Lorentz Group and Its Representations

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This article is mainly based on Prof. Dingyu Shao's lecture on relativistic QFT at Fudan University. It also summarize how these things can be found in different QFT textbooks.

1 The geometry of the Lorentz group

The **Lorentz group** is defined to be the metric-keeping
   TODO: Double covering

2 Generators of the Lorentz group

In Section 2.3 in this note, we have introduced spins with the approach in *Physics from Symmetry*. Here we redo the work in a more swift way in Peskin Section 3.1.

We already know the angular momentum operators are

$$\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\nabla}, \tag{1}$$

which form a pseudovector. According to differential geometry, we can use the Hodge star operator to map it into a rank $3 - 2 = 2$ tensor, which is

$$
\begin{aligned}
J^{ij} = \epsilon^{ijk} J^k &= -\mathrm{i}\epsilon^{ijk}\epsilon^{lmk} x^l \partial^m \\
&= -\mathrm{i}(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}) x^l \partial^m \\
&= -\mathrm{i}(x^i \partial^j - x^j \partial^i).
\end{aligned}
\tag{2}
$$

We can try to extend this antisymmetric tensor into a 4-antisymmetric tensor, which is

$$J^{\mu\nu} = \mathrm{i}(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{3}$$

Note that $\partial^\mu = (\partial_t, -\boldsymbol{\nabla})$. Now we find J^{0i} are just the boost generators. Therefore, $J^{\mu\nu}$ - an antisymmetric tensor with 6 independent components - gives all rotational and boost generators of the Lorentz group. The J^{0i} components are boost generators, and the J^{ij} components are rotational generators. A general Lorentz transformation is

$$U = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu} J^{\mu\nu}}, \tag{4}$$

where $\omega_{\mu\nu}$ is an antisymmetric matrix.

(3) is actually the scalar field representation of the Lie algebra of the Lorentz group. We need to find the commutation rules from (3), to find more representations of the Lorentz group. We have

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right). \tag{5}$$

The derivation of this equation is just to expand the LHS and sort all terms according to the metric factor.

The most frequently used representation is the representation on 4-vectors, which is

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = \mathrm{i}(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha). \tag{6}$$

3 Dirac algebra

Consider we have a representation $S^{\mu\nu}$ of (5). Note that if we assume

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{7}$$

where we have the **Dirac algebra**

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{8}$$

then (5) holds. The calculation is tedious. Here we list the main steps. First we have

$$
\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}[[\gamma^\mu, \gamma^\nu], [\gamma^\rho, \gamma^\sigma]] \\
&= -\frac{1}{16}([[[\gamma^\mu, \gamma^\nu], \gamma^\rho], \gamma^\sigma] + [[[\gamma^\nu, \gamma^\rho], \gamma^\sigma], \gamma^\mu] \\
&\qquad + [[[\gamma^\rho, \gamma^\sigma], \gamma^\mu], \gamma^\nu] + [[[\gamma^\sigma, \gamma^\mu], \gamma^\nu], \gamma^\rho]) \\
&=
\end{aligned}
$$

We can then prove

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu, \tag{9}$$

and then by inserting (7)

> **Note**
>
> We have the following formulae:
>
> $\{A, BC\} = \{A, B\}C - B[A, C],$
> $\{AB, C\} = A\{B, C\} - [A, C]B,$
> $[AB, C] = A\{B, C\} - \{A, C\}B,$
> $[[A, C], [B, D]] = [[[A, B], C], D] + [[[B, C], D], A] + [[[C, D], A], B] + [[[D, A], B], C].$

$$S^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma]. \tag{10}$$

TODO: Can every representation be rewritten into the form of gamma matrices?

4 Dirac spinors

A possible of the Dirac algebra in the four-dimensional Minkowski space is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{11}$$

which is called the **Weyl** or **chiral** representation. In this representation, we have

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & F^k \end{pmatrix} =: \frac{1}{2}\epsilon^{ijk}\Sigma^k. \tag{12}$$

Dirac spinors are