Quantum Optics, Homework 4

Jinyuan Wu

January 9, 2022

Random wave function Numerically simulate a two-level atom under an external laser using the random wave function method.

Solution The code can be found with this document. The results are Figure 1 on page 2. The "mean" curves are the average value of 1000 runs.

Discussion The code in with this document has almost nothing more than numerical solution of the Schrödinger equation. What we should note is that the numerical format

$$|\psi(t+\Delta t)\rangle = \left(1 - \frac{1}{\mathrm{i}\hbar} H_{\mathrm{eff}} \Delta t\right) |\psi(t)\rangle \tag{1}$$

may not be stable, and to make sure the error is acceptable, we need to use a very small Δt.

Calculating a figure in Figure 1 on page 2 takes about 30 seconds on my computer. Considering an atom in an external optical field where spontaneous radiation is important is actually a quite complex system, we should say the random wave function method is shockingly fast. The reason why it is fast is because we manage to not calculate the (often large) density matrix directly, but encode its classical fluctuation into different random wave function trajectories.

Stochastic wave function of a Λ system Figure 2 on page 2 is a three-level Λ system. (a) Write down the effective Hamiltonian and quantum jump operators for Figure 2 on page 2. (b) Suppose $|\psi_{\mathrm{s}}(t=0)\rangle = |\mathrm{g}\rangle$. Describe how the wave function evolves using pseudocode. (c) Consider a case in which there is no quantum jump in $0 < t < t_0$. Find the time evolution of the wave function and the scattering rate

$$\gamma_1 = \langle\psi_{\mathrm{s}}|C_1^\dagger C_1|\psi_{\mathrm{s}}\rangle\,, \quad \gamma_2 = \langle\psi_{\mathrm{s}}|C_2^\dagger C_2|\psi_{\mathrm{s}}\rangle\,. \tag{2}$$

(d) Plot the time evolution of γ_1 and γ_2 under the circumstance of (i) $\Delta = 0, \Omega \gg \Gamma_1 \gg \Gamma_2$; (ii) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$; (iii) $\Delta = 0, \Omega \ll \Gamma_1, \Gamma_2$.

Solution

(a) The effective Hamiltonian is

$$\begin{aligned} H_{\mathrm{eff}} &= -\hbar\Delta\,|\mathrm{e}\rangle\langle\mathrm{e}| + \left(\frac{1}{2}\hbar\Omega\,|\mathrm{e}\rangle\langle\mathrm{g}| + \mathrm{h.c.}\right) - \frac{\mathrm{i}\hbar}{2}(C_1^\dagger C_1 + C_2^\dagger C_2) \\ &= -\hbar(\Delta + \mathrm{i}\Gamma/2)\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hbar(\Omega\,|\mathrm{e}\rangle\langle\mathrm{g}| + \mathrm{h.c.})/2, \end{aligned} \tag{3}$$

where the quantum jump operators are

$$C_1 = \sqrt{\Gamma_1}\,|\mathrm{a}\rangle\langle\mathrm{e}|\,, \quad C_2 = \sqrt{\Gamma_2}\,|\mathrm{g}\rangle\langle\mathrm{e}|\,, \tag{4}$$

and

$$\Gamma = \Gamma_1 + \Gamma_2. \tag{5}$$

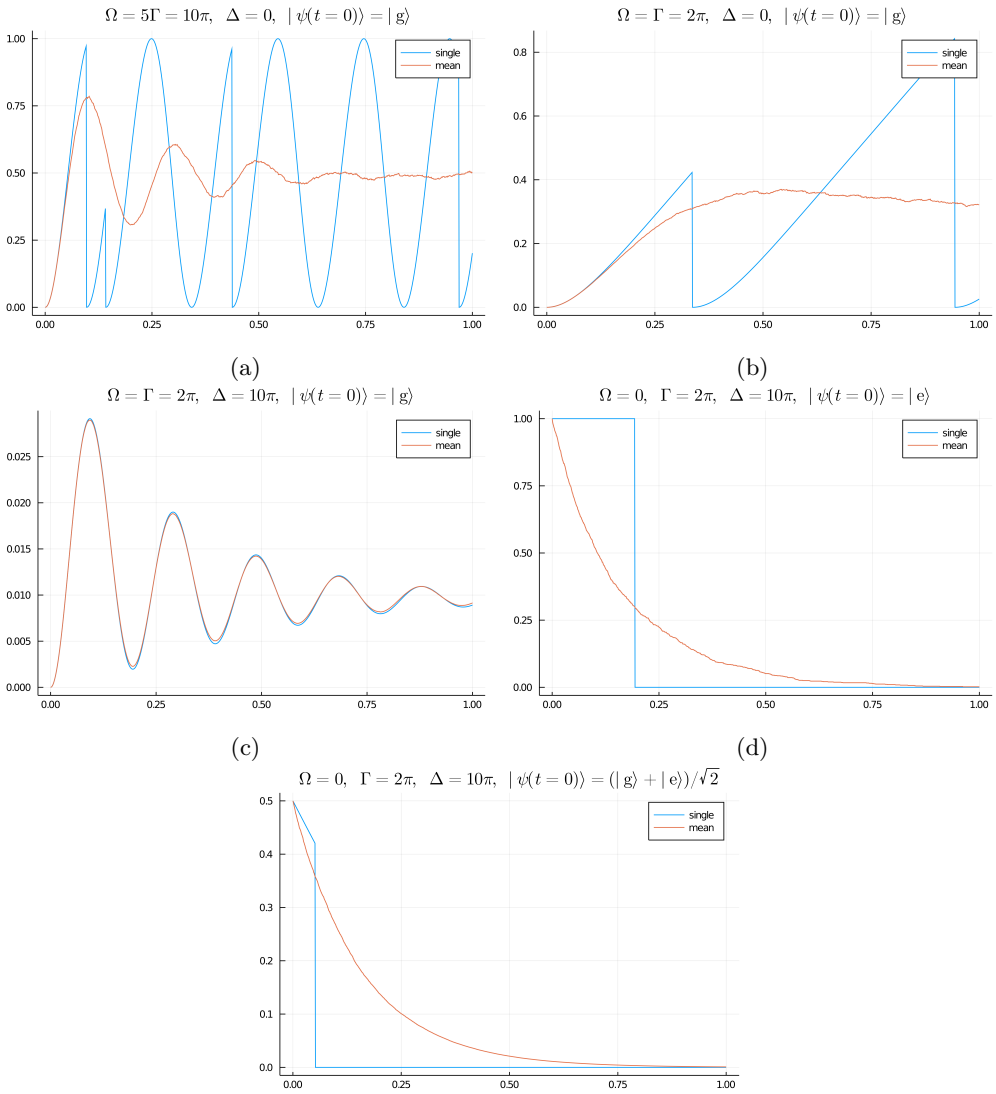


Figure 1: Numerical simulated P_{e}.

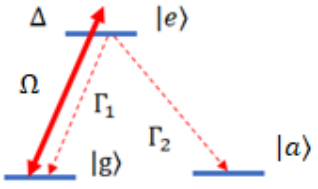


Figure 2: A three-level Λ system

(b) The time evolution can be described using the following algorithm.

input : Time step Δt, maximal time t_0
Initialize an array $\{|\psi_\mathrm{s}(t)\rangle\}_{t=n\Delta t}$ of wave functions with $t_0/\Delta t$ elements
for $t \in 0 : \Delta t : t_0$
  Pick up a uniformly distributed random number x between 0 and 1
  $P_\mathrm{g} \leftarrow \Delta t \langle\psi_\mathrm{s}(t)|C_1^\dagger C_1|\psi_\mathrm{s}(t)\rangle$
  $P_\mathrm{a} \leftarrow \Delta t \langle\psi_\mathrm{s}(t)|C_2^\dagger C_2|\psi_\mathrm{s}(t)\rangle$
  `// jumping to` $|\mathrm{g}\rangle$
  if $0 < x < P_\mathrm{g}$
    $|\psi_\mathrm{s}(t+\Delta t)\rangle \leftarrow$ normalized $C_1|\psi_\mathrm{s}(t)\rangle$
  `// jumping to` $|\mathrm{a}\rangle$
  elseif $P_\mathrm{g} < x < P_\mathrm{g} + P_\mathrm{a}$
    $|\psi_\mathrm{s}(t+\Delta t)\rangle \leftarrow$ normalized $C_2|\psi_\mathrm{s}(t)\rangle$
  `// evolution according to the effective Hamiltonian`
  else
    $|\psi_\mathrm{s}(t+\Delta t)\rangle \leftarrow$ normalized $|\psi_\mathrm{s}(t)\rangle + \frac{\Delta t}{\mathrm{i}\hbar} H_\mathrm{eff}|\psi_\mathrm{s}(t)\rangle$
  end
end

(c) The wave function in this case evolves purely according to H_eff. Since Schrödinger equation is linear, we can leave the normalization to the end of our calculation. Note that (3) actually does not contain $|\mathrm{a}\rangle$ explicitly, nor does the initial state $|\mathrm{g}\rangle$. Therefore we can work in the two-level system spanned by $|\mathrm{e}\rangle$ and $|\mathrm{g}\rangle$. The effective Hamiltonian is

$$H_\mathrm{eff} = \hbar \begin{pmatrix} 0 & \Omega^*/2 \\ \Omega/2 & -(\Delta + \mathrm{i}\Gamma/2) \end{pmatrix}, \tag{6}$$

where we let $|\mathrm{g}\rangle$ be the first component and $|\mathrm{e}\rangle$ the second. We have the decomposition

$$H_\mathrm{eff} = -\frac{\hbar}{2}(\Delta + \mathrm{i}\Gamma) + \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\Omega} = (\Omega_\mathrm{r}, \Omega_\mathrm{i}, \Delta + \mathrm{i}\Gamma/2). \tag{7}$$

Note here we cannot "shift the energy zero point" to reshape the Hamiltonian into $\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$, because the value damping rate has physical meaning. Applying (7) on $|\mathrm{g}\rangle$, we have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}H_\mathrm{eff}t/\hbar}|\mathrm{g}\rangle &= \mathrm{e}^{\mathrm{i}t(\Delta+\mathrm{i}\Gamma/2)/2}\mathrm{e}^{-\mathrm{i}t\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}/2}|\mathrm{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\sigma^0\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \frac{\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\mathrm{g}\rangle \\
&= \mathrm{e}^{-\Gamma t/4}\mathrm{e}^{\mathrm{i}\Delta t/2}\left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2}|\mathrm{g}\rangle - \left(\frac{\Omega_\mathrm{r}}{|\boldsymbol{\Omega}|}|\mathrm{e}\rangle + \frac{\mathrm{i}\Omega_\mathrm{i}}{|\boldsymbol{\Omega}|}|\mathrm{e}\rangle + \frac{\Delta+\mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}|\mathrm{g}\rangle\right)\mathrm{i}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right),
\end{aligned}$$

where

$$|\boldsymbol{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2 - \Gamma^2/4 + \mathrm{i}\Delta\Gamma}. \tag{8}$$

Note

Note that here $|\boldsymbol{n}|$ is defined as $\sqrt{\boldsymbol{n}^\top\boldsymbol{n}}$ instead of $\sqrt{\boldsymbol{n}^\dagger\boldsymbol{n}}$, because to make

$$\mathrm{e}^{\mathrm{i}\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}} = \sigma^0\cos\alpha + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\alpha$$

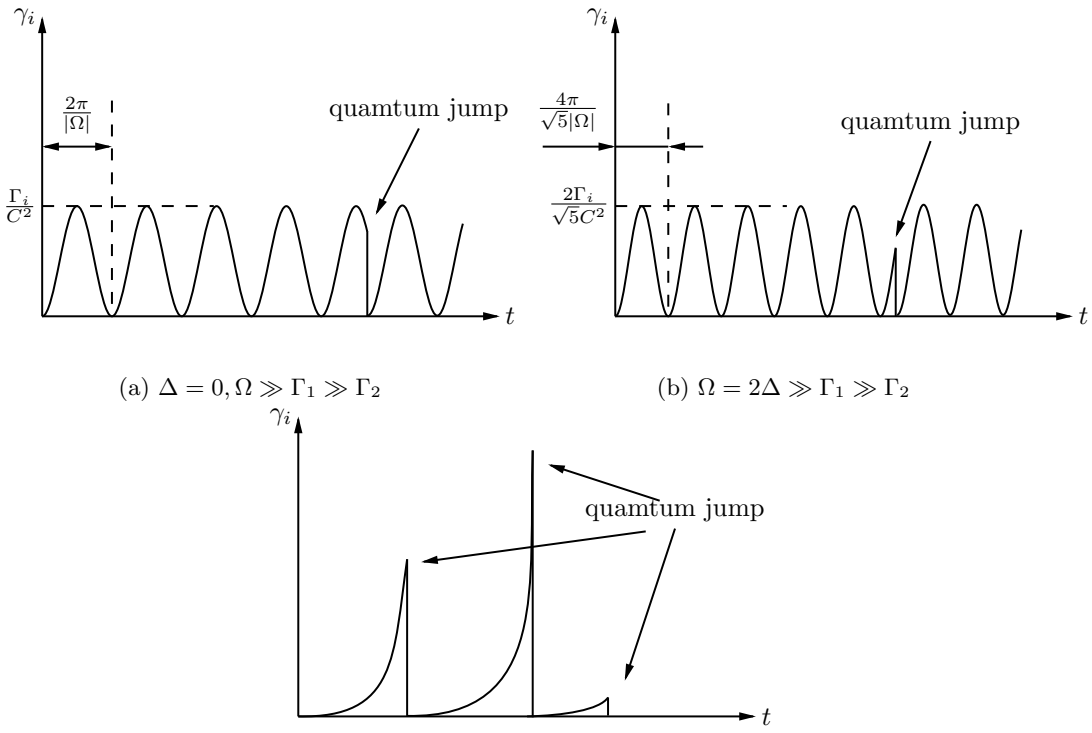
hold, it is required that

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \sigma^0,$$

which is equivalent to $\boldsymbol{n}\cdot\boldsymbol{n} = 1$, considering $\{\sigma^i, \sigma^j\} = 0$ when $i \neq j$. What is important here, therefore, is $\boldsymbol{n}\cdot\boldsymbol{n}$.

Therefore we have (we have omitted the complex factors, since they will be canceled by normalization anyway)

$$|\psi_\mathrm{s}(t)\rangle = \frac{1}{C}\left(\cos\frac{|\boldsymbol{\Omega}|\,t}{2} - \mathrm{i}\frac{\Delta+\mathrm{i}\Gamma/2}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}\right)|\mathrm{g}\rangle - \frac{\mathrm{i}\Omega}{|\boldsymbol{\Omega}|}\sin\frac{|\boldsymbol{\Omega}|\,t}{2}|\mathrm{e}\rangle, \tag{9}$$



(a) $\Delta = 0, \Omega \gg \Gamma_1 \gg \Gamma_2$

(b) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$

(c) $\Omega = 2\Delta \gg \Gamma_1 \gg \Gamma_2$

Figure 3: Time evolution of γ_1 and γ_2

the normalization constant being

$$C = \sqrt{\left|\cos\frac{|\mathbf{\Omega}|\,t}{2} - \mathrm{i}\frac{\Delta + \mathrm{i}\Gamma/2}{|\mathbf{\Omega}|}\sin\frac{|\mathbf{\Omega}|\,t}{2}\right|^2 + \frac{|\Omega|^2}{|\mathbf{\Omega}|^2}\sin^2\frac{|\mathbf{\Omega}|\,t}{2}}. \tag{10}$$

Since

$$C_1^\dagger C_1 = \Gamma_1 \left|\mathrm{e}\right\rangle\!\left\langle\mathrm{e}\right|, \quad C_2^\dagger C_2 = \Gamma_2 \left|\mathrm{e}\right\rangle\!\left\langle\mathrm{e}\right|,$$

it is then straightforward that

$$\gamma_1 = \frac{\Gamma_1}{C^2}\frac{\Omega^2}{|\mathbf{\Omega}|^2}\left|\sin\frac{|\mathbf{\Omega}|\,t}{2}\right|^2, \quad \gamma_2 = \frac{\Gamma_2}{C^2}\frac{\Omega^2}{|\mathbf{\Omega}|^2}\left|\sin\frac{|\mathbf{\Omega}|\,t}{2}\right|^2. \tag{11}$$

Note that in these equations $|\mathbf{\Omega}|^2$ means the norm, i.e. $|\mathbf{\Omega}| \cdot |\mathbf{\Omega}|^*$.

The scattering rates are calculated under the assumption that no quantum jump happened before, so what they mean are actually the probabilities that "no quantum jump happened before, and a quantum jump will happen in the next second".

(d) Note that γ_1 and γ_2 only differ with a factor, so their plots only differ in scaling.

(i) In this case $|\mathbf{\Omega}| \approx |\Omega|$, so γ_1 and γ_2 oscillate as $A\sin^2|\Omega|\,t/2$ until a quantum jump happens, and both of them drop to zero, and then another oscillation starts. See Figure 3a on page 4.

(ii) In this case

$$|\mathbf{\Omega}| = \sqrt{|\Omega|^2 + \Delta^2 - \Gamma^2/4 + \mathrm{i}\Delta\Gamma} \approx \sqrt{|\Omega|^2 + \frac{1}{4}|\Omega|^2} = \frac{\sqrt{5}}{2}|\Omega|,$$

so the curve of γ is similar to (i), but the oscillating period is now $4\pi/\sqrt{5}|\Omega|$. See Figure 3b on page 4.

(iii) In this case

$$|\mathbf{\Omega}| \approx \sqrt{-\Gamma^2/4} = \frac{\mathrm{i}}{2}\Gamma,$$

and we have

$$\begin{aligned}\gamma_i &= \frac{\Gamma_i}{C^2}\frac{\Omega^2}{\Gamma^2/4}\left|\sin\frac{\mathrm{i}\Gamma}{2}t\right|^2 \\ &= \frac{\Gamma_i}{C^2}\frac{\Omega^2}{\Gamma^2}(\mathrm{e}^{-\Gamma t/4} - \mathrm{e}^{\Gamma t/4})^2.\end{aligned}$$

Its prefactor is small considering that Ω/Γ is small, but it increases exponentially. Before it grows too large, a quantum jump will happen. See Figure 3c on page 4.

Discussion

(b) From the algorithm we can see clearly why a system described by a random wave function theory is described by both the effective Hamiltonian and the quantum jump operators. If we only have H_{eff}, we are unable to tell whether the system jumps to $|\mathrm{g}\rangle$ or $|\mathrm{a}\rangle$.

(c) (8) can also be confirmed by evaluating the eigenvalues of H_{eff}. Note that we need $\tilde{\Delta}^2 = (\Delta - \mathrm{i}\Gamma/2)^2$ under a square root instead of $|\tilde{\Delta}|^2$, because when Γ is strong, the system is overdamped, and $|\mathbf{\Omega}|$ *must be imaginary*, or in other words we need a negative number under the square root. On the other hand, $|\tilde{\Delta}|^2$ is never negative.

Note

For a non-Hermitian Hamiltonian, we need be cautious about the eigen states. They are not necessarily orthogonal to each other.

(d) Figure 3 on page 4 is not a complete description of the behavior of the atom. After a sufficiently long time, the atom is almost certain to be on $|a\rangle$, because the atom can arrive there, but since it has no coupling with the external field, the atom cannot jump out of it. We call states like $|\mathrm{a}\rangle$ a **dark state**.

It should also be noticed that in Figure 3 on page 4 quantum jumps usually occurs when γ is large.

A very strange fact in Figure 3c on page 4 is that the larger Γ is (or in other words, the shorter the lifetime of a state is), the smaller the scattering rate γ is. This is an example of **quantum Zeno effect**. Spontaneous radiation is a way to *observe* the atom, and we can see the easier we are able to observe the atom, the more impossible an actual quantum jump happens. This relation between Γ and γ is not that important, though, because it is almost impossible to keep Ω constant while changing Γ.

Cesium atom Consider a simplified cesium level diagram in Figure 4 on page 6 (We ignore Zeeman and hyperfine structure). The laser beams at 895 nm, 761 nm and 794 nm induce coupling between the ground state $6\ \mathrm{S}_{1/2} - 6\mathrm{P}_{1/2}$ (with Rabi freq Ω_1) , $6\mathrm{P}_{1/2} - 8\ \mathrm{S}_{1/2}$ (with Rabi freq Ω_2), and $8\ \mathrm{S}_{1/2} - 6\mathrm{P}_{3/2}$ (with Rabi freq Ω_3) respectively. The 1-photon detuning Δ_a, 2−photon detuning Δ_b, and 3-photon detuning Δ_c are sketched as in the diagram. We consider the case where all these detunings are at GHz level or smaller, which are tiny comparing with the optical frequencies of the lasers.

Task 1: With $|g\rangle = \left|6S_{1/2}\right\rangle, |a\rangle = \left|6P_{1/2}\right\rangle; |b\rangle = \left|8S_{1/2}\right\rangle; |c\rangle = \left|6P_{3/2}\right\rangle$, invent your own additional notations to write down the time-dependent Hamiltonian for the cesium atom in the laser fields, that explicitly have the optical frequencies of the lasers.

Task 2: Write down a time-independent Hamiltonian.

Task 3: Consider radiative life time for $6P_{1/2}, 8S_{1/2}$ and $6P_{3/2}$ are given by $\tau_a = 1/\Gamma_a, \tau_b = 1/\Gamma_b$, and $\tau_c = 1/\Gamma_c$. Write down the effective non-Hermitian Hamiltonian H_{eff} that includes an anti-Hermitian part to account for the spontaneous emissions.

Task 4: Consider the internal state of the cesium has a wavefunction $|\psi_S(t)\rangle = c_g|g\rangle + c_a|a\rangle + c_b|b\rangle + c_c|c\rangle$. Consider the weak perturbation limit (ie laser intensities are small enough

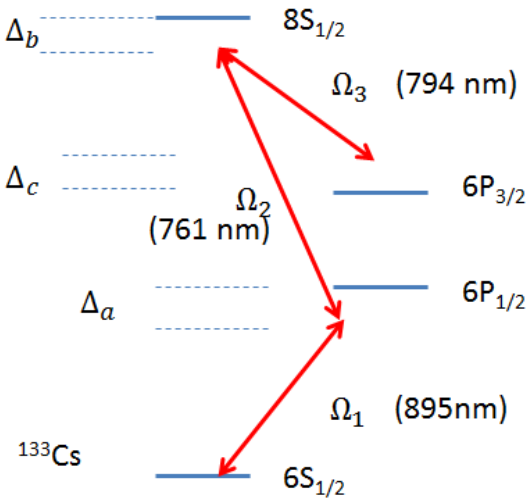


Figure 4: Cesium atom subjected to three laser fields for nearly resonant excitations

that atoms are barely excited), write down the differential equations for the coefficients and approximately solve for $|\psi_{\rm S}(t)\rangle \approx |\tilde{\psi}_{\rm S}\rangle$ with $\mathrm{H_{eff}}|\tilde{\psi}_S\rangle = 0$.

Task 5: Perturbative calculate the 3-photon scattering rate given by $\gamma_3 = |c_c|^2\,\Gamma$, by assuming $|\psi(t)\rangle$ quickly relax to eigenstate of the effective Hamiltonian with $c_g \approx 1$.

Task 6: Consider strong 761 nm and 795 nm laser, calculate the atomic polaraizability $\alpha\,(\Omega_2, \Omega_3)$ for the 852 nm laser excitation by evaluating $\langle \mathrm{d}\rangle = \left\langle \widetilde{\Psi_S}\,|d_{ag}|\,a\right\rangle \left\langle g\mid \widetilde{\psi_S}\right\rangle +$ c.c., and express it as $\alpha \mathrm{E}_1 +$ c.c..

Task 7: Discuss the validity of your method of calculating γ_3 and $\alpha\,(\Omega_2, \Omega_3)$ using an ill-defined wavefunction (since its norm cannot be unity) and an effective nonHermitian Hamitonian, instead of using a density matrix and master equations (or the full Monte-Carlo wavefunction method). Your discussion may involve Ω, Δ, Γ and the total time of observation T.

Solution

(1) The time-dependent Hamiltonian is

$$H = H_0 + H_{\rm dipole}, \tag{12}$$

where

$$H_0 = \hbar\omega_{\rm g}\,|{\rm g}\rangle\langle{\rm g}| + \hbar\omega_{\rm a}\,|{\rm a}\rangle\langle{\rm a}| + \hbar\omega_{\rm b}\,|{\rm b}\rangle\langle{\rm b}| + \hbar\omega_{\rm c}\,|{\rm c}\rangle\langle{\rm c}|\,, \tag{13}$$

and

$$\begin{aligned} H_{\rm dipole} = &- \boldsymbol{d}_{\rm ag}\cdot(\boldsymbol{E}_1 {\rm e}^{-{\rm i}\omega_1 t} + \boldsymbol{E}_1^* {\rm e}^{{\rm i}\omega_1 t})\,|{\rm a}\rangle\langle{\rm g}| - \boldsymbol{d}_{\rm ba}\cdot(\boldsymbol{E}_2 {\rm e}^{-{\rm i}\omega_2 t} + \boldsymbol{E}_2^* {\rm e}^{{\rm i}\omega_2 t})\,|{\rm b}\rangle\langle{\rm a}| \\ &- \boldsymbol{d}_{\rm bc}\cdot(\boldsymbol{E}_3 {\rm e}^{-{\rm i}\omega_3 t} + \boldsymbol{E}_3^* {\rm e}^{{\rm i}\omega_3 t})\,|{\rm b}\rangle\langle{\rm c}| + {\rm h.c.}, \end{aligned} \tag{14}$$

where we denote the external electric fields as

$$\boldsymbol{E}_i = \boldsymbol{E}_{i0}{\rm e}^{{\rm i}\omega_i t} + \boldsymbol{E}_{i0}^* {\rm e}^{-{\rm i}\omega_i t}, \quad i = 1, 2, 3. \tag{15}$$

The laser frequencies satisfy

$$\omega_1 + \Delta_{\rm a} = \omega_{\rm a} - \omega_{\rm g}, \quad \omega_2 + \Delta_{\rm b} - \Delta_{\rm a} = \omega_{\rm b} - \omega_{\rm a}, \quad \omega_3 + \Delta_{\rm b} + \Delta_{\rm c} = \omega_{\rm b} - \omega_{\rm c}, \tag{16}$$

from which we find

$$\omega_1 = \omega_{\rm a} - \omega_{\rm g} - \Delta_{\rm a}, \quad \omega_2 = \omega_{\rm b} - \omega_{\rm a} + \Delta_{\rm a} - \Delta_{\rm b}, \quad \omega_3 = \omega_{\rm b} - \omega_{\rm c} - \Delta_{\rm b} - \Delta_{\rm c}. \tag{17}$$

(2) We switch to the interaction picture, using H_0 as the free Hamiltonian, and we have

$$\begin{aligned} H = H_{\rm dipole,\ int} = &- \boldsymbol{d}_{\rm ag}\cdot(\boldsymbol{E}_1 {\rm e}^{-{\rm i}\omega_1 t} + \boldsymbol{E}_1^* {\rm e}^{{\rm i}\omega_1 t})\,|{\rm a}\rangle\langle{\rm g}|\,{\rm e}^{{\rm i}(\omega_{\rm a}-\omega_{\rm g})t} \\ &- \boldsymbol{d}_{\rm ba}\cdot(\boldsymbol{E}_2 {\rm e}^{-{\rm i}\omega_2 t} + \boldsymbol{E}_2^* {\rm e}^{{\rm i}\omega_2 t})\,|{\rm b}\rangle\langle{\rm a}|\,{\rm e}^{{\rm i}(\omega_{\rm b}-\omega_{\rm a})t} \\ &- \boldsymbol{d}_{\rm bc}\cdot(\boldsymbol{E}_3 {\rm e}^{-{\rm i}\omega_3 t} + \boldsymbol{E}_3^* {\rm e}^{{\rm i}\omega_3 t})\,|{\rm b}\rangle\langle{\rm c}|\,{\rm e}^{{\rm i}(\omega_{\rm b}-\omega_{\rm c})t} + {\rm h.c.}. \end{aligned} \tag{18}$$

We make the rotating wave approximation, i.e. omitting all terms that vibrate much faster than all detunings, and get

$$\begin{aligned} H =& \frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{a}-\omega_\mathrm{g}-\omega_1)t} + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{b}-\omega_\mathrm{a}-\omega_2)t} \\ &+ \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{b}-\omega_\mathrm{c}-\omega_3)t} + \mathrm{h.c.}, \end{aligned} \tag{19}$$

where we define

$$\frac{1}{2}\hbar\Omega_1 = -\boldsymbol{d}_{\mathrm{ag}} \cdot \boldsymbol{E}_1, \quad \frac{1}{2}\hbar\Omega_2 = -\boldsymbol{d}_{\mathrm{ba}} \cdot \boldsymbol{E}_2, \quad \frac{1}{2}\hbar\Omega_3 = -\boldsymbol{d}_{\mathrm{bc}} \cdot \boldsymbol{E}_3. \tag{20}$$

Warning

If we define the electric field as

$$\boldsymbol{E}_i = \frac{1}{2}(\boldsymbol{E}_{i0}\mathrm{e}^{\mathrm{i}\omega_i t} + \boldsymbol{E}_{i0}^*\mathrm{e}^{-\mathrm{i}\omega_i t}) = |\boldsymbol{E}_i| \cos(\omega_i t + \varphi_i), \quad i = 1, 2, 3, \tag{21}$$

then there will be no $1/2$ factor in the definition of Ω_i's. However, later we will evaluate $\langle\boldsymbol{d}\rangle$, which has the form of somethinge$^{-\mathrm{i}\omega t}$ + h.c., and if we insist on (21), an additional and easy-to-forget factor 2 must be added when we evaluate $\alpha = d/E$.

There are three phase factors, and we have four states, so it is possible to use a rotating wave transformation to eliminate them all. By

$$\begin{aligned} U\left|\mathrm{a}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_\mathrm{a}-\omega_\mathrm{g}-\omega_1)t}\left|\mathrm{a}\right\rangle = \mathrm{e}^{-\mathrm{i}\Delta_\mathrm{a}t}\left|\mathrm{a}\right\rangle, \\ U\left|\mathrm{b}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_\mathrm{b}-\omega_\mathrm{a}-\omega_2)t}\mathrm{e}^{-\mathrm{i}\Delta_\mathrm{a}t}\left|\mathrm{b}\right\rangle = \mathrm{e}^{-\mathrm{i}\Delta_\mathrm{b}t}\left|\mathrm{b}\right\rangle, \\ U\left|\mathrm{c}\right\rangle &= \mathrm{e}^{-\mathrm{i}(\omega_\mathrm{b}-\omega_\mathrm{c}-\omega_3)t}\mathrm{e}^{\mathrm{i}\Delta_\mathrm{b}t}\left|\mathrm{c}\right\rangle = \mathrm{e}^{\mathrm{i}\Delta_\mathrm{c}t}, \end{aligned} \tag{22}$$

we have

$$\begin{aligned} H \to H' &= UHU^\dagger - \mathrm{i}\hbar U\partial_t U^\dagger \\ &= \frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| + \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| + \mathrm{h.c.} + \hbar\Delta_\mathrm{a} \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{a}\right| + \hbar\Delta_\mathrm{b} \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{b}\right| - \hbar\Delta_\mathrm{c} \left|\mathrm{c}\right\rangle\!\left\langle\mathrm{c}\right|. \end{aligned} \tag{23}$$

This is the time-independent Hamiltonian we want.

(3) The effective Hamiltonian is

$$\begin{aligned} H_\mathrm{eff} = &\frac{1}{2}\hbar\Omega_1 \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{g}\right| + \frac{1}{2}\hbar\Omega_2 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{a}\right| + \frac{1}{2}\hbar\Omega_3 \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{c}\right| + \mathrm{h.c.} + \underbrace{\hbar\left(\Delta_\mathrm{a} - \frac{\mathrm{i}\Gamma_\mathrm{a}}{2}\right)}_{\tilde{\Delta}_\mathrm{a}} \left|\mathrm{a}\right\rangle\!\left\langle\mathrm{a}\right| \\ &+ \underbrace{\hbar\left(\Delta_\mathrm{b} - \frac{\mathrm{i}\Gamma_\mathrm{b}}{2}\right)}_{\tilde{\Delta}_\mathrm{b}} \left|\mathrm{b}\right\rangle\!\left\langle\mathrm{b}\right| - \underbrace{\hbar\left(\Delta_\mathrm{c} + \frac{\mathrm{i}\Gamma_\mathrm{c}}{2}\right)}_{\tilde{\Delta}_\mathrm{c}} \left|\mathrm{c}\right\rangle\!\left\langle\mathrm{c}\right|. \end{aligned} \tag{24}$$

(4) In the weak perturbation limit, $c_\mathrm{g} \approx 1$, and the equation $H_\mathrm{eff}\left|\psi_\mathrm{s}\right\rangle = 0$ is equivalent to

$$\begin{aligned} &-\frac{1}{2}\hbar\Omega_1^* c_\mathrm{a} = 0, \\ &\frac{1}{2}\hbar\Omega_1 + \hbar\left(\Delta_\mathrm{a} - \frac{\mathrm{i}\Gamma_\mathrm{a}}{2}\right) c_\mathrm{a} + \frac{1}{2}\hbar\Omega_2^* c_\mathrm{b} = 0, \\ &\frac{1}{2}\hbar\Omega_2 c_\mathrm{a} + \hbar\left(\Delta_\mathrm{b} - \frac{\mathrm{i}\Gamma_\mathrm{b}}{2}\right) c_\mathrm{b} + \frac{1}{2}\hbar\Omega_3 c_\mathrm{c} = 0, \\ &\frac{1}{2}\hbar\Omega_3^* - \hbar\left(\Delta_\mathrm{c} + \frac{\mathrm{i}\Gamma_\mathrm{c}}{2}\right) c_\mathrm{c} = 0. \end{aligned}$$

The first equation can be throw away because it merely means c_{a} is small. The solutions are therefore

$$\begin{aligned} c_{\mathrm{a}} &= -\frac{2\Omega_1\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + |\Omega_3|^2\Omega_1/2}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}}, \\ c_{\mathrm{b}} &= \frac{\Omega_1\Omega_2\tilde{\Delta}_{\mathrm{c}}}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}}, \\ c_{\mathrm{c}} &= \frac{\Omega_1\Omega_2\Omega_3^*/2}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}}. \end{aligned} \tag{25}$$

Note

What we are doing here is equivalent to the first order perturbation theory, i.e. giving the eigenstate a first order correction withing correcting the energy. We are assuming that $H_{\mathrm{eff}}|\mathrm{ground}\rangle = 0$ because before perturbation, the energy of the ground state is zero. If it is not, we need to change the RHS of the equation. This is often called **adiabatic elimination**. See the discussion below (24) here.

(5) We have

$$\gamma_3 = \Gamma_3|c_{\mathrm{c}}|^2 = \frac{|\Omega_1|^2|\Omega_2|^2|\Omega_3|^2/4}{||\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}|^2}\Gamma_3. \tag{26}$$

(6) The total unitary transformation from the original picture to the current pictures is partly given by

$$U_{\mathrm{total}}|\mathrm{g}\rangle = \mathrm{e}^{\mathrm{i}\omega_{\mathrm{g}}t}|\mathrm{g}\rangle, \quad U_{\mathrm{total}}|\mathrm{a}\rangle = \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{a}}-\omega_{\mathrm{g}}-\omega_1)t}\mathrm{e}^{\mathrm{i}\omega_{\mathrm{a}}t}|\mathrm{a}\rangle,$$

so in the current picture, we have

$$\boldsymbol{d} = U_{\mathrm{total}}\boldsymbol{d}_{\mathrm{original}}U_{\mathrm{total}}^{-1},$$

and therefore

$$\begin{aligned} \langle\boldsymbol{d}\rangle &= \langle\psi_{\mathrm{s}}|\mathrm{a}\rangle\,\mathrm{e}^{\mathrm{i}(\omega_1+\omega_{\mathrm{g}})t}\boldsymbol{d}_{\mathrm{ag}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{g}}t}\langle\mathrm{g}|\psi_{\mathrm{s}}\rangle + \mathrm{h.c.} = \mathrm{e}^{-\mathrm{i}\omega_1 t}\boldsymbol{d}_{\mathrm{ga}}\langle\psi_{\mathrm{s}}|\mathrm{g}\rangle\,\langle\mathrm{a}|\psi_{\mathrm{s}}\rangle + \mathrm{h.c.} \\ &= -\mathrm{e}^{-\mathrm{i}\omega_1 t}\boldsymbol{d}_{\mathrm{ga}}\frac{-2\boldsymbol{d}_{\mathrm{ag}}\cdot\boldsymbol{E}_1}{\hbar}\frac{2\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + |\Omega_3|^2/2}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}} + \mathrm{h.c.}, \end{aligned} \tag{27}$$

so we have

$$\overleftrightarrow{\boldsymbol{\alpha}} = \frac{\boldsymbol{d}_{\mathrm{ga}}\boldsymbol{d}_{\mathrm{ag}}}{\hbar}\frac{4\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} + |\Omega_3|^2}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} + 4\tilde{\Delta}_{\mathrm{a}}\tilde{\Delta}_{\mathrm{b}}\tilde{\Delta}_{\mathrm{c}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}}. \tag{28}$$

When the other two laser beams are very, very strong, we have approximately

$$\overleftrightarrow{\boldsymbol{\alpha}} = \frac{\boldsymbol{d}_{\mathrm{ga}}\boldsymbol{d}_{\mathrm{ag}}}{\hbar}\frac{|\Omega_3|^2}{|\Omega_3|^2\tilde{\Delta}_{\mathrm{a}} - |\Omega_2|^2\tilde{\Delta}_{\mathrm{c}}}. \tag{29}$$

(7) What we are doing is to assume (a) that quantum jump is highly impossible on the time scale that we are interested, so the main effect of spontaneous radiation is to renormalize parameters in a pure state problem, and (b) that all perturbations, be it from Ω or Γ, are small enough.

The first condition is equivalent to $\gamma T \ll 1$, where T is the time scale we are interested in, where

$$\Gamma \sim \Gamma_{\mathrm{a}}|c_{\mathrm{a}}|^2 + \Gamma_{\mathrm{b}}|c_{\mathrm{b}}|^2 + \Gamma_{\mathrm{c}}|c_{\mathrm{c}}|^2. \tag{30}$$

The second condition is equivalent to $|c_{\mathrm{a}}|^2 + |c_{\mathrm{b}}|^2 + |c_{\mathrm{c}}|^2 \ll 1$.

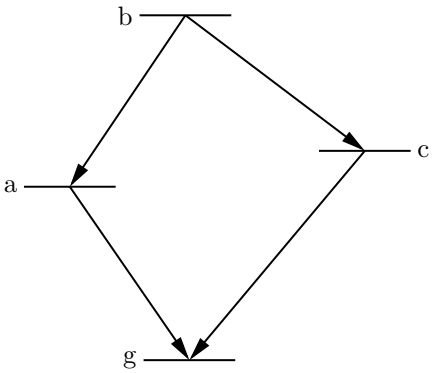


Figure 5: The energy level diagram of the diamond system that looks more like a diamond

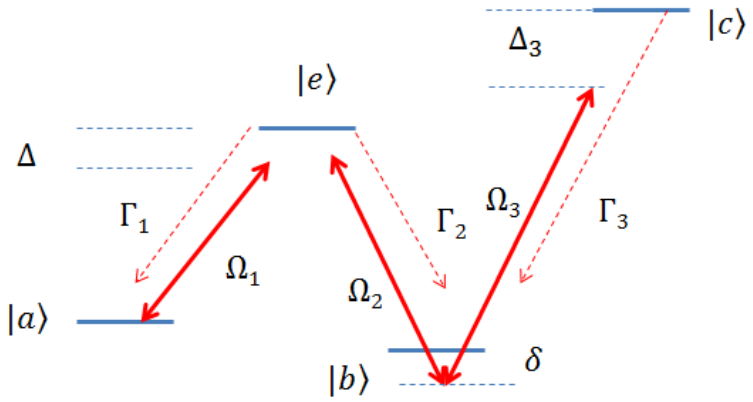


Figure 6: A three-level system coupled to an additional energy level

Discussion What we are doing in this problem is actually exactly solving a non-perturbative optical problem with perturbative techniques. When the external driving field is not that strong and the detuning is large, we can safely use perturbation theory. When the external driving field is very strong (so that electrons can be pulled out of an atom and then squeezed back), or the detuning is small, purely perturbative calculation does not work.

Now we go back to the problem. The energy level diagram that shows how spontaneous radiation happens is Figure 5 on page 9. Note that it is impossible to jump from $|\mathrm{a}\rangle$ to $|\mathrm{g}\rangle$, because if spontaneous radiation channels in Figure 5 on page 9 are possible, then $|\mathrm{b}\rangle$ and $|\mathrm{g}\rangle$ have the same parity, $|\mathrm{a}\rangle$ and $|\mathrm{c}\rangle$ have the same parity, and $|\mathrm{b}\rangle$ and $|\mathrm{a}\rangle$ have opposite parities. Therefore, there is no single-photon process from $|\mathrm{b}\rangle$ to $|\mathrm{g}\rangle$. Since we are not interested in where the atom ends in, the quantum jump operators are thrown away, and we just study H_{eff}.

Now we go on to discuss when the perturbative calculation in Task 4 works. We can expect the atom is largely on the ground state only when $\boldsymbol{E}_1$ is small and has a relatively large detuning. This imposes *no* constraints on $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$. Therefore, as long as the conditions about $\boldsymbol{E}_1$ are correct, and RWA works, the perturbative calculation in Task 4 works. Our calculation is therefore perturbative for $\boldsymbol{E}_1$ but *non* perturbative for $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$. This fact can also be seen from (25), where everything is proportion to $\Omega_1 \propto \boldsymbol{E}_1$.

Finally, we can see from Figure 4 on page 6 that what we are actually dealing with is a three-photon process that carries the atom from $|\mathrm{g}\rangle$ to $|\mathrm{c}\rangle$. This can also be seen from c_{c} in (25). If carrying the atom to $|\mathrm{c}\rangle$ is all what we care, and we are not interested by the detailed dynamics, then the condition in Task 7 can be loosen, because even though quantum jumps occur from time to time, the atom quickly relaxes to $|\psi_{\mathrm{s}}\rangle$ after one quantum jump. The atom is therefore almost constantly at $|\psi_{\mathrm{s}}\rangle$. In perturbative calculation of nonlinear effects, we sum up several diagrams like Figure 4 on page 6, but the Hamiltonian has not undergone RWA. What we are doing here is actually a resummation of these "bare" diagrams into one diagram Figure 4 on page 6, and then diagonalize it.

EIT-assisted giant Kerr effect The "lambda"-system composed of $|\mathrm{a}\rangle, |\mathrm{e}\rangle, |\mathrm{b}\rangle$ is further coupled to excited state $|\mathrm{c}\rangle$, as in Figure 6 on page 9. We consider the situation of EIT-resonance: $\delta = 0$. We further consider atomic state to be initially in $|\psi(t=0)\rangle = |\mathrm{a}\rangle$, and weak-excitation limit is satisfied ($|\Omega_1|$ small "enough"). (a) Write down the effective Hamiltonian for this problem for $\delta = 0$. (b) Obtain the approximate stochastic wavefunction in its steady state $|\tilde{\psi}_\mathrm{S}\rangle = |\mathrm{a}\rangle + c_\mathrm{e}|\mathrm{e}\rangle + c_\mathrm{b}|\mathrm{b}\rangle + c_\mathrm{c}|\mathrm{c}\rangle$ such that $H_\mathrm{eff}|\psi_\mathrm{S}\rangle \approx 0$. (c) Approximately evaluate the atomic dipole moment $\langle d\rangle = \langle\psi_S|d_{ae}|a\rangle\langle e|\psi_S\rangle + \mathrm{c.c.}$ oscillating at the E_1 frequency.

Solution

(a) Repeating procedures in Task 1 of the previous problem, after RWA, we have

$$H = H_0 + \frac{1}{2}\hbar\Omega_1 |\mathrm{e}\rangle\langle\mathrm{a}| \mathrm{e}^{-\mathrm{i}\omega_1 t} + \frac{1}{2}\hbar\Omega_2 |\mathrm{e}\rangle\langle\mathrm{b}| \mathrm{e}^{-\mathrm{i}\omega_2 t} + \frac{1}{2}\hbar\Omega_3 |\mathrm{c}\rangle\langle\mathrm{b}| \mathrm{e}^{-\mathrm{i}\omega_3 t} + \mathrm{h.c.}, \tag{31}$$

where

$$H_0 = \hbar\omega_\mathrm{a} |\mathrm{a}\rangle\langle\mathrm{a}| + \hbar\omega_\mathrm{e} |\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\omega_\mathrm{b} |\mathrm{b}\rangle\langle\mathrm{b}| + \hbar\omega_\mathrm{c} |\mathrm{c}\rangle\langle\mathrm{c}| . \tag{32}$$

Switching to the interaction picture, the Hamiltonian becomes

$$H = \frac{1}{2}\hbar\Omega_1 |\mathrm{e}\rangle\langle\mathrm{a}| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{e}-\omega_\mathrm{a}-\omega_1)t} + \frac{1}{2}\hbar\Omega_2 |\mathrm{e}\rangle\langle\mathrm{b}| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{e}-\omega_\mathrm{b}-\omega_2)t} + \frac{1}{2}\hbar\Omega_3 |\mathrm{c}\rangle\langle\mathrm{b}| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{c}-\omega_\mathrm{b}-\omega_3)t} + \mathrm{h.c.}. \tag{33}$$

The detunings are

$$\Delta + \omega_1 = \omega_\mathrm{e} - \omega_\mathrm{a}, \quad \omega_\mathrm{e} - \omega_\mathrm{b} = \omega_2, \quad \Delta_3 + \omega_3 = \omega_\mathrm{c} - \omega_\mathrm{b}, \tag{34}$$

so (21) is

$$H = \frac{1}{2}\hbar\Omega_1 |\mathrm{e}\rangle\langle\mathrm{a}| \mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2 |\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3 |\mathrm{c}\rangle\langle\mathrm{b}| \mathrm{e}^{\mathrm{i}\Delta_3 t} + \mathrm{h.c.}. \tag{35}$$

Now we do rotating wave transformation

$$U|\mathrm{e}\rangle = \mathrm{e}^{-\mathrm{i}\Delta t}|\mathrm{e}\rangle, \quad U|\mathrm{c}\rangle = \mathrm{e}^{-\mathrm{i}\Delta_3 t}|\mathrm{c}\rangle, \quad U|\mathrm{a}\rangle = |\mathrm{a}\rangle, \quad U|\mathrm{b}\rangle = |\mathrm{b}\rangle, \tag{36}$$

we have

$$\begin{aligned} H \to H' &= UHU^\dagger - \mathrm{i}\hbar U\partial_t U^\dagger \\ &= \frac{1}{2}\hbar\Omega_1 |\mathrm{e}\rangle\langle\mathrm{a}| \mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2 |\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3 |\mathrm{c}\rangle\langle\mathrm{b}| \mathrm{e}^{\mathrm{i}\Delta_3 t} + \mathrm{h.c.} + \hbar\Delta |\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\Delta_3 |\mathrm{c}\rangle\langle\mathrm{c}| . \end{aligned} \tag{37}$$

The effective Hamiltonian can be obtained by adding the damping terms, i.e.

$$\begin{aligned} H_\mathrm{eff} = &\frac{1}{2}\hbar\Omega_1 |\mathrm{e}\rangle\langle\mathrm{a}| \mathrm{e}^{\mathrm{i}\Delta t} + \frac{1}{2}\hbar\Omega_2 |\mathrm{e}\rangle\langle\mathrm{b}| + \frac{1}{2}\hbar\Omega_3 |\mathrm{c}\rangle\langle\mathrm{b}| \mathrm{e}^{\mathrm{i}\Delta_3 t} + \mathrm{h.c.} \\ &+ \hbar\Delta |\mathrm{e}\rangle\langle\mathrm{e}| + \hbar\Delta_3 |\mathrm{c}\rangle\langle\mathrm{c}| - \frac{\mathrm{i}\hbar(\Gamma_1+\Gamma_2)}{2} |\mathrm{e}\rangle\langle\mathrm{e}| - \frac{\mathrm{i}\hbar\Gamma_3}{2} |\mathrm{c}\rangle\langle\mathrm{c}| . \end{aligned} \tag{38}$$

(b) The equation $H_\mathrm{eff}|\tilde{\psi}_\mathrm{s}\rangle \approx 0$ is equivalent to

$$\begin{aligned} &\frac{1}{2}\hbar\Omega_1^* c_\mathrm{e} = 0, \\ &\frac{1}{2}\hbar\Omega_1 + \frac{1}{2}\hbar\Omega_2 c_\mathrm{b} + \hbar\left(\Delta - \frac{\mathrm{i}(\Gamma_1+\Gamma_2)}{2}\right) c_\mathrm{e} = 0, \\ &\frac{1}{2}\hbar\Omega_2^* c_\mathrm{e} + \frac{1}{2}\hbar\Omega_3^* c_\mathrm{c} = 0, \\ &\frac{1}{2}\hbar\Omega_3 c_\mathrm{b} + \hbar\left(\Delta_3 - \frac{\mathrm{i}\Gamma_3}{2}\right) c_\mathrm{c} = 0. \end{aligned}$$

We throw away the first equation because it merely requires that c_c is small, and solving the rest of the equations gives

$$\begin{aligned} c_\mathrm{b} &= -\frac{\Omega_1\Omega_2^*\tilde{\Delta}_3}{|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta}}, \\ c_\mathrm{c} &= \frac{\Omega_1\Omega_2^*\Omega_3}{2(|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta})}, \\ c_\mathrm{e} &= -\frac{\Omega_1|\Omega_3|^2}{2(|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta})}, \end{aligned} \tag{39}$$

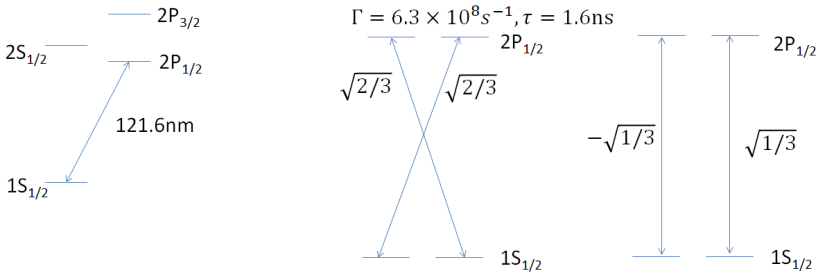


Figure 7: Left: Simplified Hydrogen levels and excitation of 1 S atom with 121.6 nm light resonant to 1 S$-$2P$_{1/2}$ transition. Right: The Clebsch-Gordon coefficients for the 1 S$_{1/2}-$2P$_{1/2}$ dipole transitions.

where we define

$$\tilde{\Delta} = \Delta - \frac{\mathrm{i}(\Gamma_1+\Gamma_2)}{2}, \quad \tilde{\Delta}_3 = \Delta_3 - \frac{\mathrm{i}\Gamma_3}{2}. \tag{40}$$

(c) The total picture transformation from the original picture to the picture we are using is

$$U_{\text{total}}\left|\mathrm{a}\right\rangle = \mathrm{e}^{\mathrm{i}\omega_\mathrm{a} t}\left|\mathrm{a}\right\rangle, \quad U_{\text{total}}\left|\mathrm{e}\right\rangle = \mathrm{e}^{-\mathrm{i}\Delta t}\mathrm{e}^{\mathrm{i}\omega_\mathrm{e} t}\left|\mathrm{e}\right\rangle = \mathrm{e}^{\mathrm{i}(\omega_1+\omega_\mathrm{a})t}, \tag{41}$$

and therefore in the current picture, we have

$$\langle \mathrm{e}|\boldsymbol{d}|\mathrm{a}\rangle = \langle \mathrm{e}|U_{\text{total}}\boldsymbol{d}_{\text{original}}U_{\text{total}}^{\dagger}|\mathrm{a}\rangle = \boldsymbol{d}_{\mathrm{ea}}\mathrm{e}^{\mathrm{i}\omega_1 t}, \quad \langle \mathrm{a}|\boldsymbol{d}|\mathrm{e}\rangle = \boldsymbol{d}_{\mathrm{ae}}\mathrm{e}^{-\mathrm{i}\omega_1 t},$$

so

$$\begin{aligned}\langle \boldsymbol{d}\rangle &= \boldsymbol{d}_{\mathrm{ae}}\mathrm{e}^{-\mathrm{i}\omega_1 t}\langle\psi_\mathrm{s}|\mathrm{a}\rangle\ \langle \mathrm{e}|\psi_\mathrm{s}\rangle + \mathrm{h.c.} \\ &= \frac{\boldsymbol{d}_{\mathrm{ae}}\boldsymbol{d}_{\mathrm{ea}}\cdot\boldsymbol{E}_{10}\mathrm{e}^{-\mathrm{i}\omega_1 t}}{\hbar}\frac{|\Omega_3|^2}{|\Omega_2|^2\tilde{\Delta}_3 + |\Omega_3|^2\tilde{\Delta}} + \mathrm{h.c.}.\end{aligned} \tag{42}$$

Discussion Kerr effect means the fact that the one beam of light can affect the refractive index of another beam of light. The system is said to be *giant* because it can be extremely sensible.

Unfortunately, the system Figure 6 on page 9 has been proven impossible to be implemented in a random gas. It is possible to be implemented in a ultracold atom system, but optics in strongly correlated systems is still a big problem.

The random wave function + perturbation theory approach used in this problem is already dangerous, because when Ω_2 is small, it is possible that the meta-stable state has a large $|\mathrm{b}\rangle$ component.

Effective Hamiltonian and Master Equation for the Hydrogen "D1" manifold This problem exercises on setting up effective Hamiltonian and master equation for multi-level system including radiative decays. We choose hydrogen atom as an example, and for simplicity we ignore the hyperfine structure.

We consider a hydrogen atom subjected to 121.6 nm laser radiation. The laser is linearly polarized, and its frequency is exactly resonant to the 2 S$_{1/2}$ $-$ 2P$_{1/2}$ transition. The Rabi frequency is defined as $\Omega = \frac{E\langle 1S_{1/2},m|d_z|2P_{1/2},m\rangle}{\hbar}$, and we choose the polarization direction of light to be along z, which is also chosen to be the quantization axis.

(a) Write down the non-Hermitian effective Hamiltonian matrix that includes both light-atom interaction and the radiative decay from 2P$_{1/2}$, in the rotating frame with no explicit time-dependence. (b) Write down the Master equation for the density matrix.

Solution

(a) After coupling the orbital angular momentum and the spin angular momentum together, for S we get $j = 1/2, m_j = \pm 1/2$, and for P we have $j = 1/2, m_j = \pm 1/2$, and $j = 3/2, m_j = \pm 1/2, \pm 3/2$. These are shown in Figure 7 on page 11, and the external laser connects 1S$_{1/2}$ and 2P$_{1/2}$. Since the direction of $\boldsymbol{E}$ is z direction, the atom-light coupling Hamiltonian is proportion to z, and since $[z, L_z] = 0$, and spins are not directly coupled to $\boldsymbol{E}$, we find $J_z = L_z + S_z$ is

conserved. Therefore, there are four states that are involved in the presence of the 121.6 nm laser radiation: $1S_{1/2}, m = \pm 1/2$, $2P_{1/2}, m = \pm 1/2$, and only states with the same m can be coupled.

To be concise we consider $1S_{1/2}$ to be g and $2P_{1/2}$ to be e. So the Hamiltonian after RWA is

$$H = -\sum_{m=\pm 1/2} \hbar\Omega_m \left|\mathrm{e}, m\right\rangle\!\left\langle \mathrm{g}, m\right| \mathrm{e}^{\mathrm{i}(\omega_\mathrm{e}-\omega_\mathrm{g}-\omega)} + \mathrm{h.c.}. \tag{43}$$

Since the detuning

$$\Delta = \omega_\mathrm{e} - \omega_\mathrm{g} - \omega \tag{44}$$

is zero, the RWA Hamiltonian is

$$H = -\sum_{m=\pm 1/2} \hbar\Omega_m \left|\mathrm{e}, m\right\rangle\!\left\langle \mathrm{g}, m\right| + \mathrm{h.c.}. \tag{45}$$

Now we include the damping terms. Note that the polarization of spontaneous radiation is not necessarily along the z axis. Emission of a σ_+ photon introduces a $1/2 \to -1/2$ decay channel. Emission of a σ_- photon introduces a $-1/2 \to 1/2$ decay channel. Emission of a π photon introduces $1/2 \to 1/2$, $-1/2 \to -1/2$ channels. Therefore

$$\begin{aligned} C_{\sigma_+} &= \sqrt{\frac{2}{3}\Gamma_\mathrm{ge}}\left|\mathrm{g}, -1/2\right\rangle\!\left\langle \mathrm{e}, 1/2\right|, \quad C_{\sigma_-} = \sqrt{\frac{2}{3}\Gamma_\mathrm{ge}}\left|\mathrm{g}, 1/2\right\rangle\!\left\langle \mathrm{e}, -1/2\right|, \\ C_\pi &= -\sqrt{\frac{1}{3}\Gamma_\mathrm{ge}}\left|\mathrm{g}, -1/2\right\rangle\!\left\langle \mathrm{e}, -1/2\right| + \sqrt{\frac{1}{3}\Gamma_\mathrm{ge}}\left|\mathrm{g}, 1/2\right\rangle\!\left\langle \mathrm{e}, 1/2\right|, \end{aligned} \tag{46}$$

and

$$\begin{aligned} H_\mathrm{eff} &= -\sum_{m=\pm 1/2} \hbar\Omega_m \left|\mathrm{e}, m\right\rangle\!\left\langle \mathrm{g}, m\right| + \mathrm{h.c.} - \frac{\mathrm{i}\hbar}{2}\sum_{p=\sigma_+,\sigma_-,\pi} C_p^\dagger C_p \\ &= -\hbar\Omega_m\left(\left|\mathrm{e}, 1/2\right\rangle\!\left\langle \mathrm{g}, 1/2\right| - \left|\mathrm{e}, -1/2\right\rangle\!\left\langle \mathrm{g}, -1/2\right|\right) + \mathrm{h.c.} \\ &\quad - \frac{\mathrm{i}\hbar}{2}\Gamma_\mathrm{ge}\left(\left|\mathrm{e}, -1/2\right\rangle\!\left\langle \mathrm{e}, -1/2\right| + \left|\mathrm{e}, 1/2\right\rangle\!\left\langle \mathrm{e}, 1/2\right|\right). \end{aligned} \tag{47}$$

Note

A question might be why we represent the $1/2 \to -1/2$ channel and the $-1/2 \to 1/2$ channel with *two* collapse operators while we represent the $1/2 \to 1/2$ and $-1/2 \to -1/2$ channels with only one collapse operator. The reason is that the collapse process is assumed to be induced by the observation of the emitted photons, and the $1/2 \to 1/2$ and $-1/2 \to -1/2$ channels generate the same kind of photons, and therefore an observation of the emitted photon will not decide whether the input state is $1/2$ or $-1/2$, and therefore the two channels are represented with a single collapse operator.
Also, note that the Wigner–Eckart theorem

$$\langle j\, m|T_q^{(k)}|j'\, m'\rangle = \langle j'\, m'\, k\, q|j\, m\rangle\langle j\|T^{(k)}\|j'\rangle \tag{48}$$

tells us that the dipole Hamiltonian is proportion to C-G coefficients, and therefore $\Omega_+ = -\Omega_-$.

(b) The master equation is just

$$\dot\rho = \frac{1}{\mathrm{i}\hbar}(H_\mathrm{eff}\rho - \rho H_\mathrm{eff}^\dagger) + \sum_{p=\sigma_+,\sigma_-,\pi} C_p^\dagger \rho C_p. \tag{49}$$

Note

Note that in a time evolution equation according to a non-Hermitian effective Hamiltonian, we should *not* use the commutator.