Introduction to Category Theory

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Contents

1 Category Theory

Q: What is Category Theory?

A:
1. Understanding math objects via relations with each other, i.e. taking an external view, so you do not inspect the internal structure of the objects.
2. A whole independent field of study.

Q: What is a category?

A: objects + morphisms

Example 1A.

Category	Objects	Morphisms
Set	sets	functions
Group	groups	group homomorphisms
Top	topological spaces	continuous functions
Prop	propositions	derivation/implication
Type	types	derivation/function
Type Theory	type theories	translations
Prog Spec	program spece	programs turning any program of one spec into a program of another spec

Counterexample: what is the category of **probablistics**?

What if two kinds of notions of morphisms are all useful? **Double category**!

Example 1B.
- **Set** - *objects*: sets; *morphisms*: functions
- **Set** - *objects*: sets; *morphisms*: relations

1.1 Definition

Definition 1.1A *(Category).* A **category** C is
- a collection of objects $\mathrm{ob}\, C$
- for every $X, Y \in \mathrm{ob}\, C$, a collection of morphisms $\hom_C(X, Y)$
- **Id**: for each $X \in \mathrm{ob}\, C$, an id morphism $\mathrm{id}_X \in \hom_C(X, X)$
- **Comp**: for each $f : X \to Y, g : Y \to Z$, a morphism $g \circ f : X \to Z$ in $\hom_C(X, Z)$, *such that*
  - $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$

  Remark 1.1A. Normally we don't differentiate left and right identity. There was one particular notion that did, but then they were shown to be equivalent.
- for any $f : X \to Y, g : Y \to Z, h : Z \to A$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Example 1.1A *(Prop Category).*
- ob Prop - the collection of propositions
- $$\hom_{\mathrm{Prop}(P,Q)} = \begin{cases} \{\top\} & \text{if } P \to Q \\ \emptyset & \text{otherwise} \end{cases}$$
- **Id**: $\mathrm{id}_P = \top$ for all $P \in \mathrm{ob}$ Prop, because $P \to P$ is always true.
- **Comp**: by modus ponens, $Q \to P$ and $P \to R$ implies $Q \to R$. Properties: trivial.

TODO this is cat but i don't see how it is useful. all morphisms are trivial.. maybe it serves as a gentle introduction to the concept of category that does not serve any real-world purpose?

1.2 Commutative Diagrams

Categories often involve lots of morphisms and relations between them. To simplify the picture, we can use *commutative diagrams* to represent the relations.

Example 1.2A. $f \circ \mathrm{id}_X = f$ can be represented as

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Notice how the two paths from X to Y yield the same morphism.

Remark 1.2A. There are also *string diagrams*, which is very similar to *proof nets*.

2 Basic Concepts

2.1 Terminal Object

Definition 2.1A *(Terminal Object).* A **terminal object** $(*)$ T in a category C is an object such that for every object $Z \in C$, there is a unique morphism $Z \xrightarrow{!} T$.

e.g.
- a singleton set ($\{\top\}$) in *Set*
- unit type (1) in *Ty*
- *A* in *a category that contains exactly one object A.*

2.2 Isomorphism

DEFINITION 2.2A *(Isomorphism).* Two objects X, Y in a category C are **isomorphic** if there exists morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

REMARK 2.2A. If $x \cong y$, we say
1. f is an *isomorphism* that $f : X \cong Y$, $f : X \xrightarrow{\sim} Y$
2. g as f^{-1}, and $f^{-1} : Y \cong X$, $f^{-1} : Y \xrightarrow{\sim} X$

e.g.
- *bijection* in *Set*
- *bi-implication* in *Prop*

LEMMA 2.2A. If $f : X \cong Y$, then for any Z, we have
- $f_* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.
- $f^* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.

Proof.

- $\Rightarrow$

  For any $g \in \hom(Z, X)$, we have $g : Z \to X$. Also we know $X \xrightarrow{f} Y$. So, $f \circ g : Z \xrightarrow{g} X \xrightarrow{f} Y$.

- $\Leftarrow$

  For any $h \in \hom(Z, Y)$, we have $h : Z \to Y$. Also we know $Y \xrightarrow{f^{-1}} X$. So, $f^{-1} \circ h : Z \xrightarrow{h} Y \xrightarrow{f^{-1}} X$. □

LEMMA 2.2B. Terminal objects are *unique up to isomorphism.*

Proof. Let T, T' be two terminal objects in a category C.
Since T is terminal, there exists a unique morphism $f : T' \xrightarrow{!'} T$. Similarly, there exists a unique morphism $g : T \xrightarrow{!} T'$.
Now $! \circ !' : T \xrightarrow{!'} T' \xrightarrow{!} T = \mathrm{id}_T$. Similarly, $!' \circ ! = \mathrm{id}_{T'}$. □

REMARK 2.2B. We know $T \xrightarrow{!'} T' \xrightarrow{!} T$ is id_T because T is terminal, which means for every object Z there is a *unique* morphism $Z \xrightarrow{!} T$. In particular, for $Z = T$, there is a *unique* morphism $T \xrightarrow{!} T$. And we know $\mathrm{id}_T : T \to T$ is one of the morphisms, so that's it.

Generally, there could be multiple morphisms to oneself, but in this case it's the terminal part that makes it unique.

2.3 Duality

DEFINITION 2.3A *(Dual Category).* Given a category C, we can define its **dual category** $C^{\{\mathrm{op}\}}$ by reversing the direction of all morphisms.

$$\mathrm{ob}\ C^{\mathrm{op}} := \mathrm{ob}\ C, \hom_C^{\{\mathrm{op}\}}(X, Y) := \hom_C(Y, X)$$

Lemma 2.3A. $\mathcal{C}^{\mathrm{op}}$ is a category.

Lemma 2.3B. If $X \cong Y$ in $\mathcal{C}$, then $X \cong Y$ in $\mathcal{C}^{\mathrm{op}}$.

2.4 Initial Object

Definition 2.4A *(Initial Object).* An **initial object** I in a category $\mathcal{C}$ is the terminal object in $\mathcal{C}^{\mathrm{op}}$.

or,

Definition 2.4B *(Initial Object, Alternative).* An **initial object** I in a category $\mathcal{C}$ is an object such that for every object $Z \in \mathcal{C}$, there is a unique morphism $I \xrightarrow{!} Z$.

Lemma 2.4A. Initial objects are *unique up to isomorphism.*

Proof. Follows dually from the uniqueness of terminal objects. □

e.g.
- empty set ($\varnothing$) in *Set*
- empty type (0) in *Ty*
- false ($\bot$) in *Prop*

3 Product

Definition 3A *(Product).* A **product** of two objects A, B in a category $\mathcal{C}$ contains
- an *object* $A \times B \in \mathrm{ob}(\mathcal{C})$
- *morphisms* $A \times B \xrightarrow{\pi_A} A$, $A \times B \xrightarrow{\pi_B} B$,

  s.t. for any $Z \in \mathrm{ob}(\mathcal{C})$ and morphisms in the diagram, we have a **unique** morphism $U_Z : Z \xrightarrow{!} A \times B$ such that the diagram commutes.

$$\begin{tikzcd} & Z \arrow[dl, "\eta_A"'] \arrow[d, dashed, "U_z"'] \arrow[dr, "\eta_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

Remark 3A. This **uniqueness** depends on the specific choice of π_A and π_B. One must realize that *product* is not just an object, but also the morphisms π_A, π_B.

Remark 3B. Only *some* categories have products.

Remark 3C. If U_Z does not need to be unique, it's called a *weak product.*

e.g.
- cartesian product in *Set*
- $\land$ in *Prop*

Theorem 3A. The product $A \times B$ $(A \times B, \pi_1, \pi_2)$ of $A, B \in \mathrm{ob}(\mathcal{C})$ is unique up to *unique isomorphism.*

Proof.

1. Recall the very definition of product (an object, and two project morphisms). First we introduce $A \times B$ and $A \times' B$:

$$
\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] & \\
A & & B
\end{tikzcd}
$$

2. Compare the previous diagram with the universal property of product. "Instantiate" the universal property of $A \times' B$ and we have a unique morphism $A \times B \xrightarrow{!f} A \times' B$:

$$
\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, dashed, "!f"] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] & \\
A & & B
\end{tikzcd}
$$

PITFALL 3A.

One might attempt to do the following thing to close the proof:

$$
\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, shift right, "!f"'] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[u, shift right, dashed, "!f'"'] & \\
A & & B
\end{tikzcd}
$$

This is not correct, because for $A \times B$ and $A \times' B$ to be isomorphic one also needs to show

$$!f' \circ !f = \mathrm{id}_{A \times B}$$

which is not depicted in the diagram.

3. For sake of reasoning, let's duplicate $A \times B$ below $A \times' B$ and exercise the universal property of $A \times B$ with "apex" $A \times' B$ to show $A \times' B \xrightarrow{!g} A \times B$:

$$
\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, "!f"] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[d, dashed, "!g"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

4. We can also "instantiate" the universal property of the lower $A \times B$ with apex being the upper $A \times B$, to show there exists a unique morphism $A \times B \to A \times B$:

$$
\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, "!f"] \arrow[dd, dashed, bend left] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[d, "!g"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

5. Note that $A \times B \xrightarrow{!g \circ !f} A \times B$, and also $A \times B \xrightarrow{\mathrm{id}_{A \times B}} A \times B$. By the previous reasoning, there exists only one morphism, so $!g \circ !f = \mathrm{id}_{A \times B}$. Similarly, $!f \circ !g = \mathrm{id}_{A \times' B}$. By uniqueness of $!f$ and $!g$, this isomorphism is *unique*.

PITFALL 3B. This *uniqueness* depends on a specific choice of π_A and π_B. In other words, if we only consider the product objects $A \times B$ and $A \times' B$, their isomorphism might not be unique because we have the freedom to choose different projection morphisms π_A, π_B and π'_A, π'_B, and they form their own unique isomorphism.

□

Definition 3B *(Product Category).* Define new category $C_{A,B}$,

1. **Objects**: diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A & & B \end{tikzcd}$$

   It can also be called a *span* from A to B.
2. **Morphisms**: f shown in the following diagram

$$\begin{tikzcd} & Z \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A & & B \\ & Y \arrow[ul, "y_A"] \arrow[ur, "y_B"'] \arrow[uu, "f"] & \end{tikzcd}$$

Theorem 3B. *Product* of A,B in C is the *terminal object* in $C_{A,B}$.

Definition 3C *(Product, Alternative).* Consider $A, B \in \mathrm{ob}(C)$, the product of A, B is an object $A \times B \in \mathrm{ob}(C)$ with the property that

$$\hom_C(Z, A \times B) \cong \hom_C(Z, A) \times \hom_C(Z, B)$$

Theorem 3C. This formation of product is equivalent to the previous two.

TODO This reminds me of *natural transformations*. Instead of producting objects (as in the first definition), we are producting morphisms now.

We can also revisit the definition of terminal object and define it as follows:

Definition 3D *(Terminal Object, Alternative).* The **terminal object** of C is an object T s.t.

$$\hom(Z, T) \cong \{\top\}$$

TODO i heard the word terminal?? pull out T?? what is a limit here???

3.1 Coproduct

Definition 3.1A *(Coproduct).* A **coproduct** $(A + B)$ of two objects A, B in a category C is the *product* of A, B in C^{op}.

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "\eta_A"] \arrow[r, "\iota_A"'] & A + B \arrow[u, dashed] & B \arrow[ul, "\eta_B"'] \arrow[l, "\iota_B"] \end{tikzcd}$$

Intuitively, one can make sense of this syntax by thinking of $A + B$ as the disjoint union of A and B.

4 Syntactic Category of STLC $\mathcal{C}_{\mathbb{T}}$

Definition 4A *(Syntactic Category of STLC).* The **syntactic category** $\mathcal{C}_{\mathbb{T}}$ for a *simply typed λ-calculus* $\mathbb{T}$ is defined as follows:
- **Objects**: types
- **Morphisms**: $x : S \vdash t : T$

One can see it satisfies:
- **Id**:

$$\frac{}{x : S \vdash x : S}\,x$$

- **Comp**:

$$\frac{x : S \vdash t : T \qquad y : T \vdash u : U}{x : S \vdash u[t/y] : U}\,\textsc{Subst}$$

Now let's add product to our $\mathbb{T}$.

$$\frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}}\times_F \qquad \frac{z : Z \vdash a : A \qquad z : Z \vdash b : B}{z : Z \vdash (a, b) : A \times B}\times_I$$

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_A p : A}\times_{E_A} \qquad \frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_B p : B}\times_{E_B}$$

and confirm it satisfies the universal property of product,

$$\begin{tikzcd}
& Z \arrow[dl, "\eta_A"'] \arrow[d, dashed, "\times_I"'] \arrow[dr, "\eta_B"] & \\
A & p : A \times B \arrow[l, "\times_{E_A}"] \arrow[r, "\times_{E_B}"'] & B
\end{tikzcd}$$

Thus, there's an *intepretation* from $\mathbb{T}$ to the category $\mathcal{C}_{\mathbb{T}}$.

Remark 4A. The *syntactic category* $\mathcal{C}_{\mathbb{T}}$ is very helpful to define alternative semantics for $\mathbb{T}$. With $\mathcal{C}_{\mathbb{T}}$, one can construct morphisms from $\mathcal{C}_{\mathbb{T}}$ to other categories to give various different semantics to $\mathbb{T}$.

5 Functors

Definition 5A *(Functor).* A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

1. A function $\operatorname{ob} F : \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$
2. A function $F_{X,Y} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(FX, FY)$ for all $X, Y \in \operatorname{ob} \mathcal{C}$

such that
1. **(Id)** $F_{X,X}\, \mathrm{id}_X = \mathrm{id}_{FX}$ for all $X \in \operatorname{ob} \mathcal{C}$
2. **(Comp)** $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ f_{X,Y}(f)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C}$

Remark 5A. $\operatorname{ob} F$ and $F_{X,Y}$ are just notations. It doesn't mean F is a category.

Theorem 5A. Functors preserve isomorphisms.

Proof.

1. From $X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$ this holds: $FX \underset{Ff^{-1}}{\overset{Ff}{\rightleftarrows}} FY$
2. So, $Ff \circ Ff^{-1} = F(f \circ f^{-1}) = F\,\mathrm{id}_Y = \mathrm{id}_{FY}$ and vice versa. □

e.g.

1. *Id functor* $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$
2. *Functor comp* $G \circ F$:
   Given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, define $G \circ F : \mathcal{C} \to \mathcal{E}$ by
   - $\mathrm{ob}\,(G \circ F) : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{E} = \lambda X.\,\mathrm{ob}\,G(\mathrm{ob}\,F(X))$ TODO $(G \circ F)\,x = G(Fx)$
   - $(G \circ F)_{X,Y} : \mathrm{hom}_{\mathcal{C}}(X, Y) \to \mathrm{hom}_{\mathcal{E}}(GFX, GFY) = G_{FX,FY} \circ F_{X,Y}$

5.1 Category of Categories

Remark 5.1A *(Category of Categories).* The **category of categories** *Cat* is informally defined as follows:

- **Objects**: categories
- **Morphisms**: functors

Now we can define various functors on *Cat*.

Example 5.1A. Consider $\mathbb{1} = \{A\}$, and $X \in \mathrm{ob}\,\mathcal{C}$, there's functor $[X] : \mathbb{1} \to \mathcal{C}$ such that

1. $\mathrm{ob}_{[X]} = \mathrm{ob}\,\mathbb{1} \to \mathrm{ob}_{\mathcal{C}}$
2. $[X]_{Y,Z} : \mathrm{hom}_{\mathbb{1}}(Y, Z) \to \mathrm{hom}_{\mathcal{C}}(X, X) = \lambda\_.\,\mathrm{id}_X$

Intuitively, this is picking one object X out of $\mathcal{C}$.

Example 5.1B. Given $\mathcal{C}$, there is a functor $! : \mathcal{C} \to \mathbb{1}$ such that

1. $\mathrm{ob}\,! : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathbb{1} \coloneqq \lambda\_.A$
2. $!_{X,Y} : \mathrm{hom}_{\mathcal{C}}(X, Y) \to \mathrm{hom}_{\mathbb{1}}(A, A) \coloneqq \lambda\_.\,\mathrm{id}_A$

Remark 5.1B.

- $\mathbb{1}$ is terminal in *Cat* because for any category $\mathcal{C}$, there is a unique functor $\mathcal{C} \xrightarrow{!} \mathbb{1}$. It is unique because the only object in $\mathbb{1}$ is A, and the only morphism is id_A.
- Similarly, $\mathbb{0}$ is initial in *Cat* because for any category $\mathcal{C}$, there is a unique empty functor $\mathbb{0} \xrightarrow{¡} \mathcal{C}$.

e.g. Set $\hookrightarrow$ *Cat*

E.g. $\mathcal{C} \times \mathcal{D}$

- **Objects**: $\mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{D}$
- **Morphisms**: $\mathrm{hom}_{\mathcal{C} \times \mathcal{D}}((C, D), (C', D')) \coloneqq \mathrm{hom}_{\mathcal{C}}(C, C') \times \mathrm{hom}_{\mathcal{D}}(D, D')$

E.g. $\mathcal{D} + \mathcal{D}$

- **Objects**: $\mathrm{ob}\,\mathcal{C} + \mathrm{ob}\,\mathcal{D}$
- **Morphisms**:

$$
\begin{aligned}
\mathrm{hom}_{C+D}(C, C') &\coloneqq \mathrm{hom}_{\mathcal{C}}(C, C') \\
\mathrm{hom}_{C+D}(D, D') &\coloneqq \mathrm{hom}_{\mathcal{D}}(D, D') \\
\mathrm{hom}_{C+D}(C, D) &\coloneqq \emptyset \\
\mathrm{hom}_{C+D}(D, C) &\coloneqq \emptyset
\end{aligned}
$$

Example 5.1C *(Maybe Functor)*. Consider $\mathcal{C}$ with $+, \top$.
We can define a **Maybe functor** $\mathrm{Maybe} : \mathcal{C} \to \mathcal{C}$ that behaves like a haskell `Maybe`:
1. $\mathrm{Maybe}\ X \coloneqq X + T$
2. $\hom_{\mathrm{Maybe}\ \mathcal{C}}(X, Y)$ is depicted in the following diagram, while f is the input morphism and $f + \mathrm{id}_T$ is the result.

$$\begin{tikzcd}
& Y + \top & \\
X \arrow[ur, "\iota_1 \circ f"] \arrow[r, "\iota_1"] & X + \top \arrow[u, dashed, "f + \mathrm{id}_T"] & \top \arrow[ul, "\iota_2 \circ \mathrm{id}"'] \arrow[l, "\iota_2"']
\end{tikzcd}$$

6 CH-Lambek Correspondence

Define a category $\mathcal{C}_{\mathbb{T}}$ (syntactic category) for a *simply typed λ-calculus* $\mathbb{T}$ with
1. structural rules
2. rules for unit type
3. rules for product type
4. constant symbols C for types
5. constant symbols c for terms

Remark 6A. Recall that *objects* are types and *morphisms* are terms in $\mathcal{C}_{\mathbb{T}}$.

Theorem 6A. There is a *functor* $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ where D is a category with *products* and *terminal object* such that
1. $F(T)$ for all $T \in C$
2. $F(t)$ for all $t \in c$
3. $F(\mathrm{unit}) \coloneqq \mathbb{1}$
4. $F(S \times T) = F(S) \times F(T)$

TODO Proof that it is a functor.

Definition 6A *(Category of Models)*. $\mathcal{M}_{\mathbb{T}}$ is the category with
1. **Objects**: categories with terminal objects, products, interpretation $[C], [c]$ for each $T \in C, t \in c$.
2. **Morphisms**: functors that preserve terminal objects, products and interpretation. *E.g.* for any $F : \mathcal{C} \to \mathcal{D}, F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}}$

Fact. $\mathcal{C}_{\mathbb{T}}$ is the initial object of $\mathcal{M}_{\mathbb{T}}$. $\mathcal{D}$ is an arbitrary object of $\mathcal{M}_{\mathbb{T}}$.

Theorem 6B *(Lambek)*. The category of extensions of *STLC* with *function types* is **equivalent** to the category of categories with terminal objects, products, and *exponent objects*.

Remark 6B.
- product - product
- unit - terminal object
- function type - exponent object

Intuitively, isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ looks like:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

where $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$.

TODO This part needs clarification.

TODO Given a $\mathbb{T}$
1. Look at objects in $\mathcal{M}_{\mathbb{T}}$
2. Look at objects ?? with morphisms $\texttt{text}_{\mathbb{T}} \to \mathcal{D}$

7 Moral

1. Type theory is category theory.
2. Category of CCC contains categories of *real* objects (sets, topological spaces, types, etc.)
3. (*A category of a certain model*) can be used to show certain statements are not provable.
4. Type theory is an *informal language* for these categories.

8 Bonus: Inductive types $\mathbb{N}$

$\mathbb{N}$ in PL is defined as:

$$O : \mathrm{unit} \to \mathbb{N}$$
$$S : \mathbb{N} \to \mathbb{N}$$

Inspired by this, to make it a category, we want to define $\mathrm{unit} + \mathbb{N} \to \mathbb{N}$.

$$\begin{tikzcd}
\mathrm{unit} \arrow[r] \arrow[dr, "O"'] & \mathrm{unit} + \mathbb{N} \arrow[d, dashed, "O+S"] & \mathbb{N} \arrow[l] \arrow[dl, "S"] \\
 & \mathbb{N} &
\end{tikzcd}$$

This looks very similar to **induction**: we can generalize it a bit and say that $\mathbb{N}$ has O, S but that functions $\mathbb{N} \to A$ corresponds to

$$Z : \mathrm{unit} \to A$$
$$O : A \to A$$

i.e. $\mathrm{unit} + A \to A$

Naturally, functor *Maybe* $(\mathcal{C} \to \mathcal{C})$ might be helpful: $A \mapsto \mathrm{unit} + A$

Remark 8A. $\mathcal{C}$ could be syntactic category for a type theory

Definition 8A *(Maybe-Algebra).* Define a category alg_{Maybe} of Maybe-algebras:

- **Objects**: (A, a) where $A \in \text{ob}\,\mathcal{C}, a : \text{Maybe } A \to A$
- **Morphisms**: $(A, a) \xrightarrow{f} (B, b)$ are morphisms $A \xrightarrow{f} B$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} \text{unit} + A \arrow[r, "a"'] \arrow[d, "\text{id}_{\text{unit}} + f"'] & A \arrow[d, "f"'] \\ \text{unit} + B \arrow[r, "b"] & B \end{tikzcd}$$

i.e. f is a morphism that preserves the structure of Maybe-algebra.

$$f \circ a = b \circ \text{Maybe } f$$

Remark 8B. f is unique regarding a given pair of (A, a) and (B, b). TODO why?

Observation. ℕ is the initial object of alg_{Maybe}.

Remark 8C. Not every functor has a initial object in its category of algebras. It's a privilege of *polynomial endo-functors.*

Recall that we can prove *Set* is a model of *MLTT.*

Similarly,

Claim. ℕ is the *genuine* natural number.

Recall we can add *co-* prefixes to concepts in category theory lol,

Define *Co-alg*$_F$ on (A, a) where $A \in \text{ob}\,\mathcal{C}, a : A \to FA$

And we take its *terminal object.* That's how we get *coinductive types.*

9 References

Lambek + Scott. Intro to higher order categorical logic. (Background required: logic) Altenkirch: Category theory for the lazy functional programmers Ahrens, Wullaert: Category theory for programmers