Homology and Homotopy Groups

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This article is mainly based on [1].

Topological field theories often emerge from condensed matter systems, and the topological invariants they give are often connected to **homotopy groups**, which, intuitively speaking, classify possible field configurations into different topological sectors.

In practice homotopy groups are hard to calculate. That is why people often seek algebraic objects that are more easy to calculated and then connect them to homotopy groups (or other algebraic objects that we are interested in). This approach is called **algebraic topology**, and one most frequently used algebraic object is the **homology group**. They do not have very intuitive meaning, but they are easier to deal with.

1 Some basic facts about Abelian groups

We use $+$ to denote the group operation of an Abelian group. The expression $x - y$ is defined as $x^{-1} \circ y$. The unit is denoted as 0.

A map between Abelian groups $f : G_1 \to G_2$ is said to be a **homomorphism** if $f(x + y) = f(x) + f(y)$, i.e. it keeps the multiplication relations. An **isomorphism** is a homomorphism that is also a bijection.

Suppose H is a subgroup of G. We have a equivalence relation $x \sim y$ if and only if $x - y \in H$. The equivalence class of x is denoted as $[x]$, and the set of all equivalence classes in G is denoted as G/H, which is the **the quotient space**. We can easily find that G/H is also a group and the group operation $+$ in G naturally induces the group operation $+$ in G/H. We have

$$G/G = \{[0]\} = \{[h]\}, \quad h \in H, \tag{1}$$

and

$$G/\{0\} = G. \tag{2}$$

Some examples of quotient spaces:

$$\mathbb{Z}/k\mathbb{Z} \simeq \mathbb{Z}_k. \tag{3}$$

If $f : G_1 \to G_2$ is a homomorphism, it can be found that $\ker f$ is a subgroup of G_1 and $\operatorname{im} f$ is a subgroup of G_2. This lemma can be proved almost directly by definition.

Now we state the **fundamental theorem of homomorphism**: for a homomorphism $f : G_1 \to G_2$, we have

$$G_1/\ker f \simeq \operatorname{im} f. \tag{4}$$

This is Theorem 3.1 in [1].

References

[1] Mikio Nakahara. *Geometry, Topology and Physics, Second Edition.* 06 2003.