Green Function in Electrodynamics by Prof. Kun Din

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1 Scalar Green function and density of states

We have already known that Green functions relate the differential form of a certain equation of motion to its integral form, that in electrodynamics the scalar Green function (i.e. the retarded potential) can be used to derive the vector Green function, and that many quantities like the $-\boldsymbol{p}/3\epsilon_0$ electric field that are usually derived using heuristic way can be derived from the vector Green function directly.

Here we show another way to derive the scalar Green function. We will derive the scalar Green function with its spacial variables in the position space and its temporal variable in the frequency domain. Suppose

$$k = \frac{\omega}{c}, \tag{1}$$

we have

$$\begin{aligned} G(\boldsymbol{R}) &= \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}}}{\boldsymbol{q}^2 - k^2 \pm \mathrm{i}0^+} \\ &= \frac{1}{(2\pi)^3} \int_0^\infty q^2 \,\mathrm{d}q \int_0^\pi \sin\theta \,\mathrm{d}\theta \int_0^\pi \mathrm{d}\phi \,\frac{\mathrm{e}^{\mathrm{i}qR\cos\theta}}{q^2 - k^2 + \mathrm{i}0^+} \\ &= \frac{1}{4\pi^2 \mathrm{i} R} \int_0^\infty \frac{q(\mathrm{e}^{\mathrm{i}qR} - \mathrm{e}^{-\mathrm{i}qR})}{q^2 - k^2 + \mathrm{i}0^+} \\ &= \frac{\mathrm{e}^{\mp\mathrm{i}(k\mp\mathrm{i}0^+)R}}{4\pi R}. \end{aligned}$$

The retarded wave is

$$G(\boldsymbol{R}) = \frac{\mathrm{e}^{\mathrm{i}kR}}{4\pi R}. \tag{2}$$

This is exactly (13) in the previous lecture. (2) is an *off-shell* propagator as $\boldsymbol{q}$ may not satisfy the momentum-energy relation $\boldsymbol{q}^2 - k^2 = 0$. It need not to be on-shell, because a Green function describes how the electromagnetic field responds to an external stimulation, not how electromagnetic waves propagate freely.

We know (in quantum field theories) that

$$\frac{1}{\boldsymbol{q}^2 - k^2 - \mathrm{i}0^+} = \mathrm{P}\,\frac{1}{\boldsymbol{q}^2 - k^2} + \mathrm{i}\pi\delta(\boldsymbol{q}^2 - k^2),$$

and we can connect (2) with the density of states. The definition of density of states for a generalized system whose dispersion relation is like $|\boldsymbol{q}|^2 = k^2$ is

$$\rho(\omega) = \frac{\mathrm{d}k^2(\omega)}{\mathrm{d}\omega} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \delta(k^2(\omega) - |\boldsymbol{q}|^2). \tag{3}$$

This can be derived using the following argument. Suppose $\mathrm{d}S$ is the area element of the isoenergetic surface in the momentum space, and there are $\mathrm{d}S\,\mathrm{d}q_\perp/((2\pi)^3/V)$ states in the volume element $\mathrm{d}S\,\mathrm{d}q_\perp$ in the momentum space. Therefore, the density of states - the number of states per energy unit per volume unit - is

$$
\begin{aligned}
\sum \frac{\mathrm{d}S\,\mathrm{d}q_\perp/((2\pi)^3/V)}{V\,\mathrm{d}\omega} &= \int \frac{\mathrm{d}S\,\mathrm{d}q_\perp}{(2\pi)^3\,\mathrm{d}\omega} \\
&= \int \frac{\mathrm{d}S}{(2\pi)^3\,\mathrm{d}\omega} \frac{\mathrm{d}(k^2-\boldsymbol{q}^2)}{|\boldsymbol{\nabla}_{\boldsymbol{q}}(k^2-\boldsymbol{q}^2)|} \\
&= \frac{\mathrm{d}k^2}{\mathrm{d}\omega} \int \frac{\mathrm{d}S}{(2\pi)^3} \frac{1}{|\boldsymbol{\nabla}_{\boldsymbol{q}}(k^2-\boldsymbol{q}^2)|} \\
&= \frac{\mathrm{d}k^2}{\mathrm{d}\omega} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \delta(k^2-\boldsymbol{q}^2).
\end{aligned}
$$

Then we find the relation between the imaginary part of the Green function and the density of states:

$$
\rho(\omega) = \frac{\mathrm{d}k(\omega)^2}{\mathrm{d}\omega} \frac{1}{\pi} \operatorname{Im} G(\boldsymbol{R}=0,\omega). \tag{4}
$$

(4) is actually more general than the case of homogeneous material we have just discussed. For a non-homogeneous, it can be generalized to connect *local density of states* with the Green function.

2 The on-shell theory of Green function

(4) involves something that is proportional to $\delta(\boldsymbol{q}^2 - p^2)$, which means the imaginary part of the Green function is given by a *on-shell* theory, where the input momentum $\boldsymbol{q}$ is on-shell. Now we discuss how to calculate the Green function only with on-shell momentum. We define

$$
k_z = \sqrt{k^2 - q_x^2 - q_y^2}, \tag{5}
$$

and we have

$$
\begin{aligned}
\frac{\mathrm{e}^{\mathrm{i}kR}}{4\pi R} = G(\boldsymbol{r}) &= \int \frac{\mathrm{d}q_x\,\mathrm{d}q_y\,\mathrm{d}q_z}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}q_x x + \mathrm{i}q_y y}\mathrm{e}^{\mathrm{i}q_z z}}{(q_z-k_z)(q_z+k_z)-\mathrm{i}0^+} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}q_x \int \mathrm{d}q_y\, 2\pi\mathrm{i} \frac{\mathrm{e}^{\mathrm{i}(q_x x + q_y y)}\mathrm{e}^{\mathrm{i}k_z z}}{2k_z}
\end{aligned}
$$

when $z > 0$. The case when $z < 0$ can be checked to be

$$
\frac{\mathrm{e}^{\mathrm{i}kR}}{4\pi R} = \frac{1}{(2\pi)^3} \int \mathrm{d}q_x \int \mathrm{d}q_y\, 2\pi\mathrm{i} \frac{\mathrm{e}^{\mathrm{i}(q_x x + q_y y)}\mathrm{e}^{-\mathrm{i}k_z z}}{2k_z}.
$$

So in the end we have

$$
\frac{\mathrm{e}^{\mathrm{i}kR}}{R} = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2\boldsymbol{q}_\perp \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}_\perp\cdot\boldsymbol{r}_\perp}\mathrm{e}^{\mathrm{i}k_z|z|}}{k_z}, \tag{6}
$$

where $\boldsymbol{q}_\perp$ and $\boldsymbol{r}_\perp$ are $\boldsymbol{q}$ and $\boldsymbol{r}$'s projection on the xy plane. (6) is called the **angular spectrum decomposition** or **Weyl decomposition**. We can see the RHS of (6) consists of all possible on-shell plane waves. This can be seen as a spacial version of Duhamel's principle, where the electromagnetic field created by an external point charge can be equivalently seen as the decomposition of several electromagnetic waves in free space. In the quantum context, it means after a scattering event, the output states are decomposition of on-shell particle states. Note that (6) contains *evanescent waves*: q_x and q_y are integrated over all possible values, which means k_z may be imaginary. Note that we require

$$
\operatorname{Re} k_z > 0, \quad \operatorname{Im} k_z > 0, \tag{7}
$$

to get outward waves.

(6) can be further simplified. Suppose

$$
\boldsymbol{q}_\perp = q_\perp(\cos\alpha, \sin\alpha), \quad \boldsymbol{r}_\perp = r_\perp(\cos\phi, \sin\phi),
$$

we have

$$
\frac{\mathrm{e}^{\mathrm{i}kR}}{R} = \frac{\mathrm{i}}{2\pi} \int q_\perp\,\mathrm{d}q_\perp \int \mathrm{d}\alpha\, \frac{\mathrm{e}^{\mathrm{i}q_\perp r_\perp\cos(\alpha-\phi)}\mathrm{e}^{\mathrm{i}k_z|z|}}{k_z},
$$

and since

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}x\cos(\alpha-\phi)},$$

we have

$$\frac{\mathrm{e}^{\mathrm{i}kR}}{R} = \mathrm{i}\int \mathrm{d}q_\perp\, \frac{q_\perp}{k_z} J_0(q_\perp r_\perp)\mathrm{e}^{\mathrm{i}k_z|z|}. \tag{8}$$

This is called the **Sommerfield identity**.

(6) and (8) are all on-shell versions of the scalar Green function, as we have constraints like (5) so that k_z cannot vary freely.

3 On-shell properties of the vector Green function

Take the second order derivative of (6), we have

$$\begin{aligned}\frac{\partial^2}{\partial z^2}G(\boldsymbol{r}) &= \left(-\frac{\mathrm{i}}{4\pi}\int \mathrm{d}^2\boldsymbol{q}_\perp\, \frac{k_z}{2}\mathrm{e}^{\mathrm{i}q_\perp r_\perp}\mathrm{e}^{\mathrm{i}k_z|z|}\right)\mathrm{sgn}(z)^2 + \left(-\frac{1}{4\pi^2}\int \mathrm{d}^2\boldsymbol{q}_\perp\, \mathrm{e}^{\mathrm{i}q_\perp r_\perp}\mathrm{e}^{\mathrm{i}k_z|z|}\right)\delta(z) \\ &= -\delta^{(3)}(\boldsymbol{r}) - \frac{\mathrm{i}}{4\pi^2}\int \mathrm{d}^2\boldsymbol{q}_\perp\, \frac{k_z}{2}\mathrm{e}^{\mathrm{i}q_\perp r_\perp}\mathrm{e}^{\mathrm{i}k_z|z|}.\end{aligned}$$

Repeating this process we get

$$\boldsymbol{\nabla}\boldsymbol{\nabla}G(\boldsymbol{r}) = -\frac{\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}}{k^2}\delta(\boldsymbol{r}) + \frac{1}{8\pi^2}\int \frac{\mathrm{d}^2\boldsymbol{q}_\perp}{k_z}\begin{cases}\left(\overleftrightarrow{\boldsymbol{I}} - \frac{\boldsymbol{k}_+\boldsymbol{k}_+}{k^2}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}, & z>0, \\ \left(\overleftrightarrow{\boldsymbol{I}} - \frac{\boldsymbol{k}_-\boldsymbol{k}_-}{k^2}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_-\cdot\boldsymbol{r}}, & z<0,\end{cases} \tag{9}$$

where

$$\boldsymbol{k}_+ = (q_x, q_y, k_z), \quad \boldsymbol{k}_- = (q_x, q_y, -k_z). \tag{10}$$

We define

$$\hat{\boldsymbol{e}} = \frac{\boldsymbol{k}\times\hat{\boldsymbol{z}}}{|\boldsymbol{k}\times\hat{\boldsymbol{z}}|}, \quad \hat{\boldsymbol{h}} = \frac{\boldsymbol{k}\times(\boldsymbol{k}\times\hat{\boldsymbol{z}})}{|\boldsymbol{k}\times(\boldsymbol{k}\times\hat{\boldsymbol{z}})|}, \quad \hat{\boldsymbol{l}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{11}$$

and it can be seen that they form a complete and orthogonal basis of $\mathbb{R}^3$. We then find that

$$\overleftrightarrow{\boldsymbol{I}} - \frac{\boldsymbol{k}_-\boldsymbol{k}_-}{k^2} = \hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \hat{\boldsymbol{h}}\hat{\boldsymbol{h}}, \tag{12}$$

4 Mode expansion of the Green function

From

$$(\mathcal{L}-\lambda)G(\boldsymbol{r}-\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

we have

$$G(\boldsymbol{r}-\boldsymbol{r}') = \sum_n \frac{u_n^*(\boldsymbol{r}')u_n(\boldsymbol{r})}{\lambda_n - \lambda}. \tag{13}$$