QFT I, Homework 3

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Feynman propagator in position space Calculate the Feynman propagator in position space. To get the pole structure correct, you may find it helpful to use Schwinger parameters (see Schwartz Appendix B). Take the $m \to 0$ limit of your result to find [This is problem 6.1 on p. 77 of Schwartz.]

$$\langle 0 \left| \mathcal{T} \left\{ \phi_0\left(x_1\right) \phi_0\left(x_2\right) \right\} \right| 0 \rangle = -\frac{1}{4\pi^2} \frac{1}{\left(x_1 - x_2\right)^2 - i\varepsilon}. \tag{1}$$

Solution The Feynman propagator in the momentum space is $\mathrm{i}/(p^2 - m^2 + \mathrm{i}0^+)$, and by Fourier transformation we have

$$\mathcal{T} \langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2)} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}0^+}. \tag{2}$$

By Schwinger parametrization

$$\frac{\mathrm{i}}{A} = \int_0^\infty \mathrm{d}u \, \mathrm{e}^{\mathrm{i}uA}$$

we have

$$\begin{aligned} \mathcal{T} \langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2)} \int_0^\infty \mathrm{d}u \, \mathrm{e}^{\mathrm{i}u(p^2 - m^2 + \mathrm{i}0^+)} \\ &= \int_0^\infty \mathrm{d}u \, \mathrm{e}^{\mathrm{i}u(-m^2 + \mathrm{i}0^+)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2) + \mathrm{i}up^2}. \end{aligned}$$

By the n-dimensional Gaussian integral

$$\int \mathrm{d}^n \mathbf{x} \, \mathrm{e}^{-\frac{1}{2}\mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{b}^\top \mathbf{x}} = \sqrt{\frac{(2\pi)^n}{\det \mathbf{A}}} \mathrm{e}^{\frac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{b}}$$

we have

$$\begin{aligned} \int \mathrm{d}^4 p \, \mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2) + \mathrm{i}up^2} &= \sqrt{\frac{(2\pi)^4}{\det(-2\mathrm{i}u\eta_{\mu\nu})}} \mathrm{e}^{\frac{1}{2}(-\mathrm{i}(x_1 - x_2)_\mu) \frac{1}{-2\mathrm{i}u} \eta^{\mu\nu} (-\mathrm{i}(x_1 - x_2)_\nu)} \\ &= \frac{(2\pi)^2}{\mathrm{i}(2u)^2} \mathrm{e}^{-\frac{\mathrm{i}}{4u}(x_1 - x_2)^2}, \end{aligned}$$

where we set

$$\mathbf{x} = p^\mu, \quad \mathbf{A} = -2\mathrm{i}u\eta_{\mu\nu}, \quad \mathbf{b} = -\mathrm{i}(x_1 - x_2)_\mu.$$

The Feynman propagator is now

$$\begin{aligned} \mathcal{T} \langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle &= \int_0^\infty \mathrm{d}u \, \mathrm{e}^{\mathrm{i}u(-m^2 + \mathrm{i}0^+)} \frac{1}{(2\pi)^4} \times \frac{(2\pi)^2}{\mathrm{i}(2u)^2} \mathrm{e}^{-\frac{\mathrm{i}}{4u}(x_1 - x_2)^2} \\ &= -\frac{\mathrm{i}}{16\pi^2} \int_0^\infty \frac{\mathrm{d}u}{u^2} \mathrm{e}^{-\frac{\mathrm{i}}{4u}(x_1 - x_2)^2 - \mathrm{i}u(m^2 - \mathrm{i}0^+)} \\ &= -\frac{\mathrm{i}}{16\pi^2} \int_0^\infty \frac{\mathrm{d}u}{u^2} \mathrm{e}^{-\mathrm{i}\left(\frac{1}{4u}(x_1 - x_2)^2 + m^2 u\right) - u0^+}. \end{aligned}$$

The integral in the last line is actually a modified Bessel function. Section 3.324 in [1] gives

$$\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right) \mathrm{d}x \, x = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}) \quad \text{where } \operatorname{Re}\beta \geq 0, \quad \operatorname{Re}\gamma > 0,$$

and by integration by substitution we have

$$\int_0^\infty \exp\left(-At - \frac{B}{4t}\right)\frac{\mathrm{d}t}{t^2} = 4\sqrt{\frac{A}{B}}K_1(\sqrt{AB}), \tag{3}$$

where $\mathrm{Re}\,A \geq 0$ and $\mathrm{Re}\,B > 0$. By rewriting the Feynman propagator into

$$\mathcal{T}\,\langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle = -\frac{\mathrm{i}}{16\pi^2}\int_0^\infty \frac{\mathrm{d}u}{u^2}\mathrm{e}^{-\mathrm{i}\left(\frac{1}{4u}(x_1-x_2)^2+m^2u\right)-\frac{1}{4u}0^+}$$

and taking

$$A = \mathrm{i}m^2, \quad B = \mathrm{i}(x_1 - x_2)^2 + 0^+,$$

we have

$$\begin{aligned}
\mathcal{T}\,\langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle &= -\frac{\mathrm{i}}{16\pi^2} \times \lim_{\epsilon\to 0} 4\sqrt{\frac{\mathrm{i}m^2}{\mathrm{i}(x_1-x_2)^2+\epsilon}}K_1(\sqrt{\mathrm{i}m^2(\mathrm{i}(x_1-x_2)^2+\epsilon)}) \\
&= -\frac{\mathrm{i}}{4\pi^2}\lim_{\epsilon\to 0}\sqrt{\frac{m^2}{(x_1-x_2)^2-\mathrm{i}\epsilon}}K_1(\sqrt{-m(x_1-x_2)^2+\mathrm{i}\epsilon}),
\end{aligned}$$

so we obtain the Feynman propagator with the pole structure taken into account:

$$\mathcal{T}\,\langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle = \frac{m}{4\pi^2\sqrt{-(x_1-x_2)^2+\mathrm{i}0^+}}K_1(\sqrt{-m(x_1-x_2)^2+\mathrm{i}0^+}). \tag{4}$$

The expansion of the Bessel K function can be obtained using Mathematica. We have

$$K_1(z) = \frac{1}{z} + \mathcal{O}(z),$$

so the massless limit is

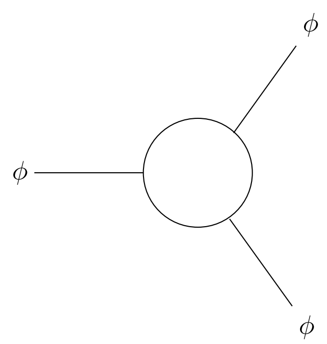
$$\begin{aligned}
\mathcal{T}\,\langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle &= \frac{m}{4\pi^2\sqrt{-(x_1-x_2)^2+\mathrm{i}0^+}}\left(\frac{1}{\sqrt{-m(x_1-x_2)^2+\mathrm{i}0^+}} + \mathcal{O}(\sqrt{m})\right) \\
&= \frac{m}{4\pi^2}\frac{1}{-m(x_1-x_2)^2+\mathrm{i}0^+} + \mathcal{O}(m^{3/2}) \\
&\to -\frac{1}{4\pi^2(x_1-x_2)^2-\mathrm{i}0^+} \quad \text{as } m \to 0,
\end{aligned}$$

which is just (1).

ϕ^3 theory Consider the Lagrangian for ϕ^3 theory, [This is problem 7.1 on p. 103 of Schwartz.]

$$\mathcal{L} = -\frac{1}{2}\phi\left(\Box + m^2\right)\phi + \frac{g}{3!}\phi^3$$

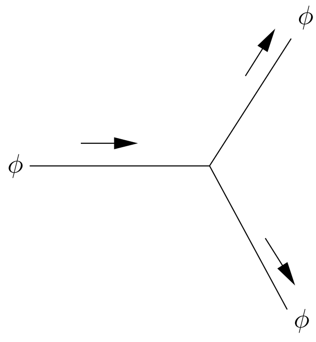
(a) Draw a tree-level Feynman diagram for the decay $\phi \to \phi\phi$. Write down the corresponding amplitude using the Feynman rules. (b) Now consider the one-loop correction, given by



Write down the corresponding amplitude using the Feynman rules. (c) Now start over and write down the diagram from part (b) in position space, in terms of integrals over the intermediate points and Wick contractions, represented with factors of D_F. (d) Show that after you apply LSZ, what you got in (c) reduces to what you got in (b), by integrating the phases into δ-functions, and integrating over those δ-functions.

Solution

(a) There is only one tree-level diagram for $\phi \to \phi\phi$ which is



$= \mathrm{i}g.$

The tree-level amplitude is therefore g since $\mathrm{i}\mathcal{M} = \mathrm{i}g$.

Example of differential cross section Use the Lagrangian [This is problem 7.6 on p. 104 of Schwartz.]

$$\mathcal{L} = -\frac{1}{2}\phi_1\Box\phi_1 - \frac{1}{2}\phi_2\Box\phi_2 + \frac{\lambda}{2}\phi_1\left(\partial_\mu\phi_2\right)\left(\partial_\mu\phi_2\right) + \frac{g}{2}\phi_1^2\phi_2$$

to calculate the differential cross section

$$\frac{d\sigma}{d\Omega}\left(\phi_1\phi_2 \to \phi_1\phi_2\right)$$

at tree level.
Solution

Decay of a scalar particle This is problem 4.2 on p. 127 of Peskin. Consider the following Lagrangian, involving two real scalar fields Φ and ϕ:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\Phi\right)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi$$

The last term is an interaction that allows a Φ particle to decay into two ϕ 's, provided that $M > 2m$. Assuming that this condition is met, calculate the lifetime of the Φ to lowest order in μ.
Solution

References

[1] Daniel Zwillinger, Victor Moll, I.S. Gradshteyn, and I.M. Ryzhik, editors. *Table of Integrals, Series, and Products.* Academic Press, Boston, seventh edition edition, 2007.