Quantum Optics by Prof. Saijun Wu

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1 The rotational wave approximation and the Bloch sphere

We can also think of the RWT in an intuitive way illustrated in Figure 1 on page 1, where we start from a $|g\rangle$ state, and then a photon comes in, and the system is brought to the $|g, n = 1\rangle$ state (shown as a dotted line), and then an absorption process happens, and this $|g, n = 1\rangle$ state is turned into an $|e\rangle$ state, and the whole process can be summarized as a $|g\rangle$ state being turned into an $|e\rangle$ state (the process is illustrated using a blue arrow). RWT can be thought as redefining $|g\rangle$ into $|g, n = 1\rangle$, the energy of which is $\omega_g + \omega$.

Under the rotational wave approximation and the corresponding RWA transformation, the Hamiltonian of a two-level system is

$$H = \frac{\hbar}{2} \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}, \tag{1}$$

where the norm of the **Rabi vector** $\boldsymbol{\Omega} = (\Omega_\mathrm{r}, \Omega_\mathrm{i}, \Delta)$ is

$$|\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2}, \tag{2}$$

$$\Omega = \frac{-2E_0^+ \hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{d}_{eg}}{\hbar} \tag{3}$$

is the **Rabi frequency** and Δ the **detuning**. The wave function is always in the form of

$$|\psi\rangle = \cos\frac{\theta}{2} \mathrm{e}^{\mathrm{i}\varphi/2} |g\rangle + \sin\frac{\theta}{2} \mathrm{e}^{-\mathrm{i}\varphi/2} |e\rangle, \tag{4}$$

and the density matrix is

$$\rho = \frac{1}{2}(1 + \boldsymbol{n} \cdot \boldsymbol{\sigma}), \tag{5}$$

where

$$\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{6}$$

It is natural to put $\boldsymbol{n}$ on a sphere, which is known as **Bloch sphere**. The constructions are standard for a qubit and can be found in Section 1.1 in the quantum information note. The equation of motion is

$$\frac{\mathrm{i}\hbar}{2} \dot{\boldsymbol{n}} \cdot \boldsymbol{\sigma} = \mathrm{i}\hbar\dot{\rho} = [H, \rho] = [\frac{\hbar}{2}\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}, \frac{1}{2}\boldsymbol{n} \cdot \boldsymbol{\sigma}] = \frac{\hbar}{4} \cdot 2\mathrm{i}(\boldsymbol{\Omega} \times \boldsymbol{n}) \cdot \boldsymbol{\sigma},$$

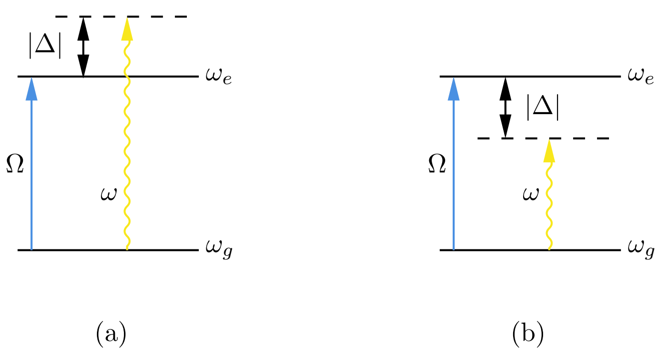

Figure 1: The energy diagram of a two-level system in an external optical field. A yellow wavy arrow is an incoming photon, and the blue lines labeled with Ω denote to the coupling between the $|g\rangle$ state and the $|e\rangle$ state. (a) $\omega - \omega_{eg} > 0$, $\Delta > 0$. (b) $\omega - \omega_{eg} < 0$, $\Delta < 0$.

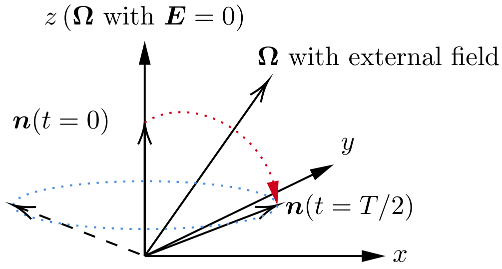

Figure 2: The consequence of a π-pulse illustrated with the Bloch sphere

and therefore we have

$$\dot{\boldsymbol{n}} = \boldsymbol{\Omega} \times \boldsymbol{n}. \tag{7}$$

We find that the Bloch vector rotates with the frequency of

$$\tilde{\Omega} = |\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2}. \tag{8}$$

We review some properties about the Bloch sphere. First, an equation of motion of a c-number defined in terms of the wave function is likely to be an equation of motion of an operator, and this is indeed the case of (7), where we may find $\boldsymbol{n} = \langle \boldsymbol{\sigma} \rangle$. Also, it can be verified that

$$|\langle \psi_1 | \psi_2 \rangle|^2 = \mathrm{tr}(\rho_1 \rho_2) = \frac{1}{2}(1 + \boldsymbol{n}_1 \cdot \boldsymbol{n}_2), \tag{9}$$

and therefore we find that two wave functions are the same or differ with only a phase factor if and only if they have the same Bloch vector, and they are orthogonal if and only if their Bloch vectors are exactly the opposite of each other.

Note that the direction of the Rabi vector is not determined. Usually we use the convention

$$\boldsymbol{\Omega} = (\Omega, 0, \Delta), \tag{10}$$

and the two Bloch vectors

$$\boldsymbol{n}_{\pm\Omega} = \pm(\sin\theta, 0, \cos\theta) \tag{11}$$

represents two eigenstates of (1), namely $|\tilde{g}\rangle$ (with eigenvalue $-\hbar|\boldsymbol{\Omega}|/2$) and $|\tilde{e}\rangle$ (with eigenvalue $\hbar|\boldsymbol{\Omega}|/2$), where

$$\theta = \arctan\frac{\Omega}{\Delta}. \tag{12}$$

The whole formulation - representing the wave function of a two-level atom as a vector on the Bloch sphere, rephrasing the EOM as (7) - is together called **Feynman–Vernon–Hellwarth representation (FVH)**. (7) directs $\boldsymbol{n}$ to rotate anticlockwise when $\boldsymbol{\Omega}$ is toward us.

2 Control a two-level system with external fields

Note that since (3), a two-level system's state can be controlled via changing the external field.

Consider we start from $|g\rangle$, i.e. $\boldsymbol{n} = \boldsymbol{e}_z$. We want to excite the atom to $|e\rangle$, i.e. $\boldsymbol{n} = -\boldsymbol{e}_z$. When there is no external field $\boldsymbol{\Omega}$ is along the z axis, and therefore $\boldsymbol{n}$ stays at $\boldsymbol{e}_z$. Now suppose we add an electric field $\boldsymbol{E}_0$, the Bloch vector starts to rotate around the new Rabi vector (depicted as the red curve in Figure 2 on page 2). If, after $t = T/2 = \pi/\tilde{\Omega}$, we remove the external field, the Bloch vector will again rotate around the z axis (depicted as the blue curve in Figure 2 on page 2). Therefore we successfully change the θ angle of $\boldsymbol{n}$. This process is called a **π-pulse**. With a series of π-pulse, we can make $\boldsymbol{n}$ arbitrarily close to $-\boldsymbol{e}_z$.

This is quite an interesting result, because it shows that the atom can be excited even with the presence of detuning. This can also be seen as an example of the E-t uncertainty principle, because what we are actually do is to use *pulses* to create some uncertainty in the energy to make up for the gap between ω and ω_{eg}.

3 Adiabatic states

When $|\mathbf{\Omega}| \gg |\langle\tilde{e}|\partial_t|\tilde{g}\rangle|$, or in other words the perturbation mix eigenstates very slowly, we can make the adiabatic approximation and we have

4 Perturbative solution

Now we go back to

$$\mathrm{i}\dot{c}_e = \Delta c_e + \frac{\Omega}{2}c_g, \quad \mathrm{i}\dot{c}_g = \frac{\Omega^*}{2}c_e. \tag{13}$$

In the first order perturbation, we can just ignore c_e in the second equation if we start from the ground state. We, therefore, have

$$c_e^{(1)}(T) = -\mathrm{i}\int_0^T \mathrm{d}t\, \frac{\Omega}{2}\mathrm{e}^{\mathrm{i}\Delta t}\,\mathrm{d}t = -\frac{\mathrm{i}}{2}\bar{\Omega}(\omega)|_{\omega=\Delta}, \tag{14}$$

where

$$\bar{\Omega}(\omega) = \int_{-\infty}^{\infty} \bar{\Omega}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\,, \tag{15}$$

and

$$\bar{\Omega}(t) = \begin{cases} \Omega(t), & 0 < t < T, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

We see that the Fourier transformation of the windowed Rabi frequency is an approximation of how strongly the two-level system will be disturbed.

5 Atomic state detection

Detecting what quantum state is an atom on has similar mathematical structure with single photon interferometers. Suppose we have a two-level system, of which we can measure directly σ^z, and we want to estimate, say, the θ angle. We have

$$\langle\sigma^z\rangle = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta, \tag{17}$$

and the standard variation is

$$\Delta\sigma^z = \sqrt{\langle(\sigma^z)^2\rangle - \langle\sigma^z\rangle^2} = \sqrt{1-\cos^2\theta} = \sin\theta. \tag{18}$$

The uncertainty of θ is therefore

$$\Delta\theta = \frac{\Delta\sigma^z}{\partial\langle\sigma^z\rangle/\partial\theta} = 1. \tag{19}$$

Therefore the measurement of θ with a single atom is extremely imprecise. If we can generate lots of atoms in the same state, we can measure the sum of σ^z of them, and we have

$$\langle\Sigma^z\rangle = N\cos\theta. \tag{20}$$

On the other hand,

$$\Delta\Sigma^z = \sqrt{N}\sin\theta, \tag{21}$$

and we have

$$\Delta\theta = \frac{1}{\sqrt{N}}, \tag{22}$$

which is the standard quantum error.

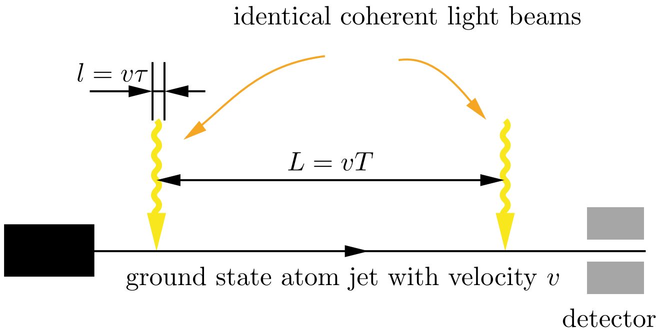

Figure 3: A atomic clock based on Ramsey's method of separated, oscillatory fields

6 Atomic clock

A light clock is discussed in the last problem of this homework, where we claimed that an atomic clock is much more precise. Now we can use the technologies mentioned in the previous few sections to build an atomic clock based on Ramsey's method of separated, oscillatory fields.

The device is illustrated as Figure 3 on page 4. We use the convention (10), and assume $\Delta \ll \Omega$, so $\tilde{\Omega} \approx \Omega$ and $\boldsymbol{\Omega} = \Omega \hat{\boldsymbol{x}}$. We carefully choose the width l of the two beams so that

$$\tilde{\Omega} \frac{l}{v} = \tilde{\Omega} \tau \approx \Omega \tau = \frac{\pi}{2}. \tag{23}$$

What happens on the Bloch vector of a single atom is shown in Figure 4 on page 5. An atom first goes through the first light beam and rotate around the x axis $90°$, then goes around the z axis and rotate $\tilde{\Omega} T$, and finally goes through the second light beam and rotate around the x axis $90°$ again. It is easy to find that if T is chose so that $\boldsymbol{n}(T+\tau) = -\hat{\boldsymbol{y}}$, then $\boldsymbol{n}(T+2\tau) = -\hat{\boldsymbol{z}}$, while if $\boldsymbol{n}(T+\tau) = \hat{\boldsymbol{y}}$, then $\boldsymbol{n}(T+2\tau) = \hat{\boldsymbol{z}}$. Therefore, we find that the probability to find the final state on $|e\rangle$ - which we denote as P_e - depends on T. It is straightforward to find that

$$\exp\left(-\mathrm{i}\frac{\Omega}{2}\sigma^x \tau\right) = \exp\left(-\frac{\pi}{4}\sigma^x\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\sigma^x,$$

and

$$\exp\left(-\mathrm{i}\frac{\Delta}{2}\sigma^z T\right) = \cos\left(\frac{T\Delta}{2}\right) - \mathrm{i}\sin\left(\frac{T\Delta}{2}\right)\sigma^z,$$

so the total transformation matrix is

$$\begin{aligned} U &= \exp\left(-\mathrm{i}\frac{\Omega}{2}\sigma^x \tau\right) \exp\left(-\mathrm{i}\frac{\Delta}{2}\sigma^z T\right) \exp\left(-\mathrm{i}\frac{\Omega}{2}\sigma^x \tau\right) \\ &= \begin{pmatrix} -\mathrm{i}\sin\frac{T\Delta}{2} & -\mathrm{i}\cos\frac{T\Delta}{2} \\ -\mathrm{i}\cos\frac{T\Delta}{2} & \mathrm{i}\sin\frac{T\Delta}{2} \end{pmatrix}, \end{aligned} \tag{24}$$

and therefore

$$|\psi\rangle (T+2\tau) = U|\psi\rangle(0) = \begin{pmatrix} -\mathrm{i}\sin\frac{T\Delta}{2} \\ -\mathrm{i}\cos\frac{T\Delta}{2} \end{pmatrix}, \tag{25}$$

and we have

$$P_e = \cos^2\frac{T\Delta}{2} = \frac{1}{2}(1+\cos(T\Delta)). \tag{26}$$

These oscillations are called **Ramsey fringes**.

Here is how to use the atomic clock: we need a laser device with adjustable frequencies, and a device which can generate a jet of atoms on the ground state whose velocity is known, and we can adjust the frequency of the laser to maximize P_e. When this is done, we know the frequency of the laser is the same as ω_{eg}. We known ω_{eg} can hardly be changed by interaction with the external environment, so after we maximize P_e, we extract the hard-to-change frequency ω_{eg} into ω, the inverse of which is a ruler of time.

It can be easily seen that the mathematical structure of the atomic clock is just the same as interferometers of light. Indeed, the two states - $|g\rangle$ and $|e\rangle$ - are just two modes of the atoms;

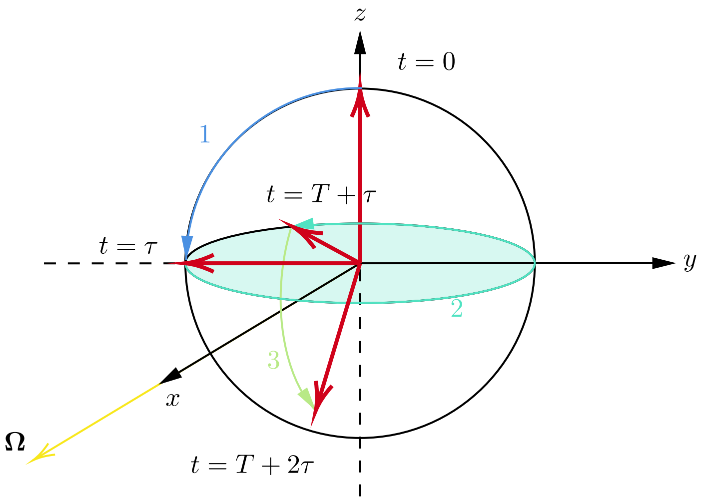

Figure 4: How the Bloch vector moves for atoms in Figure 3 on page 4

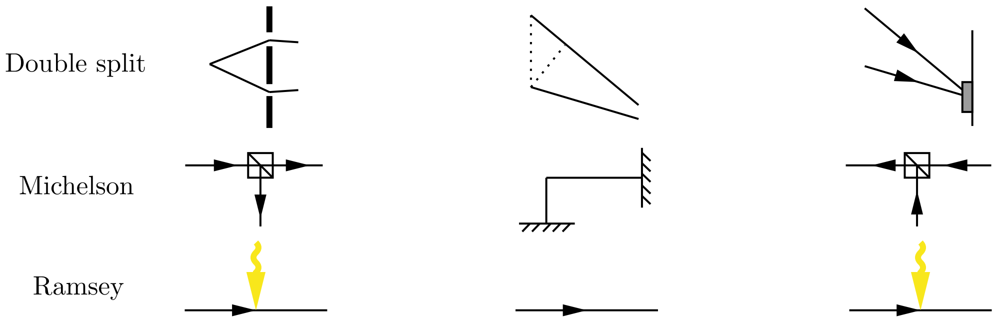

Figure 5: The common structure of a double slit experiment, a Michelson interferometer and an atomic clock

since there is coupling between the two states in the distance between two light beams, we can regard this period as two arms of a Michelson interferometer and regard the two light beams as two beam splitters. And finally, we find that both the Michelson interferometer and the atomic clock have the same mathematical structure with the most famous double slit experiment: all of them can be summarized into the following procedure:

1. One mode is split into the composition of two modes, and
2. The two modes then propagate separately, and when the propagation ends, there is a relative error between them, and
3. The two modes are again fused together.

The only differences are how these steps are implemented. Figure 5 on page 5 shows the corresponding parts of three interferometers. The double split and the Michelson interferometer use two beams with the same frequency but different propagating path, so we have a relative phase factor of the two beams, while the atomic clock uses two beams with the same propagating time and speed but different frequencies.