Quantum Optics, Homework 3

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Interference between Gaussian pulses Consider two Gaussian pulses with wave vectors $\boldsymbol{k}_{1,2} = k(\pm \sin\theta, 0, \cos\theta)$, respectively. They are incident to a plane detector on the surface $z = 0$. The intensity distributions of the two beams are all

$$|\mathcal{E}|^2 \propto e^{-\left(x^2+y^2\right)/\sigma^2}, \tag{1}$$

with $\sigma \gg \lambda$. The pulses arrive at the detector simultaneously. The detector absorbs the pulses completely and there is no reflection. Calculate $P^{(1)}(\boldsymbol{r})$ and $P^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for the following states of the optical field:

(a) $|\psi\rangle = \frac{1}{\sqrt{2^N N!}} \left(a_1^\dagger + a_2^\dagger\right)^N |V\rangle$.

(b) $|\psi\rangle = \frac{1}{N!} \left(a_1^\dagger a_2^\dagger\right)^N |V\rangle$.

(c) $|\psi\rangle = \frac{1}{\sqrt{2N!}} \left(\left(a_1^\dagger\right)^N + \left(a_2^\dagger\right)^N\right) |V\rangle$.

(d) $|\psi\rangle = D_1(\alpha) D_2(\alpha) |V\rangle, \quad D_j(\alpha) \equiv e^{\alpha a_j^\dagger - \alpha^* a_j}$.

(e) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(D_1(\alpha) + D_2(\alpha)\right) |V\rangle$.

Solution The electric field operator is

$$\boldsymbol{E} = \sum_{i=1,2} \boldsymbol{\mathcal{E}}_i \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r} - \mathrm{i}\omega t} a_i + \text{h.c.}. \tag{2}$$

(a) We define

$$b^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger),$$

and now the wave function is

$$|\psi\rangle = \frac{1}{\sqrt{N!}} (b^\dagger)^N |0\rangle .$$

We have

$$P^{(1)}(\boldsymbol{r}) = \frac{1}{N!} |\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \langle 0| b^N (a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2 - \boldsymbol{k}_1)\cdot\boldsymbol{r}} a_1^\dagger a_2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{r}} a_2^\dagger a_1)(b^\dagger)^N |0\rangle .$$

Evaluating the terms in the RHS above, we have

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_1 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_1 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

and similarly

$$\langle 0| b^N a_2^\dagger a_2 (b^\dagger)^N |0\rangle = \frac{1}{2} N^2 (N-1)!,$$

and

$$\begin{aligned}
\langle 0| b^N a_1^\dagger a_2 (b^\dagger)^N |0\rangle &= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times \text{contraction of } (N-1)\ b\text{'s and } (N-1)\ b^\dagger\text{'s} \\
&= N \langle 0| b a_1^\dagger |0\rangle \times N \langle 0| a_2 b^\dagger |0\rangle \times (N-1)! \langle 0| b b^\dagger |0\rangle \\
&= N \times \frac{1}{\sqrt{2}} \times N \frac{1}{\sqrt{2}} \times (N-1)! \times 1 = \frac{1}{2} N^2 (N-1)!,
\end{aligned}$$

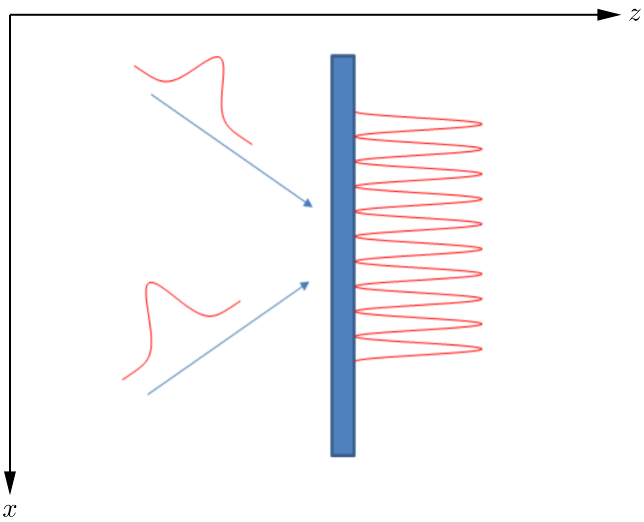

Figure 1: The two Gaussian beams incident to a detector

and similarly

$$\langle 0|b^N a_1^\dagger a_2 (b^\dagger)^N|0\rangle = \frac{1}{2}N^2(N-1)!.$$

Putting everything together we have

$$\begin{aligned} P^{(1)}(\boldsymbol{r}) &= \frac{1}{N!}|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2 \times \frac{1}{2}N^2(N-1)! \times (2 + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2))\cdot\boldsymbol{r}}) \\ &= N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}), \end{aligned}$$

so finally

$$P^{(1)}(\boldsymbol{r}) = N|\boldsymbol{\mathcal{E}}(\boldsymbol{r})|^2(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}) \propto N\mathrm{e}^{-(x^2+y^2)/\sigma^2}(1+\cos(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}). \tag{3}$$