MAT1830 Assignment 1

Question 1

Give a truth table for $\neg a \vee (a \implies \neg b) \leftrightarrow c$. State whether the sentence is a tautology, a contradiction, or neither and briefly explain why.

a	b	c	$\neg a$	$\neg b$	$a \implies \neg b$	$\neg a \vee (a \implies \neg b)$	$\neg a \vee (a \implies \neg b) \leftrightarrow c$
F	F	F	T	T	T	T	F
F	F	T	T	T	T	T	T
F	T	F	T	F	T	T	F
F	T	T	T	F	T	T	T
T	F	F	F	T	T	T	F
T	F	T	F	T	T	T	T
T	T	F	F	F	F	F	T
T	T	T	F	F	F	F	F

Table 1. truth table for the sentence $\neg a \vee (a \implies \neg b) \leftrightarrow c$

Tautologies are sentences which are true regardless of the combination of values for true and false that make up its variables.
Contradictions are sentences which are false regardless of the combination of truth values for their variables.
The sentence $\neg a \vee (a \implies \neg b) \leftrightarrow c$ is neither a tautology nor a contradiction as it has a different truth value dependent on the truth value of the variables a, b, and c.

Question 2

Use laws of logic to show that $(p \wedge \neg q) \wedge \big(q \wedge (\neg q \implies r)\big)$ is a contradiction. Explain each step fully.

By implication	$\neg q \implies r \equiv \big(\neg(\neg q)\big) \vee r$
By double negation	$\big(\neg(\neg q)\big) \vee r \equiv q \vee r$
Substituting back in	$(p \wedge \neg q) \wedge \big(q \wedge (\neg q \implies r)\big) \equiv (p \vee \neg q) \wedge \big(q \wedge (q \vee r)\big)$
By absorption	$(p \wedge \neg q) \wedge \big(q \wedge (q \vee r)\big) \equiv (p \wedge \neg q) \wedge q$
By association	$(p \wedge \neg q) \wedge q \equiv p \wedge (\neg q \wedge q)$
By inverse	$p \wedge (\neg q \wedge q) \equiv p \wedge F$
By annhilation	$p \wedge F \equiv F$

Table 2. truth table showing that $(p \wedge \neg q) \wedge \big(q \wedge (\neg q \implies r)\big)$ is a contradiction.

As the statement is always false it is therefore a contradiction.

Question 3

Consider the statement "If $gcd(x, 20) = 1$then $gcd(x, 2) = 1$ or $gcd(x, 5) = 1$." In english (not using logical symbols), write down the statement's contrapositive and then write down the statement's negation. In the negation, you may not use "if", "then", "implies" or similar words.

Contrapositive	If $gcd(x, 2) \neq 1$ and $gcd(x, 5) \neq 1$ then $gcd(x, 20) \neq 1$
Negation	When a number's greatest common divisor with 20 is 1 it will have a greatest common divisor that is not 1 with 2 or 5.

Table 3. showing contrapositive and negation of If $gcd(x, 20) = 1$ then $gcd(x, 2) = 1$ or $gcd(x, 5) = 1$

Question 4

Two cats, Felix and Sylvester, are fed every morning. On any given evening, each cat gets a medicated meal if it is sick and a normal meal otherwise. On weekends, both cats get treats as well as their meals. On weekdays, each cat gets treats as well as a meal if it has been well behaved, but otherwise does not get treats.

Let p be the proposition that Felix is sick.
Let q be the proposition that Sylvester is sick.
Let r be the proposition that Felix has been well behaved.
Let s be the proposition that Sylvester has been well behaved.
Let t be the proposition that it is a weekend.

Write down the propositions (using just p, q, r, s, t, brackets and logical connectives) that are logically equivalent to the following statements.

(i) Felix gets treats.
(ii) Sylvester gets a normal meal without treats.
(iii) Exactly one of the two cats gets treats.

(i) $r \vee t$
(ii) $\neg t \wedge \neg s \wedge \neg q$
(iii) $\neg t \wedge \neg(r \leftrightarrow s)$

Question 5

Let n be an integer such that $n \geq 3$. Suppose that a truth table is drawn for the logical sentence $(p_2 \vee p_3 \vee \ldots \vee p_n) \implies p_1$.

(i) If $n = 3$, then in how many rows would the sentence evaluate to true?
(ii) In general, how many rows would the sentence evaluate to true? (Give your answer as a function of n.)

(i) 5
(ii) Whenever p_1 is true the sentence evaluates to true, this is definitionally half the number of rows on the truth table. When p_1 is false none of the propositions on the left side of the implication arrow may be true, it is therefore the case that $T_n = 2^{n-1} + 1$

By implication	$(p_2 \vee p_3 \vee \ldots \vee p_n) \implies p_1$	$\equiv \neg(p_2 \vee p_3 \vee \ldots \vee p_n) \vee p_1$
By De Morgan's law		$\equiv (\neg p_2 \wedge \neg p_3 \wedge \ldots \wedge \neg p_n) \vee p_1$
Through distribution		$\equiv (\neg p_2 \wedge \neg p_3 \wedge \ldots \wedge \neg p_n) \vee p_1$

Table 3. $(p_2 \vee p_3 \vee \ldots \vee p_n) \implies p_1$ transformed for general solution to number of truthful rows in a truth table