Diffraction and Scattering in Electrodynamics by Prof. Kun Din

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1 Mie scattering and the partial wave expansion

In the previous lecture we discussed about Mie scattering. Here we discuss more about the partial wave expansion. First we review the scalar version of partial wave approximation. Suppose the input wave function is a plane wave

$$u_{\text{in}} = \mathrm{e}^{\mathrm{i}kz} = \sum_{n=0}^{\infty} (2n+1)\mathrm{i}^n j_n(kr)\,\mathrm{P}_n(\cos\theta). \tag{1}$$

A scattering eigen state can still be seen as a mixture of spherical functions at the infinity, and therefore it can be written as

$$u = \sum_n (2n+1)\mathrm{i}^n\,\mathrm{P}_n(\cos\theta)\frac{1}{2}(h_n^{(2)}(kr) + S_n h_n^{(1)}(kr)), \tag{2}$$

where S_n is a complex factor. We know that at the infinity we have

$$u = \mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f(\theta), \tag{3}$$

and therefore we have

$$f(\theta) = \frac{1}{2\mathrm{i}k}\sum_n (2n+1)(S_n - 1)\,\mathrm{P}_n(\cos\theta) =: \frac{1}{2\mathrm{i}k}\sum_n (2n+1)(\mathrm{e}^{\mathrm{i}2\delta_n} - 1)\,\mathrm{P}_n(\cos\theta). \tag{4}$$

The unitarity of scattering guarantees that $|S_n| = 1$, and therefore δ_n is a real number and is often called the **phase shift**.

In the $ka \to 0$ limit, we have

$$a_n = \frac{\epsilon_1 - \epsilon}{\mathrm{i}(\epsilon_1 n + \epsilon(n+1))}\frac{(ka)^{2n+1}(n+1)}{(2n+1)!!(2n-1)!!}, \quad b_n = \frac{\mu_1 - \mu}{\mathrm{i}(\mu_1 n + \mu(n+1))}\frac{(ka)^{2n+1}(n+1)}{(2n+1)!!(2n-1)!!}, \tag{5}$$

and if we just keep the a_1 term we find

$$Q_{\text{sca}} = \frac{8}{3}(ka)^4\left(\frac{\epsilon_1 - \epsilon}{\epsilon_1 + 2\epsilon}\right)^2, \tag{6}$$

which is just the Rayleigh cross section.

2 Effective medium

This is actually a way to find the effective permittivity of a electric dipole or to find the effective electric dipole of a sphere. Suppose

$$\boldsymbol{p} = \hat{\boldsymbol{e}}_x \alpha E_0 \mathrm{e}^{\mathrm{i}kz}, \tag{7}$$

and we have

$$\boldsymbol{E}(\boldsymbol{r}, \omega) = \omega^2 \mu_0 \overset{\leftrightarrow}{\boldsymbol{G}} \cdot \boldsymbol{p}, \tag{8}$$

where

$$\overset{\leftrightarrow}{\boldsymbol{G}} = \frac{\mathrm{e}^{\mathrm{i}kR}}{4\pi R}\left(\overset{\leftrightarrow}{\boldsymbol{I}} - \frac{\boldsymbol{R}\boldsymbol{R}}{\boldsymbol{R}^2}\right) \tag{9}$$

is the dyadic Green function. Then we find

$$\frac{k}{\epsilon}\,\mathrm{Im}\,\alpha = C_{\mathrm{ext}}^{\mathrm{dipole}} = C_{\mathrm{ext}}^{\mathrm{Mie}} = 4\pi k a^3\,\mathrm{Im}\left(\frac{\epsilon_1-\epsilon}{\epsilon_1+2\epsilon}\right),$$

and therefore

$$\alpha = 4\pi\epsilon a^3\frac{\epsilon_1-\epsilon}{\epsilon_1+2\epsilon}. \tag{10}$$

This is the basis of **Maxwell Garnett theory**. Suppose some particles with permittivity ϵ_1 are distributed in a matrix medium with permittivity ϵ_{m}. By the Clausius–Mossotti relation we have

$$\frac{\epsilon_{\mathrm{eff}}-1}{\epsilon_{\mathrm{eff}}+2} = \frac{N}{3\epsilon_{\mathrm{m}}}\alpha_{\mathrm{sphere}},$$

and on the other hand we have

$$\alpha_{\mathrm{sphere}} = 4\pi\epsilon_{\mathrm{m}} a^3\frac{\epsilon_1-\epsilon_{\mathrm{m}}}{\epsilon_1+2\epsilon_{\mathrm{m}}},$$

and therefore

$$\frac{\epsilon_{\mathrm{eff}}-1}{\epsilon_{\mathrm{eff}}+2} = \frac{4}{3}\pi a^3 N\frac{\epsilon_1-\epsilon_{\mathrm{m}}}{\epsilon_1+2\epsilon_{\mathrm{m}}}. \tag{11}$$

(11) only works for spheres. A more general formulation is

$$\boldsymbol{p} = V\beta(\epsilon_1-\epsilon_{\mathrm{m}})\boldsymbol{E}, \tag{12}$$

and in the case of (11) we have

$$\beta = \frac{3\epsilon_{\mathrm{m}}}{\epsilon_1+2\epsilon_{\mathrm{m}}} = \frac{\epsilon_{\mathrm{m}}}{\epsilon_{\mathrm{m}}+\frac{1}{3}(\epsilon_1-\epsilon_m)}. \tag{13}$$

More generally we have

$$\beta = \frac{\epsilon_{\mathrm{m}}}{\epsilon_{\mathrm{m}}+L(\epsilon_1-\epsilon_{\mathrm{m}})}. \tag{14}$$

Note that β and L may be tensors, and for a sphere $L=1/3$. We define

$$f_{\mathrm{r}} = NV, \tag{15}$$

where V is the total volume of the medium. It can be seen that f_{r} gives the volume fraction of the particles. Therefore we have

$$\frac{\epsilon_{\mathrm{eff}}-1}{\epsilon_{\mathrm{eff}}+2} = \frac{f_{\mathrm{r}}\beta}{3\epsilon_{\mathrm{m}}}(\epsilon_1-\epsilon_{\mathrm{m}}), \tag{16}$$

or in other words

$$\epsilon_{\mathrm{eff}} = \epsilon_{\mathrm{m}}\left(1+\frac{3f_{\mathrm{r}}\frac{\epsilon_1-\epsilon_{\mathrm{m}}}{\epsilon_1+2\epsilon_{\mathrm{m}}}}{1-f_{\mathrm{r}}\frac{\epsilon_1-\epsilon_{\mathrm{m}}}{\epsilon_1+2\epsilon_{\mathrm{m}}}}\right). \tag{17}$$

This equation works for particles in the shape of spheres, rectangles or cylinders. For tablet-like particles, for example, we have $L_{\parallel}=0, L_{\perp}=1$, and therefore

$$\beta_{\parallel}=1,\quad \beta_{\perp}=\frac{\epsilon_{\mathrm{m}}}{\epsilon_1},$$

and therefore

$$\epsilon_{\mathrm{ext},\parallel} = (1-f_{\mathrm{r}})\epsilon_{\mathrm{m}} + f_{\mathrm{r}}\epsilon_1, \tag{18}$$

and

$$\epsilon_{\mathrm{ext},\perp} = \left(\frac{1-f_{\mathrm{r}}}{\epsilon_{\mathrm{m}}}+\frac{f_{\mathrm{r}}}{\epsilon_1}\right)^{-1}. \tag{19}$$

Note that we have restricted ourselves on dielectrics, where the lowest electromagnetic modes are almost always gapless, and the effective medium approximation works well. This approximation may broke for metals, where low frequency electromagnetic waves cannot pass because

$$\epsilon = 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2}.$$

In the language of band theory, in metals, electromagnetic modes are gapped. An interesting case is when both ϵ and μ are negative. This is called **negative-index metamaterial**.

3 A brief summary

- Geometrical optics limit.
- Mie scattering.
  - Special functions: vector spherical functions, π_n, τ_n, Riccati-Bessel functions,
  - Mie coefficients, and
  - Cross sections.
- Partial wave expansion, and
- Effective permittivity.

4 Special relativity

If the form of an EOM is invariant under a certain transformation, we say that the objects involved in the equation are **covariant**. They will change under the transformation, but they change in a certain way that in an EOM their changes somehow cancel with each other, leaving the form of the EOM invariant. Sometimes we also say the equation is *covariant*, since only the form is invariant, and the actually LHS and RHS change.

Under the Galilean transformation

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t, \quad t' = t, \tag{20}$$

we have

$$\frac{\partial}{\partial \boldsymbol{r}} = \frac{\partial}{\partial \boldsymbol{r}'}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}'}, \tag{21}$$

and therefore

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} - 2\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}'} \frac{\partial}{\partial t'} + \boldsymbol{v}\boldsymbol{v} : \frac{\partial}{\partial \boldsymbol{r}'} \frac{\partial}{\partial \boldsymbol{r}'}, \tag{22}$$

so under the (passive) Galilean transformation, we see that the wave equation

$$\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \tag{23}$$

turns into

$$\left(\nabla^{2\prime} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} + \frac{2}{c^2} (\boldsymbol{v} \cdot \boldsymbol{\nabla}') \frac{\partial}{\partial t'} - \frac{1}{c^2} \boldsymbol{v}\boldsymbol{v} : \boldsymbol{\nabla}' \boldsymbol{\nabla}' \right) f = 0. \tag{24}$$

So the wave function is *not* covariant under the Galilean transformation.