Quantum Optics, Homework 3

Jinyuan Wu

November 21, 2021

Interference between Gaussian pulses Consider two Gaussian pulses with wave vectors $\boldsymbol{k}_{1,2} = k(\pm \sin\theta, 0, \cos\theta)$, respectively. They are incident to a plane detector on the surface $z = 0$. The intensity distributions of the two beams are all

$$|\mathcal{E}|^2 \propto e^{-\left(x^2+y^2\right)/\sigma^2}, \tag{1}$$

with $\sigma \gg \lambda$. The pulses arrive at the detector simultaneously. The detector absorbs the pulses completely and there is no reflection. Calculate $P^{(1)}(\boldsymbol{r})$ and $P^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for the following states of the optical field:

(a) $|\psi\rangle = \frac{1}{\sqrt{2^N N!}} \left(a_1^\dagger + a_2^\dagger\right)^N |V\rangle$.

(b) $|\psi\rangle = \frac{1}{N!} \left(a_1^\dagger a_2^\dagger\right)^N |V\rangle$.

(c) $|\psi\rangle = \frac{1}{\sqrt{2N!}} \left(\left(a_1^\dagger\right)^N + \left(a_2^\dagger\right)^N\right) |V\rangle$.

(d) $|\psi\rangle = D_1(\alpha) D_2(\alpha) |V\rangle, \quad D_j(\alpha) \equiv e^{\alpha a_j^\dagger - \alpha^* a_j}$.

(e) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(D_1(\alpha) + D_2(\alpha)\right) |V\rangle$.

Solution The electric field operator is

$$\boldsymbol{E} = \sum_{i=1,2} \boldsymbol{\mathcal{E}}_i \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r} - \mathrm{i}\omega t} a_i + \text{h.c.}. \tag{2}$$

(a) We define

$$b^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger),$$

and now the wave function is

$$|\psi\rangle = \frac{1}{\sqrt{N!}} (b^\dagger)^N |0\rangle .$$

We have

$$P^{(1)}(\boldsymbol{r}) = \frac{1}{N!} |\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0| b^N (a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2 - \boldsymbol{k}_1)\cdot\boldsymbol{r}} a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{r}} a_2^\dagger a_1)(b^\dagger)^N |0\rangle .$$

Evaluating the terms in the RHS above, we have

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_1 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

and similarly

$$\langle 0| b^N a_2^\dagger a_2 (b^\dagger)^N |0\rangle = \frac{1}{2} N^2 (N-1)!,$$

and

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_2 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

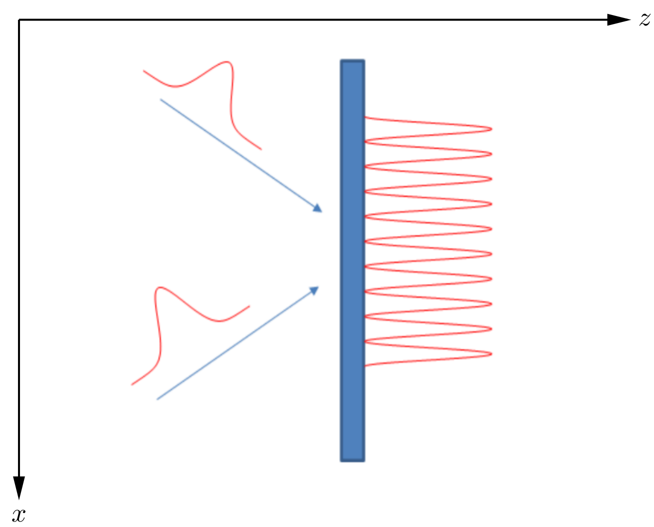

Figure 1: The two Gaussian beams incident to a detector

and similarly

$$\langle 0|b^N a_1^\dagger a_2 (b^\dagger)^N|0\rangle = \frac{1}{2}N^2(N-1)!.$$

Putting everything together we have

$$\begin{aligned}
P^{(1)}(\boldsymbol{r}) &= \eta\frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times \frac{1}{2}N^2(N-1)! \times (2+\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}+\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2))\cdot\boldsymbol{r}})\\
&= \eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r})\\
&= \eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(2k\sin\theta x))\\
&= 2\eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta),
\end{aligned}$$

so finally

$$P^{(1)}(\boldsymbol{r}) = 2N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta) \propto 2N\mathrm{e}^{-(x^2+y^2)/\sigma^2}\cos^2(kx\sin\theta). \tag{3}$$

The two-photon joint probability is

$$\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,b^N(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})\\
&\quad\times(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(b^\dagger)^N\,|0\rangle\\
&= \eta^2\frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\times N\,\langle 0|b(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})|0\rangle\\
&\quad\times(N-1)\,\langle 0|b(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})|0\rangle\\
&\quad\times N\,\langle 0|(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})b^\dagger|0\rangle\\
&\quad\times(N-1)\,\langle 0|(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})b^\dagger|0\rangle\\
&\quad\times\text{contraction between } N\ b\text{'s and } b^\dagger\text{'s}\\
&= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\frac{1}{N!}\times\frac{N}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})\times\frac{N-1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})\\
&\quad\times\frac{N-1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2}+\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})\times\frac{N}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}+\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})\times(N-2)!\\
&= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2N(N-1)(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}_1)(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}_2),
\end{aligned}$$

so

$$\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= 4\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2N(N-1)\cos^2(kx_1\sin\theta)\cos^2(kx_2\sin\theta)\\
&\propto 4\eta^2\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}N(N-1)\cos^2(kx_1\sin\theta)\cos^2(kx_2\sin\theta).
\end{aligned}\tag{4}$$

(b) We have

$$
\begin{aligned}
P^{(1)}(\boldsymbol{r}) &= \frac{\eta}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0|(a_2a_1)^N(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(a_1^\dagger a_2^\dagger)^N|0\rangle \\
&= \frac{\eta}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0|a_2^N a_1^N(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle\,.
\end{aligned}
$$

Evaluating the terms in the RHS, we have

$$
\begin{aligned}
\langle 0|a_1^N a_2^N a_1^\dagger a_1(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle &= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_1a_1^\dagger|0\rangle \\
&\qquad \times \text{contraction of } (N-1)\ a_1\text{'s and } (N-1)\ a_1^\dagger\text{'s} \\
&\qquad \times \text{contraction of } N\ a_2\text{'s and } N\ a_2^\dagger\text{'s} \\
&= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_1a_1^\dagger|0\rangle \times (N-1)!\,\langle 0|a_1a_1^\dagger|0\rangle \times N!\,\langle 0|a_2a_2^\dagger|0\rangle \\
&= N^2N!(N-1)!,
\end{aligned}
$$

and similarly we have

$$
\langle 0|a_1^N a_2^N a_2^\dagger a_2(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle = N^2N!(N-1)!.
$$

Also we have

$$
\begin{aligned}
\langle 0|a_2^N a_1^N a_1^\dagger a_2(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle &= N\langle 0|a_1a_1^\dagger|0\rangle \times N\langle 0|a_2a_2^\dagger|0\rangle \\
&\qquad \times \text{contraction of } N\ a_2\text{'s, } (N-1)\ a_1\text{'s, } N\ a_1^\dagger\text{'s and } (N-1)\ a_2^\dagger\text{'s} \\
&= 0,
\end{aligned}
$$

so it vanishes, and so does $\langle 0|a_2^N a_1^N a_2^\dagger a_1(a_1^\dagger)^N(a_2^\dagger)^N|0\rangle$. Putting everything together we have

$$
P^{(1)}(\boldsymbol{r}) = \eta\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times 2 \times N^2N!(N-1)! = 2N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2,
$$

so the single photon probability is

$$
P^{(1)}(\boldsymbol{r}) = 2\eta N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \propto 2\eta N\mathrm{e}^{-(x^2+y^2)/\sigma^2}. \tag{5}
$$

The two-photon joint probability is

$$
\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\langle 0|\,a_1^N a_2^N(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2}) \\
&\quad \times (a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(a_1^\dagger)^N(a_2^\dagger)^N\,|0\rangle \\
&= \eta^2\frac{1}{(N!)^2}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\langle 0|\,a_1^N a_2^N(a_1^\dagger a_1^\dagger a_1a_1 + a_2^\dagger a_2^\dagger a_2a_2 \\
&\quad + a_1^\dagger a_2^\dagger a_2a_1\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}_2+\boldsymbol{k}_2\cdot\boldsymbol{r}_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)} + \text{h.c.})(a_1^\dagger)^N(a_2^\dagger)^N\,|0\rangle \\
&= \eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\frac{1}{(N!)^2}(N^2(N-1)^2(N-2)!N! \times 2 \\
&\quad + N^4(N-1)!(N-1)!(\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)} + \text{h.c.})),
\end{aligned}
$$

where the second equation uses the conservation of particle numbers. So we have

$$
\begin{aligned}
P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= 2\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2(N(N-1) + N^2\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)) \\
&= 2\eta^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2(N(N-1) + N^2\cos(2k(x_1-x_2)\sin\theta)) \\
&\propto 2\eta^2\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}(N(N-1) + N^2\cos(2k(x_1-x_2)\sin\theta)).
\end{aligned} \tag{6}
$$

(c) The single photon probability is now

$$
\begin{aligned}
P^{(1)} &= \eta\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\langle 0|\,(a_1^N + a_2^N)(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1) \\
&\quad \times ((a_1^\dagger)^N + (a_2^\dagger)^N)\,|0\rangle\,.
\end{aligned}
$$

Evaluating the terms on the RHS, we have

$$\begin{aligned}\langle 0|(a_1^N + a_2^N)a_1^\dagger a_1((a_1^\dagger)^N + (a_2^\dagger)^N)|0\rangle &= \langle 0|a_1^N a_1^\dagger a_1 (a_1^\dagger)^N|0\rangle + \langle 0|a_2^N a_1^\dagger a_1 (a_2^\dagger)^N|0\rangle \\ &= N\cdot N\cdot (N-1)! + N!,\end{aligned}$$

as well as

$$\langle 0|(a_1^N + a_2^N)a_2^\dagger a_2((a_1^\dagger)^N + (a_2^\dagger)^N)|0\rangle = N\cdot N\cdot (N-1)! + N!.$$

The third term and fourth term vanish because the photon numbers in the bra and the ket is not the same. So we have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \eta\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times 2\times (N^2(N-1)! + N!) \\ &= \eta(N+1)|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \propto (N+1)\mathrm{e}^{-(x^2+y^2)/\sigma^2}.\end{aligned} \tag{7}$$

The two-photon joint probability is

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,(a_1^N + a_2^N)(a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1}) \\ &\quad\times (a_1^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})(a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_1}) \\ &\quad\times (a_1\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_2} + a_2\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2})((a_1^\dagger)^N + (a_2^\dagger)^N)\,|0\rangle \\ &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2\,\langle 0|\,(a_1^N + a_2^N)(a_1^\dagger a_1^\dagger a_1 a_1 + a_2^\dagger a_2^\dagger a_2 a_2)((a_1^\dagger)^N + (a_2^\dagger)^N)\,|0\rangle \\ &= \eta^2\frac{1}{2N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2 \times N^2(N-1)!\times 2,\end{aligned}$$

where the second equation uses conservation of particle numbers. So we have

$$\begin{aligned}P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \eta^2 N(N-1)|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_1)|^2|\boldsymbol{\mathcal{E}}(\boldsymbol{r}_2)|^2 \\ &\propto \eta^2 N(N-1)\mathrm{e}^{-(x_1^2+x_2^2+y_1^2+y_2^2)/\sigma^2}.\end{aligned} \tag{8}$$

(d) We have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\,\langle\alpha,\alpha|(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)|\alpha,\alpha\rangle \\ &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(2 + 2\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}) \\ &= \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2\cos^2(kx\sin\theta) \propto \eta\mathrm{e}^{-(x^2+y^2)/\sigma^2}\cos^2(kx\sin\theta).\end{aligned} \tag{9}$$

Similarly by the definition of coherent states we have

$$P^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2) = P^{(1)}(\boldsymbol{r}_1)P^{(1)}(\boldsymbol{r}_2). \tag{10}$$

(e) We have

$$\begin{aligned}P^{(1)}(\boldsymbol{r}) &= \frac{1}{2}\eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(\,\langle\alpha,0| + \langle 0,\alpha|)(a_1^\dagger a_1 + a_2^\dagger a_2 \\ &\qquad + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}a_2^\dagger a_1)(\,|\alpha,0\rangle + |0,\alpha\rangle) \\ &= \frac{1}{2}\eta(|\alpha|^2 + |\alpha|^2),\end{aligned}$$

so

$$P^{(1)}(\boldsymbol{r}) = \eta|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2|\alpha|^2 \propto \eta\mathrm{e}^{-(x^2+y^2)/\sigma^2}|\alpha|^2. \tag{11}$$

Similarly, all terms involving both a_1 and a_2 vanish because either of them gives 0 when acting on $|\alpha,0\rangle$ or $|0,\alpha\rangle$, and we just have (10) as well.

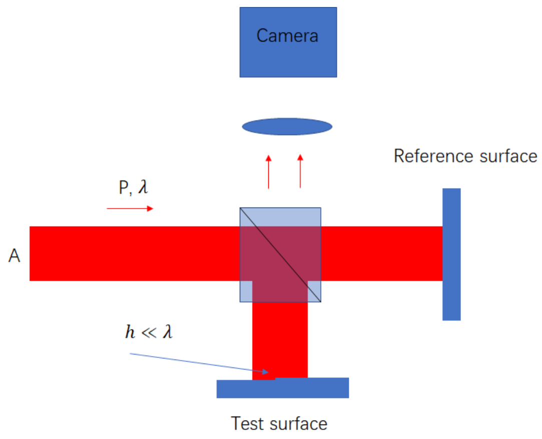

Figure 2: Surface profile measuring using lasers

Laser surface measurement

(a) Consider the Michaelson interferometer in Figure 2. Suppose that there is a step on the test surface with height $h \ll \lambda$, and that the step has no scattering effects and there is no interference between the left and the right reflected light beam. Describe the output, and estimate the necessary power P to achieve $\delta h/h = 0.1$ within time duration T.

(b) Replace the laser by a series of single photon pulses.

(c) Replace the laser by a thermal light source where

$$\bar{n} = \frac{1}{e^{\beta\hbar\omega} - 1} \gg 1. \tag{12}$$

Discuss the relation between this case and the case of coherent light.

Solution Consider Figure 3. The transformation matrix of the light propagating in the space is

$$\begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix},$$

where

$$\varphi = k(x_1 - x_2) =: kx \ll 1. \tag{13}$$

The transformation matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -i\sin\varphi/2 \\ -i\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -i\sin\varphi/2 \\ -i\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}. \tag{14}$$

By detecting $\langle b_2^\dagger b_2 \rangle$ i.e.

$$\begin{aligned} \langle n_2 \rangle = \langle b_2^\dagger b_2 \rangle &= \langle (-i\sin\varphi/2 a_1^\dagger + \cos\varphi/2 a_2^\dagger)(i\sin\varphi/2 a_1 + \cos\varphi/2 a_2) \rangle \\ &= \sin^2\varphi/2 \langle a_1^\dagger a_1 \rangle \end{aligned} \tag{15}$$

we can measure x. Also we have

$$\begin{aligned} \langle n_2^2 \rangle &= \langle ((-i\sin\varphi/2 a_1^\dagger + \cos\varphi/2 a_2^\dagger)(i\sin\varphi/2 a_1 + \cos\varphi/2 a_2))^2 \rangle \\ &= \langle (\sin^2\varphi/2 a_1^\dagger a_1 - i\sin\varphi/2\cos\varphi/2 a_1^\dagger a_2)(\sin^2\varphi/2 a_1^\dagger a_1 + i\sin\varphi/2\cos\varphi/2 a_2^\dagger a_1) \rangle \\ &= \sin^4\varphi/2 \langle a_1^\dagger a_1 a_1^\dagger a_1 \rangle + \sin^2\varphi/2\cos^2\varphi/2 \langle a_1^\dagger a_2 a_2^\dagger a_1 \rangle \\ &= \sin^4\varphi/2 \langle (a_1^\dagger)^2 a_1^2 \rangle + \sin^4\varphi/2 \langle a_1^\dagger a_1 \rangle + \sin^2\varphi\cos^2\varphi/2 \langle a_1^\dagger a_1 \rangle \\ &= \sin^4\varphi/2 \langle (a_1^\dagger)^2 a_1^2 \rangle + \sin^2\varphi/2 \langle a_1^\dagger a_1 \rangle, \end{aligned}$$

and the error is

$$\delta n_2=\sqrt{\sin^4\varphi/2\,\langle(a_1^\dagger)^2a_1^2\rangle+\sin^2\varphi/2\,\langle a_1^\dagger a_1\rangle-\sin^4\varphi/2\,\langle a_1^\dagger a_1\rangle^2}. \tag{16}$$

(a) For a coherent input on $a_1^\dagger$ mode, the measurement result is

$$\langle n_2\rangle=|\alpha|^2\sin^2\varphi/2. \tag{17}$$

By (16), the fluctuation of $\langle n_2\rangle$ is

$$\delta n_2=|\alpha|\,\sin\varphi/2,$$

and since φ is small, we have $\langle n_2\rangle\propto\varphi^2$, and therefore

$$\frac{\delta\varphi}{\varphi}=\frac{1}{2}\frac{\delta n_2}{\langle n_2\rangle}=\frac{1}{2|\alpha|\,\sin\varphi/2}. \tag{18}$$

Note that

$$|\alpha|^2=\langle\text{numbers output photons}\rangle=\frac{PT}{\hbar\omega},$$

and again by using the fact that φ is small, we obtain

$$\delta\varphi\approx\sqrt{\frac{\hbar\omega}{PT}}=\sqrt{\frac{2\pi\hbar c}{PT\lambda}}. \tag{19}$$

Now we want to measure h, which is

$$h=\frac{\varphi_{\rm L}-\varphi_{\rm R}}{k}, \tag{20}$$

so we have

$$\begin{aligned}\frac{\delta h}{h}&=\frac{\sqrt{\delta\varphi_{\rm L}{}^2+\delta\varphi_{\rm R}{}^2}}{kh}=\frac{\sqrt{2}\,\delta\varphi}{kh}\\&=\frac{\lambda}{2\pi h}\sqrt{\frac{4\pi\hbar c}{PT\lambda}},\end{aligned} \tag{21}$$

where $\varphi=\varphi_{\rm L}\approx\varphi_{\rm R}$. The condition $\delta h\,/h<0.1$ is equivalent to

$$P>\frac{100\hbar\lambda c}{\pi Th^2}. \tag{22}$$

(b) This time the input state is

$$|\psi\rangle=a_1^\dagger\,|0\rangle\,, \tag{23}$$

so

$$\langle n_2\rangle=\sin^2\varphi/2, \tag{24}$$

and

$$\delta n_2=\sin\varphi/2. \tag{25}$$

Therefore, after N pulses being measured we have

$$\frac{\delta\varphi}{\varphi}=\frac{1}{\sqrt{N}}\frac{1}{2\sin\varphi/2}. \tag{26}$$

It can be seen that the form of the equation is the same as (18). Since the derivation in (a) after (18) has nothing to do with the exact meaning of $|\alpha|$, everything should be same for (18) and (26), and we have

$$P>\frac{100\hbar\lambda c}{\pi Th^2} \tag{27}$$

again.

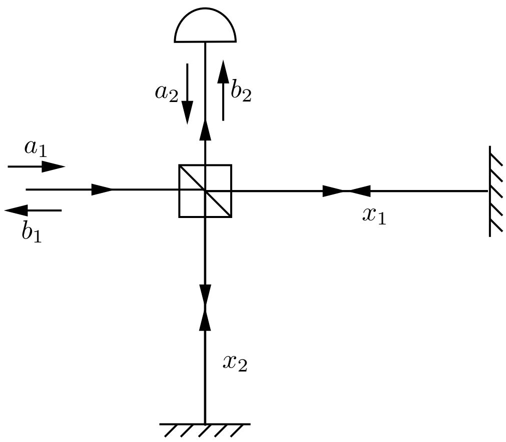

Figure 3: A standard Michaelson interferometer

(c) For a thermal optical field, we have

$$\langle n_2 \rangle = \bar{n} \sin^2 \varphi/2, \tag{28}$$

and

$$\begin{aligned}
\langle (a_1^\dagger)^2 a_1^2 \rangle &= \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{1+n}} \langle n|(a_1^\dagger)^2 a_1^2|n\rangle \\
&= \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{1+n}} n(n-1) \\
&= \frac{\alpha^2}{1+\bar{n}} \sum_{n\geq 2} \alpha^{n-2} n(n-1) \qquad (\alpha := \frac{\bar{n}}{1+\bar{n}}) \\
&= \frac{\alpha^2}{1+\bar{n}} \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \sum_{n\geq 0} \alpha^n \\
&= \frac{\alpha^2}{1+\bar{n}} \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{1}{1-\alpha} = \frac{\alpha^2}{1+\bar{n}} \frac{2}{(1-\alpha)^3} \\
&= 2\bar{n}^2,
\end{aligned}$$

so

$$\begin{aligned}
\delta n_2 &= \sqrt{\sin^4 \varphi/2 \cdot 2\bar{n}^2 + \sin^2 \varphi/2 \bar{n} - \sin^4 \varphi/2 \bar{n}^2} \\
&= \sin \varphi/2 \sqrt{\bar{n}^2 \sin^2 \varphi/2 + \bar{n}}.
\end{aligned} \tag{29}$$

From (28) and the fact that φ is small we also have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2} \frac{\delta n_2}{\langle n_2 \rangle} = \frac{1}{2 \sin \varphi/2} \sqrt{\sin^2 \varphi/2 + \frac{1}{\bar{n}}}.$$

Since $\bar{n}$ is large, we have

$$\frac{\delta\varphi}{\varphi} = \frac{1}{2}. \tag{30}$$

It can be seen that the precision of a single measurement cannot be improved unboundedly even when considering solely the shot noise. Using a thermal light source is not an effective idea.

Michaelson atomic clock The Michaelson interferometer (see again Figure 3) can also be used to measure photon frequency when $\Delta x = x_1 - x_2$ is already known. Derive its precision and compare the result with the Ramsey atomic clock.

Solution We can reuse results in the last problem. We have

$$\omega = \frac{c}{\Delta x}\varphi, \tag{31}$$

and φ is measured from $\langle n_2 \rangle$. If the input light is laser, we have (18), and from (31) we have

$$\frac{\delta\omega}{\omega} = \frac{\delta\varphi}{\varphi} = \frac{1}{2|\alpha| \sin\left(\frac{\omega\Delta x}{2c}\right)}, \tag{32}$$

or

$$\delta\omega \approx \frac{\omega}{2|\alpha| \frac{\omega\Delta x}{2c}} = \frac{1}{|\alpha| \Delta x/c}. \tag{33}$$

This is similar to the case in the Ramsey atomic clock, which is

$$\delta\omega = \frac{1}{\sqrt{N}T}, \quad T = \Delta x/v, \tag{34}$$

but here v is the speed of atoms instead of light. Therefore using Michaelson interferometer as a atomic clock is not a good idea since the precision is poor compared to a Ramsey atomic clock.