Category Theory

Q: What is Category Theory?

A:

1. Understanding math objects via relations with each other, i.e. taking an external view, so you do not inspect the internal structure of the objects.
2. A whole independent field of study.

Q: What is a category?

A: objects + morphisms

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Group** - *objects*: groups; *morphisms*: group homomorphisms
- **Top** - *objects*: topological spaces; *morphisms*: continuous functions
- **Program Spec** - *objects*: program specifications; *morphisms*: programs that trun any program metting one spec into a program meeting another spec
- **Prop** - *objects*: propositions; *morphisms*: derivation/implication
- **Type** - *objects*: types; *morphisms*: derivation/function
- **Type Theory** - *objects*: type theories; *morphisms*: translations

Counterexample: what is the category of **probablistics**?

What if two kinds of notions of morphisms are all useful? **Double category**!

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Set** - *objects*: sets; *morphisms*: relations

Definition

A **category** $\mathcal{C}$ is

- a collection of objects ob $\mathcal{C}$
- for every $X, Y \in \text{ob } \mathcal{C}$, a collection of morphisms $\hom_{\mathcal{C}}(X, Y)$
- **Id**: for each $X \in \text{ob } \mathcal{C}$, an id morphism $\mathrm{id}_X \in \hom_{\mathcal{C}}(X, X)$
- **Comp**: for each $f : X \to Y, g : Y \to Z$, a morphism $g \circ f : X \to Z$ in $\hom_{\mathcal{C}}(X, Z)$, *such that*
  - $f \bigcirc \mathrm{id}_X = f = \mathrm{id}_Y \bigcirc f$

    NOTE Normally we don't differentiate left and right identity. There was one particular notion that did, but then they were shown to be equivalent.
  - for any $f : X \to Y, g : Y \to Z, h : Z \to A, h \bigcirc (g \bigcirc f) = (h \bigcirc g) \bigcirc f$.

Commutative Diagrams

E.g. $f \bigcirc \mathrm{id}_X = f$

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Notice how the two paths from X to Y yield the same morphism.

NOTE There are also *string diagrams*, which is very similar to *proof nets*.

Let's get back to a concrete example.

Prop Category

- ob Prop - the collection of propositions
- $$\hom_{\text{Prop}(P,Q)} = \begin{cases} \{\top\} & \text{if } P \to Q \\ \emptyset & \text{otherwise} \end{cases}$$
- **Id**: $\mathrm{id}_P = \top$ for all $P \in \text{ob Prop}$, because $P \to P$ is always true.
- **Comp**: by modus ponens, $Q \to P$ and $P \to R$ implies $Q \to R$. Properties: trivial.

TODO this is cat but i don't see how it is useful. all morphisms are trivial.. maybe it serves as a gentle introduction to the concept of category that does not serve any real-world purpose?