Quantum Optics by Prof. Saijun Wu

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December 30, 2021

1 A brief overview of cavity QED

For a system where RWA and RWT work but photons emitted by the atom does not necessarily fly to infinity, the Hamiltonian is

$$H_{\text{eff}} = \hbar(\Delta_c - \mathrm{i}\kappa/2)a^\dagger a + \hbar(\Delta - \mathrm{i}\Gamma/2)\,|e\rangle\langle e| + \hbar(g_{ac}a + \Omega/2)\,|e\rangle\langle g| + \text{h.c.}, \tag{1}$$

where

$$g_{ac} = \frac{\boldsymbol{\mathcal{E}}_c \cdot \boldsymbol{d}_{eg}}{\hbar}, \quad C_1 = \sqrt{\Gamma}\,|g\rangle\langle e|\,, \quad C_2 = \sqrt{\kappa}a. \tag{2}$$

The difference between (1) and previous models is that neither the optical field nor the atom can be integrated out, leaving a Markovian evolution. (1) is a famous model in **cavity QED**, where a *local* optical field interacts with a localized atom. This is called *QED* because we are still studying both electrons and photons. When ignoring all decaying, we get

$$H_{\text{eff}} = \hbar\Delta_c a^\dagger a + \hbar\Delta\,|e\rangle\langle e| + \hbar(g_{ac}a + \Omega/2)\,|e\rangle\langle g| + \text{h.c.}. \tag{3}$$

This is **Jaynes–Cummings model**. Jaynes–Cummings model is much harder to implement in laboratories because no one can create a cavity without any decaying, but anyway, someone has implemented it.

Now we discuss when we should calculate with the complete (1), instead of Markovian approximations. Intuitively speaking, we need very, very strong coupling between the atom and the optical field in the cavity so that if the atom radiates a photon in the cavity, before it runs away, it already takes part in another process. This is the **strong coupling** condition. Note that

$$g \sim \frac{\boldsymbol{\mathcal{E}} \cdot \boldsymbol{d}_{eg}}{\hbar}, \quad \Gamma \sim \frac{\omega^3 \boldsymbol{d}_{eg}^2}{3\pi\epsilon_0 c^3},$$

what we need to do is to increase the intensity at the position of the atom, A schematic illustration of this is Figure 1 on page 1.

2 Jaynes–Cummings model

Now we discuss the Jaynes–Cummings model. From the RWA Hamiltonian

$$H = \hbar g \mathrm{e}^{-\mathrm{i}\Delta t} a\,|e\rangle\langle g| + \text{h.c.}, \tag{4}$$

after RWT, we have

$$H_{\text{JC}} = \hbar\Delta_c a^\dagger a + (\hbar g a\,|e\rangle\langle g| + \text{h.c.}). \tag{5}$$

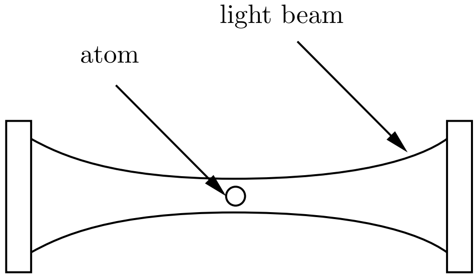

Figure 1: A cavity which focuses the light beam, and the atom is placed at the brightest position

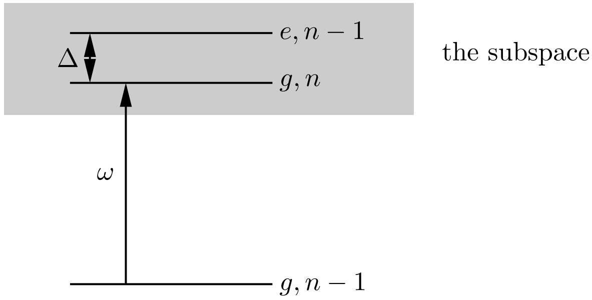

Figure 2: The energy level diagram of (6)

If we just consider the subspace spanned by $|g, n\rangle$ and $|e, n-1\rangle$, the Hamiltonian is

$$H = \begin{pmatrix} n\Delta & \sqrt{n}g \\ \sqrt{n}g & (n-1)\Delta \end{pmatrix}. \tag{6}$$

We find two *dressed* states, the energies of them being

$$\hbar\tilde{\omega}_{g,n} = \left(n - \frac{1}{2}\right)\hbar\Delta - \frac{1}{2}(\sqrt{\Delta^2 + 4ng^2} - \Delta), \tag{7}$$

and

$$\hbar\tilde{\omega}_{e,n-1} = \left(n - \frac{1}{2}\right)\hbar\Delta + \frac{1}{2}(\sqrt{\Delta^2 + 4ng^2} - \Delta). \tag{8}$$

Now the Jaynes–Cummings model where the optical field state is a single-photon state coupled with only one atom - or in other words, **single-photon Rabi oscillation** - appears to be solved completely. Things are not that simple, since the parameters g and Δ sometimes are time-dependent.

When the input state involves a non-Fock state, the time evolution can also be non-trivial, since we no longer work in Figure 2 on page 2. Suppose the input state is

$$|\psi(0)\rangle = |e, \alpha\rangle = |e\rangle \otimes \sum_n \mathrm{e}^{-|\alpha|/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{9}$$

then we have We can observe how the

This is a vivid demonstration of when the semi-classical approximation of light-atom coupling works and does not work. We can see that the time scale of the semi-classical regions is $1/g$. When $t \gg 1/g$, the state of the optical field becomes a cat state. In ordinary systems, g is small, and therefore $1/g$ is large, and long before a cat state is formed, spontaneous radiation or leaking or other quantum jumps happens. However, in Jaynes–Cummings model, there is no quantum jump and g is strong, and we can just see a cat state occurring.

3 Heisenberg picture of a leaky cavity

Now we consider the Heisenberg picture of a leaky cavity:

$$H_{\text{eff}} = \hbar(\omega - \mathrm{i}\kappa/2)a^\dagger a, \quad \kappa = \frac{2|t|^2 c}{L} \tag{10}$$

We already know that the master equation has a term $\rho H_{\text{eff}} - H_{\text{eff}}^\dagger \rho$, the † symbol coming from the non-Hermitian property of (10). Now we naively consider the Heisenberg picture of (10), *without* considering the non-Hermitian property. We have

$$\mathrm{i}\dot{a} = [a, H] = (\omega_c - \mathrm{i}\kappa/2)a,$$

and similarly

$$\mathrm{i}\dot{a}^\dagger = (-\omega_c - \mathrm{i}\kappa/2)a^\dagger.$$

These equations correctly show the damping behavior of the system, but now the commutation relation is

$$[a, a^\dagger] = \mathrm{e}^{-\kappa t},$$

which is obviously wrong, since we expect the creation and annihilation operators to always satisfy $[a, a^\dagger] = 1$.

It does not mean that a Heisenberg picture of (10) is impossible. To see clearly how, let us go back to the original Hamiltonian

$$H = \underbrace{\hbar\omega_c a^\dagger a}_{H_S} + \underbrace{\sum_k \hbar\omega_k a_k^\dagger a_k}_{H_R} + \underbrace{\sum_k \hbar g_k a a_k^\dagger + \text{h.c.}}_{H_{SR}}\,. \tag{11}$$

From the Heisenberg EOMs we have

$$\mathrm{i}\dot{a} = \sum_k g_k \mathrm{e}^{-\mathrm{i}\Delta_k t} a_k, \quad \mathrm{i}\dot{a}_k = g_k^* \mathrm{e}^{\mathrm{i}\Delta_k t} a,$$

and therefore

$$\mathrm{i}a = -\mathrm{i}\int_0^t \mathrm{d}\tau\, a(\tau) \sum_k |g|_k^2 \mathrm{e}^{\mathrm{i}\Delta_k (t-\tau)} + \sum_k g_k \mathrm{e}^{\mathrm{i}\Delta_k t} a(k).$$

Now we coarse-grain the time. When the time scale we are interested is much larger than $1/\Delta\omega$, we have

$$\mathrm{i}\dot{\tilde{a}} = (\Delta_c - \mathrm{i}\kappa/2)\tilde{a} + \tilde{F}(t), \tag{12}$$

and it can be verified that

$$\langle F(t) \rangle = 0, \tag{13}$$

and

$$\begin{aligned} \left\langle F(t) F^\dagger(t') \right\rangle &= \sum_{k,k'} \mathrm{e}^{\mathrm{i}\Delta_k t - \mathrm{i}\Delta_{k'} t} \left\langle a_k a_{k'}^\dagger \right\rangle \\ &= \sum_k |g_k|^2 \mathrm{e}^{\mathrm{i}\Delta_k (t-t')} = \kappa \delta(t - t'). \end{aligned} \tag{14}$$

Therefore, we can see (12) is actually a *quantum* Langevin equation. This is the counterpart of the wave function formalism in the Schrodinger picture. By a picture transformation we can eliminate the Δ_c term, and in this case we have

$$a(t) = a(0)\mathrm{e}^{-\kappa t/2} - \mathrm{i}\int_0^t \tilde{F}(\tau) \mathrm{e}^{\kappa(\tau - t)/2}\, \mathrm{d}\tau\,, \tag{15}$$

and since the fluctuation term has zero expectation, we have

$$\langle a(t) \rangle = \mathrm{e}^{-\kappa t/2} \langle a(0) \rangle\,. \tag{16}$$

4 Experimental realization of cavity QED

To realize a cavity QED model, we need large g and small κ and Γ. For a small Γ, we need to excite atoms to a Rydberg state. For We therefore find ordinary optical cavities are not handy. We know superconductors are impenetrable for electromagnetic waves, and therefore they can be used to create a cavity with 100 % reflection.

With quantum non-deconstructive measurement, we can even *measure* the photon number in the cavity and how quantum jump happens. These quantum jumps are not caused by measurement. Instead, they are caused by leakage of the cavity.