Category Theory

Q: What is Category Theory?

A:

1. Understanding math objects via relations with each other, i.e. taking an external view, so you do not inspect the internal structure of the objects.
2. A whole independent field of study.

Q: What is a category?

A: objects + morphisms

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Group** - *objects*: groups; *morphisms*: group homomorphisms
- **Top** - *objects*: topological spaces; *morphisms*: continuous functions
- **Program Spec** - *objects*: program specifications; *morphisms*: programs that trun any program metting one spec into a program meeting another spec
- **Prop** - *objects*: propositions; *morphisms*: derivation/implication
- **Type** - *objects*: types; *morphisms*: derivation/function
- **Type Theory** - *objects*: type theories; *morphisms*: translations

Counterexample: what is the category of **probablistics**?

What if two kinds of notions of morphisms are all useful? **Double category**!

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Set** - *objects*: sets; *morphisms*: relations

Definition

A **category** $\mathcal{C}$ is

- a collection of objects ob $\mathcal{C}$
- for every $X, Y \in \text{ob } \mathcal{C}$, a collection of morphisms $\hom_{\mathcal{C}}(X, Y)$
- **Id**: for each $X \in \text{ob } \mathcal{C}$, an id morphism $\text{id}_X \in \hom_{\mathcal{C}}(X, X)$
- **Comp**: for each $f : X \to Y, g : Y \to Z$, a morphism $g \circ f : X \to Z$ in $\hom_{\mathcal{C}}(X, Z)$, *such that*
  - $f \circ \text{id}_X = f = \text{id}_Y \circ f$

    NOTE Normally we don't differentiate left and right identity. There was one particular notion that did, but then they were shown to be equivalent.
  - for any $f : X \to Y, g : Y \to Z, h : Z \to A, h \circ (g \circ f) = (h \circ g) \circ f$.

Commutative Diagrams

E.g. $f \circ \text{id}_X = f$

$$\begin{tikzcd} X \arrow[r, "\text{id}_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Notice how the two paths from X to Y yield the same morphism.

NOTE There are also *string diagrams*, which is very similar to *proof nets*.

Let's get back to a concrete example.

Prop Category

- ob Prop - the collection of propositions
- $$\hom_{\text{Prop}(P,Q)} = \begin{cases} \{\top\} & \text{if } P \to Q \\ \emptyset & \text{otherwise} \end{cases}$$
- **Id**: $\mathrm{id}_P = \top$ for all $P \in \text{ob Prop}$, because $P \to P$ is always true.
- **Comp**: by modus ponens, $Q \to P$ and $P \to R$ implies $Q \to R$. Properties: trivial.

TODO this is cat but i don't see how it is useful. all morphisms are trivial.. maybe it serves as a gentle introduction to the concept of category that does not serve any real-world purpose?

Terminal Object

A **terminal object** $(*)$ T in a category $\mathcal{C}$ is an object such that for every object $Z \in \mathcal{C}$, there is a unique morphism $Z \xrightarrow{!} T$.

E.g.

- a singleton set $(\{\top\})$ in *Set*
- unit type (1) in *Ty*
- *A in a category that contains exactly one object A.*

Isomorphic

Two objects X, Y in a category $\mathcal{C}$ are **isomorphic** if there exists morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

NOTE If $x \cong y$, we say

1. f is an *isomorphism* that $f : X \cong Y$, $f : X \xrightarrow{\sim} Y$
2. g as f^{-1}, and $f^{-1} : Y \cong X$, $f^{-1} : Y \xrightarrow{\sim} X$

E.g.

- *bijection* in *Set*
- *bi-implication* in *Prop*

Lemma. If $f : X \cong Y$, then for any Z, we have

- $f_* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.
- $f^* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.

Proof. (f_*)

- $\Rightarrow$

  For any $g \in \hom(Z, X)$, we have $g : Z \to X$. Also we know $X \xrightarrow{f} Y$. So, $f \circ g : Z \xrightarrow{g} X \xrightarrow{f} Y$.

- $\Leftarrow$

  For any $h \in \hom(Z, Y)$, we have $h : Z \to Y$. Also we know $Y \xrightarrow{f^{-1}} X$. So, $f^{-1} \circ h : Z \xrightarrow{h} Y \xrightarrow{f^{-1}} X$.

Lemma. Terminal objects are *unique up to isomorphism.*

Proof. Let T, T' be two terminal objects in a category $\mathcal{C}$.
Since T is terminal, there exists a unique morphism $f : T' \xrightarrow{!'} T$. Similarly, there exists a unique morphism $g : T \xrightarrow{!} T'$.
Now $! \circ !' : T \xrightarrow{!'} T' \xrightarrow{!} T = \mathrm{id}_T$. Similarly, $!' \circ ! = \mathrm{id}_{T'}$.

NOTE We know $T \xrightarrow{!'} T' \xrightarrow{!} T$ is id_T because T is terminal, which means for every object Z there is a *unique* morphism $Z \xrightarrow{!} T$. In particular, for $Z = T$, there is a *unique* morphism $T \xrightarrow{!} T$. And we know $\mathrm{id}_T : T \to T$ is one of the morphisms, so that's it.

Generally, there could be multiple morphisms to oneself, but in this case it's the terminal part that makes it unique.

Duality

Given a category $\mathcal{C}$, we can define its **dual category** $\mathcal{C}^{\{\mathrm{op}\}}$ by reversing the direction of all morphisms.

$$\mathrm{ob}\ \mathcal{C}^{\mathrm{op}} := \mathrm{ob}\ \mathcal{C}, \mathrm{hom}_{\mathcal{C}}^{\{\mathrm{op}\}}(X, Y) := \mathrm{hom}_{\mathcal{C}}(Y, X)$$

Lemma. $\mathcal{C}^{\mathrm{op}}$ is a category.

Lemma. If $X \cong Y$ in $\mathcal{C}$, then $X \cong Y$ in $\mathcal{C}^{\{\mathrm{op}\}}$.

Initial Object

An **initial object** I in a category $\mathcal{C}$ is the terminal object in $\mathcal{C}^{\mathrm{op}}$.

or,

An **initial object** I in a category $\mathcal{C}$ is an object such that for every object $Z \in \mathcal{C}$, there is a unique morphism $I \xrightarrow{!} Z$.

Lemma. Initial objects are *unique up to isomorphism.*
Proof. Follows dually from the uniqueness of terminal objects.

E.g.
- empty set ($\emptyset$) in *Set*
- empty type (0) in *Ty*
- false ($\perp$) in *Prop*

Product

A **product** of two objects A, B in a category $\mathcal{C}$ contains
1. an *object* $A \times B \in \mathrm{ob}(\mathcal{C})$
2. *morphisms* $A \times B \xrightarrow{\pi_A} A, A \times B \xrightarrow{\pi_B} B$,

   s.t. for any $Z \in \mathrm{ob}(\mathcal{C})$ and morphisms in the diagram, we have a **unique** morphism $U_Z : Z \xrightarrow{!} A \times B$ such that the diagram commutes.

$$\begin{tikzcd}
 & Z \arrow[dl, "\eta_A"'] \arrow[d, dashed, "U_z"] \arrow[dr, "\eta_B"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

NOTE This **uniqueness** depends on the specific choice of π_A and π_B. One must realize that *product* is not just an object, but also the morphisms π_A, π_B.

NOTE Only *some* categories have products.

NOTE If U_Z does not need to be unique, it's called a *weak product.*

E.g.
- cartesian product in *Set*
- $\wedge$ in *Prop*

Theorem The product $A \times B$ $(A \times B, \pi_1, \pi_2)$ of $A, B \in \mathrm{ob}(\mathcal{C})$ is unique up to *unique isomorphism*.
Proof.

1. Recall the very definition of product (an object, and two project morphisms). First we introduce $A \times B$ and $A \times' B$:

$$\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] & \\
A & & B
\end{tikzcd}$$

2. Compare the previous diagram with the universal property of product. "Instantiate" the universal property of $A \times' B$ and we have a unique morphism $A \times B \xrightarrow{!f} A \times' B$:

$$\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, dashed, "!f"] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] & \\
A & & B
\end{tikzcd}$$

WARN

One might attempt to do the following thing to close the proof:

$$\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, shift right, "!f"'] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[u, dashed, shift right, "!f'"'] & \\
A & & B
\end{tikzcd}$$

This is not correct, because for $A \times B$ and $A \times' B$ to be isomorphic one also needs to show

$$!f' \circ !f = \mathrm{id}_{A \times B}$$

which is not depicted in the diagram.

3. For sake of reasoning, let's duplicate $A \times B$ below $A \times' B$ and exercise the universal property of $A \times B$ with "apex" $A \times' B$ to show $A \times' B \xrightarrow{!g} A \times B$:

$$\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, "!f"'] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[d, dashed, "!g"'] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

4. We can also "instantiate" the universal property of the lower $A \times B$ with apex being the upper $A \times B$, to show there exists a unique morphism $A \times B \to A \times B$:

$$\begin{tikzcd}
& A \times B \arrow[ddl, "\pi_A"'] \arrow[ddr, "\pi_B"] \arrow[d, "!f"'] \arrow[dd, dashed, bend left] & \\
& A \times' B \arrow[dl, "\pi'_A"] \arrow[dr, "\pi'_B"'] \arrow[d, "!g"'] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

5. Note that $A \times B \xrightarrow{!g \circ !f} A \times B$, and also $A \times B \xrightarrow{\mathrm{id}_{A \times B}} A \times B$. By the previous reasoning, there exists only one morphism, so $!g \circ !f = \mathrm{id}_{A \times B}$. Similarly, $!f \circ !g = \mathrm{id}_{A \times' B}$. By uniqueness of $!f$ and $!g$, this isomorphism is *unique*.

WARN This *uniqueness* depends on a specific choice of π_A and π_B. In other words, if we only consider the product objects $A \times B$ and $A \times' B$, their isomorphism might not be unique because we have the freedom to choose different projection morphisms π_A, π_B and π'_A, π'_B, and they form their own unique isomorphism.

Alternative Definition 1 Define new category $\mathcal{C}_{A,B}$,

1. **Objects**: diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A & & B \end{tikzcd}$$

   It can also be called a *span* from A to B.
2. **Morphisms**: f shown in the following diagram

$$\begin{tikzcd} & Z \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A & & B \\ & Y \arrow[ul, "y_A"] \arrow[ur, "y_B"'] \arrow[uu, "f"] & \end{tikzcd}$$

Theorem. *Product* of A,B in $\mathcal{C}$ is the *terminal object* in $\mathcal{C}_{A,B}$.

Alternative Definition. Consider $A, B \in \mathrm{ob}(\mathcal{C})$, the product of A, B in an object $A \times B \in \mathrm{ob}(\mathcal{C})$ with the property that

$$\hom_{\mathcal{C}}(Z, A \times B) \cong \hom_C(Z, A) \times \hom_C(Z, B)$$

Theorem. This formation of product is equivalent to the previous two.

TODO This reminds me of *natural transformations*. Instead of producting objects (as in the first definition), we are producting morphisms now.

Def The terminal object of $\mathcal{C}$ is an object T s.t.

$$\hom(Z, T) \cong \{\top\}$$

TODO i heard the word terminal?? pull out T?? what is a limit here???

Coproduct

A **coproduct** $(A + B)$ of two objects A, B in a category $\mathcal{C}$ is the *product* of A, B in $\mathcal{C}^{\mathrm{op}}$.

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "\eta_A"] \arrow[r, "\iota_A"'] & A + B \arrow[u, dashed] & B \arrow[ul, "\eta_B"'] \arrow[l, "\iota_B"] \end{tikzcd}$$

Intuitively, one can make sense of this syntax by thinking of $A + B$ as the disjoint union of A and B.

Syntactic Category of STLC

For a *simply typed λ-calculus* Π, define its **syntactic category** $\mathcal{C}_\Pi$ as follows:

- **Objects**: types

- **Morphisms**: $x : S \vdash t : T$

One can see it satisfies:

- **Id**:

$$\frac{}{x : S \vdash x : S}\, x$$

- **Comp**:

$$\frac{x : S \vdash t : T \qquad y : T \vdash u : U}{x : S \vdash u[t/y] : U}\,\textsc{Subst}$$

Now let's add product to our Π.

$$\frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}}\,\times_F \qquad \frac{z : Z \vdash a : A \qquad z : Z \vdash b : B}{z : Z \vdash (a, b) : A \times B}\,\times_I \qquad \frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_A p : A}\,\times_{E_A}$$

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_B p : B}\,\times_{E_B}$$

$$\begin{tikzcd}
& Z \arrow[dl, "\eta_A"'] \arrow[d, dashed, "\times_I"'] \arrow[dr, "\eta_B"] & \\
A & p : A \times B \arrow[l, "\times_{E_A}"] \arrow[r, "\times_{E_B}"'] & B
\end{tikzcd}$$

Thus, there's an intepretation from Π to the category $\mathcal{C}_\Pi$. After that, one can construct morphisms from $\mathcal{C}_\Pi$ to other categories to give various different semantics to Π.