Slides: https://www.irif.fr/~kesner/enseignement/mpri/ll/Proof-Nets.pdf

Why Linear Logic?

- **Purpose**: control duplication and erasure
- **Insights**: proof-nets, cbn vs cbv as logic, implicit complexity & cost models
- **Revisit**: evaluation & typing, new calculus

MLL (two-sided) sequent

$$\frac{}{A \vdash A}\,\text{Ax} \qquad \frac{\Gamma \vdash A, \Delta \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

MLL (unilateral) sequent

If we want to move lhs of $\vdash$ to rhs, we negate formulas on lhs and move them to rhs. In this process, there would be no need to have separate left and right rules.

$$\frac{}{\vdash A^{\perp}, A}\,\text{Ax} \qquad \text{TODO Cut}$$

Proof-Nets

Problem: From the same premise, there might be **multiple proofs (derivations)** to the same conclusion. Any possible derivation captures a particular constructor **history**.

Solution: **MLL Proof-Net (PN)** with conclusions $\overline{A}$, a graph defined by induction as:

TODO

How to read these graphs:

1. White squares are whatever that expose the interface shown below it.
2. Disjoint white squares are independent (parallel) proof-nets.

Terminology

Interface: leaf formulas.

1. If you just keep to the interface, you get the current proof status.
2. Interfaces can be connected and they are no longer interfaces while a new node would become an interface.
3. All proofs of the same formula can be represented as a single proof-net. The very reason is that PNs can *parallelize* different parts of the proof.
   TODO so only for reasoning systems that have *subformula property*?)

Pre Proof-Nets

Pre proof-nets are a weaker version of proof-nets that ignores all independence requirements.

E.g.

is a pre proof-net but it's not a proof-net because two inputs of the cut rule are not independent.

Correctness Criteria

But what makes a pre proof-net a proof-net?

Acyclic Connected Criteria

P^- is a Pre PN, but mark angles with dependent precedences with a red arc.

Contractibility

1. Merge two edges with a red arc if the precedence is a single common node.
2. Merge an edge with its two connected nodes into one node if the edge is the only edge connected to these two nodes.

Theorem: A pre proof-net is a proof-net iff its P^- is contractible (reduces to a single node).

Intuition: if it can collapse a single node then it's a single sequent. The only thing that stops it from collapsing is the red arc, i.e. `par`.

MELL

MLL + weakening and contraction.

If one want to weaken/contract a variable, one must mark it explicitly with a !. To allow a variable to be weakened, one can introduce a !.

NOTE **Duality** a ! in the left is a ? in the right.

Promotion rules are interesting, while when applying promotion, one allows a formula to

Introduce weakening and dereliction constructs.

Note that a graph with bare weakening is a pre PN but it's not a PN because you need something to weaken first.

NOTE There's a typo on a MLTT reduction example where two derivations don't have the same conclusion, but the PN is correct.