Advanced Electrodynamics, Homework 1

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Problem 1 Show that

$$\int_0^{2\pi}\int_0^\pi \boldsymbol{M}_{em'n'}\cdot\boldsymbol{M}_{omn}\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 0, \quad \text{for all } m, m', n, n'. \tag{1}$$

Solution The definition is

$$\begin{aligned}
\boldsymbol{M}_{emn} &= \frac{-m}{\sin\theta}\sin(m\phi)P_n^m(\cos\theta)z_n(kr)\hat{\boldsymbol{e}}_\theta - \cos(m\phi)\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta}z_n(kr)\hat{\boldsymbol{e}}_\phi, \\
\boldsymbol{M}_{omn} &= \frac{m}{\sin\theta}\cos(m\phi)P_n^m(\cos\theta)z_n(kr)\hat{\boldsymbol{e}}_\theta - \sin(m\phi)\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta}z_n(kr)\hat{\boldsymbol{e}}_\phi.
\end{aligned} \tag{2}$$

Therefore, we have

$$\begin{aligned}
&\int_0^{2\pi}\int_0^\pi \boldsymbol{M}_{em'n'}\cdot\boldsymbol{M}_{omn}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= z_{n'}(kr)z_n(kr)\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\frac{-m'm}{\sin^2\theta}\sin(m'\phi)\cos(m\phi)P_{n'}^{m'}(\cos\theta)P_n^m(\cos\theta) \\
&\quad + z_{n'}(kr)z_n(kr)\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \sin\theta\,\mathrm{d}\theta\cos(m'\varphi)\sin(m\varphi)\frac{\mathrm{d}P_{n'}^{m'}(\cos\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta}.
\end{aligned}$$

Note that

$$\int_0^{2\pi}\mathrm{d}\phi\sin(m\varphi)\cos(n\varphi) = 0$$

for all $m, n \in \mathbb{Z}$, so we have already proved (1).

Problem 2 Suppose that in the basis of vector spherical wave functions, the plane wave

$$\boldsymbol{E} = \hat{\boldsymbol{e}}_x E_0 \mathrm{e}^{\mathrm{i}kz} = \hat{\boldsymbol{e}}_x E_0 \mathrm{e}^{\mathrm{i}kr\cos\theta} \tag{3}$$

is expanded as

$$\boldsymbol{E} = \sum_{m=0}^\infty\sum_{n=m}^\infty \left(A_{emn}\boldsymbol{N}_{emn}^{(1)} + B_{emn}\boldsymbol{M}_{emn}^{(1)} + A_{omn}\boldsymbol{N}_{omn}^{(1)} + B_{omn}\boldsymbol{M}_{omn}^{(1)}\right). \tag{4}$$

The superscript means that in these vector spherical wave functions $z(kr) = j(kr)$. Prove that

$$B_{o1n} = \mathrm{i}^n E_0 \frac{2n+1}{n(n+1)}, \quad A_{e1n} = -\mathrm{i}E_0\mathrm{i}^n\frac{2n+1}{n(n+1)}. \tag{5}$$

Note that

$$\hat{\boldsymbol{e}}_x = \sin\theta\cos\phi\hat{\boldsymbol{e}}_r + \cos\theta\cos\phi\hat{\boldsymbol{e}}_\theta - \sin\phi\hat{\boldsymbol{e}}_\phi, \tag{6}$$

each term of which is proportional to $\sin\phi$ or $\cos\phi$, so non-zero vector spherical harmonic components in the expansion are all $m = 1$ ones.

Solution The angular orthogonal relation of vector spherical wave functions gives

$$A_{e1n} = \frac{\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\,\mathrm{d}\theta\,\boldsymbol{E}\cdot\boldsymbol{N}_{e1n}^{(1)}}{\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\,\mathrm{d}\theta\,|\boldsymbol{N}_{e1n}^{(1)}|^2}, \quad B_{o1n} = \frac{\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\,\mathrm{d}\theta\,\boldsymbol{E}\cdot\boldsymbol{M}_{o1n}^{(1)}}{\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\,\mathrm{d}\theta\,|\boldsymbol{M}_{o1n}^{(1)}|^2}.$$

We have

$$
\begin{aligned}
B_{o1n} &= \frac{\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta\, \boldsymbol{E}\cdot \boldsymbol{M}_{o1n}^{(1)}}{\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta\, |\boldsymbol{M}_{o1n}^{(1)}|^2} \\
&= E_0 \frac{\displaystyle\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}kr\cos\theta}\left(\frac{\cos\phi}{\sin\theta}P_n^1(\cos\theta)\cos\theta\cos\phi - \sin\phi\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}(-\sin\phi)\right)}{\displaystyle j_n(kr)\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \left(\frac{1}{\sin^2\theta}\cos^2\phi(P_n^1(\cos\theta))^2 + \sin^2\phi\left(\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\right)^2\right)}.
\end{aligned}
$$

Integrating ϕ and we have

$$
B_{o1n} = E_0 \frac{\displaystyle\int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}kr\cos\theta}\left(P_n^1(\cos\theta)\cos\theta + \frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\sin\theta\right)}{\displaystyle j_n(kr)\int_0^{\pi} \sin\theta \,\mathrm{d}\theta \left(\frac{1}{\sin^2\theta}(P_n^1(\cos\theta))^2 + \left(\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\right)^2\right)}. \tag{7}
$$

By the same way we can prove that

$$
B_{e1n} = 0 \tag{8}
$$

since the corresponding integrals contain $\int_0^{2\pi} \mathrm{d}\phi \sin\phi\cos\phi$ factors and therefore vanish. The numerator of (7) is

$$
\begin{aligned}
&\int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}kr\cos\theta}\left(P_n^1(\cos\theta)\cos\theta + \frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\sin\theta\right) \\
&= \int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}kr\cos\theta}\left(\sqrt{1-\cos^2\theta}\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\cos\theta}\cos\theta + \sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sqrt{1-\cos^2\theta}\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\cos\theta}\right)\right) \\
&= \int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}kr\cos\theta}\left(\sin\theta\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\cos\theta}\cos\theta + \sin\theta\left(\cos\theta\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\cos\theta} - \sin^2\theta\frac{\mathrm{d}^2P_n(\cos\theta)}{\mathrm{d}\cos\theta^2}\right)\right) \\
&= \int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}krx}\left(2x\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} + (x^2-1)\frac{\mathrm{d}^2P_n(x)}{\mathrm{d}x^2}\right) \\
&= n(n+1)\int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}krx}P_n(x).
\end{aligned}
$$

The last line is obtained by the definition of Legendre's polynomials. By the formula

$$
j_n(\rho) = \frac{\mathrm{i}^{-n}}{2}\int_0^{\pi} e^{\mathrm{i}\rho\cos\theta}P_n(\cos\theta)\sin\theta\mathrm{d}\theta = \frac{\mathrm{i}^{-n}}{2}\int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\rho x}P_n(x), \tag{9}
$$

the numerator of (7) can be further evaluate as

$$
\frac{2}{\mathrm{i}^{-n}}n(n+1)j_n(kr),
$$

so (7) reads

$$
\begin{aligned}
B_{o1n} &= E_0 \frac{n(n+1)j_n(kr)}{\displaystyle\frac{\mathrm{i}^{-n}}{2}j_n(kr)\int_0^{\pi} \sin\theta \,\mathrm{d}\theta \left(\frac{1}{\sin^2\theta}(P_n^1(\cos\theta))^2 + \left(\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\right)^2\right)} \\
&= E_0 \frac{n(n+1)}{\displaystyle\frac{\mathrm{i}^{-n}}{2}\int_0^{\pi} \sin\theta \,\mathrm{d}\theta \left(\frac{1}{\sin^2\theta}(P_n^1(\cos\theta))^2 + \left(\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\right)^2\right)}.
\end{aligned} \tag{10}
$$

By integral formulae

$$
\int_{-1}^{1} \mathrm{d}x \left(\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}\right)^2 = \int_{-1}^{1} \mathrm{d}x \left(\frac{1}{\sqrt{1-x^2}}P_n^1(x)\right)^2 = n(n+1)
$$

and

$$\int_{-1}^{1} \mathrm{d}x \left(\frac{\mathrm{d}P_n^1(x)}{\mathrm{d}x}\right)^2 (1-x^2) = \frac{n(n+1)(2n^2-1)}{2n+1},$$

we have

$$\begin{aligned} B_{o1n} &= \frac{2\mathrm{i}^n E_0 n(n+1)}{n(n+1)+n(n+1)\frac{2n^2-1}{2n+1}} \\ &= \frac{\mathrm{i}^n(2n+1)E_0}{n(n+1)}, \end{aligned}$$

and hence we have shown the first equation in (5).

We than evaluate A_{e1n} and A_{o1n}. The definition of $\boldsymbol{N}$ functions are

$$\begin{aligned} \boldsymbol{N}_{e1n}^{(1)} =& \frac{j_n(kr)}{kr}\cos\phi n(n+1)P_n^1(\cos\theta)\hat{\boldsymbol{e}}_r + \cos\phi\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[(kr)j_n(kr)\right]\hat{\boldsymbol{e}}_\theta \\ & -\sin\phi\frac{P_n^1(\cos\theta)}{\sin\theta}\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[(kr)z_n(kr)\right]\hat{e}_\phi, \\ \boldsymbol{N}_{o1n}^{(1)} =& \frac{j_n(kr)}{kr}\sin\phi n(n+1)P_n^1(\cos\theta)\hat{\boldsymbol{e}}_r + \sin\phi\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[(kr)j_n(kr)\right]\hat{\boldsymbol{e}}_\theta \\ & +\sin\phi\frac{P_n^1(\cos\theta)}{\sin\theta}\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[(kr)z_n(kr)\right]\hat{e}_\phi. \end{aligned} \tag{11}$$

By (6) we find that

$$A_{o1n} = 0, \tag{12}$$

as $\sin\phi$ and $\cos\phi$ are orthogonal. Now we consider the $\hat{\boldsymbol{e}}_r$ component, i.e.

$$\sin\theta\cos\phi E_0 \mathrm{e}^{\mathrm{i}kr\cos\theta} = \sum_n A_{e1n}n(n+1)\frac{j_n(kr)}{kr}\cos\phi P_n^1(\cos\theta),$$

as $\boldsymbol{M}$ functions do not have $\hat{\boldsymbol{e}}_r$ components. Therefore, we have

$$\begin{aligned} \int_{-\pi}^{\pi}\mathrm{d}\theta\sin^2\theta E_0\mathrm{e}^{\mathrm{i}kr\cos\theta}P_m^1(\cos\theta) &= \sum_n\int_{-\pi}^{\pi}\sin\theta\,\mathrm{d}\theta\,n(n+1)A_{e1n}\frac{j_n(kr)}{kr}P_n^1(\cos\theta)P_m^1(\cos\theta) \\ &= \sum_n n(n+1)A_{e1n}\frac{j_n(kr)}{kr}\int_{-1}^{1}\mathrm{d}x\,P_n^1(x)P_m^1(x) \\ &= m(m+1)A_{e1m}\frac{j_m(kr)}{kr}\frac{2m(m+1)}{2m+1}, \end{aligned} \tag{13}$$

where we have used the formula that

$$\int_{-1}^{1}P_m^1(x)P_n^1(x)\,\mathrm{d}x = \frac{2n(n+1)}{2n+1}\delta_{mn}.$$

Now we just need to evaluate the LHS and then we can find A_{e1n}. We have

$$\begin{aligned} \int_{-\pi}^{\pi}\mathrm{d}\theta\sin^2\theta E_0\mathrm{e}^{\mathrm{i}kr\cos\theta}P_m^1(\cos\theta) &= E_0\int_{-1}^{1}\mathrm{d}x\,\sqrt{1-x^2}\mathrm{e}^{\mathrm{i}krx}P_m^1(x) \\ &= E_0\int_{-1}^{1}\mathrm{d}x\,\sqrt{1-x^2}\mathrm{e}^{\mathrm{i}krx}\sqrt{1-x^2}\frac{\mathrm{d}P_m(x)}{\mathrm{d}x} \\ &= E_0\int_{-1}^{1}\mathrm{d}x\,(1-x^2)\mathrm{e}^{\mathrm{i}krx}\frac{\mathrm{d}P_m(x)}{\mathrm{d}x} \\ &= E_0\int_{-1}^{1}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}krx}m(xP_m(x)-P_{m-1}(x)) \\ &= E_0\int_{-1}^{1}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}krx}\frac{m(m+1)}{2m+1}(P_{m+1}(x)-P_{m-1}(x)), \end{aligned} \tag{14}$$

where we have used the formulae that

$$\frac{x^2-1}{n}\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} = xP_n - P_{n-1}$$

and

$$xP_n(x) = \frac{1}{2n+1}\left(nP_{n-1}(x) + (n+1)P_{n+1}(x)\right).$$

Using (9) in (14), we have

$$\begin{aligned}
&E_0\int_{-1}^{1}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}krx}\frac{n(n+1)}{2n+1}(P_{n+1}(x)-P_{n-1}(x)) \\
&= E_0\frac{n(n+1)}{2n+1}\left(\frac{2}{\mathrm{i}^{-n-1}}j_{n+1}(kr) - \frac{2}{\mathrm{i}^{-n+1}}j_{n-1}(kr)\right) \\
&= E_0\frac{n(n+1)}{2n+1}\left(\frac{2}{\mathrm{i}^{-n-1}}j_{n+1}(kr) + \frac{2}{\mathrm{i}^{-n-1}}j_{n-1}(kr)\right) \\
&= E_0\frac{n(n+1)}{2n+1}\frac{2}{\mathrm{i}^{-n-1}}\frac{2n+1}{kr}j_n(kr),
\end{aligned} \tag{15}$$

where we have used the fact that

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x}j_n(x),$$

which is a consequence of

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

and

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x).$$

Putting (13), (14) and (15) together, we have

$$m(m+1)A_{e1m}\frac{j_m(kr)}{kr}\frac{2m(m+1)}{2m+1} = E_0\frac{m(m+1)}{2m+1}\frac{2}{\mathrm{i}^{-m-1}}\frac{2m+1}{kr}j_n(kr),$$

or

$$A_{e1m} = -\mathrm{i}\frac{(2m+1)\mathrm{i}^m E_0}{m(m+1)},$$

which is the second equation in (5).

So now we have completed a proof of (5), and since we also have (4), (8) and (12), we now have expanded the plane wave (3) into

$$\boldsymbol{E} = E_0\sum_{n=1}^{\infty}\mathrm{i}^n\frac{2n+1}{n(n+1)}\left(\boldsymbol{M}^{(1)}_{o1n} - \mathrm{i}\boldsymbol{N}^{(1)}_{e1n}\right). \tag{16}$$

Problem 3 Verify numerically the expansion (16).

Solution Since visualizing a 3D vector field is hard, we will work on its three components in spherical coordinates and the real and imaginary parts one by one.

The result can be found toghether with this document, where we plot the RHS of (16) with a fixed z. The parameters - the maximum of n in the summation in (16) (labeled as m), z, which components to be shown and whether to show the real part or the imaginary part - can be changed interactively. An example can be found in (1).

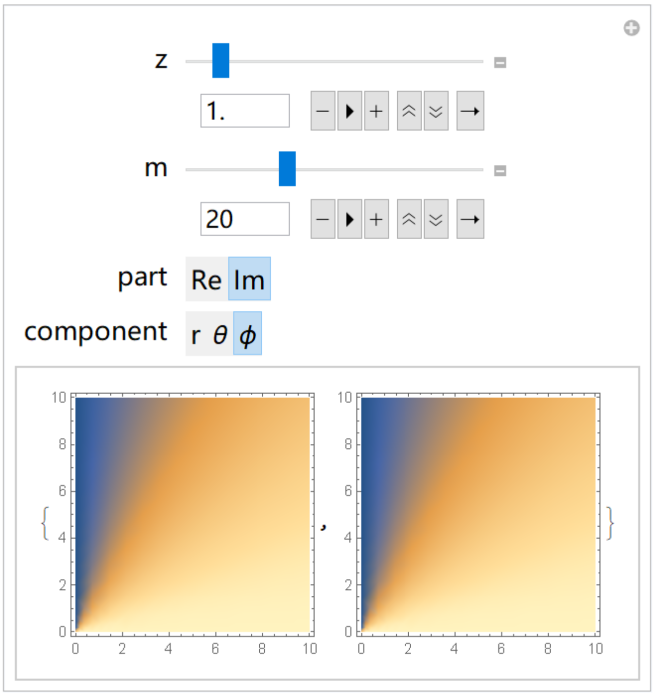

Figure 1: Numerical demonstration of (16). The left figure is the RHS and the right figure is the LHS. m should be greater than 10 to achieve a good approximation.