QFT I, Homework 4

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Scalar QED Consider the theory of a complex scalar field ϕ interacting with the electromagnetic field A^μ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

where $D_\mu = \partial_\mu + \mathrm{i} e A_\mu$ is the usual gauge covaraint derivative.

(a) Show the Lagrangian is invariant under the gauge transformations

$$\phi(x) \to \mathrm{e}^{-\mathrm{i}\alpha(x)} \phi(x), \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x). \tag{2}$$

(b) Derive the Feynman rules for the interaction between photons and scalar particles.

(c) Draw all the leading-order Feynman diagrams and compute the amplitude for the process $\gamma\gamma \to \phi\phi^*$.

(d) Compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$. You can take an average over all initial state polarizations. For simplicity, you can restrict your calculation in the limit $m = 0$.

(e) Draw all leading order Feynman diagrams, that contribute to the Compton scattering process $\gamma\phi \to \gamma\phi$ and compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$ with $m = 0$.

Solution

(a) Under the gauge transformation (2), we have

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu \left(A_\nu + \frac{1}{e} \partial_\nu \alpha \right) - \partial_\nu \left(A_\mu + \frac{1}{e} \partial_\mu \alpha \right) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu},$$

so the first term in (1) remains the same. It is obvious that under (2)

$$\phi^* \phi \to \phi'^* \phi' = \mathrm{e}^{\mathrm{i}\alpha} \phi^* \mathrm{e}^{-\mathrm{i}\alpha} \phi = \phi^* \phi,$$

so the third term in (1) is also invariant. Also we have

$$\begin{aligned} D^\mu \phi \to (\partial^\mu + \mathrm{i} e A'^\mu) \phi' &= (\partial^\mu + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \mathrm{e}^{-\mathrm{i}\alpha} \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} (\partial^\mu - \mathrm{i} \partial^\mu \alpha + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} D^\mu \phi, \end{aligned}$$

and also

$$(D^\mu \phi)^* = \mathrm{e}^{\mathrm{i}\alpha} D^\mu \phi,$$

so $D^\mu \phi (D^\mu \phi)^*$ is also invariant. Therefore (1) is invariant under (2).

(b) Expanding (2) we have

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{scalarQED}}, \tag{3}$$

where $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{vector}}$ are Lagrangians of free scalar field and free massless vector field, and

$$\begin{aligned} \mathcal{L}_{\text{scalarQED}} &= (D_\mu \phi)^* D^\mu \phi - (\partial_\mu \phi)^* \partial^\mu \phi \\ &= e^2 A_\mu A^\mu \phi \phi^* - \mathrm{i} e A_\mu \phi^* \partial^\mu \phi + \mathrm{i} e \partial_\mu \phi^* A^\mu \phi. \end{aligned} \tag{4}$$

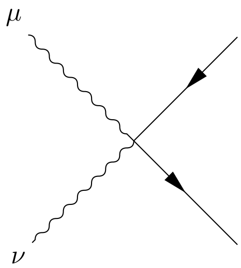
We make the following expansion of Fourier transformation. For the complex scalar field we have

$$\phi(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i} p \cdot x} + b^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i} p \cdot x}). \tag{5}$$

which was proved in (10) in Homework 2. The vector field is expanded as

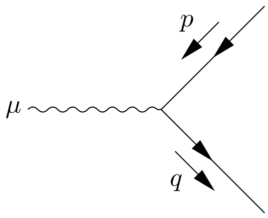
$$A_\mu(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \sum_{r=1}^{2} \epsilon^r_\mu(\boldsymbol{p}) \left(a^\dagger_{\boldsymbol{p},r} \mathrm{e}^{\mathrm{i}p\cdot x} + a_{\boldsymbol{p},r} \mathrm{e}^{-\mathrm{i}p\cdot x} \right). \tag{6}$$

The first term gives the following (momentum space) diagram:

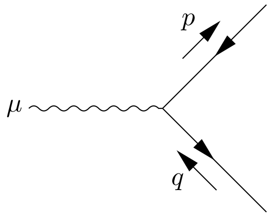


$$= \mathrm{i}e^2 \delta^\nu_\mu. \tag{7}$$

The second and the third term gives



$$= -\mathrm{i}e(p_\mu + q_\mu), \tag{8}$$



$$= \mathrm{i}e(p_\mu + q_\mu). \tag{9}$$

Note

Here we follow the notation of Peskin, i.e. using the arrow *on* a particle line to show whether this line represents a particle or a antiparticle and using the *momentum* arrow to denote whether this line represents creation or annihilation. The direction and sign of a 4-momentum is *not* represented in any arrow.

(c)