QFT I, Homework 4

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Scalar QED Consider the theory of a complex scalar field ϕ interacting with the electromagnetic field A^μ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

where $D_\mu = \partial_\mu + \mathrm{i} e A_\mu$ is the usual gauge covaraint derivative.

(a) Show the Lagrangian is invariant under the gauge transformations

$$\phi(x) \to \mathrm{e}^{-\mathrm{i}\alpha(x)} \phi(x), \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x). \tag{2}$$

(b) Derive the Feynman rules for the interaction between photons and scalar particles.

(c) Draw all the leading-order Feynman diagrams and compute the amplitude for the process $\gamma\gamma \to \phi\phi^*$.

(d) Compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$. You can take an average over all initial state polarizations. For simplicity, you can restrict your calculation in the limit $m = 0$.

(e) Draw all leading order Feynman diagrams, that contribute to the Compton scattering process $\gamma\phi \to \gamma\phi$ and compute the differential cross section $\mathrm{d}\sigma / \mathrm{d}\cos\theta$ with $m = 0$.

Solution

(a) Under the gauge transformation (2), we have

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu \left(A_\nu + \frac{1}{e} \partial_\nu \alpha \right) - \partial_\nu \left(A_\mu + \frac{1}{e} \partial_\mu \alpha \right) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu},$$

so the first term in (1) remains the same. It is obvious that under (2)

$$\phi^* \phi \to \phi'^* \phi' = \mathrm{e}^{\mathrm{i}\alpha} \phi^* \mathrm{e}^{-\mathrm{i}\alpha} \phi = \phi^* \phi,$$

so the third term in (1) is also invariant. Also we have

$$\begin{aligned} D^\mu \phi \to (\partial^\mu + \mathrm{i} e A'^\mu) \phi' &= (\partial^\mu + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \mathrm{e}^{-\mathrm{i}\alpha} \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} (\partial^\mu - \mathrm{i} \partial^\mu \alpha + \mathrm{i} e A^\mu + \mathrm{i} \partial^\mu \alpha) \phi \\ &= \mathrm{e}^{-\mathrm{i}\alpha} D^\mu \phi, \end{aligned}$$

and also

$$(D^\mu \phi)^* = \mathrm{e}^{\mathrm{i}\alpha} D^\mu \phi,$$

so $D^\mu \phi (D^\mu \phi)^*$ is also invariant. Therefore (1) is invariant under (2).

(b) We make the following expansion of Fourier transformation. For the complex scalar field we have

$$\phi(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + b^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}p\cdot x}). \tag{3}$$

which was proved in (10) in Homework 2. The vector field is expanded as

$$A_\mu(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \sum_{r=1}^{2} \epsilon^r_\mu(\boldsymbol{p}) \left(a^\dagger_{\boldsymbol{p},r} \mathrm{e}^{\mathrm{i}p\cdot x} + a_{\boldsymbol{p},r} \mathrm{e}^{-\mathrm{i}p\cdot x} \right). \tag{4}$$

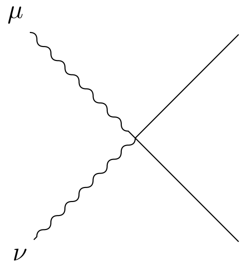
Expanding (2) we have

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{scalarQED}}, \tag{5}$$

where $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{vector}}$ are Lagrangians of free scalar field and free massless vector field, and

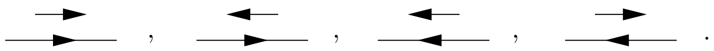
$$\begin{aligned}\mathcal{L}_{\text{scalarQED}} &= (D_\mu\phi)^* D^\mu\phi - (\partial_\mu\phi)^*\partial^\mu\phi \\ &= e^2\eta_{\mu\nu}A^\mu A^\nu\phi^*\phi - \mathrm{i}eA_\mu\phi^*\partial^\mu\phi + \mathrm{i}e\partial_\mu\phi^* A^\mu\phi.\end{aligned} \tag{6}$$

The first term has no derivatives. Therefore it gives the following (momentum space) vertex:



$$= 2\mathrm{i}e^2\eta_{\mu\nu}, \tag{7}$$

where the factor i comes from the time evolution operator and the factor 2 comes from the fact that there are two identical photon lines. The two ϕ lines can be any of the following four:



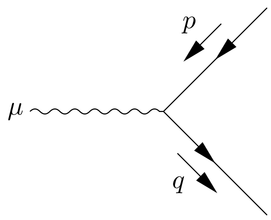
The second term gives

$$-\mathrm{i}eA_\mu\phi^*\partial^\mu\phi \sim -\mathrm{i}eA_\mu(a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x} + b_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x})(-\mathrm{i}(p'\cdot x)a_{\boldsymbol{p}'}\mathrm{e}^{-\mathrm{i}p'\cdot x} + \mathrm{i}(p'\cdot x)b^\dagger_{\boldsymbol{p}'}\mathrm{e}^{\mathrm{i}p'\cdot x}),$$

and the third term is its complex conjugate. Therefore, the $a^\dagger a$ term in the Lagrangian is

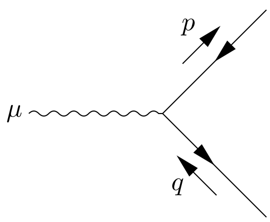
$$\sim -e(p_1+p_2)_\mu A^\mu a^\dagger_{\boldsymbol{p}_1}a_{\boldsymbol{p}_2},$$

so after adding the i factor from the time evolution operator we have



$$= -\mathrm{i}e(p_\mu + q_\mu), \tag{8}$$

and we can change the direction of a momentum line and a ϕ-particle line arbitrarily; if a momentum line goes in contrast to the corresponding particle line, then we need to add a minus sign to the corresponding momentum. For example we have



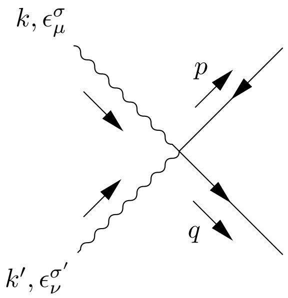
$$= \mathrm{i}e(p_\mu + q_\mu). \tag{9}$$

There are four vertices in this type in total.

Note

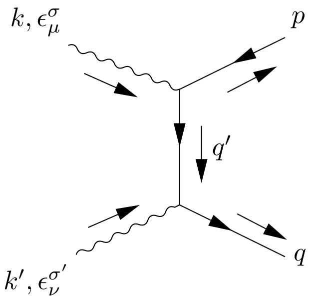
Here we follow the notation of Peskin, i.e. using the *momentum* arrow to denote whether this line represents creation or annihilation and using the arrow *on* a particle line to show whether this line represents a particle (if the direction of the particle line is parallel to the direction of the momentum line) or a antiparticle (otherwise). The real direction of a 4-momentum is *not* represented in any arrow.

(c) We enumerate over all possible diagrams. The vertex (7) itself is a diagram:



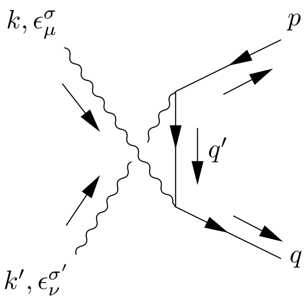
$$= (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \times 2\mathrm{i}e^2 \eta_{\mu\nu} =: \mathrm{i}\mathcal{M}_4. \tag{10}$$

Combining two (8)-type vertices we have a t-channel



$$\begin{aligned} &= \epsilon_\mu^\sigma e_\nu^{\sigma'} \times \mathrm{i}e(p-(k-p))^\mu \times \mathrm{i}e(-(k-p)-q)^\nu \times \frac{\mathrm{i}}{(k-p)^2 - m^2 + \mathrm{i}0^+} \\ &= -\mathrm{i}e^2 \epsilon_\mu^\sigma (2p-k)^\mu \epsilon_\nu^{\sigma'} (-k+p-q)^\nu \frac{1}{(k-p)^2 - m^2 + \mathrm{i}0^+} =: \mathrm{i}\mathcal{M}_t, \end{aligned} \tag{11}$$

and a u-channel



$$\begin{aligned} &= \epsilon_\mu^\sigma \epsilon_\nu^{\sigma'} \times \mathrm{i}e(p-(q-k))^\nu \times \mathrm{i}e(-q-(q-k))^\mu \times \frac{\mathrm{i}}{(q-k)^2 - m^2 + \mathrm{i}0^+} \\ &= \mathrm{i}e^2 \epsilon_\mu^\sigma (2q-k)^\mu \epsilon_\nu^{\sigma'} (p-q+k)^\nu \frac{1}{(q-k)^2 - m^2 + \mathrm{i}0^+} =: \mathrm{i}\mathcal{M}_u. \end{aligned} \tag{12}$$

Note

We *do not* need to distinguish the direction of the q' momentum line. This line can be either a particle line or an antiparticle line, but since the ordinary propagator $\mathrm{i}/(p^2 - m^2 + \mathrm{i}0^+)$ is obtained by summing up the two cases, when we write down this propagator, we have automatically considered both processes.

Summing everything up, we have

$$\begin{aligned} \mathrm{i}\mathcal{M}(\gamma\gamma \to \phi\phi^*) &= \mathrm{i}(\mathcal{M}_4 + \mathcal{M}_t + \mathcal{M}_u) \\ &= \mathrm{i}e^2 (\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \left(2\eta_{\mu\nu} + \frac{(k-2p)_\mu (k'-2q)_\nu}{t-m^2} + \frac{(k-2q)_\mu (k'-2p)_\nu}{u-m^2} \right) \\ &=: \mathrm{i}(\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu \mathcal{M}_{\mu\nu}, \end{aligned} \tag{13}$$

where

$$t = (k-p)^2, \quad u = (q-k)^2. \tag{14}$$

(d) We work in the center-of-mass frame, and therefore we have $k = (|\boldsymbol{k}|, \boldsymbol{k})$, and $k' = (|\boldsymbol{k}|, -\boldsymbol{k})$. The massless limit can be calculated with Eq. (4.85) in Peskin, which is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\mathrm{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{CM}}^2}, \tag{15}$$

What we need is $|\mathcal{M}|^2$. We have

$$
\begin{aligned}
|\mathcal{M}|^2 &= \sum_{\epsilon^\sigma, \epsilon^{\sigma'}} P(\epsilon^\sigma, \epsilon^{\sigma'})(\epsilon^\sigma)^\mu (\epsilon^{\sigma'})^\nu (\epsilon^\sigma)^{\rho *} (\epsilon^{\sigma'})^{\delta *} \mathcal{M}_{\mu\nu} \mathcal{M}^*_{\rho\delta} \\
&= \frac{1}{4} \sum_{\sigma=\pm 1} (\epsilon^\sigma)^\mu (\epsilon^\sigma)^{\rho *} \sum_{\sigma'=\pm 1} (\epsilon^{\sigma'})^\nu (\epsilon^{\sigma'})^{\delta *} \mathcal{M}_{\mu\nu} \mathcal{M}^*_{\rho\delta}.
\end{aligned}
$$

It can be verified that in a coordinate where $\boldsymbol{p}_1$ is on the $\pm\hat{\boldsymbol{z}}$ direction, we have

$$
\sum_\sigma (\epsilon^\sigma)^\mu (\epsilon^\sigma)^{\rho *} = \mathrm{diag}(0, 1, 1, 0),
$$

and in our case, it can be verified that

$$
\sum_\sigma (\epsilon^\sigma)^\mu (\epsilon^\sigma)^{\rho *} = -\eta^{\mu\rho} + \frac{k^\mu k'^\rho + k^\rho k'^\mu}{k \cdot k'}, \tag{16}
$$

and a similar result holds for the sum on σ', which can be obtained by replacing all the indices. We therefore have

$$
\begin{aligned}
\sum_\sigma (\epsilon^\sigma)^\mu (\epsilon^\sigma)^{\rho *} \mathcal{M}_{\mu\rho} = e^2 \Big(&- 2\delta^\rho_\nu - \frac{(k-2p)^\rho (k'-2q)_\nu}{t} - \frac{(k-2q)^\rho (k'-2p)_\nu}{u} + 2\frac{k_\nu k'^\rho + k'_\nu k^\rho}{k \cdot k'} \\
&+ \frac{k \cdot (k-2p) k'^\rho (k'-2q)_\nu + k' \cdot (k-2p) k^\rho (k'-2q)_\nu}{(k \cdot k')t} \\
&+ \frac{k \cdot (k-2q) k'^\rho (k'-2p)_\nu + k' \cdot (k-2q) k^\rho (k'-2p)_\nu}{(k \cdot k')u} \Big),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\sum_{\sigma'} (\epsilon^{\sigma'})^\nu (\epsilon^{\sigma'})^{\delta *} \mathcal{M}^*_{\rho\delta} = e^2 \Big(&- 2\delta^\nu_\rho - \frac{(k-2p)_\rho (k'-2q)^\nu}{t} - \frac{(k-2q)_\rho (k'-2p)^\nu}{u} + 2\frac{k^\nu k'_\rho + k'^\nu k_\rho}{k \cdot k'} \\
&+ \frac{k' \cdot (k'-2q) k^\nu (k-2p)_\rho + k \cdot (k'-2q) k'^\nu (k-2p)_\rho}{(k \cdot k')t} \\
&+ \frac{k' \cdot (k'-2p) k^\nu (k-2q)_\rho + k \cdot (k'-2p) k'^\nu (k-2q)_\rho}{(k \cdot k')u} \Big).
\end{aligned}
$$

Multiplication of the two expressions, after expansion, is

The process is too long to be displayed here. The equation above has been simplified using the fact that $k^2 = k'^2 = p^2 = q^2 = 0$.