Quantum Optics, Homework 1

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October 7, 2021

Scully 1.1 The radiation field in an empty cubic cavity of side L satisfies the wave equation

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0 \tag{1}$$

together with the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$. Show that the solution that satisfies the boundary conditions has components

$$\begin{aligned} A_x(\boldsymbol{r},t) &= A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z), \\ A_y(\boldsymbol{r},t) &= A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z), \\ A_z(\boldsymbol{r},t) &= A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z), \end{aligned} \tag{2}$$

where $\boldsymbol{A}(t)$ is independent of position and the wave vector $\boldsymbol{k}$ has components given by Eq. (1.1.21). Hence show that the integers n_x, n_y, n_z in Eq. (1.1.21) are restricted in that only one of them can be zero at a time.

Solution The boundary condition concerning the vector potential is

$$\boldsymbol{n} \times (\boldsymbol{A}_1 - \boldsymbol{A}_2) = 0, \quad \boldsymbol{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}_1 - \boldsymbol{\nabla} \times \boldsymbol{A}_2) = 0.$$

Since under the Coulomb gage condition

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

and both of them vanish outside the cavity, the vector potential is a spacial and temporal constant outside the cavity. We are free to add a global constant to $\boldsymbol{A}$, and the most convenient choice is to let $\boldsymbol{A} = 0$ outside the cavity, so the boundary condition is just

$$\boldsymbol{n} \times \boldsymbol{A} = 0, \quad \boldsymbol{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$$

inside the cavity.

The first condition reads $A_x = A_y = 0$ on the $z = 0$ plane and $z = L$ plane, $A_x = A_z = 0$ on the $y = 0$ plane and the $y = L$ plane, and $A_y = A_z = 0$ on the $x = 0$ plane and $x = L$ plane, which *do not* mix the three components of $\boldsymbol{A}$ together, so solving (1) with the boundary condition $\boldsymbol{n} \times \boldsymbol{A} = 0$ is just solving

$$\nabla^2 A_i - \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} = 0, \quad i = x, y, z \tag{3}$$

separately for all the three components. The problem for A_x is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_x - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = 0, \quad A_x|_{y=0} = A_x|_{y=L} = A_x|_{z=0} = A_x|_{z=L} = 0. \tag{4}$$

By separation of variables we seek a solution in the form of

$$A_x(\boldsymbol{r},t) = A_{x,xt}(x,t) A_{xy}(y) A_{xz}(z),$$

and the equations for $A_{x,xt}, A_{xy}, A_{xz}$ are

$$\left(\frac{\partial^2}{\partial x^2} - k_y^2 - k_z^2\right) A_{x,xt} - \frac{1}{c^2}\frac{\partial^2 A_{x,xt}}{\partial t^2} = 0, \quad \frac{\partial^2 A_{xy}}{\partial y^2} + k_y^2 = 0, \quad \frac{\partial^2 A_{xz}}{\partial z^2} + k_z^2 = 0,$$

respectively. Imposing boundary conditions to (4) to A_{xy} and A_{xz} we obtain

$$A_{xy}(y) = \text{const} \times \sin(k_y y), \quad A_{xz}(z) = \text{const} \times \sin(k_z z),$$

where

$$k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L}, \quad n_y, n_z \in \mathbb{Z}, \tag{5}$$

and we find A_x has the form

$$A_x(\boldsymbol{r}, t) = A_{x,xt}(x,t)\sin(k_{xy}y)\sin(k_{xz}z), \quad k_{xy} = \frac{2\pi n_{xy}}{L}, k_{xz} = \frac{2\pi n_{xz}}{L}, \quad n_{xy}, n_{xz} \in \mathbb{Z}. \tag{6}$$

Similarly we solve (3) for A_y and the result is

$$A_y(\boldsymbol{r}, t) = A_{y,yt}(y,t)\sin(k_{yx}x)\sin(k_{yz}z), \quad k_{yx} = \frac{2\pi n_{yx}}{L}, k_{yz} = \frac{2\pi n_{yz}}{L}, \quad n_{yx}, n_{yz} \in \mathbb{Z}, \tag{7}$$

and for A_z the result is

$$A_z(\boldsymbol{r}, t) = A_{z,zt}(z,t)\sin(k_{zx}x)\sin(k_{zy}y), \quad k_{zx} = \frac{2\pi n_{zx}}{L}, k_{zy} = \frac{2\pi n_{zy}}{L}, \quad n_{zx}, n_{zy} \in \mathbb{Z}, \tag{8}$$

Now the Coulomb gauge condition is just

$$\frac{\partial A_{x,xt}}{\partial x}\sin(k_{xy}y)\sin(k_{xz}z) + \frac{\partial A_{y,yt}}{\partial y}\sin(k_{yx}x)\sin(k_{yz}z) + \frac{\partial A_{z,zt}}{\partial z}\sin(k_{zx}x)\sin(k_{zy}y) = 0.$$

This equation holds only when

$$k_{xy} = k_{zy}, \quad k_{xz} = k_{yz}, \quad k_{yx} = k_{zx},$$

and

$$\frac{\partial A_{x,xt}}{\partial x} \propto \sin(k_{yx}x), \quad \frac{\partial A_{y,yt}}{\partial y} \propto \sin(k_{xy}y), \quad \frac{\partial A_{z,zt}}{\partial z} \propto \sin(k_{yz}z).$$

We therefore rename k_{yx} as k_x, k_{xz} as k_z, and k_{xy} as k_y, and we have

$$A_{x,xt} \propto \cos(k_x x), \quad A_{y,yt} \propto \cos(k_y y), \quad A_{z,zt} \propto \cos(k_x z),$$

so finally, we have

$$A_x(\boldsymbol{r}, t) = \text{some function of } t \times \cos(k_x x)\sin(k_y y)\sin(k_z z)$$

and similar equations for A_y and A_z,where k_x, k_y, k_z all satisfy Eq. (1.1.21), and thus we have proved (2).

Now if $n_x = n_y = n_z$, then $\boldsymbol{k} = 0$. Hence $\sin(k_i r_i), i = x, y, z$ are constantly zero, and (2) is an all-zero trivial solution. If two of the integers n_x, n_y, n_z are zero, say, $n_x = n_y = 0$, that still makes (2) an all-zero solution. That explains why only one of them can be zero at a time.

Scully 1.2 If A and B are two noncommuting operators that satisfy the conditions

$$[[A, B], A] = [[A, B], B] = 0, \tag{9}$$

then show that

$$\begin{aligned} \mathrm{e}^{A+B} &= \mathrm{e}^{-\frac{1}{2}[A,B]}\mathrm{e}^A\mathrm{e}^B \\ &= \mathrm{e}^{+\frac{1}{2}[A,B]}\mathrm{e}^B\mathrm{e}^A. \end{aligned} \tag{10}$$

This is a special case of the so-called Baker-Hausdorff theorem of group theory.

Solution To show (10) it is sufficient to prove

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}, \tag{11}$$

as $[A, B]$ commutes with both A and B and therefore commute with e^A and e^B. To prove (11), we define an operator function $G(x)$ by

$$\mathrm{e}^{xA}\mathrm{e}^{xB} = \mathrm{e}^{G(x)}. \tag{12}$$

When $x = 0$, $\mathrm{e}^{G(x)} = 1$, so $G(x) = 0$, and we have Taylor series of $G(x)$ at $x = 0$:

$$G(x) = xG_1 + x^2G_2 + \cdots.$$

Now consider the trivial equation

$$\mathrm{e}^{-xB}\mathrm{e}^{-xA}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{xA}\mathrm{e}^{xB}\right) = \mathrm{e}^{-G(x)}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{G(x)}, \tag{13}$$

where LHS is

$$\begin{aligned}\mathrm{e}^{-xB}\mathrm{e}^{-xA}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{xA}\mathrm{e}^{xB}\right) &= \mathrm{e}^{-xB}\mathrm{e}^{-xA}(A\mathrm{e}^{xA}\mathrm{e}^{xB} + \mathrm{e}^{xA}B\mathrm{e}^{xB}) \\ &= \mathrm{e}^{-xB}A\mathrm{e}^{xB} + B.\end{aligned} \tag{14}$$

Now we need an important equation:

$$\mathrm{e}^{G}A\mathrm{e}^{-G} = A + [G,A] + \frac{1}{2!}[G,[G,A]] + \cdots + \frac{1}{n!}\underbrace{[G,[G,[G,\dots[G,A]]]\dots]}_{n\text{ times}} + \cdots. \tag{15}$$

We use the proof given by [2]. By definition we have

$$\begin{aligned}\mathrm{e}^{G}A\mathrm{e}^{-G} &= \sum_{m=0}^{\infty}\frac{1}{m!}G^m A\sum_{n=0}^{\infty}\frac{1}{n!}(-G)^n \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}(-1)^n G^m A G^n \\ &= \sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(s-n)!n!}(-1)^n G^{s-n}AG^n \\ &= \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{n=0}^{\infty}\binom{s}{n}(-1)^n G^{s-n}AG^n \\ &= A + [G,A] + \cdots + \frac{1}{s!}\sum_{n=0}^{s}\binom{s}{n}(-1)^n G^{s-n}AG^n + \cdots,\end{aligned}$$

so the problem is whether

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k G^{n-k}AG^k = \underbrace{[G,[G,[G,\dots[G,A]]]\dots]}_{n\text{ times}}. \tag{16}$$

(16) can be shown recursively. The cases of $n = 1, 2$ are trivial. Suppose (16) for an arbitrary n, then

$$\begin{aligned}&\underbrace{[G,[G,[G,\dots[G,A]]]\dots]}_{n+1\text{ times}} \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k G^{n-k}[G,A]\,G^k \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k G^{n-k+1}AG^k - G^{n-k}AG^{k+1}) \\ &= \sum_{k=1}^{n}\binom{n}{k}(-1)^k (G^{n-k+1}AG^k + G^{n+1}A - \sum_{k=1}^{n}\binom{n}{k-1}(-1)^{k-1}G^{n-k+1}AG^k - (-1)^n AG^{n+1} \\ &= G^{n+1}A - (-1)^n AG^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)(-1)^k G^{n-k+1}AG^k \\ &= G^{n+1}A - (-1)^n AG^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}(-1)^k G^{n-k+1}AG^k \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k G^{n-k+1}AG^k,\end{aligned}$$

which is just (16) with n replaced by $n+1$. So (16) has been proven.

Now due to (15), the RHS of (14) is

$$B + \mathrm{e}^{-xB} A \mathrm{e}^{xB} = B + A + [-xB, A] .$$

All higher order terms contain $[B, [B, A]\,]$ and therefore vanish. The RHS of (13) can be evaluated as

$$\mathrm{e}^{-G(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{G(x)} = \mathrm{e}^{-G(x)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(m+n+1)!} G^m G' G^n ,$$

and by the integral formula

$$\int_0^1 \mathrm{d}y (1-y)^n y^m = \frac{n!m!}{(n+m+1)!}$$

we have

$$\begin{aligned}
\mathrm{e}^{-G(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{G(x)} &= \mathrm{e}^{-G(x)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \int_0^1 \mathrm{d}y\, (1-y)^n y^m G^m G' G^n \\
&= \mathrm{e}^{-G(x)} \int_0^1 \mathrm{d}y\, \mathrm{e}^{(1-y)G} G' \mathrm{e}^{yG} \\
&= \int_0^1 \mathrm{d}y\, \mathrm{e}^{-yG} G' \mathrm{e}^{yG} ,
\end{aligned}$$

and now it can be evaluated using (15) as

$$\begin{aligned}
\mathrm{e}^{-G(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{G(x)} &= \int_0^1 \mathrm{d}y \left(G' + y[G', G] + \frac{y^2}{2} [[G', G]\,, G] + \mathcal{O}(y^3) \right) \\
&= G' + \frac{1}{2}[G', G] + \frac{1}{3!} [[G', G]\,, G] + \cdots .
\end{aligned}$$

As $G(x)$ can be expanded into

$$G'(x) = G_1 + 2xG_2 + \cdots ,$$

We have

$$\mathrm{e}^{-G(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{G(x)} = G_1 + 2xG_2 + \frac{1}{2} x^2 [G_2, G_1] + \cdots .$$

So finally, by rewriting its LHS and RHS, from (13) we find that

$$B + A + x[A, B] = G_1 + 2xG_2 + \mathcal{O}(x^2),$$

and thus

$$G_1 = A + B, \quad G_2 = \frac{1}{2}[A, B], \quad G_n = 0 \text{ for } n > 2.$$

So

$$G(x) = A + B + \frac{1}{2}[A, B] ,$$

and by taking $x = 1$ in (12) we complete the proof of (11), therefore have proved (10).

Scully 1.4 If $f\left(a, a^\dagger\right)$ is a function which can be expanded in a power series of a and $a^\dagger$, then show that (a) $\left[a, f\left(a, a^\dagger\right)\right] = \frac{\partial f}{\partial a^\dagger}$, (b) $\left[a^\dagger, f\left(a, a^\dagger\right)\right] = -\frac{\partial f}{\partial a}$, (c) $\mathrm{e}^{-\alpha a^\dagger a} f\left(a, a^\dagger\right) \mathrm{e}^{\alpha a^\dagger a} = f\left(a\mathrm{e}^{\alpha}, a^\dagger \mathrm{e}^{-\alpha}\right)$ where α is a parameter.

Solution Suppose

$$f(a, a^\dagger) = \sum_{m,n \geq 0} f_{mn} a^m (a^\dagger)^n . \tag{17}$$

We do not include terms that start with $a^\dagger$, because by $[a, a^\dagger] = 1$ we can move all a operators to the left. We can also move all a operators to the right, and write down a similar expansion

$$f(a, a^\dagger) = \sum_{m,n \geq 0} g_{mn} (a^\dagger)^m a^n . \tag{18}$$

(a) We have

$$
\begin{aligned}
[a, f(a, a^\dagger)] &= \sum_{m,n\geq 0} f_{mn} a^m [a, (a^\dagger)^m] \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left([a, a^\dagger](a^\dagger)^{m-1} + a^\dagger [a, (a^\dagger)^{m-1}]\right) \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left((a^\dagger)^{m-1} + a^\dagger [a, (a^\dagger)^{m-1}]\right) \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left((a^\dagger)^{m-1} + a^\dagger \left(a^\dagger [a, (a^\dagger)^{m-2}] + [a, a^\dagger]\,(a^\dagger)^{m-2}\right)\right) \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left((a^\dagger)^{m-1} + a^\dagger \left(a^\dagger [a, (a^\dagger)^{m-2}] + (a^\dagger)^{m-2}\right)\right) \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left(2(a^\dagger)^{m-1} + (a^\dagger)^2 [a, (a^\dagger)^{m-2}]\,\right) \\
&= \cdots .
\end{aligned}
$$

As the expansion of $[a, (a^\dagger)^m]$ goes, we get a sequence of equations in the form of

$$
\begin{aligned}
[a, f(a, a^\dagger)] &= \sum_{m,n\geq 0} f_{mn} a^m \left((a^\dagger)^{m-1} + a^\dagger [a, (a^\dagger)^{m-1}]\right) \\
&= \sum_{m,n\geq 0} f_{mn} a^m \left(2(a^\dagger)^{m-1} + a^\dagger [a, (a^\dagger)^{m-2}]\right) \\
&= \cdots ,
\end{aligned}
$$

and the sequence stops at

$$
\sum_{m,n\geq 0} f_{mn} a^m \left(m(a^\dagger)^{m-1} + a^\dagger [a, (a^\dagger)^{m-m}]\,\right),
$$

so we have

$$
[a, f(a, a^\dagger)] = \sum_{m,n\geq 0} f_{mn} a^m m (a^\dagger)^{m-1},
$$

and since

$$
\frac{\partial f}{\partial a^\dagger} = \sum_{m,n\geq 0} f_{mn} a^m \frac{\partial (a^\dagger)^m}{\partial a^\dagger},
$$

we arrive at the conclusion that

$$
[a, f(a, a^\dagger)] = \frac{\partial f}{\partial a^\dagger}. \tag{19}
$$

(b) The logic is similar to (a) but this time the expansion is

$$
\begin{aligned}
[a^\dagger, f(a, a^\dagger)] &= \sum_{m,n\geq 0} [a^\dagger, a^m]\,(a^\dagger)^n \\
&= \sum_{m,n\geq 0} \left([a^\dagger, a]\,a^{m-1} + a[a^\dagger, a^{m-1}]\,\right) \\
&= \sum_{m,n\geq 0} \left(-a^{m-1} + a[a^\dagger, a^{m-1}]\,\right) \\
&= \sum_{m,n\geq 0} \left(-a^{m-1} + a\left(a[a^\dagger, a^{m-2}]\,\right) + [a^\dagger, a]\,a^{m-2}\right) \\
&= \sum_{m,n\geq 0} \left(-a^{m-1} + a\left(\left(a[a^\dagger, a^{m-2}]\,\right) - a^{m-2}\right)\right) \\
&= \sum_{m,n\geq 0} \left(-2a^{m-1} + a^2[a^\dagger, a^{m-2}]\,\right) \\
&= \cdots ,
\end{aligned}
$$

or to be concise,

$$
\begin{aligned}
[a^\dagger, f(a,a^\dagger)] &= \sum_{m,n\geq 0} \left(-a^{m-1} + a[a^\dagger, a^{m-1}]\right) \\
&= \sum_{m,n\geq 0} \left(-2a^{m-1} + a^2[a^\dagger, a^{m-2}]\right) \\
&= \cdots ,
\end{aligned}
$$

and this time the sequence stops at

$$
[a^\dagger, f(a,a^\dagger)] = \sum_{m,n\geq 0} \left(-ma^{m-1} + a^2[a^\dagger, a^{m-m}]\right),
$$

and since

$$
-ma^{m-1} = -\frac{\partial a^m}{\partial a}
$$

we have

$$
[a^\dagger, f(a,a^\dagger)] = -\frac{\partial f}{\partial a}. \tag{20}
$$

(c) We need to use results proved in Problem 1.5. By (23), we have

$$
a\mathrm{e}^{-\alpha a^\dagger a} = \mathrm{e}^{-\alpha}\mathrm{e}^{-\alpha a^\dagger a}a = (\mathrm{e}^{-\alpha}a)\mathrm{e}^{-\alpha a^\dagger a}, \tag{21}
$$

and by taking its conjugate transpose we have

$$
a^\dagger \mathrm{e}^{-\alpha^* a^\dagger a}\mathrm{e}^{-\alpha^*} = \mathrm{e}^{-\alpha^* a^\dagger a}a^\dagger,
$$

and since α is an arbitrary parameter we can redefine it and get

$$
a^\dagger \mathrm{e}^{-\alpha a^\dagger a}\mathrm{e}^{-\alpha} = \mathrm{e}^{-\alpha a^\dagger a}a^\dagger = (a^\dagger \mathrm{e}^{-\alpha})\mathrm{e}^{-\alpha a^\dagger a}. \tag{22}
$$

Substituting (18) into $\mathrm{e}^{-\alpha a^\dagger a} f(a,a^\dagger)\mathrm{e}^{\alpha a^\dagger a}$, and applying (21) and (22) iteratively, we have

$$
\begin{aligned}
\mathrm{e}^{-\alpha a^\dagger a} f(a,a^\dagger)\mathrm{e}^{\alpha a^\dagger a} &= \sum_{n,m\geq 0} g_{mn}\mathrm{e}^{-\alpha a^\dagger a}(a^\dagger)^m a^n \mathrm{e}^{\alpha a^\dagger a} \\
&= \sum_{n,m\geq 0} g_{mn}(\mathrm{e}^{-\alpha}a^\dagger)\mathrm{e}^{-\alpha a^\dagger a}(a^\dagger)^{m-1}a^{n-1}\mathrm{e}^{\alpha a^\dagger a}(a\mathrm{e}^{\alpha}) \\
&= \sum_{n,m\geq 0} g_{mn}(\mathrm{e}^{-\alpha}a^\dagger)(\mathrm{e}^{-\alpha}a^\dagger)\mathrm{e}^{-\alpha a^\dagger a}(a^\dagger)^{m-2}a^{n-2}\mathrm{e}^{\alpha a^\dagger a}(a\mathrm{e}^{\alpha})(a\mathrm{e}^{\alpha}) \\
&= \cdots \\
&= \sum_{m,n\geq 0} g_{mn}(\mathrm{e}^{-\alpha}a^\dagger)^m \mathrm{e}^{-\alpha a^\dagger a}\mathrm{e}^{\alpha a^\dagger a}(\mathrm{e}^{\alpha}a)^n \\
&= \sum_{m,n\geq 0} g_{mn}(\mathrm{e}^{-\alpha}a^\dagger)^m(\mathrm{e}^{\alpha}a)^n \\
&= f(a\mathrm{e}^{\alpha}, a^\dagger \mathrm{e}^{-\alpha}),
\end{aligned}
$$

which finishes the proof.

Scully 1.5 Show that

$$
\begin{aligned}
\left[a, \mathrm{e}^{-\alpha a^\dagger a}\right] &= \left(\mathrm{e}^{-\alpha} - 1\right)\mathrm{e}^{-\alpha a^\dagger a}a, \\
\left[a^\dagger, \mathrm{e}^{-\alpha a^\dagger a}\right] &= \left(\mathrm{e}^{\alpha} - 1\right)\mathrm{e}^{-\alpha a^\dagger a}a^\dagger,
\end{aligned} \tag{23}
$$

where α is a parameter.

Solution The most convenient way is not to invoke (19) and (20), but to check the matrix elements under the $\{|n\rangle\}$ basis. We have

$$\begin{aligned}
\left[a, \mathrm{e}^{-\alpha a^\dagger a}\right]|n\rangle &= a\mathrm{e}^{-\alpha a^\dagger a}|n\rangle - \mathrm{e}^{-\alpha a^\dagger a}a|n\rangle \\
&= a\mathrm{e}^{-\alpha n}|n\rangle - \mathrm{e}^{-\alpha a^\dagger a}\sqrt{n}|n-1\rangle \\
&= \mathrm{e}^{-\alpha n}\sqrt{n}|n-1\rangle - \mathrm{e}^{-\alpha(n-1)}\sqrt{n}|n-1\rangle \\
&= \left(\mathrm{e}^{-\alpha}-1\right)\mathrm{e}^{-\alpha(n-1)}\sqrt{n}|n-1\rangle \\
&= \left(\mathrm{e}^{-\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}\sqrt{n}|n-1\rangle \\
&= \left(\mathrm{e}^{-\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}a|n\rangle,
\end{aligned}$$

for all $n = 0, 1, \ldots$, and thus

$$\left[a, \mathrm{e}^{-\alpha a^\dagger a}\right] = \left(\mathrm{e}^{-\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}a. \tag{24}$$

The same procedure applies for the second equation. We have

$$\begin{aligned}
\left[a^\dagger, \mathrm{e}^{-\alpha a^\dagger a}\right]|n\rangle &= a^\dagger\mathrm{e}^{-\alpha a^\dagger a}|n\rangle - \mathrm{e}^{-\alpha a^\dagger a}a^\dagger|n\rangle \\
&= a^\dagger\mathrm{e}^{-\alpha n}|n\rangle - \mathrm{e}^{-\alpha a^\dagger a}\sqrt{n+1}|n+1\rangle \\
&= \mathrm{e}^{-\alpha n}\sqrt{n+1}|n+1\rangle - \mathrm{e}^{-\alpha(n+1)}\sqrt{n+1}|n+1\rangle \\
&= \left(\mathrm{e}^{\alpha}-1\right)\mathrm{e}^{-\alpha(n+1)}\sqrt{n+1}|n+1\rangle \\
&= \left(\mathrm{e}^{\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}\sqrt{n+1}|n+1\rangle \\
&= \left(\mathrm{e}^{\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}a^\dagger|n\rangle,
\end{aligned}$$

for all $n = 0, 1, \ldots$, and thus

$$\left[a^\dagger, \mathrm{e}^{-\alpha a^\dagger a}\right] = \left(\mathrm{e}^{\alpha}-1\right)\mathrm{e}^{-\alpha a^\dagger a}a^\dagger. \tag{25}$$

Scully 1.6 Show that the free-field Hamiltonian

$$\mathcal{H} = \hbar v\left(a^\dagger a + \frac{1}{2}\right) \tag{26}$$

can be written in terms of the number states as

$$\mathcal{H} = \sum_n E_n|n\rangle\langle n|,$$

and hence

$$\mathrm{e}^{\mathrm{i}\mathcal{H}t/\hbar} = \sum_n \mathrm{e}^{\mathrm{i}E_n t/\hbar}|n\rangle\langle n|. \tag{27}$$

Solution An n-photon state

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{28}$$

is an eigenstate of (26), since by applying (26) on (28) we have

$$\begin{aligned}
\mathcal{H}|n\rangle &= \hbar\nu a^\dagger a|n\rangle + \frac{1}{2}\hbar\nu|n\rangle \\
&= \hbar\nu a^\dagger\sqrt{n}|n-1\rangle + \frac{1}{2}\hbar\nu|n\rangle \\
&= \hbar\nu\sqrt{n}\sqrt{n-1+1}|n-1+1\rangle + \frac{1}{2}\hbar\nu|n\rangle \\
&= \hbar\nu\left(n+\frac{1}{2}\right)|n\rangle \\
&= E_n|n\rangle,
\end{aligned}$$

so $|n\rangle$ is an eigenstate of $\mathcal{H}$ with energy E_n. Since $|n\rangle$s are orthogonal to each other and are uniform, we have

$$\mathcal{H} = \sum_n E_n|n\rangle\langle n|. \tag{29}$$

Since $\mathcal{H}$ is diagonal under the $\{|n\rangle\}$ basis, so is $\mathrm{e}^{\mathrm{i}\mathcal{H}t/\hbar}$ and hence we have (27).

Scully 2.1 Show that

$$a^\dagger |\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha| , \tag{30}$$

and

$$|\alpha\rangle\langle\alpha| \, a = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) |\alpha\rangle\langle\alpha| . \tag{31}$$

Solution By definition

$$|\alpha\rangle\langle\alpha| = \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n(\alpha^*)^m}{\sqrt{m!n!}} |n\rangle\langle m| ,$$

so

$$\begin{aligned}
a^\dagger |\alpha\rangle\langle\alpha| &= \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n(\alpha^*)^m}{\sqrt{m!n!}} a^\dagger |n\rangle\langle m| \\
&= \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n(\alpha^*)^m}{\sqrt{m!n!}} \sqrt{n+1}\, |n+1\rangle\langle m| \\
&= \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n(\alpha^*)^m}{\sqrt{m!(n+1)!}} (n+1)\, |n+1\rangle\langle m| \\
&= \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+1)!}} \frac{\partial}{\partial\alpha}\alpha^{n+1} |n+1\rangle \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| \\
&= \mathrm{e}^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\partial}{\partial\alpha}\alpha^n\right) |n\rangle \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| .
\end{aligned}$$

Note that the factors after $\sum_m$ do not include α^*, so they can be regarded as constants. We have

$$\begin{aligned}
\mathrm{e}^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \frac{\partial}{\partial\alpha}\alpha^n |n\rangle &= \mathrm{e}^{-|\alpha|^2} \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n |n\rangle \\
&= \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\right) - \left(\frac{\partial \mathrm{e}^{-|\alpha|^2}}{\partial\alpha}\right) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle \\
&= \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\right) - (-\alpha^* \mathrm{e}^{-|\alpha|^2}) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle .
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a^\dagger |\alpha\rangle\langle\alpha| &= \mathrm{e}^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\partial}{\partial\alpha}\alpha^n\right) |n\rangle \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| \\
&= \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\right) \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| - (-\alpha^* \mathrm{e}^{-|\alpha|^2}) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| \\
&= \frac{\partial}{\partial\alpha} |\alpha\rangle\langle\alpha| + \alpha^* |\alpha\rangle\langle\alpha| ,
\end{aligned}$$

and thus we have proved that

$$a^\dagger |\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha| .$$

The scope of $\frac{\partial}{\partial\alpha}$ covers $|\alpha\rangle\langle\alpha|$. By taking the conjugate transpose we can immediately show (31), since α and α^* have exactly the same status in $|\alpha\rangle\langle\alpha|$, and taking the conjugate transpose of $f(\alpha, \alpha^*)$ is just to exchange α and α^*, $|m\rangle$ and $|n\rangle$, so after the conjugate transpose $|\alpha\rangle\langle\alpha|$ is still $|\alpha\rangle\langle\alpha|$ while $\frac{\partial}{\partial\alpha}$ turns into $\frac{\partial}{\partial\alpha^*}$, i.e.

$$\left(\frac{\partial}{\partial\alpha} |\alpha\rangle\langle\alpha|\right)^\dagger = \left(\frac{\partial}{\partial\alpha} |\alpha\rangle\langle\alpha|\right)\Bigg|_{\alpha\leftrightarrow\alpha^*, \mathrm{bra}\leftrightarrow\mathrm{ket}} = \frac{\partial}{\partial\alpha^*} |\alpha\rangle\langle\alpha| .$$

Scully 3.3 Show that the Wigner-Weyl distribution $W(\alpha,\alpha^*)$ can be expressed in terms of the P-representation $P(\alpha,\alpha^*)$ via the relation

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int \mathrm{d}^2\beta P(\beta,\beta^*)\exp\left(-2|\alpha-\beta|^2\right). \tag{32}$$

Solution Both $W(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$ can be defined in terms of integral transforms. We have

$$P(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \mathrm{e}^{-\mathrm{i}\beta\alpha^*-\mathrm{i}\beta^*\alpha}\,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\beta^* a}\mathrm{e}^{\mathrm{i}\beta a^\dagger}\rho\right)$$

and

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \mathrm{e}^{-\mathrm{i}\beta\alpha^*-\mathrm{i}\beta^*\alpha}\,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\beta a^\dagger+\mathrm{i}\beta^* a}\rho\right).$$

By the BCH formula and the completeness of coherent states we have

$$\begin{aligned}
\mathrm{Tr}\left(e^{\mathrm{i}\beta a^\dagger+\mathrm{i}\beta^* a}\rho\right) &= \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\beta^* a}\mathrm{e}^{\mathrm{i}\beta a^\dagger}\mathrm{e}^{|\beta|^2/2}\rho\right)\\
&= \mathrm{e}^{|\beta|^2/2}\,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\beta a^\dagger}\rho\mathrm{e}^{\mathrm{i}\beta^* a}\right)\\
&= \frac{1}{\pi}\mathrm{e}^{|\beta|^2/2}\int \mathrm{d}^2\gamma\ \langle\gamma|\mathrm{e}^{\mathrm{i}\beta a^\dagger}\rho\mathrm{e}^{\mathrm{i}\beta^* a}|\gamma\rangle\\
&= \frac{1}{\pi}\mathrm{e}^{|\beta|^2/2}\int \mathrm{d}^2\gamma\,\mathrm{e}^{\mathrm{i}\beta\gamma^*}\ \langle\gamma|\rho|\gamma\rangle\,\mathrm{e}^{\mathrm{i}\beta^*\gamma}\\
&= \mathrm{e}^{|\beta|^2/2}\int \mathrm{d}^2\gamma\,\mathrm{e}^{\mathrm{i}\beta\gamma^*+\mathrm{i}\beta^*\gamma}Q(\gamma,\gamma^*),
\end{aligned}$$

so

$$\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\int \mathrm{d}^2\beta\,\mathrm{e}^{-\mathrm{i}\beta\alpha^*-\mathrm{i}\beta^*\alpha}\mathrm{e}^{|\beta|^2/2}\mathrm{e}^{\mathrm{i}\beta\gamma^*+\mathrm{i}\beta^*\gamma}Q(\gamma,\gamma^*)\\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\int \mathrm{d}^2\beta\,\mathrm{e}^{|\beta|^2/2}\mathrm{e}^{\mathrm{i}\beta(\gamma^*-\alpha^*)+\mathrm{i}\beta^*(\gamma-\alpha)}Q(\gamma,\gamma^*).
\end{aligned}$$

Now by completing the squares we can integrate out the variable β, and obtain

$$\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\left(\int \mathrm{d}^2\beta\,\mathrm{e}^{|\beta|^2/2+\mathrm{i}\beta(\gamma^*-\alpha^*)+\mathrm{i}\beta^*(\gamma-\alpha)}\right)Q(\gamma,\gamma^*)\\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\,2\pi\mathrm{e}^{2(\gamma^*-\alpha^*)(\gamma-\alpha)}Q(\gamma,\gamma^*),
\end{aligned}$$

so we have proved the relation between $W(\alpha,\alpha^*)$ and $Q(\alpha,\alpha^*)$ that

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int \mathrm{d}^2\gamma\,\mathrm{e}^{2|\gamma-\alpha|^2}Q(\gamma,\gamma^*). \tag{33}$$

We also have a similar relation between $P(\alpha,\alpha^*)$ and $Q(\alpha,\alpha^*)$ that

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\int \mathrm{d}^2\beta\,P(\beta,\beta^*)\,\mathrm{e}^{-|\alpha-\beta|^2}, \tag{34}$$

which is Eq. (3.2.9) in [1]. Now by putting (34) and (33) together we can finally show (32). We have

$$\begin{aligned}
W(\alpha,\alpha^*) &= \frac{2}{\pi}\int \mathrm{d}^2\gamma\,\mathrm{e}^{2|\gamma-\alpha|^2}\frac{1}{\pi}\int \mathrm{d}^2\beta\,P(\beta,\beta^*)\,\mathrm{e}^{-|\gamma-\beta|^2}\\
&= \frac{2}{\pi^2}\int \mathrm{d}^2\beta\,P(\beta,\beta^*)\int \mathrm{d}^2\gamma\,\mathrm{e}^{|\gamma|^2+\gamma(\beta^*-2\alpha^*)+\gamma^*(\beta-2\alpha)}\mathrm{e}^{2|\alpha|^2-|\beta|^2},
\end{aligned}$$

and again, by completing the squares we have

$$\begin{aligned}
W(\alpha,\alpha^*) &= \frac{2}{\pi^2}\int \mathrm{d}^2\beta\,P(\beta,\beta^*)\times\pi\mathrm{e}^{-(\beta^*-2\alpha^*)(\beta-2\alpha)}\times\mathrm{e}^{2|\alpha|^2-|\beta|^2}\\
&= \frac{2}{\pi}\int \mathrm{d}^2\beta\,P(\beta,\beta^*)\mathrm{e}^{-2|\alpha|^2-2|\beta|^2+2\alpha^*\beta+2\alpha\beta^*},
\end{aligned}$$

so we get (32).

Q-functions for several states Derive the Q-function for Fock state $|n\rangle$, coherent state $|\alpha\rangle$ and the famous "cat" state $C(|\alpha\rangle + |-\alpha\rangle)$. Discuss the corresponding P-functions.

Solution

(a) The Q-function of $|n\rangle$ is

$$\begin{aligned}Q(\alpha,\alpha^*) &= \frac{1}{\pi}\langle\alpha|n\rangle\,\langle n|\alpha\rangle\\ &= \frac{1}{\pi}\left|\mathrm{e}^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\langle n|m\rangle\right|^2\\ &= \frac{1}{\pi}\left|\mathrm{e}^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right|^2\\ &= \frac{\mathrm{e}^{-|\alpha|^2}|\alpha|^{2n}}{\pi n!}.\end{aligned}$$

The corresponding P-function is

$$\begin{aligned}P(\alpha,\alpha^*) &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta\,\langle-\beta|n\rangle\,\langle n|\beta\rangle\,\mathrm{e}^{|\beta|^2}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta\left(\sum_{m=0}^{\infty}\mathrm{e}^{-|\beta|^2/2}\frac{(-\beta^*)^m}{\sqrt{m!}}\langle m|n\rangle\right)\left(\sum_{m=0}^{\infty}\mathrm{e}^{-|\beta|^2/2}\frac{\beta^m}{\sqrt{m!}}\langle n|m\rangle\right)\mathrm{e}^{|\beta|^2}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta\left(\mathrm{e}^{-|\beta|^2/2}\frac{(-\beta^*)^n}{\sqrt{n!}}\right)\left(\mathrm{e}^{-|\beta|^2/2}\frac{\beta^n}{\sqrt{n!}}\right)\mathrm{e}^{|\beta|^2}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta\,\mathrm{e}^{-|\beta|^2}\frac{(-1)^n|\beta|^{2n}}{n!}\mathrm{e}^{|\beta|^2}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2 n!}\int \mathrm{d}^2\beta\,(-1)^n|\beta|^{2n}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2 n!}\int \mathrm{d}^2\beta\,\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{\pi^2 n!}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\int \mathrm{d}^2\beta\,\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\\ &= \frac{\mathrm{e}^{|\alpha|^2}}{n!}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\delta^{(2)}(\alpha),\end{aligned}$$

where we have used the formula

$$\int \mathrm{d}^2\beta\,\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha} = \pi^2\delta^{(2)}(\alpha), \tag{35}$$

which can be proved by substitution of variable $\beta = \mathrm{i}\gamma$, and integrate $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ separately. It can be seen that the P-function behaves badly, proportion to the second derivative of δ-function and can be negative near $\alpha = 0$ in certain directions, which is expected since a Fock state $|n\rangle$ with completely determined photon numbers is far from a classical optical field.

(b) The Q-function of $|\alpha\rangle$ is

$$\begin{aligned}Q(\beta,\beta^*) &= \frac{1}{\pi}\langle\beta|\alpha\rangle\,\langle\alpha|\beta\rangle\\ &= \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2\\ &= \frac{1}{\pi}\mathrm{e}^{-|\alpha-\beta|^2}.\end{aligned}$$

The corresponding P-function is

$$P(\beta,\beta^*) = \delta^{(2)}(\beta-\alpha),$$

as can be directly verified with (34). This is a δ function, and again is expected because a coherent state is a quite "classical" thing, and P-function works best as a probabilistic distribution function for states resembling classical optical fields, so a coherent state must be very "clear" or "sharp" in the P-function representation.

(c) First we normalize the cat state. We have

$$
\begin{aligned}
(\langle\alpha| + \langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle) &= 1 + 1 + \langle\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle \\
&= 2 + \mathrm{e}^{-|\alpha|^2/2-|-\alpha|^2/2+\alpha^*(-\alpha)} + \mathrm{e}^{-|-\alpha|^2/2-|\alpha|^2/2+(-\alpha^*)\alpha} \\
&= 2 + 2\mathrm{e}^{-2|\alpha|^2},
\end{aligned}
$$

and hence

$$
C = \frac{1}{\sqrt{2+2\mathrm{e}^{-2|\alpha|^2}}}. \tag{36}
$$

Now we calculate the Q-function and P-function of the cat state. The Q-function is

$$
\begin{aligned}
Q(\beta,\beta^*) &= \frac{C^2}{\pi}(\langle\beta|\alpha\rangle + \langle\beta|-\alpha\rangle)(\langle\alpha|\beta\rangle + \langle-\alpha|\beta\rangle) \\
&= \frac{C^2}{\pi}(\mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2+\beta^*\alpha} + \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2-\beta^*\alpha})(\mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2+\beta\alpha^*} + \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2-\beta\alpha^*}) \\
&= \frac{C^2}{\pi}\mathrm{e}^{-|\alpha|^2-|\beta|^2}(\mathrm{e}^{\beta^*\alpha} + \mathrm{e}^{-\beta^*\alpha})(\mathrm{e}^{\beta\alpha^*} + \mathrm{e}^{-\beta\alpha^*}).
\end{aligned}
$$

The P-function is

$$
\begin{aligned}
P(\beta,\beta^*) &= C^2\frac{\mathrm{e}^{|\beta|^2}}{\pi^2}\int \mathrm{d}^2\gamma\,(\langle-\gamma|\alpha\rangle + \langle-\gamma|-\alpha\rangle)(\langle\alpha|\gamma\rangle + \langle-\alpha|\gamma\rangle)\mathrm{e}^{|\gamma|^2}\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta} \\
&= C^2\frac{\mathrm{e}^{|\beta|^2}}{\pi^2}\int \mathrm{d}^2\gamma\,\mathrm{e}^{|\gamma|^2}\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta}(\mathrm{e}^{-(|\gamma|^2+|\alpha|^2)/2-\gamma^*\alpha} + \mathrm{e}^{-(|\gamma|^2+|\alpha|^2)/2+\gamma^*\alpha}) \\
&\qquad \times (\mathrm{e}^{-(|\alpha|^2+|\gamma|^2)/2+\alpha^*\gamma} + \mathrm{e}^{-(|\alpha|^2+|\gamma|^2)/2-\alpha^*\gamma}) \\
&= \frac{C^2}{\pi^2}\mathrm{e}^{|\beta|^2-|\alpha|^2}\int \mathrm{d}^2\gamma\,\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta}(\mathrm{e}^{-\gamma^*\alpha} + \mathrm{e}^{\gamma^*\alpha})(\mathrm{e}^{\alpha^*\gamma} + \mathrm{e}^{-\alpha^*\gamma}) \\
&= \frac{C^2}{\pi^2}\mathrm{e}^{|\beta|^2-|\alpha|^2}\int \mathrm{d}^2\gamma\,(\mathrm{e}^{-\gamma^*(\alpha-\beta)+\gamma(\alpha^*-\beta^*)} + \mathrm{e}^{\gamma^*(\alpha+\beta)-\gamma(\alpha^*+\beta^*)} \\
&\qquad + (\mathrm{e}^{\gamma^*\alpha+\alpha^*\gamma} + \mathrm{e}^{-\gamma^*\alpha-\alpha^*\gamma})\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta}).
\end{aligned}
$$

By (35) we can find the first two integrals immediately:

$$
\int \mathrm{d}^2\gamma\,(\mathrm{e}^{-\gamma^*(\alpha-\beta)+\gamma(\alpha^*-\beta^*)} + \mathrm{e}^{\gamma^*(\alpha+\beta)-\gamma(\alpha^*+\beta^*)}) = \pi^2\delta^{(2)}(\alpha-\beta) + \pi^2\delta^{(2)}(\alpha+\beta).
$$

For the third and the fourth term, note that

$$
\gamma^*\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta} = \frac{\mathrm{d}}{\mathrm{d}\beta}\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta}, \quad \gamma\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta} = -\frac{\mathrm{d}}{\mathrm{d}\beta^*}\mathrm{e}^{-\gamma\beta^*+\gamma^*,\beta}
$$

and therefore

$$
\begin{aligned}
\int \mathrm{d}^2\gamma\,(\mathrm{e}^{\gamma^*\alpha+\alpha^*\gamma} + \mathrm{e}^{-\gamma^*\alpha-\alpha^*\gamma})\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta} &= \int \mathrm{d}^2\gamma\,(\mathrm{e}^{\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}-\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}} + \mathrm{e}^{-\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}+\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}})\mathrm{e}^{-\gamma\beta^*+\gamma^*\beta} \\
&= \pi^2(\mathrm{e}^{\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}-\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}} + \mathrm{e}^{-\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}+\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}})\delta^{(2)}(\beta),
\end{aligned}
$$

so we reach the final result that for a cat state

$$
P(\beta,\beta^*) = C^2\mathrm{e}^{|\beta|^2-|\alpha|^2}(\delta^{(2)}(\alpha+\beta) + \delta^{(2)}(\alpha-\beta) + (\mathrm{e}^{\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}-\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}} + \mathrm{e}^{-\alpha\frac{\mathrm{d}}{\mathrm{d}\beta}+\alpha^*\frac{\mathrm{d}}{\mathrm{d}\beta^*}})\delta^{(2)}(\beta)). \tag{37}
$$

Coherent states and dipole radiation Derive in detail the expectation of $\boldsymbol{E}$ in a system with a electric dipole placed at $\boldsymbol{r}=0$ and starting to oscillate at $t=0$, where the dipole radiation approximation works, i.e. $H_{\text{int}} = -\boldsymbol{d}\cdot\boldsymbol{E}$. Consider the $t\to\infty$ limit.

Solution We already know that the wave function of the optical field, under the interaction picture, is

$$|\psi(t)\rangle = \exp\left(\sum_n (\alpha_n a_n^\dagger - \alpha_n^* a_n)\right)|0\rangle\,, \tag{38}$$

where

$$\alpha_n = \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\,\mathcal{E}_n \boldsymbol{f}_n^* \mathrm{e}^{\mathrm{i}\omega_n t'} \cdot (\boldsymbol{d}\mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.}). \tag{39}$$

We are going to evaluate $\langle \boldsymbol{E}\rangle$ under $|\psi\rangle$.

For a free space, n is just the combination of the wave vector $\boldsymbol{k}$ and the polarization, and the energy spectrum is

$$\omega_{\boldsymbol{k}\sigma} = c|\boldsymbol{k}|, \tag{40}$$

and

$$\boldsymbol{f}_{\boldsymbol{k}\sigma}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{e}_\sigma, \tag{41}$$

where for a given $\boldsymbol{k}$, we have

$$\boldsymbol{k}\cdot\boldsymbol{e}_\sigma = 0 \tag{42}$$

to impose the transverse wave condition. Therefore we have

$$\begin{aligned}\alpha_{\boldsymbol{k}\sigma} &= \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\,\mathcal{E}_{\boldsymbol{k}\sigma}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}+\mathrm{i}\omega_{\boldsymbol{k}}t'}\boldsymbol{e}_\sigma\cdot(\boldsymbol{d}\mathrm{e}^{-\mathrm{i}\omega t'} + \text{c.c.})|_{\boldsymbol{r}=0} \\ &= \frac{1}{\hbar}\mathcal{E}_{\boldsymbol{k}\sigma}\boldsymbol{e}_\sigma\cdot\left(\boldsymbol{d}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega)t}-1}{\omega_{\boldsymbol{k}}-\omega} + \boldsymbol{d}^*\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}+\omega)t}-1}{\omega_{\boldsymbol{k}}+\omega}\right).\end{aligned} \tag{43}$$

The expectation of $\boldsymbol{E}$ under a coherent state can be written in terms of a sum of different modes in a standard way:

$$\begin{aligned}\langle\boldsymbol{E}\rangle &= \sum_{\boldsymbol{k},\sigma}\mathcal{E}_{\boldsymbol{k}\sigma}(\boldsymbol{f}_{\boldsymbol{k}\sigma}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\langle a_{\boldsymbol{k}\sigma}\rangle + \boldsymbol{f}^*_{\boldsymbol{k}\sigma}(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}\langle a^\dagger_{\boldsymbol{k}\sigma}\rangle) \\ &= \sum_{\boldsymbol{k},\sigma}\mathcal{E}_{\boldsymbol{k}\sigma}\boldsymbol{e}_\sigma(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t}\langle a_{\boldsymbol{k}\sigma}\rangle + \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}+\mathrm{i}\omega_{\boldsymbol{k}}t}\langle a^\dagger_{\boldsymbol{k}\sigma}\rangle) \\ &= \sum_{\boldsymbol{k},\sigma}\mathcal{E}_{\boldsymbol{k}\sigma}\boldsymbol{e}_\sigma(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t}\alpha_{\boldsymbol{k}\sigma} + \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}+\mathrm{i}\omega_{\boldsymbol{k}}t}\alpha^*_{\boldsymbol{k}\sigma}).\end{aligned}$$

Inserting (43) into this equation we have

$$\langle\boldsymbol{E}\rangle = \sum_{\boldsymbol{k},\sigma}\frac{c|\boldsymbol{k}|}{2\epsilon_0 V}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t}\boldsymbol{e}_\sigma\boldsymbol{e}_\sigma\cdot\left(\boldsymbol{d}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega)t}-1}{\omega_{\boldsymbol{k}}-\omega} + \boldsymbol{d}^*\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}+\omega)t}-1}{\omega_{\boldsymbol{k}}+\omega}\right) + \text{c.c.}.$$

Let φ be the phase of the dipole, so

$$\boldsymbol{d} = d\mathrm{e}^{\mathrm{i}\varphi}\hat{\boldsymbol{d}},$$

and since $\boldsymbol{e}_\sigma$ is always orthogonal to $\hat{\boldsymbol{k}}$, $\sum_\sigma \boldsymbol{e}_\sigma\boldsymbol{e}_\sigma\cdot\boldsymbol{d}$ is just $\boldsymbol{d}$ projected to the plane orthogonal to $\boldsymbol{k}$, hence we have

$$\begin{aligned}\langle\boldsymbol{E}\rangle(\boldsymbol{r},t) &= \frac{dc}{2\epsilon_0 V}\sum_{\boldsymbol{k}}|\boldsymbol{k}|\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t}(\hat{\boldsymbol{d}}-(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{d}})\hat{\boldsymbol{k}})\left(\mathrm{e}^{\mathrm{i}\varphi}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega)t}-1}{\omega_{\boldsymbol{k}}-\omega} + \mathrm{e}^{-\mathrm{i}\varphi}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}+\omega)t}-1}{\omega_{\boldsymbol{k}}+\omega}\right) + \text{c.c.} \\ &= \frac{dc}{2\epsilon_0}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}|\boldsymbol{k}|\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t}(\hat{\boldsymbol{d}}-(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{d}})\hat{\boldsymbol{k}})\left(\mathrm{e}^{\mathrm{i}\varphi}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega)t}-1}{\omega_{\boldsymbol{k}}-\omega} + \mathrm{e}^{-\mathrm{i}\varphi}\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}+\omega)t}-1}{\omega_{\boldsymbol{k}}+\omega}\right) + \text{c.c.}.\end{aligned} \tag{44}$$

It is hard to calculate the integral explicitly, but it can be observed that the non-zero frequency components are not restricted to ω, which is expected since the dipole starts to oscillate at $t = 0$ so the time translation symmetry is broken, so we do not have the strict $\omega_{\boldsymbol{k}} = \pm\omega$ relation, but as $\omega_{\boldsymbol{k}}$ approaches $\pm\omega$ the amplitude increases. In the $t\to\infty$ limit according to the standard procedure in QFT, we have

$$\frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega)t}-1}{\omega_{\boldsymbol{k}}-\omega}\to\frac{1}{\omega-\omega_{\boldsymbol{k}}+\mathrm{i}0^+},\quad \frac{\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}+\omega)t}-1}{\omega_{\boldsymbol{k}}+\omega}\to-\frac{1}{\omega+\omega_{\boldsymbol{k}}+\mathrm{i}0^+},$$

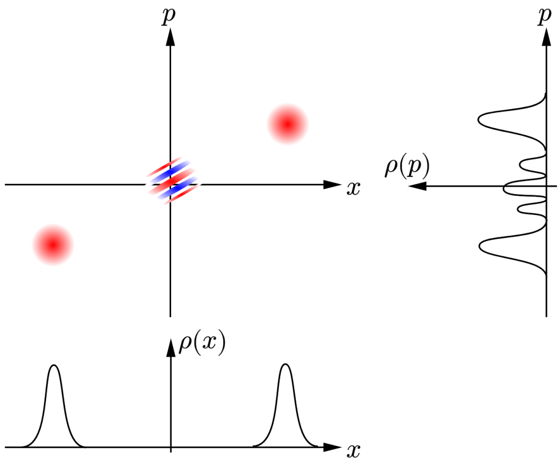

Figure 1: The Wigner function of a cat state, and the corresponding $\rho(x)$ and $\rho(p)$.

so

$$\langle \boldsymbol{E} \rangle (\boldsymbol{r}, t) = \frac{dc}{2\epsilon_0} \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3} |\boldsymbol{k}| \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega_{\boldsymbol{k}} t} (\hat{\boldsymbol{d}} - (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{d}})\hat{\boldsymbol{k}}) \left(\mathrm{e}^{\mathrm{i}\varphi} \frac{1}{\omega - \omega_{\boldsymbol{k}} + \mathrm{i}0^+} - \mathrm{e}^{-\mathrm{i}\varphi} \frac{1}{\omega + \omega_{\boldsymbol{k}} + \mathrm{i}0^+} \right) + \text{c.c.}. \tag{45}$$

The time evolution of a cat state In the Schrödinger picture, describe the time evolution of a cat state's Wigner distribution, $\rho(x)$ and $\rho(p)$ in a single mode optical field.

Solution The Wigner function of a cat state have two peaks, corresponding to α and $-\alpha$, and around $\alpha = 0$ there are a series of "squeezed" peaks and valleys, corresponding to the interference between the two components. The scheme is roughly shown in Figure 1 on page 13. $\rho(x)$ can be obtained by integrating out the p variable, so when the two peaks are close in the x coordinate, $\rho(x)$ has strong interference effect, where the two peaks cannot be distinguished clearly due to the peaks created by interference between them; on the other hand, when the two peaks are far from each other in the x coordinate, $\rho(x)$ is simply a two-peak function. Replacing x with p in the last sentence and everything holds.

In a single mode optical field the time evolution is simply to rotate (α, α^*) with angular speed ω. Therefore, the time evolution of the Wigner function of a cat state is just to rotate Figure 1 on page 13 with angular speed ω. In this process, the distance between the two peaks on the x axis goes up and down periodically, so the $\rho(x)$ function transforms between a two-peak function and a multiple-peak function where the main peak occurs at $x = 0$, the peak value decreasing as $|x|$ increases, and so does $\rho(p)$. When $\rho(x)$ is a two-peak function $\rho(p)$ is a multiple-peak function, and vice versa. This is visualized in [3].

Three-photon state Consider a three-photon state

$$|\psi\rangle = b_1^\dagger b_2^\dagger b_3^\dagger |0\rangle . \tag{46}$$

Calculate the three-photon joint probability $P^{(3)}(\boldsymbol{r}_1, t_2; \boldsymbol{r}_2, t_2; \boldsymbol{r}_3, t_3)$. Analyze the form of the three-photon coherence $g^{(3)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ when all the three modes are traveling waves with exactly the same frequency.

Solution Let η be the normalizing constant introduced by constants like $\boldsymbol{d}$ or $\hbar$ in the Fermi's golden rule. The joint probability is

$$\begin{aligned} &P^{(3)}(\boldsymbol{r}_1, t_2; \boldsymbol{r}_2, t_2; \boldsymbol{r}_3, t_3) \\ &= \eta^3 \langle E^-(\boldsymbol{r}_1, t_1) E^-(\boldsymbol{r}_2, t_2) E^-(\boldsymbol{r}_3, t_3) E^+(\boldsymbol{r}_3, t_3) E^+(\boldsymbol{r}_2, t_2) E^+(\boldsymbol{r}_1, t_2) \rangle \\ &= \eta^3 \langle 0 | b_3 b_2 b_1 E^-(\boldsymbol{r}_1, t_1) E^-(\boldsymbol{r}_2, t_2) E^-(\boldsymbol{r}_3, t_3) E^+(\boldsymbol{r}_3, t_3) E^+(\boldsymbol{r}_2, t_2) E^+(\boldsymbol{r}_1, t_2) b_1^\dagger b_2^\dagger b_3^\dagger |0\rangle . \end{aligned}$$

By Wick's theorem it can be reduced to the sum of products of two-point correlation functions. Note that since $E^+ \sim b$, it must contract with a creation operator on its right to create a non-zero two-point correlation function, or in other words all the three E^+ operators must contract with $b_1^\dagger, b_2^\dagger, b_3^\dagger$. So we have

$$\begin{aligned}P^{(3)}(\boldsymbol{r}_1,t_2;\boldsymbol{r}_2,t_2;\boldsymbol{r}_3,t_3) = \eta^3 \times &\sum \text{contractions between } b_1,b_2,b_3 \text{ and the three } E^-\text{s} \\ \times &\sum \text{contraction between the three } E^+\text{s and } b_1^\dagger, b_2^\dagger, b_3^\dagger,\end{aligned}$$

and we immediately notice that again by Wick's theorem

$$\begin{aligned}&\sum \text{contraction between the three } E^+\text{s and } b_1^\dagger, b_2^\dagger, b_3^\dagger \\ = \ &\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)b_1^\dagger b_2^\dagger b_3^\dagger|0\rangle \\ = \ &\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)|\psi\rangle\,,\end{aligned}$$

and

$$\begin{aligned}&\sum \text{contractions between } b_1,b_2,b_3 \text{ and the three } E^-\text{s} \\ = \ &\langle 0|b_3b_2b_1E^-(\boldsymbol{r}_1,t_1)E^-(\boldsymbol{r}_2,t_2)E^-(\boldsymbol{r}_3,t_3)|0\rangle \\ = \ &\langle \psi|E^-(\boldsymbol{r}_1,t_1)E^-(\boldsymbol{r}_2,t_2)E^-(\boldsymbol{r}_3,t_3)|0\rangle\,,\end{aligned}$$

so

$$P^{(3)}(\boldsymbol{r}_1,t_2;\boldsymbol{r}_2,t_2;\boldsymbol{r}_3,t_3) = \eta^3\big|\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)|\psi\rangle\big|^2. \tag{47}$$

When $t_1 = t_2 = t_3 = t$, it is just

$$P^{(3)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t) = \eta^3|\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t)|^2, \tag{48}$$

where

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t) = \langle 0|E^+(\boldsymbol{r}_3,t)E^+(\boldsymbol{r}_2,t)E^+(\boldsymbol{r}_1,t)|\psi\rangle \tag{49}$$

is the "three-photon wavefunction".

Now we evaluate $\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)|\psi\rangle$. Again by Wick's theorem

$$\begin{aligned}&\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)|\psi\rangle \\ = &\sum_{(i,j,k)\in\mathrm{Perm}(\{1,2,3\})} \langle 0|E^+(\boldsymbol{r}_i,t_i)b_1^\dagger|0\rangle\ \langle 0|E^+(\boldsymbol{r}_j,t_j)b_2^\dagger|0\rangle\ \langle 0|E^+(\boldsymbol{r}_k,t_k)b_3^\dagger|0\rangle\,,\end{aligned}$$

where

$$\begin{aligned}\langle 0|E^+(\boldsymbol{r}_m,t_m)b_n^\dagger|0\rangle &= \sum_i \mathcal{E}_i f_i(\boldsymbol{r}_m)\,\langle 0|b_i\mathrm{e}^{-\mathrm{i}\omega_i t_m}b_n^\dagger|0\rangle \\ &= \mathcal{E}_n f_n(\boldsymbol{r}_m)\mathrm{e}^{-\mathrm{i}\omega_n t_m}.\end{aligned}$$

Therefore

$$\begin{aligned}&\langle 0|E^+(\boldsymbol{r}_3,t_3)E^+(\boldsymbol{r}_2,t_2)E^+(\boldsymbol{r}_1,t_2)|\psi\rangle \\ = &\sum_{(i,j,k)\in\mathrm{Perm}(\{1,2,3\})} \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 f_1(\boldsymbol{r}_i)f_2(\boldsymbol{r}_j)f_3(\boldsymbol{r}_k)\mathrm{e}^{-\mathrm{i}(\omega_1t_i+\omega_2t_j+\omega_3t_k)}.\end{aligned} \tag{50}$$

Inserting this equation into (47), we have obtained an explicit expression of the three-photon joint probability. Particularly we have

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t) = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2+\omega_3)t}\sum_{(i,j,k)\in\mathrm{Perm}(\{1,2,3\})} f_1(\boldsymbol{r}_i)f_2(\boldsymbol{r}_j)f_3(\boldsymbol{r}_k), \quad P^{(3)} = \eta^3|\Psi|^2. \tag{51}$$

It can be observed that Ψ is just symmetrized direct projects of the three "single photon wave functions" f_1, f_2, f_3.

The definition of three-photon coherence is

$$g(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3) = \frac{\langle E^-(\boldsymbol{r}_1,t)E^-(\boldsymbol{r}_2,t)E^-(\boldsymbol{r}_3,t)E^+(\boldsymbol{r}_3,t)E^+(\boldsymbol{r}_2,t)E^+(\boldsymbol{r}_1,t)\rangle}{\langle E^-(\boldsymbol{r}_1,t)E^+(\boldsymbol{r}_1,t)\rangle\,\langle E^-(\boldsymbol{r}_2,t)E^+(\boldsymbol{r}_2,t)\rangle\,\langle E^-(\boldsymbol{r}_3,t)E^+(\boldsymbol{r}_3,t)\rangle}. \tag{52}$$

We have already known the numerator. As for the denominator, by Wick's theorem we have

$$\begin{aligned}\big\langle E^-(\boldsymbol{r}_1,t)E^+(\boldsymbol{r}_1,t)\big\rangle &= \langle 0|b_3b_2b_1E^-(\boldsymbol{r}_1,t)E^+(\boldsymbol{r}_1,t)b_1^\dagger b_2^\dagger b_3^\dagger|0\rangle \\ &= \mathcal{E}_1^2|\boldsymbol{f}_1(\boldsymbol{r}_1)|^2 + \mathcal{E}_2^2|\boldsymbol{f}_2(\boldsymbol{r}_1)|^2 + \mathcal{E}_3^2|\boldsymbol{f}_3(\boldsymbol{r}_1)|^2.\end{aligned} \tag{53}$$

Note that if E^+ contracts with $b_i^\dagger$ then E^- must contract with b_i, or otherwise we will have $\langle b_j b_k^\dagger \rangle$ factors where $j \neq k$, which evaluates to zero.

When all the three modes are propagating waves with exactly the same frequency ω, we have

$$\boldsymbol{f}_i(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}}\boldsymbol{e}_i, \quad i = 1, 2, 3,$$

and

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \mathcal{E} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}},$$

and therefore

$$\left\langle E^-(\boldsymbol{r}_1, t)E^+(\boldsymbol{r}_1, t)\right\rangle = 3\mathcal{E},$$

and

$$\begin{aligned}
&\left\langle E^-(\boldsymbol{r}_1, t)E^-(\boldsymbol{r}_2, t)E^-(\boldsymbol{r}_3, t)E^+(\boldsymbol{r}_3, t)E^+(\boldsymbol{r}_2, t)E^+(\boldsymbol{r}_1, t)\right\rangle \\
&= \left|\mathcal{E}^3 \sum_{(i,j,k)\in\mathrm{Perm}(\{1,2,3\})} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}_i+\boldsymbol{k}_2\cdot\boldsymbol{r}_j+\boldsymbol{k}_3\cdot\boldsymbol{r}_k)}\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\right|^2 \\
&= \mathcal{E}^6 \sum_{\sigma,\sigma'\in\mathrm{Perm}(\{1,2,3\})} \prod_{i=1}^{3} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_{\sigma(i)}-\boldsymbol{k}_{\sigma'(i)})\cdot\boldsymbol{r}_i}\boldsymbol{e}_{\sigma(i)}\cdot\boldsymbol{e}^*_{\sigma'(i)}.
\end{aligned}$$

Note that permuting b_1, b_2, b_3 is equivalent to permuting $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$, which justifies the third line. Putting everything together we have

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = \sum_{\sigma,\sigma'\in\mathrm{Perm}(\{1,2,3\})} \prod_{i=1}^{3} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_{\sigma(i)}-\boldsymbol{k}_{\sigma'(i)})\cdot\boldsymbol{r}_i}\boldsymbol{e}_{\sigma(i)}\cdot\boldsymbol{e}^*_{\sigma'(i)}. \tag{54}$$

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