Quantum Optics, Homework 2

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Details in the HBT experiment Figure 1 shows an experimental validation of the HBT effect. (a) Describe the expected phenomenon the experiment. Compare the expected phenomenon with the original HBT effect in astronomical observation. (b) Explain the phenomenon within classical electrodynamics. (c) Point out why the classical explanation is not enough. Construct a simplified version of the experiment and explain it with quantum optics.

Solution

(a) The integrating motor gives the averaged intensity correlation function, i.e.

$$\langle I_1 I_2 \rangle \coloneqq \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, I_1(t) I_2(t). \tag{1}$$

The expected results include that

$$\langle I_1 I_2 \rangle - \langle I_1 \rangle \langle I_2 \rangle \neq 0, \quad g^{(2)} > 1, \tag{2}$$

and that the intensity fluctuation correlation function

$$\langle \Delta I_1 \Delta I_2 \rangle = \langle I_1 I_2 \rangle - \langle I_1 \rangle \langle I_2 \rangle \tag{3}$$

reaches its peak when the optical path difference of the two beams is zero, and then drops away as the separation between the two beams increases. If the light source is laser, the relation between $g^{(2)}$ and the optical path difference is in the form of $1 + \cos(k\Delta r)$, while if the light source is thermal - as is the case of a mercury arc - the relation between $g^{(2)}$ and the optical path separation is something like Figure 2.

The experiment shown in Figure 1 is *essentially the same* with the astronomical HBT experiment. Note that the movable detector is moved *horizontally* so that the input light must have several components with different wave vector directions, and the beam from the mercury light is *never* collimated. The input light in Figure 1 is indeed *multi-mode*, where the wave vectors can take different directions. This is indeed the same with the case in astronomical HBT experiment. The "star" in Figure 1 is the aperture.

Note that the device shown in Figure 1 can be adjusted to measure the second order *temporal* correlation rather than the second order *spacial* correlation, as long as the movable detector is moved vertically instead of horizontally.

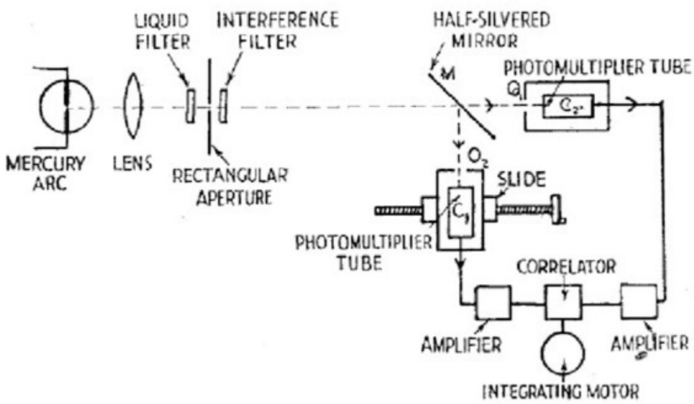

Figure 1: HBT effect in laboratory

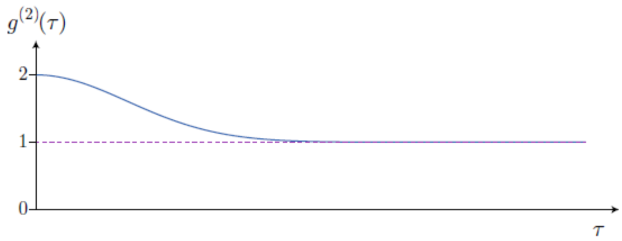

Figure 2: Second order coherence with a thermal light source in Gaussian distribution (figure taken from [1], Section 2.6.1). The maximum of $g^{(2)}$ is 2, which is a result of the optical field obeying a Gaussian distribution, where the Wick theorem holds so $\langle E_1^* E_2^* E_3 E_4 \rangle = \langle E_1^* E_3 \rangle \langle E_2^* E_4 \rangle + \langle E_1^* E_4 \rangle \langle E_2^* E_3 \rangle$, and hence $\langle I^2 \rangle = 2 \langle I \rangle^2$. Experimentally, the maximum may not be achieved due to fluctuations that destroy the correlation.

(b) The correlation function involved in the second order coherence is

$$\langle I_1 I_2 \rangle = \langle I_1(t) I_2(t) \rangle . \tag{4}$$

With a thermal light source, when the two detectors are separated far enough, E_1 and E_2 are not correlated, and we have

$$g^{(2)} = \frac{\langle I_1 I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle} \approx \frac{\langle I_1 \rangle \langle I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle} = 1.$$

When the two detectors are almost observing the same $\boldsymbol{E}$ components, however, we have

$$\langle (I_1(t) - \langle I_1 \rangle)(I_2(t) - \langle I_2 \rangle) \rangle = \left\langle (I_1(t) - \langle I_1 \rangle)^2 \right\rangle > 0,$$

and subsequently

$$\langle I_1 I_2 \rangle - \langle I_1 \rangle \langle I_2 \rangle > 0,$$

so $g^{(2)} > 1$. So we have something like Figure 2.

(c) When the classical picture of the optical field fails, the classical explanation fails as well. For example, when the light source creates a sequence of single-photon pulses, what will be observed is *photon antibunching* where $g^{(2)} = 0$ instead of bunching, because one photon cannot appear at two sites.

Discussion

(a) We did not calculate the correlation length of $g^{(2)}$ because it cannot be easily done for an arbitrary state of the optical field.

For a original light beam from a star, the correlation length given by $g^{(2)}$ is definitely affected by the temperature of the optical field generated by the star when the HBT experiment is used astronomically. The temperature determines the power distribution on different frequencies and the photon numbers on each mode. To relief ourselves from the burden to calculate $g^{(2)}$ under such a terrible state, when the HBT experiment is done, we usually *filter* the light from the star to be measured to obtain a single-frequency optical field.

The correlation length obtained from the second order coherence between two plane waves with a slightly different wave vector is λ/θ, where the angle θ is the angle between the wave vectors. The second order coherence between two plane waves is periodic, but since each point on the surface of the star in question radiates plane waves, after adding all components, we again get something like Figure 2. If we know the geometry of the surface of the star, the correlation length in Figure 2 can be obtained. The light after the monochromatic filter is still thermal, but the thermal nature of the light can be seen as adding a random phase to each plane wave component, which is not involved in the second-order coherence, the temperature of the optical field *does not* affect the correlation length.

Conditional generation of single photon pulses Many research and applications in quantum optics needs single photon pulses, that is, a wave packet of light that contains exactly one single photon. Such a single photon pulse can be generated in two ways: The deterministic approach via single atom emission, and the so-called heralded approach. This problem discusses a simplified version of the later. Consider a bi-photon generation process described by the Hamiltonian

$$H = \beta a_k^\dagger b_{k'}^\dagger + \text{h.c.}. \tag{5}$$

Here $a_k^\dagger, b_{k'}^\dagger$ are creation operators of photons into the k, k' propagation modes respectively. Such process can be realized for example in a frequency down conversion experiment, where a single photon is "split" into two in a nonlinear optical crystal, or a 4-wave mixing experiment where two incoming photons are converted into two output photons in an atomic gas. (a) Consider initially light is in vacuum state $|\psi(0)\rangle = |V\rangle$. Consider that the bi-photon generation process is switched on for time τ and then off, with $\xi = \beta\tau \ll 1$. Integrate the Schrodinger equation to obtain $|\psi(\tau)\rangle$, that is, the photon state after the interaction. (b) Consider a photon detector positioned L meters away from the bi-photon generation device along the k' propagation pathway. The time interval that the detector can detect a $b_{k'}$ photon is $[L/c, L/c+\tau]$ (we ignore any change of light speed within the experiment). For an ideal photon detector, what is the probability of detecting 1 photon, and detecting 2 photons during this time interval? If one photon is detected along k', what is the photon state in the k path? The strategy is the so called heralded single photon generation: a nearly perfect single photon pulse in the k mode is heralded by the detection of a single photon in the k'-mode.

Solution

(a) In the interaction picture, the time evolution of the state is given by

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle = h(t)\left(\beta a_k^\dagger b_{k'}^\dagger + \text{h.c.}\right)|\psi(t)\rangle,$$

where $h(t)$ is one when $t \in [0,\tau]$ and zero otherwise. Formally we have

$$|\psi(t)\rangle = \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\, h(t')(\beta a_k^\dagger b_{k'}^\dagger + \text{h.c.})\right)|\psi(0)\rangle.$$

Since $\xi \ll 1$, the operators approximately do not have time evolution, and thus we have

$$\begin{aligned} |\psi(\tau)\rangle &= \exp\left(-\mathrm{i}\tau(\beta a_k^\dagger b_{k'}^\dagger + \text{h.c.})\right)|\psi(0)\rangle \\ &= \mathrm{e}^{-\mathrm{i}\xi(a_k^\dagger b_{k'}^\dagger + a_k b_{k'})}|0\rangle. \end{aligned} \tag{6}$$

(b) Since the time interval is very short, we can view the measurement as simply measuring the state (6) as the pulse comes across the detector. Expanding (6) we have

$$\begin{aligned} |\psi(\tau)\rangle = &\ |0\rangle - \mathrm{i}\xi(a_k^\dagger b_{k'}^\dagger + \text{h.c.})|0\rangle + \frac{1}{2}(-\mathrm{i}\xi)^2(a_k^\dagger b_{k'}^\dagger + \text{h.c.})^2|0\rangle + \cdots \\ = &\left(1 - \frac{\xi^2}{2} + \cdots\right)|0\rangle - (\mathrm{i}\xi + \cdots)|n_k = 1, n_{k'} = 1\rangle - \left(\xi^2 + \cdots\right)|n_k = 2, n_{k'} = 2\rangle + \cdots. \end{aligned}$$

Taking only the leading order terms, we have

$$P(n_k = 1) = \xi^2, \quad P(n_k = 2) = \xi^4. \tag{7}$$

It can be seen that in $|\psi(\tau)\rangle$ we always have $n_k = n_{k'}$, and therefore if one photon is detected along k', the photon state in the k path is $|n_k = 1\rangle$. Therefore if we placed a baffle in path k, which is removed when $n_{k'}$ is detected to be 1, whenever a pulse is generated, it is a single-photon one.

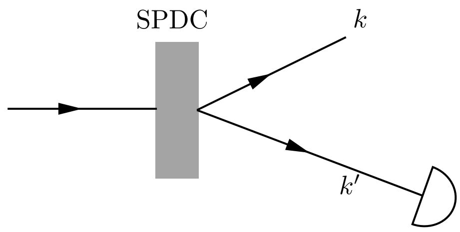

Figure 3: Light circuit in the heralded approach

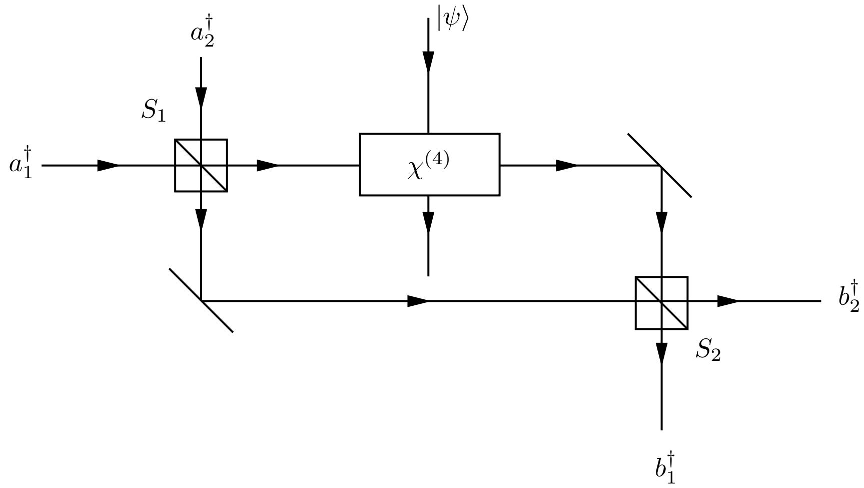

Figure 4: An example of QND, using a Mach-Zehnder interferometer

Discussion Actually nonlinear optics processes can be used to implement an unimaginable device: **quantum nondestructive (or nondemolition measurement (QND)**. In a fourth order nonlinear optical device, one input beam adds a phase factor to another beam, and the phase is determined by the strength of the first beam. Therefore, if we have a strong enough $\chi^{(4)}$, using the standard relative phase measurement approach we can measure the intensity of a light beam without disrupting it too much. Though certain disruption is unavoidable to the light beam being measured, the disruption is much less than the standard measurement, which makes the light beam in question collapse to a multi-photon state. Consider, for example, the device in Figure 4, where a Mach-Zehnder interferometer (see this document) is used to measure the phase introduced by $\chi^{(4)}$.

Details in the NLS process Analyze the NLS process in detail. The circuit is shown in Figure 5, and the two beam splitters are represented as

$$S_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad S_2 = \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \tag{8}$$

(a) Derive the output quantum state before the measurement. (b) Find the conditional quantum state with the measurement results shown in Figure 5. (c) Find when the NLS process works in terms of θ and σ, and the probability of a successful NLS.

Solution

(a) The optical circuit is linear its transformation matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sigma & \sin\sigma \\ 0 & -\sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\sigma\sin\theta & \cos\sigma\cos\theta & \sin\sigma \\ \sin\theta\sin\sigma & -\cos\theta\sin\sigma & \cos\sigma \end{pmatrix}, \tag{9}$$

and we have

$$b_j^\dagger = S_{jk} a_k^\dagger. \tag{10}$$

The quantum state is

$$|\psi\rangle = \left(\alpha + \beta a_1^\dagger + \frac{\gamma}{\sqrt{2}}(a_1^\dagger)^2\right) a_2^\dagger |0\rangle .$$

By (9) and (10) we have

$$[a_i]^\dagger = \begin{pmatrix} \cos\theta & -\cos\sigma\sin\theta & \sin\theta\sin\sigma \\ \sin\theta & \cos\theta\cos\sigma & -\cos\theta\sin\sigma \\ 0 & \sin\sigma & \cos\sigma \end{pmatrix} [b_i^\dagger],$$

and therefore we have

$$\begin{aligned} |\psi\rangle = \Big(&\alpha + \beta(b_1^\dagger\cos\theta - b_2^\dagger\sin\theta\cos\sigma + b_3^\dagger\sin\theta\sin\sigma) \\ &+ \frac{\gamma}{\sqrt{2}}(b_1^\dagger\cos\theta - b_2^\dagger\sin\theta\cos\sigma + b_3^\dagger\sin\theta\sin\sigma)^2\Big) \\ &\times (b_1^\dagger\sin\theta + b_2^\dagger\cos\theta\cos\sigma - b_3^\dagger\cos\theta\sin\sigma)|0\rangle . \end{aligned} \tag{11}$$

(b) In (11), the $b_1^\dagger(b_2^\dagger)^0$ terms are

$$\begin{aligned} &\alpha b_1^\dagger\sin\theta + \beta b_1^\dagger\cos\theta \times (-b_3^\dagger\cos\theta\sin\sigma) \\ &+ \beta b_3^\dagger\sin\theta\sin\sigma \times b_1^\dagger\sin\theta \\ &+ \frac{2\gamma}{\sqrt{2}} b_1^\dagger\cos\theta \times b_3^\dagger\sin\theta\sin\sigma \times (-b_3^\dagger\cos\theta\sin\sigma) \\ &+ \frac{\gamma}{\sqrt{2}}(b_3^\dagger)^\dagger\sin^2\theta\sin^2\sigma \times b_1^\dagger\sin\theta, \end{aligned}$$

which also reads

$$(\alpha\sin\theta + b_3^\dagger\beta\sin\sigma\cos 2\theta + \frac{1}{\sqrt{2}}(b_3^\dagger)^2\gamma(\sin^2\theta - 2\cos^2\theta)\sin\theta\sin^2\sigma)b_1,$$

and therefore the conditional quantum state is

$$|\psi\rangle_{\text{cond-out}} = \alpha\sin\theta\,|1,0,0\rangle + \beta\sin\sigma\cos 2\theta\,|1,0,1\rangle + \gamma(\sin^2\theta - 2\cos^2\theta)\sin\theta\sin^2\sigma\,|1,0,2\rangle . \tag{12}$$

Note that the state is *not* normalized. Its norm gives the probability to obtain such a state.

(c) The output state of a NLS process is

$$\alpha|0\rangle + \beta|1\rangle - \gamma|2\rangle .$$

(12) satisfied this condition if and only if

$$\sin\sigma\cos 2\theta = \sin\theta, \quad (\sin^2\theta - 2\cos^2\theta)\sin\theta\sin^2\sigma = -\sin\theta.$$

Eliminating σ we have

$$(2\cos^2\theta - \sin^2\theta)\frac{\sin^2\theta}{\cos^2 2\theta} = 1,$$

from which we find

$$\sin^2\theta = \frac{3\pm\sqrt{2}}{7}.$$

Since

$$2\cos^2\theta - \sin^2\theta > 0,$$

we throw away solution and only keep the solution $(3-\sqrt{2})/7$. Note that $\cos 2\theta > 0$, so the sign of $\sin\theta$ and σ is the same, and if we add a negative sign to both θ and σ we just get another solution. This is correct because the phase of (12) cannot be determined uniquely. Without loss of generality we take

$$\sin\theta = \sqrt{\frac{3-\sqrt{2}}{7}},$$

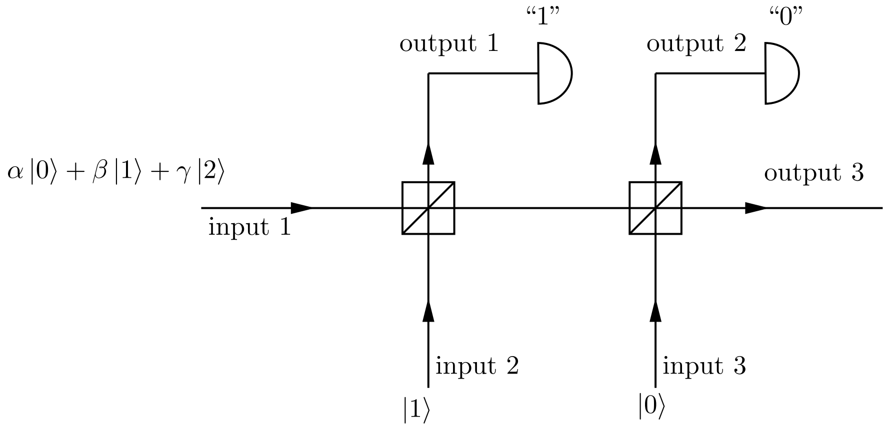

Figure 5: The NLS circuit

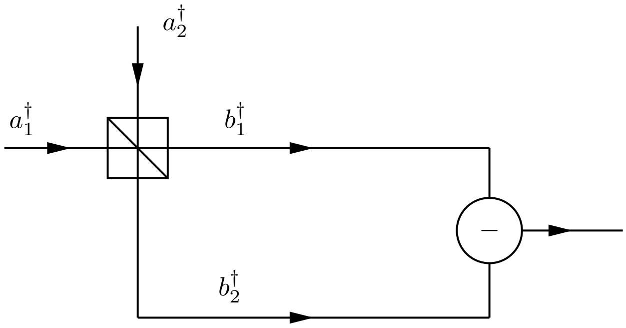

Figure 6: The optical circuit for heterodyne detection and homodyne detection

and hence

$$\sin\sigma = \frac{\sin\theta}{\cos 2\theta} = \frac{\sin\theta}{1-2\sin^2\theta} = \frac{\sqrt{21-7\sqrt{2}}}{1+2\sqrt{2}}.$$

So

$$\theta = \arcsin\sqrt{\frac{3-\sqrt{2}}{7}} = 28.42°, \quad \sigma = \arcsin\frac{\sqrt{21-7\sqrt{2}}}{1+2\sqrt{2}} = 60.49°. \tag{13}$$

Of course,

$$\theta = 180° - 28.42° = 151.58°, \quad \sigma = 180° - 60.49° = 119.51°. \tag{14}$$

When NLS is possible, we have

$$|\psi\rangle_{\text{cond-out}} = \alpha\sin\theta\,|1,0,0\rangle + \beta\sin\theta\,|1,0,1\rangle - \gamma\sin\theta\,|1,0,2\rangle\,.$$

The successful probability is the square of the norm, which is

$$P_{\text{succ}} = \sin^2\theta = \frac{3-\sqrt{2}}{7} = 22.65\,\%. \tag{15}$$

The $1/f$ noise Search for the definition of the $1/f$ noise and analyze its effect on the optical homodyne detection and the optical heterodyne detection.

Solution The $1/f$ **noise** or the so-called **pink noise** is a kind of frequently observed noise of which the power spectrum is like

$$S(f) \lesssim \frac{1}{f^\alpha},$$

where α is between 0 and 2 and is usually around 1 [2]. A main source of the pink noise is the low-frequency fluctuation in condensed matter systems.

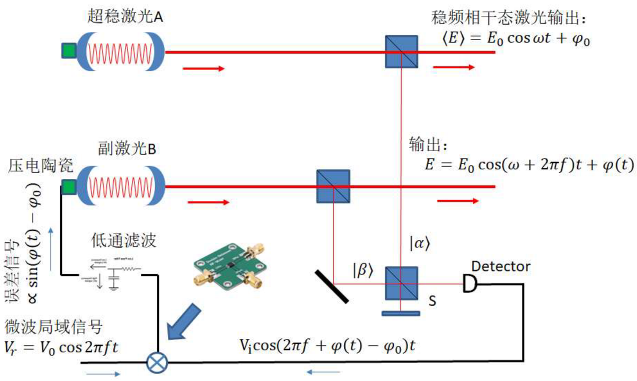

Figure 7: A device for laser phase-locking

Both the homodyne detection and the heterodyne detection can be done with the device shown in Figure 6. We have

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}.$$

The output is

$$\begin{aligned} \langle n_2 - n_1 \rangle &= \frac{1}{2} \langle (a_1^\dagger + a_2^\dagger)(a_1 + a_2) - (a_1^\dagger - a_2^\dagger)(a_1 - a_2) \rangle \\ &= \langle a_1^\dagger a_2 + a_2^\dagger a_1 \rangle , \end{aligned}$$

so after taking the time evolution

$$a_1(t) = \mathrm{e}^{-\mathrm{i}\omega_1 t} a_1(0), \quad a_2(t) = \mathrm{e}^{-\mathrm{i}\omega_2 t} a_2(0)$$

into account, we have

$$\langle n_2 - n_1 \rangle = \mathrm{e}^{\mathrm{i}(\omega_1 - \omega_2)t} \langle a_1^\dagger(0) a_2(0) \rangle + \text{h.c.}. \tag{16}$$

The $\mathrm{e}^{\mathrm{i}\Delta\omega t}$ factor is the only time dependence in (16), and the peak of the actually output of Figure 6 is around $\omega_1 - \omega_2$ in the frequency domain. Therefore the homodyne detection is more affected by the $1/f$ noise because its output is in the low frequency region where the $1/f$ noise is strong, while the heterodyne detection is less affected.

Laser phase-locking Figure 7 is a device for laser-phase locking.

Solution We label the port corresponding to $|\alpha\rangle$ as $a_1^\dagger$, the port corresponding to $|\beta\rangle$ as $a_2^\dagger$, the port corresponding to the detector $b_2^\dagger$, and the port corresponding to S as $b_1^\dagger$. Then we have

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}.$$

The expectation is

$$\begin{aligned} \langle n_2 \rangle &= \langle \alpha, \beta | b_2^\dagger b_2 | \alpha, \beta \rangle \\ &= \frac{1}{2} \langle \alpha, \beta | (a_1^\dagger + a_2^\dagger)(a_1 + a_2) | \alpha, \beta \rangle . \end{aligned}$$

We choose a picture where $a_1^\dagger$ has no time evolution and the time evolution factor of $a_2^\dagger$ is $\mathrm{e}^{\mathrm{i}2\pi f t + \mathrm{i}\varphi(t)}$, and therefore

$$\begin{aligned} \langle n_2 \rangle &= \frac{1}{2} \langle \alpha, \beta | (a_1^\dagger(0) + a_2^\dagger(0) \mathrm{e}^{\mathrm{i}2\pi f t + \mathrm{i}\varphi(t)})(a_1(0) + a_2(0) \mathrm{e}^{-\mathrm{i}2\pi f t - \mathrm{i}\varphi(t)}) | \alpha, \beta \rangle \\ &= \frac{1}{2} \langle \alpha, \beta | (\alpha^* + \beta^* \mathrm{e}^{\mathrm{i}2\pi f t + \mathrm{i}\varphi(t)})(\alpha + \beta \mathrm{e}^{-\mathrm{i}2\pi f t - \mathrm{i}\varphi(t)}) | \alpha, \beta \rangle . \end{aligned}$$

Define the phase of $\alpha\beta^*$ as φ_0 and we have

$$\langle n_2 \rangle = \frac{1}{2} (|\alpha|^2 + |\beta|^2 + 2|\alpha|\,|\beta| \cos(2\pi f t + \varphi(t) + \varphi_0)). \tag{17}$$

Don't know what to do then ...

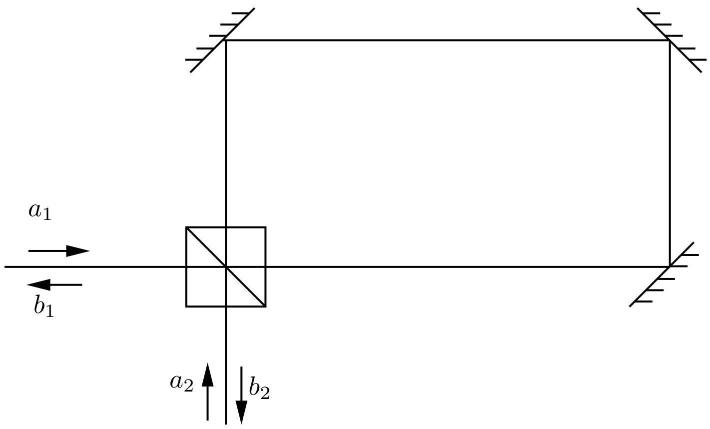

Figure 8: The Sagnac interferometer

Measuring earth rotation with a Sagnac interferometer In a Sagnac interferometer (see Figure 8), input modes a_1, a_2 are mixed by a 50% - 50% beam splitter S, with its outputs follow "time-reversal" pathways of each other so as to be "re-mixed" by the same beam splitter S into b_1 and b_2 modes. (a) Express the linear transformation matrix that couples the input a_1, a_2 and output b_1, b_2 modes. Try to argue that in absence of rotation, the 2×2 transformation matrix S is diagonal, i.e., when the a_1 port is seeded with a laser and a_2 port is left in vacuum state, then the b_2 output port is a "dark port". (b) It turns out when the interferometer is placed in a rotating frame, such as on earth, the counter-propagating light paths around the loop with area A can pick up a "Sagnac" phase,

$$\varphi_{\text{sagnac}} = \frac{4\pi\Omega \cdot A}{c\lambda}$$

Here λ is optical wavelength of the light. In presence of the Sagnac phase, derive the linear transformation matrix S again. (c) Consider a_1 mode is associated with pulsed laser input with duration $\tau = 1\,\text{ms}$. The input states $|\psi_{\text{in}}\rangle = \mathrm{e}^{\alpha a_1^\dagger - \alpha^* a_1}|V\rangle$ is a coherent state in the a_1 mode. For $A = 1\,\text{m}^2$, and let's consider the device is placed at the north pole. Suggest a detection scheme to measure the earth rotation rate (maybe quite difficult, if it is too difficult please allow yourself to be able to "control" the earth rotation). Assuming ideal detection, then how many photons are needed for the interferometer to measure earth rotation rate within 1% accuracy, using a single 1 ms pulse? (e) Provide a detailed argument that the Δn_2 shot noise for the $n_2 = b_2^\dagger b_2$ measurement in this Sagnac interferometer is a result of vacuum fluctuation with $|V\rangle$ enters from the a_2 port to the b_2 port. (f) Provide a plausible experimental arrangement to inject a "squeezed vacuum" into a_2 port, so as to improve the rotation measurement accuracy by a e^{ξ} factor.

Solution

(a) The matrix of S is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Its time reversal is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and after being reflected by all the mirrors beam 1 is at the position of the output port 2 and beam 2 is at the position of the output port 1, so the transformation matrix for the two reflected beams can be obtained by swapping the columns of $S^\dagger$, so the final matrix of the whole system is

$$S_{\text{total}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{18}$$

The matrix is indeed diagonal, so port b_2 is a dark port without input in a_2 port.

(b) Now the total matrix is

$$\begin{aligned}S_{\text{total}}(\varphi) &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cdot\begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi} & \\ & \mathrm{e}^{-\mathrm{i}\varphi}\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= \begin{pmatrix}\cos\varphi & -\mathrm{i}\sin\varphi\\ \mathrm{i}\sin\varphi & -\cos\varphi\end{pmatrix},\end{aligned} \tag{19}$$

where φ is the Sagnac phase.

(c) From

$$\begin{pmatrix}b_1^\dagger\\ b_2^\dagger\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\mathrm{i}\sin\varphi\\ \mathrm{i}\sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}a_1^\dagger\\ a_2^\dagger\end{pmatrix}$$

we find

$$\begin{pmatrix}a_1^\dagger\\ a_2^\dagger\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\mathrm{i}\sin\varphi\\ \mathrm{i}\sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}b_1^\dagger\\ b_2^\dagger\end{pmatrix}. \tag{20}$$

By substituting

$$a_1^\dagger = \cos\varphi b_1^\dagger - \mathrm{i}\sin\varphi b_2^\dagger$$

into $|\psi_{\text{in}}\rangle$, we obtain

$$\begin{aligned}|\psi_{\text{out}}\rangle &= \mathrm{e}^{\alpha(\cos\varphi b_1^\dagger - \mathrm{i}\sin\varphi b_2^\dagger) - \alpha^*(\cos\varphi b_1 + \mathrm{i}\sin\varphi b_2)}|0\rangle\\ &= |\alpha\cos\varphi, -\mathrm{i}\alpha\sin\varphi\rangle\,.\end{aligned} \tag{21}$$

We need to measure n_2 with (21). The expectation of n_2 is $|\alpha|^2\sin^2\varphi$, so the measure scheme may be measuring n_2 and calculating Ω using

$$\langle n_2\rangle = |\alpha|^2\sin^2\varphi, \quad \varphi = \frac{4\pi\Omega\cdot A}{c\lambda}. \tag{22}$$

From the properties of coherent states we also know that

$$\Delta n_2 = |\alpha|\sin\varphi = \sqrt{\langle n_2\rangle}. \tag{23}$$

Note that since φ is small, we have

$$\langle n_2\rangle \propto \varphi^2 \propto \Omega^2,$$

so we have

$$\frac{2\Delta\Omega}{\Omega} = \frac{\Delta n_2}{\langle n_2\rangle} = \frac{1}{\sqrt{\langle n_2\rangle}},$$

and therefore

$$\langle n_2\rangle = \left(\frac{\Omega}{2\Delta\Omega}\right)^2, \quad N_{\text{in}} = |\alpha|^2 = \left(\frac{c\lambda}{4\pi\Omega A}\right)^2\left(\frac{\Omega}{2\Delta\Omega}\right)^2. \tag{24}$$

Since the wavelength is not determined, the number of input photons cannot be determined. With the precision requirement $\Delta\Omega/\Omega = 1\,\%$ we need at least to detect 2500 photons at the $b_2^\dagger$ port.

Interferometric measurement of a free falling object An optical circuit is built to measure the position of a free-falling object, shown in Figure 9. Analyze the accuracy of the equipment in terms of the wavelength and the power of the light beam.

Solution The structure of the optical circuit of Figure 9 is quite like Figure 8. The main difference is that in Figure 8 a pulse may travel from one retroreflector to another over and over again.

Don't know what to do then ... If we view the whole system as a two-port system, then no information about the distance between the freely falling body and the reference retroreflector can be passed out.

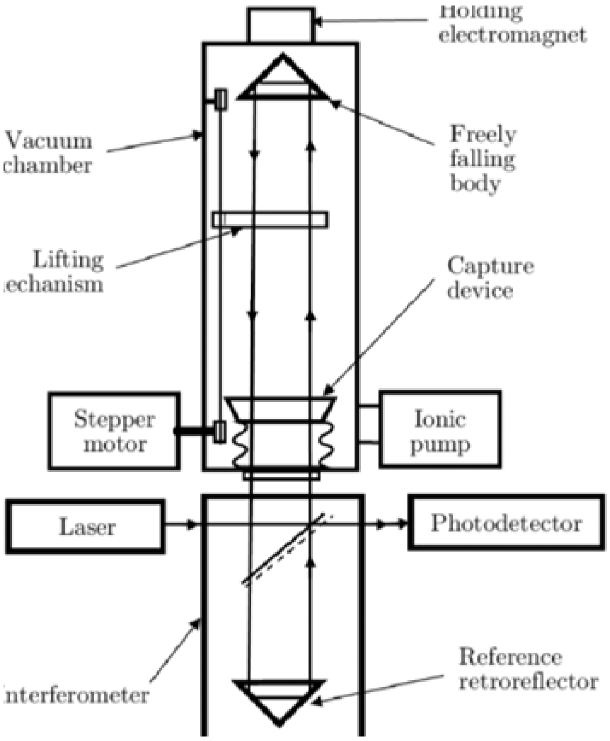

Figure 9: Measuring the position of a free-falling object

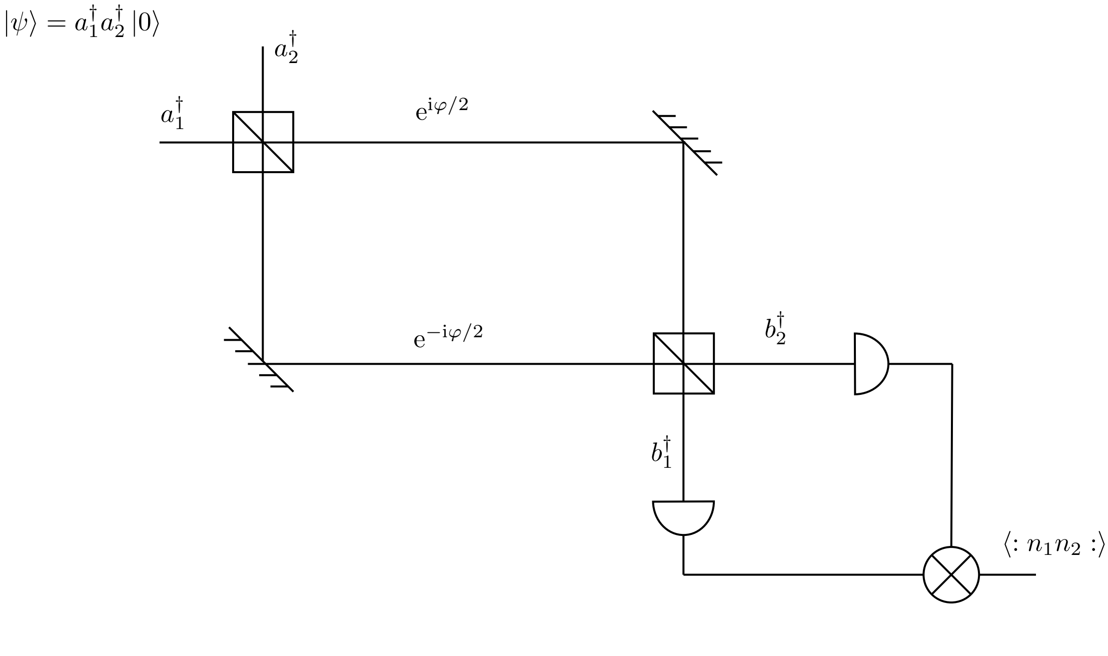

Figure 10: The M-Z interferometer

Measuring a phase shift with a double-photon interferometer A double-photon interferometer shown in Figure 10 can be used to measure the phase shift φ. Derive the output and the fluctuation.

Solution The transformation matrix of Figure 10 is

$$\begin{aligned} S &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} & \\ & \mathrm{e}^{-\mathrm{i}\varphi/2} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{i}\sin\varphi/2 & -\cos\varphi/2 \\ \cos\varphi/2 & -\mathrm{i}\sin\varphi/2 \end{pmatrix}. \end{aligned} \tag{25}$$

We therefore have

$$\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} = S^{-1}\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} -\mathrm{i}b_1^\dagger \sin\varphi/2 + b_2^\dagger \cos\varphi/2 \\ -b_1^\dagger \cos\varphi/2 + \mathrm{i}b_2^\dagger \sin\varphi/2 \end{pmatrix}. \tag{26}$$

The wave function of the optical field, therefore, is

$$\begin{aligned} |\psi\rangle &= a_1^\dagger a_2^\dagger |0\rangle = (-\mathrm{i}b_1^\dagger \sin\varphi/2 + b_2^\dagger\cos\varphi/2)(-b_1^\dagger\cos\varphi/2 + \mathrm{i}b_2^\dagger \sin\varphi/2)|0\rangle \\ &= \frac{\mathrm{i}}{\sqrt{2}}\sin\varphi\,|2,0\rangle + \frac{\mathrm{i}}{\sqrt{2}}\sin\varphi\,|0,2\rangle - \cos\varphi\,|1,1\rangle. \end{aligned} \tag{27}$$

The expectation of $: n_1 n_2 :$ is

$$\langle\psi|b_1^\dagger b_2^\dagger b_2 b_1|\psi\rangle = \cos^2\varphi \tag{28}$$

because the operators $b_1 b_2$ destroys $|0,2\rangle$ and $|2,0\rangle$, and we have

$$P(\text{result} = 1) = \cos^2\varphi, \quad P(\text{result} = 2) = \sin^2\varphi. \tag{29}$$

The standard deviation is therefore given by

$$\begin{aligned} \Delta(: n_1 n_2 :)^2 &= P(\text{result} = 1)(1 - \cos^2\varphi)^2 + P(\text{result} = 0)(0 - \cos^2\varphi)^2 \\ &= \cos^2\varphi\sin^4\varphi + \sin^2\varphi\cos^4\varphi \\ &= \sin^2\varphi\cos^2\varphi, \end{aligned}$$

or

$$\Delta(: n_1 n_2 :) = \sin\varphi\cos\varphi. \tag{30}$$

With N independent experiments (assuming N is large), we have

$$\sum \langle : n_1 n_2 : \rangle = N\cos^2\varphi,$$

and

$$\Delta\left(\sum : n_1 n_2 :\right) = \sqrt{N}\sin\varphi\cos\varphi.$$

Therefore, we have

$$\begin{aligned} \Delta\varphi &= \frac{\Delta(\sum : n_1 n_2 :)}{\partial \sum \langle : n_1 n_2 : \rangle / \partial\varphi} \\ &= \frac{\sqrt{N}\sin\varphi\cos\varphi}{N \times 2\cos\varphi\sin\varphi} \\ &= \frac{1}{2\sqrt{N}}, \end{aligned}$$

so the fluctuation of φ caused by the shot-noise error of the optical field is

$$\Delta\varphi = \frac{1}{2\sqrt{N}}. \tag{31}$$

References

[1] Daniel A. Steck. Quantum and atom optics. available online at http://steck.us/teaching, 2020.

[2] Wikipedia. Pink noise. https://en.wikipedia.org/wiki/Pink_noise. Accessed: 2021-11-06 11:39:16.