Green Function in Electrodynamics by Prof. Kun Din

Jinyuan Wu

November 17, 2021

1 A review of the theory of basic linear operators

It should be noted that the operators representing the system in linear optical systems are slightly more complicated that the case in quantum mechanics. For example, in Section 18.1.2 in the solid state physics note, we find the orthogonal relation is

$$\int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{u}_m^* \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{u}_n = \delta_{mn}$$

instead of

$$\int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{u}_m^* \cdot \boldsymbol{u}_n = \delta_{mn}.$$

This means the Maxwell equations in a general linear optical system is *not* Hermitian in the sense in quantum mechanics.

Here we review basic aspects of linear operators in an attempt to create a "generalized quantum mechanics" for photons.

Given an inner product $\langle \cdot, \cdot \rangle$, which is a positive bilinear form, we define the *adjoint* of a linear operator $\mathcal{L}$ as $\bar{\mathcal{L}}$ such that for all vectors u, v, we have

$$\langle v, \mathcal{L} u \rangle = \langle \bar{\mathcal{L}} v, u \rangle. \tag{1}$$

Then we have the following theorems:

- **Fredholm's theorem**: The orthogonal complement of the row space of $\mathcal{L}$ is the null space of $\mathcal{L}$. In other words, if there exists a vector u such that $\mathcal{L}u = f$, and we have $\bar{\mathcal{L}} v = 0$, then $\langle v, f \rangle = 0$.
- Each linear operator corresponds to a bilinear form.
- If $\mathcal{L} u_n = \lambda_n u_n$ and $\bar{\mathcal{L}} v_m = \lambda_m v_m$, then $\langle v_m, u_n \rangle = \delta_{mn}$. In other words the orthogonal relation now works for a **right eigenvector** and a **left eigenvector**.

Let us consider elementary linear algebra on $\mathbb{C}$, which is the quantum mechanics for lattice systems, like a tight-binding model. Operators now correspond to matrices, and we no longer distinguish the two concepts. We define

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\dagger \mathbf{b}, \tag{2}$$

which is a positive bilinear form. The definition of adjoints now reads

$$\mathbf{v}^\dagger (\mathcal{L}\mathbf{u}) = (\bar{\mathcal{L}}\mathbf{v})^\dagger \mathbf{u} = \mathbf{v}^\dagger (\bar{\mathcal{L}})^\dagger \mathbf{u},$$

or in other words $\bar{\mathcal{L}} = \mathcal{L}^\dagger$. If $\bar{\mathcal{L}}\mathbf{u} = \lambda \mathbf{u}$, i.e. $\mathbf{u}$ is a left eigenvector of $\mathcal{L}$, then

$$\lambda^* \mathbf{u}^\dagger = (\bar{\mathcal{L}}\mathbf{u})^\dagger = \mathbf{u}^\dagger \mathcal{L}^{\dagger\dagger} = \mathbf{u}^\dagger \mathcal{L},$$

and we see the meaning of the term "left eigenvector".

Now we go to a infinite-dimensional case: **Sturm-Liouville theory**. Consider a differential operator

$$\mathcal{L} = p_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}}{\mathrm{d}x} + p_2, \tag{3}$$

and we define

$$\langle f, g \rangle = \int_a^b \mathrm{d}x\, f(x)^* g(x). \tag{4}$$

By integration by parts we have

$$\begin{aligned}\langle v, \mathcal{L}u\rangle &= \int_a^b \mathrm{d}x\, v^* \left(p_0 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}u}{\mathrm{d}x} + p_2 u\right) \\ &= \left((p_0 v^*)\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}(p_0 v^*)}{\mathrm{d}x} u + p_1 v^* u\right)\Bigg|_a^b + \int_a^b \mathrm{d}x \left(\frac{\mathrm{d}^2(p_0 v^*)}{\mathrm{d}x^2} - \frac{\mathrm{d}(p_1 v^*)}{\mathrm{d}x} + p_2 v^*\right) u.\end{aligned}$$

We say the **formal adjoint** of $\mathcal{L}$ is defined by

$$\bar{\mathcal{L}} f = \frac{\mathrm{d}^2(p_0 f)}{\mathrm{d}x^2} - \frac{\mathrm{d}(p_1 f)}{\mathrm{d}x} + p_2 f. \tag{5}$$

If the boundary condition makes

$$\left((p_0 v^*)\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}(p_0 v^*)}{\mathrm{d}x} u + p_1 v^* u\right)\Bigg|_a^b = 0 \tag{6}$$

holds, then the formal adjoint of $\mathcal{L}$ defined in (5) is the true adjoint. We can see that (6) holds under Dirichlet boundary condition, Neumann boundary condition and Floquent boundary condition, so Sturm-Liouville theory works perfectly well for a large class of physical problems. We usually define

$$\langle v, \mathcal{L}u\rangle - \langle \bar{\mathcal{L}} v, u\rangle = J(v, u) \tag{7}$$

as the **bilinear concomitant**, where $\mathcal{L}$ is the formal adjoint in (5). (7) may be seen as the generalized version of the Green formula in electrostatics. We can see that $\mathcal{L}$ is self-adjoint if and only if

$$p_1 = \frac{\mathrm{d}p_0}{\mathrm{d}x}. \tag{8}$$

It can be soon realized that (8) does not hold in electrodynamics, where p_0 may have spacial variation (i.e. ϵ) but there is no p_1. There is one way to make up for the fact. We define

$$w(x) = \frac{1}{p_0} \exp\left(\int \mathrm{d}x\, \frac{p_1(x)}{p_0(x)}\right), \tag{9}$$

$$\bar{p}_0 = \exp\left(\int \mathrm{d}x\, \frac{p_1(x)}{p_0(x)}\right), \tag{10}$$

and

$$\bar{p}_1 = \frac{p_1}{p_0} \exp\left(\int \mathrm{d}x\, \frac{p_1(x)}{p_0(x)}\right), \tag{11}$$

then we have

$$\int_a^b \mathrm{d}x\, v^*(w(x)\mathcal{L})u = (v^* \bar{p}_0 u' - (v^*)' \bar{p}_0 u)\Big|_a^b + \int_a^b \mathrm{d}x\, (w(x)\mathcal{L}v)^* u. \tag{12}$$

So here again, we define the inner product to be

$$\langle u, v\rangle = \int_a^b \mathrm{d}x\, u^* v, \tag{13}$$

and we define the formal adjoint to be $\mathcal{L}^*$, and if the boundary condition makes

$$(v^* \bar{p}_0 u' - (v^*)' \bar{p}_0 u)\Big|_a^b = 0 \tag{14}$$

holds then $\mathcal{L}$'s adjoint is the formal adjoint.

2 Maxwell equation

We consider the wave equation

$$\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\left(\frac{\omega}{c}\right)^2\epsilon_{\mathrm{r}}(\boldsymbol{r})\right)\boldsymbol{E}=0. \tag{15}$$

This may be viewed as an eigenvalue problem, where the linear operator is

$$\mathcal{L}_E=\frac{1}{\epsilon_{\mathrm{r}}(\boldsymbol{r})}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times. \tag{16}$$

To make the equation more symmetric, we define

$$\boldsymbol{Q}(\boldsymbol{r})=\sqrt{\epsilon_{\mathrm{r}}(\boldsymbol{r})}\boldsymbol{E}(\boldsymbol{r}), \tag{17}$$

and now problem (16) reads

$$H\boldsymbol{Q}:=\frac{1}{\sqrt{\epsilon_{\mathrm{r}}(\boldsymbol{r})}}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\frac{1}{\sqrt{\epsilon_{\mathrm{r}}(\boldsymbol{r})}}\boldsymbol{Q}(\boldsymbol{r})\right)=\left(\frac{\omega}{c}\right)^2\boldsymbol{Q}. \tag{18}$$

We define inner product for $\boldsymbol{Q}$ vectors as

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle=\int\mathrm{d}^3\boldsymbol{r}\,\boldsymbol{f}^*(\boldsymbol{r})\cdot\boldsymbol{g}(\boldsymbol{r}),$$

and then we have

$$\int\mathrm{d}^3\boldsymbol{r}\left(\frac{1}{\sqrt{\epsilon_{\mathrm{r}}}}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\frac{1}{\sqrt{\epsilon_{\mathrm{r}}}}\boldsymbol{Q}_1^*\right)\right)\cdot\boldsymbol{Q}_2=\oint\mathrm{d}\boldsymbol{S}\cdot\left(\left(\boldsymbol{\nabla}\times\frac{\boldsymbol{Q}_1^*}{\sqrt{\epsilon_{\mathrm{r}}}}\right)\times\frac{\boldsymbol{Q}_2}{\sqrt{\epsilon_{\mathrm{r}}}}\right)+,$$

for Floquent boundary condition the surface term vanishes, and we fine (16) is self-adjoint. Thus, under the definition of inner product

$$\langle\boldsymbol{E}_1,\boldsymbol{E}_2\rangle=\langle\boldsymbol{Q}_1,\boldsymbol{Q}_2\rangle=\int\mathrm{d}^3\boldsymbol{r}\,\epsilon_{\mathrm{r}}(\boldsymbol{r})\boldsymbol{E}_1^*\cdot\boldsymbol{E}_2. \tag{19}$$

Now we consider the generalized linear constitutive relation

$$(\overleftrightarrow{\boldsymbol{D}}-\mathrm{i}\omega\overleftrightarrow{\boldsymbol{K}})\boldsymbol{e}=-\boldsymbol{J}, \tag{20}$$

where

$$\overleftrightarrow{\boldsymbol{D}}:=\begin{pmatrix}0 & -\mathrm{i}\boldsymbol{\nabla}\times\\ -\mathrm{i}\boldsymbol{\nabla}\times & 0\end{pmatrix},\quad \overleftrightarrow{\boldsymbol{K}}:=\begin{pmatrix}\overleftrightarrow{\boldsymbol{\epsilon}} & \overleftrightarrow{\boldsymbol{\xi}}\\ \overleftrightarrow{\boldsymbol{\eta}} & \overleftrightarrow{\boldsymbol{\mu}}\end{pmatrix},\quad \boldsymbol{e}:=\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix},\quad \boldsymbol{j}:=\begin{pmatrix}\boldsymbol{J}\\ \boldsymbol{M}\end{pmatrix}. \tag{21}$$

Let

$$U=\begin{pmatrix}1 & \\ & -1\end{pmatrix}, \tag{22}$$

and we have

3 Review

- Transforming differential equations into integral equations often makes things easier: the idea of Green functions.
- Green function as the inverse of the LHS of an inhomogeneous equation.
- From scalar Green function to the dyadic Green function.
- Retarded potential.
- Example: electric dipole.

- Green function and Fourier transformation.
- On-shell and off-shell properties.
- Weyl and Sommerfield representation.
- Fourier transformation of the dyadic Green function.
- Spectrum decomposition.
- The foundation of the spectrum decomposition: linear operators, adjoint.

These lectures do not cover 1D or 2D cases. The low dimensional cases may be derived in the same manner. The 1D, 2D and 3D Green functions are related. For example, the 2D Green function may be viewed as a generalized 3D Green function with a line source instead of a point source.