Advanced Electrodynamics, Homework 3

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2D Green function (a) Derive the 2D Green function in polar coordinates.
Solution

(a) The 2D Green function is given by the solution of the two dimensional version of Helmholtz equation with an external source:

$$(\nabla^2 + k^2)G_0(\boldsymbol{r} - \boldsymbol{r}') = -\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}'). \tag{1}$$

The solution, in terms of Fourier transformation, is

$$G_0(\boldsymbol{R}) = -\int \frac{\mathrm{d}^2\boldsymbol{p}}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{R}}}{k^2 - \boldsymbol{p}^2 + \mathrm{i}0^+}.$$

In polar coordinates where we consider the direction of $\boldsymbol{R}$ to be the $\theta = 0$ axis, we have

$$\begin{aligned}
G_0(\boldsymbol{R}) &= -\frac{1}{(2\pi)^2} \int_0^\infty p\,\mathrm{d}p \int_0^{2\pi} \mathrm{d}\theta\, \frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{R}|\cos\theta}}{k^2 - p^2 + \mathrm{i}0^+} \\
&= \frac{1}{(2\pi)^2} \int_0^\infty p\,\mathrm{d}p \times \frac{1}{p^2 - k^2 - \mathrm{i}0^+} \times 2\pi J_0(p|\boldsymbol{R}|\,) \\
&= \frac{1}{2\pi} \times K_0\left(\frac{|\boldsymbol{R}|}{\sqrt{-1/k^2}}\right) \\
&= \frac{1}{2\pi} K_0(-\mathrm{i}k|\boldsymbol{R}|\,) = \frac{1}{2\pi} \times \frac{\pi}{2}\mathrm{i}H_0^{(1)}(k|\boldsymbol{R}|\,),
\end{aligned}$$

where the third line is obtained using Mathematica, and the fourth line comes from well-known properties of Bessel functions [1]. So we get

$$G_0(\boldsymbol{R}) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{R}|\,). \tag{2}$$

Dyadic green function in Fourier space (a) Show that in vacuum the Maxwell equations can be rephrased into

$$\boldsymbol{M}^2 \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} c^2\boldsymbol{k}\cdot\boldsymbol{k} - c^2\boldsymbol{k}\boldsymbol{k} & 0 \\ 0 & c^2\boldsymbol{k}\cdot\boldsymbol{k} - c^2\boldsymbol{k}\boldsymbol{k} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \omega^2 \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix}. \tag{3}$$

(b) Find the eigenvalues and eigenvectors. (c) Derive the Green function in the Fourier space, and show why longitude modes are absent.
Solution

(a) In the Fourier space the Maxwell equations are

$$\begin{aligned}
\boldsymbol{k}\cdot\boldsymbol{E} &= 0, \\
\boldsymbol{k}\times\boldsymbol{E} &= \omega\boldsymbol{B}, \\
\boldsymbol{k}\cdot\boldsymbol{B} &= 0, \\
\boldsymbol{k}\times\boldsymbol{B} &= -\frac{1}{c^2}\omega\boldsymbol{E}.
\end{aligned}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are actually $\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega)$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{k},\omega)$, respectively. The first and the third equations have no ω dependence and therefore cannot be a part of the eigenvalue problem. From the second and the fourth equations we have

$$\begin{aligned}
\omega\boldsymbol{k}\times\boldsymbol{B} &= \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) \\
&= (\boldsymbol{k}\cdot\boldsymbol{E})\boldsymbol{k} - \boldsymbol{k}^2\boldsymbol{E} \\
&= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E},
\end{aligned}$$

and therefore

$$-\frac{\omega^2}{c^2}\boldsymbol{E} = (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}. \tag{4}$$

Similarly we have

$$\begin{aligned}\frac{\omega}{c^2}\boldsymbol{k}\times\boldsymbol{E} &= -\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{B})\\ &= -(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{k} + \boldsymbol{k}^2\boldsymbol{B}\\ &= (-\boldsymbol{k}\boldsymbol{k} + \boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B},\end{aligned}$$

and therefore

$$\frac{\omega^2}{c^2}\boldsymbol{B} = (-\boldsymbol{k}\boldsymbol{k} + \boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B}. \tag{5}$$

From (4) and (5) we have

$$\begin{pmatrix}\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}\boldsymbol{k} & \\ & \boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}\boldsymbol{k}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix} = \frac{\omega^2}{c^2}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix}. \tag{6}$$

Note that $B = \mu_0\boldsymbol{H}$, we find this is just (3).

(b) What we need to do is to solve the equation

$$\det\left(c^2\boldsymbol{k}\cdot\boldsymbol{k} - c^2\boldsymbol{k}\boldsymbol{k} - \omega^2\right)^2 = 0,$$

or in other words

$$\det\begin{pmatrix}k_y^2 + k_z^2 - \frac{\omega^2}{c^2} & -k_xk_y & -k_xk_z\\ -k_xk_y & k_x^2 + k_z^2 - \frac{\omega^2}{c^2} & -k_yk_z\\ -k_xk_z & -k_yk_z & k_x^2 + k_y^2 - \frac{\omega^2}{c^2}\end{pmatrix}^2 = 0.$$

Factoring the LHS of the equation using Mathematica, we have

$$\frac{\omega^4}{c^4}\left(k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2}\right)^4 = 0,$$

so we have six eigen values, which are

$$\left(\frac{\omega}{c}\right)_1^2 = \left(\frac{\omega}{c}\right)_2^2 = 0, \quad \left(\frac{\omega}{c}\right)_3^2 = \left(\frac{\omega}{c}\right)_4^2 = \left(\frac{\omega}{c}\right)_5^2 = \left(\frac{\omega}{c}\right)_6^2 = \boldsymbol{k}^2. \tag{7}$$

We get six modes, and now we find the eigenmodes. For mode 1 and mode 2, we have

$$\begin{pmatrix}k_y^2 + k_z^2 & -k_xk_y & -k_xk_z\\ -k_xk_y & k_x^2 + k_z^2 & -k_yk_z\\ -k_xk_z & -k_yk_z & k_x^2 + k_y^2\end{pmatrix}\begin{pmatrix}E_x\\ E_y^2\\ E_z^2\end{pmatrix} = 0$$

and the same equation holds for $\boldsymbol{H}$, and a complete set of linear independent non-zero solutions are

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \begin{pmatrix}k_x\\ k_y\\ k_z\\ 0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 0\\ k_x\\ k_y\\ k_z\end{pmatrix}. \tag{8}$$

For mode 3, 4, 5, and 6, we have

$$\begin{pmatrix}-k_z^2 & -k_xk_y & -k_xk_z\\ -k_xk_y & -k_x^2 & -k_yk_z\\ -k_xk_z & -k_yk_z & -k_y^2\end{pmatrix}\begin{pmatrix}E_x\\ E_y^2\\ E_z^2\end{pmatrix} = 0,$$

and the same equation holds for $\boldsymbol{H}$, and a complete set of linear independent non-zero solutions are

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \begin{pmatrix}k_z\\ 0\\ -k_x\\ 0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 0\\ k_z\\ 0\\ -k_x\end{pmatrix}, \begin{pmatrix}-k_y\\ k_x\\ 0\\ 0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 0\\ -k_y\\ k_x\\ 0\end{pmatrix}. \tag{9}$$

(c) We do Schmidt orthogonalization and then invoke spectrum decomposition. We let

$$\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = k \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

then the invariant space (9) may be spanned by the direct product of the orthogonal uniform basis

$$\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \tag{10}$$

and $\{(1,0),(0,1)\}$. The projector operator into (10) is

$$\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \end{pmatrix} + \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \end{pmatrix}.$$

Replacing θ, φ by k_x, k_y, k_z using the equations

$$\sin\varphi = \frac{k_y}{\sqrt{k_x^2+k_y^2}}, \quad \cos\varphi = \frac{k_x}{\sqrt{k_x^2+k_y^2}}, \quad \sin\theta = \frac{\sqrt{k_x^2+k_y^2}}{\sqrt{k_x^2+k_y^2+k_z^2}}, \quad \cos\theta = \frac{k_z}{\sqrt{k_x^2+k_y^2+k_z^2}},$$

the projector operator into (9) is the product between

$$\begin{pmatrix} \frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_xk_y}{k_x^2+k_y^2} & 0 \\ -\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_x^2}{k_x^2+k_y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} \\ \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} \\ -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} & \frac{k_x^2+k_y^2}{k_x^2+k_y^2+k_z^2} \end{pmatrix}$$

and $I^{2\times 2}$. So by spectrum decomposition

$$\overleftrightarrow{\boldsymbol{G}} = \sum_n \frac{\boldsymbol{u}_n \boldsymbol{u}_n^\dagger}{\lambda_n - \lambda}, \tag{11}$$

we have

$$\begin{aligned}
\overleftrightarrow{\boldsymbol{G}} = &-\frac{1}{\omega^2}\frac{1}{k^2} \begin{pmatrix} k_x^2 & k_xk_y & k_xk_z & & & \\ k_xk_y & k_y^2 & k_yk_z & & & \\ k_xk_z & k_yk_z & k_z^2 & & & \\ & & & k_x^2 & k_xk_y & k_xk_z \\ & & & k_xk_y & k_y^2 & k_yk_z \\ & & & k_xk_z & k_yk_z & k_z^2 \end{pmatrix} \\
&+ \frac{1}{c^2k^2-\omega^2} \begin{pmatrix} \frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_xk_y}{k_x^2+k_y^2} & 0 & & & \\ -\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_x^2}{k_x^2+k_y^2} & 0 & & & \\ 0 & 0 & 0 & & & \\ & & & \frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_xk_y}{k_x^2+k_y^2} & 0 \\ & & & -\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_x^2}{k_x^2+k_y^2} & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \\
&+ \frac{1}{c^2k^2-\omega^2} \begin{pmatrix} \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} & & & \\ \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} & & & \\ -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} & \frac{k_x^2+k_y^2}{k_x^2+k_y^2+k_z^2} & & & \\ & & & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} \\ & & & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_xk_y}{k_x^2+k_y^2} & \frac{k_z^2}{k_x^2+k_y^2+k_z^2}\frac{k_y^2}{k_x^2+k_y^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} \\ & & & -\frac{k_xk_z}{k_x^2+k_y^2+k_z^2} & -\frac{k_yk_z}{k_x^2+k_y^2+k_z^2} & \frac{k_x^2+k_y^2}{k_x^2+k_y^2+k_z^2} \end{pmatrix}
\end{aligned} \tag{12}$$

References

[1] Wikipedia. Bessel function. https://en.wikipedia.org/wiki/Bessel_function, Nov 2021.