Quantum Optics, Homework 5

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Stochastic wavefunction of a leaky cavity photon field The effective Halmitonian for the photon field in a leaky cavity is given by: $H_{\text{eff}} = \hbar\left(-\mathrm{i}\frac{\kappa}{2}\right)a^\dagger a$, with the associated quantum jump operator: $C = \sqrt{\kappa}a$. Discuss the non-Hermitian evolution of stochastic wavefunction $|\psi(\mathrm{t})\rangle$ without quantum jump, and provide wavefunction after a quantum jump at time t.

(a) For $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |1\rangle)$

(b) For $|\psi(t=0)\rangle = |\alpha\rangle$

(c) For $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle)$ (here we consider $|\alpha|^2 \gg 1$)

Solution

(a)

Stochastic wavefunction of two cavity fields We consider the setup in Figure 1 on page 1, the leak fields of two identical cavities are mixed by a beamsplitter S and then detected by the photon counter D1 and D2. Following the stochastic wavefunction method, the effective Hamiltonian for the photon field of the twocavity system can be written as $H_{\text{eff}} = \hbar\left(-i\frac{\kappa}{2}\right)(a^+a + b^+b)$. While normally we would have quantum jump operators of $C_a = \sqrt{\kappa}a$ and $C_b = \sqrt{\kappa}b$, here it is more convenient to introduce collective jump operators $C_1 = \sqrt{\kappa}(ta + rb)$ and $C_2 = \sqrt{\kappa}\left(-r^*a + tb\right)$, with r, t to be the reflective and transmission coefficients of the beamsplitter S.

2.1 Consider the initial state to be a product coherent state: $|\psi(0)\rangle = |\alpha, \beta\rangle$. Evaluate the stochastic wavefunction of the photon field, $|\psi_{\mathrm{S}}(\mathrm{t})\rangle$, for the non-Hermitian evolution without any quantum jump, and after a quantum jump by C_1 operator (D1 "click")

2.2 Consider the simple situation of a 50% beamspitter, $r = t = 1/\sqrt{2}$, continue with the first problem to derive the photon detection rate $\gamma_1(t) = \left\langle\psi(t)\left|C_1^+C_1\right|\psi(t)\right\rangle$ and $\gamma_2(t) = \left\langle\psi(t)\left|C_2^+C_2\right|\psi(t)\right\rangle$. Is it possible to properly choose non-zero α and β values, so as to have $\gamma_1 \equiv 0$?

2.3 Repeat 2.1 with the Fock initial state $|\psi(0)\rangle = |N, N\rangle$.

2.4 Continue with 2.2, again assuming $r = t = 1/\sqrt{2}$, evaluate $\gamma_1(t) = \left\langle\psi(t)\left|C_1^+C_1\right|\psi(t)\right\rangle$ and $\gamma_2(t) = \left\langle\psi(t)\left|C_2^+C_2\right|\psi(t)\right\rangle$ before there is any quantum jump, and after a quantum jump with a D_1 "click". For $N = 1$. Discuss your results in terms of the HongOu-Mandel effect.

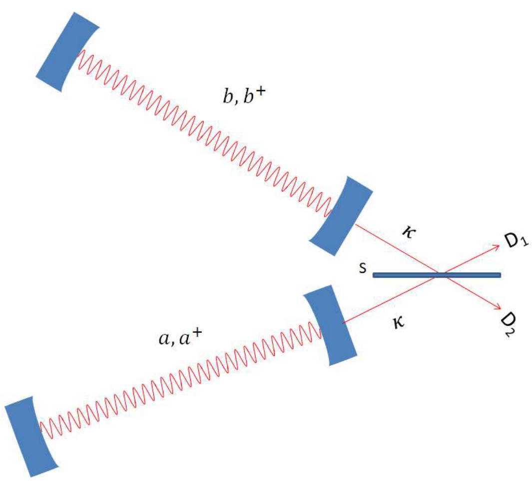

Figure 1: Output of two cavities are mixed by splitter S and detected by photon-counter D_1 and D_2

Solution

Cavity QED and single photon source As in the figure above, consider a 2-level atom coupled to a cavity field with single-photon Rabi frequency g_{ac}. The atom is in addition subjected to a laser field excitation from the side with a Rabi frequency $\Omega(\mathrm{t})$. Taking into account the radiative decay by the atom and the leak of the cavity, the effective Hamiltonian of the controlled and coupled system is given by

$$H_{\text{eff}} = \hbar\left(-\mathrm{i}\frac{\Gamma}{2}\right)|e\rangle\langle e| + \hbar\left(-\Delta - \mathrm{i}\frac{\kappa}{2}\right)a^{+}a + \left[\hbar\left(\frac{\Omega(\mathrm{t})}{2} + g_{ac}a\right)|e\rangle\langle g| + \text{ h.c. }\right], \tag{1}$$

where $\Delta = \omega - \omega_{eg}$ is the detuning of the cavity mode frequency from the atomic resonant frequency. The collapse operators are given by $C_1 = \sqrt{\Gamma}|g\rangle\langle e|$ and $C_2 = \sqrt{\kappa}a$. 3a) Consider $\Omega(\mathrm{t}) = 0$ and with system initially in $|\psi_S(0)\rangle = |g, n = 1\rangle$, that is, the atom is in the ground state and the cavity mode is in n = 1 Fock state. Consider good cavity ($\kappa \ll g, \Gamma$) and weak coupling ($g \ll \Delta, \Gamma$) limits. Expand $|\psi_S(t)\rangle$ in proper basis of choice, and to derive the Schrodinger equation for the coefficients of the stochastic wavefunction, without quantum jump. Perturbatively derive the system decay rate $\gamma_1(t) = \langle\psi_S\left|C_1^{+}C_1\right|\psi_S\rangle$ and $\gamma_2(t) = \langle\psi_S\left|C_2^{+}C_2\right|\psi_S\rangle$ (i.e., using the adiabatic elimination method which assumes $|\psi_S(t)\rangle \approx \left|\widetilde{\psi_S}\right\rangle$, with $H_{\text{eff}}\left|\widetilde{\psi_S}\right\rangle \approx -\Delta - \frac{i\kappa}{2}\left|\widetilde{\psi_S}\right\rangle\Big)$. 3b) Repeat Question 3a, but with system initially in $|\psi\rangle = |e, n = 0\rangle$ and in the bad cavity ($\kappa\rangle\rangle$ g, Γ) and weak coupling ($g \ll \Delta, \Gamma$) limit. You should arrive at a total decay rate $\gamma = \gamma_1 + \gamma_2$ that describes the Purcell effect as in the class. Discuss the condition under which $\gamma_2 \gg \gamma_1$, that is, the decay of the system more likely leading to a single photon emission into the cavity leak mode. 3c) With the system initially in $|\psi\rangle = |g, n = 0\rangle$ and with a resonant pulse $\Omega(t) = \Omega_0 \sin\left(\frac{\pi t}{\tau}\right)$ switched on and off smoothly for $0 < t < \tau$. Assuming $|\psi(t)\rangle$ to be driven by the 2-level Hamiltonian $H_a = H_{eff}(t; \Gamma, \kappa, g \to 0)$ [note: this happens effectively when $\Gamma, \kappa, g \ll 1/\tau$]. Now, putting back all the parameters into H_{eff}, Calculate $\gamma_1(t) = \left\langle\psi\left|C_1^{\dagger}C_1\right|\psi\right\rangle$ and $\gamma_2(t) = \left\langle\psi\left|C_2^{\dagger}C_2\right|\psi\right\rangle$ for stochastic wavefunction without quantum jump during 0 < t < T, with $T \gg \frac{1}{\kappa}, \frac{1}{\Gamma}$ 4d) Discuss $\Omega(t)$ and other parameters in Eq. (2), so that a single photon can be deterministically generated into the cavity leaking mode with high efficiency. Discuss the form of the single-photon wavefunction, and the fidelity of the single-photon source (how likely there is exactly one photon in the time-dependent leaky mode).

Solution