f(\boldsymbol{r}-\boldsymbol{r}_n) - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} f(\boldsymbol{r}-\boldsymbol{r}_n)) \\
&= \sum_n (q_n f(\boldsymbol{r}-\boldsymbol{r}_n) - \boldsymbol{\nabla}\cdot(\boldsymbol{p}_n f(\boldsymbol{r}-\boldsymbol{r}_n))).
\end{aligned}$$

Now we note that

$$\langle \delta(\boldsymbol{r}-\boldsymbol{r}_n) \rangle = f(\boldsymbol{r}-\boldsymbol{r}_n),$$

and we get

$$\langle \eta \rangle (\boldsymbol{r}) = \rho(\boldsymbol{r}) - \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}), \tag{2}$$

where

$$\rho(\boldsymbol{r}) = \left\langle \sum_n q_n \delta(\boldsymbol{r}-\boldsymbol{r}_n) \right\rangle, \quad \boldsymbol{P}(\boldsymbol{r}) = \left\langle \sum_n \boldsymbol{p}_n \delta(\boldsymbol{r}-\boldsymbol{r}_n) \right\rangle. \tag{3}$$

TODO: magnetic average

3 Constitutive relations

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi_\mathrm{e}) \boldsymbol{E}, \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \tag{4}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho, \tag{5}$$

4 Frequency dispersion and the Clausius-Mossotti equation

Now we work on a frequent mechanism of frequency dispersion, i.e. the response of the material depending on ω. We consider a oscillator model, where the electrons can be seen as oscillators, the EOM of each of which is

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2 \boldsymbol{r} - m\gamma \boldsymbol{r} + e\boldsymbol{E}_\text{driving}. \tag{6}$$

Note that though the electrons has no interaction, but first, since the electrons are immersed in the radiation created by themselves, m, γ, and ω_0 will be modified, and second, an electron can feel the collective electric field of other electrons, and there must be a term for this in $\boldsymbol{E}_\text{drive}$. Since $\boldsymbol{E}_\text{other electrons} + \boldsymbol{E}_\text{this electron}$ has no long scale effects and only modifies m, γ, and ω_0 (because they it modifies the lattice potential and therefore change ω_0, and it also modifies the phonon spectrum, thus modifying γ, and since we know the radiation of a moving charge has a term proportion to $\ddot{\boldsymbol{d}}$, m is also modified), we can say that

$$\boldsymbol{E}_\text{driving} = \boldsymbol{E}_\text{external} - \boldsymbol{E}_\text{self}.$$

Here and after we write $\boldsymbol{E}_\text{external}$ as $\boldsymbol{E}$, and we have

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2 \boldsymbol{r} - m\gamma \boldsymbol{r} + e(\boldsymbol{E} - \boldsymbol{E}_\text{self}). \tag{7}$$

Note that this can also be seen as "using $\boldsymbol{E}_\text{self}$ to modify m, ω_0 and γ and subtracting $\boldsymbol{E}_\text{sef}$ from $\boldsymbol{E}$". A more generalized discussion of the modifications can be found in discussions around (16.12) in this optics note. What we are doing is actually a self-energy correction.

Suppose $\boldsymbol{r} \propto \mathrm{e}^{-\mathrm{i}\omega t}$, and we have

$$e\boldsymbol{r} =: \boldsymbol{d} = \alpha \boldsymbol{E}, \quad \alpha = \frac{e^2}{m(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega)}. \tag{8}$$

The long range radiation is summarized as

$$\boldsymbol{P} = N\boldsymbol{d} = N\alpha(\boldsymbol{E} - \boldsymbol{E}_\text{self}), \tag{9}$$

where N is the number of electrons per volume unit, and if we approximate the self field as

$$\boldsymbol{E}_\text{sef} = -\frac{\boldsymbol{P}}{3\epsilon_0}, \tag{10}$$

which is the field strength in a empty cavity in a homogeneously polarized insulator, and more generally, is the self field of a large number of dipoles (see (25) in this note), then we get

$$\boldsymbol{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}} \boldsymbol{E} =: \epsilon_0 \chi_\mathrm{e} \boldsymbol{E}, \tag{11}$$

$$\chi_\mathrm{e} = \frac{N\alpha/\epsilon_0}{1 - N\alpha/(3\epsilon_0)}, \quad \frac{\epsilon_\mathrm{r} - 1}{\epsilon_\mathrm{r} + 2} = \frac{N\alpha}{3\epsilon_0}. \tag{12}$$

This is the so-called **Clausius-Mossotti equations**.

Now we put (8) into (12), and get

$$\epsilon_\mathrm{r} = 1 + \chi_\mathrm{e} = 1 + \frac{1}{\left(\frac{Ne^2}{m\epsilon_0}\right)^{-1} (\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega) - \frac{1}{3}},$$

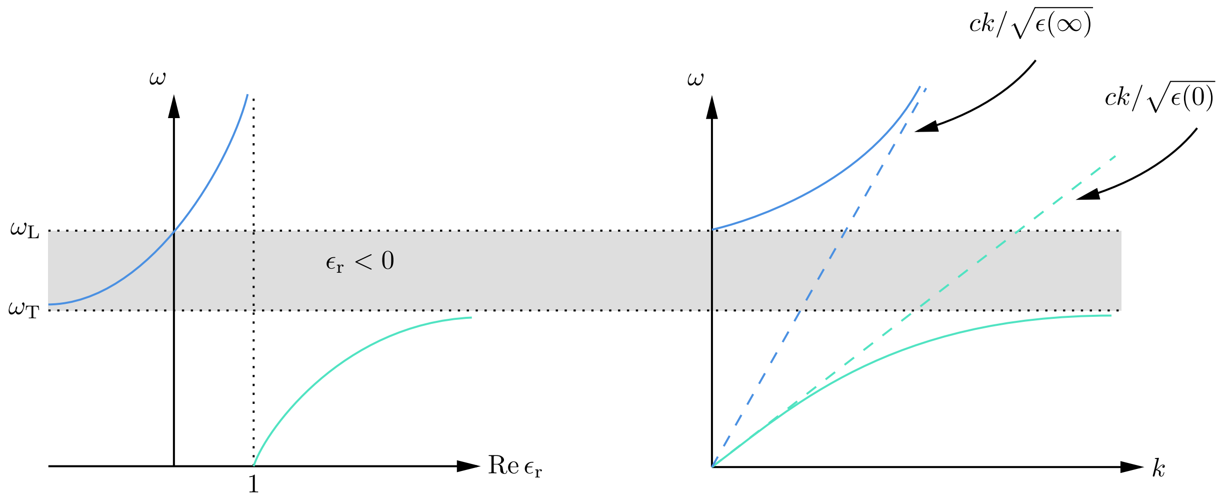

Figure 1: The photon band structure of (13)

and therefore after introducing the plasma frequency we get

$$\epsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}} = 1 + \frac{\omega_{\mathrm{p}}^2}{\tilde{\omega}_0^2 - \omega^2 - \mathrm{i}\gamma\omega}, \quad \tilde{\omega}_0^2 = \omega_0^2 - \frac{1}{3}\omega_{\mathrm{p}}^2, \quad \omega_{\mathrm{p}}^2 = \frac{Ne^2}{m\epsilon_0}. \tag{13}$$

This equation means a non-trivial photon band structure. Now we assume $\gamma = 0$, and because we have

$$k^2 = \left(\frac{\omega}{c}\right)^2 \epsilon_{\mathrm{r}}(\omega), \tag{14}$$

it is possible that $\epsilon_{\mathrm{r}} < 0$ in some frequencies. The photon band structure, therefore, has a forbidden band. This is visualize in Figure 1 on page 3. It is easy to verify that

$$\omega_{\mathrm{L}} = \sqrt{\tilde{\omega}_0^2 + \omega_{\mathrm{p}}^2}, \quad \omega_{\mathrm{T}} = \tilde{\omega}_0. \tag{15}$$

The subscripts have meanings. When $\omega = \omega_{\mathrm{L}}$, $\epsilon_{\mathrm{r}} = 0$, and the equation

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{D} = \epsilon_{\mathrm{r}} \boldsymbol{\nabla} \cdot \boldsymbol{E}$$

is trivial, and it is possible that $\boldsymbol{\nabla} \cdot \boldsymbol{E} \neq 0$. So we see the subscript L means "longitude mode". When $\omega = \omega_{\mathrm{T}} = \tilde{\omega}_0$, the oscillators oscillate in its eigen mode and are coupled strongly to the electromagnetic field. Both of them vibrate transversely. Since there is no interaction between oscillators, the dispersion relation of density wave modes of these oscillators is a flat line, independent of $\boldsymbol{k}$, with the frequency ω_{T}. That is the meaning of the subscript T: the "transverse density mode".