Solution to Topological Quantum: Lecture Notes and Proto-Book

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Introduction: History of Topology, Knots, Peter Tait and Lord Kelvin

Chapter 2

Kauffman Bracket Invariant and Relation to Physics

2.1 Trefoil Knot and the Kauffman Bracket

Using the Kauffman rules, calculate the Kauffman bracket invariant of the right and left handed trefoil knots shown in Fig. 2.1. Conclude these two knots are topologically inequivalent. While this statement appears obvious on sight, it was not proved mathematically until 1914 (by Max Dehn). It is trivial using this technique!

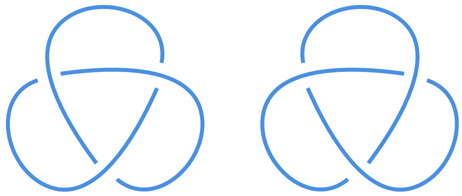
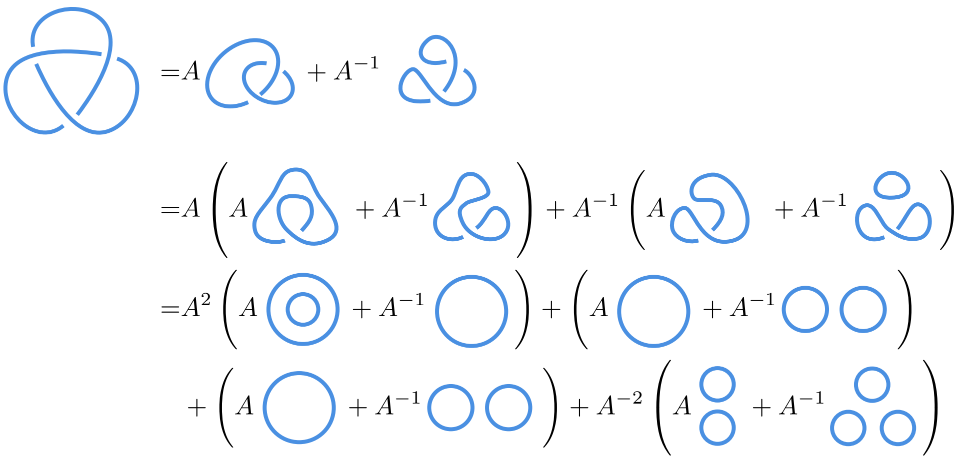


Figure 2.1: Left and Right Handed Trefoil Knots (on the left and right respectively)

Remark The word "trefoil" is from the plant trifolium, or clover, which has compound trifoliate leaves.

Answer We try to figure out the right trefoil first:



$$=A^2(Ad^2+A^{-1}d)+(Ad+A^{-1}d^2)+(Ad+A^{-1}d^2)+A^{-2}(Ad^2+A^{-1}d^3)$$
$$=-A^{-9}+A^{-1}+A^3+A^7.$$

Similar calculation gives

$$=-A^9+A^1+A^{-3}+A^{-7}.$$

We can see the mirror symmetry gives $A \to A^{-1}$. In other notations, we can set $d^i \to d^{i-1}$, then the trivial knot(i.e. a circle) give 1. Then we have

$$= A^{-7} - A^5 - A^3.$$

2.2 Abelian Kauffman Anyons

Anyons described by the Kauffman bracket invariant with certain special values of the constant A are abelian anyons - meaning that an exchange introduces only a simple phase as shown in Fig. 2.2.

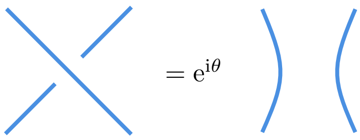


Figure 2.2: For abelian anyons, exchange gives a phase $\mathrm{e}^{\mathrm{i}\theta}$.

(a) For $A = \pm \mathrm{e}^{\mathrm{i}\pi/3}$ (and the complex conjugates of these values), show that the anyons are bosons or fermions respectively (i.e., $\mathrm{e}^{\mathrm{i}\theta} = \pm 1$).

(b) For $A = \pm \mathrm{e}^{\mathrm{i}\pi/6}$ (and the complex conjugates of these values) show the anyons are semions (i.e., $\mathrm{e}^{\mathrm{i}\theta} = \pm \mathrm{i}$). In fact these are precisely the anyons that arise for the $\nu = 1/2$ fractional quantum Hall effect of bosons (We will discuss this later in this book (See chapter 37). This particular phase of quantum Hall matter has been produced experimentally(Clark et al. [2020]), but only in very small puddles so far and it has not been possible to measure braiding statistics as of yet.

HINT: For (a) and (b) show first the identity shown in Fig. 2.3.

If you can't figure it out, try evaluating the Kauffman bracket invariant for a few knots with these values of A and see how the result arises.

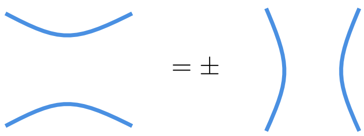
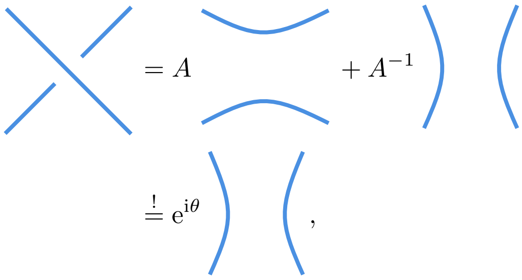
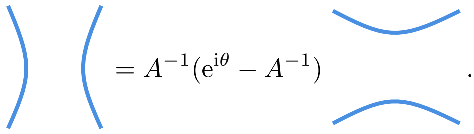


Figure 2.3: For bosons or fermions the sign in this figure is $+$, for semions the sign is $-$.

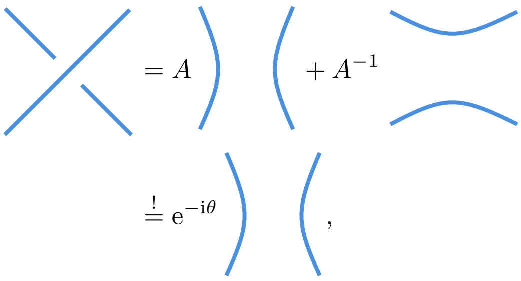
Answer Using the definition of Kauffman bracket:



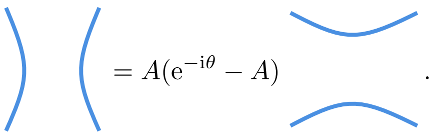
which means we have:



At the same time, we also have



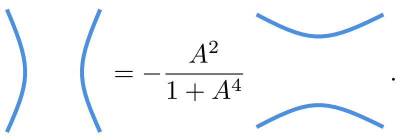
Because exchange two particles in one certain direction gives a phase factor $e^{i\theta}$. Then



So we have the equation

$$A(e^{-i\theta} - A) = A^{-1}(e^{i\theta} - A^{-1}) \Rightarrow e^{i\theta} = -A^3 \text{ or } A^{-1}.$$

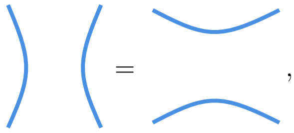
Where A^{-1} is a trivial solution. By the way, we also have



(a) For $A = \pm e^{i\pi/3}$, we have

$$e^{i\theta} = \pm 1,$$

which gives bosons for fermions. And



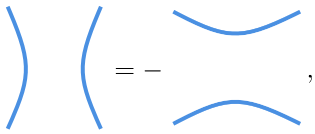
(2.1)

which are fermions and bosons.

(b) For $A = \pm e^{i\pi/6}$, we have

$$e^{i\theta} = \mp i,$$

which gives semions. And



(2.2)

which are semions.

Such illustration can also be viewed as if we first acknowledge equ. 2.1 and equ. 2.2, then $A = \pm e^{i\pi/3}$ will give $A^2 + A^{-2} = \pm 1$, and $A = \pm e^{i\pi/6}$ will give $A^2 - A^{-2} = \pm i$.

2.3 Reidermeister moves and the Kauffman Bracket

Show that the Kauffman bracket invariant is unchanged under application of Reidermeister move of type II and type III. Thus conclude that the Kauffman invariant is an invariant of regular isotopy.

Answer (a) The Reidermeister move of type II is given by Fig.2.4. Then according to the

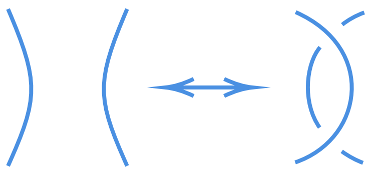
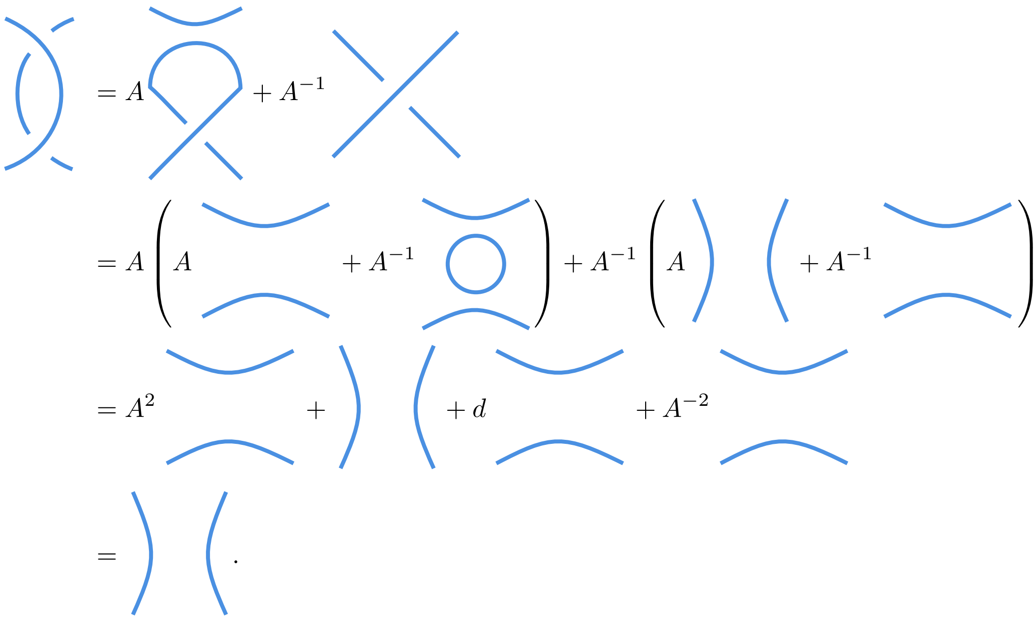


Figure 2.4: Reidermeister move of type II

definition of the Kauffman bracket, we have:



In the last line we use $d = -A^2 - A^{-2}$. Now we can see type II is invariant.

(b) The Reidermeister move of type III is given by Fig.2.5. Then according to the definition

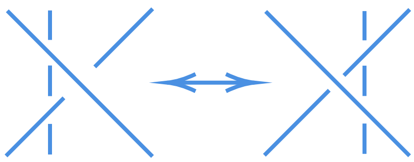
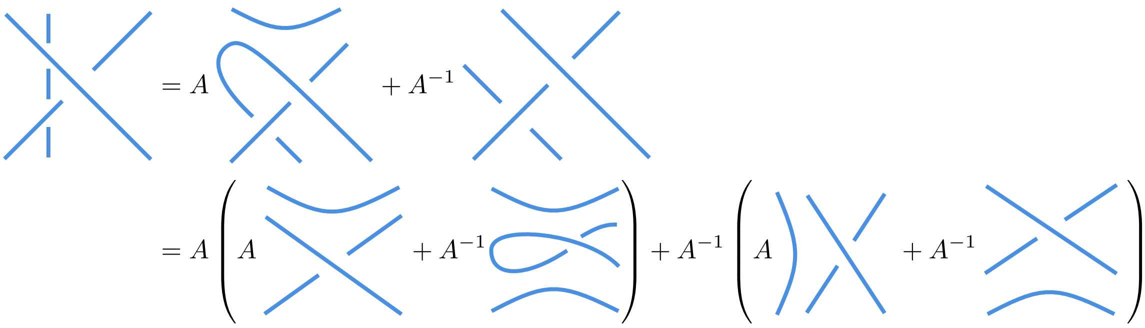
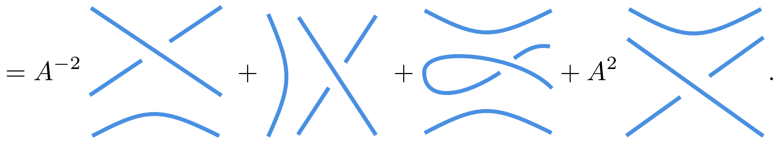


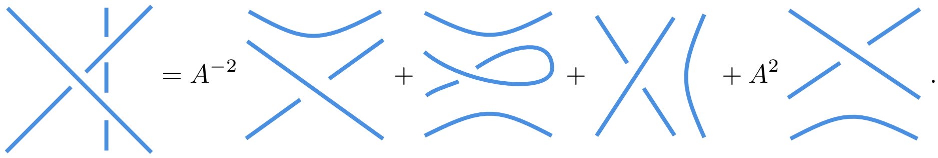
Figure 2.5: Reidermeister move of type III

of the Kauffman bracket, we have:

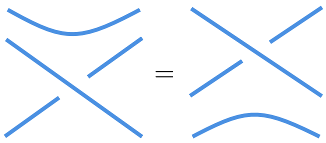




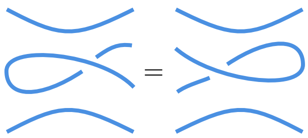
Similarly, we can also work out that



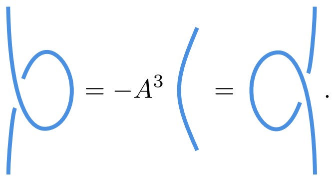
If we expand, we will find



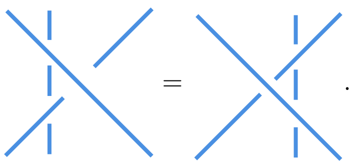
trivially and



using



So we have



Now we can see type III is invariant. Thus we can conclude that the Kauffman invariant is an invariant of regular isotopy.

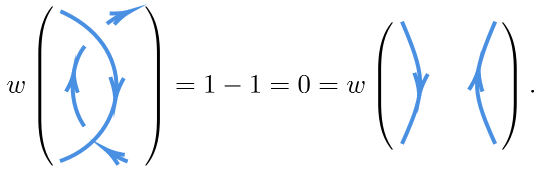
2.4 Jones polynomial

Let us define the Jones polynomial of an oriented knot as

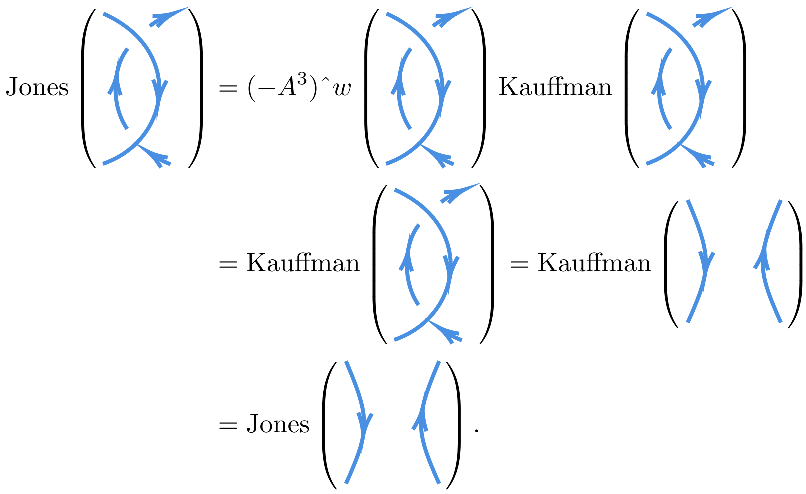
$$\text{Jones(knot)} = (-A^3)^{w(\text{knot})}\,\text{Kauffman(knot)}$$

where w is the writhe (We must first orient the knot, meaning we arrows on the strands, in order to define a writhe). Show that this quantity is an invariant of ambient isotopy - that is, it is invariant under all three Reidermeister moves.

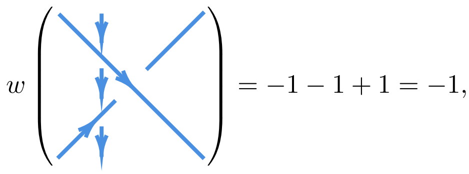
Answer We first consider the regular isotopy. For type II move:



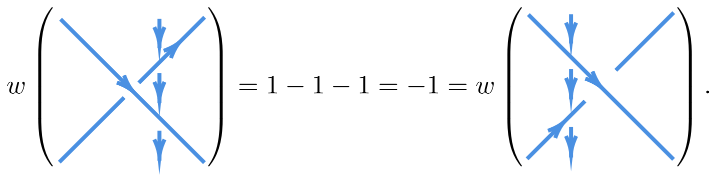
So



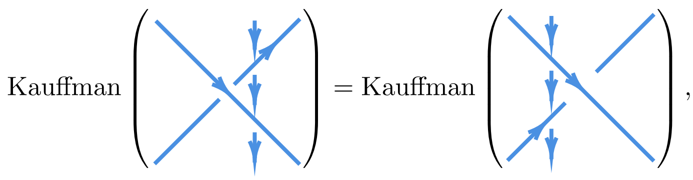
For type III, we have



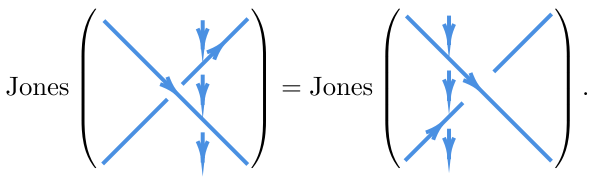
while



Since



we also have



Note here the initial direction of the lines doesn't matter, but the relative direction of the initial points and final points between two knots should be the same.

Now we can see Jones polynomial is invariant under regular isotopy. Then we consider the Reidermeister move of type I in Fig. 2.6.

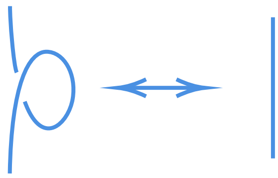


Figure 2.6: Reidermeister move of type I

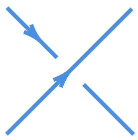
However:

w () $= 1,$ (2.3)

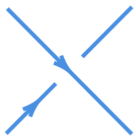
while

w () $= 0.$

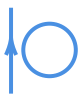
Note that in (2.3), the direction is different from

 $= -1,$

but same as

 $= 1$

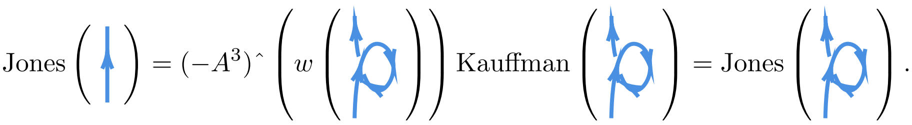
with a $90°$ rotation. Then using

Kauffman () = Kauffman (A ) + Kauffman (A^{-1} )

= Kauffman ($(A + A^{-1}d)$ ) $= -A^{-3}$ Kauffman ()

$= (-A^3)^{\wedge}$ ($-w$ ()) Kauffman ()

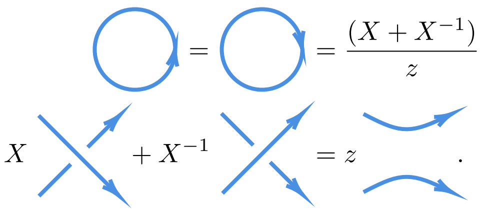
$= (-A^3)^{\wedge}$ ($-w$ ()) Jones (), which means



So Jones polynomial is invariant of ambient isotopy.

2.5 HOMFLY Polynomial

The HOMFLY polynomial is a generalization of the Jones polynomial which has two variables X and z rather than just one variable. To define the HOMFLY polynomial we must first orient the strings in our knot or link (meaning we put arrows on the lines). The HOMFLY polynomial (Freyd et al. [1985]; Przytycki and Traczyk [1987]) of an oriented link is then defined in terms of two variables X and z by the two rules



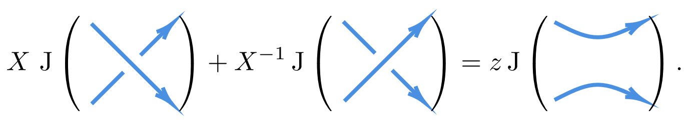
(2.4)

(a) Given the definition of the Jones polynomial in Exercise 2.4, for what value of X and z does the HOMFLY polynomial become the Jones polynomial?

(b) Calculate the HOMFLY polynomial of the right and left handed trefoil knots (shown in Fig.2.1).

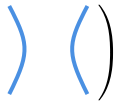
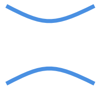
Remark HOMFLY is an acronym of the names of the inventors of this polynomial. Sometimes credit is even more distributed and it is called HOMFLYPT.

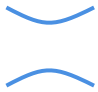
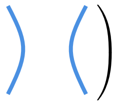
Answer (a) For simplicity, we use H(Knot) to indicate the HOMFLY polynomial, J(Knot) to indicate the Jones polynomial and K(Knot) to refer the Kauffman polynomial. Now, if H(Knot) = J(Knot), we must have

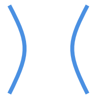
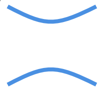


Left hand side gives

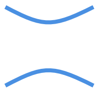
$$\text{LHS} = X(-A^3)^1\mathrm{K}\Big($$  $$\Big) + X^{-1}(-A^3)^{-1}\mathrm{K}\Big($$  $$\Big)$$

$$= -XA^3\Big(A^{-1}\mathrm{K}\Big($$  $$\Big) + A\mathrm{K}\Big($$  $$\Big)\Big)$$

$$- X^{-1}A^{-3}\Big(A^{-1}\mathrm{K}\Big($$  $$\Big) + A\mathrm{K}\Big($$  $$\Big)\Big)$$

$$= -(XA^2 + X^{-1}A^{-2})\mathrm{K}\Big($$  $$\Big) - (XA^4 + X^{-1}A^{-4})\mathrm{K}\Big($$  $$\Big).$$

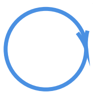
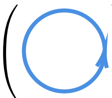
While

$$\text{RHS} = z\mathrm{K}\Big($$  $$\Big),$$

So we must have

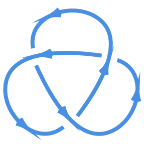
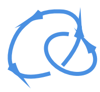
$$\begin{cases} XA^4 + X^{-1}A^{-4} = 0 \\ XA^2 + X^{-1}A^{-2} = -z \end{cases} \Rightarrow \begin{cases} X = \mathrm{i}A^{-4} \\ z = \mathrm{i}(A^2 - A^{-2}) \end{cases}. \tag{2.5}$$

We can check in this case:

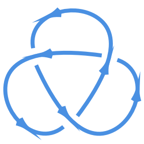
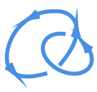
$$\mathrm{H}\Big($$  $$\Big) = \mathrm{H}\Big($$  $$\Big) = \frac{(X + X^{-1})}{z} = -A^2 - A^{-2} = \mathrm{J}\Big($$  $$\Big).$$

So (3) is indeed the condition that HOMFLY polynomial becomes the Jones polynomial.

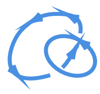
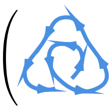
(b) We first assign a direction to the right handed trefoil. Then the second skein relation gives:

$$X\mathrm{H}\Big($$  $$\Big) + X^{-1}\mathrm{H}\Big($$  $$\Big) = z\mathrm{H}\Big($$  $$\Big)$$

But the second term in the left hand side is a trivial knot, i.e. a circle, so

$$\mathrm{H}\Big($$  $$\Big) = zX^{-1}\mathrm{H}\Big($$  $$\Big) - \frac{X^{-2}(X + X^{-1})}{z}.$$

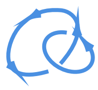
Then:

$$X\mathrm{H}\Big($$  $$\Big) + X^{-1}\mathrm{H}\Big($$  $$\Big) = z\mathrm{H}\Big($$  $$\Big).$$

But note here the right hand side is trivial, while

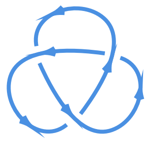
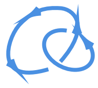
$\mathrm{H}\Big($  $\Big) = \mathrm{H}\Big($  $\Big) = \dfrac{(X+X^{-1})^2}{z^2},$

We have

$\mathrm{H}\Big($  $\Big)$

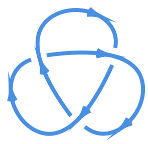
$$= X^{-1}\left((X+X^{-1}) - X^{-1}\frac{(X+X^{-1})^2}{z^2}\right) = \frac{\left(X^2+1\right)\left(X^2z^2-X^2-1\right)}{X^4z^2}$$

Finally:

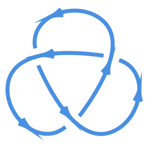
$\mathrm{H}\Big($  $\Big) = zX^{-1}\mathrm{H}\Big($  $\Big) - \dfrac{X^{-2}(X+X^{-1})}{z}$

$$= \frac{(1+X^2)(-1+X^2(-2+z^2))}{X^5z}.$$

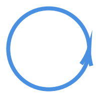
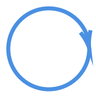
Similarly, we can get for the left trefoil:

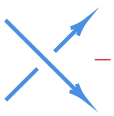
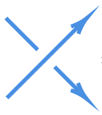
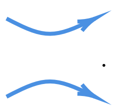
$\mathrm{H}\Big($  $\Big) = -\dfrac{X(X^2+1)(X-z^2+2)}{z}.$

With condition 2.5, we can also get the Jones polynomial:

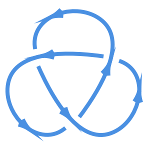
$\mathrm{J}\Big($  $\Big) = -((1-\mathrm{i})A^{14}) + \mathrm{i}A^{10} - A^2.$

However, in common normalization, we always set

 $=$  $= 1$ (2.6)

X  $-X^{-1}$  $= z$  .

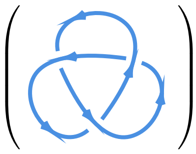
In this case:

$\mathrm{H}\Big($  $\Big) = 2X^{-2} - X^{-4} + X^{-2}z^2.$

Now equation 2.5 becomes

$$\begin{cases} X = A^{-1} \\ z = A^{1/2} - A^{-1/2} \end{cases},$$

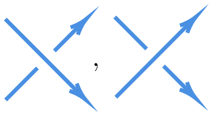
then the standard result of Jones polynomial gives:

J  $= A + A^3 - A^4.$

We also note the mirror symmetry of HOMFLY polynomial:

$$\mathrm{H}(\text{Knot}; X, z) = \mathrm{H}(\text{mirror image}; X^{-1}, z). \tag{2.7}$$

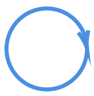
This comes from a simple observation: exchange

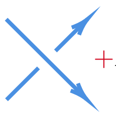
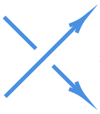
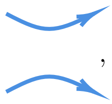


in 2.4 has the same effect of $X \to X^{-1}$. This means for the relation 2.6, the symmetry 2.7 becomes

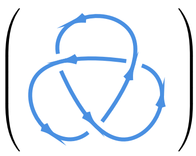
$$\mathrm{H}(\text{Knot}; X, z) = \mathrm{H}(\text{mirror image}; X^{-1}, -z).$$

Another common skein relation is that

 $=$  $= 1$

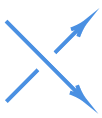
X  $+X^{-1}$  $= z$  ,

in this case:

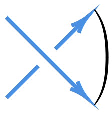
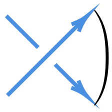
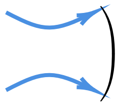
H  $= X^{-2}z^2 - 2X^{-2} - X^{-4}.$

2.6 Knot sum

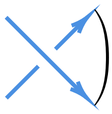
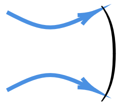
We can imagine that a bigger knot contains such two knots with a crossing labeled by arrows:



then we can use HOMFLY the second rule to derive that:

$XH\Big($  $\Big) + X^{-1}H\Big($  $\Big) = zH\Big($  $\Big)$

Now set X, z in equ. 2.5, using Kauffman rule, it's easy to see that the two knots in the left side are the same (writhe are not important now), then we have:

$(A^{-4} - A^{4})K\Big($  $\Big) = (A^{2} - A^{-2})K\Big($  $\Big)$

It's easy to read that:

$$\begin{aligned} -(A^2 + A^{-2})K(K_1\#K_2) &= K(K_1)K(K_2) \\ K(K_1\#K_2) &= d^{-1}K(K_1)K(K_2) \end{aligned} \tag{2.8}$$

Part I

Anyons and Topological Quantum Field Theories

Chapter 3

Particle Quantum Statistics

3.1 About the Braid Group

(a) Convince yourself geometrically that the defining relations of the braid group on M particles B_M are:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & 1 \le i \le M-2 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ for } \quad |i-j|>1, & 1 \le i,j \le M-1 \end{aligned} \tag{3.1}$$

(b) Instead of thinking about particles on a plane, let us think about particles on the surface of a sphere. In this case, the braid group of M strands on the sphere is written as $B_M(S^2)$. To think about braids on a sphere, it is useful to think of time as being the radial direction of the sphere, so that braids are drawn as in Fig.3.1.

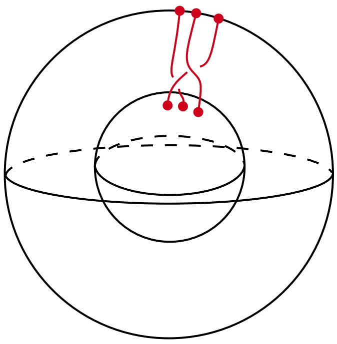


Figure 3.1: An element of the braid group $B_3(S^2)$. The braid shown here is $\sigma_1\sigma_2^{-1}$

The braid generators on the sphere still obey (3.1), but they also obey one additional identity

$$\sigma_1\sigma_2 \ldots \sigma_{M-2}\sigma_{M-1}\sigma_{M-1}\sigma_{M-2} \ldots \sigma_2\sigma_1 = I \tag{3.2}$$

where I is the identity (or trivial) braid. What does this additional identity mean geometrically? [In fact, for understanding the properties of anyons on a sphere,(3.2) is not quite enough. We will try to figure out below why this is so by using Ising Anyons as an example.]

Answer The second one of (3.1) is trivial, while the first one is due to the Yang-Baxter equation. Take B_3 for example, the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ equals to Fig.3.2.

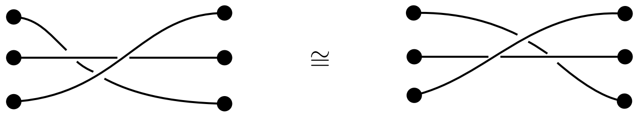


Figure 3.2: The configuration of $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

(b) We also take $B_3(S^2)$ for example. Relation (3.2) now becomes

$$\sigma_1\sigma_2\sigma_2\sigma_1 = I,$$

here we define σ_1 to be as Fig.3.3 shows.

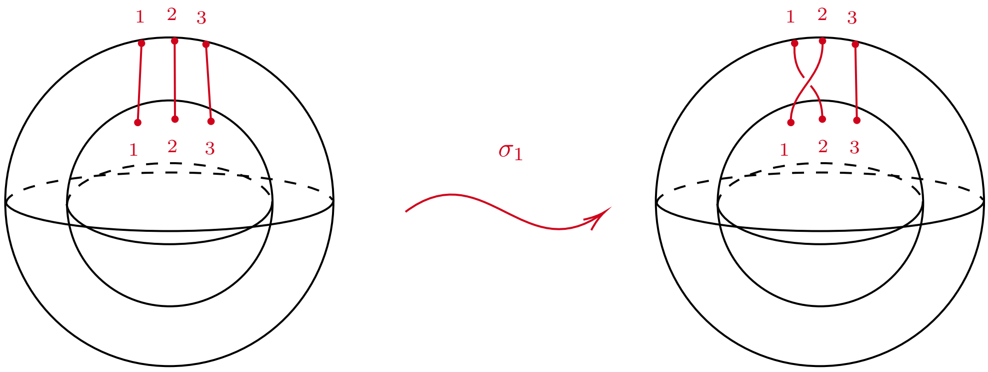


Figure 3.3: The effect of σ_1 on S^2.

Then the $\sigma_1\sigma_2\sigma_2\sigma_1$ becomes Fig.3.4. This means the line can get cross the sphere and return

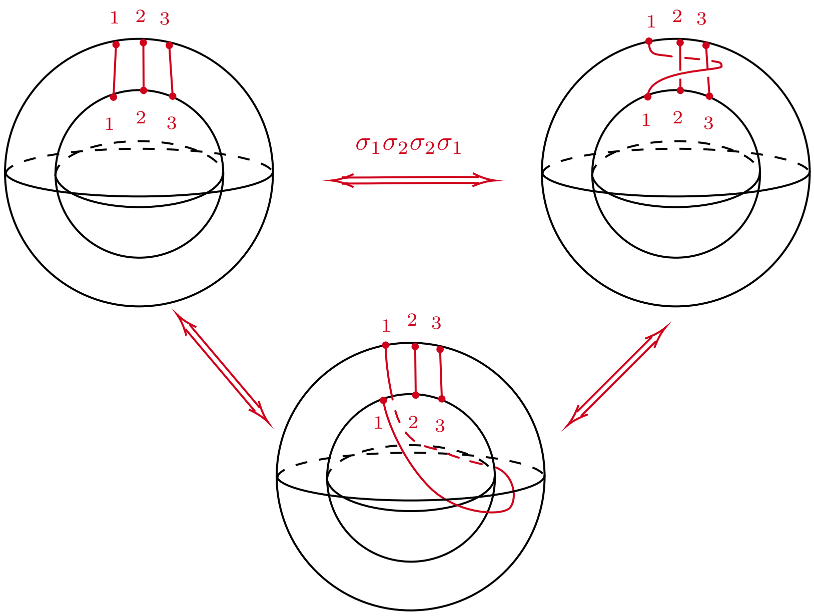


Figure 3.4: Effect of $\sigma_1\sigma_2\sigma_2\sigma_1$.

to trivial configuration.

3.2 About the Symmetric Group

Show that (3.1) also hold for the generators of the symmetric group S_M on M particles, where σ_i exchanges particle i and $i+1$. In the symmetric group we have the additional condition that $\sigma_i^2 = 1$. Prove the statement used in section 3.4.1 that there are only two one-dimensional representations of the symmetric group. Hint: The proof is just a few lines. Use $\rho(\sigma_i)\rho(\sigma_j) = \rho(\sigma_i\sigma_j)$ where ρ is a representation.

Answer (a) For the element in symmetric group S_M, we can write σ_i as $(i, i+1)$. Then

$$\sigma_i\sigma_j = (i, i+1)(j, j+1) = (j, j+1)(i, i+1) = \sigma_j\sigma_i, \quad |i-j| > 1,$$

is trivial. The first one gives:

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= (i, i+1)(i+1, i+2)(i, i+1) \\
&= (i, i+1, i+2)(i, i+1) \\
&= (i+2, i, i+1)(i, i+1) \\
&= (i+2, i)
\end{aligned}$$

while

$$\begin{aligned}
\sigma_{i+1}\sigma_i\sigma_{i+1} &= (i+1, i+2)(i, i+1)(i+1, i+2) \\
&= (i+2, i+1, i)(i+1, i+2) \\
&= (i, i+2, i+1)(i+1, i+2) \\
&= (i, i+2) = \sigma_i\sigma_{i+1}\sigma_i,
\end{aligned}$$

which also holds.

(b) Suppose $\rho(\sigma_1) = c$, which means

$$\rho(\sigma_1)\rho(\sigma_1) = c^2 = \rho(\sigma_1\sigma_1) = \rho(I) = 1,$$

thus $c^2 = 1$ have only two possibilities $c = \pm 1$. Suppose there exist i, such that $\rho(\sigma_i) = 1$ while $\rho(\sigma_{i+1}) = -1$, then according to (3.1):

$$\rho(\sigma_i\sigma_{i+1}\sigma_i) = -1 \neq 1 = \rho(\sigma_i\sigma_{i+1}\sigma_i).$$

So $\rho(\sigma_i)$ and $\rho(\sigma_{i+1})$ have the same sign, either 1 or -1, then all $\rho(\sigma_i)$ have the same sign. We also know any permutation can be generate by $\sigma_i, i \in \{1, \cdots, M\}$, we can see there are only two kinds of 1-d representation of S_M.

3.3 Ising Anyons and Majorana Fermions

The most commonly discussed type of nonabelian anyon is the Ising anyon (we will discuss this in more depth later). Ising anyons occurs in the Moore-Read quantum Hall state ($\nu = 5/2$), as well as in any chiral p-wave superconductor and in recently experimentally relevant so called "Majorana" systems.

The nonabelian statistics of these anyons may be described in terms of Majorana fermions by attaching a Majorana operator to each anyon. The Hamiltonian for these Majoranas is zero - they are completely noninteracting. In case you haven't seen them before, Majorana Fermions γ_j satisfy the anticommutation relation

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$$

as well as being self conjugate $\gamma_i^\dagger = \gamma_i$.

(a) Show that the ground state degeneracy of a system with $2N$ Majoranas is 2^N if the Hamiltonian is zero. Thus conclude that each *pair* of Ising anyons is a two-state system. Hint: Construct a regular (Dirac) fermion operator from two Majorana fermion operators. For example,

$$c^\dagger = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2)$$

will then satisfy the usual fermion anti-commutation $\{c, c^\dagger\} = cc^\dagger + c^\dagger c = \mathbb{1}$. (If you haven't run into fermion creation operators yet, you might want to read up on this first!) There is more discussion of this transformation in later exercises 9.7 and 10.2.

(b) When anyon i is exchanged clockwise with anyon j, the unitary transformation that occurs on the ground state is

$$U_{ij} = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}[1 + \gamma_i\gamma_j], i < j. \tag{3.3}$$

for some real value of α. Show that these unitary operators form a representation of the braid group. (Refer back to the previous problem, "About the Braid Group"). In other words we must show that replacing σ_i with $U_{i,i+1}$ in (3.1) yields equalities. This representation is 2^N dimensional since the ground state degeneracy is 2^N.

(c) Consider the operator

$$\gamma^{\text{FIVE}} = (\mathrm{i})^N \gamma_1\gamma_2 \ldots \gamma_{2N}$$

(the notation five is in analogy with the γ^5 of the Dirac gamma matrices). Show that the eigenvalues of γ^{FIVE} are ± 1. Further show that this eigenvalue remains unchanged under any braid operation. Conclude that we actually have two 2^{N-1} dimensional representations of the braid group. We will assume that any particular system of Ising anyons is in one of these two representations.

(d) Thus, 4 Ising anyons on a sphere comprise a single 2-state system, or a qubit. Show that by only braiding these four Ising anyons one cannot obtain all possible unitary operation on this qubit. Indeed, braiding Ising anyons is not sufficient to build a quantum computer. [Part (d) is not required to solve parts (e) and (f)]

(e) (bit harder)Now consider $2N$ Ising anyons on a sphere (See above problem "About the braid group" for information about the braid group on a sphere). Show that in order for either one of the 2^{N-1} dimensional representations of the braid group to satisfy the sphere relation, (3.2), one must choose the right abelian phase α in (3.3). Determine this phase.

(f) (a bit harder) The value you just determined is not quite right. It should look a bit unnatural as the abelian phase associated with a braid depends on the number of anyons in the system. Go back to (3.2) and insert an additional abelian phase on the right hand side which will make the final result of part (e) independent of the number of anyons in the system. In fact, there should be such an additional factor - to figure out where it comes from, go back and look again at the geometric "proof" of (3.2). Note that the proof involves a self-twist of one of the anyon world lines. The additional phase you added is associated with one particle twisting around itself. The relation between self-rotation of a single particle and exchange of two particles is a generalized spin-statistics theorem.

Answer We can use two Majorana fermions to construct a normal Dirac fermion by the transformation

$$c_i^\dagger = \frac{1}{2}(\gamma_{2i-1} + \mathrm{i}\gamma_{2i}), \quad c_i = \frac{1}{2}(\gamma_{2i-1} - \mathrm{i}\gamma_{2i}),$$

where $i = 1, \cdots, N$. We can check:

$$\begin{aligned}\{c_i, c_j^\dagger\} = c_i c_j^\dagger + c_j^\dagger c_i &= \frac{1}{4}(\gamma_{2i-1} - \mathrm{i}\gamma_{2i})(\gamma_{2j-1} + \mathrm{i}\gamma_{2j}) + \frac{1}{4}(\gamma_{2j-1} + \mathrm{i}\gamma_{2j})(\gamma_{2i-1} - \mathrm{i}\gamma_{2i}) \\ &= \frac{1}{4}(\gamma_{2i-1}\gamma_{2j-1} - \mathrm{i}\gamma_{2i}\gamma_{2j-1} + \mathrm{i}\gamma_{2i-1}\gamma_{2j} + \gamma_{2i}\gamma_{2j} + (i \leftrightarrow j, \mathrm{i} \leftrightarrow -\mathrm{i})) \\ &= \delta_{ij}.\end{aligned}$$

Therefore, the fermion number operator is given by $\sum_{i=1}^N n_i = \sum_{i=1}^N c_i^\dagger c_i = \sum_i^N (1+\mathrm{i}\gamma_{2i}\gamma_{2i-1})/2$. Obviously it is a projection operator $n_i^2 = n_i$ so it only has eigenvalue 1 or 0, i.e. the occupied and empty state. Therefore, the ground state of the system has the degeneracy of 2^N because there are N fermions.

(b) We just need to show the map $\sigma_i \mapsto U_{i,i+1}$ induce an isomorphism. Here we prove the second equality first:

$$\begin{aligned}\sigma_i\sigma_j &\mapsto U_{i,i+1}U_{j,j+1} \\ &= \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{2}(1 + \gamma_i\gamma_{i+1})(1 + \gamma_j\gamma_{j+1}) \\ &= \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{2}(1 + \gamma_j\gamma_{j+1} + \gamma_i\gamma_{i+1} + \gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}).\end{aligned}$$

While for

$$\begin{aligned}\sigma_j\sigma_i &\mapsto U_{j,j+1}U_{i,i+1} \\ &= \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{2}(1 + \gamma_j\gamma_{j+1})(1 + \gamma_i\gamma_{i+1}) \\ &= \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{2}(1 + \gamma_i\gamma_{i+1} + \gamma_j\gamma_{j+1} + \gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1}).\end{aligned}$$

According to the anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} = 0 = \gamma_i\gamma_j + \gamma_j\gamma_i \Rightarrow \gamma_i\gamma_j = -\gamma_j\gamma_i,$$

So $\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1} = -\gamma_j\gamma_i\gamma_{j+1}\gamma_{i+1} = \gamma_i\gamma_j\gamma_{j+1}\gamma_{i+1} = -\gamma_i\gamma_j\gamma_{i+1}\gamma_{j+1} = \gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}$. Therefore we have

$$U_{i,i+1}U_{j,j+1} = U_{j,j+1}U_{i,i+1}.$$

For the first identity:

$$\begin{aligned}\sigma_i\sigma_{i+1}\sigma_i \mapsto &\, U_{i,i+1}U_{i+1,i+2}U_{i,i+1}\\ &= \frac{\mathrm{e}^{3i\alpha}}{2\sqrt{2}}(1+\gamma_i\gamma_{i+1})(1+\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1})\\ &= \frac{\mathrm{e}^{3i\alpha}}{2\sqrt{2}}(1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_i\gamma_{i+1}\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1}).\end{aligned}$$

According to the anti-commutation relation:

$$\{\gamma_i, \gamma_i\} = 2\delta_{ii} = 2 = 2\gamma_i\gamma_i,$$

So

$$\begin{aligned}\sigma_i\sigma_{i+1}\sigma_i \mapsto &\frac{\mathrm{e}^{3i\alpha}}{2\sqrt{2}}(1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_i\gamma_{i+2})(1+\gamma_i\gamma_{i+1})\\ &= \frac{\mathrm{e}^{3i\alpha}}{\sqrt{2}}(\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}).\end{aligned}$$

For the second half:

$$\begin{aligned}\sigma_{i+1}\sigma_i\sigma_{i+1} \mapsto &\, U_{i+1,i+2}U_{i,i+1}U_{i+1,i+2}\\ &= \frac{\mathrm{e}^{3i\alpha}}{2\sqrt{2}}(1+\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1})(1+\gamma_{i+1}\gamma_{i+2})\\ &= \frac{\mathrm{e}^{3i\alpha}}{2\sqrt{2}}(1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_{i+2}\gamma_i)(1+\gamma_{i+1}\gamma_{i+2})\\ &= \frac{\mathrm{e}^{3i\alpha}}{\sqrt{2}}(\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}).\end{aligned}$$

So we have checked

$$U_{i,i+1}U_{i+1,i+2}U_{i,i+1} = U_{i+1,i+2}U_{i,i+1}U_{i+1,i+2}.$$

Therefore, since σ_i are the generator of the braid group, $\sigma_i \mapsto U_{i,j}$ induce an isomorphism, thus a representation.

(c) We can easily see

$$\begin{aligned}(\gamma^{\mathrm{FIVE}})^2 &= \gamma_1\gamma_2\dots\gamma_{2N}\gamma_1\gamma_2\dots\gamma_{2N}\\ &= (-1)^N(-1)^{(2N-1)+(2N-2)+\dots+1}\\ &= (-1)^N(-1)^{(2N-1+1)(2N-1)/2} = 1.\end{aligned}$$

So γ^{FIVE} have eigenvalue ± 1. Next, to prove eigenvalue remains unchanged under any braid operation $U_{i,j}$, we consider

$$\begin{aligned}
[\gamma^{\mathrm{FIVE}}, U_{ij}] &= \frac{\mathrm{e}^{\mathrm{i}\alpha}(\mathrm{i})^N}{\sqrt{2}}[\gamma_1\gamma_2\cdots\gamma_i\cdots\gamma_j\cdots\gamma_{2N}, \gamma_i\gamma_j] \\
&= \frac{\mathrm{e}^{\mathrm{i}\alpha}(\mathrm{i})^N}{\sqrt{2}}(\gamma_1\gamma_2\cdots\gamma_i\cdots\gamma_j\cdots\gamma_{2N}\gamma_i\gamma_j - \gamma_i\gamma_j\gamma_1\gamma_2\cdots\gamma_i\cdots\gamma_j\cdots\gamma_{2N}) \\
&= \frac{\mathrm{e}^{\mathrm{i}\alpha}(\mathrm{i})^N}{\sqrt{2}}(\gamma_1\gamma_2\cdots\gamma_i\cdots\gamma_j\cdots\gamma_{2N}\gamma_i\gamma_j - (-1)^{(2N-1)\cdot 2}\gamma_1\gamma_2\cdots\gamma_i\cdots\gamma_j\cdots\gamma_{2N}\gamma_i\gamma_j) \\
&= 0.
\end{aligned}$$

So the eigenvalue remains unchanged. Thus the overall parity of this system is conserved, which means we have 2^{N-1} dimensional Hilbert space.

(d) With four Ising anyons, we can construct two creation-annihilation operator $c_1, c_1^\dagger, c_2, c_2^\dagger$. We label the state by $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. We have two possible parity, with parity 1, the states are $|0,0\rangle, |1,1\rangle$; and with parity -1, the states are $|0,1\rangle, |1,0\rangle$. Although in each qubit, we have only two states, i.e. the Hilbert space should be two dimensional. This means we in fact should set $|0,0\rangle = (1,0), |1,1\rangle = (1,0)$ for the first case and similar for the second, however, we can still use the tensor product Hilbert space in four dimensional, and for each system, we just use half of it. So we can consider the first fermion:

$$\begin{aligned}
&c_1^\dagger |0\rangle = |1\rangle = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2)|1\rangle \\
&c_1^\dagger |1\rangle = 0,
\end{aligned}$$

which means under basis $|0\rangle = (1,0), |1\rangle = (0,1)$, the matrix elements of $c_1^\dagger$ are:

$$c_1^\dagger = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

similarly:

$$c_1 = \frac{1}{2}(\gamma_1 - \mathrm{i}\gamma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

Thus, we can specify the effect on U_{ij} on each state:

$$U_{12}|n_1, n_2\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}(1 + \gamma_i\gamma_j)|n_1, n_2\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}(1 + (-1)^{n_1+1}\mathrm{i})|n_1, n_2\rangle,$$

which means the braiding inside each pair gives an overall factor. Now consider

$$U_{23}|n_1, n_2\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}(|n_1, n_2\rangle + \mathrm{i}(-1)^{n_1}|n_1, n_2\rangle).$$

Therefore, we can see the braiding between Ising anyons, i.e. U_{ij} cannot make the unitary operatoration such as "magic gate" like

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

so it cannot obtain all possible unitary operation on this qubit.

(e) With $\sigma_i \mapsto U_{i,i+1}$, the relation (3.2) can be written as:

$$\prod_{i=1}^{2N-1} U_{i,i+1} = \prod_{i=1}^{2N-1} U_{i,i+1}^{\dagger}$$

because $U_{i,i+1}$ is a unitary transformation(which can be checked by its definition easily). Then according to its definition:

$$e^{i(2N-1)\alpha} \prod_{i=1}^{2N-1} (1 + \gamma_i\gamma_{i+1}) = e^{-i(2N-1)\alpha} \prod_{i=1}^{2N-1} (1 - \gamma_i\gamma_{i+1}).$$

Let's list the above identity with small N. For $N = 1$:

$$e^{i\alpha}(1 + \gamma_1\gamma_2) = e^{-i\alpha}(1 - \gamma_1\gamma_2).$$

If $\gamma_1\gamma_2$ have eigenvalue $-i$, i.e. with parity 1,then $\alpha = \pi/4$, if $\gamma_1\gamma_2$ have eigenvalue i, i.e. with parity -1, then $\alpha = 3\pi/4$.

For $N = 2$:

$$\begin{aligned} \text{LHS} &= e^{i3\alpha}(1 + \gamma_1\gamma_2)(1 + \gamma_2\gamma_3)(1 + \gamma_3\gamma_4) \\ &= e^{i3\alpha}(1 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_3 + \gamma_3\gamma_4 + \gamma_2\gamma_4 + \gamma_1\gamma_4 + \gamma_1\gamma_2\gamma_3\gamma_4) \\ \text{RHS} &= e^{-i3\alpha}(1 - \gamma_1\gamma_2)(1 - \gamma_2\gamma_3)(1 - \gamma_3\gamma_4) \\ &= e^{-i3\alpha}(1 - \gamma_1\gamma_2 - \gamma_2\gamma_3 + \gamma_1\gamma_3 - \gamma_3\gamma_4 + \gamma_2\gamma_4 - \gamma_1\gamma_4 + \gamma_1\gamma_2\gamma_3\gamma_4). \end{aligned}$$

This is an over-determined equation, we just need the condition that

For $N = 3$:

$$\begin{aligned} \text{LHS} &= e^{i5\alpha}(1 + \gamma_1\gamma_2)(1 + \gamma_2\gamma_3)(1 + \gamma_3\gamma_4)(1 + \gamma_4\gamma_5)(1 + \gamma_5\gamma_6) \\ &= e^{i3\alpha}(1 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_3 + \gamma_3\gamma_4 + \gamma_2\gamma_4 + \gamma_1\gamma_4 + \gamma_1\gamma_2\gamma_3\gamma_4)(1 + \gamma_4\gamma_5)(1 + \gamma_5\gamma_6) \\ &= 1 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_3 + \gamma_3\gamma_4 + \gamma_2\gamma_4 + \gamma_1\gamma_4 + \gamma_3\gamma_5 + \gamma_2\gamma_5 + \gamma_1\gamma_5 + \gamma_4\gamma_5 \\ &+ \gamma_1\gamma_2\gamma_3\gamma_4 + \gamma_1\gamma_2\gamma_4\gamma_5 + \gamma_2\gamma_3\gamma_4\gamma_5 + \gamma_1\gamma_3\gamma_4\gamma_5 + \gamma_1\gamma_2\gamma_3\gamma_5 \\ &+ \gamma_5\gamma_6 + \gamma_1\gamma_2\gamma_5\gamma_6 + \gamma_2\gamma_3\gamma_5\gamma_6 + \gamma_1\gamma_3\gamma_5\gamma_6 + \gamma_3\gamma_4\gamma_5\gamma_6 + \gamma_2\gamma_4\gamma_5\gamma_6 + \gamma_1\gamma_4\gamma_5\gamma_6 \\ &+ \gamma_3\gamma_6 + \gamma_2\gamma_6 + \gamma_1\gamma_6 + \gamma_4\gamma_6 \\ &+ \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 + \gamma_1\gamma_2\gamma_4\gamma_6 + \gamma_2\gamma_3\gamma_4\gamma_6 + \gamma_1\gamma_3\gamma_4\gamma_6 + \gamma_1\gamma_2\gamma_3\gamma_6 \end{aligned}$$

If we only consider the γ^{FIVE} term:

$$e^{i(2N-1)\alpha}(1 + (-i)^N\gamma^{\text{FIVE}}) = e^{-i(2N-1)\alpha}(1 + (i)^N\gamma^{\text{FIVE}}),$$

whcih means

$$\alpha = \frac{\pi}{4}\frac{N}{2N-1}$$

for even parity, and

$$\alpha = \frac{\pi}{4}\frac{N+2}{2N-1}$$

for odd parity.

[(Unsolved! This isn't over. I will be back.] From my perspective, we can only focus on theγ^{FIVE} term.

(f) Such additional phase factor $e^{i\phi}$ could be added to the right hand side of equ. 3.3, we can obtain that $\alpha = \frac{\pi}{8}\frac{2N+4\phi/\pi}{2N-1}$, thus $\phi = -\pi/4$ for even parity, $\phi = -\pi/2$ for odd parity, finally we get $\alpha = \pi/8$.

3.4 Small Numbers of Anyons on a Sphere

On the plane, the braid group of two particles is an infinite group (the group of integers describing the number of twists!). However, this is not true on a sphere.

First review the problem "About the Braid Group" about braiding on a sphere.

(a) Now consider the case of two particles on a sphere. Determine the full structure of the braid group. Show it is a well known finite discrete group. What group is it?

(b) (Harder) Now consider three particles on a sphere. Determine the full structure of the braid group. Show that it is a finite discrete group. [Even Harder] What group is it? It is "well known" only to people who know a lot of group theory. But you can google to find information about it on the web with some work. It may be useful to list all the subgroups of the group and the multiplication table of the group elements.

(c) Suppose we have two (or three) anyons on a sphere. Suppose the ground state is two-fold degenerate (or more generally N-fold degenerate for some finite N). Since the braid group is discrete, conclude that no type of anyon statistics can allow us to do arbitrary $SU(2)$ (or $SU(N)$) rotations on this degenerate ground state by braiding.

Answer We have only one generator σ_1 and the equation (3.2) tells us that

$$\sigma_1^2 = I$$

which means that $B_2(S^2) \cong \mathbb{Z}_2$.

(b) Since this is a finite group, every element have a finite order. So the simplest way is to apply an element as many times as possible to feel the group structure. Consider σ_1(same as σ_2) first, as the Fig.3.4 shows.

From continuous deformation, we can see

$$\sigma_1^2 = \sigma_2^{-2} = \sigma_1^{-2} = \sigma_2^2.$$

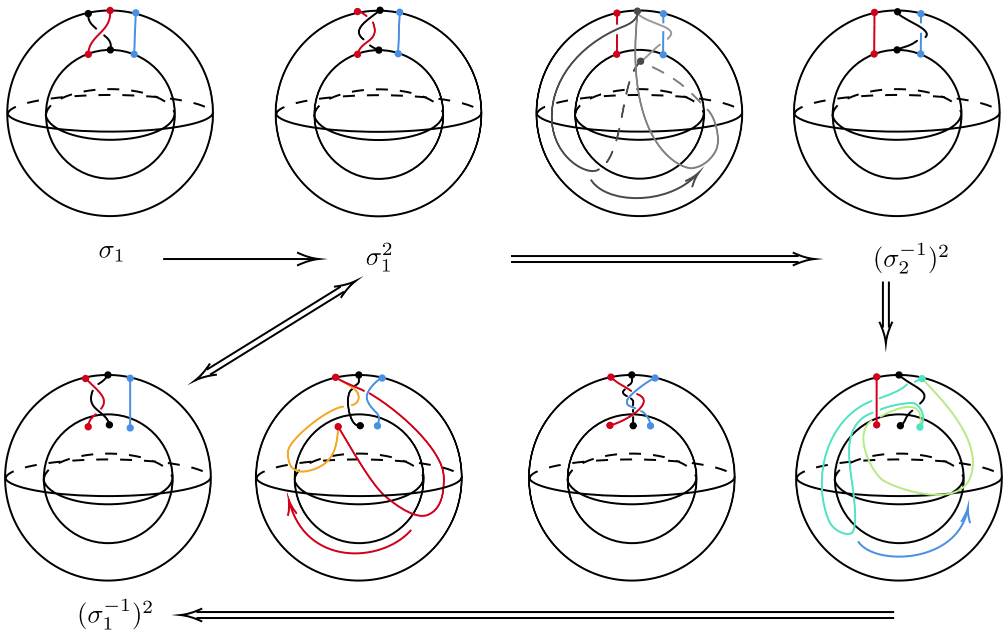


Figure 3.5: The action of a series of σ_1.

Thus,

$$\sigma_1^3 = \sigma_1^{-1}, \quad \sigma_1^4 = I.$$

So we have found three subgroups:

$$\{I\}, \{I, \sigma_1^2 = \sigma_2^2\}, \{I, \sigma_1, \sigma_1^2, \sigma_1^{-1}\}, \{I, \sigma_2, \sigma_2^2, \sigma_2^{-1}\}.$$

Now we try other elements. We try the scheme(this is arbitrary) that apply σ_1 first, then σ_2, then σ_1, then σ_2... to get identity. This is from relation (3.1). For simplicity, we call $\sigma_1\sigma_2 = a, \sigma_1\sigma_2\sigma_1 = x$. Then we try to list all elements:

We have 12 group elements: $\{I, a^i, a^i x\}, i = 1, \cdots, 5$. We can list its group multiplication

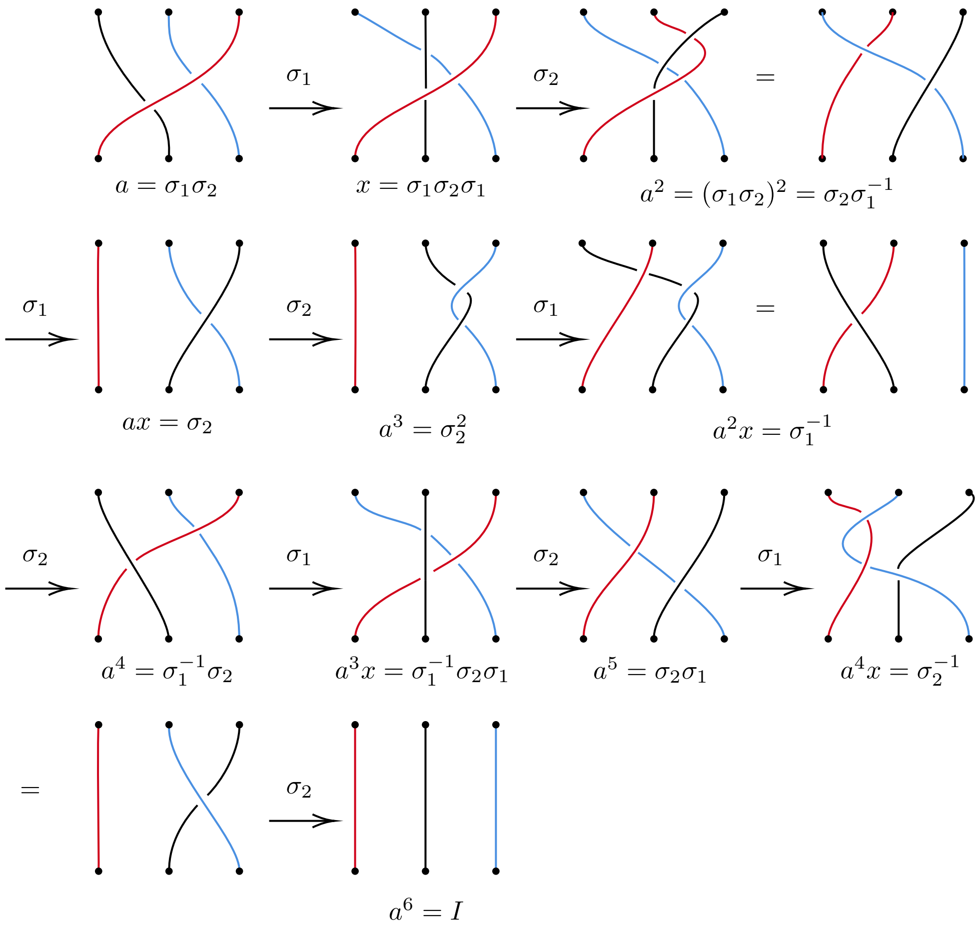


table and gives:

	1	a	a^2	a^3	a^4	a^5	x	ax	a^2x	a^3x	a^4x	a^5x
1	1	a	a^2	a^3	a^4	a^5	x	ax	a^2x	a^3x	a^4x	a^5x
a	a	a^2	a^3	a^4	a^5	1	a^5x	x	ax	a^2x	a^3	a^4x
a^2	a^2	a^3	a^4	a^5	1	a	a^4x	a^5x	x	ax	a^2x	a^3x
a^3	a^3	a^4	a^5	1	a	a^2	a^3x	a^4x	a^5x	x	ax	a^2x
a^4	a^4	a^5	1	a	a^2	a^3	a^2x	a^3x	a^4x	a^5x	x	ax
a^5	a^5	1	a	a^2	a^3	a^4	ax	a^2x	a^3x	a^4x	a^5x	x
x	x	ax	a^2x	a^3x	a^4x	a^5x	a^3	a^4	a^5	1	a	a^2
ax	ax	a^2x	a^3x	a^4x	a^5x	x	a^2	a^3	a^4	a^5	1	a
a^2x	a^2x	a^3x	a^4x	a^5x	x	ax	a	a^2	a^3	a^4	a^5	1
a^3x	a^3x	a^4x	a^5x	x	ax	a^2x	1	a	a^2	a^3	a^4	a^5
a^4x	a^4x	a^5x	x	ax	a^2x	a^3x	a^5	1	a	a^2	a^3	a^4
a^5x	a^5x	x	ax	a^2x	a^3x	a^4x	a^4	a^5	1	a	a^2	a^3

This is closed and we can see it is generated by σ_1, σ_2, which it is actually $B_3(S^2)$. Actually, we can also prove it in an algebraic way. Here using the definition of a, x, we have

$$\sigma_1 = a^{-1}x, \quad \sigma_2 = x^{-1}a^2.$$

Then the group representation is given by

$$B_3(S^2) \cong \langle a, x | x^2 = a^3, x^{-1}ax = a^{-1} \rangle.$$

We can also see

$$a^6 = x^2a^3 = x(xa^3x^{-1})x = xa^{-3}x = x^{1-2+1} = I.$$

This subgroup $\mathbb{Z}_6$ generate by a is a normal subgroup because $xa^nx^{-1} = a^{-n}$. Besides, the quotient group gives

$$B_3(S^2)/\mathbb{Z}_6 \cong \langle x | x^2 = I \rangle \cong \mathbb{Z}_2,$$

which means it is order 12.

Now let's look at its group structure. It is non-trivial subgroups are

$$\{1, a^3\}, \{1, a^2, a^4\}, \{1, a^i, i = 1, \cdots 5\}, \{1, a^3, x, a^3x\}, \{1, a^3, ax, a^4x\}, \{1, a^3, a^2x, a^5x\}.$$

It is the well known ZS-metacyclic group of order 12, which is also the dicyclic group Dic_3. The dicyclic group is defined by:

$$\mathrm{Dic}_n = \langle a, x | a^{2n} = I, x^2 = a^n, x^{-1}ax = a^{-1} \rangle,$$

which is exactly our case. It is called metacyclic group because it has a cyclic normal subgroup N, such that the quotient G/N is also cyclic. In this case, $B_3(S^2)$ has the normal subgroup $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, and $B_3(S^2)/\mathbb{Z}_6 \cong \mathbb{Z}_2$. Also, it can be written as

$$B_3(S^2) = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$$

from its subgroups.

An interesting fact is that for $n \geq 4$, $B_n(S^2)$ is infinite[GG13], but the element $\sigma_1\sigma_2 \cdots \sigma_{n-1}$ still have order $2n$.

(c) For n-fold degeneracy, we should consider n-dimensional representation of the braid group. In two particle case, all two dimensional representation of $\mathbb{Z}_2$ are similar to $\{I_2, \sigma_z\}$, which means the $SU(2)$ rotation is not allowed. For three particle case, we have

$$12 = 1 + 1 + 1 + 1 + 2^2 + 2^2,$$

which means we have four 1 dimensional irreducible representation and two 2 dimensional representation. Non of them can cover all $SU(2)$ rotations. For $n \geq 4$, $B_4(S^2)$ is infinite and may allow $SU(2)$ rotations.

Now let's try to figure out the irreducible representation of $B_3(S^2)$. First, its conjugacy classes are given by:

$$c_1 = \{1\}, c_2 = \{2,6\}, c_3 = \{3,5\}, c_4 = \{4\}, c_5 = \{7,9,11\}, c_6 = \{8,10,12\}.$$

Then from its subgroups and the great orthogonality theorem, we can get its character table:

	c_1	c_2	c_3	c_4	c_5	c_6
λ_1	1	1	1	1	1	1
λ_2	1	1	1	1	-1	-1
λ_3	1	-1	1	-1	i	$-$i
λ_4	1	-1	1	-1	$-$i	i
λ_5	2	-1	-1	2	0	0
λ_6	2	1	-1	-2	0	0

For the rep λ_5, it is orthogonal lifted from S_3, which means the representation of its generators are given by:

$$a \to \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, b \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and another representation is quaternionic representation:

$$a \to \begin{pmatrix} \mathrm{e}^{\mathrm{i}\pi/3} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\pi/3} \end{pmatrix}, b \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Chapter 4

Aharanov-Bohm Effect and Charge-Flux Composite

4.1 Abelian Anyon Vacuum on a Two-Handle Torus

Using similar technique as in section 4.3, show that the ground state vacuum degeneracy on a two handle torus is m^2 for a system of abelian anyons with statistical angle $\theta = \pi p/m$ for integers p and m relatively prime. Hint: Consider what the independent nontrivial cycles are on a two-handled torus and determine the commutation relations for operators T_i that take anyon-antianyon pairs around these cycles.

Answer There are four kinds of non-trivial loop in a genus-2 surface, as the Fig. 4.1 below shows.

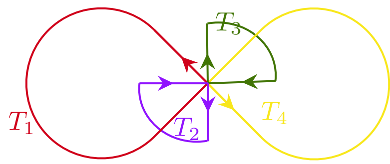


Figure 4.1: Nontrivial loops on genus-2 surface.

This time, we cannot use only one complex number to label the state, since T_1 and T_4 don't commute. We call

$$T_1|\alpha,\beta\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\alpha,\beta\rangle; \quad T_4\,|\alpha,\beta\rangle = \mathrm{e}^{\mathrm{i}\beta}|\alpha,\beta\rangle.$$

Now we can repeat the same deduction in section 4.3. The commutator gives

$$T_2T_1 = \mathrm{e}^{-2\mathrm{i}\theta}T_1T_2; \quad T_3T_4 = \mathrm{e}^{-2\mathrm{i}\theta}T_4T_3$$

and

$$\begin{aligned} T_1(T_2|\alpha,\beta\rangle) &= \mathrm{e}^{2\mathrm{i}\theta}T_2T_1\,|\alpha,\beta\rangle = \mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\alpha}(T_2|\alpha,\beta\rangle) \\ T_4(T_3|\alpha,\beta\rangle) &= \mathrm{e}^{2\mathrm{i}\theta}T_3T_4\,|\alpha,\beta\rangle = \mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\beta}(T_3|\alpha,\beta\rangle). \end{aligned}$$

Now, for T_2 we have m basis states:

$$|\alpha,\beta\rangle, |\alpha+2\pi p/m,\beta\rangle, \cdots, |\alpha+2\pi(m-1)p/m,\beta\rangle.$$

However, this time, each state of these is a new "generator" for T_3, for example

$$|\alpha+2\pi p/m,\beta\rangle \to |\alpha+2\pi p/m,\beta+2\pi p/m\rangle \to \cdots \to |\alpha+2\pi p/m,\beta+2\pi(m-1)p/m\rangle.$$

So there are totally m^2 basis states for a genus 2 surface. For genus g surface, the GSD is m^g.

Chapter 5

Chern-Simons Theory Basics

5.1 Polyakov Representation of the Linking Number

Consider a link made of two strands, L_1 and L_2. Consider the double line integral

$$\Phi(L_1, L_2) = \frac{\epsilon_{ijk}}{4\pi} \oint_{L_1} \mathrm{d}x^i \oint_{L_2} \mathrm{d}y^j \frac{x^k - y^k}{|\boldsymbol{x} - \boldsymbol{y}|^3}$$

(a) Show that Φ is equal to the phase accumulated by letting a unit of flux run along one strand, and moving a unit charged particle along the path of the other strand.

(b) Show that the resulting phase is the topological invariant known as the linking number - the number of times one strand wraps around the other, see section 2.6.2.

This integral representation of linking was known to Gauss.

Answer (a) According to the text, the path integral of the system is given by:

$$Z = \sum_{\text{paths } \{\boldsymbol{x}(t)\}} \int \mathcal{D}a_\mu \exp\left(\frac{\mathrm{i}}{\hbar} S_{CS} + \frac{\mathrm{i}q}{\hbar} \int \mathrm{d}\ell^\alpha a_\alpha\right).$$

Integrate out a_μ, we get a phase

$$\sum_{\text{paths } \{\boldsymbol{x}(t)\}} \mathrm{e}^{\mathrm{i}S_0/\hbar} \mathrm{e}^{\mathrm{i}\theta W(\text{paths})}.$$

We want to prove that

$$2\pi\Phi(L_1, L_2) \propto \theta W(L_1, L_2).$$

In this case, the path dependent terms is given by

$$\begin{aligned} Z &= \int \mathcal{D}a_\mu \exp\left[\frac{\mathrm{i}}{\hbar} \oint_{L_1} \mathrm{d}\ell_1^\alpha a_\alpha\right] \exp\left[\frac{\mathrm{i}}{\hbar} \oint_{L_2} \mathrm{d}\ell_2^\alpha a_\alpha\right] \mathrm{e}^{\mathrm{i}S_{CS}/\hbar}. \\ &\equiv \int \mathcal{D}a_\mu W_1 W_2 \mathrm{e}^{\mathrm{i}S_{CS}/\hbar}. \end{aligned}$$

Here W_1, W_2 denote the Wilson loop. The EOM we have got in the text is

$$j^\alpha = \epsilon^{\alpha\beta\gamma} \partial_\beta a_\gamma.$$

In Lorenz gauge, the solution is obtained in classical electrodynamics:

$$a_\alpha(x) = \int \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma} \frac{\partial^\beta j^\gamma(y)}{|\boldsymbol{x} - \boldsymbol{y}|}.$$

In this case, the flux is

$$j_a^\alpha = \oint_{L_a} \mathrm{d}x_a^\alpha \delta(x - x_a(t)),$$

then

$$a_\alpha(x) = \sum_{a=1}^{2} \oint_{L_a} \mathrm{d}x_a^\beta \epsilon_{\alpha\beta\gamma} \frac{(x - x_a)^\rho}{|\boldsymbol{x} - \boldsymbol{x}_a|^3}.$$

Therefore, the phase is given by:

$$\begin{aligned} Z &= \langle W_1 W_2 \rangle \\ &= \exp(\mathrm{i} S_{CS}[a_\alpha(x)]) \\ &= \exp\left(\frac{\mathrm{i}}{2\hbar} \oint_{L_1} \mathrm{d}x^i \oint_{L_2} \mathrm{d}y^j \epsilon_{ijk} \frac{x^k - y^k}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) = \exp\left(\frac{2\pi\mathrm{i}}{\hbar}\Phi(L_1, L_2)\right). \end{aligned}$$

(b) Consider this two strands case. Using Stokes' theorem:

$$\begin{aligned} \Phi &= \frac{1}{4\pi} \oint_{L_1} \mathrm{d}x^i \oint_{L_2} \mathrm{d}y^j \epsilon_{ijk} \frac{\partial}{\partial y_k}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) \\ &= \frac{1}{4\pi} \oint_{L_1} \mathrm{d}x^i \int_{\Sigma(L_2)} \mathrm{d}^2 y_i \boldsymbol{\nabla}^2 \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) \\ &= \oint_{L_1} \mathrm{d}x^i \int_{\Sigma(L_2)} \mathrm{d}^2 y_i \delta(\boldsymbol{x}-\boldsymbol{y}). \end{aligned}$$

Here $\Sigma(L_2)$ is an arbitrary surface bounded by L_2. Using the limit definition of δ function:

$$\delta(\boldsymbol{x}-\boldsymbol{y}) = \lim_{\epsilon\to 0} \delta_\epsilon(\boldsymbol{x}-\boldsymbol{y}) = \lim_{\epsilon\to 0} \frac{1}{(2\pi\epsilon)^{3/2}} \mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{y})^2/\epsilon}.$$

Parameterize the $\epsilon \to 0$ region, we have

$$\Phi = \frac{1}{2\pi} \oint_{L_1} \mathrm{d}x^i(t) \epsilon_{ijk} n^j \dot{n}^k,$$

where

$$\dot{n}^i = \frac{\mathrm{d}n^i}{\mathrm{d}t},$$

and n^i is the normal vector along the surface $\Sigma(L_2)$. Note here the integral is equal to $\mathrm{d}\boldsymbol{x}\cdot(\boldsymbol{n}\times\dot{\boldsymbol{n}})$, which means the volume of the parallelepiped spanned by $\mathrm{d}\boldsymbol{x}, \boldsymbol{n}, \dot{\boldsymbol{n}}$, as Fig.5.1 shows.

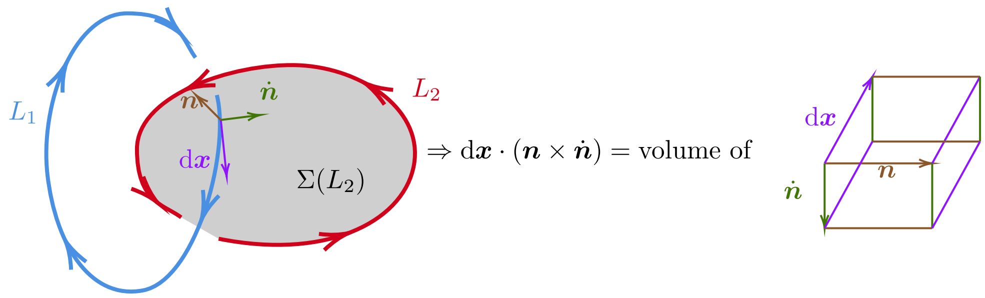


Figure 5.1: The geometric meaning of Φ.

Therefore due to $\boldsymbol{n} \perp \dot{\boldsymbol{n}}$ in this two loop case, we have

$$\Phi = \frac{1}{2\pi} \oint_{L_1} \epsilon \mathrm{d}\theta = \epsilon,$$

where $\epsilon = 1$ or -1 depend on the relative direction whether $\mathrm{d}\boldsymbol{x}$ is in the same direction as $\boldsymbol{n} \times \dot{\boldsymbol{n}}$. In the case of Fig.5.1, $\epsilon = 1$ according to the right handed rule. This is exactly the definition using the sign of the crossing:$(\epsilon(L_1) + \epsilon(L_2)))/2$.

For the prove in the general case, you can refer to [RN11].

5.2 Gauge Transforming the Chern-Simons Action

Make the gauge transform Eq. 5.13 on the Chern-Simons action 5.9 and show that it results in the change 5.15. Note that there will be an additional term that shows up which it a total derivative and will therefore vanish when integrated over the whole manifold $\mathcal{M}$.

Answer We first give the transformation rule of the general Chern-Simons action. Under the gauge transformation:

$$\mathrm{d}a_\mu \to \mathrm{d}(U^{-1}\mathrm{d}U + U^{-1}a_\mu U) = \mathrm{d}U^{-1}\wedge \mathrm{d}U + \mathrm{d}U^{-1} \wedge a_\mu U + U^{-1}\mathrm{d}a_\mu U - U^{-1}a_\mu \wedge \mathrm{d}U.$$

Using $\mathrm{d}U^{-1} = -U^{-1}\mathrm{d}U \wedge U^{-1}$, we have:

$$\begin{aligned}\mathrm{d}a_\mu &\to -U^{-1}\mathrm{d}U \wedge U^{-1} \wedge \mathrm{d}U - U^{-1}\mathrm{d}U \wedge U^{-1} \wedge a_\mu U + U^{-1}\mathrm{d}a_\mu U - U^{-1}a_\mu \wedge \mathrm{d}U \\ &= -U^{-1}\mathrm{d}U \wedge a_\mu + U^{-1}\mathrm{d}a_\mu U - U^{-1}a_\mu \wedge \mathrm{d}U.\end{aligned}$$

Under this transformation, consider the difference

$$\begin{aligned}&\mathrm{tr}(a'_\mu \wedge \mathrm{d}a'_\mu) - \mathrm{tr}(a_\mu \wedge \mathrm{d}a_\mu) \\ \to &\,\mathrm{tr}\left[(U^{-1}\mathrm{d}U + U^{-1}a_\mu U) \wedge (-U^{-1}\mathrm{d}U \wedge a_\mu + U^{-1}\mathrm{d}a_\mu U - U^{-1}a_\mu \wedge \mathrm{d}U) - a_\mu \wedge \mathrm{d}a_\mu\right] \\ = &\,\mathrm{tr}[-(U^{-1}\mathrm{d}U)^2 \wedge a'_\mu - U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U \wedge a'_\mu + U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}a_\mu U + U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}a_\mu U \\ &-U^{-1}\mathrm{d}U \wedge U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U - U^{-1}a_\mu U \wedge U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U - a_\mu \wedge \mathrm{d}a_\mu] \\ = &\,\mathrm{tr}\left[-(U^{-1}\mathrm{d}U)^2 \wedge (a'_\mu + U^{-1}a_\mu U) - U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U \wedge (a'_\mu + U^{-1}a_\mu U) + U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}a_\mu U\right].\end{aligned}$$

The second term gives:

$$\begin{aligned}&\frac{2}{3}\,\mathrm{tr}({a'_\mu}^3 - a_\mu^3) \\ =&\frac{2}{3}\,\mathrm{tr}\left[(U^{-1}\mathrm{d}U + U^{-1}a_\mu U)^3 - {a_\mu}^3\right] \\ =&\,\mathrm{tr}\left[\frac{2}{3}(U^{-1}\mathrm{d}U)^3 + 2(U^{-1}\mathrm{d}U)^2 \wedge U^{-1}a_\mu U + 2U^{-1}\mathrm{d}U \wedge (U^{-1}a_\mu U)^2\right].\end{aligned}$$

So the two terms gives:

$$\begin{aligned}
&\mathrm{tr}\left(a'_\mu \wedge \mathrm{d}a'_\mu + \frac{2}{3}{a'_\mu}^3\right) - \mathrm{tr}\left(a_\mu \wedge \mathrm{d}a_\mu + \frac{2}{3}a_\mu^3\right) \\
=&\,\mathrm{tr}\left\{\frac{2}{3}(U^{-1}\mathrm{d}U)^3 + (U^{-1}\mathrm{d}U)^2\wedge(U^{-1}a_\mu U - a'_\mu) - U^{-1}a_\mu U \wedge U^{-1}\mathrm{dU}\wedge(U^{-1}a_\mu U - a'_\mu)\right. \\
&\left. + U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}a_\mu U\right\} \\
=& -\frac{1}{3}\,\mathrm{tr}(U^{-1}\mathrm{d}U)^3 + \mathrm{tr}\left[U^{-1}\mathrm{d}U \wedge (U^{-1}\mathrm{d}a_\mu U - U^{-1}\mathrm{d}U \wedge U^{-1}a_\mu U)\right] \\
=& -\frac{1}{3}\,\mathrm{tr}(U^{-1}\mathrm{d}U)^3 + \mathrm{tr}\left[\mathrm{d}(U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U)\right].
\end{aligned}$$

So we have:

$$S_{CS} \to S_{CS} + \frac{k}{4\pi}\int_M \mathrm{d}\,\mathrm{tr}(U^{-1}a_\mu U \wedge U^{-1}\mathrm{d}U) - \frac{k}{12\pi}\int_M \mathrm{tr}((U^{-1}dU)^3). \tag{5.1}$$

Now, we can specify that $M = S^3$, therefore, the second term vanishes because S^3 is closed. Then we consider the third term. Suppose $G = SU(2) \cong S^3$, then we consider a function $U : S^3 \to S^3$:

$$(t_1, t_2, t_3, t_4) \mapsto (x_1, x_2, x_3, x_4).$$

Then we can use complex numbers $z = x_1 + \mathrm{i}x_2, w = x_3 + \mathrm{i}x_4$ to characterize them. We can see U can be expressed as:

$$U = \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}, |z|^2 + |w|^2 = 1.$$

Noting that $\mu = U^{-1}\mathrm{d}U$ is the Maurer-Cartan form and $\mathrm{tr}(\mu \wedge \mu \wedge \mu)/6$ is the left-invariant volume form of $SU(2)$, so the third term is propotional to the mapping degree of U, which indicate that it is an integer! Now let's start our proof.

The Maurer-Cartan form $U^{-1}\mathrm{d}U$ can be parametrized as:

$$U^{-1}\mathrm{d}U = \begin{pmatrix} \bar{z}\mathrm{d}z + w\mathrm{d}\bar{z} & -\bar{z}\mathrm{d}w + w\mathrm{d}\bar{z} \\ -\bar{w}\mathrm{d}z + z\mathrm{d}\bar{w} & \bar{w}\mathrm{d}w + z\mathrm{d}\bar{z} \end{pmatrix}.$$

The condition $|z|^2 + |w|^2 = 1$ gives

$$\bar{w}\mathrm{d}w + z\mathrm{d}\bar{z} = -(\bar{z}\mathrm{d}z + w\mathrm{d}\bar{w}).$$

If we call the basis $T^i \equiv \sigma^i/2$, and call $\alpha \equiv -\bar{z}\mathrm{d}w + w\mathrm{d}z$, then we have:

$$U^{-1}\mathrm{d}U = (\alpha - \bar{\alpha})T^1 + \mathrm{i}(\alpha + \bar{\alpha})T^2 + 2(\bar{z}\mathrm{d}z + w\mathrm{d}\bar{w})T^3 \equiv a_i T^i \in \mathfrak{su}(2),$$

which means the Maurer-Cartan form $U^{-1}\mathrm{d}U$ is an element of $\mathfrak{su}(2)$ Lie algebra. Then we have:

$$\begin{aligned}
\mathrm{tr}(U^{-1}\mathrm{d}U)^3 &= a^a \wedge a^b \wedge a^c\,\mathrm{tr}(T^a T^b T^c) \\
&= a^a \wedge a^b \wedge a^c \frac{1}{2}\,\mathrm{tr}[([T^a, T^b] + \{T^a, T^b\})T^c] \\
&= \frac{1}{2}a^a \wedge a^b \wedge a^c\,\mathrm{tr}\left[\left(\mathrm{i}\epsilon^{abc}T^d + \frac{\delta^{ab}}{2}\right)T^c\right] \\
&= \frac{\mathrm{i}}{2}\epsilon^{abd}a^a \wedge a^b \wedge a^c = \frac{3\mathrm{i}}{2}a^1 \wedge a^2 \wedge a^3.
\end{aligned}$$

In terms of z and w, we have:

$$\begin{aligned}
\mathrm{tr}(U^{-1}\mathrm{d}U)^3 =& \frac{3\mathrm{i}}{2}(\alpha-\bar{\alpha})\wedge \mathrm{i}(\alpha+\bar{\alpha})\wedge 2(\bar{z}\mathrm{d}z+w\mathrm{d}\bar{w})\\
=&6(w\mathrm{d}\bar{z}-\bar{z}\mathrm{d}w)\wedge(z\mathrm{d}\bar{w}-\bar{w}\mathrm{d}z)\wedge(\bar{z}\mathrm{d}z+w\mathrm{d}\bar{w})\\
=&6(w\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{w}\wedge\mathrm{d}z+\bar{z}\mathrm{d}w\wedge\mathrm{d}z\wedge\mathrm{d}\bar{w})\\
\xlongequal[w=x_3+\mathrm{i}x_4]{z=x_1+\mathrm{i}x_2}& -12\mathrm{i}[(x_3+\mathrm{i}x_4)\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge(\mathrm{d}x_3-\mathrm{i}\mathrm{d}x_4)-(x_1-\mathrm{i}x_2)\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge(\mathrm{d}x_1+\mathrm{i}\mathrm{d}x_2)]\\
=&12\sum_{i=1}^{4}(-1)^i x_i\mathrm{d}x_1\wedge\cdots\cdots\wedge\mathrm{d}x_4\\
&-12\mathrm{i}[\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge(x_3\mathrm{d}x_3+x_4\mathrm{d}x_4)-(x_1\mathrm{d}x_1+x_2\mathrm{d}x_2)\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4].
\end{aligned}$$

But use

$$|z|^2+|w|^2=1\Rightarrow\sum_{i=1}^{4}x_i\mathrm{d}x_i=0\Rightarrow x_1\mathrm{d}x_1+x_2\mathrm{d}x_2=-(x_3\mathrm{d}x_3+x_4\mathrm{d}x_4),$$

which means

$$\mathrm{tr}(U^{-1}\mathrm{d}U)^3=12\sum_{i=1}^{4}(-1)^i x_i\mathrm{d}x_1\wedge\cdots\mathrm{d}\hat{x}_i\cdots\wedge\mathrm{d}x_4=12U^*\tilde{\omega}.$$

Where $\tilde{\omega}$ is the volume form of S^3, U^* is the pull back of U. So according to the definition of mapping degree:

$$\begin{aligned}
\frac{k}{12\pi}\int_{S^3}\mathrm{tr}(U^{-1}\mathrm{d}U)^3 &= \frac{k}{\pi}\int_{S^3}U^*\tilde{\omega}\\
&=\frac{k}{\pi}\deg U\cdot\int_{S^3}\tilde{\omega}\\
&=\frac{k}{\pi}\deg U\int_{D^4}4\mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\mathrm{d}y^3\wedge\mathrm{d}y^4\\
&=2\pi k\deg U.
\end{aligned}$$

Call $\deg U=\nu\in\mathbb{Z}$, we finally arrive our goal:

$$S_{CS}\to S_{CS}+2\pi\nu k.$$

Furthermore, we can argue that the level k **must** be an integer in order for the partition function to be well-defined. Because a'_μ and a_μ are connected by the **gauge** transformation, which means the partition function should not change:

$$\mathrm{e}^{\mathrm{i}S[a_\mu]}\overset{!}{=}\mathrm{e}^{\mathrm{i}S[a'_\mu]}\Rightarrow S[a_\mu]=S[a'_\mu]+2k\pi,k\in\mathbb{Z}.$$

This gives $k\cdot\deg U\in\mathbb{Z}\Rightarrow k\in\mathbb{Z}$.

5.3 Winding Numbers of Groups in Manifolds

Consider the mapping of $U(x) \in SU(2) \to S^3$. Construct an example of a map with winding number n for arbitrary n. I.e., find a representative of each group element of $\Pi_3(SU(2))$ (See note 17).

Answer Let's construct a kind of famous map: the hedgehog ansatz. This is a map with spherical symmetry:

$$U_H(r) = \exp(\mathrm{i}F(r)\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}) \in SU(2),$$

Here $F(r)$ is an arbitrary function about r. In nucleon theory, this ansatz correspond to the pion field with spherical symmetry:

$$\boldsymbol{\pi}(r) = F_\pi F(r)\hat{\boldsymbol{r}},$$

as the Fig.5.2 shows, so this map is called the hedgehog ansatz.

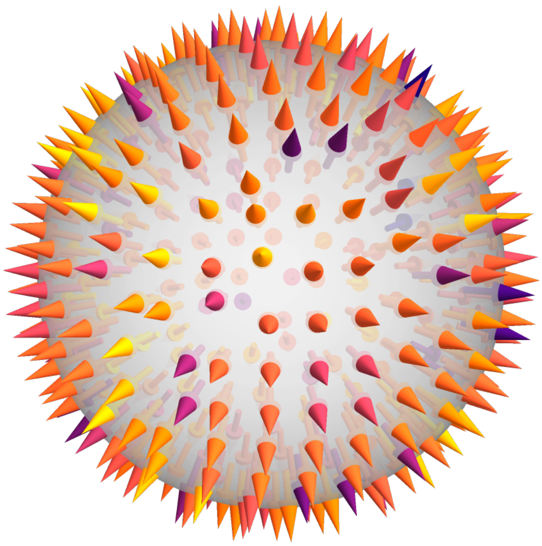


Figure 5.2: The hedgehog ansatz, the figure shows the diagram presentation of $\boldsymbol{\pi}$ field, the length of the arrow is proportional to $F(r)$.

Now we try to determine the boundary condition of $F(r)$. We know $SU(2) \cong S^3 = \mathbb{R}^3 \cup \{\infty\}$, we can set $U(|\boldsymbol{x}| \to \infty) = \mathbb{1}$(this means the field becomes vacuum at the infinity physically), then we have

$$F(\infty) = 0.$$

We can then expand U_H:

$$U_H(r) = \cos F(r)\mathbb{1} + \mathrm{i}\sin F(r)\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma},$$

which means F is actually the azimuthal angle of the group space. At the origin, our field cannot have any 'out-pointing' component, due to continuity. Therefore, our boundary condition must impose that $\sin F(0) = 0$, which implies

$$F(0) = n\pi, \quad n \in \mathbb{Z}.$$

With these two boundary condition, we can actually calculate the topological charge, i.e. the additional term in the gauge transformation (5.1) of S_{CS}:

$$\mathcal{B} \equiv \operatorname{tr}((U^{-1}\mathrm{d}U)^3) = \frac{1}{2\pi^2 r^2} F'(r)\sin^2(F(r)),$$

and the integration gives:

$$\begin{aligned} B[U_H] &\equiv \frac{1}{24\pi^2}\epsilon_{ijk}\int \mathrm{d}^3x \operatorname{tr}((U^{-1}\mathrm{d}U)^3) \\ &= -\frac{2}{\pi}\int_0^\infty F'(r)\sin^2 F\mathrm{d}r \\ &= \frac{1}{\pi}\left[\frac{1}{2}\sin 2F - F\right]\Bigg|_{r=0}^{r\to\infty} = n \in \mathbb{Z}. \end{aligned} \tag{5.2}$$

Of course, we can also use the geometry method to compute B. Note that S^3 can be parameterized by three angles (F, Θ, Φ), and B is the winding number of the map, which means we just need to pull back the normalized volume element on S^3 to the real space. The volume form of S^3 in $\mathbb{R}^4$ is given by

$$\mathrm{d}\Omega = \sin^2 F \sin\Theta \mathrm{d}F \wedge \mathrm{d}\Theta \wedge \mathrm{d}\Phi,$$

and the integration gives

$$\int \mathrm{d}\Omega = \int_0^\pi \sin^2 F\mathrm{d}F \int_0^\pi \sin\Theta\mathrm{d}\Theta \int_0^{2\pi} \mathrm{d}\Phi = 2\pi^2,$$

i.e. the normalized volume form is

$$\mathrm{d}\hat{\Omega} = \frac{1}{2\pi^2}\sin^2 F\sin\Theta\mathrm{d}F \wedge \mathrm{d}\Theta \wedge \mathrm{d}\Phi.$$

Then we define the pull back to be:

$$\begin{cases} F \to F(r) \Rightarrow \mathrm{d}F = F'\mathrm{d}r \\ \Theta \to \theta \\ \Phi \to \phi, \end{cases},$$

then the integral in real space equals to

$$\int_{\mathbb{R}^3} \mathrm{d}\tilde{\hat{\Omega}} = \frac{1}{2\pi^2}\int_\infty^0 F'\sin^2 F\mathrm{d}r \int_0^\pi \sin\theta\mathrm{d} \int_0^{2\pi} \mathrm{d}\Phi,$$

which will also give (5.2).

5.4 Quantization of Winding Number

Let us consider the manifold S^3 which we consider as $\mathbb{R}^3$ plus a point at infinity. Consider the gauge transform function defined

$$U(\boldsymbol{x}) = \exp\left(\frac{\mathrm{i}\pi N \boldsymbol{x}\cdot\boldsymbol{\sigma}}{\sqrt{|\boldsymbol{x}|^2+R^2}}\right)$$

where $\boldsymbol{x}$ is a point in $\mathbb{R}^3$, and $\boldsymbol{\sigma}$ represents the Pauli matrices with R an arbitrary length scale. Show the winding number Eq. 5.16 gives the integer N. Why does N need to be an integer here?

Answer We can just repeat our calculation in our last calculation again. To throw away N, we consider $B[U_1U_2]$. Write it down:

$$B[U_1U_2] = \frac{1}{24\pi^2}\epsilon_{ijk}\int \mathrm{d}^3x\,\mathrm{tr}(U_2^\dagger U_1^\dagger \partial_i(U_1U_2)\partial_j(U_2^\dagger U_1^\dagger)\partial_k(U_1U_2)).$$

where we have used the identity $(U_1U_2)^\dagger \equiv U_2^\dagger U_1$.(Noting that U is unitary, we can use $U^\dagger$ instead of U^{-1}). Expanding the traced part out yields

$$\begin{aligned}
&U_2^\dagger U_1^\dagger(\partial_i U_1)U_2(\partial_j U_2^\dagger)U_1^\dagger(\partial_k U_1)U_2 && \text{term 1}\\
&+U_2^\dagger U_1^\dagger(\partial_i U_1)U_2(\partial_j U_2^\dagger)U_1^\dagger U_1(\partial_k U_2) && \text{term 2}\\
&+\cancel{U_2^\dagger}U_1^\dagger(\partial_i U_1)\cancel{U_2U_2^\dagger}(\partial_j U_1^\dagger)(\partial_k U_1)\cancel{U_2} && B[U_1]\\
&+U_2^\dagger U_1^\dagger(\partial_i U_1)U_2U_2^\dagger(\partial_j U_1^\dagger)U_1(\partial_k U_2) && \text{term 4}\\
&+U_2^\dagger U_1^\dagger U_1(\partial_i U_2)(\partial_j U_2^\dagger)U_1^\dagger(\partial_k U_1)U_2 && \text{term 5}\\
&+U_2^\dagger \cancel{U_1^\dagger U_1}(\partial_i U_2)(\partial_j U_2^\dagger)\cancel{U_1^\dagger U_1}(\partial_k U_2) && B[U_2]\\
&+U_2^\dagger U_1^\dagger U_1(\partial_i U_2)U_2^\dagger(\partial_j U_1^\dagger)(\partial_k U_1)U_2 && \text{term 7}\\
&+U_2^\dagger U_1^\dagger U_1(\partial_i U_2)U_2^\dagger(\partial_j U_1^\dagger)U_1(\partial_k U_2). && \text{term 8}
\end{aligned}$$

Now we can integrate by part, for example, the red term in term 2 gives $U_1(\partial_k U_2) \to -(\partial_k U_1)U_2$, which cancels term 1. Similarly, term 7 cancels term 8. Finally, in term 5 we can integrate by parts three times, which cancels term 4. Which means we have the additive property

$$B[U_1U_2] = B[U_1] + B[U_2].$$

Then we can see

$$B[U(\boldsymbol{x})] = B[U_1(\boldsymbol{x})^N] = NB[U_1(\boldsymbol{x})],$$

where

$$U_1(\boldsymbol{x}) = \exp\left(\frac{\mathrm{i}\pi \boldsymbol{x}\cdot\boldsymbol{\sigma}}{\sqrt{|\boldsymbol{x}|^2+R^2}}\right).$$

Then we try to compute $B[U_1]$. Expand first:

$$U_1(\boldsymbol{x}) = \cos\left(\frac{\mathrm{i}\pi|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2+R^2}}\right)\mathbb{1} + \mathrm{i}\sin\left(\frac{\mathrm{i}\pi|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2+R^2}}\right)\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

Where $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$. We also embed the base space S^3 into $\mathbb{R}^4$ like before, while this time, we can use the coordinate

$$y^0 \equiv \cos\left(\frac{\mathrm{i}\pi|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2 + R^2}}\right), \quad \boldsymbol{y} = \sin\left(\frac{\mathrm{i}\pi|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2 + R^2}}\right)\boldsymbol{n}.$$

Then

$$U_1 \equiv y^0\mathbb{1} + \boldsymbol{y}\cdot\boldsymbol{\sigma},$$

with the condition $(y^0)^2 + \boldsymbol{y}^2 = 1$. The winding number clearly doesn't depend on the coordinate, we can use the point $y^0 = 1$, i.e. the north pole to evaluate B. Now

$$\begin{aligned}\mathcal{B}[U_1] &= \mathrm{tr}((U_1^{-1}\mathrm{d}U_1)^3) = \mathrm{tr}[(\mathrm{i}\mathrm{d}\boldsymbol{y}\cdot\boldsymbol{\sigma})^3] \\ &= \mathrm{i}^3 3!\,\mathrm{tr}(\sigma_1\sigma_2\sigma_3)\mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\mathrm{d}y^3 \\ &= 12\mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\mathrm{d}y^3.\end{aligned}$$

Then the integration over $\mathbb{R}^3$ gives

$$B[U_1] = \frac{1}{24\pi^2}\times 12\times 2\pi^2 = 1.$$

Finally, we can see

$$B[U(\boldsymbol{x})] = NB[U_1] = N.$$

Besides, at the infinity, we have

$$U(\boldsymbol{x}\to\infty) = \exp(\mathrm{i}\pi N\boldsymbol{n}\cdot\boldsymbol{\sigma}),$$

which should be independent of $\boldsymbol{n}$(or more physically, $U\to\mathbb{1}$), which means $\sin(N\pi) = 0$, thus N must be an integer.

Chapter 6

Short Digression on Quantum Gravity

Chapter 7

Defining Topological Quantum Field Theory

Part II

Anyon Basics

Chapter 8

Fusion and Structure of Hilbert Space

8.1 Quantum Dimension

Let N_{ab}^c be the fusion multiplicity matrices of a TQFT

$$a \times b = \sum_c N_{ab}^c c$$

meaning that N_{ab}^c is the number of distinct ways that a and b can fuse to c. (In many, or even most, theories of interest all N 's are either 0 or 1).

The quantum dimension d_a of a particle a is defined as the largest eigenvalue of the matrix $[N_a]_b^c$ where this is now thought of as a two dimensional matrix with a fixed and b, c the indices.

Show that

$$\mathsf{d}_a \mathsf{d}_b = \sum_c N_{ab}^c \mathsf{d}_c.$$

We will prove this formula algebraically in Chapter 17. However there is a simple and much more physical way to get to the the result: Imagine fusing together M anyons of type a and M anyons of type b where M gets very large and determine the dimension of space that results. Then imagine fusing together $a \times b$ and do this M times and then fuse together all the results.

Answer We first consider the fusion of M anyons of type a, as the Fig.8.1 shows. Suppose the

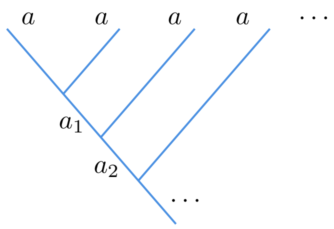


Figure 8.1: Fusion of M anyons of type a.

final result is a_{M-1}, then the dimension of fusing M anyons of type a to get a_f is:

$$\dim(a_{M-1}) = \sum_{a_1, a_2 \cdots a_{M-2}} N_{aa}^{a_1} N_{a_1 a}^{a_2} \cdots N_{a_{M-2} a}^{a_{M-1}} = [(N_a)^{M-1}]_a^{a_{M-1}}.$$

Similarly, the fusiong of M anyons of type b gives

$$\dim(b_{M-1}) = [(N_b)^{M-1}]_b^{b_{M-1}}.$$

Then the fusion of a_{M-1} and b_{M-1} gives

$$\dim(a_{M-1} \times b_{M-1}) = \sum_{a_{M-1} b_{M-1}} [(N_b)^{M-1}]_b^{b_{M-1}} [(N_a)^{M-1}]_a^{a_{M-1}} N_{a_{M-1} b_{M-1}}^{c_{M-1}}.$$

In large M limit, we have

$$\dim(a_{M-1} \times b_{M-1}) \sim (\mathsf{d}_b \mathsf{d}_a)^{M-1}.$$

However, if we fuse a, b first to give c, then fuse M anyons of type c, as the Fig.8.2 shows, we get

$$\dim(c_{M-1}) = [(N^c_{ab} N_c)^{M-1}]_c^{c_{M-1}} N^c_{ab} \sim (N^c_{ab} \mathsf{d}_c)^{M-1} = \dim(a_{M-1} \times b_{M-1}).$$

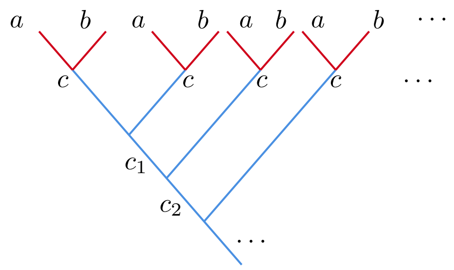


Figure 8.2: Fuse a, b first to give c, then fuse M anyons of type c.

So we have

$$\mathsf{d}_a \mathsf{d}_b = \sum_c N^c_{ab} \mathsf{d}_c.$$

8.2 Fusion and Ground State Degeneracy

To determine the ground state degeneracy of a 2-manifold in a $2+1$ dimensional TQFT one can cut the manifold into pieces and sew back together. One can think of the open "edges" or connecting tube-ends as each having a label given by one of the particle types (i.e., one of the anyons) of the theory. Really we are labeling each edge with a basis element of a possible Hilbert space. The labels on two tubes that have been connected together must match (label a on one tube fits into label $\bar{a}$ on another tube.) To calculate the ground state degeneracy we must keep track of all possible ways that these assembled tubes could have been labeled. For example, when we assemble a torus as in Fig.8.3, we must match the quantum number on one open end to the (opposite) quantum number on the opposite open end. The ground state degeneracy is then just the number of different possible labels, or equivalently the number of different particle types.

For more complicated 2-d manifolds, we can decompose the manifold into so-called pants diagrams that look like Fig.8.4. When we sew together pants diagrams, we should include a factor of the fusions multiplicity N^c_{ab} for each pants which has its three tube edges labeled with a, b and $\bar{c}$.

(a) Write a general formula for the ground state degeneracy of an M-handled torus in terms of the N matrices.

(b) For the Fibonacci anyon model, find the ground state degeneracy of a 4-handled torus.

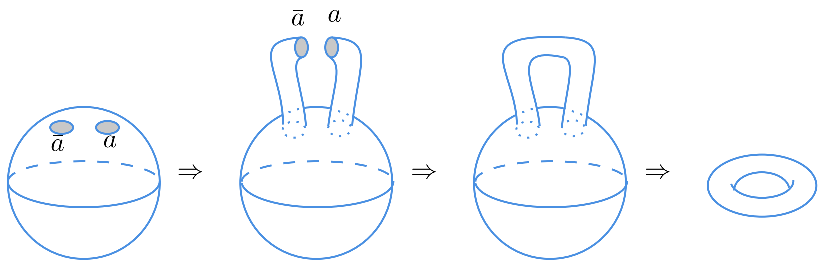


Figure 8.3: Surgering the twice punctured sphere into a torus. This is the gluing axiom in action. Note that we are implicitly assuming the system is trivial in the "time" direction, which we assume to form a circle S^1_{time}.

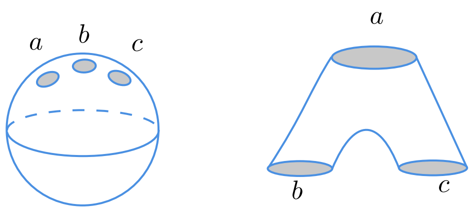


Figure 8.4: A three-times punctured sphere is known as a "pair of pants".

(c) Show that in the limit of large number of handles M the ground state degeneracy scales as $\sim \mathcal{D}^{2M}$ where $\mathcal{D}^2 = \sum_a \mathsf{d}_a^2$.

Answer

8.3 Consistency of Fusion Rules

Show by using commutativity and associativity of fusion along with identity (8.1)

$$N^c_{ab} = N^{\bar{c}}_{\bar{a}\bar{b}}, \tag{8.1}$$

that no anyon theory can have a particle a such that $a \times a = a$ meaning a fuses to a to form only a and nothing else.

Answer Suppose there is a theory that $a \times a = a$ and $a \neq I$, then consider the process of fusing a, a and $\bar{a}$. We can simplicity, we can set the fusion rule between $a, \bar{a}$ is

$$a \times \bar{a} = I + c_1 + \cdots + c_n. \tag{8.2}$$

As the Fig.8.5 shows, if we fuse $a \times a = a$ first, then fuse $a \times \bar{a}$, we will get $I + c_1 + \cdots + c_n$ finally. However, if we change the order of fusion, fuse $a \times \bar{a}$ first and get $I + c_1 + \cdots + c_n$, then fuse a, we will get

$$a \times \bar{a} = a + a \times c_1 + \cdots + a \times c_n.$$

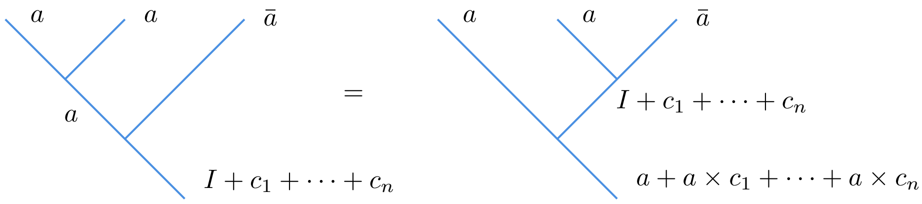


Figure 8.5: The fusion of $a, a, \bar{a}$.

In order to get the same result, we know that $\{a, a \times c_1, \cdots, a \times c_n\}$ must be a permutation of $\{I, c_1, \cdots, c_n\}$, which means $a \times c_i$ **cannot have multiple fusion channels** because each fusion channel have at least one result. Furthermore, in first case, we have I in the final result, so $\exists i$, s.t. $c_i = \bar{a}$. Now the rule that $a \times c_i$ cannot have multiple fusion channel contradicts with (8.2), so no anyon theory can have a particle a such that $a \times a = a$.

Chapter 9

Change of Basis and F-Matrices

9.1 F-gauge choice

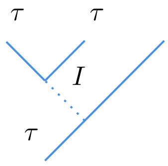
(a) Explain why in the Fibonacci theory, $[F_\tau^{\tau\tau\tau}]_{\tau\tau}$ is gauge independent but $[F_\tau^{\tau\tau\tau}]_{I\tau}$ is gauge dependent.

(b) Explain why in the Ising theory is $[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma}$ is gauge independent, but $[F_\psi^{\sigma\psi\sigma}]_{\sigma\sigma}$ is gauge dependent.

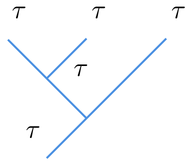
Answer (a) In Fibonacci theory, consider the process τ, τ, τ fuse to τ. We have:

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \begin{pmatrix} [F_\tau^{\tau\tau\tau}]_{II} & [F_\tau^{\tau\tau\tau}]_{I\tau} \\ [F_\tau^{\tau\tau\tau}]_{\tau I} & [F_\tau^{\tau\tau\tau}]_{\tau\tau} \end{pmatrix} \begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix},$$

where

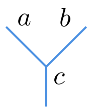


$= |0\rangle,$

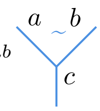


$= |1\rangle.$

Now we consider a gauge transformation on the vertices. According to the transformation rule of F, if the on one vertex, the gauge transformation is given by:



$= u_c^{ab}$



,

then

$$[\tilde{F}_e^{abc}]_{df} = \frac{u_e^{af} u_f^{bc}}{u_d^{ab} u_e^{dc}} [F_e^{abc}]_{df}.$$

Also note that if one of the upper legs is the identity, we typically do not allow a gauge transform of this type of vertex, i.e.

$$u_\tau^{I\tau} = 1.$$

Then

$$[\tilde{F}_\tau^{\tau\tau\tau}]_{\tau\tau} = \frac{u_\tau^{\tau\tau} u_\tau^{\tau\tau}}{u_\tau^{\tau\tau} u_\tau^{\tau\tau}} [F_\tau^{\tau\tau\tau}]_{\tau\tau} = [F_\tau^{\tau\tau\tau}]_{\tau\tau},$$

which means $[F_\tau^{\tau\tau\tau}]_{\tau\tau}$ is gauge independent. But for

$$[\tilde{F}_\tau^{\tau\tau\tau}]_{I\tau} = \frac{u_\tau^{\tau\tau} u_\tau^{\tau\tau}}{u_I^{\tau\tau} u_\tau^{\tau\tau}} [F_\tau^{\tau\tau\tau}]_{I\tau} = u_\tau^{\tau\tau} [F_\tau^{\tau\tau\tau}]_{I\tau},$$

which means $[F_\tau^{\tau\tau\tau}]_{I\tau}$ is gauge dependent.

(b) For Ising, the case is the same:

$$[\tilde{F}_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} = \frac{u_\sigma^{\psi\sigma} u_\sigma^{\sigma\psi}}{u_\sigma^{\psi\sigma} u_\sigma^{\sigma\psi}} [F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} = [F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma},$$

which means $[F_{\sigma}^{\psi\sigma\psi}]_{\sigma\sigma}$ is gauge independent. While

$$[\tilde{F}_{\psi}^{\sigma\psi\sigma}]_{\sigma\sigma} = \frac{u_{\psi}^{\sigma\sigma} u_{\sigma}^{\psi\sigma}}{u_{\sigma}^{\sigma\psi} u_{\psi}^{\sigma\sigma}} [F_{\psi}^{\sigma\psi\sigma}]_{\sigma\sigma} = \frac{u_{\sigma}^{\psi\sigma}}{u_{\sigma}^{\sigma\psi}} [F_{\psi}^{\sigma\psi\sigma}]_{\sigma\sigma},$$

which means $[F_{\psi}^{\sigma\psi\sigma}]_{\sigma\sigma}$ is gauge dependent. Note that $u_c^{ab} \neq u_c^{ba}$ in general.

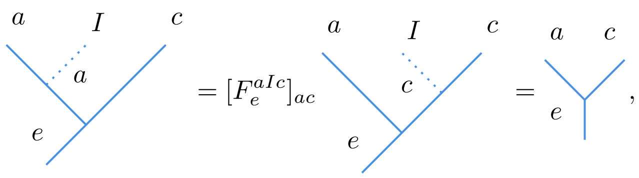
9.2 F's with the vacuum field I

Explain why $[F_e^{aIc}]_{ac} = [F_d^{abI}]_{db} = [F_e^{Ibc}]_{be} = 1$.

Answer We first prove $[F_e^{aIc}]_{ac}$ is gauge invariant:

$$[\tilde{F}_e^{aIc}]_{ac} = \frac{u_e^{ac} u_c^{Ic}}{u_a^{aI} u_e^{ac}} [F_e^{aIc}]_{ac} = [F_e^{aIc}]_{ac}.$$

Then draw the tree:



we can see the identity can be omitted, which means they are actually the same process, namely

$$[F_e^{aIc}]_{ac} = 1.$$

For $[F_d^{abI}]_{db}, [F_e^{Ibc}]_{be}$, the argument is the same.

9.3 Ising Pentagon

Consider a system of Ising anyons. Given the fusion rules, F_w^{xyz} will be a 2 by 2 matrix in the case of $x = y = z = w = \sigma$ (given by Eq. 9.4) and is a simply a scalar otherwise. One might hope that these scalars can all be taken to be unity. Unfortunately this is not the case. By examining the pentagon equation, Eq. 9.7 in the case of $a = b = c = \sigma$ and $d = f = \psi$ show that taking the scalar to always be unity is not consistent. Show further that choosing $[F_{\sigma}^{\psi\sigma\psi}]_{\sigma\sigma} = -1$ (and leaving the other scalars to be unity) allows a consistent solution of the pentagon for $a = b = c = \sigma$ and $d = f = \psi$.

Answer We give the pentagon identity:

$$[F_e^{fcd}]_{gl} [F_e^{abl}]_{fk} = \sum_h [F_g^{abc}]_{fh} [F_e^{ahd}]_{gk} [F_k^{bcd}]_{hl}.$$

with $a = b = c = \sigma$ and $d = f = \psi$, we have:

$$[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} \cdot [F_\sigma^{\sigma\sigma\sigma}]_{\psi k} = \sum_h [F_\sigma^{\sigma\sigma\sigma}]_{\psi h} [F_\sigma^{\sigma h\psi}]_{\sigma k} [F_k^{\sigma\sigma\psi}]_{h\sigma}.$$

However, for process $F_\sigma^{\sigma h\psi}$, we must have $h = \psi$ because if $h = \sigma$, $\sigma \times \sigma = I + \psi$, while $(I+\psi)\times\psi$ cannot give σ. So

$$[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} \cdot [F_\sigma^{\sigma\sigma\sigma}]_{\psi k} = [F_\sigma^{\sigma\sigma\sigma}]_{\psi\psi} [F_\sigma^{\sigma\psi\psi}]_{\sigma k} [F_k^{\sigma\sigma\psi}]_{\psi\sigma}.$$

For the term $F_\sigma^{\sigma\psi\psi}$, the only possible k is $k = I$ because $\psi \times \psi = I$. In this case:

$$\text{LHS} = [F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} \cdot [F_\sigma^{\sigma\sigma\sigma}]_{\psi I} = [F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} \cdot \frac{1}{\sqrt{2}},$$

while

$$\text{RHS} = -\frac{1}{\sqrt{2}} \cdot [F_\sigma^{\sigma\psi\psi}]_{\sigma I} \cdot [F_I^{\sigma\sigma\psi}]_{\psi\sigma}.$$

If we choose $[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} = [F_\sigma^{\sigma\psi\psi}]_{\sigma I} = [F_I^{\sigma\sigma\psi}]_{\psi\sigma}$(because $F_I^{\sigma\sigma\psi}, F_\sigma^{\sigma\psi\psi}$ are also scalar), the equality cannot hold, which means that taking the scalar to always be unity is not consistent.

Now, if we leave the scalar $F_I^{\sigma\sigma\psi}, F_\sigma^{\sigma\psi\psi}$ to be unity and choose $[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} = -1$, then

$$\text{LHS} = [F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}},$$

while

$$\text{RHS} = -\frac{1}{\sqrt{2}} \cdot [F_\sigma^{\sigma\psi\psi}]_{\sigma I} \cdot [F_I^{\sigma\sigma\psi}]_{\psi\sigma} = -\frac{1}{\sqrt{2}} = \text{LHS},$$

which is consistent, thus allowing a consistent solution of the pentagon for $a = b = c = \sigma$ and $d = f = \psi$.

9.4 Fibonacci Pentagon

In the Fibonacci anyon model, there are two particle types which are usually called I and τ. The only nontrivial fusion rule is $\tau \times \tau = I + \tau$. With these fusion rules, the F-matrix is completely fixed up to a gauge freedom (corresponding to adding a phase to some of the kets). If we choose all elements of the F-matrix to be real, then the F-matrix is completely determined by the pentagon up to one sign (gauge) choice. Using the pentagon equation determine the F-matrix. (To get you started, note that in Fig. 9.7 the variables a, b, c, d, e, f, g, h can only take values I and τ. You only need to consider the cases where a, b, c, d are all τ).

If you are stuck as to how to start, part of the calculation is given in Nayak et al. [2008].

Answer Before we look at the pentagon identity, consider the only nontrivial matrix $F_\tau^{\tau\tau\tau}$:

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} (F_\tau^{\tau\tau\tau})_{II} & (F_\tau^{\tau\tau\tau})_{I\tau} \\ (F_\tau^{\tau\tau\tau})_{\tau I} & (F_\tau^{\tau\tau\tau})_{\tau\tau} \end{pmatrix} \equiv \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}.$$

We can choose this basis transform to be unitary and real, i.e.

$$\begin{pmatrix} F_{00}^2 + F_{01}^2 & F_{00}F_{10} + F_{01}F_{11} \\ F_{10}F_{00} + F_{11}F_{01} & F_{10}^2 + F_{11}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we consider the case $a = b = l = f = c = d = g = k = \tau$, while $e = I$, this gives

$$[F_I^{\tau\tau\tau}]_{\tau\tau}[F_I^{\tau\tau\tau}]_{\tau\tau} = [F_\tau^{\tau\tau\tau}]_{\tau\tau}[F_I^{\tau\tau\tau}]_{\tau\tau}[F_\tau^{\tau\tau\tau}]_{\tau\tau} + [F_\tau^{\tau\tau\tau}]_{\tau I}[F_I^{\tau I\tau}]_{\tau\tau}[F_\tau^{\tau\tau\tau}]_{I\tau}.$$

However, we know $F_I^{\tau\tau\tau} = F_I^{\tau I\tau} = 1$ because in the second case, one of the upper indice is the identity. This gives

$$1 = F_{11}^2 + F_{10}F_{01}.$$

Now with

$$F_{10}^2 + F_{11}^2 = 1,$$

we have $F_{10} = F_{01}$, which means we also have

$$F_{00}^2 = F_{11}^2 = 1 - F_{01}^2.$$

Now we take $a = b = c = d = \tau$, and choose $e = f = g = k = l = \tau$, which gives

$$(F_\tau^{\tau\tau\tau})^2 = (F_\tau^{\tau\tau\tau})^2_{\tau\tau}(F_\tau^{\tau\tau\tau})_{\tau\tau} + (F_\tau^{\tau\tau\tau})_{\tau I}(F_\tau^{\tau\tau\tau})_{\tau\tau}(F_\tau^{\tau\tau\tau})_{I\tau},$$

i.e.

$$F_{11}^2 = F_{11}^3 + F_{01}^2 = F_{11}^3 + 1 - F_{11}^2.$$

This equation have three roots: $F_{11} = 1$, $F_{11} = (\sqrt{5}+1)/2$ or $F_{11} = (-\sqrt{5}+1)/2$. According to the reality condition and $F_{01}^2 = 1 - F_{11}^2$, the second root is ruled out.

Finally we choose $e = f = k = \tau$, while $g = l = I$, we can see

$$F_{00} = F_{01}^2.$$

Now with $F_{00}^2 = F_{11}^2$, the root $F_{11} = 1$ can be also excluded. So we finally have

$$F_{00} = -F_{11} = \frac{\sqrt{5}-1}{2},$$

while $F_{01} = F_{10} = F_{00}^{1/2}$ for one gauge, i.e.

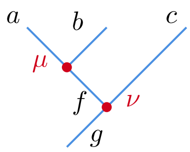
$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi \end{pmatrix},$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio.

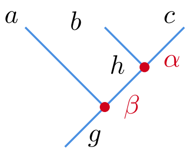
9.5 Pentagon and Fusion Multiplicities

Consider the case of Appendix 9.5.3 where there are fusion multiplicies $N_{ab}^c > 1$. Write the generalization of the pentagon equation Eq. 9.7.

Answer With $N_{ab}^c > 1$, the F matrix between these three anyons are given by:



$$= \sum_{h,\alpha,\beta} [F_g^{abc}]_{(f\mu\nu)(h\alpha\beta)}$$



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Then the pentagon diagram looks like Fig.9.1.

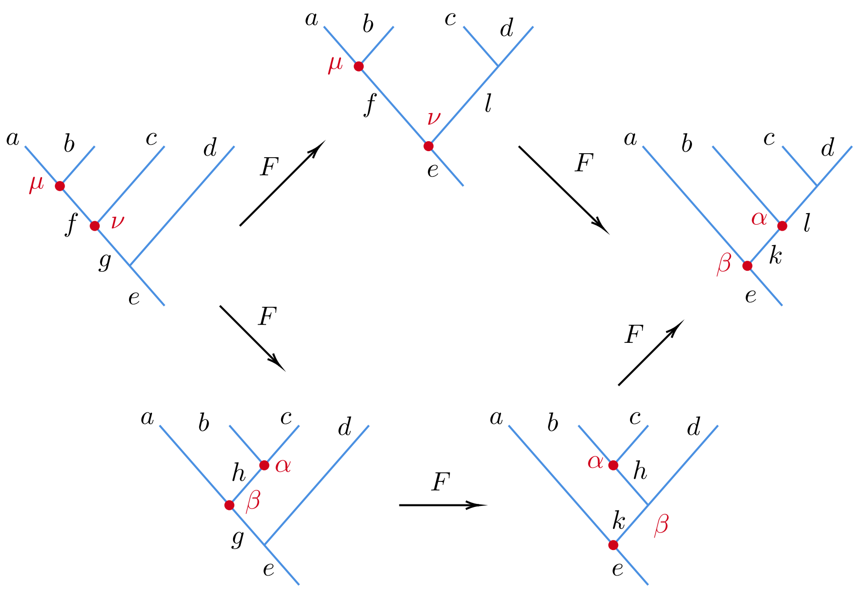


Figure 9.1: Pentagon Identity with fusion multiplicity.

Therefore, the new pentagon identity is now given by:

$$[F_e^{fcd}]_{(g\nu)(l\nu)}[F_e^{abl}]_{(f\mu\nu)(k\alpha\beta)} = \sum_h [F_g^{abc}]_{(f\mu\nu)(h\alpha\beta)}[F_e^{ahd}]_{(g\beta)(k\beta)}[F_k^{bcd}]_{(h\alpha)(l\alpha)}. \tag{9.1}$$

9.6 Gauge Change

(a) (i) Confirm that the F-matrix transforms under gauge change as indicated in Eq. 9.8.

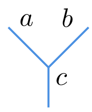
(ii) Show that a solution of the pentagon equation remains a solution under any gauge transformation.

(b) (Harder) Now consider the case of Appendix 9.5.3 where there are fusion multiplicies $N_{ab}^c > 1$.

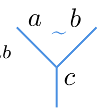
(i) Analogous to (a.i) Confirm Eq. 9.9.

(ii) Analogous to (a.ii) show that a solution of the pentagon equation remains a solution under any gauge transformation. (You will need to solve problem 9.5 first!)

Answer (a.i) Under gauge transformation



$= u_c^{ab}$



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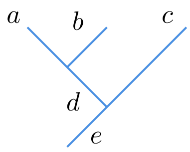
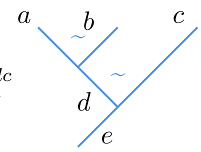
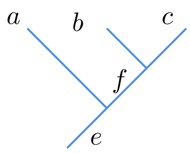
the original tree transforms like:



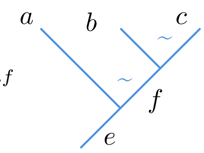
$\to u_d^{ab} u_e^{dc}$



,



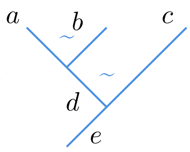
$\to u_f^{bc} u_e^{af}$



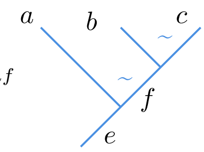
.

Therefore, the F matrix transforms like:

$u_d^{ab} u_e^{dc}$



$= \sum_f [F_e^{abc}]_{df} u_f^{bc} u_e^{af}$



,

Namely

$$[\tilde{F}_e^{abc}]_{df} = \frac{u_e^{af} u_f^{bc}}{u_d^{ab} u_e^{dc}} [F_e^{abc}]_{df}.$$

(a.ii) Under gauge transformation, the LHS of the pentagon identity transforms as:

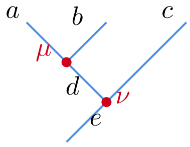
$$\begin{aligned}\text{LHS} =& [F_e^{fcd}]_{gl} [F_e^{abl}]_{fk} \\ \to& [\tilde{F}_e^{fcd}]_{gl} [\tilde{F}_e^{abl}]_{fk} \\ =& \frac{u_e^{ak} u_k^{bl} u_l^{cd}}{u_f^{ab} u_g^{cf} u_e^{dg}} [F_e^{abl}]_{fk} [F_e^{cdf}]_{gl},\end{aligned}$$

while

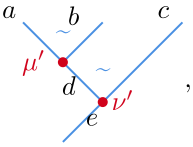
$$\begin{aligned}\text{RHS} =& \sum_h [F_g^{abc}]_{fh} [F_e^{ahd}]_{gk} [F_k^{bcd}]_{hl} \\ \to& \sum_h \frac{u_e^{ak} u_k^{bl} u_l^{cd}}{u_f^{ab} u_g^{cf} u_e^{dg}} [F_g^{abc}]_{fh} [F_e^{adh}]_{gk} [F_k^{bcd}]_{hl}.\end{aligned}$$

We can see now LHS = RHS, which means a solution of the pentagon equation remains a solution under any gauge transformation.

(b.i) If there are fusion multiplicies $N^c_{ab} > 1$, the original tree transforms like:



$$\to \sum_{\mu',\nu'} (u^{ab}_d)_{\mu\mu'}(u^{dc}_e)_{\nu\nu'}$$



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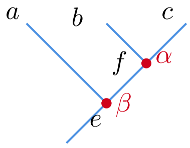
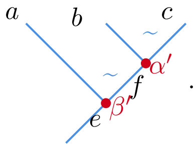


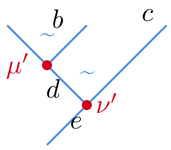
$$\to \sum_{\alpha',\beta'} (u^{bc}_f)_{\alpha\alpha'}(u^{af}_e)_{\beta\beta'}$$



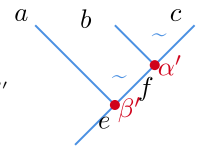
.

Then the F matrix transforms like:

$$\sum_{\mu',\nu'} (u^{ab}_d)_{\mu\mu'}(u^{dc}_e)_{\nu\nu'}$$



$$= \sum_{f,\alpha',\beta'} [F^{abc}_e]_{(d\mu\nu)(f\alpha\beta)}(u^{bc}_f)_{\alpha\alpha'}(u^{af}_e)_{\beta\beta'}$$



,

Namely

$$[\tilde{F}^{abc}_e]_{(d\mu'\nu')(f\alpha'\beta')} = \sum_{\alpha,\beta,\mu,\nu} ([u^{ab}_d]^{-1})_{\mu'\mu}([u^{dc}_e]^{-1})_{\nu'\nu}[F^{abc}_e]_{(d\mu\nu)(f\alpha\beta)}[u^{af}_e]_{\beta\beta'}[u^{bc}_f]_{\alpha\alpha'}.$$

(b.ii) In this case, for pentagon identity (9.1) we have:

$$\begin{aligned}
\text{LHS} &= [F^{fcd}_e]_{(g\nu)(l\nu)}[F^{abl}_e]_{(f\mu\nu)(k\alpha\beta)} \\
&\to \frac{(u^{ak}_e)_{\beta\beta'}(u^{bl}_k)_{\alpha\alpha'}u^{cd}_l(u^{fl}_e)_{\nu\nu'}}{(u^{ab}_f)_{\mu'\mu}(u^{fl}_e)_{\nu'\nu}(u^{cf}_g)_{\nu\nu'}u^{dg}_e}[F^{abl}_e]_{(f\mu'\nu')(k\alpha'\beta')}[F^{cdf}_e]_{(g\nu')(l\nu')} \\
&= \frac{(u^{ak}_e)_{\beta\beta'}(u^{bl}_k)_{\alpha\alpha'}u^{cd}_l}{(u^{ab}_f)_{\mu'\mu}(u^{cf}_g)_{\nu\nu'}u^{dg}_e}[F^{abl}_e]_{(f\mu'\nu')(k\alpha'\beta')}[F^{cdf}_e]_{(g\nu')(l\nu')},
\end{aligned}$$

while

$$\begin{aligned}
\text{RHS} &= \sum_h [F^{abc}_g]_{(f\mu\nu)(h\alpha\beta)}[F^{ahd}_e]_{(g\beta)(k\beta)}[F^{bcd}_k]_{(h\alpha)(l\alpha)} \\
&\to \sum_h \frac{(u^{ah}_g)_{\beta\beta'}(u^{bc}_h)_{\alpha\alpha'}}{(u^{ab}_f)_{\mu\mu'}(u^{cf}_g)_{\nu\nu'}}\frac{(u^{ak}_e)_{\beta\beta'}u^{hd}_k}{u^{dg}_e(u^{ah}_g)_{\beta\beta'}}\frac{(u^{bl}_k)_{\alpha\alpha'}u^{cd}_l}{(u^{bc}_h)_{\alpha\alpha'}u^{dh}_k}[F^{abc}_g]_{(f\mu\nu)(h\alpha\beta)}[F^{ahd}_e]_{(g\beta)(k\beta)}[F^{bcd}_k]_{(h\alpha)(l\alpha)} \\
&= \sum_h \frac{(u^{ak}_e)_{\beta\beta'}(u^{bl}_k)_{\alpha\alpha'}u^{cd}_l}{(u^{ab}_f)_{\mu\mu'}(u^{cf}_g)_{\nu\nu'}u^{dg}_e}[F^{abc}_g]_{(f\mu\nu)(h\alpha\beta)}[F^{ahd}_e]_{(g\beta)(k\beta)}[F^{bcd}_k]_{(h\alpha)(l\alpha)} = \text{LHS}.
\end{aligned}$$

For the multiplicity index, you can refer to Fig.9.1. Therefore, a solution of the pentagon equation with multiplicity remains a solution under any gauge transformation.

9.7 Ising F-matrix

[Hard] As discussed in the earlier problem, "Ising Anyons and Majorana Fermions" (Ex, 3.3), one can express Ising anyons in terms of Majorana fermions which are operators γ_i with anticommutations $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. As discussed there we can choose any two Majoranas and construct a fermion operator

$$c^\dagger_{12} = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2)$$

then the corresponding fermion orbital can be either filled or empty. We might write this as $|0_{12}\rangle = c_{12}|1_{12}\rangle$ and $|1_{12}\rangle = c^\dagger_{12}|0_{12}\rangle$. The subscript 12 here meaning that we have made the orbital out of Majoranas number 1 and 2. Note however, that we have to be careful that $|0_{12}\rangle = \mathrm{e}^{\mathrm{i}\phi}|1_{21}\rangle$ where ϕ is a gauge choice which is arbitrary (think about this if it is not obvious already).

Let us consider a system of 4 Majoranas, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Consider the basis of states

$$\begin{aligned}
|a\rangle &= |0_{12}0_{34}\rangle \\
|b\rangle &= |0_{12}1_{34}\rangle \\
|c\rangle &= |1_{12}0_{34}\rangle \\
|d\rangle &= |1_{12}1_{34}\rangle
\end{aligned}$$

rewrite these states in terms of basis of states

$$\begin{aligned}
|a'\rangle &= |0_{41}0_{23}\rangle \\
|b'\rangle &= |0_{41}1_{23}\rangle \\
|c'\rangle &= |1_{41}0_{23}\rangle \\
|d'\rangle &= |1_{41}1_{23}\rangle
\end{aligned}$$

Hence determine the F-matrix for Ising anyons. Be cautious about fermionic anticommutations: $c^\dagger_x c^\dagger_y = -c^\dagger_y c^\dagger_x$ so if we define $|1_x 1_y\rangle = c^\dagger_x c^\dagger_y |0_x 0_y\rangle$ with the convention that $|0_x 0_y\rangle = |0_y 0_x\rangle$ then we will have $|1_x 1_y\rangle = -|1_y 1_x\rangle$. Note also that you have to make a gauge choice of some phases (analogous to the mentioned gauge choice above). You can choose F to be always real.

Answer

Chapter 10

Exchanging Identical Particles

10.1 Calculating Exchanges

(a) Use Eq. 10.1 to confirm Eq. 10.11

(b) Use Eq. 10.1 to confirm Eq. 10.7

(c) Confirm the braiding relation $\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1 = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2$ in both cases. What does this identity mean geometrically. See exercise 3.1.

Answer (a) Eq.10.1 is given by:

$$\hat{\sigma}_2|c;f\rangle = \sum_{g,z}[F_f^{aaa}]_{cg}R_g^{aa}[(F_f^{aaa})^{-1}]_{gz}|z;f\rangle.$$

For Ising anyons, the F matrix is given by:

$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = [F_\sigma^{\sigma\sigma\sigma}]^{-1},$$

and the R matrix is given by:

$$\begin{aligned}R_I^{\sigma\sigma} &= \mathrm{e}^{-\mathrm{i}\pi/8},\\ R_\psi^{\sigma\sigma} &= \mathrm{e}^{\mathrm{i}3\pi/8}.\end{aligned}$$

So we have directly

$$\begin{aligned}\hat{\sigma}_2|I;\sigma\rangle =&\sum_g([F_\sigma^{\sigma\sigma\sigma}]_{Ig}R_g^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{gI}|I;\sigma\rangle + [F_\sigma^{\sigma\sigma\sigma}]_{Ig}R_g^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{g\psi}|\psi;\sigma\rangle)\\ =&([F_\sigma^{\sigma\sigma\sigma}]_{II}R_I^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{II} + [F_\sigma^{\sigma\sigma\sigma}]_{I\psi}R_\psi^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{\psi I})|I;\sigma\rangle\\ &+([F_\sigma^{\sigma\sigma\sigma}]_{II}R_I^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{I\psi} + [F_\sigma^{\sigma\sigma\sigma}]_{I\psi}R_\psi^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{\psi\psi})|\psi;\sigma\rangle\\ =&\frac{1}{2}(\mathrm{e}^{-\mathrm{i}\pi/8} + \mathrm{e}^{\mathrm{i}3\pi/8})|I;\sigma\rangle + \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\pi/3} - \mathrm{e}^{\mathrm{i}3\pi/8})|\psi;\sigma\rangle\\ =&\frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}(|I;\sigma\rangle - \mathrm{i}|\psi;\sigma\rangle).\end{aligned}$$

For the second basis vector this is the same:

$$\begin{aligned}\hat{\sigma}_2|\psi;\sigma\rangle =&\sum_g([F_\sigma^{\sigma\sigma\sigma}]_{\psi g}R_g^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{gI}|I;\sigma\rangle + [F_\sigma^{\sigma\sigma\sigma}]_{\psi g}R_g^{\sigma\sigma}[(F_\sigma^{\sigma\sigma\sigma})^{-1}]_{g\psi}|\psi;\sigma\rangle)\\ =&\frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}(-\mathrm{i}|I;\sigma\rangle + |\psi;\sigma\rangle).\end{aligned}$$

Therefore, we have

$$\hat{\sigma}_2\begin{pmatrix}|I;\sigma\rangle\\ |\psi;\sigma\rangle\end{pmatrix} = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}|I;\sigma\rangle\\ |\psi;\sigma\rangle\end{pmatrix},$$

i.e.

$$\hat{\sigma}_2 = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}.$$

(b) For Fibonacci Anyons, the non-trivial F matrix is given by:

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} = [F_\tau^{\tau\tau\tau}]^{-1}.$$

For $|0\rangle = |I;\tau\rangle$, we have

$$\begin{aligned}\hat{\sigma}_2|0\rangle =&(F_{00'}R_I^{\tau\tau}[F^{-1}]_{0'0} + F_{01'}R_\tau^{\tau\tau}[F^{-1}]_{10})|0\rangle \\ &+(F_{00'}R_I^{\tau\tau}[F^{-1}]_{0'1} + F_{01'}R_\tau^{\tau\tau}[F^{-1}]_{1'1})|1\rangle \\ =&\phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5}|0\rangle + \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5}|1\rangle,\end{aligned}$$

and for $|1\rangle = |\tau;\tau\rangle$, the procedure is the same:

$$\hat{\sigma}_2|1\rangle = \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5}|0\rangle - \phi^{-1}|1\rangle.$$

For the single state $|N\rangle = |\tau;I\rangle$, the F matrix is just a number, which means

$$\hat{\sigma}_2|N\rangle = R_\tau^{\tau\tau}|N\rangle = \mathrm{e}^{3\pi\mathrm{i}/5}.$$

Therefore, we have

$$\hat{\sigma}_2 = \begin{pmatrix} \mathrm{e}^{3\pi\mathrm{i}/5} & 0 & 0 \\ 0 & \phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5} & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} \\ 0 & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} & -\phi^{-1} \end{pmatrix}.$$

(c) In first case:

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1 = \frac{\mathrm{e}^{-\mathrm{i}\pi/8}}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2.$$

In the second case

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1 = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}/5} & 0 & 0 \\ 0 & -\phi^{-1}\mathrm{e}^{\pi\mathrm{i}/5} & \phi^{-1/2}\phi^{-4\mathrm{i}\pi/5} \\ 0 & \phi^{-1/2}\phi^{-4\mathrm{i}\pi/5} & \phi^{-1}\mathrm{e}^{\mathrm{i}\pi/5} \end{pmatrix} = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2.$$

Geometrically, this means the Yang-Baxter Equation.

10.2 Ising Anyons Redux

In exercise 3.3 we introduced a representation for the exchange matrices for Ising anyons which, for three anyons, would be of the form

$$\begin{aligned}\hat{\sigma}_1 &= \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}(1+\gamma_1\gamma_2) \\ \hat{\sigma}_2 &= \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{2}}(1+\gamma_2\gamma_3)\end{aligned}$$

where the γ 's are Majorana operators defined by

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$$

with $\gamma_i = \gamma_i^\dagger$.

Show that the exchange matrices in Eq. 10.11 are equivalent to this representation. How does one represent the $|0\rangle$ and $|1\rangle$ state of the Hilbert space in this language? The answer may not be unique.

Answer In this case, we first write

$$\hat{\sigma}_1 = \mathrm{e}^{-\mathrm{i}\pi/8}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix} = \mathrm{e}^{\mathrm{i}\pi/8}\mathrm{e}^{-\mathrm{i}\pi/4}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix} = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}\begin{pmatrix}1-\mathrm{i} & 0\\ 0 & \mathrm{i}+1\end{pmatrix}.$$

Then we have directly $\alpha = \pi/8$ and

$$\gamma_1\gamma_2 = \begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}, \quad \gamma_2\gamma_3 = \begin{pmatrix}0 & -\mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}.$$

For example, if we take

$$\gamma_1 = \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma_2 = -\sigma_2 = \begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}, \quad \gamma_3 = \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

we can the the result. In this case, the creation operator is

$$c_{12}^\dagger = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

and we have:

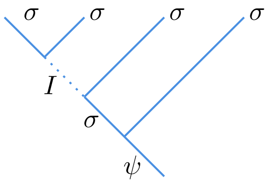
$$c_{12}^\dagger|0\rangle = |1\rangle$$
$$c_{12}^\dagger|1\rangle = 0.$$

10.3 Exchanging More Particles

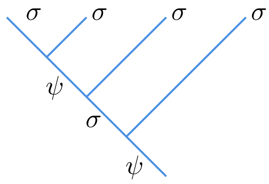
(a) Consider a system of 4 identical Ising anyons. Use the F - and R-matrices to calculate the braid matrices $\hat{\sigma}_1, \hat{\sigma}_2$, and $\hat{\sigma}_3$. (You should be able to check your answer using the Majorana representation of exercise 3.3.)

(b) (Harder) Consider a system of 4 identical Fibonacci anyons. Use the F and R-matrices to calculate the braid matrices $\hat{\sigma}_1, \hat{\sigma}_2$, and $\hat{\sigma}_3$.

Answer (a) For four σ anyons, we have two kinds of results: I or ψ. Each type have two possibilities, for example we have $|0;\psi\rangle \equiv |I;\psi\rangle$ and $|1;\psi\rangle = |\psi;\psi\rangle$, as the Fig.10.1 shows.



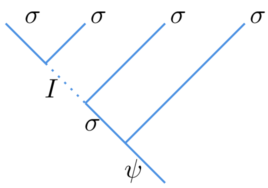
$|0;\psi\rangle$

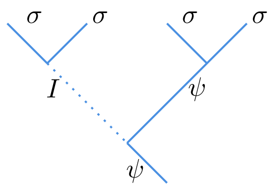


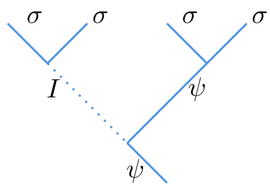
$|1;\psi\rangle$

Figure 10.1: Fusing channel of four identical Ising anyons.

We just need to figure out the effect of $\hat{\sigma}_3$. Take $|0;\psi\rangle$ as example, we first change into basis that the third σ and fourth σ fuse first:

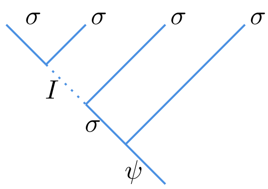


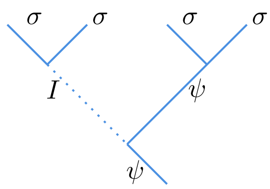
$= [F^{I\sigma\sigma}_{\sigma}]_{\sigma\psi}$


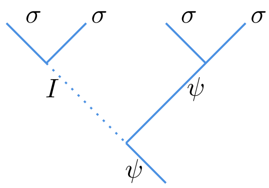
$=$


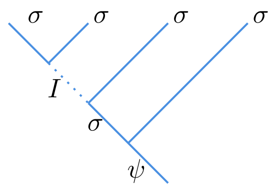
.

Then braiding gives

$\hat{\sigma}_3$


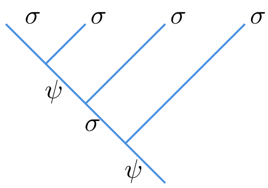
$= R^{\sigma\sigma}_{\psi}$


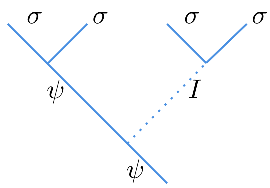
$= \mathrm{e}^{3\pi\mathrm{i}/8}$


$= \mathrm{e}^{3\pi\mathrm{i}/8}$


.

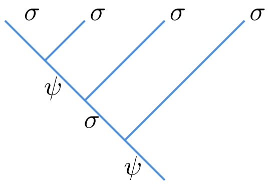
For the second case:

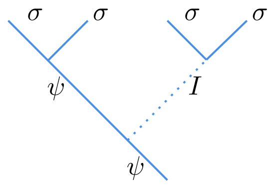


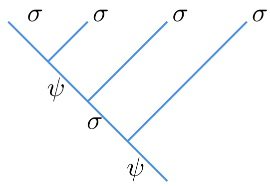
$= [F^{\psi\sigma\sigma}_{\psi}]_{\sigma I}$


,

Then braiding gives

$\hat{\sigma}_3$ 

$= [F_{\psi}^{\psi\sigma\sigma}]_{\sigma I} R_I^{\sigma\sigma}$ 

$= \mathrm{e}^{-\mathrm{i}\pi/8}$ 

.

This means we have

$$\hat{\sigma}_3 \begin{pmatrix} |0;\psi\rangle \\ |1;\psi\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{3\pi\mathrm{i}/8}|0;\psi\rangle \\ \mathrm{e}^{-\mathrm{i}\pi/8}|1;\psi\rangle \end{pmatrix},$$

i.e.

$$\hat{\sigma}_3 = \begin{pmatrix} \mathrm{e}^{3\pi\mathrm{i}/8} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\pi/8} \end{pmatrix}.$$

If we try $|0;I\rangle, |1;I\rangle$, we will get the same result. In Majorana representation, we can see

$$\hat{\sigma}_3 = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}} \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} \end{pmatrix} = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{2}}(1+\gamma_3\gamma_4),$$

with $\gamma_3 = \sigma_3$, we have

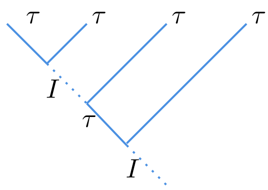
$$\gamma_4 = \begin{pmatrix} \mathrm{i} & \\ & \mathrm{i} \end{pmatrix},$$

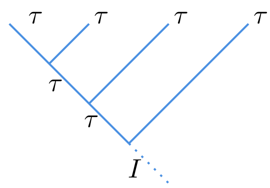
which satisfies the condition $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. So for the full space,in the order $|0;I\rangle, |1;I\rangle, |0;\psi\rangle, |1;\psi\rangle$, we have

$$\hat{\sigma}_{1,\mathrm{full}} = \begin{pmatrix} \hat{\sigma}_1 & \\ & \hat{\sigma}_1 \end{pmatrix}, \quad \hat{\sigma}_{2,\mathrm{full}} = \begin{pmatrix} \hat{\sigma}_2 & \\ & \hat{\sigma}_2 \end{pmatrix}, \quad \hat{\sigma}_{3,\mathrm{full}} = \begin{pmatrix} \hat{\sigma}_3 & \\ & \hat{\sigma}_3 \end{pmatrix}.$$

(b) With four τ, we have 5 possibilities, as the Fig.10.2 shows.

We look at the simplier, final state I first. Call

$|0;I\rangle \equiv$ 

, $|1;I\rangle \equiv$ 

.

For $\hat{\sigma}_1$, it is trivial:

$$\hat{\sigma}_1|0;I\rangle = R_I^{\tau\tau}|0;I\rangle = \mathrm{e}^{-4\pi\mathrm{i}/5}|0;I\rangle,$$
$$\hat{\sigma}_1|1;I\rangle = R_\tau^{\tau\tau}|1;I\rangle = \mathrm{e}^{3\pi\mathrm{i}/5}.$$

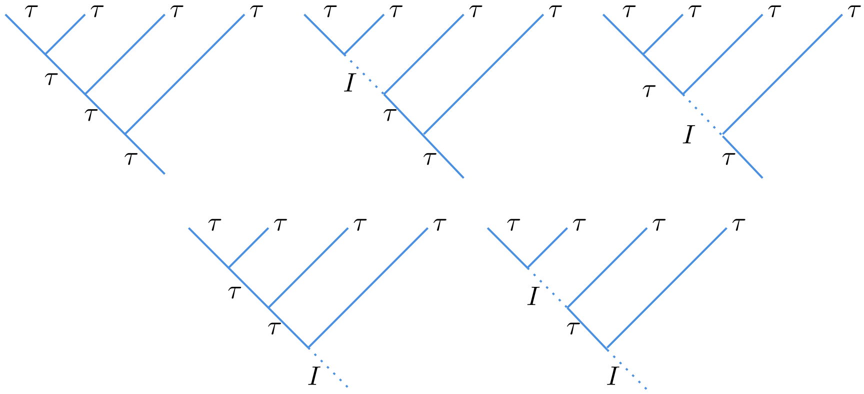


Figure 10.2: Fusing channel of four identical Fibonacci anyons.

So

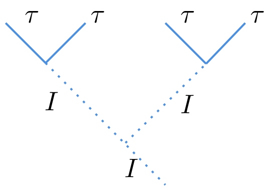
$$\hat{\sigma}_1 = \begin{pmatrix} \mathrm{e}^{-4\pi\mathrm{i}/5} & 0 \\ 0 & \mathrm{e}^{3\pi\mathrm{i}/5} \end{pmatrix}.$$

For $\hat{\sigma}_2$, it is same as the original one, i.e.

$$\hat{\sigma}_2 = \begin{pmatrix} \phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5} & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} \\ \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} & -\phi^{-1} \end{pmatrix}.$$

For $\hat{\sigma}_3$:

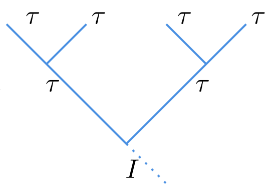
$$\hat{\sigma}_3|0;I\rangle = \hat{\sigma}_3 F_I^{I\tau\tau}$$



$$= R_I^{\tau\tau}|0;I\rangle = \mathrm{e}^{-4\pi\mathrm{i}/5}|0;I\rangle,$$

while

$$\hat{\sigma}_3|1;I\rangle = \hat{\sigma}_3 F_I^{\tau\tau\tau}$$

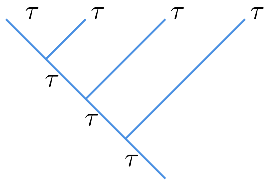


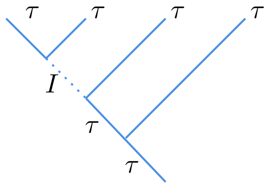
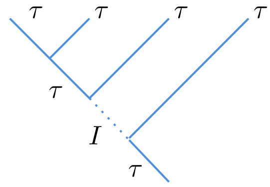
$$= R_\tau^{\tau\tau}|0;I\rangle = \mathrm{e}^{3\pi\mathrm{i}/5}|0;I\rangle.$$

So

$$\hat{\sigma}_3 = \begin{pmatrix} \mathrm{e}^{-4\pi\mathrm{i}/5} & 0 \\ 0 & \mathrm{e}^{3\pi\mathrm{i}/5} \end{pmatrix}.$$

Now we consider the subspace with final result τ. We call

$|0;\tau\rangle \equiv$  ,

$|-1;\tau\rangle \equiv$  , $|1;\tau\rangle \equiv$  .

Then for $\hat{\sigma}_1$, the result is trivial, i.e.

$$\hat{\sigma}_1 \begin{pmatrix} |-1;\tau\rangle \\ |0;\tau\rangle \\ |1;\tau\rangle \end{pmatrix} = \begin{pmatrix} R_I^{\tau\tau}|-1;\tau\rangle \\ R_\tau^{\tau\tau}|0;\tau\rangle \\ R_\tau^{\tau\tau}|1;\tau\rangle \end{pmatrix} \Rightarrow \hat{\sigma}_1 = \mathrm{diag}(\mathrm{e}^{-4\pi\mathrm{i}/5}, \mathrm{e}^{3\pi\mathrm{i}/5}, \mathrm{e}^{3\pi\mathrm{i}/5}).$$

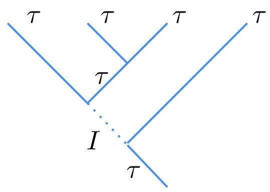
For $\hat{\sigma}_2$:

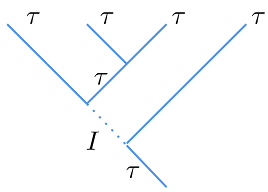
$$\begin{aligned}\hat{\sigma}_2|0;\tau\rangle =&([F_\tau^{\tau\tau\tau}]_{\tau\tau}R_\tau^{\tau\tau}[F_\tau^{\tau\tau\tau}]_{\tau\tau} + [F_\tau^{\tau\tau\tau}]_{\tau I}R_I^{\tau\tau}[F_\tau^{\tau\tau\tau}]_{I\tau})|0;\tau\rangle \\ &+([F_\tau^{\tau\tau\tau}]_{\tau\tau}R_\tau^{\tau\tau}[F_\tau^{\tau\tau\tau}]_{\tau I} + [F_\tau^{\tau\tau\tau}]_{\tau I}R_I^{\tau\tau}[F_\tau^{\tau\tau\tau}]_{II})|-1;\tau\rangle \\ =& -\phi^{-1}|0;\tau\rangle + \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5}|-1;\tau\rangle.\end{aligned}$$

This is same as the case with two τs, so we also have

$$\hat{\sigma}_2|-1;\tau\rangle = \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5}|0;\tau\rangle + \phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5}|1;\tau\rangle.$$

For $|1;\tau\rangle$:

$\hat{\sigma}_2|1;\tau\rangle = \hat{\sigma}_2 F_I^{\tau\tau\tau}$ 

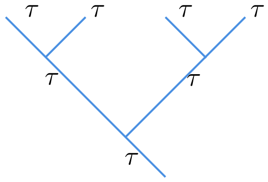
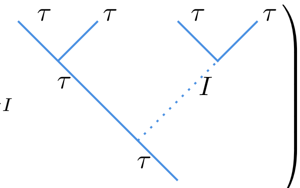
$= R_\tau^{\tau\tau}$ 

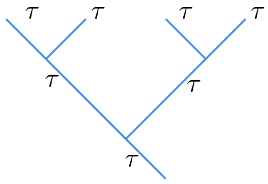
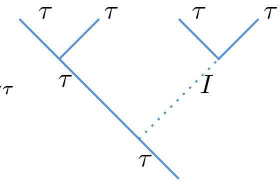
$= \mathrm{e}^{3\pi\mathrm{i}/5}|1;\tau\rangle.$

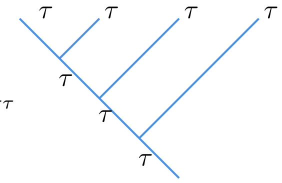
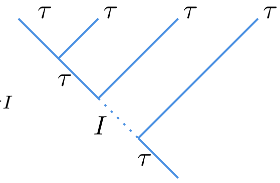
So for $\hat{\sigma}_2$, the result is same as $|N\rangle, |0\rangle, |1\rangle$, i.e. in order $|-1;\tau\rangle, |0;\tau\rangle; |1;\tau\rangle$, $\hat{\sigma}_2$ is given by

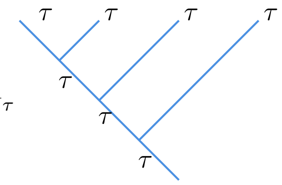
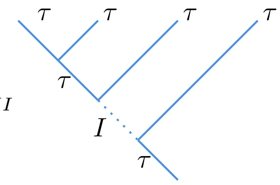
$$\hat{\sigma}_2 = \begin{pmatrix} \phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5} & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} & \\ \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} & -\phi^{-1} & \\ & & \mathrm{e}^{3\pi\mathrm{i}/5} \end{pmatrix}.$$

Now we look at $\hat{\sigma}_3$. Same process gives

$$\hat{\sigma}_3|0;\tau\rangle = \hat{\sigma}_3 \Big([F_\tau^{\tau\tau\tau}]_{\tau\tau}$$

$$+ [F_\tau^{\tau\tau\tau}]_{\tau I}$$

$$\Big)$$

$$= [F_\tau^{\tau\tau\tau}]_{\tau\tau} R_\tau^{\tau\tau}$$

$$+ [F_\tau^{\tau\tau\tau}]_{\tau I} R_I^{\tau\tau}$$


$$= [F_\tau^{\tau\tau\tau}]_{\tau\tau} R_\tau^{\tau\tau} \Big([F_\tau^{\tau\tau\tau}]_{\tau\tau}$$

$$+ [F_\tau^{\tau\tau\tau}]_{\tau I}$$

$$\Big)$$

$$+ [F_\tau^{\tau\tau\tau}]_{\tau I} R_I^{\tau\tau} \Big([F_\tau^{\tau\tau\tau}]_{I\tau}$$

$$+ [F_\tau^{\tau\tau\tau}]_{II}$$

$$\Big)$$

$$\begin{aligned}
&= ([F_\tau^{\tau\tau\tau}]_{\tau\tau} R_\tau^{\tau\tau} [F_\tau^{\tau\tau\tau}]_{\tau\tau} + [F_\tau^{\tau\tau\tau}]_{\tau I} R_I^{\tau\tau} [F_\tau^{\tau\tau\tau}]_{I\tau})|0;\tau\rangle \\
&+ ([F_\tau^{\tau\tau\tau}]_{\tau\tau} R_\tau^{\tau\tau} [F_\tau^{\tau\tau\tau}]_{\tau I} + [F_\tau^{\tau\tau\tau}]_{\tau I} R_I^{\tau\tau} [F_\tau^{\tau\tau\tau}]_{II})|1;\tau\rangle \\
&= -\phi^{-1}|0;\tau\rangle + \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5}|1;\tau\rangle.
\end{aligned}$$

Which change $|-1;\tau\rangle$ to $|1;\tau\rangle$ just as in $\hat{\sigma}_2$ case. We can directly write:

$$\hat{\sigma}_3 = \begin{pmatrix} \mathrm{e}^{3\pi\mathrm{i}/5} & & \\ & -\phi^{-1} & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} \\ & \phi^{-1/2}\mathrm{e}^{-3\pi\mathrm{i}/5} & \phi^{-1}\mathrm{e}^{4\pi\mathrm{i}/5} \end{pmatrix}.$$

Thus, in the order $|0;I\rangle, |1;I\rangle, |-1;\tau\rangle, |0;\tau\rangle; |1;\tau\rangle$, the braid matrices are

$$\hat{\sigma}_1 = \begin{pmatrix} \mathrm{e}^{-4\pi\mathrm{i}/5} & & & & \\ & \mathrm{e}^{3\pi\mathrm{i}/5} & & & \\ & & \mathrm{e}^{-4\pi\mathrm{i}/5} & & \\ & & & \mathrm{e}^{3\pi\mathrm{i}/5} & \\ & & & & \mathrm{e}^{3\pi\mathrm{i}/5} \end{pmatrix},$$

$$\hat{\sigma}_2 = \begin{pmatrix} \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} & & & \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} & & & \\ & & \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} & \\ & & \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} & \\ & & & & e^{3\pi i/5} \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} e^{-4\pi i/5} & & & & \\ & e^{3\pi i/5} & & & \\ & & e^{3\pi i/5} & & \\ & & & -\phi^{-1} & \phi^{-1/2}e^{-3\pi i/5} \\ & & & \phi^{-1/2}e^{-3\pi i/5} & \phi^{-1}e^{4\pi i/5} \end{pmatrix}. \tag{10.1}$$

10.4 Determinant and Trace of Braid Matrices

Consider a system of N-identical anyons with a total Hilbert space dimension D. The braid matrix $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N-1}$ are all D-dimensional. Show that each of these matrices has the same determinant, and each of these matrices has the same trace. Hint: This is easy if you think about it right!

Answer For the determinant, we just use the defining relation

$$\det(\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1) = \det(\hat{\sigma}_1)^2 \cdot \det(\hat{\sigma}_2) = \det(\hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2) = \det(\hat{\sigma}_1) \cdot \det(\hat{\sigma}_2)^2,$$

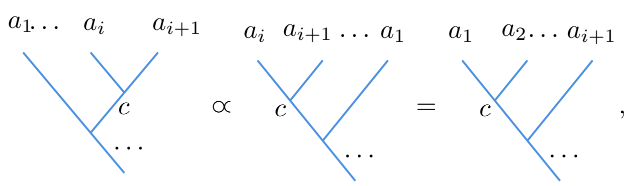
which means

$$\det\hat{\sigma}_1 = \det\hat{\sigma}_2 = \cdots .$$

For the trace, say $\hat{\sigma}_i$ which braids a_i, a_{i+1}, we can use the F matrix to change the basis that a_i and a_{i+1} fuse directly together, apply $\hat{\sigma}_i$, then use F^{-1} to change the basis back. Note that this is actually a similarly transformation, which preserves trace, we know that

$$\mathrm{tr}(\hat{\sigma}_i) = \sum_c m_c^i R_c^{aa},$$

where m_c^i is the multiplicity of c appears in the diagrams. Now the important thing is that, the braidng between c and a_1 only involves a factor:



which means $m_c^i = m_c^1$, so

$$\mathrm{tr}(\hat{\sigma}_i) = \mathrm{tr}(\hat{\sigma}_1) = \cdots .$$

10.5 Checking the locality constraint

(Easy) Consider Fig.10.3 The braid on the left can be written as $\hat{b}_3 = \hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2$.

(a) For the Fibonacci theory with $a = \tau$ check that the matrix $\hat{b}_3$ gives just a phase, which is dependent on the fusion channel c. I.e., show the matrix $\hat{b}_3$ is a diagonal matrix of complex phases. Show further that these phases are the same as the phase that would be accumulated for taking a single τ particle around the particle c.

(b) Consider the same braid for the Ising theory with $a = \sigma$. Show again that the result is a c-dependent phase.

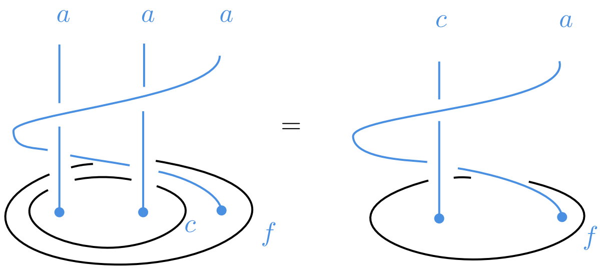


Figure 10.3: The locality constraint of a, a, a.

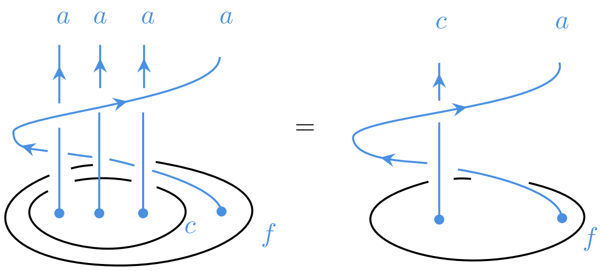


Figure 10.4: The locality constraint of a, a, a, a.

(Hard) Consider the braid shown on the left of Fig.10.4. The braid can be written as $\hat{b}_4 = \hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2\hat{\sigma}_3$.

(c) Consider Ising anyons where $a = \sigma$, Use the F and R-matrices to calculate $\hat{\sigma}_3$ (See exercise 10.3.a). Since the fusion of three σ anyons always gives $c = \sigma$, calculate $\hat{b}_4$, show this is a phase times the identity matrix, and show that the phase matches the phase of taking a single σ all the way around another σ.

(d) Consider Fibonacci anyons with $a = \tau$, Use the F and R-matrices to calculate $\hat{\sigma}_3$. (See exercise 10.3.b). Check that $\hat{b}_4$ is a diagonal matrix of phases. Check the phases match the two possible phases accumulated by wrapping a single τ all the way around a single particle c which can be I or τ.

Answer (a) For Fibonacci anyons, these two braiding operators are given by

$$\hat{\sigma}_1 = \begin{pmatrix} e^{3\pi i/5} & & \\ & e^{-4\pi i/5} & \\ & & e^{3\pi i/5} \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} e^{3\pi i/5} & 0 & 0 \\ 0 & \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ 0 & \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}.$$

Direct calculation gives

$$\hat{b}_3 = \hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2 = \begin{pmatrix} e^{2\pi i/5} & & \\ & 1 & \\ & & e^{-4\pi i/5} \end{pmatrix},$$

which is a diagonal matrix of complex phases. Note that in this process, c only have two possibilities, i.e. I, τ, and the final result f corresponds I, τ, we know that

$$\begin{aligned}
\hat{b}_3|N\rangle &= (R_I^{c\tau})^2 = (R_I^{\tau\tau})^2 = e^{-8\pi i/5} = e^{2\pi i/5}, \\
\hat{b}_3|0\rangle &= (R_\tau^{c\tau})^2 = (R_\tau^{I\tau})^2 = 1, \\
\hat{b}_3|1\rangle &= (R_\tau^{c\tau})^2 = (R_\tau^{\tau\tau})^2 = e^{6\pi i/5} = e^{-4\pi i/5}.
\end{aligned}$$

So these phases are the same as the phase that would be accumulated for taking a single τ particle around the particle c.

(b) For Ising anyons, the two braiding operators are given by

$$\hat{\sigma}_1 = e^{-i\pi/8}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \hat{\sigma}_2 = \frac{e^{i\pi/8}}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Direct calculation gives

$$\hat{b}_3 = \hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Which indicates that actually $(R_\sigma^{\psi\sigma})^2 = \hat{b}_3|1\rangle = -1$.

(c) For Ising anyons, we have already calculate that

$$\hat{\sigma}_3 = \begin{pmatrix} e^{3\pi i/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}.$$

Therefore, we have

$$\hat{b}_4 = \hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2\hat{\sigma}_3 = e^{3\pi i/4}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

which is a phase times the identity matrix. We know that four Ising anyons can only fusion to ψ, so the phase must equals to $(R_\psi^{\sigma\sigma})^2 = (e^{3\pi i/8})^2 = e^{3\pi i/4}$, which is exactly the case.

(d) According to our calculation result (10.1):

$$\hat{b}_4 = \hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2\hat{\sigma}_3 = \mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{-4\pi i/5}, e^{-4\pi i/5}, 1).$$

This is actually the case:

$$\begin{aligned}
\hat{b}_4|0;I\rangle = \hat{b}_4|1;I\rangle &= (R_I^{\tau\tau})^2 = e^{2\pi i/5}, \\
\hat{b}_4|-1;I\rangle = \hat{b}_4|0;I\rangle &= (R_\tau^{\tau\tau})^2 = e^{-4\pi i/5}, \\
\hat{b}_4|1;I\rangle &= (R_\tau^{\tau I})^2 = 1.
\end{aligned}$$

10.6 Enforcing the locality constraint

The locality constraint shown in Fig.10.3 turns out to be extremely powerful. In this exercise we will use this constraint to (almost) derive the possible values for the R-matrix for Fibonacci anyons given the known F-matrix.

Consider an anyon theory with Fibonacci fusion rules and Fibonacci F matrix as in Eq. 9.2.

(a) (Easy) Confirm the locality constraint shown in Fig.10.3 (see also Fig. 10.4) given the values of R given in Eq. 10.2. Make sure to confirm the equality for all three cases $f = I, c = \tau$ and $f = \tau, c = I$ and $f = \tau, c = \tau$.

Note that on the left of Fig. 10.16 is the braiding operation $\hat{O} = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1\hat{\sigma}_2$. whereas the operation on the right is σ^2.

(b) Show that the locality constraint of Fig.10.3 would also be satisfied by

$$R_I^{\tau\tau} \to -R_I^{\tau\tau} \quad R_\tau^{\tau\tau} \to -R_\tau^{\tau\tau}$$

It will turn out (See *** below) that this additional solution is spurious, as there are other consistency conditions it does not satisfy.

(c) In addition to right and left handed Fibonacci anyons and the two additional spurious solutions provided by Eq. 10.14, there are four additional possible sets of R-matrices that are consistent with the F-matrices of the Fibonacci theory given the locality constraint of Fig. 10.16. These additional solutions are all fairly trivial. Can you guess any of them?

If we cannot guess the additional possible R-matrices, we can derive them explicitly (and show that no others exist). Let us suppose that we do not know the values of the R-matrix elements $R_I^{\tau\tau}$ and $R_\tau^{\tau\tau}$.

(d) For the case of $f = I$ and $c = \tau$ show that Fig. 10.16 implies

$$[R_\tau^{\tau\tau}]^4 = [R_I^{\tau\tau}]^2$$

(e) (Harder) For the case of $f = \tau$ we have a two-dimensional Hilbert space spanned by the two values of $c = I$ or $c = \tau$. Any linear operator on this Hilbert space should be a 2 by 2 matrix. Thus the locality constraint Eq. 10.16 is actually an equality of 2 by 2 matrices. Derive this equality.

(f) Use this result, in combination with Eq. 10.15 to find all possible R matrices that satisfy the locality constraint. You should find a total of eight solutions. Six of these are spurious as we will see in section 13.3.

The calculation you have just done is equivalent to enforcing the so-called hexagon condition which we will discuss in section 13.3 below.

Answer (a) As we have checked before:

$$\begin{aligned}\hat{b}_3|N\rangle &= (R_f^{c\tau})^2 = (R_I^{\tau\tau})^2 = \mathrm{e}^{-8\pi\mathrm{i}/5} = \mathrm{e}^{2\pi\mathrm{i}/5},\\ \hat{b}_3|0\rangle &= (R_f^{c\tau})^2 = (R_\tau^{I\tau})^2 = 1,\\ \hat{b}_3|1\rangle &= (R_f^{c\tau})^2 = (R_\tau^{\tau\tau})^2 = \mathrm{e}^{6\pi\mathrm{i}/5} = \mathrm{e}^{-4\pi\mathrm{i}/5},\end{aligned}$$

which gives

$$\hat{b}_3 = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}/5} & & \\ & 1 & \\ & & \mathrm{e}^{-4\pi\mathrm{i}/5} \end{pmatrix}.$$

(b) From the Fig.10.3, we can see that τ braids around f twice, which means the result will always be $(R_f^{c\tau})^2$. So it is invariant under the transformation $R_f^{c\tau} \to -R_f^{c\tau}$.

(c) A fairly trivial guess is that

$$R_\tau^{I\tau} = R_\tau^{\tau I} = 1.$$

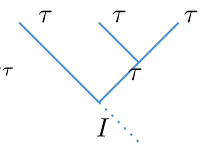
This can be checked by the hexagon identity using $F_\tau^{\tau\tau I} = F_I^{\tau\tau\tau} = 1$.

(d) We have already figure out that the right part of Fig.10.3 is:

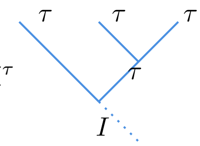
$$\hat{b}_3|N\rangle = (R_I^{\tau\tau})^2.$$

While the left part of Fig.10.3 is

$$\hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1\hat{\sigma}_2|N\rangle = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1\hat{\sigma}_2 F_I^{\tau\tau\tau}$$



$$= \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1 R_\tau^{\tau\tau}$$



$$\begin{aligned}&= R_\tau^{\tau\tau}\hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1|N\rangle\\ &= (R_\tau^{\tau\tau})^3\hat{\sigma}_2|N\rangle = (R_\tau^{\tau\tau})^4.\end{aligned}$$

So without knowing the exact value of $R_\tau^{\tau\tau}$ and $R_I^{\tau\tau}$, we can give the relation $(R_\tau^{\tau\tau})^4 = (R_I^{\tau\tau})^2$ directly from the locality constraints.

(e) Without knowing the exact value of $R_I^{\tau\tau}, R_\tau^{\tau\tau}$, we can just write

$$\hat{\sigma}_1 = \begin{pmatrix} R_I^{\tau\tau} & \\ & R_\tau^{\tau\tau} \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} \phi^{-2}R_I^{\tau\tau} + \phi^{-1}R_\tau^{\tau\tau} & \phi^{-3/2}(R_I^{\tau\tau} - R_\tau^{\tau\tau}) \\ \phi^{-3/2}(R_I^{\tau\tau} - R_\tau^{\tau\tau}) & \phi^{-1}R_I^{\tau\tau} + \phi^{-2}R_\tau^{\tau\tau} \end{pmatrix}.$$

We further know that with $f = \tau, c = I, \tau$

$$\hat{b}_3 = \begin{pmatrix} (R_I^{\tau\tau})^2 & \\ & (R_\tau^{\tau\tau})^2 \end{pmatrix} \stackrel{!}{=} \hat{\sigma}_2\hat{\sigma}_1^2\hat{\sigma}_2.$$

This gives three equations:

$$\begin{aligned}
(R_I^{\tau\tau})^2 &= \frac{(R_I^{\tau\tau})^2(\phi R_\tau^{\tau\tau} + R_I^{\tau\tau})^2 + \phi (R_\tau^{\tau\tau})^2 (R_I^{\tau\tau} - R_\tau^{\tau\tau})^2}{\phi^4}, \\
0 &= \frac{(R_\tau^{\tau\tau} - R_I^{\tau\tau})(\phi (R_I^{\tau\tau})^2 R_\tau^{\tau\tau} + \phi R_I^{\tau\tau} (R_\tau^{\tau\tau})^2 + (R_I^{\tau\tau})^3 + (R_\tau^{\tau\tau})^3)}{\phi^{7/2}}, \\
(R_\tau^{\tau\tau})^2 &= \frac{\phi (R_I^{\tau\tau})^2 (R_I^{\tau\tau} - R_\tau^{\tau\tau})^2 + (R_\tau^{\tau\tau})^2 (\phi R_I^{\tau\tau} + R_\tau^{\tau\tau})^2}{\phi^4}.
\end{aligned}$$

(f) The second equation can be reduced to:

$$(R_I^{\tau\tau} - R_\tau^{\tau\tau})(R_I^{\tau\tau} + R_\tau^{\tau\tau})((R_I^{\tau\tau})^2 + \phi^{-1} R_I^{\tau\tau} R_\tau^{\tau\tau} + (R_\tau^{\tau\tau})^2) = 0.$$

We have solutions

$$\frac{R_I^{\tau\tau}}{R_\tau^{\tau\tau}} = \pm 1, \mathrm{e}^{\pm 7\pi \mathrm{i}/5}.$$

With $(R_\tau^{\tau\tau})^4 = (R_I^{\tau\tau})^2$, we have eight solutions

$$(R_I^{\tau\tau}, R_\tau^{\tau\tau}) = (\pm 1, \pm 1), (\pm 1, \mp 1), (\mathrm{e}^{\pm \mathrm{i}\pi/5}, \mathrm{e}^{\mp 2\pi \mathrm{i}/5}), (\mathrm{e}^{\mp 4\pi \mathrm{i}/5}, \mathrm{e}^{\pm 3\pi \mathrm{i}/5}).$$

We can plug these solutions into other two equation to examine these solutions are correct. The last two solutions are physical, which is the left and right handed Fibonacci anyons.

Chapter 11

Computing with Anyons

11.1 Ising Nonuniversality

The braiding matrices for Ising anyons are given by Eqs. 10.10 and 10.11. Demonstrate that any multiplication of these matrices, and their inverses will only produce a finite number of possible results. Thus conclude that Ising anyons are not universal for quantum computation. Hint: write the braiding matrices as $e^{i\alpha}U_i$ where U_i is unitary with unit determinant, i.e., is an element of $SU(2)$. Then note that any $SU(2)$ matrix can be thought of as a rotation $\exp(i\hat{n}\cdot\boldsymbol{\sigma}\theta/2)$ where here θ is an angle of rotation $\hat{n}$ is the axis of rotation and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices.

Answer We are given that

$$\hat{\sigma}_1 = e^{-i\pi/8}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} = e^{i\pi/8}\begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix} \equiv e^{i\pi/8}U_1,$$
$$\hat{\sigma}_2 = \frac{e^{i\pi/8}}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix} = e^{i\pi/8}\begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ -i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \equiv e^{i\pi/8}U_2.$$

Here $\det U_1 = \det U_2 = 1$. Now we can write U_1,U_2 in the form of generators of $SU(2)$:

$$U_1 = \begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix} = \exp(-i\pi\sigma_z/4),$$

and

$$U_2 = \begin{pmatrix}\cos(-\pi/4) & i\sin(-\pi/4)\\ i\sin(-\pi/4) & \cos(-\pi/4)\end{pmatrix} = \exp(-i\pi\sigma_x/4).$$

Now we try to prove the group generated by U_1, U_2 is finite. We can see

$$U_1U_2U_1^{-1} = U_3 \equiv \exp(-i\pi\sigma_y/4),$$

This is to say:

$$U_1U_2 = U_3U_1.$$

Now any element generated by U_1, U_2 can be written in the form

$$U_2^mU_3^nU_1^k,$$

where $0 \le m, n, k \le 8$. For example:

$$U_1U_2U_1 = U_3U_1^2, U_2^2U_1U_2 = U_2^2U_3U_1.$$

By listing all its elements, we can find it is a finite group of order 48. It has a normal subgroup[GAP22]

$$\left\langle \frac{1}{2}\begin{pmatrix}-1+i & -1-i\\ 1-i & -1-i\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\rangle \cong SL(2,3),$$

and

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong \mathbb{Q}_8,$$

and

$$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong \mathbb{Z}_2.$$

Actually, the group generated by $\langle U_1, U_2 \rangle$ is a generalization of the famous **Von Dyck group**[Sto20]. The general Von Dyck group is defined by the presentation:

$$D(l, m, n) = \langle x, y | x^l = y^m = (xy)^n = 1 \rangle.$$

This kind of group is **finite if and only if**

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1.$$

If we try to construct:

$$x \equiv U_1 U_2 U_1, \quad y \equiv U_1 U_2,$$

then we will find easily:

$$x^2 = y^3 = (xy)^4 = -\mathbb{1}_2.$$

The case $D(2,3,4)$ is just S_4, and our group $G \equiv \langle U_1, U_2 \rangle$ is actually the group extension of $D(2,3,4)$. We can construct the short exact sequence

$$\mathbb{1} \to \mathbb{Z}_2 \to G \to S_4 \to \mathbb{1},$$

which is a non-split exact sequence. As the presentation above shows, it can be considered as “double copy” of $D(2,3,4) \cong S_4$.

11.2 Brute Force Search

Given the braid matrices for Fibonacci anyons in Eq. 10.6 and 10.7, write a computer program for brute-force searching braids up to length 10.

Ignoring the noncomputational state $|N\rangle$, and ignoring the overall phase as usual, determine the closest approximation to the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Partial Answer: A braid of length 10 exists with phase-invariant distance to target dist ≈ 0.084

Answer After brute force search, the closest braiding is given by

$$U_1 = \hat{\sigma}_2^{-1}\hat{\sigma}_1^{-1}\hat{\sigma}_2^{-2}\hat{\sigma}_1^{-1}\hat{\sigma}_2^{-2}\hat{\sigma}_1^{-1}\hat{\sigma}_2^{-1} = \begin{pmatrix} 0.190983 + 0.587785\mathrm{i} & 0.242934 + 0.747674\mathrm{i} \\ 0.242934 + 0.747674\mathrm{i} & -0.190983 - 0.587785\mathrm{i} \end{pmatrix},$$

$$U_2 = \hat{\sigma}_2^{-1}\hat{\sigma}_2^{-3}\hat{\sigma}_1^{-1}\hat{\sigma}_2^{-1}\hat{\sigma}_1^{3} = \begin{pmatrix} 0.190983 + 0.587785\mathrm{i} & 0.242934 + 0.747674\mathrm{i} \\ 0.242934 + 0.747674\mathrm{i} & -0.190983 - 0.587785\mathrm{i} \end{pmatrix}.$$

Their distances are given by:

$$\mathrm{dist}(H, U_1) = \mathrm{dist}(H, U_2) \approx 0.08421.$$

The mathematica code is given in the attached file.

11.3 Scaling of Kitaev-Solovay Algorithm

Given the discussion just above Eq. 11.8, prove Eqs. 11.8 and 11.9.

Answer Suppose we want to get a target gate $U^{(0)}_{\text{target}}$. Using brute-force search, we can get $U^{(0)}_{\text{approx}}$ in t_0 steps within the distance ϵ_0. Then we define

$$U^{(1)}_{\text{target}} \equiv [U^{(0)}_{\text{approx}}]^{-1} U^{(0)}_{\text{target}},$$

which close to identity within distance ϵ_0. With decomposition

$$U^{(1)}_{\text{target}} = VWV^{-1}W^{-1},$$

and W and V being unitary operations close to the identity within $\epsilon_0^{1/2}$, we can approximate $U^{(0)}_{\text{target}}$ with accuracy $\epsilon_0^{3/2}$. Now the total length is $5t_0$, and by n iteration, the accuracy is

$$\epsilon = \epsilon_0^{(2/3)^n},$$

with length

$$t = 5^n t_0.$$

So for large n:

$$\ln[\log(1/\epsilon)]/\ln(3/2) \sim \ln(t)/\ln(5),$$

which means

$$t \sim \mathcal{O}\left(\log(1/\epsilon)^{\ln(5)/\ln(3/2)}\right).$$

For classical computation, the time consumed by n iteration is

$$T = 3^n T_0,$$

which means

$$T \sim \mathcal{O}\left(\log(1/\epsilon)^{\ln(3)/\ln(3/2)}\right).$$

11.4 About the Injection Weave

One might wonder why we choose to work with an injection weave in Fig. 11.11 which moves the red strand from the far right at the bottom all the way to the far left on the top. Show that for three Fibonacci anyons, there does not exist any injection weave that moves the (red) strand from the far right on the bottom to the middle on the top, even up to an overall phase. I.e., show that no weave exists starting on the bottom far left ending in the middle on the top whose effect on the three dimensional Hilbert space is $\mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_{3\times 3}$ for any phase ϕ.

Answer Suppose such operation exists, i.e. exits any injection weave that moves the red strand from the far right on the bottom to the middle on the top, which equals to $U = \mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_3$. Assume $\hat{\sigma}_1$ appears m times in the $U(\hat{\sigma}_1^{-1}$ contributes -1 to $m)$ and $\hat{\sigma}_2$ appears n times in the sequence. Note that for $|N\rangle$, the braiding matrix is block-diagonal, which means

$$\exp(3(m+n)\pi\mathrm{i}/5) = \exp(\mathrm{i}\phi),$$

i.e.

$$\phi = 3(m+n)\pi/5 + 2k\pi.$$

We can take det of eq $U = \mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_3$, using $\det\hat{\hat{\sigma}}_1 = \det\hat{\hat{\sigma}}_2 = \mathrm{e}^{2\pi\mathrm{i}/5}$, we will have

$$\exp(2(m+n)\pi\mathrm{i}/5) = \exp(3\mathrm{i}\phi),$$

i.e.

$$3\phi = 2(m+n)\pi\mathrm{i}/5 + 2k'\pi.$$

Combine two equations:

$$m+n = \frac{10}{7}(k' - 3k).$$

However, note that if $m+n$ is an integer, it must be even according to this formula, but our operation is an odd permutation, which means $m+n$ should be odd, contradicts. So such injection weave doesn't exist.

11.5 Universal Weaving and the Injection Weave

Consider injection weaves as described in Fig. 11.11. Let us assume that we can construct an injection weave of arbitrary precision. Given such an (approximately) perfect injection weave show that for any number of anyons $N > 3$, a weave can be constructed that performs the same unitary operation on the Hilbert space as any given braid. A more general mathematical proof of the universality of weaving is also given in Simon et al. [2006].

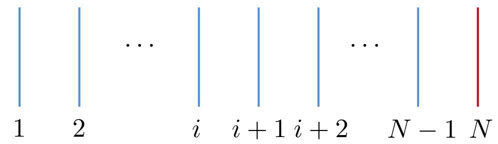


Figure 11.1: N anyons with number 1 to N.

Answer Suppose there are N anyons, as the Fig. 11.1 shows.

Any braiding can be generated by $\hat{\sigma}_i$, $1 \leq i \leq N-1$. Suppose we want to perform $\hat{\sigma}_i$, we can view strand $i+2$ to $N-1$ as a single strand c according to locality. Then we perform the injection weave between $i+1, c$ and N, as the Fig.11.2 shows. After that, we do $\hat{\sigma}_i$ between i and the red line. Finally, we view strand $i+2$(blue) to N(green) as a single strand c' and perform the inverse injection. This is a weave and has the effect of $\hat{\sigma}_i$ purely.

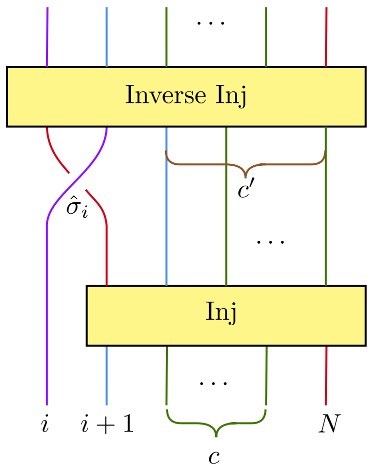


Figure 11.2: The method to realize $\hat{\sigma}_i$, where $1 \leq i < N-1$.

For $i < N-1$, this scheme applies to every i, however, when $i = N-1$, things gets different, we should use the scheme as the Fig.11.3 shows.

Now that we can realize every $\hat{\sigma}_i$, we have proven that a weave can be constructed that performs the same unitary operation on the Hilbert space as any given braid.

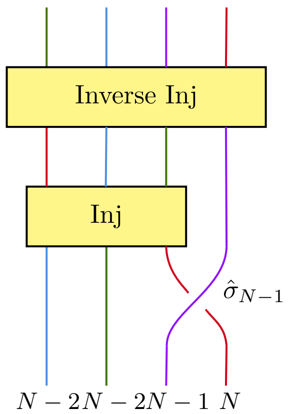


Figure 11.3: The method to realize $\hat{\sigma}_{N-1}$.

Part III

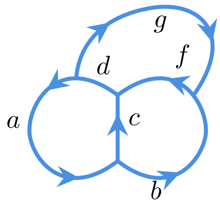
Anyon Diagrammatics (in detail)

Chapter 12

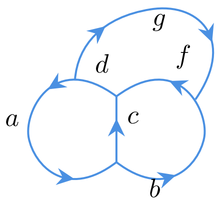
Planar Diagrams

12.1 Evaluating diagrams with F-matrices

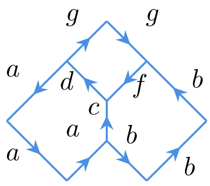
Evaluate the following diagram, writing the result in terms of F 's.



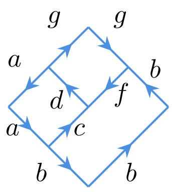
Answer Direct computation gives:



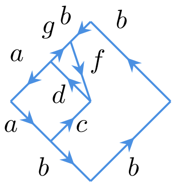
$=$



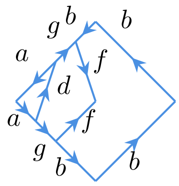
$= [F^{a\bar{a}\bar{b}}_{\bar{b}}]_{Ic}$



$= [F^{a\bar{a}\bar{b}}_{\bar{b}}]_{Ic}[F^{\bar{b}f\bar{g}}_{I}]_{gb}$



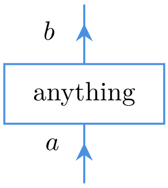
$= [F^{a\bar{a}\bar{b}}_{\bar{b}}]_{Ic}[F^{\bar{b}f\bar{g}}_{I}]_{gb}[F^{\bar{a}d\bar{f}}_{\bar{b}}]^{*}_{cg}$



$= [F^{a\bar{a}\bar{b}}_{\bar{b}}]_{Ic}[F^{\bar{b}f\bar{g}}_{I}]_{gb}[F^{\bar{a}d\bar{f}}_{\bar{b}}]^{*}_{cg}.$

12.2 Locality Principle

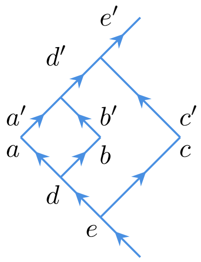
Show that the locality principle (Fig.12.1) is derivable from our other rules for evaluating diagrams, and is not therefore an independent assumption.



$= 0$ unless $a = b$

Figure 12.1: The locality, or no-transmutation, principle as in Fig. 8.7.

Answer Proof scheme: change every diagram into the standard basis, and use the orthogonality of the tree, such as the Eq. (12.1) below:



$$= \delta_{aa'}\delta_{bb'}\delta_{cc'}\delta_{dd'}\delta_{ee'} \tag{12.1}$$

Firstly, we notice that the planar diagram must have the property that each vertex can only be linked to three edges at most, because we don't allow three anyons fusion at the same time, such as $a \times b \times c = d$. After changing it into standard basis, we can use the orthogonality relation to eliminate the bubbles. The algorithm of changing into standard basis is as follows:

(a) Start from the incoming anyon.

(b) Travel along the right line at the bottom

(c) For every following vertex, choose the right path, and use the basis changing to eliminate the left path

(d) Reach the outcoming anyon finally

(e) Travel along the left path from the last crossing which we haven't chosen the left path

(f) Repeat (c) to (e) for every vertex until every vertex in the diagram is traversed.

We will look at several examples. Firstly, if $\boxed{\text{anything}}$ is planar diagram K_4 or butterfly graph as the Fig.12.2 below shows, this is not allowed, because every vertex in K_4 are linked to three edges, and there is four-valent vertex in the butterfly graph.

Then we take two examples, as the Fig. 12.3 below shows.

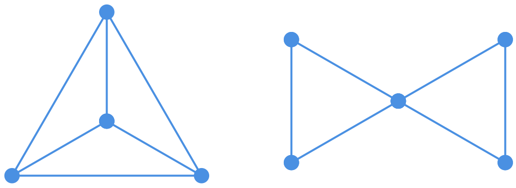


Figure 12.2: Planar diagram K_4 and the butterfly graph.

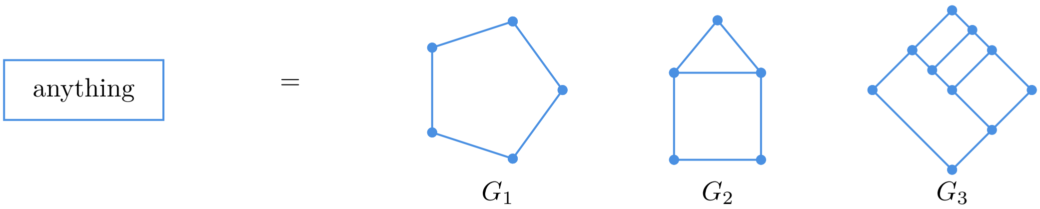
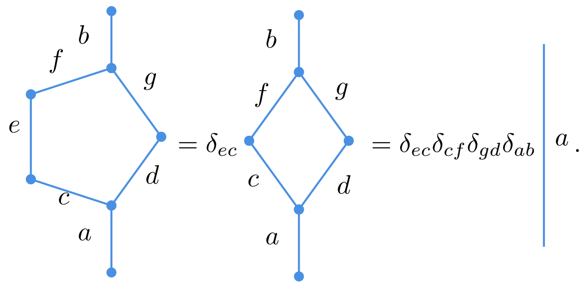
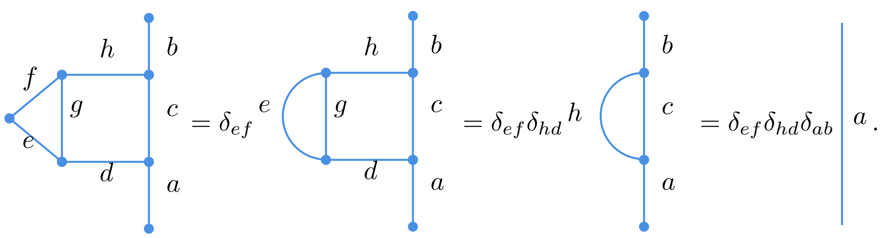


Figure 12.3: Three examples of the algorithm.

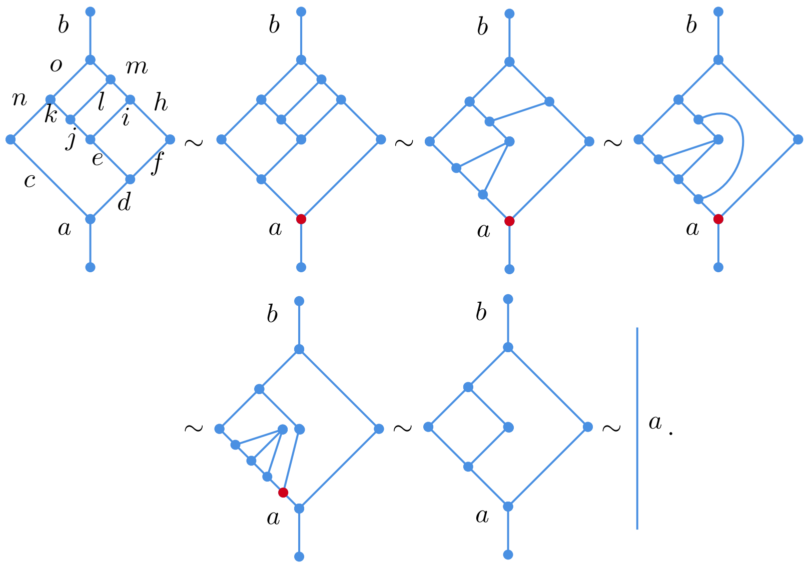
For graph G_1:



For graph G_2:



For graph G_3:



Of course, there are more efficient route, here we just use the systematic way to demonstrate our algorithm is universal.

Chapter 13

Braiding Diagrams

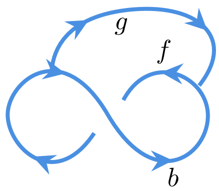
13.1 Fibonacci Hexagon

Once F-matrices are defined for a TQFT, consistency of the R-matrix is enforced by the so-called hexagon equations as shown in the figure diagramatically by Fig. 13.11. or the Fibonacci anyon theory, once the F-matrix is fixed as in Eq. 9.3, the R-matrices are defined up to complex conjugation (i.e., there is a right and left handed Fibonacci anyon theory - both are consistent). Derive these R-matrices. Confirm Eqs. 10.2 as one of the two solutions and show no other solutions exist.

Answer

13.2 Evaluation of a Diagram

Consider the following diagram: Evaluate this diagram in terms of R's and F's. Hint: First



reduce the diagram to that shown in exercise 12.1.

Answer

13.3 Gauge transform of R and Hexagon

(a) Confirm the gauge transform Eq. 13.3.

(b) Show that a set of F-matrices and R-matrices satisfying the hexagon equations, Eq. 13.1 and 13.2 remains a solution after a gauge transformation. Remember that both R - and F-transform.

Answer

13.4 Reidermeister Moves

(a) Use the R-matrix, and the completeness relationship, to derive the equivalence shown on the left of Fig. 13.9.

(b) How does the hexagon equation imply the equivalence shown in Fig.13.1. Hint: This is very subtle, but is almost trivial.

(c) Use Fig.13.1 to show the equality on the right of Fig. 13.9.

(d) Use the result of Fig. 13.14 along with completeness and the R-matrix to demonstrate Fig. 13.10.

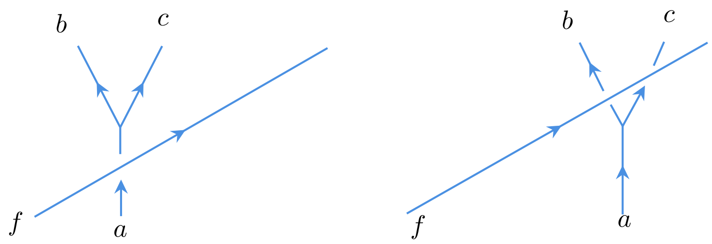


Figure 13.1: This move is implied by the hexagon equation. (Similar with the straight line f going under the other two, and similar if the left-to-right slope of f is negative instead of positive.).

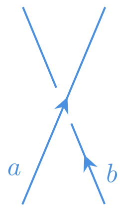
This exercise shows that equalities like those shown in Fig. 13.9 and 13.10 are not independent assumptions but can be derived from the planar algebra and the definition of an R-matrix satisfying the hexagon.

Chapter 14

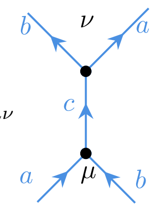
Seeking Isotopy

14.1 Higher Fusion Multiplicities

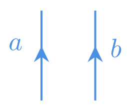
Derive Eq.(14.1).



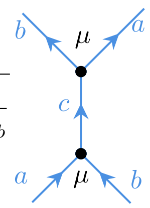
$$= \sum_{c,\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} [R^{ab}_c]_{\mu\nu} \tag{14.1}$$

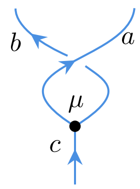


Answer According to

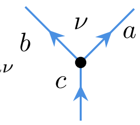


$$= \sum_{c,\mu} \sqrt{\frac{d_c}{d_a d_b}}$$





$$= \sum_{\nu} [R^{ab}_c]_{\mu\nu}$$



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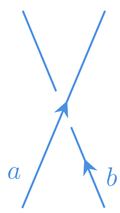
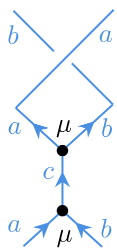
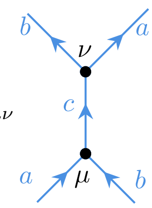
we can see



$$= \sum_{c,\mu} \sqrt{\frac{d_c}{d_a d_b}}$$



$$= \sum_{c,\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} [R^{ab}_c]_{\mu\nu}$$



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Chapter 15

Twists

15.1 Fibonacci Twists

For Fibonacci anyons, the twist factor is $\theta_\tau = \mathrm{e}^{\pm 4\pi \mathrm{i}/5}$ (With \pm being for right or left-handed theories respectively). Check that these twist factors agree with the R-matrices (Eq. 10.2) and Eq. 15.2 and Eq. 15.3.

Answer

15.2 Using Geometric Moves I

(a) Using the allowed moves in Fig. 13.9, show the equivalence of the left and right of Fig. 15.4.

(b) Similarly, show the equivalence of the left and right of Fig. 15.5.

(c) Similarly show the equivalence of the middle two figures in Fig. 15.8.

Answer

15.3 Using Geometric Moves II

Demonstrate the middle step of Fig. 15.9 by using allowed geometric moves such as Fig. 13.9 and Fig. 13.10 and Fig. 13.14. You may also need the pivotal identity Fig. 14.27.

Answer

15.4 Gauge Independence of Ribbon Identity

Show that the ribbon identity Eq. 15.4 is gauge independent.

Answer

15.5 Higher Fusion Multiplicities

Derive Eq. 15.5 and Eq. 15.6.

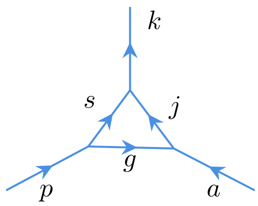
Answer

Chapter 16

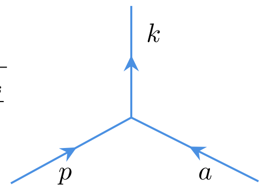
Theories with Tetrahedral Symmetry (or Full Isotopy)

16.1 Triangle Bubble Collapse

A useful lemma is the collapsing of a triangular bubble

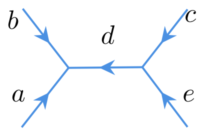


$$= F^{a\bar{j}g}_{\bar{s}p\bar{k}}\sqrt{\frac{d_j d_s}{d_k}}$$

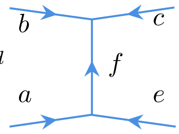


Derive this lemma.

Answer According to the definition of the F matrix for fully isotopy invariant theories:



$$= \sum_f F^{bad}_{ecf}$$



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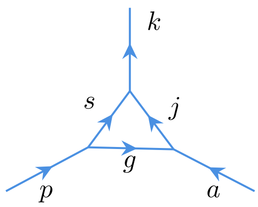
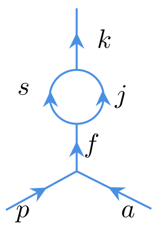
We can see



$$= \sum_f F^{a\bar{j}g}_{\bar{s}p\bar{f}}$$



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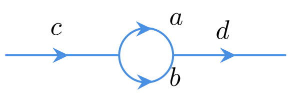
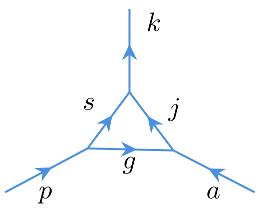
Then according to the contraction rule of a bubble for fully isotopy invariant theories:



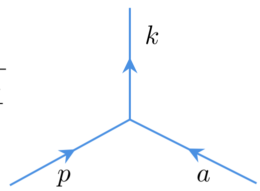
$$= \delta_{cd}\sqrt{\frac{d_a d_b}{d_c}}$$

,

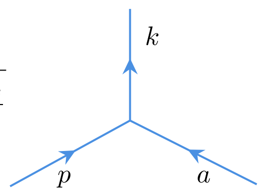
we can see



$$= \sum_f F^{a\bar{j}g}_{\bar{s}p\bar{f}}\delta_{fk}\sqrt{\frac{d_j d_s}{d_k}}$$



$$= F^{a\bar{j}g}_{\bar{s}p\bar{k}}\delta_{fk}\sqrt{\frac{d_j d_s}{d_k}}$$



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Chapter 17

Further Structure

17.1 Fibonacci S-matrix

Recall the Fibonacci theory which we introduced in sections 8.2.1 and 10.2.1.

(a) First let us pretend that we have not calculated the R-matrices or θ_τ, i.e., we do not know the braiding phases or the twist factors. We only know the fusion rules $\tau \times \tau = I + \tau$. Using the quantum dimensions, we can obtain three out of four elements of the 2 by 2S-matrix. Determine the remaining elment of the S-matrix by enforcing unitarity.

(b) Given the twist factor $\theta_\tau = \mathrm{e}^{\pm 4\pi \mathrm{i}/5}$ (With $\pm$ being for right or lefthanded theories), calculate the S-matrix explicitly by using Eq. 17.20.

Answer

17.2 Using the pivotal property

Use the pivotal property (Section 14.8.1) to demonstrate the identity shown in Fig. 17.1.

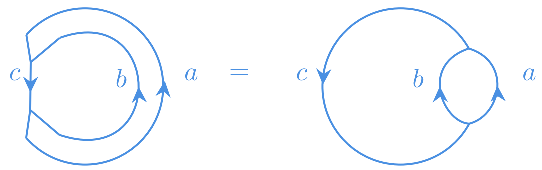


Figure 17.1: This identity can be shown without full isotopy invariance by using the pivotal property.

You should not assume full isotopy invariance. Nor should you assume $\epsilon = +1$ for any of the particles.

Answer

17.3 Symmetries of S

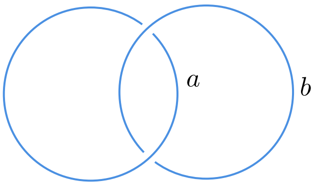
Use isotopy of diagrams and Hermitian conjugation of diagrams to show the identities in Eq. 17.7.

Answer

17.4 Evaluation of the S-link

(a) Use the R-matrices and Eq. 15.4 to derive the value of the matrix $\tilde{S}_{ax}$ (See Fig. 17.2) in terms of fusion multiplicities, twist factors θ_a, and the quantum dimensions d_a.

(b) From your result show that



$$= \sum_c N^c_{a\bar{b}} \frac{\theta_c}{\theta_a \theta_b} \mathsf{d}_c$$

Note that this diagram differs from S_{ab} by a factor of $Z(S^3) = 1/\mathcal{D}$.

Answer

17.5 Theories with one nontrivial particle

Consider an anyon theory with only the identity and one nontrivial particle type a having twist factor θ_s. The only possible fusion rules are $s \times s = I + ms$ for some integer m (the semion model is $m = 0$ the Fibonacci model is $m = 1$). Calculate d_s and $\mathcal{D}$ from the fusion rules. Use Eq. 17.20 to calculate the S matrix in terms of θ_s. Show that this matrix cannot be unitary for any $m > 1$. This justifies that on table 17.1 there are only two types of theory with one nontrivial particle.

Answer

17.6 Product theories[Easy]

Given two anyon theories A and B with corresponding S-matrices S_A and S_B

(a) Show that the product theory $A \times B$ has S-matrix $S_A \otimes S_B$.

(b) Show that $A \times B$ is modular if and only if both A and B are modular.

(c) Show that the central charge of the product theory is the sum of the central charges of the constituent theories. I.e.,

$$c_{A \times B} = (c_A + c_B) \text{ mod } 8$$

In fact, central charges strictly add in product theories. However, we have only defined the central charge mod 8 so far!

Answer

17.7 Probability of Fusion Channels

Consider a modular anyon theory on a sphere with a very large number of quasiparticles of all types.

(a) Divide these anyons randomly into two large groups. Show that the probability that the two groups have overall fusion channels a and $\overline{a}$ is given by

$$p(a, \bar{a}) = \mathsf{d}_a^2 / \mathcal{D}^2$$

Hint: You are counting the total number of fusion trees. Use the strategy of section 8.3 along with the Verlinde formula, and the knowledge that $S_{0b}/S_{00} \geq |S_{xb}/S_{xb}|$ for all x. (You may start by assuming this is strictly $>$ and worry about the $\geq$ case later).

(b) Instead divide the anyons randomly into three large groups. Show that the probability that the groups have overall fusion channels a, b, c is given by

$$p(a, b, c) = N_{abc} \mathsf{d}_a \mathsf{d}_b \mathsf{d}_c / \mathcal{D}^4$$

(c) Finally try four large groups. Show that the probability that the groups have overall fusion channels a, b, c, f is given by

$$p(a, b, c, f) = N_{abcf} \mathsf{d}_a \mathsf{d}_b \mathsf{d}_c \mathsf{d}_f / \mathcal{D}^6$$

where N_{abcf} is the number of ways a, b, c and f can fuse to the vacuum.

Answer

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