Special Relativity by Prof. Kun Din

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In the previous lecture we discussed the covariant form of the Maxwell equations. We also need to have a theory about *matters* that are coupled to the electromagnetic field. This can be done by a

$$\frac{\mathrm{d}p^\nu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu =: f^\mu, \tag{1}$$

where u_μ is the four-velocity.

The dual field strength tensor, when written as a functional of the scalar and vector potentials

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -(\boldsymbol{\nabla}\times\boldsymbol{A})_x & -(\boldsymbol{\nabla}\times\boldsymbol{A})_y & -(\boldsymbol{\nabla}\times\boldsymbol{A})_z \\ (\boldsymbol{\nabla}\times\boldsymbol{A})_x & 0 & -\partial_z(c^{-1}\phi) - \partial_{ct}A_z & \partial_y(c^{-1}\phi) + \partial_{ct}A_y \\ (\boldsymbol{\nabla}\times\boldsymbol{A})_y & \partial_z(c^{-1}\phi) + \partial_{ct}A_z & 0 & -\partial_x(c^{-1}\phi) - \partial_{ct}A_x \\ (\boldsymbol{\nabla}\times\boldsymbol{A})_z & -\partial_y(c^{-1}\phi) - \partial_{ct}A_y & \partial_x(c^{-1}\phi) + \partial_{ct}A_x & 0 \end{pmatrix}, \tag{2}$$

can be verified to satisfy the condition $\partial_\mu \tilde{F}^{\mu\nu} = 0$. Therefore, when we are dealing with a theory about potentials instead of electromagnetic fields, we can completely ignore the two sourceless Maxwell equations, and focus on $\partial_\mu F^{\mu\nu} = J^\nu$ and (1).

The Lagrangian of a particle is

$$L_\mathrm{m} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}. \tag{3}$$

The total Lagrangian of the relativistic particle coupled to an electromagnetic field is

$$L = L_\mathrm{m} - q\phi + q\boldsymbol{v}\cdot\boldsymbol{A}. \tag{4}$$

We need to verify whether (4) gives (1). The Euler-Lagrangian equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-mc^2\frac{-\boldsymbol{v}/c^2}{\sqrt{1-v^2/c^2}}\right) - (-q\boldsymbol{\nabla}\phi)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}. \tag{5}$$

We go on to write down a Lagrangian for the electromagnetic field. What we need to do is to construct a scalar bilinear using A_μ, $\partial_\mu A^\mu$, and $\partial_\mu A^\nu$. The most general case is

$$\mathcal{L} = \alpha(\partial_\mu A^\mu)^2 + \beta(\partial_\mu A^\nu)(\partial^\mu A_\nu) + \gamma(\partial_\mu A^\nu)(\partial_\nu A^\mu) + \delta A^\mu A_\mu,$$

and we need to add a coupling term

$$\mathcal{L}_\text{couple} = -j^\mu A_\mu.$$

Note that $\tilde{F}^{\mu\nu}$ is a pseudotensor and therefore cannot appear linearly in the Lagrangian. For example, we know

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{4}{c}\boldsymbol{E}\cdot\boldsymbol{B}$$

is a pseudoscalar, and therefore cannot be a term of the Lagrangian. Theories about axions has a term like

$$\operatorname{tr}\overset{\leftrightarrow}{\boldsymbol{k}}\, F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

We do not discuss such theories here. The Euler-Lagrangian equation is now

$$-\beta\partial_\nu\partial^\nu A^\mu - (\alpha+\gamma)\partial_\nu\partial^\mu A^\nu + \delta A^\mu = \frac{1}{2}j^\mu.$$

Comparing the equation with $\partial_\mu F^{\mu\nu} = j^\nu$, we find

$$\delta = 0, \quad \beta = -\frac{1}{2\mu_0}, \quad \alpha + \gamma = \frac{1}{2\mu_0}.$$

If we impose the Lorenz gauge, we have $\alpha = 0$, and in this way the Lagrangian can be written concisely as

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu. \tag{6}$$

The fact that $\delta = 0$ means photons are massless. Weak interaction has a Lagrangian similar to the one of electrodynamics, but the bosons are massive due to Higgs mechanism.

We can see the gauge symmetry in the Lagrangian. Under a gauge transformation of the potential, we have

$$\begin{aligned} j_\mu A^\mu &\longrightarrow j_\mu (A^\mu + \partial^\mu \Lambda) \\ &= j_\mu A^\mu + \partial^\mu (j_\mu \Lambda) - \Lambda \partial_\mu j^\mu, \end{aligned}$$

where the second term is a boundary term and the third term is zero because of charge conservation. Therefore, we find $j_\mu A^\mu$ invariant under the gauge transformation of A^μ. Actually j_μ itself may change because of a phase factor in the matter field, and in QED we find it cancels the change of $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Now we derive the conservation of energy and momentum. Under a small coordinate translation, we have

$$\delta x^\mu = x'^\mu = x^\mu,$$

and it can be verified that if the Lagrangian has temporal and spacial translational invariance, then

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \Theta_{\mu\nu} = \frac{\partial L}{\partial(\partial^\mu A_\sigma)} \partial_\nu A_\sigma - g^{\mu\nu} \mathcal{L}. \tag{7}$$

Note that here we fix j_μ, and therefore whether the Lagrangian has temporal or spacial translational invariance depends on j^μ. We then verify this for (6). We have

$$\frac{\partial F_{\alpha\beta} F^{\alpha\beta}}{\partial(\partial_\mu A_\sigma)} = 4 F^{\mu\sigma},$$

and therefore

$$\Theta^{\mu\nu} = \frac{1}{4\mu_0} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{\mu_0} F^{\mu\sigma} \partial^\nu A_\sigma + \eta^{\mu\nu} j_\sigma A^\sigma, \tag{8}$$

and

$$\partial_\mu \Theta^{\mu\nu} = A^\sigma \partial^\nu j_\sigma. \tag{9}$$

Therefore, if j_σ depends on t, or in other words $\partial^0 j_\sigma \neq 0$, then $\partial_\mu \Theta^{\mu 0} \neq 0$. So we see that we have energy conservation if and only if the system has time translational symmetry. Similarly, a system has conserved momentum if and only if the system has space translational symmetry.

The definition of $\Theta^{\mu\nu}$ as a energy-momentum tensor is flawed. We see that

$$\Theta^{00} = \frac{\epsilon_0}{2} \boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{2\mu_0} \boldsymbol{B} \cdot \boldsymbol{B} + \epsilon_0 \boldsymbol{\nabla} \cdot (\phi \boldsymbol{E}). \tag{10}$$

Here we see a useless boundary term, which is absent in the usual definition of electromagnetic energy. Nor is $\Theta^{\mu\nu}$ gauge invariant. What we need to do is to *symmetrize* the tensor, hopping this can solve these two flaws. We will find the **Belinfante symmetrization**

$$\begin{aligned} T^{\mu\nu} &= \Theta^{\mu\nu} + \frac{1}{\mu_0} \partial_\sigma (F^{\mu\sigma} A^\nu) \\ &= \frac{1}{4\mu_0} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{\mu_0} F^{\mu\sigma} F_\sigma{}^\mu + \eta^{\mu\nu} j_\sigma A^\sigma - j^\mu A^\nu, \end{aligned} \tag{11}$$

is a good choice. We find

$$T^{00} = \frac{\epsilon_0}{2} \left(\boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{c^2} \boldsymbol{B} \cdot \boldsymbol{B} \right) - \boldsymbol{j} \cdot \boldsymbol{A}, \tag{12}$$

and

$$T^{0k} = \frac{1}{\mu_0} F^{0\sigma} F_\sigma{}^{k} = \tag{13}$$

We turn to consider the conserved flow associated with Lorentz boost and rotation. We have

$$\begin{aligned} x'^\mu &= x^\mu + \delta\omega^{\mu\nu}\, x_\nu, \\ A'^\mu &= A^\mu + \frac{1}{2}\,\delta\omega_{\alpha\beta}\,(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu})A_\nu(x), \end{aligned} \tag{14}$$

and we have

$$M^{\mu\nu\lambda} = \Theta^{\sigma\lambda}x^\nu - \Theta^{\sigma\nu}x^\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\sigma A^\sigma)}(\eta^{\nu\sigma}\eta^{\lambda\tau} - \eta^{\nu\tau}\eta^{\lambda\sigma})A_\tau. \tag{15}$$

Especially, the conserved charge $M^{\nu\lambda} \coloneqq M^{0\nu\lambda}$ is

$$M^{\nu\lambda} = \int \mathrm{d}^3\boldsymbol{x} \left(\Theta^{0\lambda}x^\nu - \Theta^{0\nu}x^\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\sigma)}(\eta^{\nu\sigma}\eta^{\lambda\tau} - \eta^{\nu\tau}\eta^{\lambda\sigma})A_\tau\right). \tag{16}$$

We can see that we have an orbital angular momentum term

$$L^{\nu\lambda} = \int \mathrm{d}^3\boldsymbol{x}\,\left(\Theta^{0\lambda}x^\nu - \Theta^{0\nu}x^\lambda\right), \tag{17}$$

and a spin angular momentum term

$$S^{\nu\lambda} = \int \mathrm{d}^3\boldsymbol{x}\,\frac{\partial\mathcal{L}}{\partial(\partial_0 A^\sigma)}(\eta^{\nu\sigma}\eta^{\lambda\tau} - \eta^{\nu\tau}\eta^{\lambda\sigma})A_\tau. \tag{18}$$