

# Category Theory

Q: What is Category Theory?

A:

1. Understanding math objects via relations with each other, i.e. taking an external view, so you do not inspect the internal structure of the objects.
2. A whole independent field of study.

Q: What is a category?

A: objects + morphisms

Example:

- **Set** - *objects*: sets; *morphisms*: functions
- **Group** - *objects*: groups; *morphisms*: group homomorphisms
- **Top** - *objects*: topological spaces; *morphisms*: continuous functions
- **Program Spec** - *objects*: program specifications; *morphisms*: programs that turn any program meeting one spec into a program meeting another spec
- **Prop** - *objects*: propositions; *morphisms*: derivation/implication
- **Type** - *objects*: types; *morphisms*: derivation/function
- **Type Theory** - *objects*: type theories; *morphisms*: translations

Counterexample: what is the category of **probabilistics**?

What if two kinds of notions of morphisms are all useful? **Double category!**

Example:

- **Set** - *objects*: sets; *morphisms*: **functions**
- **Set** - *objects*: sets; *morphisms*: **relations**

## Definition

A **category**  $\mathcal{C}$  is

- a collection of objects  $\text{ob } \mathcal{C}$
- for every  $X, Y \in \text{ob } \mathcal{C}$ , a collection of morphisms  $\text{hom}_{\mathcal{C}}(X, Y)$
- **Id**: for each  $X \in \text{ob } \mathcal{C}$ , an id morphism  $\text{id}_X \in \text{hom}_{\mathcal{C}}(X, X)$
- **Comp**: for each  $f : X \rightarrow Y, g : Y \rightarrow Z$ , a morphism  $g \circ f : X \rightarrow Z$  in  $\text{hom}_{\mathcal{C}}(X, Z)$ , such that
  - $f \circ \text{id}_X = f = \text{id}_Y \circ f$

**NOTE** Normally we don't differentiate left and right identity. There was one particular notion that did, but then they were shown to be equivalent.

- for any  $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow A, h \circ (g \circ f) = (h \circ g) \circ f$ .

## Commutative Diagrams

E.g.  $f \circ \text{id}_X = f$

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ & \searrow f & \downarrow f \\ & & Y \end{array}$$

Notice how the two paths from  $X$  to  $Y$  yield the same morphism.

**NOTE** There are also *string diagrams*, which is very similar to *proof nets*.

Let's get back to a concrete example.

### Prop Category

- **ob Prop** - the collection of propositions

- $$\text{hom}_{\text{Prop}}(P, Q) = \begin{cases} \{\top\} & \text{if } P \rightarrow Q \\ \emptyset & \text{otherwise} \end{cases}$$

- **Id**:  $\text{id}_P = \top$  for all  $P \in \text{ob Prop}$ , because  $P \rightarrow P$  is always true.
- **Comp**: by modus ponens,  $Q \rightarrow P$  and  $P \rightarrow R$  implies  $Q \rightarrow R$ . Properties: trivial.

**TODO** this is cat but i don't see how it is useful. all morphisms are trivial.. maybe it serves as a gentle introduction to the concept of category that does not serve any real-world purpose?

### Terminal Object

A **terminal object**  $(*) T$  in a category  $\mathcal{C}$  is an object such that for every object  $Z \in \mathcal{C}$ , there is a unique morphism  $Z \rightarrow T$ .

E.g.

- a singleton set  $(\{\top\})$  in *Set*
- unit type  $(1)$  in *Ty*
- $A$  in a category that contains exactly one object  $A$ .

### Isomorphic

Two objects  $X, Y$  in a category  $\mathcal{C}$  are **isomorphic** if there exists morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ .

**NOTE** If  $x \cong y$ , we say

1.  $f$  is an *isomorphism* that  $f : X \cong Y, f : X \xrightarrow{\sim} Y$
2.  $g$  as  $f^{-1}$ , and  $f^{-1} : Y \cong X, f^{-1} : Y \xrightarrow{\sim} X$

E.g.

- *bijection* in *Set*
- *bi-implication* in *Prop*

**Lemma.** If  $f : X \cong Y$ , then for any  $Z$ , we have

- $f_* : \text{hom}(Z, X) \cong_{\text{Set}} \text{hom}(Z, Y)$ .
- $f^* : \text{hom}(Z, X) \cong_{\text{Set}} \text{hom}(Z, Y)$ .

**Proof.** ( $f_*$ )

- $\Rightarrow$

For any  $g \in \text{hom}(Z, X)$ , we have  $g : Z \rightarrow X$ . Also we know  $X \xrightarrow{f} Y$ . So,  $f \circ g : Z \xrightarrow{g} X \xrightarrow{f} Y$ .

- $\Leftarrow$

For any  $h \in \text{hom}(Z, Y)$ , we have  $h : Z \rightarrow Y$ . Also we know  $Y \xrightarrow{f^{-1}} X$ . So,  $f^{-1} \circ h : Z \xrightarrow{h} Y \xrightarrow{f^{-1}} X$ .

**Lemma.** Terminal objects are *unique up to isomorphism*.

**Proof.** Let  $T, T'$  be two terminal objects in a category  $\mathcal{C}$ .

Since  $T$  is terminal, there exists a unique morphism  $f : T' \xrightarrow{!'} T$ . Similarly, there exists a unique morphism  $g : T \xrightarrow{!} T'$ .

Now  $! \circ !' : T \xrightarrow{!'} T' \xrightarrow{!} T = \text{id}_T$ . Similarly,  $!' \circ ! = \text{id}_{T'}$ .

**NOTE** We know  $T \xrightarrow{!} T' \xrightarrow{!} T$  is  $\text{id}_T$  because  $T$  is terminal, which means for every object  $Z$  there is a *unique* morphism  $Z \xrightarrow{!} T$ . In particular, for  $Z = T$ , there is a *unique* morphism  $T \xrightarrow{!} T$ . And we know  $\text{id}_T : T \rightarrow T$  is one of the morphisms, so that's it.

Generally, there could be multiple morphisms to oneself, but in this case it's the terminal part that makes it unique.

## Duality

Given a category  $\mathcal{C}$ , we can define its **dual category**  $\mathcal{C}^{\{\text{op}\}}$  by reversing the direction of all morphisms.

$$\text{ob } \mathcal{C}^{\text{op}} := \text{ob } \mathcal{C}, \text{hom}_{\mathcal{C}^{\{\text{op}\}}}(X, Y) := \text{hom}_{\mathcal{C}}(Y, X)$$

**Lemma.**  $\mathcal{C}^{\text{op}}$  is a category.

**Lemma.** If  $X \cong Y$  in  $\mathcal{C}$ , then  $X \cong Y$  in  $\mathcal{C}^{\{\text{op}\}}$ .

## Initial Object

An **initial object**  $I$  in a category  $\mathcal{C}$  is the terminal object in  $\mathcal{C}^{\text{op}}$ .

or,

An **initial object**  $I$  in a category  $\mathcal{C}$  is an object such that for every object  $Z \in \mathcal{C}$ , there is a unique morphism  $I \rightarrow Z$ .

**Lemma.** Initial objects are *unique up to isomorphism*.

**Proof.** Follows dually from the uniqueness of terminal objects.

E.g.

- empty set ( $\emptyset$ ) in *Set*
- empty type ( $()$ ) in *Ty*
- false ( $\perp$ ) in *Prop*

## Product

A **product** of two objects  $A, B$  in a category  $\mathcal{C}$  contains

1. an *object*  $A \times B \in \text{ob}(\mathcal{C})$
2. *morphisms*  $A \times B \xrightarrow{\pi_A} A, A \times B \xrightarrow{\pi_B} B$ ,

s.t. for any  $Z \in \text{ob}(\mathcal{C})$  and morphisms in the diagram, we have a **unique** morphism  $U_Z : Z \xrightarrow{!} A \times B$  such that the diagram commutes.

$$\begin{array}{ccccc} & & Z & & \\ & \eta_A \swarrow & & \searrow \eta_B & \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array}$$

**NOTE** This **uniqueness** depends on the specific choice of  $\pi_A$  and  $\pi_B$ . One must realize that *product* is not just an object, but also the morphisms  $\pi_A, \pi_B$ .

**NOTE** Only *some* categories have products.

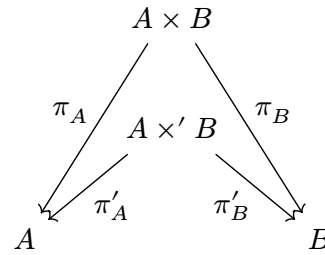
**NOTE** If  $U_Z$  does not need to be unique, it's called a *weak product*.

E.g.

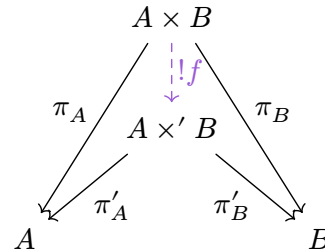
- cartesian product in *Set*
- $\wedge$  in *Prop*

**Theorem** The product  $A \times B$  ( $A \times B, \pi_1, \pi_2$ ) of  $A, B \in \text{ob}(\mathcal{C})$  is unique up to *unique isomorphism*.  
**Proof.**

1. Recall the very definition of product (an object, and two project morphisms). First we introduce  $A \times B$  and  $A \times' B$ :



2. Compare the previous diagram with the universal property of product. “Instantiate” the universal property of  $A \times' B$  and we have a unique morphism  $A \times B \xrightarrow{!f} A \times' B$ :



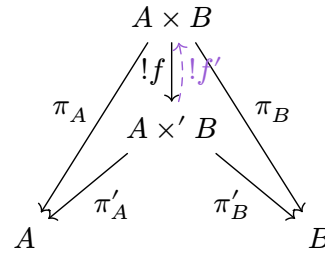
#### WARN

One might attempt to do the following thing to close the proof:

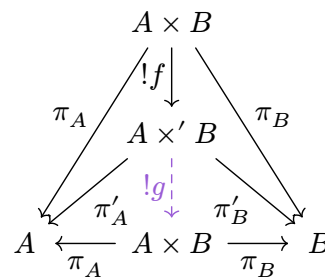
This is not correct, because for  $A \times B$  and  $A \times' B$  to be isomorphic one also needs to show

$$!f' \circ !f = \text{id}_{A \times B}$$

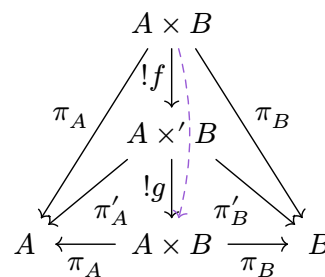
which is not depicted in the diagram.



3. For sake of reasoning, let's duplicate  $A \times B$  below  $A \times' B$  and exercise the universal property of  $A \times B$  with “apex”  $A \times' B$  to show  $A \times' B \xrightarrow{!g} A \times B$ :



4. We can also “instantiate” the universal property of the lower  $A \times B$  with apex being the upper  $A \times B$ , to show there exists a unique morphism  $A \times B \rightarrow A \times' B$ :

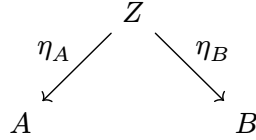


5. Note that  $A \times B \xrightarrow{!g \circ !f} A \times B$ , and also  $A \times B \xrightarrow{\text{id}_{A \times B}} A \times B$ . By the previous reasoning, there exists only one morphism, so  $!g \circ !f = \text{id}_{A \times B}$ . Similarly,  $!f \circ !g = \text{id}_{A \times' B}$ . By uniqueness of  $!f$  and  $!g$ , this isomorphism is *unique*.

**WARN** This *uniqueness* depends on a specific choice of  $\pi_A$  and  $\pi_B$ . In other words, if we only consider the product objects  $A \times B$  and  $A \times' B$ , their isomorphism might not be unique because we have the freedom to choose different projection morphisms  $\pi_A, \pi_B$  and  $\pi'_A, \pi'_B$ , and they form their own unique isomorphism.

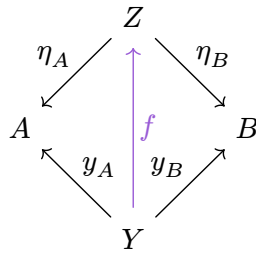
**Alternative Definition 1** Define new category  $\mathcal{C}_{A,B}$ ,

1. **Objects:** diagrams of the form



It can also be called a *span* from  $A$  to  $B$ .

2. **Morphisms:**  $f$  shown in the following diagram



**Theorem.** Product of  $A, B$  in  $\mathcal{C}$  is the *terminal object* in  $\mathcal{C}_{A,B}$ .

**Alternative Definition.** Consider  $A, B \in \text{ob}(\mathcal{C})$ , the product of  $A, B$  in an object  $A \times B \in \text{ob}(\mathcal{C})$  with the property that

$$\text{hom}_{\mathcal{C}}(Z, A \times B) \cong \text{hom}_{\mathcal{C}}(Z, A) \times \text{hom}_{\mathcal{C}}(Z, B)$$

**Theorem.** This formation of product is equivalent to the previous two.

**TODO** This reminds me of *natural transformations*. Instead of producing objects (as in the first definition), we are producing morphisms now.

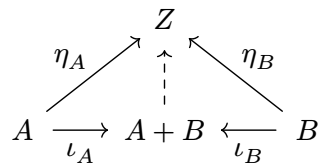
**Def** The terminal object of  $\mathcal{C}$  is an object  $T$  s.t.

$$\text{hom}(Z, T) \cong \{\top\}$$

**TODO** i heard the word terminal?? pull out T?? what is a limit here???

## Coproduct

A **coproduct** ( $A + B$ ) of two objects  $A, B$  in a category  $\mathcal{C}$  is the *product* of  $A, B$  in  $\mathcal{C}^{\text{op}}$ .



Intuitively, one can make sense of this syntax by thinking of  $A + B$  as the disjoint union of  $A$  and  $B$ .

## Syntactic Category of STLC

For a *simply typed*  $\lambda$ -calculus  $\Pi$ , define its **syntactic category**  $\mathcal{C}_{\Pi}$  as follows:

- **Objects:** types

- **Morphisms:**  $x : S \vdash t : T$

One can see it satisfies:

- **Id:**

$$\frac{}{x : S \vdash x : S} x$$

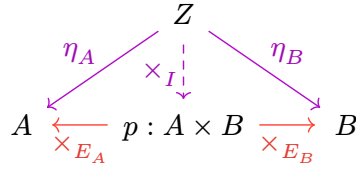
- **Comp:**

$$\frac{x : S \vdash t : T \quad y : T \vdash u : U}{x : S \vdash u[t/y] : U} \text{SUBST}$$

Now let's add product to our  $\Pi$ .

$$\frac{A \text{ type} \quad B \text{ type}}{A \times B \text{ type}} \times_F \quad \frac{z : Z \vdash a : A \quad z : Z \vdash b : B}{z : Z \vdash (a, b) : A \times B} \times_I \quad \frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_A p : A} \times_{E_A}$$

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_B p : B} \times_{E_B}$$



Thus, there's an interpretation from  $\Pi$  to the category  $\mathcal{C}_\Pi$ . After that, one can construct morphisms from  $\mathcal{C}_\Pi$  to other categories to give various different semantics to  $\Pi$ .