Category Theory

Q: What is Category Theory?

A:

1. Understanding math objects via relations with each other, i.e. taking an external view, so you do not inspect the internal structure of the objects.
2. A whole independent field of study.

Q: What is a category?

A: objects + morphisms

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Group** - *objects*: groups; *morphisms*: group homomorphisms
- **Top** - *objects*: topological spaces; *morphisms*: continuous functions
- **Program Spec** - *objects*: program specifications; *morphisms*: programs that trun any program metting one spec into a program meeting another spec
- **Prop** - *objects*: propositions; *morphisms*: derivation/implication
- **Type** - *objects*: types; *morphisms*: derivation/function
- **Type Theory** - *objects*: type theories; *morphisms*: translations

Counterexample: what is the category of **probablistics**?

What if two kinds of notions of morphisms are all useful? **Double category**!

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Set** - *objects*: sets; *morphisms*: relations

Definition

A **category** $\mathcal{C}$ is

- a collection of objects ob $\mathcal{C}$
- for every $X, Y \in \text{ob } \mathcal{C}$, a collection of morphisms $\hom_{\mathcal{C}}(X, Y)$
- **Id**: for each $X \in \text{ob } \mathcal{C}$, an id morphism $\text{id}_X \in \hom_{\mathcal{C}}(X, X)$
- **Comp**: for each $f : X \to Y, g : Y \to Z$, a morphism $g \circ f : X \to Z$ in $\hom_{\mathcal{C}}(X, Z)$, *such that*
  - $f \bigcirc \text{id}_X = f = \text{id}_Y \bigcirc f$

    NOTE Normally we don't differentiate left and right identity. There was one particular notion that did, but then they were shown to be equivalent.
  - for any $f : X \to Y, g : Y \to Z, h : Z \to A$, $h \bigcirc (g \bigcirc f) = (h \bigcirc g) \bigcirc f$.

Commutative Diagrams

E.g. $f \bigcirc \text{id}_X = f$

$$\begin{tikzcd} X \arrow[r, "\text{id}_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Notice how the two paths from X to Y yield the same morphism.

NOTE There are also *string diagrams*, which is very similar to *proof nets*.

Let's get back to a concrete example.

Prop Category

- ob Prop - the collection of propositions
- $$\hom_{\text{Prop}(P,Q)} = \begin{cases} \{\top\} & \text{if } P \to Q \\ \emptyset & \text{otherwise} \end{cases}$$
- **Id**: $\text{id}_P = \top$ for all $P \in \text{ob Prop}$, because $P \to P$ is always true.
- **Comp**: by modus ponens, $Q \to P$ and $P \to R$ implies $Q \to R$. Properties: trivial.

TODO this is cat but i don't see how it is useful. all morphisms are trivial.. maybe it serves as a gentle introduction to the concept of category that does not serve any real-world purpose?

Terminal Object

A **terminal object** $(*)$ T in a category $\mathcal{C}$ is an object such that for every object $Z \in \mathcal{C}$, there is a unique morphism $Z \xrightarrow{!} T$.

E.g.

- a singleton set ($\{\top\}$) in *Set*
- unit type (1) in *Ty*
- *A in a category that contains exactly one object A.*

Isomorphic

Two objects X, Y in a category $\mathcal{C}$ are **isomorphic** if there exists morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

NOTE If $x \cong y$, we say
1. f is an *isomorphism* that $f : X \cong Y$, $f : X \xrightarrow{\sim} Y$
2. g as f^{-1}, and $f^{-1} : Y \cong X$, $f^{-1} : Y \xrightarrow{\sim} X$

E.g.

- *bijection* in *Set*
- *bi-implication* in *Prop*

Lemma. If $f : X \cong Y$, then for any Z, we have

- $f_* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.
- $f^* : \hom(Z, X) \cong_{Set} \hom(Z, Y)$.

Proof. (f_*)

- $\Rightarrow$

  For any $g \in \hom(Z, X)$, we have $g : Z \to X$. Also we know $X \xrightarrow{f} Y$. So, $f \circ g : Z \xrightarrow{g} X \xrightarrow{f} Y$.
- $\Leftarrow$

  For any $h \in \hom(Z, Y)$, we have $h : Z \to Y$. Also we know $Y \xrightarrow{f^{-1}} X$. So, $f^{-1} \circ h : Z \xrightarrow{h} Y \xrightarrow{f^{-1}} X$.

Lemma. Terminal objects are *unique up to isomorphism.*

Proof. Let T, T' be two terminal objects in a category $\mathcal{C}$.
Since T is terminal, there exists a unique morphism $f : T' \xrightarrow{!'} T$. Similarly, there exists a unique morphism $g : T \xrightarrow{!} T'$.
Now $! \circ !' : T \xrightarrow{!'} T' \xrightarrow{!} T = \text{id}_T$. Similarly, $!' \circ ! = \text{id}_{T'}$.

NOTE We know $T \xrightarrow{!'} T' \xrightarrow{!} T$ is id_T because T is terminal, which means for every object Z there is a *unique* morphism $Z \xrightarrow{!} T$. In particular, for $Z = T$, there is a *unique* morphism $T \xrightarrow{!} T$. And we know $\mathrm{id}_T : T \to T$ is one of the morphisms, so that's it.

Generally, there could be multiple morphisms to oneself, but in this case it's the terminal part that makes it unique.

Duality

Given a category $\mathcal{C}$, we can define its **dual category** $\mathcal{C}^{\{\mathrm{op}\}}$ by reversing the direction of all morphisms.

$$\mathrm{ob}\ \mathcal{C}^{\mathrm{op}} := \mathrm{ob}\ \mathcal{C}, \mathrm{hom}_{\mathcal{C}}^{\{\mathrm{op}\}}(X, Y) := \mathrm{hom}_{\mathcal{C}}(Y, X)$$

Lemma. $\mathcal{C}^{\mathrm{op}}$ is a category.

Lemma. If $X \cong Y$ in $\mathcal{C}$, then $X \cong Y$ in $\mathcal{C}^{\{\mathrm{op}\}}$.

Initial Object

An **initial object** I in a category $\mathcal{C}$ is the terminal object in $\mathcal{C}^{\mathrm{op}}$.

or,

An **initial object** I in a category $\mathcal{C}$ is an object such that for every object $Z \in \mathcal{C}$, there is a unique morphism $I \xrightarrow{!} Z$.

Lemma. Initial objects are *unique up to isomorphism.*
Proof. Follows dually from the uniqueness of terminal objects.

E.g.
- empty set ($\emptyset$) in *Set*
- empty type (0) in *Ty*
- false ($\perp$) in *Prop*