Field Theories for Anyons

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1 Abelian anyons and Chern-Simons theory

This section reviews what is covered in [1]. Briefly speaking, what we are going to do is to encode the phases of anyons into a gauge theory. We expect a gauge theory will do the job because we can always mimic the exchange phases of anyons by considering the anyons as particles created by a bosonic field coupled to a gauge field. The particles created by the bosonic field carry both a flux and a charge, and thus the Aharonov–Bohm effect can reproduce the correct self and mutual phases. What we are going to talk about in this section are Abelian anyons, so things will not be too hard. We claim after the exchange phases are correctly encoded into a gauge theory, we can obtained a *complete* low energy effective theory of a topological order, since we are usually not interested in the scattering of anyons, and the low energy dynamics of an anyon system has nothing more than the exchange phases.

Why we use a gauge field theory instead of something like a Coulomb interaction is discussed in Section 3.1 in [1]. In summary, it is because with a Coulomb-like Hamiltonian we often run into duplicate phase factors. Another approach is to construct anyon field operators. It is possible, but the resulting theories are usually highly complicated and technical.

Before discussing what gauge field theory we want to use we can first narrow down the candidates. We would better use a *topological quantum field theory (TQFT)*, which has no non-trivial particle dynamics. A type of well-studied TQFT is **Chern-Simons theories** - and they are what we are going to use. We consider, for example, the following theory where Schrödinger bosons are coupled to a single gauge field, and there is a Chern-Simons term for the gauge field itself:

$$\mathcal{L} = \mathrm{i}\bar{\phi} D_0 \phi + \frac{1}{2m}\bar{\phi}\boldsymbol{D}^2\phi + \underbrace{\frac{1}{4\theta}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda}_{\mathcal{L}_{\text{CS}}}, \tag{1}$$

where

$$D_\mu = \partial_\mu + \mathrm{i} a_\mu. \tag{2}$$

We have not proved that (1) is indeed gauge invariant yet. Under a local transformation

$$a_\mu \longrightarrow a_\mu + \partial_\mu h,$$

we have

$$\begin{aligned}
\frac{1}{4\theta}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \longrightarrow & \frac{1}{4\theta}\epsilon^{\mu\nu\lambda}(a_\mu + \partial_\mu h)\partial_\nu(a_\lambda + \partial_\lambda h) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu h \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu \partial_\lambda h + \epsilon^{\mu\nu\lambda}\partial_\mu h \partial_\nu \partial_\lambda h) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu (h \partial_\nu a_\lambda) - \epsilon^{\mu\nu\lambda} h \partial_\nu \partial_\mu a_\lambda) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu (h \partial_\nu a_\lambda)),
\end{aligned}$$

where the last two terms in the second line vanish because $\epsilon^{\mu\nu\lambda} a_\mu a_\nu = 0$, and this also how we get the last line. Note that though after the transformation the Lagrangian *density* changes, after integrating over the whole space the total Lagrangian does *not* change because the second term in the last line above is a boundary term. So we find that in (1), the Chern-Simons part is indeed gauge invariant. The coupling term in (1) transforms in the same way of QED, where the bosonic field undergoes a local $U(1)$ transformation. Thus, (1) is a $U(1)$ *Chern-Simons theory.*

Now we evaluate the EOM of (1). The Chern-Simons terms gives

$$\begin{aligned}
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_\sigma} - \frac{\partial \mathcal{L}}{\partial a_\sigma} &= \frac{1}{4\theta}\partial_\rho \epsilon^{\mu\rho\sigma} a_\mu - \frac{1}{4\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda \\
&= -\frac{1}{2\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda.
\end{aligned}$$

Another way to find the above equation is to use integration by parts to rephrase all terms containing $a_\mu \partial_\rho a_\sigma$ into $-\partial_\rho a_\mu a_\sigma$, so the first term in the LHS of the first line vanishes, and we get a 2 factor. In the Schrödinger field part, terms involving a_μ are

$$
\begin{aligned}
&-\bar{\phi}a_0\phi + \frac{\mathrm{i}}{2m}\bar{\phi}\partial_i(a_i\phi) + \frac{\mathrm{i}}{2m}\bar{\phi}a_i\partial_i\phi - \frac{1}{2m}a_i^2\bar{\phi}\phi \\
=& -\bar{\phi}a_0\phi - \frac{\mathrm{i}}{2m}(\partial_i\bar{\phi})a_i\phi + \frac{\mathrm{i}}{2m}\bar{\phi}a_i\partial_i\phi - \frac{1}{2m}a_i^2\bar{\phi}\phi,
\end{aligned}
$$

and we have

$$
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_0} - \frac{\partial \mathcal{L}}{\partial a_0} = \bar{\phi}\phi,
$$

and

$$
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_i} - \frac{\partial \mathcal{L}}{\partial a_i} = \frac{a_i}{m}\bar{\phi}\phi + \frac{\mathrm{i}}{2m}(\phi\partial_i\bar{\phi} - \bar{\phi}\partial_i\phi).
$$

We soon find that the RHSs of the above two equations are just conservation flows of a Schrödinger field coupled to a gauge field, as is the case in QED. So the EOM of a_μ is just

$$
j^\sigma - \frac{1}{2\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda = 0. \tag{3}
$$

Specifically, the temporal part of the EOM is

$$
\epsilon^{ij}\partial_i a_j = 2\theta\rho, \quad \rho = \bar{\phi}\phi. \tag{4}
$$

References

[1] Susanne F. Viefers. Field theory of anyons and the fractional quantum hall effect. https://inis.iaea.org/collection/NCLCollectionStore/_Public/30/061/30061237.pdf, 1997.