Special Relativity by Prof. Kun Din

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So the problem is whether the wave equation of light does transform under Galilean transformation. An experiment shown in Figure 1 on page 2 can be used to verify the claim. If Galilean transformation works well, then the two light beams undergo the following velocities change:

$$w_+ = \frac{c}{n} + v, \quad w_- = \frac{c}{n} - v, \tag{1}$$

where v is the speed of the liquid. What was actually observed, however, was

$$w_+ = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{2}$$

(2) means things are more complicated than a pure Galilean transformation. This was not considered as a big problem, though, because at that time the only wave phenomenon observed was mechanical wave. Therefore, (2) was initially thought as a piece of evidence for **Luminiferous aether**, something that bore light and could have interaction with ordinary materials.

The claim of existence of aether, nonetheless, was rejected by the Michelson–Morley experiment.

Lorentz found that if the transformation (now named **Lorentz transformation**)

$$\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}, \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{3}$$

was the correct rule for light, then everything was find. We can verify that

$$\begin{aligned}
\frac{\partial}{\partial t} &= -\beta\gamma c \frac{\partial}{\partial z'} + \gamma \frac{\partial}{\partial t'}, \\
\frac{\partial^2}{\partial t^2} &= \beta^2\gamma^2 c^2 \frac{\partial^2}{\partial z'^2} - 2\beta c \gamma^2 \frac{\partial}{\partial t'}\frac{\partial}{\partial z'} + \gamma^2 \frac{\partial^2}{\partial t'^2}, \\
\frac{\partial}{\partial z} &= \gamma = \frac{\partial}{\partial z'} - \frac{\beta\gamma}{c}\frac{\partial}{\partial t'}, \\
\frac{\partial^2}{\partial z^2} &= \gamma^2 \frac{\partial^2}{\partial z'^2} - \frac{2\beta\gamma^2}{c}\frac{\partial}{\partial t'}\frac{\partial}{\partial z'} + \frac{\beta^2\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2},
\end{aligned}$$

and therefore

$$\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}. \tag{4}$$

Therefore, the negative result in Michelson–Morley experiment can be explained.

If we accept (3) to be the correct time-space transformation, we need to define an inner product that is invariant under it. Since we have (4), a wise idea is to define

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \mathbf{b}. \tag{5}$$

This is called the **Lorentz metric**. Usually, we use a instead of $\mathbf{a}$ to denote a 4-vector. Note that here we need to pay attention to *covariant* and *contravariant* vectors. A covariant vector transforms like

$$a^\mu \to a'^\mu = \Lambda^\mu{}_\nu a_\nu, \tag{6}$$

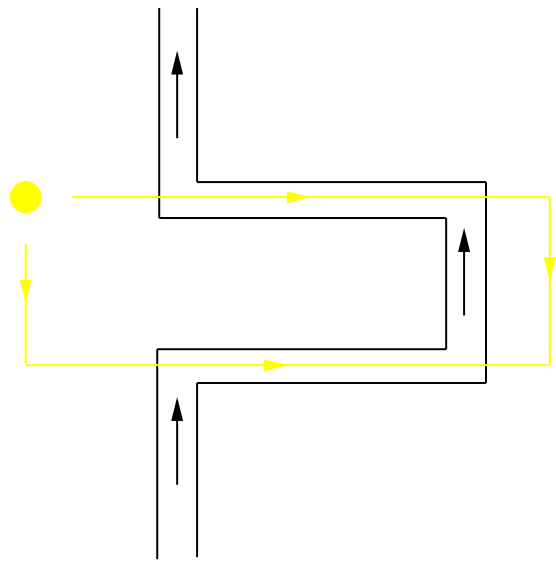

Figure 1: Light in moving liquid

while a contravariant vector transforms like

$$a_\mu \to a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu. \tag{7}$$

A inner product is always performed between a contravariant vector and a covariant vector.

Now we are able to find all possible Lorentz transformations: they are just matrices that keep the Lorentz metric.[1] A detailed derivation can be found in Jackson Chapter 17. There we will find

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta^\top \\ -\gamma\beta^\top & \mathbf{I}_3 + (\gamma - 1)\frac{\beta\beta^\top}{\beta^2} \end{pmatrix}, \tag{8}$$

where

$$\beta = (\beta_1, \beta_2, \beta_3). \tag{9}$$

Now we rewrite electrodynamics in a relativistic way. We have

$$\partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \boldsymbol{\nabla} \end{pmatrix}, \quad \partial^\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\boldsymbol{\nabla} \end{pmatrix}, \tag{10}$$

and therefore the continuity equation is

$$\partial_\mu j^\mu = 0, \quad j^\mu = (\rho, \boldsymbol{j}). \tag{11}$$

We define

$$A^\mu = \begin{pmatrix} \frac{\phi}{c} \\ \boldsymbol{A} \end{pmatrix}, \tag{12}$$

and we will find that the Lorentz gauge can now be written in a concise way:

$$\partial_\mu A^\mu = 0. \tag{13}$$

Under the Lorentz gauge, the wave equations can be rewritten as

$$\partial^\mu \partial_\mu A^\nu = \mu_0 j^\nu. \tag{14}$$

The generalized (i.e. gauge independent) wave equation is

$$\partial_\mu (\underbrace{\partial^\mu A^\nu - \partial^\nu A^\mu}_{F^{\mu\nu}}) = \mu_0 j^\nu. \tag{15}$$

[1] To be exact, we just need to keep the Lorentz metrics before and after the transformation *congruent*, because a Lorentz transformation can rescale the coordinates. What really matter are the *inertia indices*.

We can verify that the components of this equation reduce to Maxwell equations. We have

$$F^{0(1,2,3)} = \frac{\partial}{\partial(ct)} A^{(1,2,3)} - \left(-\frac{\partial}{\partial x}(c^{-1}\phi) \right) = -\frac{1}{c} E^{(1,2,3)},$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \left(-\frac{\partial}{\partial x} \right) A_y - \left(-\frac{\partial}{\partial y} \right) A_x = -B_z$$

and similarly

$$F^{23} = -B_x, \quad F^{31} = -B_y,$$

and therefore we find the equation

$$\partial_\mu F^{\mu 0} = \mu_0 j^0$$

is equivalent to

$$\boldsymbol{\nabla} \cdot (c^{-1} \boldsymbol{E}) = \mu_0 \rho c,$$

which is the first Maxwell equation. The equation

$$\partial_\mu F^{\mu 1} = \mu_0 j^1$$

is equivalent to

$$\frac{\partial}{\partial(ct)}(-c^{-1}E_x) + \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y = \mu_0 j_x,$$

which is the fourth Maxwell equation. The other two Maxwell equations - two the sourceless equations - cannot be derived from (15). They actually come from the fact that $F^{\mu\nu}$ is generated from antisymmetrized $\partial^\mu A^\nu$ and the consequent Bianchi identity. Therefore, (15) is actually not exactly equivalent to all Maxwell equations. The tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -c^{-1}E_x & -c^{-1}E_y & -c^{-1}E_z \\ c^{-1}E_x & 0 & -B_z & B_y \\ c^{-1}E_y & B_z & 0 & -B_x \\ c^{-1}E_z & -B_y & B_x & 0 \end{pmatrix} \tag{16}$$

or in other words

$$F_{\mu\nu} = \begin{pmatrix} 0 & c^{-1}E_x & c^{-1}E_y & c^{-1}E_z \\ -c^{-1}E_x & 0 & -B_z & B_y \\ -c^{-1}E_y & B_z & 0 & -B_x \\ -c^{-1}E_z & -B_y & B_x & 0 \end{pmatrix} \tag{17}$$

is called the **field strength tensor**.

Note

Matrix representation of tensors may be misleading. For vectors, we can always use a column vector to represent components of a covariant vector and use a row vector to represent components of a contravariant vector, but since $T^{\mu\nu}, T_{\mu\nu}, T^\mu{}_\nu$ and $T_\mu{}^\nu$ are all written into a rectangular matrix, it is impossible to distinguish which index is covariant or contravariant.

We can find that if we define

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ -B_x & 0 & c^{-1}E_z & -c^{-1}E_y \\ -B_y & -c^{-1}E_z & 0 & c^{-1}E_x \\ -B_z & c^{-1}E_y & -c^{-1}E_x & 0 \end{pmatrix}, \tag{18}$$

the equation

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{19}$$

gives the two sourceless Maxwell equations. Now we can find a duality between the electric field and the magnetic field: when there is no source in the space, we have

$$\partial_\mu \mathcal{F}^{\mu\nu} = \partial_\mu F^{\mu\nu} = 0 = 0,$$

and therefore the transformation

$$-c^{-1}E_i \longrightarrow B_i \tag{20}$$

leaves every equation the same.